ROBUSTNESS OF INFINITE DIMENSIONAL STOCHASTIC SYSTEMS

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Abstract

The subject of this thesis is to study the problems of robust stability and robust stabilization for linear deterministic systems on real Hilbert spaces which are subjected to Lipschitzian stochastic structured multi-perturbations. within the framework of stability radii.

First we consider the case where the operators describing the structure of the perturbations are bounded. We establish characterizations of the stability radius in terms of a Lyapunov equation and the corresponding inequalities. These characterizations are used to obtain a computational formula for this radius.

Then, we study the problem of maximizing the stability radius by state feedback. We establish conditions for the existence of suboptimal controllers in terms of a Riccati equation. We showed also how the supremal stability radius can be determined in terms of this equation.

Finally, we investigate the robustness of stability in the case where the operators structure are unbounded. We show how we can generalize the results established in the bounded case for this case. We characterize the stability radius in terms of a Lyapunov equation similar to the one used in the bounded case. These characterizations enable us to determine a lower bound for the stability radius.

Key words: Wiener process, Stochastic differential equation, Exponential stability, Mean square stability, Robustness, Stability radius, Lyapunov equation, Riccati equation.

Contents

List of abbreviations

We list the notations that will be used in the thesis. Let H, U, V be real separable Hilbert spaces.

$L(U,H)$	The space of bounded linear operators form U to H.
$L(H)$	The space of bounded linear operators form H to H.
$\|.\|$	The norm in H.
$\langle ., \ .\rangle$	The inner product in H.
$P \succ 0$	$P \in L(H)$ is positive ($\langle Pz, \ z\rangle > 0$, for all $z \in H$).
$P \succeq 0$	$P \in L(H)$ is nonnegative ($\langle Pz, \ z\rangle \geq 0$, for all $z \in H$).
$L^{+}(H)$	The set of self-adjoint linear bounded operators $P \in L(H)$ such that $P \succeq 0$.
$Lip\,(Y,U)$	The set of Lipschitzian functions Δ, such that $\Delta : Y \longrightarrow U$ and $\Delta(0) = 0$.
$L^{p}\left((0,T)\,,\ H\right), \, p \geq 1$	The space of strongly measurable functions $f(t)$ with $\lvert f(t)\rvert^{p}$ is integrable.
$C^{1}\left((0,T)\,,\ H\right)$	The space of strongly continuously differentiable functions on $(0,T)$ with values in H.
$L^{2}\left(\Omega, \mu, \ H\right)$	The space of square integrable $H-$valued functions on the probability space $(\Omega, \mathcal{F}, \mu)$
$\mathcal{E}\ (x)$	The expectation of x.

Chapter 1

Introduction

In this thesis, we study the robust stability and robust stabilization problems for linear infinite dimensional systems, subjected to structured stochastic perturbations. We investigate these problems using the stability radii approach. In this chapter, we give an overview of the extensive research efforts in this approach and our main contributions in this subject.

1.1 Robust stability and stabilization problems of linear systems subjected to deterministic perturbations

The first step in most applications of mathematics is to determine a mathematical model for the system under investigation. The model may be used in a number of different ways. For example a mathematical and computational analysis of the model often leads to a better understanding of real physical system it represents. From a more practical viewpoint the model can be used to make predictions about the future behavior of the system, or to design algorithms of automatic control which ensure that the system behaves in some desirable fashion. However, in each of these applications it is of fundamental importance to keep in mind that the model is only a model, its behavior and that of the real system might be quite different. The origins and causes of this possible discrepancy are many and in the systems theory literature are collectively referred to as model uncertainties:

1. **Parameter uncertainty**: The model may depend on some physical parameters which are not known precisely.

2. **Unknown inputs and neglected dynamics:** A system is usually in dynamic interaction with its environment and it is often not clear where the boundary of the system should be drawn. Uncertainties arise if parts of the real system dynamics are not accounted for the model and if the inputs to the system from the environment are not accurately known.

3. **Model simplification**: Although an accurate complex model of the real physical system may be available, it is often necessary to simplify this for the purpose of analysis and design. For example, nonlinearities and time-variations are neglected.

4. **Discretization and rounding errors**: If simulations are carried out on

a computer, discretization methods must be applied and rounding errors are introduced which will lead to unknown nonlinear model perturbations.

To be true it has to be determined that the approximate model adequately describes the features of the system one is interested in. A special case of this problem is to determine whether the nominal system adequately describes stability properties of the actual system, or whether the expected difference between the real system and the nominal model is small enough such that stability of the model implies stability of the real system. If this is the case one could say that the stability of the model is sufficiently robust.

1.1.1 Stability radii theory

The issue of robustness of stability has been prominent in the literature on control theory over the last two decades. An important state-space approach to robustness analysis is the stability radii theory. The notion of stability radius was first introduced by Hinrichsen and Pritchard [34], although the idea behind it can be found in many different fields, see e.g.[70]. The distance from instability has been analyzed by [75] and Hinrichsen and Pritchard [34] for time-invariant finite-dimensional systems. It is the size of the smallest perturbation Δ which results a time-invariant system $\dot{x}(t) = (A + D\Delta E)\,x(t)$, $t \geq 0$, that is not exponentially stable. An important advantage of this approach is that it introduces concepts and techniques which can be generalized to different classes of systems and different kinds of perturbations.

For state space systems of the form $\dot{x}(t) = Ax(t)$ or $x(t + 1) = Ax(t)$,

$A \in \pounds(K^n)$, $K = \mathbb{R}$ or $\mathbb{C}$, linear perturbations of the form

1. $A \to A_\Delta = A + \Delta$, $\Delta \in L(K^n)$: unstructured perturbations

2. $A \to A_\Delta = A + D\Delta E$, $\Delta \in L(K^q, K^l)$: simple structured perturbations

3. $A \to A_\Delta = A + \sum_{i=1}^{N} D_i \Delta_i E_i$, $\Delta_i \in L(K^q, K^l)$: multi-perturbations

   has been considered, where D, E, D_i, E_i are given operators defining the perturbation structure.

Depending upon whether complex or real disturbances Δ are considered the stability radius is called complex or real, respectively. It is important to note that these two stability radii are in general distinct. In the complex case a fairly theory of the complex stability radius $r_{\mathbb{C}}$ is available for perturbations of the form (1) and (2), for finite dimensional systems with deterministic perturbations for both continuous and discrete-time systems (see the survey [32]). It also extends to infinite dimensional systems described by semigroups of operators on a complex Banach space, see [66]. The problem of computing the stability radius of positive linear systems under multi-perturbations has been solved recently in [62].

For time-varying systems the results are less satisfactory and there is no computational formula available for the stability radius, although there are lower bounds which can be significantly improved by scaling techniques (see [41], [33]). A formula for the stability radius of time-varying systems with respect to linear dynamical causal perturbations has been developed in [50]. Stability radii for a wide class of linear infinite-dimensional time-varying systems under structured multi-perturbations have been studied in [37]. Wirth [78] extends their results for discrete-time systems.

The problem of calculation the real stability radius for simple structured perturbations has been solved in [69]. However, when it comes to either real perturbations of more complicated structured or to any type of perturbations of time-varying systems the theory is far from complete.

1.1.2 Robust stabilization

The problem of stabilization is to design a state feedback control law that assures stability of systems with respect to some structured perturbations. This stabilization problem for perturbed linear systems has received considerable attention. A prominent method which has attracted many researches is the H_∞-method [30]. The standard H_∞-control problem was first formulated for finite-dimensional time-invariant systems and several approaches to solve this problem were adopted (see [18], [72]). The H_∞-control problem for infinite dimensional systems have been studied in several papers (see [74], [59], [49]). In [36], It is shown that there is a close relationship between the theories of stability radii and the H_∞-theory for the special case where stability radii with respect to complex perturbations are considered. In fact, in this case the problem of maximizing the stability radius by state feedback control is equivalent to a singular H_∞-control problem. The problem of maximizing the stability radius of a linear discrete-time system has been considered in [51]. Pritchard and Townely [65] analyzed similar problem for infinite dimensional systems with unbounded perturbations.

1.2 Robust stability and stabilization problem of systems subjected to stochastic perturbations

In this section we present an overview of some existing results on the robustness problems of linear stochastic systems. The interest in this topic is motivated by the variety of random phenomena arising in physical, engineering, biological, and social processes. The study of stochastic systems has a long history, but two distinct classes of such systems drew much attention in the control literature, namely stochastic systems subjected to white noise perturbations and systems with Markovian jumping. At the same time, the remarkable progress in recent decades in the control theory of deterministic systems strongly influenced the research effort in the stochastic area. Thus, the modern treatments of stochastic systems include optimal control, robust stability and robust stabilization for both stochastic systems corrupted with white noise and systems with jump Markov perturbations.

1.2.1 Robust stability

In the finite dimensional case there has been a good deal of research on the robustness stability of stochastic systems with multiplicative noise, for both continuous and discrete-time systems (see [20] and references in). El Bouhtouri and Pritchard [21] introduced the notion of stability radii for continuous-time systems subjected to simple structured stochastic perturbation. They show that the stability radii can be characterized in terms of a Laypunov equation. Hinrichsen and Pritchard [39] considered continuous-time systems subjected to stochastic structured multi-perturbations. They derived a precise characterization of the corresponding stability radius via scaling techniques (i.e. multiplying E_j, by a positive α_j and D_j by α_j^{-1}). Moreover, they showed that the real and complex stability radii coincide for stochastic perturbations of the above kind. El Bouhtouri, Hinrichsen and Pritchard [25] investigated the corresponding problem for discrete-time systems. Some results for the stability radius for stochastic systems with both deterministic and stochastic parameter uncertainties can be found in [23] and [24]. Time-varying stochastic systems with multiplicative white noise are considered in [60], and systems with Markovian jumping in [61]. In the case of stochastic systems with state multiplicative noise and jump Markov perturbations, some estimations on the stability radius are given in [19] and [20].

For infinite dimensional systems, there are few papers dealing with robustness issues for this class of systems. Brusin and Ugrinnovskii [5] introduced stochastic infinite dimensional counterparts of the Kalman-Yakubovich Lemma to provide conditions for stability of infinite dimensional systems with nonlinear and stochastic uncertainties. Hinrichsen and Pritchard [38] characterized a stability radius for linear deterministic systems subjected to structured multi-perturbations.

1.2.2 Robust stabilization

The stabilization of stochastic systems with multiplicative white noise has been studied since the late sixties, particularly in the context of linear quadratic optimal control; see, e.g.[77]. The subject of robust stabilization is of more recent vintage. Recently, a number of papers have been published which deal with robust stabilization problems in the spirit of H_∞-control or the stability radius approach. State feedback H_∞-control for linear systems with multiplicative white noise has been studied in several works. Among them we cite [73], [26], [40], for time invariant systems. For time varying systems, corresponding results can be found in [19]. In the Markovian situation, the problem has been addressed in [61] and [20]. The maximization of stability radius via state feedback, of deterministic systems subjected to stochastic single perturbation was considered in [22]. For systems with multi-perturbations the problem of maximizing the stability radii by dynamic output feedback was studied in [39] and [25]. To our knowledge, this problem has not been considered for infinite dimensional systems.

1.3 Main contributions of the thesis

Stochastic differential equations in infinite dimensional spaces are motivated by the development of analysis and the theory of stochastic processes itself such as stochastic partial differential equations and stochastic delay differential equations on the one hand, and by such topics as stochastic control, population biology and turbulence in applications on the other. Stochastic stability for linear evolution equations in Hilbert spaces has been treated extensively in the literature (see references [4], [7], [44], [46], [45], [57]). However, there are few papers dealing with robustness issues for this class of systems [5], [1]. In this thesis we use the framework of stability radii to study robust stability and robust stabilization problems for infinite dimensional systems with stochastic structured multi-perturbations.

We consider the system

$$dx(t) = Ax(t)dt + \sum_{i=1}^{N} D_i \Delta_i \left(E_i x(t) \right) dw_i(t) \tag{1.1}$$

where A is the infinitesimal generator of an exponentially stable semigroup $S(t)$ on a real Hilbert space H, $(w_i(t))_{t \geq 0}$, $i \in \{1, ..., N\}$, are independent real Wiener processes and D_i, E_i, $i \in \{1, ..., N\}$, are linear bounded operators defining the structure of the perturbations. We first establish, by adapting the approach used in [39], characterizations of the stability radius in terms of a Lyapunov equation and the corresponding inequalities. Then we show how we can combine these characterizations with scaling techniques to obtain a computational formula for this radius . This result was stated in [38] without proof.

The second contribution of this thesis concerns the problem of maximizing the stability radius of systems of the form

$$dx(t) = Ax(t)dt + \sum_{i=1}^{N} D_i \Delta_i \left(E_i x(t) \right) dw_i(t) + Bu(t)dt \tag{1.2}$$

by static feedback. Following the method developed in [22] we establish conditions for the existence of suboptimal controllers. These are expressed in terms of a parametrized Riccati equation and a number of linear operator inequalities. From these conditions we characterize the supremal stability radius.

Our last contribution concerns the robustness of stability for system (1.1) when the generator A is subjected to stochastic unbounded perturbations. We consider the case where A is the generator of an analytic semigroup and the perturbation are of single type. This abstract model covers the case of parabolic equations with boundary and pointwise noise. Ichikawa [47] proposed a semigroup model for parabolic equations with boundary and pointwise noise and obtained existence, uniqueness and regularity of their solution. Semigroup models for boundary noise can be also found in [58], [14], [16], [29], [54],[2]. The stability of these systems was studied in [48]. He established the equivalence of mean square stability and the existence of a Lyapunov type equation. We first establish an existence and uniqueness theorem. It is proved by a standard fixed point argument along the lines of [47]. Then, we investigate the robustness problem. We characterize the stochastic stability radius in terms of a Lyapunov equation similar to the one obtained for the bounded structure case. and different from the one used in [48]. These characterizations enable us to determine a lower bound for the stability radius under perturbation of unbounded structure. The main problem here is to construct the smallest destabilizing perturbation Δ. Under an additional assumption we are able to prove that the lower bound is in fact equal to the stability radius.

1.4 Organization of the thesis

To facilitate the reading of the thesis, we give a brief description of the material contained in the succeeding chapters.

Chapter 2: This chapter contains some material which will be used in the thesis such that: Semigroup theory, concepts of mild and strong solutions of deterministic evolution equations, exponential stability, stabilizability, random variables, Wiener processes, stochastic integrals in Hilbert spaces, concepts of the solution of stochastic evolution equations and mean square stability.

Chapter 3: The purpose of this chapter is to study robust stability of system (1.1) by the stability radius approach. First, we define the corresponding stability radius, then we establish characterizations of this radius. Our starting point is analogous to the setting in the finite dimensional case. We define an input-output operator associated with the perturbed system and obtain a formula for its norm. A computable formula for the stability radius is given in terms of a Lyapunov equation. This is carried out first by obtaining a lower bound in terms of the norm of the input-output operator of a scaled system and then constructing a nonlinear perturbation which destabilizes the system and whose norm is equal to the lower bound. These characterizations enable us to obtain a computational formula for the stability radius. Characterizations of the stability radius in terms of a Lyapunov inequality are also given. We conclude the chapter with some examples where we illustrate the calculus of the stochastic stability radius and we compare our results with some existing results. We consider two examples of infinite-dimensional systems which occur most frequently in the applications; the heat and the wave equations. Delay systems are also considered. We calculate the stability radii of a one and two dimensional systems which are subjected to a single perturbation.

Chapter 4: In this chapter we investigate how the stability radius of a stochastic perturbed system can be improved by state feedback. We consider controlled stochastic systems described by (1.2). Following the approach developed in [22], we obtain necessary and sufficient conditions for the existence of state feedback controllers that stabilize the system and achieve a stability radius larger than a specific bound. Stabilizing state-feedback with this property are constructed by solving a parametrized Riccati equation, and it is shown that the supremal stability radius can be determined via this equation. Examples are given in which we apply the obtained result to calculate the supremal stability radius.

Chapter 5: We recall that in Chapter 3 our basic model was

$$\begin{aligned} dx(t) &= Ax(t)dt + \sum_{i=1}^{N} D_i\Delta_i\left(E_i x(t)\right)dw_i(t),\ t > 0 \\ \|\Delta_i\|_L &< \sigma,\ i \in \overline{N}. \end{aligned}$$

where $\left((D_i, E_i)_{i\in\overline{N}}\right)$ is a given family of linear bounded operators. However, the assumptions that D_i, E_i are bounded operators is very restrictive and does not allow us to consider many examples of practical importance such that boundary perturbations for systems described by partial differential equations. In this chapter we show how we can extend the theory of Chapter 3 to a class of unbounded perturbations. We begin with recalling the notion of fractional powers of closed operators which has an important interest in this chapter. Under some assumptions on the perturbation structure, we show that we can establish characterizations of the stability radius via a Lyapunov equation similar to the one used in the bounded case. We give a lower bound for the corresponding stability radius, and we show that we can obtain a computational formula for the stability radius under extra assumptions. We Illustrate the theory by some examples, investigated in [48].

Chapter 2

Stochastic evolution equations

2.1 Introduction

In this chapter we recall some basic definitions and properties which will be used in this thesis. At first, we collect basic results from the theory of semigroups. Then we consider stochastic evolution equations. We recall definitions concerning random variables, Wiener processes and stochastic integrals in Hilbert spaces. Different concepts of the solution of stochastic evolution equations are defined. Mean square stability is also considered. Much material in Section 1 is classical and taken mainly from [10], [68], [8]. For stochastic systems, much material is taken from [8], [44], [57] and [13].

2.2 Semigroup theory

Semigroups naturally arise when we wish to extend exponential functions to infinite dimensions. The semigroup theory enables us to present a unified treatment of a wide class of infinite-dimensional systems and finite-dimensional ones. In this section we recall some basic properties of semigroups which will be used later.

2.2.1 Definition and properties

Let H be a real separable Hilbert space. We recall at first the definition of a semigroup.

Definition 2.1 *a strongly continuous semigroup is an operator -valued function $S(t)$ from $\mathbb{R}^+$ to $L(H)$ that satisfies the following properties:*

1. $S(t+s) = S(t)S(s)$ *for any* s, $t \geq 0$
2. $S(0) = I_H$,
3. $\|S(t)z - z\| \to 0$ *as* $t \to 0^+$, *for any* $z \in H$.

We shall use the standard abbreviation C_0-semigroup for a strongly continuous semigroup.

Example 2.2 *Let $A \in L(H)$, then*

$$S(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{fact\,(n)}$$

is a C_0-semigroup..

Some elementary properties of semigroups are given in the following theorem [10].

Theorem 2.3 *A strongly continuous semigroup on a Hilbert space H $S(t)$ has the following properties*

1. $\|S(t)\|$ *is bounded on every finite subinterval of* $[0, \infty)$;

2. $S(t)$ *is strongly continuous for all* $t \in [0, \infty)$;

3. *If* $\omega_0 = \inf_{t>0} \frac{1}{t} (\log \|S(t)\|)$, *then* $\omega_0 = \lim_{t \to +\infty} \frac{1}{t} (\log \|S(t)\|) < +\infty$,

4. *For all* $z \in H$ *we have that*

$$\frac{1}{t} \int_0^t S(s) z ds \to z \text{ as } t \to 0^+$$

5. *for any* $\omega > \omega_0$, *there exists a constant* M_ω *such that* $\|S(t)\| \leq M e^{\omega t}$ *for all* $t \geq 0$.

   This constant ω_0 *is called the growth bound of the semigroup.*

Let $D(A)$ denote the subspace of all elements such that $(S(t)z - z)/t$ converge in H as $t \to 0^+$ and define the operator on $D(A)$:

$$Az = \lim_{t \to 0^+} (S(t)z - z)/t \tag{2.1}$$

Definition 2.4 *The operator A given by (2.1) is the infinitesimal generator of the semigroup $S(t)$.*

The result below which is known as the Hille-Yosida Theorem provide a complete characterization of infinitesimal generators [10].

Theorem 2.5 (Hille-Yosida Theorem) *A necessary and sufficient condition for a closed, densely defined, linear operator A on a Hilbert space H to be the infinitesimal generator of a C_0-semigroup is that there exist real numbers M, ω, such that for all real number $\lambda > \omega$, $\lambda \in \rho(A)$, the resolvent set of A, and*

$$\|R(\lambda,\ A)^r\| \leq \frac{M}{(\lambda - \omega)^r}, \quad \text{for all } r \geq 1,$$

where $R(\lambda,\ A) = (\lambda I - A)^{-1}$ is the resolvent operator. In this case

$$\|S(t)\| \leq Me^{\omega t}$$

2.2.2 Inhomogenuous differential equations

Consider the Cauchy problem

$$\begin{aligned} \frac{dz(t)}{dt} &= Az(t),\ t \geq 0, \\ z(0) &= z_0 \end{aligned}$$

If A is the infinitesimal generator of a C_0-semigroup $S(t)$, the unique solution of this homogeneous problem is given by $z(t) = S(t)z_0$. Now let us consider the abstract inhomogenuous Cauchy problem

$$\frac{dz(t)}{dt} = Az(t) + f(t),\ t \geq 0,\ z(o) = z_0, \tag{2.2}$$

where for the moment we shall assume that $f \in C([0,\ T];\ H)$. First we have to define what we mean by a solution of (2.2), and we begin with the notion of a classical solution.

Theorem 2.6 *[10]If A is the infinitesimal generator of a C_0-semigroup $S(t)$ on a Hilbert space H , $f \in C^1([0,\ T];\ H)$ and $z_0 \in D(A)$, then*

$$z(t) = S(t)z_0 + \int_0^t S(t-s)f(s)ds \tag{2.3}$$

is continuously differentiable on $[0,T]$ and it is the unique classical solution of (2.2).

The conditions of Theorem 2.6 are too strong for control applications, where in general we do not wish to assume that $f \in C^1([0,T];\ H)$. So we introduce the following weaker concept of a solution of (2.2).

Definition 2.7 *If $f \in L^p([0,T];\ H)$ for a $p \geq 1$ and $z_0 \in H$, then we call (2.3) a mild solution of (2.2) on $[0,T]$.*

2.2.3 Stability and stabilizability

One of the most important aspects of systems theory is that of stability. Consider the linear equation

$$\frac{dz(t)}{dt} = Az(t),\ z(0) = z_0 \in H. \tag{2.4}$$

where A is the infinitesimal generator of a C_0- semigroup $S(t)$, $t > 0$. We shall be concerned with the following concept of stability.

Definition 2.8 *A C_0-semigroup $S(t)$, on a Hilbert space H is exponentially stable if there exist positive constants M and ω such that*

$$\|S(t)\| \leq Me^{-\omega t} \text{ for } t \geq 0.$$

ω is called the decay rate, and the supremum over all possible values of ω is the stability margin of $S(t)$.

The following theorem established in [10], is an extension of well-known finite dimensional results proved by Datko [12].

Theorem 2.9 *Suppose that A is the infinitesimal generator of the C_0-semigroup $S(t)$ on the Hilbert space H. The following statements are equivalent.*

1. *$S(t)$ is exponentially stable,*

2. *There exists a self-adjoint nonnegative operator $P \in L(H)$ which satisfies the Lyapunov equation*

$$\langle Az,\ Pz\rangle + \langle Pz,\ Az\rangle = -\langle z,\ z\rangle \text{ for all } z \in D(A) \tag{2.5}$$

3. *For every $z \in H$ there exists a positive constant $\gamma_z > 0$ such that*

$$\int_0^{+\infty} \|S(t)z\|^2\, dt \leq \gamma_z.$$

Remark 2.10 *If $S(t)$ is exponentially stable,we said that the system (2.4) is L^2-stable.*

Explicit formula for the solution of a Lyapunov equation is given in the following lemma given in [44].

Lemma 2.11 *Let $S(t)$ be an exponentially stable semigroup on H with infinitesimal generator A and let $Q \in L(H)$ be a nonnegative operator. Then the operator P defined by*

$$Pz = \int_0^{+\infty} S^*(t)QS(t)zdt$$

is well-defined and nonnegative and satisfies the equation

$$\langle Az,\ Pz\rangle + \langle Pz,\ Az\rangle + \langle Qz,\ z\rangle = 0 \text{ for all } z \in D(A) \tag{2.6}$$

Conversely, if P is self-adjoint and satisfies the equation (2.6), P is represented by the above integral.

Let U and V be Hilbert spaces and B, C, R linear bounded operators belonging respectively to the spaces $L(U,H)$, $L(H,V)$ and $L(U,U)$, where R is assumed to be an invertible positive operator. Consider the system

$$\frac{dz(t)}{dt} = Az(t) + Bu(t),\ z(0) = z_0 \in H.$$

where A is the infinitesimal generator of a C_0- semigroup $S(t)$, $t > 0$, on the Hilbert space H and $u \in L^2(0,\infty;\ U)$.We recall now the definitions of the stabilizability and detectability.

Definition 2.12 *If there exists an $F \in L(H,\ U)$ such that $A + BF$ generates an exponentially stable C_0-semigroup $S_{BF}(t)$, then we say that $(A,\ B)$ is exponentially stabilizable.*

Definition 2.13 *If there exists $L \in L(V,\ H)$ such that $A+LC$ generates an exponentially stable C_0-semigroup $S_{LC}(t)$, then we say that (A, C) is exponentially detectable.*

We have the following result proved in [80].

Theorem 2.14 *If the pair (A, C) is detectable then the Riccati equation*

$$\langle Az,\ Pz\rangle + \langle Pz,\ Az\rangle - \left\langle PBR^{-1}B^{*}Pz,\ z\right\rangle + \langle Cz, C\ z\rangle\ = 0,\ z \in D(A), \tag{2.7}$$

has at most one nonnegative solution and if P is the solution then the operator $A - BR^{-1}B^*P$ is stable. If, in addition, the pair $(A,\ B)$ is stabilizable then the equation (2.7) has exactly one solution.

2.3 Stochastic processes and stochastic differential equations

A measurable space is a pair $(\Omega,\ \mathcal{F})$ where Ω is a set and $\mathcal{F}$ is a σ-field, also called a σ-algebra, of subsets of Ω. This mean that the family $\mathcal{F}$ contains the set Ω and is closed under the operation of taking complements and countable unions of its elements.

A probability measure on a measurable space $(\Omega,\ \mathcal{F})$ is a σ-additive function μ from $\mathcal{F}$ into $[0,\ 1]$ such that $\mu(\Omega) = 1$. The triplet $(\Omega,\ \mathcal{F},\ \mu)$ is called a probability space. If $(\Omega,\ \mathcal{F},\ \mu)$ is a probability space, we set

$$\overline{\mathcal{F}} = \{A \subset \Omega :\ \exists\ B,\ C \in \mathcal{F};\ B \subset A \subset C,\ \mu(B) = \mu(C)\}$$

Then $\overline{\mathcal{F}}$ is a σ-field, called the completion of $\mathcal{F}$. If $\mathcal{F} = \overline{\mathcal{F}}$, the probability space $(\Omega,\ \mathcal{F},\ \mu)$ is said to be complete.

Let $(\Omega,\ \mathcal{F},\ \mu)$ a complete probability space. A family $\{\mathcal{F}_t\}$, $t \geq 0$, for which the $\mathcal{F}_t$ are sub-σ-fields of $\mathcal{F}$ and form an increasing family of σ-fields, is called a filtration if $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $s \leq t$.

2.3.1 Random variables

In this section we shall collect definitions and basic results on random variables in Hilbert spaces. Let $(\Omega,\ \mathcal{F},\ \mu)$ be a complete probability space.

Definition 2.15 *The Borel σ-field of H, $\mathcal{B}(H)$ is the smallest σ-field containing all closed (or open) subsets of H*

Definition 2.16 *An H-valued random variable is a map $x : \Omega \to H$ which is measurable from $(\Omega,\ \mathcal{F})$ to $(H,\ \mathcal{B}(H))$. If x is integrable on Ω we define its expectation by*

$$\mathcal{E}(x) = \int_{\Omega} x d\mu$$

Let $x : \Omega \to H$ be a square integrable random variable,.i.e., $x \in L^2(\Omega, H)$.

Definition 2.17 *The covariance operator of x is*

$$cov(x) = \mathcal{E}((x - \mathcal{E}(x)) \circ (x - \mathcal{E}(x)))$$

where $u \circ v \in L(H)$ is defined for each $u,\ v \in L(H)$ by

$$(u \circ v)(h) = u \langle v, h \rangle,\ \ h \in H$$

The covariance operator $cov(x)$ is a self-adjoint nonnegative and nuclear operator and

$$tr\,(cov(x)) = \mathcal{E}\,\langle x, x \rangle,\ \ \mathcal{E}\,\langle Rx, x \rangle = trRcov(x),\ \ R \in L(H)$$

where $tr.$ denotes the trace of an operator.

Definition 2.18 *H-valued random variables x and y are independent if $\{\omega;\ x(\omega) \in A\}$ and $\{\omega;\ y(\omega) \in B\}$ are independent sets for any Borel sets A, B in $\mathcal{B}(H)$. If x and y are in $L^1(\Omega, H)$ and are independent, then*

$$\mathcal{E}(\langle x, y\rangle) = \langle \mathcal{E}(x), \mathcal{E}(y)\rangle$$

Definition 2.19 *A real-valued random variable x is said to be Gaussian (or normal), if it has a probability distribution function*

$$F_x(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\left[(\xi-m)^2/2\sigma^2\right]} d\xi,$$

for some constants m and $\sigma > 0$.

The constants m and σ represent the mean and the standard deviation, respectively, of the random variable. A Gaussian random variable with mean m and variance $\sigma^2 > 0$ will be denoted $x \in \aleph\left(m,\ \sigma^2\right)$.

Definition 2.20 *A random variable $x \in L^2(\Omega, \mu, H)$ is Gaussian if $\langle x, \Phi_i\rangle$ is a real Gaussian random variable for all i where $\{\Phi_i\}$; $i = 1,\ 2, ..$ is a complete orthonormal basis for H.*

We have the following inequalities for a Gaussian random variable

Lemma 2.21 *Let x be a real Gaussian random variable in H with zero mean and covariance operator Q. Then*

$$\mathcal{E}\left|x\right|^{2n} \leq fact\,(2n-1)\,trace(Q)^n,\ \ n = 1,\ 2, ...$$

where $fact\,(2n-1) = (2n-1)\,(2n-3)\,...3.1$ and the equality holds for $n = 1$.

Of the many types of convergence one can introduce for random variables the following

Definition 2.22 *A sequence (x_n) of H-valued random variables converges to x*

1. *in probability if $P(\|x_n - x\| > 0) \to 0$ as $n \to +\infty$*

2. *with probability one (w.p.1) if*

$$\|x_n - x\| \to 0 \text{ as } n \to +\infty \text{ except on a set of measure zero}$$

3. *in mean square if*

$$\mathcal{E}\left(\|x_n - x\|^2\right) \to 0 \text{ as } n \to +\infty$$

4. *in distribution if for every $f \in C(H)$, the space of bounded real valued continues functions on H,*

$$\int_H f d\mu_n \to \int_H f d\mu \text{ as } n \to +\infty$$

where μ_n and μ are the measures induced on $\mathcal{B}(H)$ by x_n and x respectively.

2.3.2 Stochastic processes

Definition 2.23 *An H-valued stochastic process is a map $x : [t_0,\ T] \times \Omega \to H$ which is measurable in the product measure on $[t_0,\ T] \times \Omega$.*

Continuity of stochastic processes is given in the following definition

Definition 2.24 *Let x be an H-valued stochastic process on $[t_0,\ T]$, then*

1. x is continuous in probability if

$$P\left(\|x(t+\delta)-x(t)\|>0\right)\to 0 \text{ as } \delta\to 0$$

2. x is continuous in probability one (w.p.1) if

$$\|x(t+\delta)-x(t)\|\to 0 \text{ as } \delta\to 0 \text{ w.p.1}$$

3. x is continuous in mean square if

$$\mathcal{E}\left(\|x(t+\delta)-x(t)\|^2\right)\to 0 \text{ as } \delta\to 0 \text{ w.p.1}$$

4. x has continuous sample paths if

$$P\left(\sup_{t_0\le t\le T}\|x(t+\delta)-x(t)\|\right)\to 0 \text{ as } \delta\to 0$$

Wiener processes are used for modelling white noise disturbances in engineering systems. The following is one of several equivalent definitions.

Definition 2.25 *$w(t)$ is an H- valued Wiener process on $[0,\ T]$ if it is an H-valued process on $[0,\ T]$, such that*

$$w(t)-w(s)\in L^2(\Omega,\ \mu,\ H) \textit{ for all } s,\ t\in[0,\ T] \textit{ and}$$

1. $\mathcal{E}\left(w(t)-w(s)\right)=0$,

2. *$cov(w(t)-w(s))=(t-s)\,W$, where $W\in L(H)$ nonnegative and nuclear,*

3. *$w(s_4)-w(s_3)$ and $w(s_2)-w(s_1)$ are independent whenever $0\le s_1\le s_2\le s_3\le s_4\le T$,*

4. *$w(t)$ has continuous sample paths on $[0,\ T]$.*

A particular useful representation for a Wiener process is the following expansion

Lemma 2.26 *If $w(t)$ is an H-valued Wiener process, then there exists a complete orthonormal basis $\{e_i\}_{i=0}^{\infty}$ for H, such that*

$$w(t) = \sum_{i=0}^{+\infty} \beta_i(t) e_i \qquad w.p.1 \tag{2.8}$$

where $\beta_i(t)$ are mutually independent real Wiener process with incremental covariance λ_i and $\sum_{i=0}^{+\infty} \lambda_i < +\infty$.

Lemma 2.27 *We have*

1. *$w(t)$ is Gaussian for each $t \in [0, T]$,*

2. *$\mathcal{E} \|w(t) - w(s)\|^{2n} \leq fact\,(2n-1)\,(t-s)^n\,(trW)^n$, where the equality holds for $n = 1$.*

Definition 2.28 *Any linear transformation x from Hilbert space H into $L^2(\Omega; \mathcal{F}, \mu)$ with values being Gaussian random variables and such that*

$$\mathcal{E}x(h)x(g) = \langle h, g \rangle, \;\; h, \;\; g \in H$$

is called white noise.

We have the following theorem established in [6] for $H = \mathbb{R}^n$.

Theorem 2.29 *[6] A white noise process $x(t)$, $t \in [0, T)$ with $T \leq \infty$, is the generalized derivative of a general Wiener process, $w(t)$, in the sense of distributions:*

$$\frac{d}{dt} w(t) = x(t) \;\; \text{or} \;\; w(t) = \int_0^t x(s) ds,$$

which has a covariance function given by the delta function

$$R_x(s,t) = \delta(t-s),\ \ s,\ \ t \in [0,T)\,.$$

Consequently, a white noise has the property that the values $x(s)$ and $x(t)$ are independent for all $s \neq t$ in $[0,T)$.

We conclude this section with the following definitions.

Definition 2.30 *The σ-field, $\mathcal{P}_\infty$, generated by sets of the form:*

$$(s,t] \times \mathcal{F},\ \ 0 \leq s < t < \infty,\ \ \mathcal{F} \in \mathcal{F}_s \text{ and } \{0\} \times \mathcal{F},\ \mathcal{F} \in \mathcal{F}_0.$$

is called predictable and its elements are called predictable sets. The restriction of the σ-field $\mathcal{P}_\infty$ to $[0,\ T] \times \Omega$, $0 \leq T \leq \infty$, will be denoted by $\mathcal{P}_T$.

Definition 2.31 *An arbitrary measurable mapping from $([0,\ T] \times \Omega,\ \mathcal{P}_\infty)$ or $([0,\ T] \times \Omega,\ \mathcal{P}_T)$ into $(H,\ \mathcal{B}(H))$ is called a predictable process.*

2.3.3 Stochastic integration

Let $\{\mathcal{F}_t\}$ be an increasing family of σ-fields of $\mathcal{F}$. If for arbitrary $t \in [0,\ T]$ the random variable $x(t)$ is $\mathcal{F}_t$-measurable then x is said to be adapted (to the family $\mathcal{F}_{tt}$). Let Y be a real Hilbert space and let $B_2\left([0,\ T] \times \Omega; L(Y,H)\right)$ be the space of $L(Y,\ H)$-valued stochastic processes $G(t)$ such that $G(t)h$, for any $h \in Y$, is strongly measurable and adapted to the σ-field $\mathcal{F}_t$ and $\mathcal{E} \int_0^T \|G(t)\|^2_{L(Y,H)}\, dt < \infty$. We shall define the stochastic integral $\int_0^T G(t)dw(t)$ using the representation (2.8) of $w(t)$.

Let $w_n(t) = \sum\limits_{i=1}^{n} e_i \beta_i(t)$, then the stochastic integral

$$y_n = \int_0^T G(t) dw_n(t) = \sum_{i=1}^{n} \int_0^T G(t) e_i d\beta_i(t)$$

is well defined just as in the scalar case. Then $\{y_n\}$ is a Cauchy sequence in $L_2(\Omega, \mu, Y)$. In fact for any $m > n$ we have

$$\mathcal{E} \|y_m - y_n\|^2 = \sum_{i=n+1}^{m} \lambda_i \int_0^T \mathcal{E} \|G(t)e_i\|^2 dt \leq \sum_{i=n+1}^{m} \lambda_i \int_0^T \mathcal{E} \|G(t)\|^2 dt$$

Thus we define the stochastic integral as follows

$$\int_0^T G(t) dw(t) = \lim_{n \to +\infty} \int_0^T G(t) dw_n(t) = \sum_{i=1}^{+\infty} \int_0^T G(t) e_i d\beta_i(t)$$

where the limit is understood in $L^2(\Omega, \mu, Y)$. By an obvious modification $\int\limits_{t_0}^{T} G(t) dw(t)$ can be defined for any $0 \leq t_0 \leq t \leq T$. Now we give some properties of the stochastic integral.

Lemma 2.32 *[44]Let* $G(t) \in B_2([0,\ T] \times \Omega;\ L(Y,\ H))$. *Then*

1. $\mathcal{E}\left(\int\limits_0^T G(t) dw(t)\right) = 0$

2. $\mathcal{E}\left\|\int\limits_0^T G(t) dw(t)\right\|^2 = \begin{cases} \int\limits_0^T \mathcal{E}\left(trG(t)WG^*(t)\right) dt \\ \int\limits_0^T \mathcal{E}\left(trG^*(t)G(t)W\right) dt \end{cases} \leq trW \int\limits_0^T \mathcal{E} \|G(t)\|^2 dt$

Lemma 2.33 *[44]Let* $G(t,\ s)$ *is an* $L(Y, H)$ *valued process defined on* $[0,\ T] \times [0,\ T]$ *such that* $G(t,\ s)h$ *is strongly measurable for any* $h \in Y$ *and* $\mathcal{F}_s$*-measurable for any* t *and* h *with*

$$\int_0^T \int_0^T \mathcal{E} \|G(t,\ s)\|^2 dsdt < +\infty$$

We have

$$\int_0^T \int_0^T G(t,\ s)dw(s)dt = \int_0^T \int_0^T G(t,\ s)dtdw(s) \qquad w.p.1$$

where the right hand side is understood in the sense

$$\sum_{i=1}^{+\infty} \int_0^T \int_0^T G(t,\ s)e_i dt d\beta_i(s)$$

Remark 2.34 *The stochastic integrals can be defined for those $G(t)$ with only $\int_0^T \|G(t)\|^2 dt < +\infty$ w.p.1.*

2.3.4 Stochastic evolution equations

Now let Y, H be real separable Hilbert spaces. We consider the stochastic evolution equation

$$dz(t) = (Az(t) + f(z(t)))\, dt + D(z(t))dw(t),\ z(t_0) = z_0 \tag{2.9}$$

where A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on H, $w(t)$ is a Wiener process with covariance operator W in Y , $f : H \to H$ and $D(.)$:$H \to L(Y, H)$ satisfy Lipschitz conditions

$$\begin{aligned} \|f(y) - f(z)\| &\leq c_1 |y - z|,\ y,\ z \in H,\ c_1 > 0, \\ \|D(y) - D(z)\| &\leq c_2 |y - z|,\ y,\ z \in H,\ c_2 > 0 \end{aligned} \tag{2.10}$$

Let $\mathcal{F}_{tt}$ be the σ-field given above. We introduce two concepts of a solution of (2.9).

Definition 2.35 *$z(t)$ is a strong solution of (2.9) on $[t_0,\ T]$ if $z(t) \in D(A)$ with probability one for all most all t and satisfies the following:*

1. *$z(t)$ is in $C\left((t_0,\ T),\ L^2(\Omega,\ \mu,\ Y)\right)$ and $\int_{t_0}^{T}\|Az(t)\|\,dt < +\infty$ w.p.1.*

2. *$z(t)$ is adapted to F_t and has continuous sample paths.*

3.

$$z(t) = z_0 + \int_{t_0}^{t} Az(s)ds + \int_{t_0}^{t} f\left(z(s)\right)ds + \int_{t_0}^{t} D\left(z(s)\right)dw(s) \quad w.p.1$$

This concept is very strong, so we introduce a weaker concept.

Definition 2.36 *$z(t)$ is a mild solution on $[t_0,\ T]$ if $z(t) \in C\left((t_0,\ t_1),\ L^2(\Omega,\ \mu,\ H)\right)$ and adapted to F_t such that*

$$z(t) = S(t-t_0)z_0 + \int_{t_0}^{t} S(t-s)f\left(z(s)\right)ds + \int_{t_0}^{t} S(t-s)D\left(z(s)\right)dw(s) \qquad (2.11)$$

The following theorem has been established in [8].

Theorem 2.37 *Let $T > t_0 > 0$ be arbitrary and let z_0 be measurable relative to $\mathcal{F}_{tt}$ with $\mathcal{E}\left|z_0\right|^2 < \infty$. Then there exists a unique solution of (2.11) in $C\left((t_0,\ t_1),\ L^p(\Omega,\ \mu,\ H)\right)$, $p = 2, 4$ which is adapted to F_t.*

We conclude this section with the following inequality known as Bukholder-Davis-Gundy inequality.

Theorem 2.38 ([15]) *For arbirary $p > 0$, there exists a constant $C = C_p > 0$, such that for any $\Phi \in L^p_\omega\left((0,T),\ H\right)$, $T > 0$*

$$\mathcal{E}\left(\int_0^T \|\Phi(s)dw(s)\|^p\right) \leq C_p\mathcal{E}\left(\int_0^T \left(\|\Phi(s)\|^2\right)ds\right)^{\frac{p}{2}}$$

2.3.5 Stochastic stability

We consider the stochastic differential equation

$$dz(t) = Az(t)dt + D(z(t))dw(t), \; z(0) = z_0 \in H \tag{2.12}$$

and its integrated version

$$z(t) = S(t)z_0 + \int_{t_0}^{t} S(t-r)D(z(r))dw(r)$$

where $D \in L(H,\, L(Y,\, H))$ and $w(t)$ is a Wiener process in Y with covariance operator W.

There are a number of possible stability concepts in the study of deterministic systems and we expect many more in the stochastic case. Since the problem of stability is essentially a problem of convergence for each deterministic stability definition we find that there are at least four corresponding stochastic stability definitions. These are generated by the four modes of convergence. In this thesis we consider only the stability concept based upon convergence in mean square criterion.

Definition 2.39 *The system (2.12) is said to be mean square stable if there exist constants $M > 0$ and $\omega > 0$ such that*

$$\mathcal{E} \left\| z(t) \right\|^2 \leq Me^{-\omega t} \left\| z_0 \right\|^2, \; \textit{for any } z_0 \in H, \; t \geq 0.$$

The following theorem ([12]) is a stochastic version of Datko's result.

Theorem 2.40 *The following statements are equivalent*

1. *There exist* $M > 0$ *and* $\omega > 0$ *such that*

$$\mathcal{E}\left\|z(t)\right\|^2 \leq Me^{-\omega t}\left\|z_0\right\|^2, \ \textit{for any } z_0 \in H, \ t \geq 0.$$

2. *There exists a nonnegative linear operator* $P \in L(H)$ *such that*

$$2\left\langle Az, \ Pz\right\rangle + \left\langle \Delta(P)z, \ z\right\rangle = -\left\langle z, \ z\right\rangle, \quad z \in D(A)$$

   where $\left\langle \Delta(P)z, \ z\right\rangle = trD^*(z)PD(z)W$.

3. *For any* $z_0 \in H$ *we have* $\mathcal{E}\int_0^{+\infty}\left\|z(t)\right\|^2 dt < +\infty$.

Remark 2.41 *As in the deterministic case, we say that the system (2.12) is* L^2*-stable when it is mean square stable. If there exists* $p \geq 2$ *such that* $\mathcal{E}\int_0^{+\infty}\left\|z(t)\right\|^p dt < +\infty$, *system (2.12) is said to be* L^p*-stable.*

Chapter 3

Stability radii: Definition and characterizations

3.1 Introduction

In this chapter we consider infinite dimensional systems which are subjected to N stochastic structured perturbations. At first we define the stability radii of these systems, then we use scaling techniques to derive characterizations of these radii. These characterizations are given in terms of a Lyapunov equation which satisfies a number of operator inequalities and the corresponding Lyapunov inequalities. A computable formula is given for these radii in terms of this equation. We conclude the chapter by some illustrative examples.

3.2 System description

Let $N \in \mathbb{N}$ and suppose that A is the infinitesimal generator of an exponentially stable semigroup $S(t)$ on a real separable Hilbert space H. Moreover let $(D_i, E_i)_{i\in\overline{N}}$ be a given family of operators $D_i \in L(U_i, H)$, $E_i \in L(H, Y_i)$, where U_i, Y_i are also real separable Hilbert spaces, $i \in \overline{N}$, $\overline{N} = \{1, ..., N\}$. We will consider infinite dimensional uncertain systems described by Ito equations of the form

$$\begin{aligned} dx(t) &= Ax(t)dt + \sum_{i=1}^{N} D_i \Delta_i \left(E_i x(t)\right) dw_i(t),\ t > 0 && (3.1) \\ x(0) &= x_0 && (3.2) \\ \|\Delta_i\|_{Lip} &< \sigma,\ i \in \overline{N}. \end{aligned}$$

where x_0 varies in H and $\Delta_1, \ldots, \Delta_N$ are unknown Lipschitzian nonlinearities, $(w_i(t))_{t\in\mathbb{R}_+}$, $i \in \overline{N}$, are independent zero mean real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ relative to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of $\sigma-$algebras $(\mathcal{F}_t)_{t\in\mathbb{R}_+} \subset \mathcal{F}$. Thus if $\theta_i > 0$ denotes the variance of $(w_i(t))_{t\in\mathbb{R}_+}$, we have, for all t, $s \in \mathbb{R}^+$, $t > s$,

$$\begin{cases} \mathcal{E}\left(w_i(t)\right) = 0 \\ \mathcal{E}\left(w_i(t) - w_i(s)\right)\left(w_j(t) - w_j(s)\right) = \delta_{ij}\theta_i\left(t - s\right),\ i,\ j \in \overline{\mathbb{N}}, \end{cases}$$

where δ_{ij} is the Kronecker symbol. The disturbances Δ_i vary in $Lip\left(Y_i, U_i\right)$. The unknown Δ_i represent uncertainty in the state-dependent gains through which the stationary white noise processes $dw_i(t)$ affect the evolution of the system. The family $(D_i, E_i)_{i\in\overline{N}}$ determines the structure of the perturbations and $\sigma > 0$ indicates the overall level of the stochastic disturbances. Altogether (3.1) describes a set of stochastic systems parametrized by $\Delta_i \in Lip\,(Y_i, U_i)$, $i \in \overline{N}$ with norms $\|\Delta_i\|_{Lip} < \sigma$.

The size of each $\Delta \in Lip\,(Y, U)$ is measured by the Lipschitz norm

$$\|\Delta\|_{Lip} = \inf\left\{\gamma > 0;\ \forall\ y,\ \hat{y} \in Y : \|\Delta\,(y) - \Delta\,(\hat{y})\|_U \leq \gamma\,\|y - \hat{y}\|_Y\right\}$$

Let $L^2_\omega\left(\mathbb{R}^+,\ L^2\,(\Omega,\ H)\right)$ the space of predictable stochastic processes $z(t) = (z(t))_{t\in\mathbb{R}^+}$ with respect to the $\sigma-$algebras $(\mathcal{F}_t)_{t\in\mathbb{R}_+} \subset \mathcal{F}$ satisfying

$$\|z(.)\|^2_{L^2_\omega} = \mathcal{E}\int_0^{+\infty} \|z(t)\|^2_H\,dt = \int_0^{+\infty} \mathcal{E}\left(\|z(t)\|^2_H\right)dt < +\infty$$

For arbitrary $\Delta_i \in Lip\,(Y_i, U_i)$, $i \in \overline{N}$, and any initial state $x_0 \in H$ there exists a unique solution $x(.)$ of (3.1) such that $x(0) = x_0$ [13]. This solution $x(.)$ is continuous predictable stochastic process such that :

$$\int_0^T \mathcal{E}\left(\|x(t)\|^2_H\right)dt < +\infty,\ T > 0$$

3.3 Characterizations of the stochastic stability radius

Our aim is to determine which bounds σ of the perturbations Δ_i ensure that the stability of the deterministic system $dx(t) = Ax(t)dt$ is preserved under additive stochastic perturbations of the form $\sum_{i=1}^{N} D_i\Delta_i\,(E_i x(t))\,dw_i(t)$. Let Δ denote the combined perturbation operator

$$\Delta = \bigoplus_1^N \Delta_i,\ Y = \bigoplus_1^N Y_i,\ U = \bigoplus_1^N U_i$$

The Lipschitz norm of Δ is given by

$$\|\Delta\|_{Lip} = \max_{i\in\overline{N}} \|\Delta_i\|_{Lip}$$

The maximal $\sigma > 0$ for which all the systems in (3.1) are L^2-stable is called the stability radius of (3.1) .

Definition 3.1 *The stochastic stability radius of A with respect to the perturbation structure $(D_i, E_i)_{i\in\overline{N}}$ and the Wiener process $(w_i(t))_{t\in\mathbb{R}_+}$ is*

$$r^w\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) = \inf\left\{\begin{array}{c} \left\|\bigoplus_1^N \Delta_i\right\|_{Lip};\ \Delta_i \in Lip\,(Y_i, U_i)\ \textit{such that (3.1)} \\ \textit{is not stable in mean square} \end{array}\right\}$$

The approach used in this thesis to characterize the stochastic stability radius $r^w\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right)$ is based on the following lemma.

Lemma 3.2 *Suppose that $E \in L(H, Y)$ and*

$$y(t) = ES(t)x_0 + \sum_{i=1}^{N} \int_0^t ES(t-s)D_i\left(v_i(s)\right)dw_i(s)$$

where $v_i \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U_i\right)\right),\ i \in \overline{N},$ and $x_0 \in H$. Then

$$\mathcal{E}\left(\|y(t)\|^2\right) = \|ES(t)x_0\|^2 + \sum_{i=1}^{N} \theta_i \int_0^t \mathcal{E}\left(\|ES(t-s)D_iv_i(s)\|^2\right)ds,\ t > 0$$

Moreover, $y(.) \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y\right)\right)$ and

$$\begin{aligned} \|y\|^2_{L^2_w} &= \int_0^{+\infty} \mathcal{E}\left(\|y(t)\|^2\right)dt \\ &= \int_0^{+\infty} \|ES(t)x_0\|^2\,dt + \sum_{i=1}^{N} \theta_i \int_0^{+\infty} \mathcal{E}\left(\langle D_iv_i(s), PD_iv_i(s)\rangle\right)ds. \end{aligned}$$

where $P \in L(H)$ is a self-adjoint nonnegative operator satisfying

$$2\langle Px, Ax\rangle + \langle Ex, Ex\rangle = 0, x \in D(A) \tag{3.3}$$

Proof. We have

$$\begin{aligned}
\|y(t)\|^2 &= \langle y(t),\ y(t)\rangle \\
&= \left\langle \begin{array}{c} ES(t)x_0 + \sum\limits_{i=1}^{N} \int_0^t ES(t-s)D_i\left(v_i(s)\right)dw_i(s),\ ES(t)x_0 \\ + \sum\limits_{i=1}^{N} \int_0^t ES(t-s)D_i\left(v_i(s)\right)dw_i(s) \end{array} \right\rangle
\end{aligned}$$

Set $G_i(s) = ES(t-s)D_i\left(v_i(s)\right)$, then

$$\begin{aligned}
\|y(t)\|^2 &= \|ES(t)x_0\|^2 + \left\langle ES(t)x_0,\ \sum_{i=1}^{N}\int_0^t G_i(s)dw_i(s)\right\rangle \\
&\quad + \left\langle \sum_{i=1}^{N}\int_0^t G_i(s)dw_i(s),\ ES(t)x_0\right\rangle + \left\|\sum_{i=1}^{N}\int_0^t G_i(s)dw_i(s)\right\|^2
\end{aligned}$$

We have

1. $\mathcal{E}\,\|ES(t)x_0\|^2 = \|ES(t)x_0\|^2$

2. $\mathcal{E}\left\langle ES(t)x_0,\ \sum\limits_{i=1}^{N}\int_0^t G_i(s)dw_i(s)\right\rangle = 0$

3. $\mathcal{E}\left(\left\|\sum\limits_{i=1}^{N}\int_0^t G_i(s)dw_i(s)\right\|^2\right) = \mathcal{E}\left(\left\langle \sum\limits_{i=1}^{N}\int_0^t G_i(s)dw_i(s),\ \sum\limits_{j=1}^{N}\int_0^t G_j(s)dw_j(s)\right\rangle\right)$

   Since the Wiener processes are independent we have

   For $i = j$,

$$\mathcal{E}\left(\left\langle\int_0^t G_i(s)dw_i(s),\ \int_0^t G_i(s)dw_i(s)\right\rangle\right) = \theta_i\int_0^t \mathcal{E}\,\|G_i(s)\|^2\,ds$$

   For $i \neq j$,

$$\mathcal{E}\left(\left\langle\int_0^t G_i(s)dw_i(s),\ \int_0^t G_j(s)dw_j(s)\right\rangle\right) = 0$$

It follows then that

$$\mathcal{E}\left(\left\|\sum_{i=1}^{N}\int_0^t G_i(s)dw_i(s)\right\|^2\right) = \sum_{i=1}^{N}\theta_i\int_0^t \mathcal{E}\,\|G_i(s)\|^2\,ds$$

Hence

$$\mathcal{E}\left(\|y(t)\|^2\right) = \|ES(t)x_0\|^2 + \sum_{i=1}^{N}\theta_i \int_0^t \mathcal{E}\,\|ES(t-s)D_i\,(v_i(s))\|^2\,ds$$

We have

$$\begin{aligned}
\|y\|^2_{L^2_\omega} &= \int_0^{+\infty} \mathcal{E}\left(\|y(t)\|^2\right) dt \\
&= \int_0^{+\infty} \|ES(t)x_0\|^2\,dt + \int_0^{+\infty}\left(\sum_{i=1}^{N}\theta_i \int_0^t \mathcal{E}\,\|ES(t-s)D_i\,(v_i(s))\|^2\,ds\right) dt
\end{aligned}$$

Since $S(t)$ is an exponentially stable semigroup, there exist positive constants ω and M such that

$$\|S(t)\| \leq Me^{-\omega t}$$

Thus

$$\int_0^{+\infty} \|ES(t)x_0\|^2\,dt < +\infty \tag{3.4}$$

For the second term we have

$$\begin{aligned}
\int_0^t \mathcal{E}\,\|ES(t-s)D_i\,(v_i(s))\|^2\,ds &\leq \int_0^t \|ES(t-s)\|^2\,\|D_i\|^2\,\mathcal{E}\,\|(v_i(s))\|^2\,ds \\
&\leq M^2\,\|E\|^2\,\|D_i\|^2 \int_0^t e^{-2\omega(t-s)}\mathcal{E}\,\|v_i(s)\|^2\,ds
\end{aligned}$$

Set $M_i = M^2\,\|E\|^2\,\|D_i\|^2$. It follows then that

$$\begin{aligned}
&\int_0^{+\infty} \sum_{i=1}^{N}\theta_i \int_0^t \mathcal{E}\,\|ES(t-s)D_i\,(v_i(s))\|^2\,dsdt \\
\leq\ & \sum_{i=1}^{N}\theta_i \int_0^{+\infty} M_i\left(\int_0^t e^{-2\omega(t-s)}\mathcal{E}\,\|v_i(s)\|^2\,ds\right) dt \\
\leq\ & \sum_{i=1}^{N}\theta_i M_i \int_0^{+\infty}\left(\int_s^{+\infty} e^{-2\omega(t-s)}\mathcal{E}\,\|v_i(s)\|^2\,dt\right) ds
\end{aligned}$$

hence

$$
\begin{aligned}
& \int_0^{+\infty} \sum_{i=1}^{N} \theta_i \int_0^t \mathcal{E} \left\| ES(t-s)D_i\left(v_i(s)\right)\right\|^2 dsdt \\
\leq \; & \sum_{i=1}^{N} \theta_i M_i \int_0^{+\infty} e^{2\omega s} \mathcal{E} \left\| v_i(s)\right\|^2 \left(\int_s^{+\infty} e^{-2\omega t} dt \right) ds \\
\leq \; & \frac{1}{2\omega} \max_{i \in \{1,\ldots,N\}} \left(\theta_i M_i\right) \int_0^{+\infty} \mathcal{E} \left(\sum_{i=1}^{N} \left\| \left(v_i(s)\right)\right\|^2 \right) ds
\end{aligned}
$$

Thus

$$
\sum_{i=1}^{N} \theta_i \int_0^{+\infty} \int_0^t \mathcal{E} \left\| ES(t-s)D_i\left(v_i(s)\right)\right\|^2 dsdt \leq \widetilde{M} \left\| v \right\|_{L^2_\omega}^2 , \; \widetilde{M} > 0
$$

Using this inequality and inequality (3.4) we deduce that $y(.) \in L^2_\omega\left(\mathbb{R}^+, \; L^2\left(\Omega, \; Y\right)\right)$.

Now we will show the second statement of the Lemma using Fubbini's Theorem.

We have

$$
\begin{aligned}
& \int_0^{+\infty} \int_0^t \mathcal{E} \left\| ES(t-s)D_i\left(v_i(s)\right)\right\|^2 dsdt \\
= \; & \int_0^{+\infty} \int_s^{+\infty} \mathcal{E} \left\| ES(t-s)D_i\left(v_i(s)\right)\right\|^2 dtds \\
= \; & \int_0^{+\infty} \mathcal{E} \left\langle D_i\left(v_i(s)\right), \; \left(\int_s^{+\infty} S^*(t-s)E^*ES(t-s)dt \right) D_i\left(v_i(s)\right) \; \right\rangle ds \\
= \; & \int_0^{+\infty} \mathcal{E} \left\langle D_i\left(v_i(s)\right), \; PD_i\left(v_i(s)\right) \; \right\rangle ds
\end{aligned}
$$

where P is the solution of the Lyapunov equation (3.3) and it is given by

$$
Px = \int_s^{+\infty} S^*(t-s)E^*ES(t-s)xdt, \; x \in H
$$

■

The second lemma will be given in terms of the input-output operator

$$
L : L^2_\omega\left(\mathbb{R}^+, \; L^2\left(\Omega, \; U\right)\right) \longrightarrow L^2_\omega\left(\mathbb{R}^+, \; L^2\left(\Omega, \; Y\right)\right)
$$

defined by

$$(Lv)(t) = \left(L \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \right)(t) = \sum_{i=1}^{N} \int_0^t \left(ES(t-s) D_i v_i(s) \right) dw_i(s) \qquad (3.5)$$

By the previous lemma, $Lv \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y\right)\right)$ for all $v \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$.

Lemma 3.3 *The linear map L defined by (3.5) has the operator norm*

$$\|L\| = \left(\max_{i \in \overline{N}} \left(\theta_i \|D_i^* P D_i\| \right) \right)^{\frac{1}{2}}$$

where P satisfies (3.3).

Proof. Let $v \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$. By the previous Lemma we have

$$\begin{aligned} \|Lv\|^2_{L^2_\omega} &= \sum_{i=1}^{N} \theta_i \int_0^{+\infty} \mathcal{E}\left(\langle D_i v_i(s), P D_i v_i(s) \rangle \right) ds \\ &\leq \sum_{i=1}^{N} \theta_i \int_0^{+\infty} \mathcal{E}\left(\|D_i^* P D_i\| \|v_i(s)\|^2 \right) ds \end{aligned}$$

Thus

$$\begin{aligned} \|Lv\|^2_{L^2_\omega} &\leq \sum_{i=1}^{N} \theta_i \|D_i^* P D_i\| \int_0^{+\infty} \mathcal{E}\left(\|v_i(s)\|^2 \right) ds \\ &\leq \max_{i \in \{1,\ldots,N\}} \left(\theta_i \|D_i^* P D_i\| \right) \int_0^{+\infty} \sum_{i=1}^{N} \mathcal{E}\left(\|v_i(s)\|^2 \right) ds \\ &= \max_{i \in \{1,\ldots,N\}} \left(\theta_i \|D_i^* P D_i\| \right) \|v\|^2_{L^2_\omega} \end{aligned}$$

and hence

$$\frac{\|Lv\|^2_{L^2_\omega}}{\|v\|^2_{L^2_\omega}} \leq \max_{i \in \{1,\ldots,N\}} \left(\theta_i \|D_i^* P D_i\| \right), \text{ for all } v \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$$

which implies that

$$\|L\| \leq \left(\max_{i \in \{1,\ldots,N\}} \left(\theta_i \|D_i^* P D_i\| \right) \right)^{\frac{1}{2}}$$

Now, we shall show that there exists $v^0 \in L^2_\omega\left(\mathbb{R}^+,\ L^2(\Omega,\ U)\right)$ such that

$$\|Lv^0\|_{L^2_\omega} = \left(\max_{i\in\{1,\ldots,N\}} (\theta_i \|D_i^* P D_i\|)\right)^{\frac{1}{2}}$$

or

$$\sum_{i=1}^{N} \theta_i \int_0^{+\infty} \mathcal{E}\left(\langle D_i v_i^0(s), P D_i v_i^0(s)\rangle\right) ds = \max_{i\in\{1,\ldots,N\}} (\theta_i \|D_i^* P D_i\|)$$

Now suppose that

$$\left(\max_{i\in\{1,\ldots,N\}} (\theta_i \|D_i^* P D_i\|)\right) = \theta_{i_0} \left\|D_{i_0}^* P D_{i_0}\right\|$$

Since $D_{i_0}^* P D_{i_0}$ is a self adjoint operator it follows that

$$\left\|D_{i_0}^* P D_{i_0}\right\| = \left(\max_{\|u\|_U = 1} \langle u, D_{i_0}^* P D_{i_0} u\rangle_U\right) = \left\langle v_{i_0}, D_{i_0}^* P D_{i_0} v_{i_0}\right\rangle,\ \ \|v_{i_0}\|_U = 1$$

We define v^0 as follows

$v_i^0(t) = 0,\ t \in \Re^+,\ i \neq i_0,\ v_{i_0}^0(t) = \beta(.) v_{i_0},$ where $\beta(.) \in L^2\left(\Re^+,\ \Re\right)$ and $\|\beta(.)\|_{L^2} = 1$

Then $v^0(.) = \left(v_i^0(.)\right)_{i\in\overline{N}} \in L^2_\omega\left(\mathbb{R}^+,\ L^2(\Omega,\ U)\right)$, and

$$\begin{aligned}
\left\|v^0(.)\right\|^2_{L^2_\omega} &= \sum_{i=1}^{N} \int_0^{+\infty} \mathcal{E}\left\|v_i^0(s)\right\|^2 ds \\
&= \sum_{i=1}^{N} \int_0^{+\infty} \left\|v_i^0(s)\right\|^2 ds \\
&= \int_0^{+\infty} \|\beta(s) v_{i_0}\|^2 ds \\
&= \|v_{i_0}\|^2 \int_0^{+\infty} |\beta(s)|^2 ds \\
&= \|v_{i_0}\|^2 = 1
\end{aligned}$$

and

$$\begin{aligned}
\left\|Lv^0\right\|_{L^2_\omega}^2 &= \sum_{i=1}^{N}\theta_i \int_0^{+\infty} \mathcal{E}\left(\left\langle D_i v_i^0(s), PD_i v_i^0(s)\right\rangle\right) ds \\
&= \sum_{i=1}^{N}\theta_i \int_0^{+\infty} \left(\left\langle D_i v_i^0(s), PD_i v_i^0(s)\right\rangle\right) ds \\
&= \theta_{i_0} \int_0^{+\infty} \left(\left\langle \beta(s) v_{i_0}, D_{i_0}^* PD_{i_0}\beta(s) v_{i_0}\right\rangle\right) ds \\
&= \theta_{i_0} \int_0^{+\infty} |\beta(s)|^2 \left\|D_{i_0}^* PD_{i_0}\right\| ds \\
&= \theta_{i_0} \left\|D_{i_0}^* PD_{i_0}\right\| \int_0^{+\infty} |\beta(s)|^2 ds
\end{aligned}$$

Thus

$$\left\|Lv^0\right\|_{L^2_\omega}^2 = \theta_{i_0}\left\|D_{i_0}^* PD_{i_0}\right\| = \left(\max_{i\in\{1,\ldots,N\}}\left(\theta_i \left\|D_i^* PD_i\right\|\right)\right)$$

We conclude that

$$\|L\| = \frac{\left\|Lv^0\right\|_{L^2_\omega}}{\left\|v^0\right\|_{L^2_\omega}} = \left(\max_{i\in\overline{N}}\left(\theta_i \left\|D_i^* PD_i\right\|\right)\right)^{\frac{1}{2}}$$

■

Let $\alpha \in (0,+\infty)^N$, $\alpha = (\alpha_1,\ \alpha_2, ..., \alpha_N)$, then the perturbed system (3.1) remains unchanged if we replace D_i, E_i, Δ_i with $D_i^{\alpha_i}$, $E_i^{\alpha_i}$, $\Delta_i^{\alpha_i}$, where

$$D_i^{\alpha_i} = \alpha_i^{-1} D_i,\ E_i^{\alpha_i} = \alpha_i E_i,\ \Delta_i^{\alpha_i}(.) = \alpha_i \Delta_i(\alpha_i^{-1}.),\ i \in \overline{N}. \qquad (3.6)$$

More precisely, every solution of the perturbed system (3.1) is also a solution of the scaled perturbed system

$$x(t) = S(t)x_0 + \sum_{i=1}^{N}\int_0^t S(t-s) D_i^{\alpha_i}\Delta_i^{\alpha_i}\left(E_i^{\alpha_i} x(s)\right) dw_i(s),\ t > 0. \qquad (3.7)$$

The input-output operator $L^\alpha : L^2_\omega\left(\mathbb{R}^+,\ L^2(\Omega,\ U)\right) \longrightarrow L^2_\omega\left(\mathbb{R}^+,\ L^2(\Omega,\ Y)\right)$ of the system $\left(A; (D_i^{\alpha_i},\ E_i^{\alpha_i})_{i\in\overline{N}}\right)$ is given by

$$\begin{aligned}(L^\alpha v)(t) &= \left(L^\alpha \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}\right)(t) \\ &= \sum_{i=1}^N \int_0^t \begin{pmatrix} E_1^{\alpha_1} \\ \vdots \\ E_n^{\alpha_n} \end{pmatrix} S(t-s) D_i^{\alpha_i} v_i(s) dw_i(s),\ t>0\end{aligned}$$

Although the solutions of the perturbed system (3.1) and of the scaled system (3.7) are the same and do not depend on α, the input-output operator of the scaled system L^α does change with α and we will use this added freedom to get a complete characterization of the stability radius. First we obtain a lower bound.

Theorem 3.4 *Suppose that there exist* $\alpha = (\alpha_i)_{i\in\overline{N}} \in (0,+\infty)^N$, $P(\alpha) \in L^+(H)$ *satisfying*

$$2\langle Px,\ Ax\rangle + \langle E(\alpha)x,\ E(\alpha)x\rangle = 0,\ x \in D(A) \tag{3.8}$$

$$I - (\sigma/\alpha_j)^2 \theta_j D_j^* P(\alpha) D_j \succeq 0,\ j \in \overline{N} \tag{3.9}$$

where $\langle E(\alpha)x,\ E(\alpha)x\rangle = \sum_{i=1}^N \alpha_i^2 \langle E_i x, E_i x\rangle$. *Then* $r^w\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) \geq \sigma$

Proof. Let $\Delta = \bigoplus_1^N \Delta_i$ such that $\Delta_i \in Lip(Y_i, U_i)$, $i \in \overline{N}$, and $\|\Delta\|_{Lip} < \sigma$, and suppose that $\alpha \in (0,+\infty)^N$, $P(\alpha) \in L^+(H)$ are such that (3.8) and (3.9) hold. The unique solution (3.1) with initial condition $x(0) = x_0$ satisfies the scaled equation (3.7). Let $u_i^{\alpha_i}(t) = \Delta_i^{\alpha_i}\left(y_i^{\alpha_i}(t)\right),\ y_i^{\alpha_i}(t) = E_i^{\alpha_i}x(t)$, $t > 0$ and

$$\begin{aligned} E(\alpha) &= \begin{pmatrix} E_1^{\alpha_1} \\ \vdots \\ E_n^{\alpha_n} \end{pmatrix},\ y^{\alpha}(t) = \begin{pmatrix} y_1^{\alpha_1}(t) \\ \vdots \\ y_n^{\alpha_n}(t) \end{pmatrix}, \\ u^{\alpha}(t) &= \begin{pmatrix} u_1^{\alpha_1}(t) \\ \vdots \\ u_n^{\alpha_n}(t) \end{pmatrix},\ \Delta^{\alpha} = \bigoplus_1^N \Delta_i^{\alpha_i},\ t > 0. \end{aligned}$$

We have

$$y^{\alpha}(t) = E\left(\alpha\right)S(t)x_0 + \sum_{i=1}^{N}\int_0^t E\left(\alpha\right)S(t-s)D_i^{\alpha_i}u_i^{\alpha_i}(s)dw_i(s),\ t > 0. \quad (3.10)$$

For every T>0, define the truncations $u_{i,\,T}^{\alpha_i} \in L_\omega^2\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U_i\right)\right),\ i \in \overline{N}$, and $u_T^{\alpha} \in L_\omega^2\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$ by

$$u_{i,\,T}^{\alpha_i}(t) = \begin{cases} u_i^{\alpha_i}(t) = \Delta_i^{\alpha_i}\left(y_i^{\alpha_i}(t)\right) & \text{if } t \in [0,\ T], \\ 0 & \text{if } t > T, \end{cases} \qquad u_T^{\alpha}(t) = \begin{pmatrix} u_{1,T}^{\alpha_1}(t) \\ \vdots \\ u_{n,T}^{\alpha_n}(t) \end{pmatrix}.$$

Then

$$\begin{aligned} \|u_T^{\alpha}\|_{L_w^2}^2 &= \int_0^{+\infty} \mathcal{E}\left(\|u_T^{\alpha}(t)\|\right)^2 dt = \int_0^T \mathcal{E}\left(\|u_T^{\alpha}(t)\|\right)^2 dt \\ &= \int_0^T \mathcal{E}\left(\sum_{i=1}^N \left\|u_{i,T}^{\alpha}(t)\right\|^2\right) dt \\ &= \int_0^T \left(\sum_{i=1}^N \int_\Omega \left\|u_{i,T}^{\alpha}(t,w)\right\|^2 d\mu(\omega)\right) dt \\ &= \int_0^T \left(\sum_{i=1}^N \int_\Omega \left\|\Delta_i^{\alpha_i}\left(y_i^{\alpha_i}(t,w)\right)\right\|^2 d\mu(\omega)\right) dt \end{aligned}$$

It follows that

$$
\begin{aligned}
\|u_T^\alpha\|_{L_w^2}^2 &\leq \int_0^T \left(\sum_{i=1}^N \int_\Omega \|\Delta_i^{\alpha_i}\|_{Lip}^2 \|y_i^{\alpha_i}(t,w)\|^2 \, d\mu(\omega) \right) dt \\
&\leq \int_0^T \left(\sum_{i=1}^N \|\Delta_i^{\alpha_i}\|_{Lip}^2 \int_\Omega \|y_i^{\alpha_i}(t,w)\|^2 \, d\mu(\omega) \right) dt \\
&\leq \max_{i\in\{1,\ldots,N\}} \|\Delta_i^{\alpha_i}\|_{Lip}^2 \int_0^T \left(\sum_{i=1}^N \mathcal{E} \|y_i^{\alpha_i}(t)\|^2 \right) dt \\
&= \|\Delta^\alpha\|_{Lip}^2 \int_0^T \left(\mathcal{E} \|y^\alpha(t)\|^2 \right) dt
\end{aligned}
$$

Hence

$$
\|u_T^\alpha\|_{L_w^2}^2 \leq \|\Delta^\alpha\|_{Lip}^2 \int_0^{+\infty} \mathcal{E} \left(\|y^\alpha(t)\|\right)^2 dt \tag{3.11}
$$

Now define y_T^α as the output of the system $\left(A; (D_i^{\alpha_i},\ E_i^{\alpha_i})_{i\in\overline{N}}\right)$ generated by the input u_T^α with initial condition $x(0) = x_0$. Then

$$
y_T^\alpha(t) = E\left(\alpha\right) S(t) x_0 + L^\alpha u_T^\alpha(t),\ t > 0. \tag{3.12}
$$

which implies that

$$
\begin{aligned}
\left(\int_0^T \mathcal{E} \|y^\alpha(t)\|^2 \, dt \right)^{\frac{1}{2}} &\leq \|y_T^\alpha\|_{L_w^2} \\
&\leq \|E\left(\alpha\right) S(t) x_0\| + \|L^\alpha\| \, \|u_T^\alpha\|_{L_w^2}
\end{aligned}
$$

Using (3.11) we obtain

$$
\left(\int_0^T \mathcal{E} \|y^\alpha(t)\|^2 \, dt \right)^{\frac{1}{2}} \leq \|E\left(\alpha\right) S(t) x_0\| + \|L^\alpha\| \, \|\Delta^\alpha\| \left(\int_0^T \mathcal{E} \left(\|y^\alpha(t)\|\right)^2 dt \right)^{\frac{1}{2}} \tag{3.13}
$$

now condition (3.9) implies that

$$
1 - \left(\frac{\sigma}{\alpha_j} \right)^2 \theta_j \left\| D_j^* P\left(\alpha\right) D_j \right\| \geq 0,\ \text{for all } j \in \overline{N}.
$$

Thus

$$\max_{j\in\{1,\ldots,N\}}\theta_j\left\|\left(D_j^{\alpha_j}\right)^*P\left(\alpha\right)\left(D_j^{\alpha j}\right)\right\|\leq\sigma^{-2}$$

By Lemma 3.3, it follows that

$$\left\|L^\alpha\right\|^2\leq\sigma^{-2}$$

Now since $\|\Delta\|_{Lip}=\|\Delta^\alpha\|_{Lip}<\sigma$, the operator $L^\alpha\Delta^\alpha$ is a truncation on $L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y\right)\right)$ with ν=$\|L^\alpha\|\,\|\Delta^\alpha\|_{Lip}<1$. From (3.13) we get that

$$\left(\int_0^T\mathcal{E}\left\|y^\alpha(t)\right\|^2dt\right)^{\frac{1}{2}}\leq(1-\nu)^{-1}\left\|E\left(\alpha\right)S(t)x_0\right\|\ \text{ for all } T>0,$$

Therefore $y^\alpha\in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y\right)\right)$ and $u^\alpha=\Delta^\alpha(y^\alpha)\in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$. By Lemma 3.2, the solution $x(.)$ of (3.7) belongs to $L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ H\right)\right)$. We conclude then that $r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right)\geq\sigma$. ■

Remark 3.5 *1. From the above it follows that*

$$r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right)\geq\sup_{\alpha\in(0,\ +\infty)^N}\left(\max_{j\in\overline{N}}\left\|\left(\theta_j/\alpha_j^2\right)D_j^*P\left(\alpha\right)D_j\right\|\right)^{-1/2}.$$

2. If instead of $\succeq$ in (3.9) , the condition is satisfies with $\succ$, then we obtain $r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right)>\sigma$.

A similar result can be obtained if we replace the Lyapunov equation (3.8) by the corresponding Lyapunov inequality.

Corollary 3.6 *Suppose that there exist $\alpha\in(0,+\infty)^N$, $P\in L^+(H)$ satisfying*

$$2\left\langle Px,\ Ax\right\rangle+\left\langle E(\alpha)x,\ E(\alpha)x\right\rangle\leq0,\ x\in D(A)\qquad(3.14)$$

$$I-(\sigma/\alpha_j)^2\theta_jD_j^*PD_j\succeq0\ \textit{(resp. } I-(\sigma/\alpha_j)^2\theta_jD_j^*PD_j\succ0\textit{)},\ j\in\overline{N}\qquad(3.15)$$

Then $r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right)\geq\sigma$ (resp. $r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right)>\sigma$). In this case the Lyapunov equation (3.8) has a solution $P_0\in L^+(H)$ such that $P\succeq P_0$.

Proof. Because $S(t)$ is exponentially stable there exists a solution P_0 of the Lyapunov equation (3.8). Set $X=P-P_0$, then

$$2\left\langle Xx,\ Ax\right\rangle\preceq 0, x\in D(A)$$

Applying Lemma 2.1 in [9] we obtain that $X\succeq 0$, thus $P\succeq P_0$. Hence conditions (3.8) and (3.9) are satisfied. By applying Theorem 3.4 we deduce that $r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right)\geq\sigma$. ∎

For all $i,\ j\in\overline{N}$, define the operator $H_{ij}\in L(U_j)$ by

$$H_{ij}u_j=\theta_j\int_0^{+\infty}D_j^*S^*(t)E_i^*E_iS(t)D_ju_jdt.$$

Let $J\subset\overline{N}$ and $\alpha^J\in(0,\ \infty)^J$ where $(0,\ \infty)^J=\{(\alpha_j)_{j\in J},\alpha_j\in(0,\ \infty)\text{ for all }j\in J\}$.

The operator

$$P(\alpha^J)=\int_0^{+\infty}S^*(t)\left(\sum_{i\in J}\alpha_i^2E_i^*E_i\right)S(t)dt.$$

is the unique solution of the Lyapunov equation

$$2\left\langle Px,\ Ax\right\rangle+\left\langle x,\sum_{i\in J}\alpha_i^2E_i^*E_ix\right\rangle=0,\ x\in D(A). \tag{3.16}$$

Then

$$(\theta_j/\alpha_j^2)D_j^*P\left(\alpha^J\right)D_ju_j=\sum_{i\in J}\left(\alpha_i/\alpha_j\right)^2H_{ij}u_j,\ j\in J.$$

An important characterization of the stability radius $\left(r^\omega\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right)\right)$ will be given in the next theorem. This is done by adopting an approach similar to the one used in the finite dimensional case [39]. This approach is based on a result on a minimax problem for quadratic forms. Since the structure of the perturbations is bounded then this result can be generalized to the infinite dimensional case and as a consequence of this result we have the following lemma (for the proof see Appendix B).

Lemma 3.7 *Let* $\hat{\mu} = \inf_{\alpha \in (0,\ +\infty)^N} \max_{j \in \overline{N}} \left\| \left(\sum_{i=1}^{N} (\alpha_i/\alpha_j)^2 H_{ij} \right) \right\|$. *There exist a subset* $J \subset \overline{N}$ *and, for every* $\delta > 0$, *a vector* $\alpha \in (0, +\infty)^N$, *such that*

$$\begin{aligned} \left\| \left(\sum_{i=1}^{N} (\alpha_i/\alpha_j)^2 H_{ij} \right) \right\| &\leq \hat{\mu} + \delta, \quad j \in \overline{N}, \\ \left\| \left(\sum_{i \in J} (\alpha_i/\alpha_j)^2 H_{ij} \right) \right\| &= \hat{\mu}, \quad j \in J. \end{aligned}$$

By constructing a destabilizing perturbation Δ which norm is equal to $\hat{\mu}^{-1/2}$ we will show that $r^w \left(A; (D_i,\ E_i)_{i \in \overline{N}} \right) = \hat{\mu}^{-1/2}$. Notice that this result was given in [38] without proof.

Theorem 3.8 *Let* $\left(A; (D_i,\ E_i)_{i \in \overline{N}} \right)$ *and* $(w_i(t))_{i \in \overline{N}}$ *be as above. Then the associated stability radius is given by*

$$r^w \left(A; (D_i,\ E_i)_{i \in \overline{N}} \right) = \sup_{\alpha \in (0,\ +\infty)^N} \left(\max_{j \in \overline{N}} \left\| \left(\theta_j / \alpha_j^2 \right) D_j^* P(\alpha) D_j \right\| \right)^{-1/2} \tag{3.17}$$

where $P(\alpha)$ *is the unique solution of (3.8). If* $r^w \left(A; (D_i,\ E_i)_{i \in \overline{N}} \right) < +\infty$ *there exists a minimum norm destabilizing perturbation* $\Delta = \bigoplus_1^N \Delta_i$, $\|\Delta\|_{Lip} = r^w \left(A; (D_i,\ E_i)_{i \in \overline{N}} \right)$. *Moreover there exist a subset* $J \subset \overline{N}$ *and a scaling vector* $\alpha^J \in (0,\ \infty)^J$ *such that*

$$\begin{aligned} r^w \left(A; (D_i,\ E_i)_{i \in \overline{N}} \right) &= r^w \left(A; (D_i,\ E_i)_{i \in J} \right) \\ &= \left(\max_{j \in J} \left\| \left(\theta_j / \alpha_j^2 \right) D_j^* P\left(\alpha^J\right) D_j \right\| \right)^{-1/2} \end{aligned} \tag{3.18}$$

where $P(\alpha^J) \in L^+(H)$ is the unique solution of (3.16).

Proof. We have $\hat{\mu} = \inf_{\alpha\in(0,\ +\infty)^N} \max_{j\in\bar{N}} \left\| \left(\theta_j/\alpha_j^2\right) D_j^* P\left(\alpha\right) D_j \right\|$.

1. If $\hat{\mu} = 0$, it follows from Theorem 3.4 that $r^w\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) = +\infty$. Hence (3.17) is satisfied. Moreover

$$E_j S(t) D_j = 0,\ t > 0,\ j \in \overline{N}.$$

so (3.18) is satisfied for every singleton $J = \{j\} \subset \overline{N}$ and all $\alpha^J \in (0,\ \infty)^J$.

2. If $\hat{\mu} > 0$, we show that $r^w\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) \leq \hat{\mu}^{-1/2}$ by constructing a destabilizing perturbation Δ. For every $\alpha \in (0,\ \infty)^N$, let $\sigma(\alpha)$ be the largest σ for which (3.8) has a solution $P(\alpha) \in L^+(H)$ satisfying (3.9). For all $\sigma \leq \sigma(\alpha)$, we have

$$I - (\sigma/\alpha_j)^2 \theta_j D_j^* P(\alpha) D_j \succeq 0,\ j \in \overline{N}$$

this implies that

$$\sigma \leq \left(\max_{j\in\overline{N}} \left(\theta_j/\alpha_j^2\right) \left\| D_j^* P D_j \right\| \right)^{-1/2},\ \text{for any } j \in \overline{N}$$

Hence

$$\sigma(\alpha) = \left(\max_{j\in\overline{N}} \left\| \left(\theta_j/\alpha_j^2\right) D_j^* P\left(\alpha\right) D_j \right\| \right)^{-1/2}$$

Thus

$$\sup_{\alpha\in(0,\ +\infty)^N} \sigma^2(\alpha) = \hat{\mu}^{-1}$$

By Lemma 3.7, there exist $J \subset \overline{N}$ and a vector $\alpha^J = (\alpha_j)_{j\in J} \in (0,\ \infty)^J$ such that

$$\hat{\mu} = \left\| \sum_{i\in J} \left(\alpha_i/\alpha_j\right)^2 H_{ij} \right\| = \left\| \left(\theta_j/\alpha_j^2\right) D_j^* P\left(\alpha^J\right) D_j \right\|,\ j \in J.$$

where $P(\alpha^J) \in L^+(H)$ is the unique solution of (3.16). Let $v_j^0 \in U$, $\left\|v_j^0\right\|_U = 1$, $j \in J$ such that

$$\begin{aligned}\left\|\sum_{i\in J}\left(\alpha_i/\alpha_j\right)^2 H_{ij}\right\| &= \left\langle v_j^0,\ \sum_{i\in J}\left(\alpha_i/\alpha_j\right)^2 H_{ij}v_j^0\right\rangle \\ &= \left\langle v_j^0,\ \left(\theta j/\alpha_j^2\right) D_j^* P\left(\alpha^J\right) D_j v_j^0\right\rangle \\ &= \hat{\mu}\end{aligned}$$

It follows that

$$\left\langle v_j^0,\ \left(\theta j/\alpha_j^2\right)\hat{\mu}^{-1} D_j^* P\left(\alpha^J\right) D_j v_j^0\right\rangle = 1,\ \text{for all } j \in J$$

Setting $\hat{\sigma} = \hat{\mu}^{-\frac{1}{2}}$, we obtain

$$\left\langle v_j^0,\ \left(\theta j/\alpha_j^2\right)\hat{\sigma}^2 D_j^* P\left(\alpha^J\right) D_j v_j^0\right\rangle = 1,\ j \in J \tag{3.19}$$

Define for $j \in \overline{N}$ the perturbation $\Delta_j \in Lip\left(Y_j, U_j\right)$ by

$$\begin{cases}\Delta_j(y_j) = \hat{\sigma}\left\|y_j\right\| v_j^0, & j \in J, \quad y_j \in Y_j \\ \Delta_j(y_j) = 0\ , & j \in \overline{N}\backslash J.\end{cases}$$

Then $\left\|\Delta_j\right\|_{Lip} = \hat{\sigma}$, $j \in J$. It follows then that $\left\|\Delta\right\|_{Lip} = \hat{\sigma}$ where $\Delta = \bigoplus_1^N \Delta_i$. We will show that for this Δ the system (3.1) cannot be stable. Assume that this is not the case. Let $x_0 \in H$ the solution $x(.)$ of (3.1) satisfies

$$x(t) = S(t)x_0 + \sum_{j\in J}\int_0^t S(t-s) D_j^{\alpha_i}\Delta_j^{\alpha_i}\left(E_j^{\alpha j}x(s)\right) dw_j(s),\ t > 0. \tag{3.20}$$

where $D_j^{\alpha_j}$, $E_j^{\alpha_j}$, and $\Delta_j^{\alpha j}$ are defined in (3.6). Set $y_j^{\alpha_j} = E_j^{\alpha_j}x$, then $y_j^{\alpha_j} \in L_\omega^2\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y_j\right)\right)$, $j \in J$. Now for all $j \in J$ we have

$$\begin{aligned}\Delta_j^{\alpha_j}(y_j) &= \alpha_j \Delta_j(\alpha_j^{-1} y_j) \\ &= \hat{\sigma}\left\|y_j\right\| v_j^0,\ y_j \in Y_j,\ j \in J.\end{aligned}$$

Define y^{α_J} and E^{α_J} by

$$y^{\alpha_J} = \left(y_j^{\alpha_j}\right)_{j\in J}, \; E^{\alpha_J}x = \left(E_j^{\alpha_j}x\right)_{j\in J},$$

Then

$$\begin{aligned} y^{\alpha_J}(t) &= E^{\alpha_J}S(t)x_0 + \sum_{j\in J}\int_0^t E^{\alpha_J}S(t-s)D_j^{\alpha_j}\Delta_j^{\alpha_j}\left(y_j^{\alpha_j}(s)\right)dw_j(s),\; t>0 \\ &= E^{\alpha_J}S(t)x_0 + \hat{\sigma}\sum_{j\in J}\int_0^t E^{\alpha_J}S(t-s)D_j^{\alpha j}v_j^0\left\|y_j^{\alpha j}(s)\right\|dw_j(s),\; t>0 \end{aligned}$$

By applying Lemma 3.2 to this equation we obtain

$$\begin{aligned} \int_0^\infty \mathcal{E}\left\|y^{\alpha_J}(t)\right\|^2 dt &= \int_0^\infty \left\|E^{\alpha_J}S(t)x_0\right\|^2 dt \\ &\quad +\hat{\sigma}^2\sum_{j\in J}\theta_j\int_0^t \mathcal{E}\left\langle D_j^{\alpha_j}v_j^0\left\|y_j^{\alpha j}(s)\right\|,\; P\left(\alpha^J\right)D_j^{\alpha_j}v_j^0\left\|y_j^{\alpha j}(s)\right\|\right\rangle ds \\ &= \int_0^\infty \left\|E^{\alpha_J}S(t)x_0\right\|^2 dt \\ &\quad +\hat{\sigma}^2\sum_{j\in J}\theta_j/\alpha_j^2\left\langle v_j^0,\; D_j^*P\left(\alpha^J\right)D_jv_j^0\right\rangle\int_0^\infty \mathcal{E}\left\|y_j^{\alpha j}(s)\right\|^2 ds \end{aligned}$$

By (3.19) we get

$$\int_0^\infty \mathcal{E}\left\|y^{\alpha_J}(t)\right\|^2 dt = \int_0^\infty \left\|E^{\alpha_J}S(t)x_0\right\|^2 dt + \int_0^\infty \mathcal{E}\left\|y^{\alpha_J}(s)\right\| ds$$

for all $x_0 \in H$. This identity implies that $E_j = 0$ for every $j \in J$, hence $P(\alpha^J) = 0$, thus $\hat{\mu} = 0$. Therefore neither of the stochastic system (3.1) nor (3.20) is L^2-stable. It follows that

$$\begin{aligned} r^w\left(A;(D_i,\; E_i)_{i\in\overline{N}}\right) &\leq r^w\left(A;(D_i,\; E_i)_{i\in J}\right) \\ &\leq \left(\max_{j\in J}\left\|\left(\theta_j/\alpha_j^2\right)D_j^*P\left(\alpha^J\right)D_j\right\|\right)^{-1/2} \\ &= \hat{\mu}^{-1/2} \end{aligned}$$

By Remark 3.5 it follows that $r^w\left(A;(D_i,\ E_i)_{i\in J}\right) \geq \hat{\mu}^{-1/2}$. In conclusion we have

$$\begin{aligned} r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right) &\leq r^w\left(A;(D_i,\ E_i)_{i\in J}\right) \\ &\leq \left(\max_{j\in J}\left\|\left(\theta_j/\alpha_j^2\right) D_j^* P\left(\alpha^J\right) D_j\right\|\right)^{-1/2} \\ &\leq r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right) \end{aligned}$$

■

Remark 3.9 *1. Using Lemma 3.3, we deduce from (3.17) that*

$$r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right) = \sup_{\alpha\in(0,\ +\infty)^N}\left\|L^\alpha\right\|^{-1} = r^w\left(A;(D_i^{\alpha_i},\ E_i^{\alpha_i})_{i\in\overline{N}}\right)$$

2. *In the deterministic case, the stability radii for complex and real multi-perturbations are, in general different. This is not the case in the stochastic one. From the above proof, we see that the destabilizing perturbation does not depend on the choice of the field K. Thus the real and stability radii coincide.*

3. *In the deterministic case, the complex stability radius was characterized in terms of a parametrized Riccati equation (see [35]). In the stochastic case, we proved in the above theorem that the stability radius is characterized just in terms of a single Lyapunov equation without parameters.*

The above Theorem enables us to obtain the following result.

Corollary 3.10 *Let $\sigma > 0$ such that $r^w\left(A;(D_i,\ E_i)_{i\in\overline{N}}\right) > \sigma$. Then there exist*

$\alpha_i > 0$, $i \in \overline{N}$, and $P \in L^+(H)$ such that

$$\begin{aligned} 2\langle Px,\ Ax\rangle + \langle E(\alpha)x,\ E(\alpha)x\rangle &= 0,\ x \in D(A), \\ I - (\sigma/\alpha_j)^2\theta_j D_j^* P D_j &\succ 0,\ j \in \overline{N}. \end{aligned} \tag{3.21}$$

Proof. Let $\sigma' \in \left]\sigma,\ r^w\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right)\right[$. It follows from Theorem 3.8, that there exists $\alpha \in (0,\ \infty)^N$ such that

$$\sigma' < \left(\max_{j\in\overline{N}} \left\|\left(\theta_j/\alpha_j^2\right) D_j^* P\left(\alpha\right) D_j\right\|\right)^{-1/2}$$

where $P\left(\alpha\right)$ is the solution of the equation (3.8). This implies that

$$\sigma^{-2} > \left\|\left(\theta_j/\alpha_j^2\right) D_j^* P\left(\alpha\right) D_j\right\|,\ \text{for all } j \in \overline{N}.$$

Thus

$$\theta_j\left(\sigma/\alpha_j\right)^2 D_j^* P(\alpha) D_j \prec I,\ j \in \overline{N}\,.$$

Therefore $P(\alpha)$ satisfies the equation (3.8) and the condition (3.21). ■

The following corollary gives a characterization of the stability radius with the strict Lyapunov inequality

$$2\langle Px,\ Ax\rangle + \langle E(\alpha)x,\ E(\alpha)x\rangle < 0,\ x \in D(A) \tag{3.22}$$

Corollary 3.11 *Let $\sigma > 0$ such that $r^w\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) > \sigma$. Then the inequality (3.22) has a solution $P \in L^+(H)$, $P \succ 0$ for some $\alpha \in (0, +\infty)^N$ which satisfies $I - (\sigma/\alpha_j)^2\theta_j D_j^* P D_j \succ 0$, $j \in \overline{N}$.*

Proof. Let $\sigma > 0$ such that $r^w\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) > \sigma$. By the previous proof there exists $\alpha \in (0,\ \infty)^N$ such that

$$\theta_j\left(\sigma/\alpha_j\right)^2 D_j^* P(\alpha) D_j \prec I,\ j \in \overline{N},$$

where $P(\alpha)$ is the solution of the equation (3.8). Hence

$$\theta_j \left(\sigma/\alpha_j\right)^2 D_j^* \left[\int_0^{+\infty} S^*(t)E^*(\alpha)E(\alpha)S(t)dt\right] D_j \prec I,\ j \in \overline{N}\,.$$

This implies that there exists $\varepsilon > 0$ such that

$$\theta_j \left(\sigma/\alpha_j\right)^2 D_j^* \left[\int_0^{+\infty} S^*(t)\left(E^*(\alpha)E(\alpha) + \varepsilon I\right) S(t)dt\right] D_j \prec I,\ j \in \overline{N}\,.$$

Set $P = \int_0^{+\infty} S^*(t)\left(E^*(\alpha)E(\alpha) + \varepsilon I\right) S(t)dt$, then $P \succ 0$ satisfies

$$\begin{aligned} 2\langle Px,\ Ax\rangle + \langle x,\ \left(E^*(\alpha)E(\alpha) + \varepsilon I\right) x\rangle &= 0,\ x \in D(A) \\ I - \theta_j \left(\sigma/\alpha_j\right)^2 D_j^* P D_j &\succ 0,\ j \in \overline{N}. \end{aligned}$$

■

3.4 Examples

In this section we give some examples in which we illustrate the useful of Theorem 3.8 to calculate the stochastic stability radius.

3.4.1 Stability radius of some partial differential equations

Example 3.12 *Consider the stochastic heat equation*

$$\begin{aligned} dz(t,x) &= \frac{\partial^2 z(t,x)}{\partial x^2}dt + \sum_{i=1}^{N} \Delta_i(z(t,x))d\omega_i(t), \qquad 0 < x < 1, t > 0 \qquad (3.23) \\ z(0,x) &= z_0(x), \qquad 0 < x < 1 \\ z(t,0) &= z(t,1) = 0, \qquad t \geq 0 \end{aligned}$$

where $\Delta_i \in Lip(L^2(0,1))$ and ω_i are independent Wiener processes of variance θ_i.System (3.23) can be formulated as an abstract differential equation on the *space* $H = L^2(0,1)$ *of the form*

$$\begin{aligned} dz(t) &= Az(t)dt + \sum_{i=1}^{N} \Delta_i(z(t))d\omega_i(t), \qquad t > 0 \\ z(0) &= z_0 \end{aligned} \tag{3.24}$$

where the operator A *defined by*

$$Ah = \frac{\partial^2 h}{\partial x^2} \quad \textit{with} \quad D(A) = H^2(0,\ 1) \cap H_0^1(0,\ 1)$$

is the infinitesimal generator of an exponentially stable C_0*-semigroup* $S(t)$ *given by (see [8])*

$$S(t)h = \sum_{n=1}^{+\infty} e^{\lambda_n t} \langle h, \Phi_n \rangle \Phi_n,\ \ \lambda_n = -n^2\pi^2 \ \textit{and}\ \Phi_n = \sqrt{2}\sin(n\pi x)\,,\ n \geqslant 1. \tag{3.25}$$

Since in this example $D_i = E_i = I$, *for all* $i \in \{1, ..., N\}$, *it follows from Theorem (3.8) that*

$$r^{\omega}\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) = \sup_{\alpha\in(0,+\infty)^N} \left(\max_{j\in\overline{N}} \left(\theta_j/\alpha_j^2\right) \|P(\alpha)\| \right)^{-\frac{1}{2}} \tag{3.26}$$

where $P(\alpha)$ *is the unique solution of the Lyapunov equation*

$$2\langle PAz, z\rangle + \sum_{i=1}^{N} \alpha_i^2 \langle z, z\rangle = 0,\ \ z \in D(A).$$

For all $z \in D(A)$ *we have*

$$2\langle PAz, z\rangle + \sum_{i=1}^{N} \alpha_i^2 \langle z, z\rangle = 0 \Leftrightarrow 2\langle PAz, z\rangle + \beta\langle z, z\rangle = 0, \quad \beta = \sum_{i=1}^{N} \alpha_i^2$$

The unique solution of this equation is

$$P(\alpha)z = \beta \int_0^{+\infty} S^*(t)S(t)z dt,\ \ z \in H$$

Since A is self-adjoint then

$$P(\alpha)z = \beta \int\limits_{0}^{+\infty} S^2(t)zdt = \beta \int\limits_{0}^{+\infty} S(2t)zdt, \ \textit{for all } z \in H.$$

Using (3.25) we obtain

$$\begin{aligned} P(\alpha)\, z &= \beta \int\limits_{0}^{+\infty} \sum_{n=1}^{+\infty} e^{2\lambda_n t} \langle z, \Phi_n \rangle \Phi_n dt \\ &= \beta \sum_{n=1}^{+\infty} \left(\int_0^{+\infty} e^{2\lambda_n t} dt \right) \langle z, \Phi_n \rangle \Phi_n \\ &= -\beta \sum_{n=1}^{+\infty} \frac{1}{2\lambda_n} \langle z, \Phi_n \rangle \Phi_n \end{aligned}$$

Hence

$$P(\alpha)z = \left(\sum_{i=1}^{n} \alpha_i^2 \right) \sum_{n=1}^{+\infty} \frac{1}{n^2\pi^2} \sin(n\pi x) \int_0^1 z(s) \sin(n\pi s)\, ds$$

From (3.26) we get

$$r^{\omega}\left(A, (D_i,\ E_i)_{i\in\overline{N}}\right) = \sup_{\alpha\in(0,+\infty)^N} \min_{j\in\{1,\dots,N\}} \left(\left(\alpha_j/\sqrt{\theta_j}\right) \|P(\alpha)\|^{-\frac{1}{2}} \right)$$

We can show that

$$\|P(\alpha)\| = \langle P(\alpha)\Phi_1, \Phi_1 \rangle = \frac{\beta}{2\pi^2}$$

Hence

$$\begin{aligned} r^{\omega}\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) &= \sup_{\alpha\in(0,+\infty)^N} \left(\min_{j\in\{1,\dots,N\}} \left(\left(\alpha_j/\sqrt{\theta_j}\right) \sqrt{\frac{2\pi^2}{\beta}} \right) \right) \\ &= \sup_{\alpha\in(0,+\infty)^N} \left(\min_{j\in\{1\dots,N\}} \left(\left(\alpha_j/\sqrt{\sum_{i=1}^{N} \alpha_i^2} \right) \frac{\sqrt{2}\pi}{\sqrt{\theta_j}} \right) \right) \\ &= \min_{j\in\{1,\dots,N\}} \left(\frac{\sqrt{2}\pi}{\sqrt{\theta_j}} \right) \end{aligned}$$

Therefore

$$r^{\omega}\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) = \min_{i\in\{1,\dots,N\}} \left\{ \frac{\sqrt{2}\pi}{\sqrt{\theta_i}} \right\}$$

If $N = 1, \theta_1 = 1$ and $\Delta(z(t)) = cz(t)$ where c is an uncertain real parameter, then we deduce from the above analysis that the system (3.24) is mean square stable provided that $|c| < \sqrt{2}\pi$. The same result was obtained by Ichikawa [44] using a stochastic version of Lyapunov Theorem.

Example 3.13 *We consider the following second order system*

$$\begin{aligned}
\frac{\partial^2 y(t,x)}{\partial t^2} &= \frac{\partial^2 y(t,x)}{\partial x^2} - \gamma \frac{\partial y(t,x)}{\partial t} + \Delta_1(y(t,x))\dot{\omega_1}(t) + \Delta_2(\frac{\partial y(t,x)}{\partial t})\dot{\omega_2}(t), \\
t &> 0, \quad 0 < x < 1 \qquad (3.27) \\
y(0,x) &= y_0(x), \ \frac{\partial y(0,x)}{\partial t} = y_1(x), \qquad 0 < x < 1 \\
y(t,0) &= y(t,1) = 0, \qquad t > 0
\end{aligned}$$

*where γ is a given positive constant, $\Delta_1 \in Lip(H_0^1(0,1), L^2(0,1))$, $\Delta_2 \in Lip(L^2(0,1), L^2(0,1))$ and $\omega_i(t)$, $i = 1,2$, are two independent Wiener processes with variance θ_i. In order to represent the system (**??**) as an abstract ordinary differential equation, we define*

$$z = \begin{pmatrix} y \\ \frac{\partial y}{\partial t} \end{pmatrix}, \qquad H = H_0^1(0,1) \times L^2(0,1)$$

Let I_1 and I_2 be respectively the identity operator in $L(H_0^1(0,1))$ and $L(L^2(0,1))$. Let $A : L^2(0,1) \to L^2(0,1)$ be defined as

$$Ay = -\frac{\partial^2 y}{\partial x^2}, \qquad \text{for } y \in D(A) = H_0^1(0,1) \cap H^2(0,1)$$

Next we introduce the operators $A_\gamma : H \to H$ defined by

$$A_\gamma z = \begin{pmatrix} 0 & I_2 \\ -A & -\gamma I_2 \end{pmatrix} z, \qquad \text{for } z \in D(A_\gamma) = D(A) \times H_0^1(0,1)$$

$E_1 : H \to H_0^1(0,1)$ defined by

$$E_1 = \begin{pmatrix} I_1 & 0 \end{pmatrix}$$

$E_2 : H \to L^2(0,1)$ defined by

$$E_2 = \begin{pmatrix} 0 & I_2 \end{pmatrix}$$

$D : L^2(0,1) \to H$ defined by

$$D = \begin{pmatrix} 0 \\ I_2 \end{pmatrix}$$

*With the above notations, we rewrite (**??**) in the following abstract form on the space $H = H_0^1\,(0,1) \times\, L^2\,(0,1)$*

$$dz(t) = A_\gamma z(t)dt + \sum_{i=1}^{2} D\Delta_i(E_i z(t))d\omega_i(t) \tag{3.28}$$

A_γ generates an exponentially stable semigroup $S_\gamma(t)$ (see [67]). Applying Theorem 3.8 we get

$$r^\omega\left(A_\gamma; (D_i,\ E_i)_{i=1,2}\right) = \sup_{\alpha\in(0,+\infty)^2}\left(\max_{j\in\{1,2\}}\left(\left\|\left(\theta_j/\alpha_j^2\right) D^* P(\alpha) D\right\|\right)\right)^{-\frac{1}{2}}$$

where $P(\alpha)$ is the unique solution of the Lyapunov equation

$$2\left\langle A_\gamma z, P\ z\right\rangle + \sum_{i=1}^{2}\alpha_i^2\left\langle E_i z, E_i z\right\rangle = 0,\ \ z \in D(A_\gamma)$$

At first we shall solve this equation. Let $z \in D(A_\gamma)$ and set $z = \begin{pmatrix} x \\ y \end{pmatrix}$, we have

$$2\left\langle A_\gamma z, P\ z\right\rangle + \sum_{i=1}^{2}\alpha_i^2\left\langle E_i z, E_i z\right\rangle = 0 \Leftrightarrow 2\left\langle A_\gamma z, P\ z\right\rangle + \alpha_1^2\left\langle Ax, x\right\rangle + \alpha_2^2\left\langle y, y\right\rangle = 0$$

Set $P(\alpha) = \begin{pmatrix} P_1 & P_2 \\ & \\ P_3 & P_4 \end{pmatrix}$. *Since* $P(\alpha)$ *is self-adjoint then* P_1, P_4 *are self-adjoint operators and* $P_3 = P_2^* A$. *Then*

$$\begin{aligned}
\langle A_\gamma z, P\ z\rangle_H &= \left\langle \begin{pmatrix} 0 & I \\ -A & -\gamma I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} P_1 & P_2 \\ P_2 A & P_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_H \\
&= \left\langle \begin{pmatrix} y \\ -Ax - \gamma y \end{pmatrix}, \begin{pmatrix} P_1 x + P_2 y \\ P_2 A x + P_4 y \end{pmatrix} \right\rangle_H \\
&= \left\langle A^{1/2} y, A^{1/2} \left(P_1 x + P_2 y\right) \right\rangle_{L^2(0,1)} + \left\langle -Ax - \gamma y, P_2 A x + P_4 y \right\rangle_{L^2(0,1)} \\
&= \langle Ay, P_1 x\rangle_{L^2(0,1)} + \langle Ay, P_2 y\rangle_{L^2(0,1)} - \langle Ax, P_2 A x\rangle_{L^2(0,1)} \\
&\quad - \langle Ax, P_4 y\rangle_{L^2(0,1)} - \gamma \langle y, P_2 A x\rangle_{L^2(0,1)} - \gamma \langle y, P_4 y\rangle_{L^2(0,1)}
\end{aligned}$$

Replacing in the Lyapunov equation we obtain

$$\begin{aligned}
&2(\langle Ay, P_1 x\rangle_{L^2(0,1)} + \langle Ay, P_2 y\rangle_{L^2(0,1)} - \langle Ax, P_2 A x\rangle_{L^2(0,1)} - \langle Ax, P_4 y\rangle_{L^2(0,1)} \\
&-\gamma \langle y, P_2 A x\rangle_{L^2(0,1)} - \gamma \langle y, P_4 y\rangle_{L^2(0,1)}) + \alpha_1^2 \langle Ax, x\rangle_{L^2(0,1)} + \alpha_2^2 \langle y, y\rangle_{L^2(0,1)} = 0
\end{aligned}$$

hence

$$\begin{aligned}
&\left(\alpha_1^2 \langle Ax, x\rangle_{L^2(0,1)} - 2 \langle Ax, P_2 A x\rangle_{L^2(0,1)}\right) + 2(\langle Ay, P_1 x\rangle_{L^2(0,1)} - \langle Ax, P_4 y\rangle_{L^2(0,1)} - \\
&\gamma \langle y, P_2 A x\rangle_{L^2(0,1)}) - 2\gamma \langle y, P_4 y\rangle_{L^2(0,1)} + 2 \langle Ay, P_2 y\rangle_{L^2(0,1)} + \alpha_2^2 \langle y, y\rangle_{L^2(0,1)} = 0
\end{aligned} \tag{3.29}$$

For

$$\begin{cases} P_1 = \left(\alpha_1^2/2\right) \gamma A^{-1} + \left(\left(\alpha_1^2 + \alpha_2^2\right)/2\gamma\right) I \\ P_2 = \left(\alpha_1^2/2\right) A^{-1} \\ P_4 = \left(\left(\alpha_1^2 + \alpha_2^2\right)/2\gamma\right) I \end{cases}$$

equation (3.29) holds. We deduce that

$$P(\alpha) = \begin{pmatrix} \left(\alpha_1^2/2\right)\gamma A^{-1} + \left(\left(\alpha_1^2+\alpha_2^2\right)/2\gamma\right) I & \left(\alpha_1^2/2\right) A^{-1} \\ \left(\alpha_1^2/2\right) I & \left(\left(\alpha_1^2+\alpha_2^2\right)/2\gamma\right) I \end{pmatrix}$$

Since $D^*P(\alpha)D = P_4$, it follows that

$$r^{\omega}\left(A_\gamma; (E_i,\ D_i)_{i=1,2}\right) = \sup_{\alpha\in(0,+\infty)^2}\left(\max_{j\in\{1,2\}}\left(\left\|\left(\theta_j/\alpha_j^2\right)\left(\left(\alpha_1^2+\alpha_2^2\right)/2\gamma\right) I\right\|\right)\right)^{-\frac{1}{2}}$$

Hence

$$r^{\omega}\left(A_\gamma; (E_i,\ D_i)_{i=1,2}\right) = \min_{i\in\{1,2\}}\left(\sqrt{2\gamma}/\sqrt{\theta_i}\right)$$

*In the particular case where $\Delta_2(\frac{\partial y(t,x)}{\partial t}) = 0$, $\Delta_1(y(t,x)) = c\frac{\partial y(t,x)}{\partial t}$ where c is an uncertain real parameter and $\theta_1 = 1$, we deduce from the value of r^{ω} $\left(A_\gamma; (E_i,\ D_i)_{i=1,2}\right)$ that the system (**??**) is L^2-stable for all c satisfying $|c| < \sqrt{2\gamma}$. This bound is larger than*

$$\left(\frac{4\pi^2\gamma}{4\pi^2 + \gamma\left(\gamma + \sqrt{\gamma^2+4\pi^2}\right)}\right)^{\frac{1}{2}}$$

obtained by Curtain in [7].

3.4.2 Stability radius of some delay equations

Let b be a positive number and $r_1, ..., r_k$, be real numbers satisfying

$$-b = r_k < r_{k-1} < ... < r_1 < r_0 = 0$$

Consider the perturbed linear delay differential equation

$$\begin{cases} dx(t) = \left(A_0 x(t) + \sum_{i=1}^{k} A_i x(t+r_i)\right) dt + \Delta\left(x(t)\right) dw(t),\ t > 0 \\ x(0) = v, \\ x(t) = h\left(t\right),\ \ -b \le t < 0. \end{cases} \tag{3.30}$$

where $A_i \in L(\mathbb{R}^N)$, $i = 1, ..., k$, $v \in \mathbb{R}^N$, $h \in L^2\left(-b,\ 0;\ \mathbb{R}^N\right)$, $\Delta \in L(\mathbb{R}^N)$ and $w(t)$ is a real Wiener process. Taking the space $H = M_2\left(-b,\ 0;\ \mathbb{R}^N\right) = \mathbb{R}^N \times L^2\left(-b,\ 0;\ \mathbb{R}^N\right)$ and the new state $z(t) = \begin{pmatrix} x(t) \\ x(t+r) \end{pmatrix}$ in H and the operator

$$Az = A\begin{pmatrix} y(0) \\ y \end{pmatrix} = \begin{pmatrix} A_0 y(0) + \sum\limits_{i=1}^{k} A_i y(r_i) \\ \frac{dy}{dr} \end{pmatrix}$$

with domain

$$D(A) = \left\{ \begin{pmatrix} y(0) \\ y \end{pmatrix};\ y,\ \text{abs. cont. and } y' \in L^2\left(-b,\ 0;\ \mathbb{R}^N\right) \right\}$$

the system (3.30) can be formulated as an abstract differential equation

$$\begin{cases} dz(t) = Az(t)dt + D\Delta E(z(t))dw(t), \\ z(0) = z_0 \end{cases} \tag{3.31}$$

on the state space H, where $D = \begin{bmatrix} I_{\mathbb{R}^N} \\ 0 \end{bmatrix}$, $E = \begin{bmatrix} I_H & 0 \end{bmatrix}$.

Suppose that A generates an exponentially stable semigroup. Our objective is to calculate the stability radius $r^w\left(A, (D,\ E)\right)$. By Theorem 3.8

$$r^\omega\left(A, (D,\ E)\right) = \sup_{\alpha \in (0, +\infty)} \left(\left(\theta/\alpha^2\right)\|D^* P(\alpha) D\|\right)^{-\frac{1}{2}}$$

where $P(\alpha)$ is the unique solution of the Lyapunov equation

$$2\left\langle P\left(\alpha\right)Az, z\right\rangle + \alpha^2\left\langle E^* Ez, z\right\rangle = 0,\ \ z \in D(A).$$

Using Theorem A.4, the nonnegative solution $P(\alpha) \in \mathcal{L}(H)$ is given by

$$P(\alpha) \;=\; \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}, \text{ where } P_{00} \in L(\mathbb{R}^N),\ P_{01} \in L\left(L^2\left(-b,\ 0;\ \mathbb{R}^N\right),\ \mathbb{R}^N\right),$$

$$P_{10} \;\in\; L\left(\mathbb{R}^N,\ L^2\left(-b,\ 0;\ \mathbb{R}^N\right)\right) \text{ and } P_{11} \in L\left(L^2\left(-b,\ 0;\ \mathbb{R}^N\right)\right).$$

P_{00} is characterized by the equation

$$P_{00}A_0 + A_0^* P_{00} + P_{10}(0) + P_{10}(0)^* + \alpha^2 I = 0,\ P_{00} = (P_{00})^* \succeq 0. \quad (3.32)$$

P_{10} is characterized in the following way:

$$\left(P_{10}y^0\right)(r) = P_{10}(r)y^0,\ y^0 \in \mathbb{R}^N \quad (3.33)$$

where the map

$$r \longmapsto P_{10}(r) : [-b,\ 0] \longrightarrow L(\mathbb{R}^N) \quad (3.34)$$

is piecewise absolutely continuous with jumps at $\alpha = r_i$ of height $A_i^* P_{00}$, $i = 1, ..., k-1$. Moreover the map (3.34) is itself characterized by the differential equation

$$\begin{aligned} \frac{dP_{10}}{dr}(r) &= P_{10}(r)A_0 + \sum_{i=1}^{k-1} A_i^* P_{00}\delta(r - r_i) + P_{11}(r,\ 0), \quad (3.35) \\ &\quad \text{ae. in } \ [-b,\ 0], \\ P_{10}(-b) &= A_k^* P_{00}, \end{aligned}$$

where $\delta(r - r_i)$ is the δ-function at $r = r_i$.

P_{01} is obtained from P_{10} :

$$P_{01}y^1 = \int_{-b}^{0} P_{10}(r)^* y^1(r)dr,\ y^1 \in L^2\left(-b,\ 0;\ \mathbb{R}^N\right). \quad (3.36)$$

P_{11} is characterized in the following way:

$$\left(P_{11}y^1\right)(r) = \int_{-b}^{0} P_{11}(r,\beta)y^1(\beta)d\beta$$

where the map

$$(r,\ \beta) \longmapsto P_{11}(r,\ \beta) : [-b,\ 0] \times [-b,\ 0] \longrightarrow L(\mathbb{R}^N) \tag{3.37}$$

is piecewise absolutely continuous in each variable with jumps of height $A_i^* P_{10}(\beta)^*$at $r = r_i,\ i = 1, ..., k-1$ (resp. $P_{10}(r) A_j$ at $\beta = r_j,\ j = 1, ..., k-1$). Moreover the map $P_{11}(r,\ \beta)$ is the solution of

$$\left(\frac{\partial}{\partial r} + \frac{\partial}{\partial \beta}\right) P_{11}(r,\beta) = \sum_{i=1}^{k-1} A_i^* P_{10}(\beta)^* \delta(r - r_i) + \sum_{i=1}^{k-1} P_{10}(r) A_i \delta(\beta - r_j) \tag{3.38}$$

with boundary conditions

$$P_{11}(-b,\ \beta) = A_k^* P_{10}(\beta)^*,\ P_{11}(r, -b\ ) = P_{10}(r) A_k, \tag{3.39}$$

and symmetry property $P_{11}(r,\ \beta) = P_{11}(\beta,\ r)^*$. The solution of the above differential system is

$$\begin{aligned} P_{11}(r,\ \beta) \quad &= \quad \begin{cases} P_{10}(r-\beta-b)A_k,\ r \geqslant \beta \\ A_k^* P_{10}(\beta-r-b)^*,\ r < \beta \end{cases} \\ &+ \sum_{i=1}^{k-1} \begin{cases} A_i^* P_{10}(\beta - r + r_i)^*, & -b \leq \beta - r + r_i,\ r_i < r \\ 0, & \text{otherwise} \end{cases} \\ &+ \sum_{j=1}^{k-1} \begin{cases} P_{10}(r - \beta + r_j)A_j, & -b \leq r - \beta + r_j,\ r_j < \beta \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

For $\beta = 0$, we obtain

$$\begin{aligned}
P_{11}(r,\ 0) &= A_k^* P_{10}(-r-b)^* \\
&+\sum_{i=1}^{k-1}\begin{cases} A_i^* P_{10}(-r+r_i)^*, & -b \le -r+r_i,\ r_i < r \\ 0, & \text{otherwise} \end{cases} \\
&+\sum_{j=1}^{k-1}\begin{cases} P_{10}(r+r_j)A_j, & -b \le r+r_j,\ r_j < 0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \tag{3.40}$$

P_{10} is then the solution of the differential equation

$$\begin{aligned}
\frac{dP_{10}}{dr}(r) &= P_{10}(r)A_0 + \sum_{i=1}^{k-1} A_i^* P_{00}\delta(r-r_i) + A_k^* P_{10}(-r-b)^* \\
&+\sum_{i=1}^{k-1}\begin{cases} A_i^* P_{10}(-r+r_i)^*, & -b \le -r+r_i,\ r_i < r \\ 0, & \text{otherwise} \end{cases} \qquad (3.41) \\
&+\sum_{j=1}^{k-1}\begin{cases} P_{10}(r+r_j)A_j, & -b \le r+r_j,\ r_j < 0 \\ 0, & \text{otherwise} \end{cases}, \\
P_{10}(-b) &= A_k^* P_{00},
\end{aligned}$$

Because of the difficulty in resolving this differential equation we shall calculate the stability radius $r^w(A,(D,\ E))$ in the case of one delay for a scalar and a two dimensional system.

Example 3.14 *Consider the scalar delay system*

$$\begin{cases} dx(t) = (-a_0 x(t) + a_1 x(t-b))\,dt + \Delta(x(t))dw(t), & t \ge 0 \\ x(0) = x_0,\ x(s) = 0, & -b \le s < 0 \end{cases} \tag{3.42}$$

where $a_0, a_1, b > 0$, Δ is a Lipschitzian function and $w(t)$ is a scalar wiener process with variance θ. If $a_0 > a_1$, then the zero of

$$\Delta(\lambda) = \lambda + a_0 - a_1 e^{-\lambda b}$$

is negative, so the operator A, corresponding to this system, generates an exponentially stable semigroup (see Theorem A.2). The differential equation (3.41) corresponding to this system is

$$\begin{aligned}\frac{dP_{10}}{dr}(r) &= -a_0P_{10}(r) + a_1P_{10}(-r-b),\ \ in\ [-b,\ 0] \qquad (3.43)\\ P_{10}(-b) &= a_1P_{00} \qquad (3.44)\end{aligned}$$

By deriving (3.43) we get

$$\begin{aligned}\frac{d^2P_{10}}{dr^2}(r) &= -a_0\left(-a_0P_{10}(r) + a_1P_{10}(-r-b)\right)\\ &\quad -a_1\left(-a_0P_{10}(-r-b) + a_1P_{10}(-(-r-b)-b)\right)\\ &= \left(a_0^2 - a_1^2\right)P_{10}(r)\end{aligned}$$

We obtain then the following second order boundary problem

$$\begin{aligned}\frac{d^2P_{10}}{dr^2}(r) &= \left(a_0^2 - a_1^2\right)P_{10}(r) \qquad (3.45)\\ \frac{dP_{10}}{dr}(0) &= -a_0P_{10}(0) + a_1P_{10}(-b) \qquad (3.46)\\ P_{10}(-b) &= a_1P_{00} \qquad (3.47)\end{aligned}$$

$P_{10}(0)$ *is then given by*

$$P_{10}(0) = a_1\left(\frac{2\tau e^{\tau b} + a_1e^{2\tau b} - a_1}{(\tau - a_0) + (\tau + a_0)\,e^{2\tau b}}\right)P_{00},\ \ \tau = \sqrt{(a_0^2 - a_1^2)} \qquad (3.48)$$

Equation (3.32) is then equivalent to

$$-2a_0P_{00} + 2P_{10}(0) + \alpha^2 = 0$$

Using (3.48) we obtain

$$-2a_0P_{00} + 2a_1P_{00}\left(\frac{2\tau e^{\tau b} + a_1e^{2\tau b} - a_1}{(\tau - a_0) + (\tau + a_0)\,e^{2\tau b}}\right) + \alpha^2 = 0$$

Hence

$$P_{00} = -\frac{\alpha^2}{2}K^{-1}, \quad K = \left(-a_0 + a_1 \frac{2\tau e^{\tau b} + a_1 e^{2\tau b} - a_1}{(\tau - a_{0'}) + (\tau + a_0)\, e^{2\tau b}}\right)$$

The stability radius is then given by

$$r^{\omega}\left(A; (D,\ E)\right) = \sup_{\alpha \in (0, +\infty)} \left(\left(\theta/\alpha^2\right) \|D^* P(\alpha) D\|\right)^{-\frac{1}{2}} = \sup_{\alpha \in (0, +\infty)} \left(\left(\theta/\alpha^2\right) \|P_{00}\|\right)^{-\frac{1}{2}}$$

It follows that

$$r^{\omega}\left(A; (D,\ E)\right) = \frac{\sqrt{2}}{\sqrt{\theta}} K^{\frac{1}{2}}$$

We conclude that the stability radius of the system (3.42) is given by

$$r^{\omega}\left(A; (D,\ E)\right) = \frac{\sqrt{2}}{\sqrt{\theta}} K^{\frac{1}{2}}, \quad K = \left(-a_0 + a_1 \frac{2\tau e^{\tau b} + a_1 e^{2\tau b} - a_1}{(\tau - a_0) + (\tau + a_0)\, e^{2\tau b}}\right).$$

Hence for all Δ *such that* $\|\Delta\| < \frac{\sqrt{2}}{\sqrt{\theta}} K^{\frac{1}{2}}$, *the system (3.42) is stable. Let* Δ *such that* $\Delta x(t) = hg(cx(t))$, *where* g *is a Lipschitzian function with Lipschitz constant* M, $h, c \in \mathbb{R}$ *and* $\theta = 1$. *If* $M^2 < 2\left(a_0 - a_1\right)/h^2 c^2$, *we can show that* $\|\Delta\|^2 < 2K$, *it follows that the system (3.42) is stable, which is the same result established in [46].*

Now we consider the following system treated in [76].

Example 3.15 *Consider the following delay system*

$$\begin{cases} dx(t) = A_0 x(t) + A_1 x(t-1) + \Delta(x(t)) dw(t) \\ x(t) = \varphi(t),\ t \in [-1, 0] \end{cases} \tag{3.49}$$

where $A_0 = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix}$, $A_1 = \begin{bmatrix} -0.1 & -0.5 \\ 0.2 & 0.6 \end{bmatrix}$,*and* Δ *is a Lipschitzian function. We have*

$$\Delta(\lambda) = \det\left(\lambda I - A_0 - A_1 e^{-\lambda}\right)$$

The zero of $\det(\Delta(\lambda))$ is negative, so the operator A , corresponding to this system, generates an exponentially stable semigroup. The differential equation (3.41) corresponding to this system is equivalent to

$$\begin{aligned} \frac{dP_{10}}{dr}(r) &= P_{10}(r)A_0 + A_1^* P_{10}^*(-r-1), \text{ in } [-1,\ 0] \\ P_{10}(-1) &= A_1^* P_{00} \end{aligned} \tag{3.50}$$

By deriving (3.50) we get

$$\frac{d^2P_{10}}{dr^2}(r) = P_{10}(r)A_0^2 + A_1^* P_{10}(r)A_1 + A_1^* P_{10}(-r-1)A_0 - A_1^* A_0^* P_{10}^*(-r-1) \tag{3.51}$$

Set $P_{10}(r) = \begin{bmatrix} f_1(r) & f_2(r) \\ f_2(r) & f_3(r) \end{bmatrix}$, and suppose that $f_2 = \frac{f_1+f_3}{2}$, then

$$P_{10}(-r-1)A_0 - A_0^* P_{10}^*(-r-1) = 0$$

Equation (3.51) is then equivalent to:

$$\frac{d^2P_{10}(r)}{dr^2} = \begin{bmatrix} 2.99f_1 + 6.04f_2 - 0.04f_3 & -6.05f_1 + 15.16f_2 - 0.12f_3 \\ -0.05f_1 + 3.16f_2 + 5.88f_3 & -0.25f_1 - 5.4f_2 + 14.64f_3 \end{bmatrix} \tag{3.52}$$

Since $f_2 = \frac{f_1+f_3}{2}$, equation (3.52) yields:

$$\begin{cases} \frac{d^2f_1(r)}{dr^2} = 6.01f_1 + 2.98f_3 \\ \frac{d^2f_3(r)}{dr^2} = -2.95f_1 + 11.94f_3 \end{cases}$$

Set $z(r) = \begin{pmatrix} f_1(r) \\ f_3(r) \end{pmatrix}$, hence

$$\frac{d^2 z(r)}{dr^2} = Mz(r) \tag{3.53}$$

where $M = \begin{bmatrix} 6.01 & 2.98 \\ -2.95 & 11.94 \end{bmatrix}$. The matrix M can be decomposed as follows

$$M = P_1 D P_1^{-1}; \text{ where } D = \begin{bmatrix} 8.96 & 0 \\ 0 & 8.99 \end{bmatrix} \text{ and } P_1 = \begin{bmatrix} 1.0102 & 1 \\ 1 & 1 \end{bmatrix}$$

Therefore (3.53) is equivalent to

$$\frac{d^2 z(r)}{dr^2} = P_1 D P_1^{-1} z(r)$$

Set $y(r) = \begin{pmatrix} y_1(r) \\ y_2(r) \end{pmatrix} = P_1^{-1} z(r)$, then $y(r)$ satisfy the following second order differential equation

$$\frac{d^2 y(r)}{dr^2} = Dy(r)$$

Thus

$$\begin{cases} y_1(r) = c_1 e^{\sqrt{8.96}} + c_2 e^{-\sqrt{8.96}} \\ y_2(r) = c_3 e^{\sqrt{8.99}} + c_4 e^{-\sqrt{8.99}} \end{cases} \tag{3.54}$$

To obtain c_1, c_2, c_3,and c_4, we use the following boundary conditions

$$\begin{cases} P_{10}(-1) = A_1^* P_{00} \\ \frac{dP_{10}(0)}{dr} = P_{10}(0) A_0 + A_1^* P_{10}^* (-1) \end{cases}$$

By setting $P_{00} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$, these conditions yield the linear system $SC = V$, where

$$S = \begin{bmatrix} 5.0632 \times 10^{-2} & 20.155 & 0.04987 & 20.052 \\ 0.05012 & 19.952 & 0.04987 & 20.052 \\ 6.0494 & 1.6422 \times 10^{-3} & 5.9983 & 1.6671 \times 10^{-3} \\ 5.9882 & 1.5741 \times 10^{-3} & 5.9983 & 1.6671 \times 10^{-3} \end{bmatrix}, \ C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$ and

$$V = \begin{pmatrix} 4.2858 \times 10^{-2} a_1 + 5.7142 \times 10^{-2} a_3 \\ -0.35715 a_1 + 0.45715 a_3 \\ 4.5715 \times 10^{-2} a_3 - 3.5715 \times 10^{-2} a_1 \\ 0.14572 a_3 - 0.13572 a_1 \end{pmatrix}$$

The solution of this system is given by

$$C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1.6319 a_1 - 1.6319 a_3 \\ 1.9664 a_1 - 1.9664 a_3 \\ -1.6517 a_1 + 1.6534 a_3 \\ -1.9743 a_1 + 1.9793 a_3 \end{pmatrix}$$

Using (3.54), we get

$$\begin{cases} f_1(r) = 1.0102 \left(c_1 e^{\sqrt{8.96} r} + c_2 e^{-\sqrt{8.96} r} \right) + c_3 e^{\sqrt{8.99} r} + c_4 e^{-\sqrt{8.99} r} \\ f_3(r) = c_1 e^{\sqrt{8.96} \alpha} + c_2 e^{-\sqrt{8.96} r} + c_3 e^{\sqrt{8.99} r} + c_4 e^{-\sqrt{8.99} r} \end{cases} \tag{3.55}$$

Substituting the expressions of c_1, c_2, c_3 and c_4 in (3.55) we get

$$\begin{cases} f_1(r) = 1.010\,2\left((1.631\,9a_1 - 1.631\,9a_3)\,e^{\sqrt{8.96}r} + (1.966\,4a_1 - 1.966\,4a_3)\,e^{-\sqrt{8.96}r}\right) \\ \qquad + (-1.651\,7a_1 + 1.653\,4a_3)\,e^{\sqrt{8.99}r} + (-1.974\,3a_1 + 1.979\,3a_3)\,e^{-\sqrt{8.99}r} \\ f_3(r) = (1.631\,9a_1 - 1.631\,9a_3)\,e^{\sqrt{8.96}r} + (1.966\,4a_1 - 1.966\,4a_3)\,e^{-\sqrt{8.96}r} \\ \qquad + (-1.651\,7a_1 + 1.653\,4a_3)\,e^{\sqrt{8.99}r\alpha} + (-1.974\,3a_1 + 1.979\,3a_3)\,e^{-\sqrt{8.99}r} \end{cases}$$

From which we obtain

$$\begin{cases} f_1(0) = 0.009\,a_1 - 0.002\,3a_3 \\ f_3(0) = 0.034\,4a_3 - 0.027\,7a_1 \end{cases} \tag{3.56}$$

and since $f_2 = \frac{f_1+f_3}{2}$, it follows that

$$f_2(0) = 0.016\,05a_3 - 0.009\,35a_1 \tag{3.57}$$

These expressions will be used to obtain P_{00} the solution of the following algebraic equation:

$$P_{00}A_0 + A_0^*P_{00} + P_{10}(0) + P_{10}^*(0) + \alpha^2E^*E = 0$$

Using (3.56) and (3.57) we obtain

$$a_{1=}0.167\,04\alpha^2,\ a_2 = 0.167\,04\alpha^2,\ a_3 = 0.167\,04\alpha^2$$

We deduce that :

$$P_{00} = 0.167\,04\alpha^2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Let us now calculate the stability radius associated to this system

$$
\begin{aligned}
r^{w}\left(A;(D,\ E)\right) &= \sup_{\alpha>0}\left(\left(\theta/\alpha^{2}\right)\left\|D^{*}P(\alpha)D\right\|\right)^{\frac{-1}{2}} \\
&= \sup_{\alpha>0}\left(\alpha/\sqrt{\theta}\right)\left(\left\|\begin{pmatrix} I & 0 \end{pmatrix}\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}\begin{pmatrix} I \\ 0 \end{pmatrix}\right\|\right)^{\frac{-1}{2}}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
r^{w}\left(A;(D,\ E)\right) &= \frac{1}{\sqrt{\theta}}\left(0.167\,04\right)^{-\frac{1}{2}}\left(2\right)^{-\frac{1}{2}} \\
&= 1.\,223\,4\frac{\sqrt{2}}{\sqrt{\theta}}
\end{aligned}
$$

Hence for any Δ such that $\|\Delta\| < 1.\,223\,4\left(\sqrt{2}/\sqrt{\theta}\right)$, the system (3.49) is stable. For Δ such that $\Delta\left(x(t)\right)=\begin{bmatrix} -0.1\cos x_{1}(t) \\ -0.1\cos x_{2}(t) \end{bmatrix}$ and $\theta = 1$, we have $\|\Delta\|_{Lip} \leq 0.1 < \left(\sqrt{2}/\sqrt{\theta}\right)1.\,223\,4$, we deduce that system (3.49) is stable. We can show the same result using the results of [76].

Chapter 4

Maximizing the stochastic stability radius by state feedback

4.1 Introduction

In this chapter we consider controlled stochastic systems described by Ito equations of the form

$$\begin{aligned} dx(t) &= Ax(t)dt + \sum_{i=1}^{N} D_i \Delta_i \left(E_i x(t)\right) dw_i(t) + Bu(t)dt, \ t \in \Re^+ \quad (4.1) \\ x(0) &= x_0 \in H \quad (4.2) \end{aligned}$$

where u takes its values in the real separable Hilbert space Z, $B \in L(Z, H)$ and the other operators are as in the previous Chapter. In addition we assume that $(A,\ B)$ is stabilizable. Our aim is to characterize the supremum of the stability radii which can be achieved by linear state feedback $u = Fx$, where $F \in L(H, Z)$. Let

$$\overline{\mathcal{F}} = \left\{ \begin{array}{c} F \in L(H, Z);\ A + BF \text{ is the infinitesimal generator of} \\ \text{an exponentially stable semigroup } S_F(t) \end{array} \right\}$$

and define

$$\overline{r^{\omega}} \left(A; (D_i,\ E_i)_{i \in \overline{N}} \right) = \sup \left\{ r^{\omega} \left(A + BF; (D_i,\ E_i)_{i \in \overline{N}} \right);\ F \in\ \overline{\mathcal{F}} \right\}$$

We follow the approach developed in [22] to investigate this problem. For all $F \in \overline{\mathcal{F}}$, $\alpha \in (0,\ +\infty)^N$ and $\varepsilon > 0$ consider the Lyapunov inequality

$$2 \langle P(A + BF)x,\ x \rangle + \langle E(\alpha)x,\ E(\alpha)x \rangle + \varepsilon^2 \langle Fx,\ Fx \rangle \preceq 0,\ x \in D(A) \tag{4.3}$$

with

$$I - \sigma^2 \left(\theta_j / {\alpha_j}^2 \right) D_j^* P D_j \succeq 0,\ j \in \overline{N} \tag{4.4}$$

where

$$\langle E(\alpha)x,\ E(\alpha)x \rangle = \sum_{i=1}^{N} \alpha_i^2 \langle E_i x,\ E_i x \rangle$$

4.2 Conditions of suboptimality

In order to establish conditions for the existence of suboptimal controllers $u(t) = Fx(t)$ such that $F \in \overline{\mathcal{F}}$ and $\sigma < r^{\omega} \left(A + BF; (D_i,\ E_i)_{i \in \overline{N}} \right)$, for $\sigma > 0$, we need the following lemmas.

This lemma is of technical interest.

Lemma 4.1 *Let $\alpha \in (0,\ \infty)^N$ and $\varepsilon > 0$. If there exists $P \in L^+(H)$ such that*

$$2\left\langle Px,\ \left(A - \varepsilon^{-2}BB^*P\right)x\right\rangle + \varepsilon^{-2}\left\langle PBB^*Px,\ x\right\rangle +$$

$$\langle E(\alpha)x,\ E(\alpha)x\rangle \leq 0, x \in D(A) \tag{4.5}$$

$$I - \theta_j\left(\sigma/\alpha_j\right)^2 D_j^* P D_j \succeq 0,\ j \in \overline{N}, \tag{4.6}$$

then $A_\varepsilon = A - \varepsilon^{-2}BB^*P$ generates an exponentially stable semigroup and $\sigma \leq r^\omega\left(A_\varepsilon;\ (D_i,\ E_i)_{i\in\overline{N}}\right)$.

Proof. Consider the initial value problem

$$\begin{cases} \frac{d}{dt}x(t) = A_\varepsilon x(t) \\ x(0) = x_0 \end{cases} \tag{4.7}$$

For $x_0 \in D(A_\varepsilon)$, $V(x) = \langle x,\ Px\rangle$ is differentiable and

$$\frac{d}{dt}V\left(x(t)\right) = 2\left\langle PA_\varepsilon x, x\right\rangle$$

From the inequality (4.5) we obtain

$$\begin{aligned} \frac{d}{dt}V\left(x(t)\right) &\leq -\varepsilon^{-2}\left\langle PBB^*Px,\ x\right\rangle - \left\langle E(\alpha)x, E(\alpha)\ x\right\rangle \\ &\leq -\varepsilon^{-2}\left\langle PBB^*Px,\ x\right\rangle \end{aligned}$$

Thus

$$\int_0^T \dot{V}(x(t))dt \leq -\varepsilon^{-2}\int_0^T \left\langle PBB^*Px(t),\ x(t)\right\rangle dt$$

Hence

$$V\left(x(T)\right) - V\left(x(0)\right) \leq -\varepsilon^{-2}\int_0^T \left\|B^*Px(t)\right\|^2 dt$$

Using the fact that $P \succeq 0$ we get

$$\varepsilon^{-2} \int_0^T \|B^* Px(t)\|^2 \, dt \leq V(x_0), \text{ for all } T > 0$$

Therefore

$$\varepsilon^{-2} \int_0^{+\infty} \|B^* Px(t)\|^2 \, dt \leq k \|x_0\|^2$$

which implies that $B^* Px(t) \in L^2(\mathbb{R}^+, Z)$. The solution $x(t)$ of the system (4.7) is given by

$$x(t) = S(t)x_0 - \varepsilon^{-2} \int_0^t S(t-s)BB^* Px(s)ds$$

We have

$$\begin{aligned}
\|x(t)\| &\leq \|S(t)x_0\| + \varepsilon^{-2} \left\| \int_0^t S(t-s)BB^* Px(s)ds \right\| \\
&\leq Me^{-\omega t} \|x_0\| + \varepsilon^{-2} M \|B\| \int_0^t e^{-\omega(t-s)} \|B^* Px(s)\| \, ds
\end{aligned}$$

from which we get

$$\begin{aligned}
\|x(t)\|^2 &\leq 2M^2 \|x_0\|^2 e^{-2\omega t} + 2\varepsilon^{-4} M^2 \|B\|^2 \left[\int_0^t e^{-\omega(t-s)} \|B^* Px(s)\| \, ds \right]^2 \\
&\leq K_1 e^{-2\omega t} + K_2 \int_0^t e^{-2\omega(t-s)} \|B^* Px(s)\|^2 \, ds
\end{aligned}$$

where

$$K_1 = 2M^2 \|x_0\|^2, \ K_2 = 2\varepsilon^{-4} M^2 \|B\|^2$$

It follows then that

$$\int_0^{+\infty} \|x(t)\|^2 \, dt \leq K_1 \int_0^{+\infty} e^{-2\omega t} dt + K_2 \int_0^{+\infty} \left(\int_0^t e^{-2\omega(t-s)} \|B^* Px(s)\|^2 \, ds \right) dt$$

Thus

$$\begin{aligned}
\int_0^{+\infty} \|x(t)\|^2 \, dt &\leq K_1/2\omega + K_2 \int_0^{+\infty} \left(\int_s^{+\infty} e^{-2\omega(t-s)} \|B^* Px(s)\|^2 \, dt \right) ds \\
&\leq K_1/2\omega + K_2 \int_0^{+\infty} \|B^* Px(s)\|^2 e^{2\omega s} \left(\int_s^{+\infty} e^{-2\omega t} dt \right) ds
\end{aligned}$$

which implies that

$$\int_0^{+\infty} \|x(t)\|^2 \, dt \leq K_1/2\omega + \frac{K_2}{2\omega} \int_0^{+\infty} \|B^* P x(s)\|^2 \, ds$$

Since $B^* P x(t) \in L^2(\mathbb{R}^+, Z)$, we deduce that $x(t)$ belongs to the space $L^2(\mathbb{R}^+, H)$.

Now inequality (4.5) implies that

$$2 \langle P A_\varepsilon x,\ x \rangle + \langle E(\alpha) x, E(\alpha)\ x \rangle \leq -\varepsilon^{-2} \|B^* P x(t)\|^2$$

Therefore, we have

$$2 \langle P A_\varepsilon x,\ x \rangle + \langle E(\alpha) x, E(\alpha)\ x \rangle \leq 0$$

$$I - \sigma^2 \left(\theta_j / {\alpha_j}^2\right) D_j^* P D_j \succeq 0,\ j \in \overline{N}$$

$$F_\varepsilon = -\varepsilon^{-2} B^* P \in \overline{\mathcal{F}}$$

Applying Corollary 3.6 we get that $\sigma \leq r^\omega \left(A_\varepsilon; (D_i,\ E_i)_{i \in \overline{N}}\right)$ and if the inequality (4.6) is strict, we obtain $\sigma < r^\omega \left(A_\varepsilon; (D_i,\ E_i)_{i \in \overline{N}}\right)$. ■

The following lemma is of basic importance for the approach used to investigate the maximization problem.

Lemma 4.2 *Let $\alpha \in (0, +\infty)^N$, $\varepsilon > 0$ and $F \in \overline{\mathcal{F}}$. If the inequality (4.3) has a solution $P_0 \in L^+(H)$ satisfying condition (4.4), then $F_0 = -\varepsilon^{-2} B^* P_0 \in \overline{\mathcal{F}}$ and $\sigma \leq r^\omega \left(A + BF_0; (D_i, E_i)_{i \in \overline{N}}\right)$. Moreover, there exists $P_1 \in L^+(H)$ such that*

$$2 \langle P_1 (A + BF_0) x,\ x \rangle + \langle E(\alpha) x,\ E(\alpha) x \rangle + \varepsilon^{-2} \langle P_0 B B^* P_0 x,\ x \rangle = 0,\ x \in D(A)$$

$$I - \sigma^2 \left(\theta_j / {\alpha_j}^2\right) D_j^* P_1 D_j \succeq 0,\ j \in \overline{N}$$

$$P_1 \preceq P_0$$

Proof. For all $x \in D(A)$, we have

$$\begin{aligned} & 2\langle P_0 (A+BF) x,\ x\rangle + \langle E(\alpha)x, E(\alpha)\ x\rangle + \varepsilon^2 \langle Fx,\ Fx\rangle \\ = \ & 2\langle P_0 Ax,\ x\rangle + \langle E(\alpha)x,\ E(\alpha)x\rangle + 2\langle P_0 BFx,\ x\rangle + \varepsilon^2 \langle Fx,\ Fx\rangle \end{aligned}$$

Set $F' = \varepsilon F + \varepsilon^{-1} B^* P_0$. Then

$$\langle F'x, F'x\rangle = \varepsilon^2 \langle Fx,\ Fx\rangle + \langle Fx,\ B^*P_0x\rangle + \langle B^*P_0x,\ Fx\rangle + \varepsilon^{-2}\langle B^*P_0x,\ B^*P_0x\rangle$$

Since P_0 is a solution of (4.3) it follows that

$$2\langle P_0Ax,\ x\rangle + \langle E(\alpha)x,\ E(\alpha)x\rangle - \varepsilon^{-2}\langle B^*P_0x,\ B^*P_0x\rangle + \langle F'x,\ F'x\rangle \le 0 \quad (4.8)$$

Set $A_0 = A + BF_0$, where $F_0 = -\varepsilon^{-2}B^*P_0$, then

$$2\langle P_0A_0x,\ x\rangle = 2\langle P_0Ax,\ x\rangle - 2\varepsilon^{-2}\langle P_0BB^*P_0x,\ x\rangle$$

The inequality (4.8) implies that

$$2\langle P_0A_0x,\ x\rangle \preceq -\langle E(\alpha)x, E(\alpha)\ x\rangle - \varepsilon^{-2}\langle P_0BB^*P_0x,\ x\rangle - \langle F'x, F'x\rangle$$

Thus

$$2\langle P_0A_0x,\ x\rangle + \varepsilon^{-2}\langle P_0BB^*P_0x,\ x\rangle + \langle E(\alpha)x, E(\alpha)x\rangle \le 0 \qquad (4.9)$$

Applying Lemma 4.1 we conclude that $F_0 \in \overline{\mathcal{F}}$ and $\sigma \le r^{\omega}\left(A_{0;}\,(D_i, E_i)_{i\in\overline{N}}\right)$. Now since P_0 is a solution of the inequality (4.9), then it satisfies the following inequality

$$2\langle P_0A_0x,\ x\rangle + \left\langle \tilde{E}(\alpha)x, \tilde{E}(\alpha)\ x\right\rangle \le 0$$

where $\tilde{E}(\alpha) = \begin{bmatrix} E(\alpha) \\ \varepsilon^{-1}B^*P_0 \end{bmatrix}$. By Corollary 3.6, there exists $P_1 \in L^+(H)$ such that

$$2\langle P_1A_0x,\ x\rangle + \left\langle \tilde{E}(\alpha)x, \tilde{E}(\alpha)\ x\right\rangle = 0$$

with $P_1 \preceq P_0$. Therefore

$$2\left\langle P_1 A_0 x,\ x\right\rangle + \varepsilon^{-2}\left\langle x, P_0 B B^* P_0\ x\right\rangle + \left\langle E(\alpha)x, E(\alpha)\ x\right\rangle = 0$$

and

$$I - \sigma^2\left(\theta_j/{\alpha_j}^2\right) D_j^* P_1 D_j \succeq 0,\ j \in \overline{N}$$

■

Applying this lemma iteratively we show in the following theorem that there exists $P \in L^+(H)$ such that

$$\begin{array}{rcll} 2\left\langle Ax, Px\right\rangle + \left\langle E(\alpha)x, E(\alpha)x\right\rangle - \varepsilon^{-2}\left\langle x, PBB^*Px\right\rangle & = & 0, & ((\mathrm{SARE})_\varepsilon) \\ x & \in & D(A) & \\ I - \sigma^2\left(\theta_j/\alpha_j^2\right) D_j^* P D_j & \succeq & 0,\ j \in \overline{N} & \end{array}$$

Theorem 4.3 *Let $F \in \overline{\mathcal{F}}$. Suppose that there exist $\alpha \in (0, +\infty)^N$ and $\varepsilon > 0$ such that the Lyapunov inequality (4.3) has a solution $P_0 \in L^+(H)$ which satisfies condition (4.4). Then the Riccati equation (SARE)$_\varepsilon$ has a solution $P \in L^+(H)$ satisfying*

$$I - \sigma^2\left(\theta_j/{\alpha_j}^2\right) D_j^* P D_j \succeq 0,\ j \in \overline{N}$$

$$F_\varepsilon = -\varepsilon^{-2} B^* P \in \overline{\mathcal{F}}$$

$$\sigma \leq r^\omega\left(A - \varepsilon^{-2} BB^*P; (D_i,\ E_i)_{i\in\overline{N}}\right)$$

Proof. Applying the above lemma iteratively we construct a sequence of linear operators $(P_k)_{k\in\aleph} \in L^+(H)$ which satisfy

$$2\langle P_{k+1}A_k x,\ x\rangle + \langle E(\alpha)x, E(\alpha)\ x\rangle + \varepsilon^{-2}\langle x, P_k BB^* P_k\ x\rangle = 0,\ x \in D(A)$$

$$I - \sigma^2\left(\theta_j/{\alpha_j}^2\right) D_j^* P_{k+1} D_j \succeq 0,\ j \in \overline{N}$$

$$P_{k+1} \preceq P_k$$

where P_0 is the solution of the inequality (4.3) and $A_k = A - \varepsilon^{-2}BB^*P_k$. Since $(P_k)_{k\in\aleph}$ is a decreasing sequence and it is bounded from below by 0, the limit as $k \to +\infty$ exists (see [53]). Let $P = \lim\limits_{k\to+\infty} P_k$, then

$$2\langle PA_\varepsilon x,\ x\rangle + \langle E(\alpha)x, E(\alpha)\ x\rangle + \varepsilon^{-2}\langle x, PBB^*P\ x\rangle = 0,\ x \in D(A)$$

$$I - \sigma^2\left(\theta_j/{\alpha_j}^2\right) D_j^* P D_j \succeq 0,\ j \in \overline{N}$$

where $A_\varepsilon = A - \varepsilon^{-2}BB^*P$. Using Lemma 4.1 we deduce that $F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathcal{F}}$ and $\sigma \le r^\omega\left(A - \varepsilon^{-2}BB^*P; (D_i,\ E_i)_{i\in\overline{N}}\right)$. Finally since

$$\begin{aligned} & 2\langle PA_\varepsilon x,\ x\rangle + \langle E(\alpha)x, E(\alpha)\ x\rangle + \varepsilon^{-2}\langle x, PBB^*P\ x\rangle \\ = \ & 2\langle PAx,\ x\rangle + \langle E(\alpha)x, E(\alpha)\ x\rangle - \varepsilon^{-2}\langle x, PBB^*P\ x\rangle \end{aligned}$$

then P satisfies the Riccati equation $(\text{SARE})_\varepsilon$. ■

Now we will use the above results to establish conditions for the existence of suboptimal controllers.

Proposition 4.4 *Let $\sigma > 0$. Suppose that there exists $F \in \overline{\mathcal{F}}$ such that $\sigma < r^\omega\left(A + BF; (D_i,\ E_i)_{i\in\overline{N}}\right)$. Then there exist $\alpha \in (0,\ \infty)^N$, $\varepsilon > 0$ such that the Riccati equation (SARE)$_\varepsilon$ has a solution $P \in L^+(H)$ satisfying*

$$I - \sigma^2\left(\theta_j/{\alpha_j}^2\right) D_j^* P D_j \succeq 0,\ j \in \overline{N}$$

$$F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathcal{F}}$$

Proof. Since $\sigma < r^{\omega}\left(A+BF;(D_i,\ E_i)_{i\in\overline{N}}\right)$, there exists σ' such that $\sigma < \sigma' < r^{\omega}\left(A+BF;(D_i,\ E_i)_{i\in\overline{N}}\right)$. Hence there exists $\alpha\in(0,+\infty)^N$ such that

$$\sigma' \leq \left(\max_{j\in\overline{N}}\left\|\left(\theta_j/\alpha_j^2\right)D_j^*P\left(\alpha\right)D_j\right\|\right)^{-1/2}$$

where $P(\alpha)$ is the solution of the equation

$$2\left\langle P(A+BF)x,\ x\right\rangle+\left\langle E(\alpha)x,E(\alpha)\ x\right\rangle=0,\ x\in D(A).$$

Thus

$$\sigma < \left(\max_{j\in\overline{N}}\left\|\left(\theta_j/\alpha_j^2\right)D_j^*P\left(\alpha\right)D_j\right\|\right)^{-1/2}$$

or

$$\sigma^{-2} > \left(\max_{j\in\overline{N}}\left\|\left(\theta_j/\alpha_j^2\right)D_j^*P\left(\alpha\right)D_j\right\|\right)$$

Hence for all $j\in\overline{N}$, we have

$$\sigma^{-2} > \left\|\left(\theta_j/\alpha_j^2\right)D_j^*P\left(\alpha\right)D_j\right\|$$

from which we deduce that

$$I-\sigma^2\left(\theta_j/{\alpha_j}^2\right)D_j^*P(\alpha)D_j\succ 0,\ j\in\overline{N,}$$

This implies that

$$\theta_j\left(\sigma/\alpha_j\right)^2D_j^*\left[\int_0^{+\infty}S_F^*(t)E^*(\alpha)E(\alpha)S_F(t)dt\right]D_j\prec I,\ j\in\overline{N}\,.$$

where $S_F(t)$ is the semigroup generated by $(A+BF)$. Let $\beta_j=\theta_j\left(\sigma/\alpha_j\right)^2$ and P_0 the solution of the Lyapunov equation

$$2\left\langle PA_Fx,\ x\right\rangle+\left\langle Fx,F\ x\right\rangle=0,\ x\in D(A)$$

For all $\varepsilon > 0$ such that $\varepsilon^2 < (1 - \beta M) / \beta M_0$, with $M = \max_{j \in \overline{N}} \left\| D_j^* P(\alpha) D_j \right\|$ and $M_0 = \max_{j \in \overline{N}} \left\| D_j^* P_0 D_j \right\|$, we have

$$\varepsilon^2 \beta \max_{j \in \overline{N}} \left\| D_j^* P_0 D_j \right\| < 1 - \beta \max_{j \in \overline{N}} \left\| D_j^* P(\alpha) D_j \right\|$$

It follows that

$$\varepsilon^2 \beta \left\| D_j^* P_0 D_j \right\| < 1 - \beta \left\| D_j^* P(\alpha) D_j \right\|, \text{ for all } j \in \overline{N}.$$

Using the fact that

$$\left\| D_j^* P(\alpha) D_j \right\| = \sup_{u \neq 0} \frac{\left\langle D_j^* P(\alpha) D_j u, \ u \right\rangle}{\|u\|^2} \text{ and } \left\| D_j^* P_0 D_j \right\| = \sup_{u \neq 0} \frac{\left\langle D_j^* P_0 D_j u, \ u \right\rangle}{\|u\|^2}$$

we get, for all $j \in \overline{N}$, that

$$\|u\|^2 > \beta \left(\left\langle D_j^* P(\alpha) D_j u, \ u \right\rangle + \varepsilon^2 \left\langle D_j^* P_0 D_j u, \ u \right\rangle \right), \text{ for all } u \neq 0.$$

Thus

$$\|u\|^2 > \beta \left\langle D_j^* \left(P(\alpha) + \varepsilon^2 P_0 \right) D_j u, \ u \right\rangle, \text{ for all } u \neq 0.$$

Set $P_\varepsilon = P(\alpha) + \varepsilon^2 P_0$. Then

$$\|u\|^2 > \beta \left\langle D_j^* P_\varepsilon D_j u, \ u \right\rangle, \text{ for all } u \neq 0.$$

Hence

$$I - \beta D_j^* P_\varepsilon D_j \succ 0, \text{ for all } j \in \overline{N}.$$

We deduce that there exists $\varepsilon > 0$ such that $P_\varepsilon \succeq 0$ and

$$2 \left\langle P_\varepsilon (A + BF) x, \ x \right\rangle + \left\langle x, \ \left(E^*(\alpha) E(\alpha) + \varepsilon^2 F^* F \right) x \right\rangle = 0, x \in D(A),$$

$$I - \theta_j \left(\sigma / \alpha_j \right)^2 D_j^* P_\varepsilon D_j \succ 0, \ j \in \overline{N}.$$

Applying Theorem 4.3 to deduce that there exists $X \in L^{+}(H)$ which satisfies

$$2\langle Ax,\ Xx\rangle + \langle E(\alpha)x,\ E(\alpha)x\rangle - \varepsilon^{-2}\langle x, XBB^{*}X\ x\rangle = 0, x \in D(A),$$

$$I - \theta_j\left(\sigma/\alpha_j\right)^2 D_j^{*}XD_j \succ 0,\ j \in \overline{N}.$$

and for which $F_\varepsilon = -\varepsilon^{-2}B^{*}X \in \overline{\mathcal{F}}$. ■

Proposition 4.5 *Let* σ, $\varepsilon > 0$. *Suppose that the Riccati equation (SARE)* $_\varepsilon$ *has a solution* P *in* $L^{+}(H)$ *such that* $I - \sigma^2\left(\theta_j/{\alpha_j}^2\right)D_j^{*}PD_j \succeq 0$, $j \in \overline{N}$, *for some* $\alpha \in (0, +\infty)^N$. *Then* $F_\varepsilon = -\varepsilon^{-2}B^{*}P \in \overline{\mathcal{F}}$ *and* $\sigma \leq r^{\omega}\left(A + BF_\varepsilon; (D_i,\ E_i)_{i\in\overline{N}}\right)$.

Proof. Since P is a solution of the Riccati equation (SARE) $_\varepsilon$, then

$$2\left\langle P\left(A - \varepsilon^{-2}BB^{*}P\right)x,\ x\right\rangle + \langle E(\alpha)x, E(\alpha)\ x\rangle$$

$$+\,\varepsilon^{-2}\langle x, PBB^{*}P\ x\rangle = 0,\ x \in D(A),$$

$$I - \sigma^2\left(\theta_j/{\alpha_j}^2\right)D_j^{*}PD_j \succeq 0,\ j \in \overline{N}.$$

From Lemma 4.1 we obtain $F_\varepsilon = -\varepsilon^{-2}B^{*}P \in \overline{\mathcal{F}}$ and $\sigma \leq r^{\omega}\left(A + BF_\varepsilon, (D_i,\ E_i)_{i\in\overline{N}}\right)$.

■

As a consequence of the above propositions we characterize the supremal achievable stability radius via the Riccati equation (SARE) $_\varepsilon$ as follows.

Corollary 4.6 *We have*

$$\overline{r^{\omega}}\left(A; (D_i,\ E_i)_{i\in\overline{N}}\right) = \sup\left\{\begin{array}{c} \sigma > 0; \textit{There exist } \alpha \in (0, +\infty)^N \textit{ and } \varepsilon > 0 \\ \textit{such that (SARE) }_\varepsilon \textit{ has a solution } P \in L^{+}(H) \\ \textit{with } I - \sigma^2\left(\theta_j/\alpha_j^2\right)D_j^{*}PD_j \succeq 0 \textit{ for all } j \in \overline{N} \end{array}\right\}$$

4.3 Examples

Example 4.7 *Consider the controlled heat equation corresponding to (3.24)*

$$dz(t) = Az(t)dt + \sum_{i=1}^{N} D_i \Delta_i(E_i(z(t)))d\omega_i(t) + Bu(t)dt, \qquad 0 < x < 1, \quad t > 0$$

$$z(0) = z_0$$

where $B = I$ *and* $u(.) \in L^2_{loc}(0, +\infty; H)$

Let $\alpha \in (0, +\infty)^N$ *and* $\varepsilon > 0$. *The Riccati equation (SARE)* $_\varepsilon$ *corresponding to this system is*

$$2\langle Az, P\ z\rangle + \sum_{i=1}^{N} \alpha_i^2 \langle z, z\rangle - \varepsilon^{-2}\langle Pz, P\ z\rangle = 0,\ z \in D(A) \tag{4.10}$$

Suppose we can express the solution P *of (4.10) by*

$$Pz = \sum_{n,j=1}^{+\infty} P_{nj}\langle z, \Phi_n\rangle \Phi_j,\ z \in H$$

Then since

$$Az = \sum_{n=1}^{+\infty} \lambda_n \langle z, \Phi_n\rangle \Phi_n,\ z \in D(A)$$

(4.10) becomes

$$2\sum_{n,j=1}^{+\infty} \lambda_j P_{nj}\langle z, \Phi_n\rangle\langle z, \Phi_j\rangle + \left(\sum_{i=1}^{N}\alpha_i^2\right)\sum_{n=1}^{+\infty}\langle z, \Phi_n\rangle^2$$
$$-\varepsilon^{-2}\sum_{n,j=1}^{+\infty}\sum_{l=1}^{+\infty} P_{nj}P_{lj}\langle z, \Phi_n\rangle\langle z, \Phi_l\rangle = 0,\ z \in D(A)$$

For $z = \Phi_k$, $k \geq 1$, *we get*

$$2\lambda_k P_{kk} + \beta - \varepsilon^{-2}P_{kk}^2 = 0,\ \beta = \left(\sum_{i=1}^{N}\alpha_i^2\right).$$

Because we search for the solutions of (4.10) in $L^+(H)$, we can assume P to be of the form

$$Pz = \sum_{k=1}^{+\infty} P_k \langle z, \Phi_k \rangle \Phi_k,\ \ z \in H;\ \ \textit{where } P_k = \varepsilon^2 \left(\lambda_k + \sqrt{\Lambda_k} \right) \ \textit{ and } \Lambda_k = \lambda_k^2 + \varepsilon^{-2}\beta.$$

Now we show that $I - \sigma^2 \left(\theta_j / {\alpha_j}^2 \right) P \succeq 0$, $\forall j \in \{1, ..., n\}$, for any $\sigma > 0$. Let $z \in H$, we have

$$\left\langle \left(I - \sigma^2 \left(\theta_j / {\alpha_j}^2 \right) P \right) z, z \right\rangle = \sum_{i=1}^{+\infty} \left(1 - \sigma^2 \left(\theta_j / {\alpha_j}^2 \right) P_i \right) \langle z, \Phi_i \rangle^2$$

hence $I - \sigma^2 \left(\theta_j / {\alpha_j}^2 \right) P \succeq 0$ if and only if $1 - \sigma^2 \left(\theta_j / {\alpha_j}^2 \right) P_i \geq 0$, $\forall i \geq 1$. Let $j \in \{1, ..., n\}$, for all $i \geq 1$ we have

$$1 - \sigma^2 \left(\theta_j / {\alpha_j}^2 \right) P_i \geq 0 \Leftrightarrow \sigma^2 \leq \varepsilon^{-2} \left(\alpha_j^2 / \theta_j \right) \left(\lambda_i + \sqrt{\lambda_i^2 + \varepsilon^{-2}\beta} \right)^{-1}, \ \textit{for any } \varepsilon > 0.$$

But

$$\left(\lambda_i + \sqrt{\lambda_i^2 + \varepsilon^{-2}\beta} \right)^{-1} = \frac{\left(\lambda_i - \sqrt{\lambda_i^2 + \varepsilon^{-2}\beta} \right)}{\left(\lambda_i^2 - \lambda_i^2 - \varepsilon^{-2}\beta \right)} = -\varepsilon^2 \beta^{-1} \left(\lambda_i - \sqrt{\lambda_i^2 + \varepsilon^{-2}\beta} \right)$$

Thus

$$1 - \sigma^2 \left(\theta_j / {\alpha_j}^2 \right) P_i \geq 0 \Leftrightarrow \sigma^2 \leq \beta^{-1} \left(\alpha_j^2 / \theta_j \right) \left(-\lambda_i + \sqrt{\lambda_i^2 + \varepsilon^{-2}\beta} \right)$$

which implies that $\overline{r^\omega} = +\infty$.

This result can also be established directly as follows. Let $A_\gamma = A - \gamma I_H$, $\gamma > 0$. we have

$$\langle A_\lambda z, z \rangle = \langle (A - \lambda I) z, z \rangle = \langle Az, z \rangle - \lambda \langle z, z \rangle \leq -\pi^2 \|z\|^2 - \lambda \|z\|^2$$

thus $\langle A_\lambda z, z \rangle \leq -\left(\pi^2 + \lambda \right) \|z\|^2$, $\forall z \in D(A)$. By Corollary 1 in [44], there exists $\omega > 0$ such that $\|S_\lambda(t)\|^2 \leq \exp(-\omega t)$, $t > 0$, hence S_λ is exponentially stable.

Therefore, proceeding as for Example 3.12 we can show that

$$r^{\omega}(A_\gamma;(D_i,\ E_i)_{i=1,n}) = \min_{j\in\overline{N}}\sqrt{\frac{2\,(\gamma+\pi^2)}{\theta_j}}$$

Now since $\{F_\gamma = -\gamma I,\ \gamma>0\}\subset\overline{\mathcal{F}}$, *we have*

$$\sup\left\{r^{\omega}(A_\gamma;(D_i,\ E_i)_{i\in\overline{N}}),\ \gamma>0\right\}\le\overline{r^{\omega}}(A;(D_i,\ E_i)_{i\in\overline{N}})$$

and this implies $\overline{r^{\omega}}(A;(D_i,\ E_i)_{i\in\overline{N}}) = +\infty$.

Example 4.8 *Consider the controlled version of Example 3.13*

$$dz(t) = A_\gamma z(t)dt + \sum_{i=1}^{2} D\Delta_i(E_i z(t))d\omega_i(t) + Bu(t),\ \gamma>0$$

where $B=\begin{pmatrix}0\\ I_2\end{pmatrix}$. *Let*

$$\mathcal{K}=\left\{F_\eta : F_\eta = \begin{pmatrix}0 & -\eta I\end{pmatrix},\ \eta>0\right\},\ \text{and } A_0 = A_\gamma - BF_\gamma$$

Since $\mathcal{K}\subset\overline{\mathcal{F}}$ *then*

$$\sup\left\{r^{\omega}\left(A_\eta;(D_i,\ E_i)_{i=1,2}\right),\ \eta>0\right\}\le\overline{r^{\omega}}(A_0;(D_i,\ E_i)_{i=1,2})$$

From Example 3.13, we have

$$r^{\omega}\left(A_\eta;(D_i,\ E_i)_{i=1,2}\right) = \min\left\{\sqrt{2\eta}/\sqrt{\theta_1},\ \sqrt{2\eta}/\sqrt{\theta_2}\right\}$$

Thus

$$\sup_{\eta>0}\left\{\min\left\{\sqrt{2\eta}/\sqrt{\theta_1},\ \sqrt{2\eta}/\sqrt{\theta_2}\right\}\right\} = +\infty$$

Hence $\overline{r^{\omega}}(A_0;(D_i,\ E_i)_{i\in\overline{N}}) = +\infty$.

We will prove this result using Corollary 4.6. The Riccati equation $(SARE)_{\varepsilon}$ *for this example is*

$$2\langle A_\gamma z, P\ z\rangle + \sum_{i=1}^{2}\alpha_i^2\langle E_i z, E_i z\rangle - \varepsilon^{-2}\langle Pz, BB^*P\ z\rangle = 0,\ \ z \in D(A_\gamma). \quad (4.11)$$

Suppose that a solution P *of (4.11) can be expressed as*

$$P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$$

Since $P \in L^+(H)$ *then* P_1 *and* P_2 *are self-adjoint and* $P_3 = P_2^* A$. *Let* $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $x \in D\left(A^{1/2}\right)$ *and* $y \in L^2(0,1)$, *we will calculate each term of the Riccati equation (4.11). For the first term we have*

$$\begin{aligned}
\langle A_\gamma z, Pz\rangle_H &= \left\langle \begin{pmatrix} y \\ -Ax-\gamma y \end{pmatrix}, \begin{pmatrix} P_1x + P_2y \\ P_3x + P_4y \end{pmatrix} \right\rangle_H \\
&= \left\langle A^{1/2}y, A^{1/2}\left(P_1x+P_2y\right)\right\rangle_{L^2(0,1)} + \langle -Ax-\gamma y, P_3x + P_4y\rangle_{L^2(0,1)} \\
&= \langle Ay, P_1x\rangle_{L^2(0,1)} + \langle Ay, P_2y\rangle_{L^2(0,1)} - \langle Ax, P_3x\rangle_{L^2(0,1)} \\
&\quad -\gamma\langle y, P_4y\rangle_{L^2(0,1)} - \langle Ax, P_4y\rangle_{L^2(0,1)} - \gamma\langle y, P_3x\rangle_{L^2(0,1)}
\end{aligned}$$

The second term is equivalent to

$$\begin{aligned}
\sum_{i=1}^{2}\alpha_i^2\langle E_i z, E_i z\rangle &= \alpha_1^2\langle x, x\rangle_{D\left(A^{1/2}\right)} + \alpha_2^2\langle y, y\rangle_{L^2(0,1)} \\
&= \alpha_1^2\langle Ax, x\rangle_{L^2(0,1)} + \alpha_2^2\langle y, y\rangle_{L^2(0,1)}
\end{aligned}$$

and the third term is

$$\begin{aligned}\langle Pz, BB^*P\ z\rangle &= \left\langle \begin{pmatrix} P_1x + P_2y \\ \\ P_3x + P_4y \end{pmatrix}, \begin{pmatrix} 0 \\ \\ P_3x + P_4y \end{pmatrix} \right\rangle \\ &= \langle P_3x + P_4y, P_3x + P_4y\rangle \\ &= \langle P_3x, P_3x\rangle + 2\langle P_3x, P_4y\rangle + \langle P_4y, P_4y\rangle\end{aligned}$$

Hence the Riccati equation (4.11) is equivalent to

$$2\langle Ay, P_1x\rangle + 2\langle Ay, P_2y\rangle - 2\langle Ax, P_3x\rangle - 2\gamma\langle y, P_4y\rangle$$

$$-2\gamma\langle y, P_3x\rangle - 2\langle Ax, P_4y\rangle + \alpha_1^2\langle Ax, x\rangle + \alpha_2^2\langle y, y\rangle$$

$$-\varepsilon^{-2}\langle P_3x, P_3x\rangle - 2\varepsilon^{-2}\langle P_3x, P_4y\rangle - \varepsilon^{-2}\langle P_4y, P_4y\rangle = 0$$

It follows that

$$2\langle Ay, P_1x\rangle + 2\langle P_3y, y\rangle - 2\langle Ax, P_3x\rangle - 2\gamma\langle y, P_4y\rangle$$

$$-2\gamma\langle y, P_3x\rangle - 2\langle Ax, P_4y\rangle + \alpha_1^2\langle Ax, x\rangle + \alpha_2^2\langle y, y\rangle$$

$$-\varepsilon^{-2}\langle P_3x, P_3x\rangle - 2\varepsilon^{-2}\langle P_3x, P_4y\rangle - \varepsilon^{-2}\langle P_4y, P_4y\rangle = 0$$

For $z = \begin{pmatrix} x \\ \\ 0 \end{pmatrix}$, $x \in D(A)$, *we get*

$$-2\langle Ax, P_3x\rangle + \alpha_1^2\langle Ax, x\rangle - \varepsilon^{-2}\langle P_3x, P_3x\rangle = 0 \tag{4.12}$$

For $z = \begin{pmatrix} 0 \\ \\ y \end{pmatrix}$, $y \in D(A^{1/2})$, *we get*

$$2\langle P_3y, y\rangle - 2\gamma\langle y, P_4y\rangle + \alpha_2^2\langle y, y\rangle - \varepsilon^{-2}\langle P_4y, P_4y\rangle = 0 \tag{4.13}$$

It follows that

$$2\langle Ay, P_1x\rangle - 2\gamma\langle y, P_3x\rangle - 2\langle Ax, P_4y\rangle - 2\varepsilon^{-2}\langle P_3x, P_4y\rangle = 0, \text{ For all } (x,y) \in D(A)\times D(A^{1/2}) \tag{4.14}$$

Assume that we can represent the operators $P_1,\ P_2,\ P_4$ *in the basis* $\Phi_i(x) = \sqrt{2}\sin(\pi i x)$, *as follows*

$$\begin{aligned} P_1x &= \sum_{i,j=0}^{+\infty} \beta_{ij}\langle x, \Phi_j\rangle \Phi_i \\ P_3x &= \sum_{i,j=0} \eta_{ij}\langle x, \Phi_j\rangle \Phi_i \\ P_4x &= \sum_{i,j=0}^{+\infty} \delta_{ij}\langle x, \Phi_j\rangle \Phi_i \end{aligned}$$

Let $x \in D(A)$, *we have*

$$P_3x = \sum_{i,j=0}^{+\infty} \eta_{ij}\langle x, \Phi_j\rangle \Phi_i$$

and since $Ax = \sum\limits_{n=0}^{+\infty} -\lambda_n\langle x, \Phi_n\rangle \Phi_n,\ \lambda_n = -n^2\pi^2,\ n \geq 0,$ *it follows that*

$$\langle Ax,\ P_3x\rangle = \sum_{i,j=0}^{+\infty} -\eta_{ij}\lambda_j\langle x, \Phi_j\rangle\langle x, \Phi_i\rangle$$

$$\langle Ax, x\rangle = \sum_{j=0}^{+\infty} -\lambda_j\langle x, \Phi_j\rangle^2$$

$$\langle P_3x,\ P_3x\rangle = \sum_{i,j,k=0}^{+\infty} \eta_{ik}\eta_{ij}\langle x, \Phi_j\rangle\langle x, \Phi_k\rangle$$

Replacing these expressions in the equation (4.12) we obtain

$$+2\sum_{i,j=0}^{+\infty}\eta_{ij}\lambda_j\langle x, \Phi_j\rangle\langle x, \Phi_i\rangle - \alpha_1^2\sum_{j=0}^{+\infty}\lambda_j\langle x, \Phi_j\rangle^2 - \varepsilon^{-2}\sum_{i,j,k=0}^{+\infty}\eta_{ik}\eta_{ij}\langle x, \Phi_j\rangle\langle x, \Phi_k\rangle = 0$$

For $x = \Phi_n,\ n \geq 0,$ *we get*

$$2\eta_{nn}\lambda_n - \alpha_1^2\lambda_n - \varepsilon^{-2}\sum_{i=0}^{+\infty}\eta_{in}\eta_{in} = 0$$

Let P_3 such that $\eta_{in} = 0$, for $i \neq n$, then η_{nn} satisfies

$$2\eta_{nn}\lambda_n - \alpha^2\lambda_n - \varepsilon^{-2}\eta_{nn}^2 = 0$$

The solutions of this equation are

$$\eta_{nn}^{(1)} = \varepsilon^2\left(\lambda_n + \sqrt{\Lambda'_n}\right),\ \eta_{nn}^{(2)} = \varepsilon^2\left(\lambda_n - \sqrt{\Lambda'_n}\right), \textit{with } \Lambda'_n = \lambda_n^2 - \varepsilon^{-2}\alpha_1^2\lambda_n$$

Since the solution $P \in L^+(H)$, it follows that

$$\eta_{nn} = \varepsilon^2\left(\lambda_n + \sqrt{\Lambda'_n}\right), n \geq 1$$

Now we calculate P_4. Let $y \in D\left(A^{1/2}\right)$, we have

$$P_3 y = \sum_{i=0}\eta_{ii}\left\langle y, \Phi_i\right\rangle \Phi_i,\ \ \left\langle P_4 y,\ y\right\rangle = \sum_{i,j=0}^{+\infty}\delta_{ij}\left\langle y, \Phi_j\right\rangle\left\langle y, \Phi_i\right\rangle$$

and

$$\left\langle P_4 y,\ P_4 y\right\rangle = \left\langle \sum_{i,j=0}^{+\infty}\delta_{ij}\left\langle y, \Phi_j\right\rangle \Phi_i, \sum_{l,k=0}^{+\infty}\delta_{lk}\left\langle y, \Phi_k\right\rangle \Phi_l\right\rangle = \sum_{i,j,k=0}^{+\infty}\delta_{ik}\delta_{ij}\left\langle y, \Phi_j\right\rangle\left\langle y, \Phi_k\right\rangle$$

Replacing these expressions in the equation (4.13) we obtain

$$2\sum_{i=0}\eta_{ii}\left\langle y, \Phi_i\right\rangle^2 - 2\gamma\sum_{i,j=0}^{+\infty}\delta_{ij}\left\langle y, \Phi_j\right\rangle\left\langle y, \Phi_i\right\rangle + \alpha_2^2\sum_{i=0}^{+\infty}\left\langle y, \Phi_i\right\rangle^2 - \varepsilon^{-2}\sum_{i,j,k=0}^{+\infty}\delta_{ik}\delta_{ij}\left\langle y, \Phi_j\right\rangle\left\langle y, \Phi_k\right\rangle = 0$$

For $x = \Phi_n$, $n \geq 1$,we get

$$2\eta_{nn} - 2\gamma\delta_{nn} + \alpha_2^2 - \varepsilon^{-2}\sum_{i=0}^{+\infty}\delta_{in}\delta_{in} = 0$$

Let P_4 such that $\delta_{in} = 0$, for $i \neq n$, then δ_{nn} satisfies

$$2\eta_{nn} - 2\gamma\delta_{nn} + \alpha_2^2 - \varepsilon^{-2}\delta_{nn}^2 = 0$$

The solutions of this equation are

$$\delta_{nn}^{(1)} = -\varepsilon^2\left(\gamma + \sqrt{\Lambda''_n}\right),\ \delta_{nn}^{(2)} = \varepsilon^2\left(-\gamma + \sqrt{\Lambda''_n}\right)\ \textit{ with } \Lambda''_n = \gamma^2 + \varepsilon^{-2}\left(2\eta_{nn} + \alpha_2^2\right)$$

Since the solution $P \in L^{+}(H)$, it follows that

$$P_4 y = \sum_{i=1}^{+\infty} \varepsilon^2 \left(-\gamma + \sqrt{\Lambda_n''}\right) \langle y, \Phi_i \rangle$$

It remains to find P_1. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in D(A) \times D(A^{1/2})$, *we have*

$$\langle Ay,\ P_1 x \rangle = -\sum_{i=0}^{+\infty} \beta_{ij} \lambda_i \langle y, \Phi_i \rangle \langle x, \Phi_j \rangle$$

$$\langle Ax,\ P_4 y \rangle = -\sum_{i=0}^{+\infty} \delta_{ii} \lambda_i \langle x, \Phi_i \rangle \langle y, \Phi i \rangle$$

$$\langle y,\ P_3 x \rangle = \left\langle y, \sum_{i=0}^{+\infty} \eta_{ii} \langle x, \Phi_i \rangle \Phi_i \right\rangle = \sum_{i=0}^{+\infty} \eta_{ii} \langle x, \Phi_i \rangle \langle y, \Phi i \rangle$$

and

$$\langle P_3 x,\ P_4 y \rangle = \sum_{i=0}^{+\infty} \eta_{ii} \delta_{ii} \langle x, \Phi_i \rangle \langle y, \Phi_i \rangle$$

Replacing these expressions in the equation (4.14) we obtain

$$\begin{aligned} &-2\sum_{i,j=0}^{+\infty} \beta_{ij} \lambda_i \langle y, \Phi_i \rangle \langle x, \Phi_j \rangle - 2\gamma \sum_{i=0}^{+\infty} \eta_{ii} \langle x, \Phi_i \rangle \langle y, \Phi i \rangle \\ &+2\sum_{i=0}^{+\infty} \delta_{ii} \lambda_i \langle x, \Phi_i \rangle \langle y, \Phi i \rangle - 2\varepsilon^{-2} \sum_{i=0}^{+\infty} \eta_{ii} \delta_{ii} \langle x, \Phi_i \rangle \langle y, \Phi_i \rangle = 0 \end{aligned}$$

For $x = y = \Phi_n$, $n \geq 1$, we obtain

$$-2\beta_{nn}\lambda_n - 2\gamma\eta_{nn} + 2\delta_{nn}\lambda_n - 2\varepsilon^{-2}\eta_{nn}\delta_{nn} = 0$$

which implies that

$$\beta_{nn} = -\left(\gamma\eta_{nn} - \delta_{nn}\lambda_n + \varepsilon^{-2}\eta_{nn}\delta_{nn}\right)/\lambda_n, n \geq 1$$

Hence

$$P_1 x = \sum_{n=1}^{+\infty} \left(-\gamma\eta_{nn} + \delta_{nn}\lambda_n - \varepsilon^{-2}\eta_{nn}\delta_{nn}\right)/\lambda_n \langle x, \Phi_n \rangle \Phi_n$$

We conclude that the solution P of the Riccati equation (4.11) is given by

$$P = \begin{pmatrix} P_1 & A^{-1}P_3^* \\ & \\ P_3 & P_4 \end{pmatrix}$$

where

$$P_1 x = \sum_{n=1}^{+\infty} \left(-\gamma\eta_n + \delta_n\lambda_n - \varepsilon^{-2}\eta_n\delta_n\right)/\lambda_n \langle x, \Phi_n\rangle \Phi_n$$

$P_3 x = \sum_{n=1}^{+\infty} \eta_n \langle x, \Phi_n\rangle \Phi_n$, *where* $\eta_n = \varepsilon^2\left(\lambda_n + \sqrt{\Lambda_n'}\right)$ *and* $\Lambda_n' = \lambda_n^2 - \varepsilon^{-2}\alpha_1^2\lambda_n$

$P_4 y = \sum_{n=1}^{+\infty} \delta_n \langle y, \Phi_n\rangle \Phi_n$, *where* $\delta_n = \varepsilon^2\left(-\gamma + \sqrt{\Lambda_n^{\prime\prime}}\right)$, *and* $\Lambda_n^{\prime\prime} = \gamma^2 + \varepsilon^{-2}\left(2\eta_n + \alpha_2^2\right)$

It remains to check that

$$I - \sigma^2\left(\theta_j/\alpha_j^2\right) D_j^* P D_j \succeq 0, \qquad j \in \{1,2\}$$

holds for some $\sigma > 0$.

From the definitions of D_j, P and δ_n, we see that this is equivalent to

$$\varepsilon^{-2}\sigma^{-2}\left(\alpha_j^2/\theta_j\right) + \gamma \geq \sqrt{\Lambda''}_n, \text{ for all } n \geq 1 \text{ and } j \in \{1,2\}$$

Let σ be so that $\varepsilon^{-2}\sigma^{-4}\left(\alpha_j^4/\theta_j^2\right) \geq \alpha_1^2 + \alpha_2^2$. *Then*

$$\varepsilon^{-2}\sigma^{-4}\left(\alpha_j^4/\theta_j^2\right) \geq \frac{2}{1 + \sqrt{1 + \varepsilon^{-2}\left(\alpha_1^2/-\lambda_n\right)}}\alpha_1^2 + \alpha_2^2,$$

But

$$\frac{\alpha_1^2}{1 + \sqrt{1 + \varepsilon^{-2}\left(\alpha_1^2/-\lambda_n\right)}} = \varepsilon^2\left(\lambda_n + \sqrt{\Lambda_n'}\right),$$

Thus

$$\varepsilon^{-2}\sigma^{-4}\left(\alpha_j^4/\theta_j^2\right) \geq 2\eta_n + \alpha_2^2,$$

Hence

$$\varepsilon^{-2}\sigma^{-2}\left(\alpha_j^2/\theta_j\right) + \gamma \geq \sqrt{\Lambda''}_n$$

Now recalling Proposition 4.5 we obtain

$$\sqrt{\varepsilon^{-1}\theta_j^{-1}\left(\alpha_j^2/\sqrt{\alpha_1^2+\alpha_2^2}\right)} \leq \overline{r^{\omega}}\left(A_\gamma;(D_i,E_i)_{i=1,2}\right), \ \textit{for all } \varepsilon > 0$$

We conclude therefore that $\overline{r^{\omega}}\left(A_\gamma;(D_i,E_i)_{i=1,2}\right) = +\infty.$

Chapter 5

Stability radii of linear systems subjected to unbounded perturbations

5.1 Introduction

We recall that in Chapter 3 our basic model was

$$\begin{aligned} dx(t) &= Ax(t)dt + \sum_{i=1}^{N} D_i\Delta_i\left(E_i x(t)\right) dw_i(t),\ t > 0 \\ \|\Delta_i\|_L &< \sigma,\ i \in \overline{N}. \end{aligned} \tag{5.1}$$

where $\left((D_i, E_i)_{i\in\overline{N}}\right)$ is a given family of linear bounded operators. However, the assumptions that D_i, E_i are bounded operators is very restrictive and does not allow us to consider many examples of practical importance such that boundary perturbations for systems described by partial differential equations. In this chapter we will show how we can extend the theory of the Chapter 3 to a class of unbounded perturbations.

Several works have been devoted to stochastic partial equations with noise at the boundary (for instance see [47], [58], [16], [54], [29]). However there are only few papers which deal with the stability of this class of systems.

Ichikawa [47] used an approach based on semigroup theory, to establish existence, uniquenesss and regularity results for parabolic equations with boundary and pointwise noise. The stability of this class of systems was investigated in [48], where he obtained an equivalence result between mean square stability and the existence of the solution to a Lyapunov type equation.

Maslowski [58] considered a more general class of stochastic semilinear equation with boundary and pointwise noise. He adopted an approach similar to the one in [48]. In contrast, the conditions on the noise terms are fairly general and cover Wiener processes with values in the basic state space. He established an existence and uniqueness theorem and studied the stability in the mean under the assumption of compacity of A^{-1}

Our main objective in this chapter is to investigate the robustness of stability for system (5.1) when A is subjected to stochastic structured unbounded perturbations. We consider the case where A is the generator of an analytic semigroup and the perturbation are of single type. This abstract model covers the case of parabolic equations with boundary and pointwise noise. We first establish an existence and uniqueness theorem . This Theorem is proved by a standard fixed point argument along the lines of some existing results (for instance, [46], [58]). Then we investigate the robustness problem. We give some characterizations of the stochastic stability radius. These characterizations are in terms of a Lyapunov equation similar to the one of the bounded structure case and different of the one used by [48]. These characterizations enable us to determine a lower bound for the stability radius under perturbation of unbounded structure. The central problem here is to construct the smallest destabilizing perturbation Δ. Under an additional assumption we are able to prove that the lower bound is in fact equal to the stability radius. In the end, we illustrate the theory by three examples investigated by [48].

The chapter is divided into four sections. Section 1 is devoted to briefly recalling some facts about fractional powers of closed operators which we shall use throughout the chapter. Section 2 contains the system description and some basic properties of the solution. In Section 3 we derive the main results of this chapter. We give characterizations of the stability radius in terms of the input -output map and the associated Lyapunov equation. These results are the counterparts of those of the bounded structure case. In the last section the applicability of the abstract conditions and results are illustrated by three examples.

5.2 Fractional powers of closed operators

In this section we recall the definition of analytic semigroups and some basic properties of fractional powers of closed linear operators. The contain of this section is taken mainly from [68], [11], [28], [13], [3].

For any $\omega \in \mathbb{R}$ and $\theta \in]0,\ \pi/2[$ we denote by $S_{\omega,\ \theta}$ the sector in $\mathbb{C}$:

$$S_{\omega,\ \theta} = \{\lambda \in \mathbb{C} \backslash \{\omega\} : \theta \leq |\arg(\lambda - \omega)| \leq \pi\}.$$

Definition 5.1 *We call a linear operator A in a Hilbert space H a sectorial operator if it is closed densely defined operator such that, for some θ in $\left(0, \frac{\pi}{2}\right)$ and some $M \geq 1$ and a real ω, the sector $S_{\omega,\ \theta}$ is in the resolvant set of A and*

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|} \text{ for all } \lambda \in S_{\omega,\ \theta}$$

A particular important class of strongly continuous semigroups is analytic semigroups

Definition 5.2 *An analytic semigroup on a Hilbert space H is a family of continuous linear operators on H, $(S(t))_{t\geq 0}$, satisfying*

1. $S(0) = I$ *and* $S(t_1 + t_2) = S(t_1)S(t_2)$ *for all* $t_1,\ t_2 \geq 0$.
2. *The map* $t \longrightarrow S(t)z$ *is real analytic on* $0 < t < \infty$ *for each* $z \in H$.
3. $\lim_{t\to 0^+} S(t)z = z$ *for all* $z \in H$.

We have the following relation ship between analytic semigroups and sectorial operators obtained from [11].

Theorem 5.3 *1. If $(-A)$ is a sectorial operator, then A is the infinitesimal generator of an analytic semigroup $(S(t))_{t\geq 0}$, where*

$$S(t) = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} e^{\lambda t} R(\lambda,\ \text{-}A)\, d\lambda,\ t > 0,\ \theta \in \left]\frac{\pi}{2}, \pi\right[$$

whith

$$\begin{aligned}
\gamma_{\varepsilon,\theta} &= \gamma^{+}_{\varepsilon,\theta} \cup \gamma^{-}_{\varepsilon,\theta} \cup \gamma^{0}_{\varepsilon,\theta}, \\
\gamma^{\pm}_{\varepsilon,\theta} &= \left\{ z \in \mathbb{C} : z = \omega + re^{\pm i\theta},\ r \geq \varepsilon \right\}, \\
\gamma^{0}_{\varepsilon,\theta} &= \left\{ z \in \mathbb{C} : z = \omega + re^{\pm i\eta},\ |\eta| \leq \theta \right\}
\end{aligned}$$

2. If A generates an analytic semigroup, then $(-A)$ is sectorial.

Assume that $(-A)$ is a sectorial operator and $\mathrm{Re}(\sigma(-A)) > 0$.

Definition 5.4 *For $\alpha > 0$, the bounded linear operator $(-A)^{-\alpha}$ is defined as follows*

$$(-A)^{-\alpha} = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} (-\lambda)^{-\alpha} R(\lambda,\ A)\, d\lambda,\ t > 0,\ x \in H, \tag{5.2}$$

We shall denote by $(-A)^{\alpha}$ the inverse of $(-A)^{-\alpha}$ and by $D\left((-A)^{\alpha}\right)$ its domain.

Definition 5.5 *The operators $(-A)^{\alpha}$ are called fractional powers of $(-A)$.*

In the next theorem we collect some simple properties of $(-A)^{\alpha}$.

Theorem 5.6 *We have:*

1. $(-A)^{\alpha}$ is a closed operator with domain $D((-A)^{\alpha}) = rg((-A)^{-\alpha})$.

2. $\alpha \geq \beta > 0$ *implies* $D((-A)^{\alpha}) \subset D((-A)^{\beta})$.

3. $\overline{D((-A)^{\alpha})} = H$ *for every* $\alpha \geq 0$.

4. *If* α, β *are real then*

$$(-A)^{\alpha+\beta} = (-A)^{\alpha} (-A)^{\beta}$$

on $D((-A)^{\gamma})$ *where* $\gamma = \max(\alpha, \ \beta, \ \alpha+\beta)$.

We conclude this section with some results relating $(-A)^{\alpha}$ and the analytic semigroup $S(t)$.

Theorem 5.7 *Assume that there exists* $\delta > 0$ *such that* $Re(\sigma(-A)) > \delta > 0$.

1. $S(t) : H \longrightarrow D((-A)^{\alpha})$ *for every* $t > 0$ *and* $\alpha > 0$.

2. *For every* $x \in D((-A)^{\alpha})$ *we have*

$$S(t) (-A)^{\alpha} x = (-A)^{\alpha} S(t)x, \ t > 0 \tag{5.3}$$

3. *For* $\alpha \geq 0$ *there exists* M_{α} *such that*

$$\|(-A)^{\alpha} S(t)\| \leq M_{\alpha} t^{-\alpha} e^{-\delta t}, \ t > 0. \tag{5.4}$$

4. *Let* $0 < \alpha \leq 1$ *and there exist* $C_{\alpha} > 0$ *such that*

$$\|S(t)x - x\| \leq C_{\alpha} t^{\alpha} \|A^{\alpha} x\|, \ \text{for any } x \in D((-A)^{\alpha}) \tag{5.5}$$

5.3 System description

Let A be the infinitesimal generator of an analytic semigroup $(S(t))_{t\geq 0}$ on the real separable Hilbert space H. We assume that the semigroup $(S(t))_{t\geq 0}$ is exponentially stable. Then, the fractional powers $(-A)^{\alpha}$, $0 < \alpha < 1$, are well defined. We want to consider perturbations of the operator A which have an unbounded structure.

We will consider infinite dimensional uncertain systems described by Ito equations of the form

$$\begin{aligned} dx(t) &= Ax(t)dt + D\Delta\left(Ex(t)\right)dw(t),\ t > 0, \left\|\Delta\right\|_{Lip} < \sigma \\ x\left(0\right) &= x_0,\ x_0 \in H \end{aligned} \tag{5.6}$$

where:

1. D is a linear operator: $U \longrightarrow H$, (D is generally unbounded as an operator from U to H), such that $(-A)^{-\gamma} D \in L(U, H)$ for some fixed γ, $0 \leq \gamma < 1$, where U is a real separable Hilbert space.
2. $E \in L(D\left((-A)^{\delta}\right), Y)$ with $\delta < \min\left\{1/2 - \gamma,\ 1/2\right\}$, such that $E\left(-A\right)^{\delta} \in L(H, Y)$, where Y is a real separable Hilbert space.
3. $\Delta \in Lip\left(Y, U\right)$.
4. $(w(t))_{t \in \mathbb{R}_+}$ is a real Wiener process on a probability space $(\Omega, \mathcal{F}, \mu)$ with variance θ.

5.3.1 Existence and uniqueness

In this theorem we establish the existence and uniqueness of the solution to the problem (5.6).

Theorem 5.8 *For any $T > 0$, there exists a unique mild solution of the equation (5.6) in $\mathbb{C}\left([0,\ T]\,; L^2\left(\Omega,\ H\right)\right)$ satisfying the initial condition $x(0) = x_0$.*

Proof. The proof is based on the classical fixed point theorem for contractions. Set

$$\Sigma = \mathbb{C}\left([0,\ T]\,; L^2\left(\Omega,\ H\right)\right)$$

and define the corresponding norm by

$$\|x\|_\Sigma = \left(\sup_{t\in[0,T]} \mathcal{E}\left\|x(t)\right\|^2\right)^{\frac{1}{2}} < +\infty.$$

The solution of the system (5.6) is formally given by:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)D\Delta\left(E(x(s))\right)dw(s)$$

We have

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-s)\left(-A\right)^\gamma\left(-A\right)^{-\gamma} D\Delta\left(E\left(-A\right)^{-\delta}\left(-A\right)^\delta\left(x(s)\right)\right)dw(s) \\ &= S(t)x_0 + \int_0^t S(t-s)\left(-A\right)^\gamma \widetilde{D}\Delta\left(\widetilde{E}\left(-A\right)^\delta x(s)\right)dw(s) \end{aligned}$$

where $\widetilde{D} = (-A)^{-\gamma} D \in L(U, H)$ and $\widetilde{E} = E\left(-A\right)^{-\delta} \in L(H, Y)$. It follows that

$$\left(-A\right)^\delta x(t) = \left(-A\right)^\delta S(t)x_0 + \left(-A\right)^\delta \int_0^t S(t-s)\left(-A\right)^\gamma \widetilde{D}\Delta\left(\widetilde{E}\left(-A\right)^\delta\left(x(s)\right)\right)dw(s)$$

By setting $z(t) = \left(-A\right)^\delta x(t)$ we obtain:

$$z(t) = S(t)z_0 + \int_0^t S(t-s)\left(-A\right)^\delta\left(-A\right)^\gamma \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s)$$

which yields

$$z(t) = S(t)z_0 + \int_0^t S(t-s)\left(-A\right)^\alpha \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s) \tag{5.7}$$

where $\alpha = \delta + \gamma$.

In order to establish existence and uniqueness for (5.7), we establish the existence and uniqueness for (5.6). We proceed in three steps.

Step1 Let $\Gamma : \Sigma \longrightarrow \Sigma$ be the mapping defined as:

$$\Gamma(z)(t) = S(t)z_0 + \int_0^t S(t-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}\,(z(s))\right)dw(s),\ t \in [0,\ T],\ z \in \Sigma.$$

At first we show that Γ is well defined as a mapping $\Sigma \longrightarrow \Sigma$. Let $z(.) \in \Sigma$, we have:

$$\Gamma(z)(t) = S(t)z_0 + \int_0^t S(t-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s)$$

Since $(S(t))_{t\geq 0}$ is an analytic exponentially stable semigroup there exist positive constants, M, M_α α, and ω such that

$$\|S(t)\| \leq Me^{-\omega t},\ t > 0,\ \omega > 0 \text{ and } \|A^{\alpha}S(t)\| \leq M_\alpha t^{-\alpha}e^{-\omega t}.$$

Now since $\widetilde{D} = (-A)^{-\gamma} D \in L(U,\ H)$ and $\widetilde{E} = E(-A)^{-\delta} \in L(H,\ Y)$, there exist constants $M\gamma$ and M_δ such that

$$\left\|\widetilde{D}\right\|_{L(U,H)} \leq M\gamma \text{ and } \left\|\widetilde{E}\right\|_{L(H,Y)} \leq M_\delta,$$

From a version of Burkholder-Davis-Gundy inequality we have that

$$\mathcal{E}\left\|\Gamma(z)(t)\right\|^2 \leq \|S(t)z_0\|^2 + C\int_0^t \mathcal{E}\left\|S(t-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^2 ds,\ C > 0$$

By Lipschitzianity of Δ we obtain

$$\mathcal{E}\left\|\Gamma(z)(t)\right\|^2 \leq M^2e^{-2\omega t}\|z_0\|^2 + CM\gamma^2M_\delta^2M_\alpha^2K^2\int_0^t (t-s)^{-2\alpha}e^{-2w(t-s)}\mathcal{E}\left(\|z(s)\|^2\right)ds$$

It follows that

$$\begin{aligned}
\mathcal{E}\left\|\Gamma(z)(t)\right\|^{2} &\leq M^{2}e^{-2\omega t}\left\|z_{0}\right\|^{2}+M'\int_{0}^{t}(t-s)^{-2\alpha}e^{-2\omega(t-s)}\mathcal{E}\left(\left\|z(s)\right\|^{2}\right)ds,\ M'>0\\
&\leq M^{2}e^{-2\omega t}\left\|z_{0}\right\|^{2}+M'\left(\sup_{s\in[0,\ T]}\mathcal{E}\left(\left\|z(s)\right\|^{2}\right)\right)\int_{0}^{t}(t-s)^{-2\alpha}e^{-2\omega(t-s)}ds
\end{aligned}$$

But

$$\int_{0}^{t}(t-s)^{-2\alpha}e^{-2\omega(t-s)}ds\leq\frac{1}{1-2\alpha}t^{1-2\alpha}$$

Therefore

$$\mathcal{E}\left\|\Gamma(z)(t)\right\|^{2}\leq M^{2}\left\|z_{0}\right\|^{2}+\frac{M'}{1-2\alpha}\left(\sup_{s\in[0,\ T]}\mathcal{E}\left(\left\|z(s)\right\|^{2}\right)\right)T^{1-2\alpha}$$

We conclude that Γ is well defined on Σ.

Step 2 Now we show that Γ maps Σ into Σ. For $h\in[0,\ T]$, $t\in[0,\ T-h]$ we

have

$$\begin{aligned}
\Gamma(z)(t+h)-\Gamma(z)(t) &= S(t+h)z_{0}+\int_{0}^{t+h}S(t+h-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s)\\
&\quad -S(t)z_{0}-\int_{0}^{t}S(t-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s)\\
&= S(t+h)z_{0}-S(t)z_{0}\\
&\quad +\int_{0}^{t}S(t+h-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s)\\
&\quad +\int_{t}^{t+h}S(t+h-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s)\\
&\quad -\int_{0}^{t}S(t-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s)
\end{aligned}$$

Hence

$$\begin{aligned}
\Gamma(z)(t+h)-\Gamma(z)(t) &= \left(S(t+h)z_{0}-S(t)z_{0}\right)\\
&\quad +\int_{0}^{t}\left(S(t+h-s)-S(t-s)\right)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s)\\
&\quad +\int_{t}^{t+h}S(t+h-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)dw(s)
\end{aligned}$$

From a version of Burkholder-Davis-Gundy inequality, there exist positive constants C, C' such that

$$\begin{aligned}
\mathcal{E}\left\|\Gamma(z)(t+h)-\Gamma(z)(t)\right\|^{2} \leq\; & \left\|(S(t+h)z_{0}-S(t)z_{0})\right\|^{2} \\
& +C\int_{0}^{t}\mathcal{E}\left\|\left(S(t+h-s)-S(t-s)\right)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^{2}ds \\
& +C'\int_{t}^{t+h}\mathcal{E}\left\|S(t+h-s)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^{2}ds
\end{aligned} \tag{5.8}$$

Since $S(t)$ is strongly continuous we have

$$\lim_{h\longrightarrow 0}\left\|(S(t+h)z_{0}-S(t)z_{0})\right\|=0 \tag{5.9}$$

Now, since

$$\begin{aligned}
& \left(S(t+h-s)-S(t-s)\right)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right) \\
= \;& \left(S(h)-I\right)(-A)^{\alpha}S(t-s)\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& \int_{0}^{t}\mathcal{E}\left\|\left(S(t+h-s)-S(t-s)\right)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^{2}ds \\
\leq \;& \left\|(S(h)-I)\right\|^{2}\int_{0}^{t}\mathcal{E}\left\|(-A)^{\alpha}S(t-s)\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^{2}ds
\end{aligned}$$

Using (5.4), there exist $M'>0$ such that

$$\begin{aligned}
& \int_{0}^{t}\mathcal{E}\left\|\left(S(t+h-s)-S(t-s)\right)(-A)^{\alpha}\widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^{2}ds \\
\leq \;& M'\left\|(S(h)-I)\right\|^{2}\int_{0}^{t}(t-s)^{-2\alpha}e^{-2\omega(t-s)}\mathcal{E}\left(\left\|z(s)\right\|^{2}\right)ds,\ M'>0. \\
\leq \;& M'\left\|(S(h)-I)\right\|^{2}\sup_{s\in[0,T]}\mathcal{E}\left(\left\|z(s)\right\|^{2}\right)\int_{0}^{t}(t-s)^{-2\alpha}e^{-2\omega(t-s)}ds,\ M'>0.
\end{aligned}$$

But

$$\int_{0}^{t}(t-s)^{-2\alpha}e^{-2\omega(t-s)}ds\leq\frac{1}{1-2\alpha}T^{1-2\alpha}$$

It follows that

$$\int_0^t \mathcal{E}\left\| \left(S(t+h-s) - S(t-s)\right)(-A)^\alpha \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^2 ds \tag{5.10}$$

$$\leq \frac{M'}{1-2\alpha}\left\|(S(h)-I)\right\|^2 \sup_{s\in[0,T]} \mathcal{E}\left(\|z(s)\|^2\right) T^{1-2\alpha},\ M'>0.$$

For the last term on the right-hand side of (5.8), we obtain from (5.4)

$$\int_t^{t+h} \mathcal{E}\left\| S(t+h-s)(-A)^\alpha \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^2 ds$$

$$\leq M' \sup_{s\in[0,T]} \mathcal{E}\left(\|z(s)\|^2\right) \int_t^{t+h} (t+h-s)^{-2\alpha} e^{-2\omega(t+h-s)} ds,\ M'>0.$$

But

$$\int_t^{t+h} (t+h-s)^{-2\alpha} e^{-2\omega(t+h-s)} ds \leq \frac{1}{1-2\alpha} h^{1-2\alpha}$$

It follows then that

$$\mathcal{E}\left\| \int_t^{t+h} S(t+h-s)(-A)^\alpha \widetilde{D}\Delta\left(\widetilde{E}(z_2(s))\right) dw(s)\right\|^2 \leq \frac{M'}{1-2\alpha} \sup_{s\in[0,T]} \mathcal{E}\left(\|z(s)\|^2\right) h^{1-2\alpha} \tag{5.11}$$

From (5.9), (5.10) and the estimate (5.11) we deduce that

$$\lim_{h\longrightarrow 0^+} \mathcal{E}\left\|\Gamma(z)(t+h) - \Gamma(z)(t)\right\|^2 = 0,\ t\in[0,\ T]$$

In order to prove the left continuity of $\Gamma(z)$ we have, for all $h\in[0,\ t]$, $t\in[h,\ T]$

$$\begin{aligned} \Gamma(z)(t-h) - \Gamma(z)(t) &= (S(t-h)z_0 - S(t)z_0) \\ &+ \int_0^{t-h} \left(S(t-h-s) - S(t-s)\right)(-A)^\alpha \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right) dw(s) \\ &- \int_{t-h}^{t} S(t-s)(-A)^\alpha \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right) dw(s) \end{aligned} \tag{5.12}$$

By the strong continuity of the semigroup $S(t)$ we have

$$\lim_{h\longrightarrow 0} \left\|(S(t-h)z_0 - S(t)z_0)\right\| = 0 \tag{5.13}$$

For the second term of the right-hand side of (5.12) we have

$$\begin{aligned}
&\int_0^{t-h} \left(S(t-h-s) - S(t-s)\right)(-A)^{\alpha}\, \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right) dw(s) \\
= \quad & -\left(S(h)-I\right)\int_0^{t-h} S(t-h-s)\,(-A)^{\alpha}\, \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right) dw(s)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathcal{E}\left\|\int_0^{t-h} \left(S(t-h-s) - S(t-s)\right)(-A)^{\alpha}\, \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right) dw(s)\right\|^2 \\
\leq \quad & \widetilde{C}\left\|\left(S(h)-I\right)\right\|^2 \int_0^{t-h} \mathcal{E}\left\|S(t-h-s)\,(-A)^{\alpha}\, \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^2 ds
\end{aligned}$$

Using (5.4), we get

$$\begin{aligned}
&\int_0^{t-h} \mathcal{E}\left\|S(t-s)\,(-A)^{\alpha}\, \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^2 ds \\
\leq \quad & M'\int_0^{t-h} (t-h-s)^{-2\alpha}\, e^{-2\omega(t-h-s)}\mathcal{E}\left(\|z(s)\|^2\right) ds,\ M' > 0
\end{aligned}$$

We deduce that there is a constant M'' such that

$$\begin{aligned}
&\mathcal{E}\left\|\int_0^{t-h} \left(S(t-h-s) - S(t-s)\right)(-A)^{\alpha}\, \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right) dw(s)\right\|^2 \\
\leq \quad & \frac{M''}{1-2\alpha}\left\|\left(S(h)-I\right)\right\|^2 \sup_{s\in[0,T]} \mathcal{E}\left(\|z(s)\|^2\right) T^{1-2\alpha}
\end{aligned} \tag{5.14}$$

Now for the last term of the right-hand side of (5.12) we have

$$\begin{aligned}
&\mathcal{E}\left\|\int_{t-h}^{t} S(t-h-s)\,(-A)^{\alpha}\, \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right) dw(s)\right\|^2 \\
\leq \quad & C_1\int_{t-h}^{t} \mathcal{E}\left\|S(t-h-s)\,(-A)^{\alpha}\, \widetilde{D}\Delta\left(\widetilde{E}(z(s))\right)\right\|^2 ds \\
\leq \quad & M' \sup_{s\in[0,T]} \mathcal{E}\left(\|z(s)\|^2\right)\left|\int_{t-h}^{t} (t-h-s)^{-2\alpha}\, e^{-2\omega(t-h-s)} ds\right|
\end{aligned}$$

But

$$\left|\int_{t-h}^{t} (t-h-s)^{-2\alpha}\, e^{-2\omega(t-h-s)} ds\right| \leq \frac{1}{1-2\alpha} h^{1-2\alpha}$$

Hence

$$\mathcal{E}\left\|\int_{t-h}^{t} S(t-h-s)\,(-A)^{\alpha}\,\widetilde{D}\Delta\left(\widetilde{E}(z_2(s))\right)dw(s)\right\|^2 \leq \frac{M'}{1-2\alpha}h^{1-2\alpha}\sup_{s\in[0,T]}\mathcal{E}\left(\|z(s)\|^2\right) \tag{5.15}$$

From (5.13), (5.14) and (5.15) it follows that

$$\lim_{h\longrightarrow 0^+}\mathcal{E}\left\|\Gamma(z)(t)-\Gamma(z)(t-h)\right\|^2=0,\ t\in[0,\ T]$$

step 3 It remains to verify that $\Gamma:\Sigma\longrightarrow\Sigma$ is a contraction. Let z_1 and z_2 be arbitrary processes from Σ; then

$$\begin{aligned}\Gamma(z_1(t))-\Gamma(z_2(t)) &= \int_0^t S(t-s)\,(-A)^{\alpha}\,\widetilde{D}\Delta\left(\widetilde{E}(z_1(s)\right)dw(s)\\ &\quad-\int_0^t S(t-s)\,(-A)^{\alpha}\,\widetilde{D}\Delta\left(\widetilde{E}(z_2(s)\right)dw(s)\\ &= \int_0^t S(t-s)\,(-A)^{\alpha}\,\widetilde{D}\left(\Delta\left(\widetilde{E}(z_1(s)\right)-\Delta\left(\widetilde{E}(z_2(s))\right)\right)dw(s)\end{aligned}$$

Using the Burkholder-Davis-Gundy inequality, there exists $C_2>0$ such that

$$\mathcal{E}\left(\|\Gamma(z_1(t))-\Gamma(z_2(t))\|^2\right)\leq C_2\int_0^t\|S(t-s)\,(-A)^{\alpha}\|^2\,\mathcal{E}\left\|\widetilde{D}\left(\Delta\left(\widetilde{E}\ (z_1(s)\right)-\Delta\left(\widetilde{E}\ (z_2(s))\right)\right)\right\|^2 ds$$

Using the fact that $\widetilde{D}\in L(U,\,H)$ and the Lipschitzianity of Δ we get

$$\mathcal{E}\left(\|\Gamma(z_1(t))-\Gamma(z_2(t))\|^2\right)\leq C_2\int_0^t\frac{M_\alpha^2}{(t-s)^{2\alpha}}M_\gamma^2K^2\mathcal{E}\left\|\left(\widetilde{E}\ (z_1(s))-\widetilde{E}(z_2(s))\right)\right\|^2 ds$$

Now since $\widetilde{E}\in L(H,\,Y)$, it follows that

$$\mathcal{E}\left(\|\Gamma(z_1(t))-\Gamma(z_2(t))\|^2\right)\leq M_\gamma^2K^2C_2\int_0^t\frac{M_\alpha^2}{(t-s)^{2\alpha}}M_\delta^2\mathcal{E}\left(\|z_1(s)-z_2(s)\|^2\right)ds$$

We deduce that

$$\begin{aligned}\mathcal{E}\left(\|\Gamma(z_1(t))-\Gamma(z_2(t))\|^2\right) &\leq C_2\int_0^t\frac{M'}{(t-s)^{2\alpha}}\mathcal{E}\left(\|z_1(s)-z_2(s)\|^2\right)ds,\ M'>0.\\ &\leq \sup_{s\in[0,T]}\mathcal{E}\left(\|z_1(s)-z_2(s)\|^2\right)\int_0^t\frac{M'}{(t-s)^{2\alpha}}ds\end{aligned}$$

Hence

$$\mathcal{E}\left(\|\Gamma(z_1(t))-\Gamma(z_2(t))\|^2\right)\le M'\frac{T^{1-2\alpha}}{1-2\alpha}\sup_{s\in[0,T]}\mathcal{E}\left(\|z_1(s)-z_2(s)\|^2\right)$$

We conclude that there exists a constant $M(T)$ such that

$$\sup_{s\in[0,T]}\mathcal{E}\left(\|\Gamma(z_1(t))-\Gamma(z_2(t))\|^2\right)\le M(T)\sup_{s\in[0,T]}\mathcal{E}\left(\|z_1(s)-z_2(s)\|^2\right)$$

Therefore Γ is contractive for enough small $T>0$. For large T we can proceed in a usual way by considering the equation on intervals, $\left[0,\widetilde{T}\right]$, $\left[\widetilde{T},2\widetilde{T}\right]$, ...with $\widetilde{T}$ enough small.

■

5.4 Characterizations of the stability radius

In this section we extend some results of Chapter 3 to the unbounded structure case.

Lemma 5.9 *Let*

$$y(t)=ES(t)x_0+E\int_0^t S(t-s)Dv(s)dw(s) \tag{5.16}$$

where $v\in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$ *and* $x_0\in H$. *Then*

$$\mathcal{E}\left(\|y(t)\|_Y^2\right)=\|ES(t)x_0\|_Y^2+\theta\int_0^t\mathcal{E}\left(\left\|\widetilde{E}S(t-s)\left(-A\right)^{\alpha}\widetilde{D}v(s)\right\|_Y^2\right)ds,\ t>0$$

where $\widetilde{E}=E\left(-A\right)^{-\delta}$ *and* $\widetilde{D}=\left(-A\right)^{-\gamma}D$. *Moreover,* $y(.)\in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y\right)\right)$ *and*

$$\begin{aligned}\|y\|^2_{L^2_w} &= \int_0^{+\infty}\mathcal{E}\left(\|y(t)\|_Y^2\right)dt\\ &= \int_0^{+\infty}\|ES(t)x_0\|_Y^2\,dt+\theta\int_0^{+\infty}\mathcal{E}\left(\left\langle Dv(s),\widetilde{P}Dv(s)\right\rangle\right)ds\end{aligned}$$

where $\widetilde{P} \in L(H)$ is a self-adjoint nonnegative operator satisfying

$$2\left\langle \widetilde{P}x, Ax \right\rangle + \langle Ex, Ex\rangle = 0, x \in D(A) \tag{5.17}$$

Proof. We have

$$\begin{aligned} E\int_0^t S(t-s)Dv(s)dw(s) &= E\int_0^t (-A)^{-\delta}(-A)^{\delta} S(t-s)(-A)^{\gamma}(-A)^{-\gamma} Dv(s)dw(s) \\ &= \widetilde{E}\int_0^t (-A)^{\delta} S(t-s)(-A)^{\gamma} \widetilde{D}v(s)dw(s) \\ &= \int_0^t \widetilde{E}S(t-s)(-A)^{\delta+\gamma} \widetilde{D}v(s)dw(s) \end{aligned}$$

Hence

$$E\int_0^t S(t-s)Dv(s)dw(s) = \int_0^t \widetilde{E}S(t-s)(-A)^{\alpha} \widetilde{D}v(s)dw(s)$$

Therefore

$$y(t) = ES(t)x_0 + \int_0^t \widetilde{E}S(t-s)(-A)^{\alpha} \widetilde{D}v(s)dw(s)$$

Since $\widetilde{D}$ and $\widetilde{E}$ are linear bounded operators the proof of the first statement of this Lemma is similar to the one of Lemma (3.2). For the second statement, we have

$$\int_0^{+\infty} \mathcal{E}\left(\|y(t)\|_Y^2\right)dt = \int_0^{+\infty} \|ES(t)x_0\|_Y^2\, dt + \theta\int_0^{+\infty}\int_0^t \mathcal{E}\left(\|ES(t-s)Dv(s)\|_Y^2\right)dsdt \tag{5.18}$$

The first term of the R.H.S of (5.18) is equivalent to

$$\|ES(t)x_0\|_Y^2 = \left\|\widetilde{E}(-A)^{\delta} S(t)x_0\right\|^2$$

Since $\widetilde{E} \in L(H,Y)$, there exists $M_\delta > 0$, such that

$$\|ES(t)x_0\|_Y^2 \le M_\delta^2 t^{-2\delta} e^{-2\omega t}\|x_0\|^2$$

Therefore

$$\int_0^{+\infty} \|ES(t)x_0\|_Y^2 \, dt \le M_\delta^2 \|x_0\|^2 \int_0^{+\infty} t^{-2\delta} e^{-2\omega t} dt$$

and since $\delta < \frac{1}{2}$, it follows that

$$\int_0^{+\infty} \|ES(t)x_0\|_Y^2 \, dt < +\infty$$

For the second term of the R.H.S. of (5.18) we obtain from the Fubini theorem

$$\begin{aligned}
& \int_0^{+\infty} \int_0^t \mathcal{E} \left\| \widetilde{E} S(t-s) (-A)^\alpha \widetilde{D} v(s) \right\|_Y^2 ds dt \\
= & \int\limits_0^{+\infty} \left(\int_s^{+\infty} \mathcal{E} \left\| \widetilde{E} S(t-s) (-A)^\alpha \widetilde{D} v(s) \right\|_Y^2 dt \right) ds \\
= & \int\limits_0^{+\infty} \mathcal{E} \left(\int_s^{+\infty} \left\langle \widetilde{D} v(s), \; S^*(t-s) \left((-A)^\alpha \right)^* \widetilde{E}^* \widetilde{E} (-A)^\alpha S(t-s) \widetilde{D} v(s) \right\rangle_H dt \right) ds \\
= & \int\limits_0^{+\infty} \mathcal{E} \left(\left\langle \widetilde{D} v(s), \; \left(\int_s^{+\infty} S^*(t-s) \left((-A)^\alpha \right)^* \widetilde{E}^* \widetilde{E} (-A)^\alpha S(t-s) dt \right) \widetilde{D} v(s) \right\rangle_H \right) ds
\end{aligned}$$

Using the fact that $\widetilde{D} = A^{-\gamma} D$, it follows that

$$\begin{aligned}
& \int\limits_0^{+\infty} \mathcal{E} \left(\left\langle \widetilde{D} v(s), \; \left(\int_s^{+\infty} S^*(t-s) \left((-A)^\alpha \right)^* \widetilde{E}^* \widetilde{E} (-A)^\alpha S(t-s) dt \right) \widetilde{D} v(s) \right\rangle_H \right) ds \\
= & \int\limits_0^{+\infty} \mathcal{E} \left(\left\langle (-A)^{-\gamma} D v(s), \; \left(\int_s^{+\infty} S^*(t-s) \left((-A)^\alpha \right)^* \widetilde{E}^* \widetilde{E} (-A)^\alpha S(t-s) dt \right) (-A)^{-\gamma} D v(s) \right\rangle_H \right) ds \\
= & \int\limits_0^{+\infty} \mathcal{E} \left(\left\langle \begin{array}{c} D v(s), \\ \left(\int_s^{+\infty} S^*(t-s) \left((-A)^{-\gamma} \right)^* \left((-A)^\alpha \right)^* \widetilde{E}^* \widetilde{E} (-A)^\alpha (-A)^{-\gamma} S(t-s) dt \right) D v(s) \end{array} \right\rangle \right) ds
\end{aligned}$$

and since $\widetilde{E} = E\,(-A)^{-\delta}$ and $\alpha = \delta + \gamma$, we get

$$\int\limits_0^{+\infty} \mathcal{E}\left(\left\langle \begin{array}{c} Dv(s), \\ \left(\int_s^{+\infty} S^*(t-s)\left((-A)^{-\gamma}\right)^*((-A)^{\alpha})^*\,\widetilde{E}^*\widetilde{E}\,(-A)^{\alpha}\,(-A)^{-\gamma}\,S(t-s)ds\right)Dv(s) \end{array}\right\rangle\right) dt$$

$$= \int\limits_0^{+\infty} \mathcal{E}\left(\left\langle \begin{array}{c} Dv(s), \\ \left(\int_s^{+\infty} S^*(t-s)E^*ES(t-s)dt\right)Dv(s) \end{array}\right\rangle\right) ds$$

Set

$$\widetilde{P}z = \int_0^{+\infty} S^*(r)E^*ES(r)zdr, \quad z \in H$$

We show that $\widetilde{P}$ is a bounded operator. Let $z \in H$, we have

$$\begin{aligned} \left\langle \widetilde{P}z,\ z\right\rangle &= \left\langle \int_0^{+\infty} S^*(r)\left((-A)^{-\gamma}\right)^*((-A)^{\alpha})^*\,\widetilde{E}^*\widetilde{E}\,(-A)^{\alpha}\left((-A)^{-\gamma}\right)S(r)zdr, z\right\rangle \\ &= \left\langle \int_0^{+\infty} S^*(r)\left((-A)^{\delta}\right)^*\widetilde{E}^*\widetilde{E}\,(-A)^{\delta}\,S(r)zdr,\ z\right\rangle \end{aligned}$$

Thus

$$\left\langle \widetilde{P}z,\ z\right\rangle = \int_0^{+\infty}\left\langle \widetilde{E}\,(-A)^{\delta}\,S(r)z,\ \widetilde{E}\,(-A)^{\delta}\,S(r)z\right\rangle dr \tag{5.19}$$

Hence

$$\left\langle \widetilde{P}z,\ z\right\rangle \leq M_{\delta}{}^2\left\|\widetilde{E}\right\|^2\left(\int_0^{+\infty}\frac{e^{-2\omega r}}{r^{2\delta}}dr\right)\|z\|^2$$

Therefore

$$\left\langle \widetilde{P}z,\ z\right\rangle \leq M'_{\delta}M_{\delta}{}^2\left(\int_0^{+\infty}\frac{e^{-2\omega r}}{r^{2\delta}}dr\right)\|z\|^2$$

Since $\delta < 1/2$, the integral $\left(\int_0^{+\infty}\frac{e^{-2\omega r}}{r^{2\delta}}dr\right)$ is bounded. We deduce then that $\widetilde{P} \in L(H)$

Since

$$\int_0^{+\infty}\mathcal{E}\int_0^t\left\|\widetilde{E}S(t-s)\,(-A)^{\alpha}\,\widetilde{D}v(s)\right\|^2 dsdt = \int\limits_0^{+\infty}\mathcal{E}\left(\left\langle Dv(s),\ \widetilde{P}Dv(s)\right\rangle\right)dt$$

It follows that

$$\int_0^{+\infty} \mathcal{E}\left(\|y(t)\|^2\right) dt = \int_0^{+\infty} \|ES(t)x_0\|^2\, dt + \theta \int_0^{+\infty} \mathcal{E}\left(\left\langle Dv(s), \widetilde{P}Dv(s)\right\rangle\right) ds$$

■

As in Chapter 3 we consider the input-output linear map

$$\widetilde{L} : L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right) \longrightarrow L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y\right)\right)$$

defined by

$$(\widetilde{L}v)(t) = E\int_0^t S(t-s)Dv(s)dw(s) \tag{5.20}$$

By the previous lemma, $\widetilde{L}v \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y\right)\right)$ for all $v \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$.

Lemma 5.10 *The linear map $\widetilde{L}$ defined by (5.20) has the operator norm*

$$\begin{aligned}\left\|\widetilde{L}\right\|_{L^2_\omega} &= \left(\left(\theta\left\|D^*\left[\int_0^{+\infty} S^*(t)E^*ES(t)dt\right]D\right\|\right)^{\frac{1}{2}}\right)\\ &= \left(\theta\left\|D^*\widetilde{P}D\right\|\right)^{\frac{1}{2}}\end{aligned}$$

where $\widetilde{P}$ satisfies (5.17).

Proof. Let $v \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$. By the previous Lemma we have

$$\int_0^{+\infty} \mathcal{E}\left\|(\widetilde{L}v)(t)\right\|^2 dt = \theta\int_0^{+\infty} \mathcal{E}\left(\left\langle Dv(s), \widetilde{P}Dv(s)\right\rangle\right) ds$$

Now we have:

$$\begin{aligned}\left\|\widetilde{L}v\right\|^2_{L^2_\omega} &= \int_0^{+\infty} \mathcal{E}\left\|(\widetilde{L}v)(t)\right\|^2 dt\\ &= \theta\int_0^{+\infty} \mathcal{E}\left(\left\langle Dv(s), \widetilde{P}Dv(s)\right\rangle\right) ds\\ &\leq \theta\int_0^{+\infty} \mathcal{E}\left(\left\|D^*\widetilde{P}D\right\|\|v(s)\|^2\right) ds\\ &\leq \theta\left\|D^*\widetilde{P}D\right\|\int_0^{+\infty} \mathcal{E}\left(\|v(s)\|^2\right) ds\end{aligned}$$

It follows that

$$\left\|\widetilde{L}v\right\|_{L^2_\omega}^2 \leq \theta \left\|D^*\widetilde{P}D\right\| \int_0^{+\infty} \mathcal{E}\left(\|v(s)\|^2\right) ds$$

Hence

$$\left\|\widetilde{L}v\right\|_{L^2_\omega}^2 \leq \theta \left\|D^*\widetilde{P}D\right\| \|v\|_{L^2_\omega}^2$$

Therefore

$$\frac{\left\|\widetilde{L}v\right\|_{L^2_\omega}^2}{\|v\|_{L^2_\omega}^2} \leq \theta \left\|D^*\widetilde{P}D\right\|, \text{ for all } v \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$$

which implies that

$$\left\|\widetilde{L}\right\|_{L^2_\omega} \leq \left(\theta \left\|D^*\widetilde{P}D\right\|\right)^{\frac{1}{2}}$$

Now, we will show that there exists $v \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$ such that

$$\left\|\widetilde{L}v\right\|_{L^2_\omega} = \left(\theta \left\|D^*\widetilde{P}D\right\|\right)^{\frac{1}{2}}$$

which is equivalent to

$$\theta \int_0^{+\infty} \mathcal{E}\left(\left\langle Dv(s), \widetilde{P}Dv(s)\right\rangle\right) ds = \left(\theta \left\|D^*\widetilde{P}D\right\|\right)^{\frac{1}{2}}$$

Now suppose that

$$\left\|D^*\widetilde{P}D\right\|_U = \left(\max_{\|v\|=1} \left\langle v, D^*\widetilde{P}Dv\right\rangle_U\right) = \left\langle v_0, D^*\widetilde{P}Dv_0\right\rangle_U,\ \ \|v_0\|_U = 1$$

We define v_1 as follows

$$v_1(t) = \beta(.)v_0,\ \text{where } \beta(.) \in L^2\left(\Re^+,\ \Re\right) \text{ and } \|\beta(.)\|_{L^2(\Re^+,\ \Re)} = 1$$

Then

$$
\begin{aligned}
\|v_1(.)\|^2_{L^2_\omega} &= \int_0^{+\infty} \mathcal{E}\, \|v_1(s)\|^2\, ds \\
&= \int_0^{+\infty} \|\beta(s) v_0\|^2\, ds \\
&= \|v_0\|^2 \int_0^{+\infty} \|\beta(s)\|^2\, ds
\end{aligned}
$$

Therefore

$$
\|v_1(.)\|^2_{L^2_\omega} = \|v_0\|^2 = 1
$$

and

$$
\begin{aligned}
\|Lv_1\|^2_{L^2_\omega} &= \theta \int_0^{+\infty} \mathcal{E}\left(\left\langle Dv_1(s), \widetilde{P} D v_1(s)\right\rangle\right) ds \\
&= \theta \int_0^{+\infty} \left(\left\langle \beta(s) v_0, D^* \widetilde{P} D v_0\right\rangle\right) ds \\
&= \theta \left\| D^* \widetilde{P} D \right\| \int_0^{+\infty} |\beta(s)|^2\, ds \\
&= \theta \left\| D^* \widetilde{P} D \right\|
\end{aligned}
$$

Which concludes the proof. ■

In the following theorem we give an important characterization of the stability radius $r^w\left(A; (D,\ E)\right)$ in terms of the solution of the Lyapunov equation (5.17).

Theorem 5.11 *Suppose that there exists* $\widetilde{P} \in L^+(H)$ *satisfying*

$$
2\left\langle \widetilde{P}x,\ Ax\right\rangle + \langle Ex,\ Ex\rangle = 0,\ x \in D(A) \tag{5.21}
$$

$$
I - \sigma^2 \theta D^* \widetilde{P} D \succeq 0 \tag{5.22}
$$

Then $r^w\left(A; (D,\ E)\right) \geq \sigma$.

Proof. Let $\Delta \in Lip\,(Y,\ U)$ such that $\|\Delta\|_{Lip} < \sigma$, and suppose that $\widetilde{P} \in L^{+}(H)$ is such that (5.21) and (5.22) hold. Let $x(t)$ such that

$$x(t) = S(t)x_0 + \int_0^t S(t-s)D\Delta\left(E(x(s))\,dw(s)\right.$$

set $y(t) = Ex(t)$ and $u(t) = \Delta\left(y(t)\right),\ t > 0$. We have

$$y(t) = ES(t)x_0 + E\int_0^t S(t-s)Du(s)dw(s),\ \ t > 0. \tag{5.23}$$

For every $T > 0$, define the truncations $u_T \in L^2_\omega\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$ by

$$u_T(t) = \begin{cases} u(t) = \Delta\left(y(t)\right) & \text{if } t \in [0,\ T], \\ 0 & \text{if } t > T, \end{cases}$$

Then

$$\begin{aligned} \|u_T\|^2_{L^2_w} &= \int_0^{+\infty} \mathcal{E}\left(\|u_T(t)\|\right)^2 dt \\ &= \int_0^T \mathcal{E}\left(\|u(t)\|\right)^2 dt \\ &= \int_0^T \mathcal{E}\left(\|\Delta y(t)\|\right)^2 dt \\ &\le \|\Delta\|^2_{Lip} \int_0^T \left(\mathcal{E}\,\|y(t)\|^2\right) dt \end{aligned}$$

Hence

$$\|u_T\|^2_{L^2_w} \le \|\Delta\|^2_{Lip} \int_0^T \mathcal{E}\left(\|y(t)\|\right)^2 dt \tag{5.24}$$

Now define y_T as the output of the system $(A; (D,\ E))$ generated by the input u_T with initial condition $x(0) = x_0$. Then

$$y_T(t) = ES(t)x_0 + \widetilde{L}u_T(t),\ \ t > 0. \tag{5.25}$$

From (5.23)-(5.25), we get

$$\begin{aligned}\left(\int_0^T \mathcal{E}\left\|y(t)\right\|^2 dt\right)^{\frac{1}{2}} &\leq \left\|y_T\right\|_{L_w^2} \\ &\leq \left\|ES(t)x_0\right\| + \left\|\widetilde{L}\right\| \left\|u_T\right\|_{L_w^2}\end{aligned}$$

Thus

$$\left(\int_0^T \mathcal{E}\left\|y(t)\right\|^2 dt\right)^{\frac{1}{2}} \leq \left\|ES(t)x_0\right\| + \left\|\widetilde{L}\right\| \left\|\Delta\right\|_{Lip} \left(\int_0^T \mathcal{E}\left(\left\|y(t)\right\|\right)^2 dt\right)^{\frac{1}{2}} \quad (5.26)$$

Condition (5.22) implies that

$$1 - \sigma^2\theta \left\|D^*\widetilde{P}D\right\| \geq 0$$

Thus

$$\theta \left\|D^*\widetilde{P}D\right\| \leq \sigma^{-2}$$

By the previous lemma, it follows that

$$\left\|\widetilde{L}\right\|^2 \leq \sigma^{-2}$$

Now since $\left\|\Delta\right\|_{Lip} < \sigma$, the operator $\widetilde{L}\Delta$ is a contraction on $L_\omega^2\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y\right)\right)$ with $\beta{=}\left\|\widetilde{L}\right\| \left\|\Delta\right\|_{Lip} < 1$. From (5.26) we get that

$$\left(\int_0^T \mathcal{E}\left\|y(t)\right\|^2 dt\right)^{\frac{1}{2}} \leq (1-\beta)^{-1}\left\|ES(t)x_0\right\| \text{ for all } T > 0,$$

Therefore $y \in L_\omega^2\left(\mathbb{R}^+,\ L^2\left(\Omega,\ Y\right)\right)$ and $u = \Delta(y) \in L_\omega^2\left(\mathbb{R}^+,\ L^2\left(\Omega,\ U\right)\right)$. By Lemma 5.9, the solution $x(.)$ of (5.6) belongs to $L_\omega^2\left(\mathbb{R}^+,\ L^2\left(\Omega,\ H\right)\right)$. We conclude then that $r^w\left(A;(D,\ E)\right) \geq \sigma$. ■

As a consequence of this theorem we have the following corollary which enables us to obtain a lower bound for the stability radius.

Corollary 5.12 *Suppose that there exists $\widetilde{P} \in L^{+}(H)$ a solution of the Lyapunov equation (5.17). Then,*

$$r^{w}(A;(D,\ E)) \geq \left(\theta \left\|D^{*}\widetilde{P}D\right\|\right)^{-\frac{1}{2}} \tag{5.27}$$

Proof. If $\left\|D^{*}\widetilde{P}D\right\| = 0$, then $I - \sigma^{2}\theta D^{*}\widetilde{P}D \succeq 0$, for all $\sigma > 0$. From the above Theorem, it follows that $r^{w}(A;(D,\ E)) \geq \sigma$, for all $\sigma > 0$. From which we deduce that $r^{w}(A;(D,\ E)) = +\infty$.

If $\left\|D^{*}\widetilde{P}D\right\| \neq 0$, we obtain from the fact that $\left\|D^{*}\widetilde{P}D\right\| = \sup\limits_{u\neq 0}\frac{\left\langle D^{*}\widetilde{P}Du,\ u\right\rangle}{\|u\|^{2}}$

$$\|u\|^{2} - \left\|D^{*}\widetilde{P}D\right\|^{-1}\left\langle D^{*}\widetilde{P}Du,\ u\right\rangle \geq 0,\ \text{for all } u \in U$$

Hence

$$\|u\|^{2} - \left(\left(\theta \left\|D^{*}\widetilde{P}D\right\|\right)^{-1/2}\right)^{2}\theta\left\langle D^{*}\widetilde{P}Du,\ u\right\rangle \geq 0,\ \text{for all } u \in U$$

By the previous Theorem we deduce that

$$r^{w}(A;(D,\ E)) \geq \left(\theta \left\|D^{*}\widetilde{P}D\right\|\right)^{-\frac{1}{2}}$$

■

We will show that the equality holds in (5.27) when E is a bounded operator.

Proposition 5.13 *Assume that E is a bounded operator. If there exists $P \in L^{+}(H)$ a solution of the Lyapunov equation (5.17), then*

$$r^{w}(A;(D,\ E)) = \left(\theta \left\|D^{*}\widetilde{P}D\right\|\right)^{-\frac{1}{2}} \tag{5.28}$$

Proof. From the above Corollary, it remains to show that $r^{w}(A;(D,\ E)) \leq \left(\theta \left\|D^{*}\widetilde{P}D\right\|\right)^{-\frac{1}{2}}$. Let $u_{0} \in U$ with $\|u_{0}\| = 1$ such that

$$\left\|D^{*}\widetilde{P}D\right\| = \left\langle D^{*}\widetilde{P}Du_{0},\ u_{0}\right\rangle = \kappa \tag{5.29}$$

1. If $\kappa = 0$ then $D^*\widetilde{P}D = 0$. It follows from the above corollary that $r^w(A;(D,\ E)) = +\infty$.

2. Suppose now that $\kappa \neq 0$. Define the perturbation Δ as follows

$$\Delta(y) = (\theta\kappa)^{-1/2} \|y\| u_0,\ y \in Y$$

Then $\|\Delta\|_{Lip} = (\theta\kappa)^{-1/2}$. We will show that for this Δ the system (5.6) cannot be stable. Assume that this is not the case. The solution $x(.)$ of (5.6) satisfies

$$x(t) = S(t)x_0 + \int_0^t S(t-s)D\Delta(Ex(s))\,dw(s),\ t > 0.$$

Set $y = Ex$, then $y \in L^2_\omega\left(\mathbb{R}^+,\ L^2(\Omega,\ Y)\right)$. We have

$$\begin{aligned} y(t) &= ES(t)x_0 + E\int_0^t S(t-s)D\Delta(y(s))\,dw(s),\ t>0 \\ &= ES(t)x_0 + (\theta\kappa)^{-1/2} E\int_0^t S(t-s)Du_0 \|y(s)\|\,dw(s),\ t>0 \end{aligned}$$

By applying Lemma 5.9 to this equation we obtain

$$\begin{aligned} \int_0^\infty \mathcal{E}\|y(t)\|^2\,dt &= \int_0^\infty \|ES(t)x_0\|^2\,dt \\ &\quad + (\theta\kappa)^{-1}\theta\int_0^{+\infty}\mathcal{E}\left\langle Du_0\|y(s)\|,\ \widetilde{P}Du_0\|y(s)\|\right\rangle ds \\ &= \int_0^\infty \|ES(t)x_0\|^2\,dt \\ &\quad +\kappa^{-1}\left\langle u_0,\ D^*\widetilde{P}Du_0\right\rangle \int_0^\infty \mathcal{E}\|y(s)\|^2\,ds \end{aligned}$$

From (5.29) we get

$$\int_0^\infty \mathcal{E}\|y(t)\|^2\,dt = \int_0^\infty \|ES(t)x_0\|^2\,dt + \int_0^\infty \mathcal{E}\|y(s)\|^2\,ds$$

This identity implies that $\int_0^\infty \|ES(t)x_0\|^2 dt = 0$. From (5.19) we have

$$\begin{aligned} \left\langle \widetilde{P}x_0,\ x_0 \right\rangle &= \int_0^{+\infty} \left\langle \widetilde{E}\,(-A)^\delta\, S(s)x_0,\ \widetilde{E}\,(-A)^\delta\, S(s)x_0 \right\rangle ds \\ &= \int_0^{+\infty} \|ES(s)x_0\|^2\, ds \end{aligned} \tag{5.30}$$

We deduce then that $\widetilde{P} = 0$, which implies that $\kappa = \left\| D^*\widetilde{P}D \right\| = 0$. Therefore the stochastic system (5.6) can not be L^2-stable. We conclude that

$$\|\Delta\| = \left(\left\| \theta D^* \widetilde{P} D \right\| \right)^{-1/2} \geq r^w\left(A;(D,\ E)\right)$$

■

5.5 Examples

Example 5.14 ([48]) *Consider the parabolic equation with Newman boundary condition*

$$\begin{cases} \partial y(x,\ t)/\partial t = \partial^2 y(x,\ t)/\partial x^2 - y(x,t),\ x \in [0,\ 1],\ t \in [0,\ T] \\ y(x,\ 0) = y_0(x), \\ \partial y(0,\ t)/\partial x = -c\,\langle f, y(x,t)\rangle\, \dot{w}(t),\ \partial y(1,\ t)/\partial x\ = 0,\ c \in \mathbb{R},\ f \in L^2\,(0,1) \end{cases} \tag{5.31}$$

To put the problem (5.31) into the abstract setting we introduce the self-adjoint operator $Ah = \frac{\partial^2 h}{\partial x^2} - h$ in the Hilbert space $H = L^2\,(0,1))$ with

$$D(A) = \left\{ z \in L_2(0,\ 1),\ \frac{\partial^2 z}{\partial x^2} \in L_2(0,\ 1),\ \frac{\partial z}{\partial x} = 0 \text{ at } x = 0,\ 1 \right\}$$

The operator A generates an analytic semigroup $S(t)$ (see [44]).The eigenvalues of A are $\lambda_n = -\left(1 + n^2\pi^2\right)$, and eigenvectors are $\varphi_n = \frac{1}{\sqrt{2}}\cos n\pi x,\ n \geq 1$

$\varphi_0 = 1$. Let N (Newman map) be the map $L_2(\{0,\ 1\}) \longrightarrow L_2(0,1)$ defined by $y = Ng$ where y is the solution of

$$Ay = 0,\ \frac{\partial y}{\partial x}(0) = g,\ \frac{\partial y}{\partial x}(1) = 0$$

Now define the operator

$$Du = A^* Nu$$

Let $\Phi \in D(A)$, we have

$$\begin{aligned} \langle Du,\ \Phi\rangle_{L^2(0,\ 1)} &= \langle Nu,\ A\Phi\rangle_{L^2(0,\ 1)} \\ &= \int_0^1 Nu(x)\frac{\partial^2 \Phi}{\partial x^2}(x)dx - \int_0^1 Nu(x)\Phi(x)dx \end{aligned}$$

We obtain from the Green's formula:

$$\begin{aligned} \langle Du,\ \Phi\rangle_{L^2(0,\ 1)} &= Nu(1)\frac{\partial \Phi}{\partial x}(1) - Nu(0)\frac{\partial \Phi}{\partial x}(0) \\ &\quad - \int_0^1 \frac{\partial Nu(x)}{\partial x}\frac{\partial \Phi}{\partial x}(x)dx - \int_0^1 Nu(x)\Phi(x)dx \\ &= - \int_0^1 \frac{\partial Nu(x)}{\partial x}\frac{\partial \Phi}{\partial x}(x)dx - \int_0^1 Nu(x)\Phi(x)dx \end{aligned}$$

Applying again the Green's formula we obtain

$$\begin{aligned} \int_0^1 \frac{\partial Nu(x)}{\partial x}\frac{\partial \Phi}{\partial x}(x)dx &= \frac{\partial Nu(1)}{\partial x}\Phi(1) - \frac{\partial Nu(0)}{\partial x}\Phi(0) - \int_0^1 \frac{\partial^2 Nu(x)}{\partial x^2}\Phi(x)dx \\ &= u(x)\Phi(0) - \int_0^1 \frac{\partial^2 Nu(x)}{\partial x^2}\Phi(x)dx \end{aligned}$$

From which we obtain

$$\begin{aligned} \langle Du,\ \Phi\rangle_{L^2(0,\ 1)} &= -u(x)\Phi(0) + \int_0^1 \frac{\partial^2 Nu(x)}{\partial x^2}\Phi(x)dx - \int_0^1 Nu(x)\Phi(x)dx \\ &= -u(x)\delta^*\Phi(x) + \int_0^1 \left(\frac{\partial^2 Nu(x)}{\partial x^2} - Nu(x)\right)\Phi(x)dx \\ &= -u(x)\delta^*\Phi(x) - \int_0^1 \langle ANu(x),\ \Phi(x)\rangle\, dx \end{aligned}$$

where δ is the Dirac Delta function at zero. But since $ANu = 0$ it follows that

$$\langle Du,\ \Phi\rangle_{L^2(0,\ 1)} = -\langle \delta u(x),\ \Phi(x)\rangle_{L^2(0,\ 1)}$$

We deduce that $D \in L\left(\mathbb{R}, H^{-\frac{1}{2}-\epsilon}(0,\ 1)\right)$ is defined by $Du = -\delta u$. Setting $Ez = \langle f,\ z\rangle$, $\Delta = k \in R$, We can present (5.31) as follows

$$\begin{cases} dz(t) = Az(t)dt + D\Delta E(z(t))dw(t) \\ z(0) = z_0 \end{cases} \tag{5.32}$$

Since $\alpha = \gamma + \delta = \frac{1}{4} + \frac{\varepsilon}{2} < \frac{1}{2}$, for ε such that $0 < \varepsilon < \frac{1}{2}$, then there exists a unique solution of (5.32). In order to get an explicit formula for the stability radius we need at first to solve the following Lyapunov equation

$$2\left\langle \widetilde{P}z, Az\right\rangle + \langle Ez, Ez\rangle = 0,\ z \in D(A) \tag{5.33}$$

Suppose we can express the solution $\widetilde{P}$ of (5.33) by

$$\widetilde{P}z = \sum_{n,j=0}^{+\infty} P_{nj}\langle z, \varphi_n\rangle \varphi_j,\ z \in H$$

Then since

$$Az = \sum_{n=0}^{+\infty}\lambda_n \langle z, \varphi_n\rangle \varphi_n,\ z \in D(A)$$

It follows that

$$\begin{aligned} \left\langle \widetilde{P}z, Az\right\rangle &= \left\langle \sum_{i,,j=0}^{+\infty} P_{ij}\langle z, \varphi_i\rangle \varphi_j,\ \sum_{n=0}^{+\infty}\lambda_n \langle z, \varphi_n\rangle \varphi_n \right\rangle \\ &= \sum_{i,,n=0}^{+\infty} P_{in}\lambda_n \langle z, \varphi_n\rangle^2 \end{aligned}$$

Now since

$$\begin{aligned} \langle f,\ z\rangle &= \left\langle f,\ \sum_{n=0}^{+\infty} \langle z, \varphi_n\rangle \varphi_n \right\rangle \\ &= \sum_{n=0}^{+\infty} \langle z, \varphi_n\rangle \langle f,\ \varphi_n\rangle \end{aligned}$$

It follows that

$$\begin{aligned}\langle Ez, Ez\rangle &= |\langle f,\ z\rangle|^2 \\ &= \left(\sum_{n=0}^{+\infty} \langle z, \varphi_n\rangle \langle f,\ \varphi_n\rangle\right)^2\end{aligned}$$

Equation (5.33) becomes

$$2\sum_{i,n=0}^{+\infty} P_{in}\lambda_n \langle z, \varphi_n\rangle^2 + \left(\sum_{n=0}^{+\infty} \langle z, \varphi_n\rangle \langle f,\ \varphi_n\rangle\right)^2 = 0,\ z \in D(A)$$

Assume that $P_{in} = 0$ *for* $i \neq n$. *For* $z = \varphi_k$, $k \geq 0$, *we get*

$$2\lambda_k P_{kk} + \langle f,\ \varphi_k\rangle^2 = 0$$

Therefore

$$P_{kk} = -\langle f,\ \varphi_k\rangle^2 / 2\lambda_k$$

We deduce that the solution of (5.33) is given by

$$\widetilde{P}z = \sum_{k=0}^{+\infty} P_k \langle z, \varphi_k\rangle \varphi_k,\ z \in H;\ \textit{where } P_k = \langle f,\ \varphi_k\rangle^2 / 2\left(1 + k^2\pi^2\right),\ k \geq 1.$$

For all $u \in U$ *we have*

$$\left\| D^*\widetilde{P}D \right\| = \sup_{u\in U} \left\langle D^*\widetilde{P}Du,\ u\right\rangle$$

But

$$\left\langle D^*\widetilde{P}Du,\ u\right\rangle = \left\langle \widetilde{P}Du,\ Du\right\rangle$$

Since

$$\begin{aligned}Du &= \sum_{k=0}^{+\infty} \langle Du, \varphi_k\rangle \varphi_k = \sum_{k=0}^{+\infty} \langle u, \delta^*\varphi_k\rangle \varphi_k \\ &= -\sum_{k=0}^{+\infty} u(s)\varphi_k(0)\varphi_k = -\frac{1}{\sqrt{2}} \sum_{k=0}^{+\infty} u(s)\varphi_k\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{P}Du &= \sum_{k=0}^{+\infty} P_k \left\langle Du, \varphi_k \right\rangle \varphi_k \\
&= -\frac{1}{\sqrt{2}} \sum_{k=0}^{+\infty} P_k u(s) \varphi_k
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left\langle D^* \widetilde{P} Du,\ u \right\rangle &= \left\langle -\frac{1}{\sqrt{2}} \sum_{k=0}^{+\infty} P_k u(s)\varphi_k, -\frac{1}{\sqrt{2}} \sum_{n=0}^{+\infty} u(s)\varphi_n \right\rangle \\
&= \frac{1}{2} \left\langle \sum_{k=0}^{+\infty} P_k u(s)\varphi_k,\ \ \sum_{n=0}^{+\infty} u(s)\varphi_n \right\rangle \\
&= \frac{1}{2} \left| u(s) \right|^2 \sum_{k=0}^{+\infty} \sum_{n=0}^{+\infty} P_k \left\langle \varphi_k,\ \varphi_n \right\rangle
\end{aligned}
$$

we get than

$$
\left\langle D^* P D u,\ u \right\rangle = \frac{1}{2} \left| u(s) \right|^2 \sum_{k=0}^{+\infty} P_k
$$

From which we deduce that

$$
\left\| D^* \widetilde{P} D \right\| = \sup_{\substack{u \in U \\ \|u\|=1}} \left\langle D^* \widetilde{P} Du,\ u \right\rangle = \frac{1}{2} \sum_{k=0}^{+\infty} P_k
$$

But

$$
\sum_{k=0}^{+\infty} P_k = \sum_{k=0}^{+\infty} \frac{\left\langle f,\ \varphi_k \right\rangle^2}{2\left(1 + k^2\pi^2\right)} \tag{5.34}
$$

If we assume that $\langle f,\ \varphi_k \rangle = 1$, *for any* $k \in \mathbb{N}$, *then (5.34) yields*

$$
\sum_{k=0}^{+\infty} P_k = \sum_{k=0}^{+\infty} \frac{1}{2\left(1 + k^2\pi^2\right)} = \frac{1}{4} + \frac{1}{4}\coth(1) \simeq 0.578\,26
$$

Thus

$$
\left\| D^* \widetilde{P} D \right\| = \left(\frac{1}{8} + \frac{1}{8}\coth(1) \right) \simeq 0.28913
$$

Therefore

$$
r^{\omega}\left(A,\ D,\ E\right) = \left\| D^* \widetilde{P} D \right\|^{-1/2} \simeq 1.8597
$$

We conclude that for all $c^2 < 3.4587$, *the system (5.31) is stable. This bound is larger than* $3/2$ *obtained by Ichikawa in [48].*

Example 5.15 *Consider the stochastic parabolic equation*

$$\begin{cases} dy(x,t) = \left[\partial^2 y(x,t)/\partial x^2\right] dt + cb(x)y(\xi_0,t)dw(t),\ 0 < x,\ \xi_0 < 1. \\ y(x,0) = y_0(x),\ y(0,t) = y(1,\ t) = 0. \end{cases} \tag{5.35}$$

In this example we take $H = L^2(0,\ 1)$ *and* $A = d^2/dx^2$, $D(A) = H_0^1(0,\ 1) \cap H^2(0,\ 1)$, $D = b \in L^2(0,\ 1)$, $\Delta = c \in \mathbb{R}$ *and* $E \in L\left(H^{\frac{1}{2}+\varepsilon}(0,\ 1),\ \mathbb{R}\right)$, *such that* $Ez = z(\xi_0)$. *In the abstract form, the system (5.35) can be presented as follows*

$$\begin{cases} dz(t) = Az(t)dt + D\Delta\left(Ez(t)\right)dw(t) \\ z(0) = z_0 \end{cases} \tag{5.36a}$$

For this system, we have $\gamma = 0$ *and* $\delta = \frac{1}{4} + \frac{\varepsilon}{2}$, $\varepsilon > 0$, *so there exists* $\varepsilon > 0$ *such that* $\alpha < \frac{1}{2}$.*We deduce that equation (5.36a) has a unique solution.*

Now we solve the Lyapunov equation

$$2\left\langle \widetilde{P}z, Az\right\rangle + \langle Ez, Ez\rangle = 0, z \in D(A). \tag{5.37}$$

As in the above example we can show that

$$\left\langle \widetilde{P}z, Az\right\rangle = \sum_{i,,n=1}^{+\infty} P_{in}\xi_n \langle z, \psi_n\rangle^2$$

where ξ_n *and* ψ_n *are, respectively, the eigenvalues and the eigenvectors of the operator* A. *They are given by [8]*

$$\xi_n = -n^2\pi^2,\ \psi_n(x) = \sqrt{2}\sin(n\pi x),\ n \geq 1$$

For the second term of the Lyapunov equation (5.37) we have

$$\langle Ez, Ez \rangle = |z(\xi_0)|^2$$

Equation (5.37) is then equivalent to

$$2 \sum_{i,,n=1}^{+\infty} P_{in} \xi_n \langle z, \psi_n \rangle^2 + |z(\xi_0)|^2 = 0$$

Assume that $P_{in} = 0$ *for* $i \neq n$. *For* $z = \psi_k$, $k \geq 1$, *we get*

$$2P_{kk}\xi_k + |\psi_k(\xi_0)|^2 = 0$$

From which we obtain

$$P_{kk} = -|\psi_k(\xi_0)|^2 / 2\xi_k$$

We deduce that the solution of (5.37) is given by

$$\widetilde{P}z = \sum_{k=1}^{+\infty} P_k \langle z, \psi_k \rangle \psi_k, \; z \in H; \; \textit{where } P_k = |\psi_k(\xi_0)|^2 / 2k^2\pi^2, \; k \geq 1.$$

We have

$$D^* \widetilde{P} D = b^2 \widetilde{P}$$

To obtain a sufficient condition for stability we assume $\|b\| = 1$. *Then*

$$\left\| D^* \widetilde{P} D \right\|_U = b^2 \left\| \widetilde{P} \right\| = \sum_{k=0}^{+\infty} P_k = \sum_{k=1}^{+\infty} \frac{\left| \sqrt{2} \sin(k\pi\xi_0) \right|^2}{2k^2\pi^2}$$

Since $\left| \sqrt{2} \sin(n\pi\xi_0) \right| \leq \sqrt{2}$, *for all* $k \in N^*$, *we obtain*

$$\left\| D^* \widetilde{P} D \right\| \leq \sum_{k=1}^{+\infty} \frac{1}{k^2\pi^2}$$

But

$$\sum_{k=1}^{+\infty} \frac{1}{k^2\pi^2} = \frac{1}{\pi^2} \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{1}{6}$$

We deduce that $\left\| D^* \widetilde{P} D \right\| \leq 1/6$, *and thus* $\left\| D^* \widetilde{P} D \right\|^{-1} \geq 6$. *Hence if* $c^2 < 6$, *then* $c^2 < \left\| D^* \widetilde{P} D \right\|^{-1} \leq \left(r^{\omega}\left(A,\ D,\ E \right) \right)^2$ *from which we conclude that the system (5.35) is stable The same result was obtained by Ichikawa in [48].*

Example 5.16 *Consider the stochastic parabolic equation*

$$\begin{cases} dy(x,t) = \left[\partial^2 y(x,t)/\partial x^2 \right] dt + c\delta(x-\xi) \left\langle f, y \right\rangle dw(t),\ 0 < x,\ \xi < 1,\ f \in L^2\left(0,1\right). \\ \qquad y(x,0) = y_0(x),\ y(0,t) = y(1,\ t) = 0. \end{cases} \tag{5.38}$$

In this case we take $H = L^2\left(0,\ 1\right)$ *and* $A = d^2/dx^2$, $D(A) = H_0^1\left(0,\ 1\right) \cap H^2\left(0,\ 1\right)$, $Du = \delta_\xi u$, *where* δ_ξ *is the Dirac Delta function at* ξ *and* $Ez = \left\langle f,\ z \right\rangle$, $\Delta = c \in R$. *Then* $D \in L\left(\mathbb{R}, H^{-\frac{1}{2}-\epsilon}\left(0,\ 1\right) \right)$, $\varepsilon > 0$ *and* $E \in L\left(H, \mathbb{R}\right)$. *In the abstract form, the problem (5.38) can be formulated as follows*

$$\begin{cases} dz(t) = Az(t) + D\Delta E(z(t))dw(t) \\ \qquad z(0) = z_0 \end{cases} \tag{5.39}$$

For this system, we have $\gamma = \frac{1}{4} + \frac{\varepsilon}{2}$, $\varepsilon > 0$, *and* $\delta = 0$, *so there exists* $\varepsilon > 0$ *such that* $\alpha < \frac{1}{2}$. *We deduce that equation (5.39) has a unique solution.*

In order to get an explicit formula for the stability radius we solve at first the following Lyapunov equation

$$2\left\langle \widetilde{P}z, Az \right\rangle + \left\langle Ez, Ez \right\rangle = 0, z \in D(A). \tag{5.40}$$

As in Example (5.15) we obtain

$$\widetilde{P}z = \sum_{k=1}^{+\infty} P_k \left\langle z, \psi_k \right\rangle \psi_k,\ z \in H;\ \text{where } P_k = \left\langle f,\ \psi_k \right\rangle^2 / 2k^2\pi^2\ k \geq 1.$$

here ξ_n *and* ψ_n *are, respectively, the eigenvalues and the eigenvectors of the* *operator A. They are given by [8]*

$$\xi_n = -n^2\pi^2,\ \ \psi_n(x) = \sqrt{2}\sin(n\pi x),\ \ n \geq 1$$

For all $u \in U$ we have

$$\left\| D^* \widetilde{P} D \right\| = \sup_{u\in U} \left\langle D^* \widetilde{P} Du,\ u \right\rangle$$

But

$$\left\langle D^* \widetilde{P} Du,\ u \right\rangle = \left\langle \widetilde{P} Du,\ Du \right\rangle$$

Since

$$\begin{aligned} Du &= \sum_{k=1}^{+\infty} \langle Du, \psi_k \rangle \psi_k \\ &= \sqrt{2} \left(\sum_{k=1}^{+\infty} \sin(k\pi\xi)\psi_k \right) u(s) \end{aligned}$$

and

$$\begin{aligned} PDu &= \sum_{k=1}^{+\infty} P_k \langle Du, \psi_k \rangle \psi_k \\ &= \sqrt{2} \left(\sum_{k=1}^{+\infty} P_k \sin(k\pi\xi)\psi_k \right) u(s) \end{aligned}$$

It follows that

$$\left\langle D^* \widetilde{P} Du,\ u \right\rangle = 2\left|u(s)\right|^2 \sum_{k=1}^{+\infty}\sum_{n=1}^{+\infty} \sin(n\pi\xi) P_k \left\langle \psi_k,\ \psi_n \right\rangle$$

we get that

$$\left\langle D^* \widetilde{P} Du,\ u \right\rangle = 2\left|u(s)\right|^2 \sum_{k=1}^{+\infty} \sin^2(k\pi\xi) P_k$$

From which we deduce that

$$\left\| D^* \widetilde{P} D \right\| = \sup_{\|u\|=1} \left\langle D^* \widetilde{P} Du,\ u \right\rangle = 2\sum_{k=1}^{+\infty} \sin^2(k\pi\xi) P_k$$

But

$$\sum_{k=0}^{+\infty} \sin^2(k\pi\xi) P_k = \sum_{k=0}^{+\infty} \sin^2(k\pi\xi) \frac{\langle f,\ \psi_k\rangle^2}{2k^2\pi^2} \tag{5.41}$$

If we assume that $\langle f,\ \psi_k\rangle = 1$, *for any* $k \in \mathbb{N}$, *then (5.41) yields*

$$\sum_{k=1}^{+\infty} \sin^2(k\pi\xi) P_k = \sum_{k=1}^{+\infty} \sin^2(k\pi\xi) \frac{1}{2k^2\pi^2} \leq \sum_{k=1}^{+\infty} \frac{1}{2k^2\pi^2}$$

We have

$$\sum_{k=1}^{+\infty} \frac{1}{2k^2\pi^2} = \frac{1}{12}$$

Thus

$$\left\| D^* \widetilde{P} D \right\| \leq \frac{1}{6}$$

We deduce that

$$r^{\omega}\left(A,\ D,\ E\right) = \left\| D^* \widetilde{P} D \right\|^{-1/2} \geq \sqrt{6}$$

Hence if $c^2 < 6$, *the system (5.38) is stable. The same result was obtained by Ichikawa in [48]. In the particular case* $\xi = \frac{1}{2}$, *we obtain*

$$\left\| D^* \widetilde{P} D \right\| = \sum_{k=1}^{+\infty} \frac{1}{k^2\pi^2} \sin^2(\frac{k\pi}{2})$$

But

$$\begin{aligned} \sum_{k=1}^{+\infty} \frac{1}{k^2\pi^2} \sin^2\left(\frac{k\pi}{2}\right) &= \frac{1}{\pi^2} \sum_{n=0}^{+\infty} \frac{1}{\left(2n+1\right)^2} \sin^2\left(\frac{\left(2n+1\right)\pi}{2}\right) \\ &= \frac{1}{\pi^2} \sum_{n=0}^{+\infty} \frac{\left(-1\right)^{2n}}{\left(2n+1\right)^2} \end{aligned}$$

Hence

$$\left\| D^* \widetilde{P} D \right\| \simeq 0.125$$

In this case the stability radius is given by

$$r^{\omega}\left(A;\ D,\ E\right) = \left\| D^* \widetilde{P} D \right\|^{-\frac{1}{2}} \simeq 2.828\,4$$

Chapter 6

Conclusion

Motivated by many applications in control engineering, problems of robust stability of dynamic systems have attracted a lot of attention of researchers during the last twenty years. In the study of these problems, the notion of stability radius was proved to be a very effective tool. In this thesis, we have used the framework of stability radii to investigate the problems of robust stability and robust stabilization for linear deterministic systems on real Hilbert spaces which are subjected to Lipschitzian stochastic structured multi-perturbations. Naturally, the research in this area is not completed by this thesis and several interesting open problems remain.

6.1 Summary of the obtained results

First we considered the case where the operators describing the structure of the perturbations are bounded. We established, by adapting the approach used in [39], characterizations of the stability radius in terms of a Lyapunov equation and the corresponding inequalities. These characterizations are used to obtain a computational formula for this radius.

In order to improve the stability margin, we studied the problem of maximizing the stability radius with respect to state feedback. We established conditions for the existence of suboptimal controllers in terms of a Riccati equation. We showed also how the supremal stability radius can be determined in terms of this equation.

Our last contribution concerns the robustness of stability in the case where the operators structure are unbounded. This case is more important in the applications because it covers the case of partial differential equations with boundary and pointwise noise. We showed how we can generalize the results established in the bounded case for this case. We characterized the stability radius in terms of a Lyapunov equation similar to the one used in the bounded case. These characterizations enables us to determine a lower bound for the stability radius. It is shown, under an additional assumption, that this lower bound is equal to the stability radius.

Several examples are given in the thesis which illustrate the different results obtained.

6.2 Open problems

In this section we discuss some open problems, which appear to be interesting.

1. Since time delays are encountered in various physical and engineering systems, robust stability problems of linear time-delay systems have received much attention. In Chapter 3, we illustrated the calculus of the stability radius for the class of delay systems. We considered just the one and two dimensional cases. It is important to establish formula for computing the corresponding stability radius. We notice that the problem of computing the stability radius for linear time-delay systems under deterministic multi-perturbations has just been solved recently in [43] and only for positive systems.

2. The problem of maximizing the stability radius is studied assuming that the full state is available for measurement. However, for some control systems this may not be a valid hypothesis. Therefore, it would be interesting to consider the problem of optimizing the stability radius by static or dynamic output feedback.

3. In the case where the operators structure are unbounded, there seems to be many interesting open problems.

   (a) In this case, we have obtained just a lower bound for the stability radius. We believe that we can obtain a result similar to the bounded case. However, this requires an investigation.

(b) It would be interesting to generalize the obtained results for the multi-peturbations case.

(c) The results of Chapter 5 are established assuming that the semi-group describing the dynamics is analytic. A study which drops this hypothesis is desirable.

(d) The maximization problem has not been considered. This can be an area for future research.

4. In the case of stochastic systems with Markov jump perturbations, some estimations on the stability radius are given in [20]. It seems to be interesting to generalize there results to infinite dimensional systems.

This seems to be an appropriate end of this thesis but not of the research.

Appendix A

In this appendix we give some results, established in [17], concerning the resolution of the Lyapunov equation for delay systems.

Let b be a positive number and $\theta_1, ..., \theta_k$, be real numbers satisfying

$$-b = \theta_k < \theta_{k-1} < ... < \theta_1 < \theta_0 = 0$$

Consider the linear delay differential equation

$$\begin{cases} \frac{dx(t)}{dt} = A_0 x(t) + \sum\limits_{i=1}^{k} A_i x(t+\theta_i) \\ x(0) = r, \\ \quad x(\theta) = h(\theta), \quad -b \leq \theta < 0. \end{cases} \tag{A.1}$$

where $A_i \in L(\mathbb{R}^N)$, $i = 1, ..., k$, $r \in \mathbb{R}^N$ and $h \in L^2\left(-b,\ 0;\ \mathbb{R}^N\right)$. Taking the space $H = M_2\left(-b,\ 0;\ \mathbb{R}^N\right) = \mathbb{R}^N \times L^2\left(-b,\ 0;\ \mathbb{R}^N\right)$ endowed with the inner product

$$\langle z,\ z' \rangle_H = \left\langle \begin{pmatrix} y(0) \\ y \end{pmatrix},\ \begin{pmatrix} y'(0) \\ y' \end{pmatrix} \right\rangle_H = \left[\langle y(0),\ y'(0) \rangle + \int_{-b}^{0} \langle y(\theta),\ y'(\theta) \rangle \, d\theta \right]$$

and the new state $z(t) = \begin{pmatrix} x(t) \\ x(t+\theta) \end{pmatrix}$ in H, it can be shown [10] that the operator

$$Az = A \begin{pmatrix} y(0) \\ y \end{pmatrix} = \begin{pmatrix} A_0 y(0) + \sum_{i=1}^{k} A_i y(\theta_i) \\ \frac{dy}{d\theta} \end{pmatrix}$$

with domain

$$D(A) = \left\{ \begin{pmatrix} y(0) \\ y \end{pmatrix};\ y,\ \text{abs. cont. and } \dot{y} \in L^2\left(-b,\ 0;\ \mathbb{R}^N\right) \right\}$$

generates a strongly continuous semigroup in H. The System (A.1) can be formulated as the abstract differential equation

$$\begin{cases} \dot{z}(t) = Az(t), \\ z(0) = z_0 \end{cases} \tag{A.2}$$

on the state space H.

We have the following definition of the L^2-stability.

Definition A.1 *The system (A.1) is said to be L^2-stable if*

$$\lim_{t\to\infty} \int_0^t \langle z(s),\ z(s)\rangle_H \, ds < \infty,$$

where $z(s)$ is the solution of the system (A.2).

We have the following conditions for L^2-stability, established in [17].

Theorem A.2 *Let $Q \succ 0$ in $L(\mathbb{R}^N)$. The following statements are equivalent.*

1. *The system (A.1) is L^2-stable.*

2. There exists a nonnegative self-adjoint operator $P \in L(H)$ such that

$$\langle Az,\ Pz' \rangle + \langle Pz,\ Az' \rangle + \left\langle z,\ \widetilde{Q}z' \right\rangle = 0,\ \text{for all } z, z' \in D(A) \qquad \text{(A.3)}$$

where $\widetilde{Q}z = (Qy_0,\ 0)$; *for* $z = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in \mathbb{R}^N \times L^2\left(-b,\ 0;\ \mathbb{R}^N\right)$.

3. There exists an $\alpha < 0$ such that the spectrum $\sigma(A)$ of A lies entirely in $\{\lambda \in \mathbb{C} / \operatorname{Re}\lambda \leq \alpha\}$, where $\sigma(A) = \{\lambda \in \mathbb{C} / \det \Delta(\lambda) = 0\}$ and $\det \Delta(\lambda)$ is the determinant of the matrix

$$\Delta(\lambda) = \lambda I - A_0 - \sum_{i=1}^{k} A_i e^{\lambda \theta_i}.$$

Let $\Phi(t)$ be the semigroup generated by A. By [17], the semigroup $\Phi(t)$ satisfy the following identity for all $z = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$ in H

$$[\Phi(t)z]_0 = \Phi^0(t)y_0 + \Phi^1(t)y_1. \qquad \text{(A.4)}$$

where

$$\Phi^1(t)y_1 = \int_{-b}^{0} \Phi^1(t,\alpha)y_1(\alpha)d\alpha \qquad \text{(A.5)}$$

and

$$\Phi^1(t,\alpha) = \sum_{j=1}^{k} \begin{cases} \Phi^0(t-\alpha+\theta_i)A_i, & t \geqslant \alpha - \theta_i \geqslant 0 \\ 0 & , \ \text{otherwise} \end{cases} \qquad \text{(A.6)}$$

In the following proposition, established in [17], we give a characterization of the solution of the Lyapunov equation (A.3) in terms of $\Phi(t)$.

Proposition A.3 *If the Lyapunov equation (A.3) has a nonnegative solution in $L(H)$, it is unique and for all z, z' in H*

$$\langle Pz, z' \rangle_H = \int_0^{+\infty} \left\langle \Phi(t)z, \widetilde{Q}\Phi(t)z' \right\rangle dt;\ \ z,\ z' \in H$$

More characterizations of the solution P will be given in the following theorem.

Theorem A.4 *Let $P \succeq 0$ in $L(H)$ be the solution of the Lyapunov equation (A.3). It is completely characterized by its matrix of operators*

$$\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}, \qquad \begin{aligned} P_{00} &\in L(\mathbb{R}^N),\ P_{01} \in L\left(L^2\left(-b,\ 0;\ \mathbb{R}^N\right),\ \mathbb{R}^N\right), \\ P_{10} &\in L\left(\mathbb{R}^N,\ L^2\left(-b,\ 0;\ \mathbb{R}^N\right)\right),\ P_{11} \in L\left(L^2\left(-b,\ 0;\ \mathbb{R}^N\right)\right). \end{aligned}$$

P_{00} is characterized by the equation

$$P_{00}A_0 + A_0^* P_{00} + P_{10}(0) + P_{10}(0)^* + Q = 0,\ P_{00} = (P_{00})^* \succeq 0. \tag{A.7}$$

P_{10} is characterized in the following way:

$$\left(P_{10}y^0\right)(\alpha) = P_{10}(\alpha)y^0,\ y^0 \in \mathbb{R}^N \tag{A.8}$$

where the map

$$\alpha \longmapsto P_{10}(\alpha) : [-b,\ 0] \longrightarrow L(\mathbb{R}^N) \tag{A.9}$$

is piecewise absolutely continuous with jumps at $\alpha = \theta_i$ of height $A_i^* P_{00}$, $i = 1, ..., k-1$. Moreover the map (A.9) is itself characterized by the differential equation

$$\begin{aligned} \frac{dP_{10}}{d\alpha}(\alpha) &= P_{10}(\alpha)A_0 + \sum_{i=1}^{k-1} A_i^* P_{00}\delta(\alpha - \theta_i) + P_{11}(\alpha,\ 0),\ ae.\ in\ \ [-b,\ 0], \\ P_{10}(-b) &= A_k^* P_{00}, \end{aligned} \tag{A.10}$$

where $\delta(\alpha - \theta_i)$ is the δ-function at $\alpha = \theta_i$.

P_{01} is obtained from P_{10} :

$$P_{01}y^1 = \int_{-b}^{0} P_{10}(\alpha)^* y^1(\alpha) d\alpha,\ \ y^1 \in L^2\left(-b,\ 0;\ \mathbb{R}^N\right). \tag{A.11}$$

P_{11} is characterized in the following way:

$$\left(P_{11}y^1\right)(\alpha) = \int_{-b}^{0} P_{11}(\alpha,\ \beta) y^1(\beta) d\beta, \tag{A.12}$$

where the map

$$(\alpha,\ \beta) \longmapsto P_{11}(\alpha,\ \beta) : [-b,\ 0] \times [-b,\ 0] \longrightarrow L(\mathbb{R}^N) \tag{A.13}$$

is piecewise absolutely continuous in each variable with jumps of height $A_i^* P_{10}(\beta)^*$ at $\alpha = \theta_i,\ i = 1, ..., k-1$ (resp. $P_{10}(\alpha) A_j$ at $\beta = \theta_j,\ j = 1, ..., k-1$). Moreover the map $P_{11}(\alpha,\ \beta)$ is the solution of

$$\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right) P_{11}(\alpha, \beta) = \sum_{i=1}^{k-1} A_i^* P_{10}(\beta)^* \delta(\alpha - \theta_i) + \sum_{i=1}^{k-1} P_{10}(\alpha) A_i \delta(\beta - \theta_j) \tag{A.14}$$

with boundary conditions

$$P_{11}(-b,\ \beta) = A_k^* P_{10}(\beta)^*,\ \ P_{11}(\alpha, -b\ ) = P_{10}(\alpha) A_k, \tag{A.15}$$

and symmetry property $P_{11}(\alpha,\ \beta) = P_{11}(\beta,\ \alpha)^*$.

The solution of the above differential system is

$$\begin{aligned} P_{11}(\alpha,\ \beta) \quad = \quad & \begin{cases} P_{10}(\alpha - \beta - b) A_k, & \alpha \geqslant \beta \\ A_k^* P_{10}(\beta - \alpha - b)^*, & \alpha < \beta \end{cases} \\ & + \sum_{i=1}^{k-1} \begin{cases} A_i^* P_{10}(\beta - \alpha + \theta_i)^*, & -b \leq \beta - \alpha + \theta_i,\ \theta_i < \alpha \\ 0, & \textit{otherwise} \end{cases} \\ & + \sum_{j=1}^{k-1} \begin{cases} P_{10}(\alpha - \beta + \theta_j) A_j, & -b \leq \alpha - \beta + \theta_j,\ \theta_j < \beta \\ 0, & \textit{otherwise} \end{cases} \end{aligned} \tag{A.16}$$

In order to show how we can get equation (A.7) and the differential equations (A.10) and (A.14) we will give a detailed proof of this theorem similar to the one given in [17].

Proof. The proof will be decomposed in three parts. In the first part we use the proposition (A.3) to study P_{00} and the kernels $P_{10}(\alpha)$ and $P_{11}(\alpha,\beta)$ of the operators P_{10} and P_{11}. In the second part we use the results of the first one to derive equation (A.7) and the differential equations for $P_{10}(\alpha)$ and $P_{11}(\alpha, \beta)$. Explicit expression of $P_{11}(\alpha,\beta)$ in terms of $P_{10}(\alpha)$ is given in the third part.

Part1: By proposition (A.3) we have

$$\langle Pz, z'\rangle_H = \int_0^{+\infty} \langle \Phi(t)z, Q\Phi(t)z'\rangle\, dt,\ z, z' \in H.$$

(i) Let $z = \begin{pmatrix} h_0 \\ 0 \end{pmatrix}$ and $z' = \begin{pmatrix} f_0 \\ 0 \end{pmatrix}$ such that $h_0,\ f_0 \in \mathbb{R}^N$. Then

$$\left\langle \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \begin{pmatrix} h_0 \\ 0 \end{pmatrix}, \begin{pmatrix} f_0 \\ 0 \end{pmatrix} \right\rangle_H = \int_0^{+\infty} \left\langle \Phi^0(t)h_0, \widetilde{Q}\Phi^0(t)f_0 \right\rangle dt$$

which implies that

$$\langle h_0, P_{00}f_0\rangle = \left\langle h_0, \left(\int_0^{+\infty} \Phi^0(t)^* Q \Phi^0(t)dt\right) f_0 \right\rangle, \text{ for all } h_0,\ f_0 \text{ in } \mathbb{R}^n.$$

We deduce that $P_{00} = \left(\int_0^{+\infty} \Phi^0(t)^* Q\Phi^0(t)dt\right)$.

(ii) Let $z = \begin{pmatrix} 0 \\ h_1 \end{pmatrix}$ and $z' = \begin{pmatrix} f_0 \\ 0 \end{pmatrix}$ such that $h_1 \in L^2\left(-b,\ 0;\ \mathbb{R}^N\right)$ and $f_0 \in \mathbb{R}^N$. Then

$$\left\langle \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \begin{pmatrix} 0 \\ h_1 \end{pmatrix}, \begin{pmatrix} f_0 \\ 0 \end{pmatrix} \right\rangle_H = \int_0^{+\infty} \left\langle \Phi^1(t)h_1, Q\Phi^0(t)f_0 \right\rangle dt.$$

Since $\Phi^1(t)h_1 = \int_{-b}^{0} \Phi^1(t,\alpha)h_1(\alpha)d\alpha$, it follows that

$$\left\langle \begin{pmatrix} P_{01}h_1 \\ P_{11}h_1 \end{pmatrix}, \begin{pmatrix} f_0 \\ 0 \end{pmatrix} \right\rangle_H = \int_0^{+\infty} \left\langle \int_{-b}^{0} \Phi^1(t,\alpha)h_1(\alpha)d\alpha, \widetilde{Q}\Phi^0(t)f_0 \right\rangle dt$$

which yields

$$\langle P_{01}h_1, f_0\rangle = \int_{-b}^{0} \left\langle \left(\int_0^{+\infty} \Phi^0(t)^* \widetilde{Q}^* \Phi^1(t,\alpha)dt \right) h_1(\alpha),\ f_0 \right\rangle d\alpha.$$

Using (A.11) we obtain

$$\left\langle \int_{-b}^{0} P_{10}(\alpha)^* h_1(\alpha)d\alpha,\ f_0 \right\rangle = \left\langle \int_{-b}^{0} \left(\int_0^{+\infty} \Phi^0(t)^* Q^* \Phi^1(t,\alpha)dt \right) h_1(\alpha)d\alpha,\ f_0 \right\rangle$$

from which we deduce that

$$P_{10}(\alpha) = \int_0^{+\infty} \Phi^1(t,\alpha)^* Q \Phi^0(t)dt$$

We now substitute for $\Phi^1(t,\alpha)$ the expression

$$\sum_{i=1}^{k} \begin{cases} \Phi^0(t-\alpha+\theta_i)A_i,\ t \geqslant \alpha - \theta_i \geqslant 0 \\ 0, \quad \text{otherwise} \end{cases} \tag{A.17}$$

We get

$$P_{10}(\alpha) = \sum_{i=1}^{k} \begin{cases} A_i^* \int_{\alpha-\theta_i}^{+\infty} \Phi^0(t-\alpha+\theta_i)^* Q \Phi^0(t)dt,\ \theta_i \leq \alpha \\ 0 \qquad\qquad , \theta_i > \alpha \end{cases}$$

From this expression we see that $P_{10}(\alpha)$ has jumps at $\alpha = \theta_i$, $i = 1, ..., k-1$, of respective heights $A_i^* P_{00}$. Moreover $P_{10}(-b) = A_k^* P_{00}$.

(iii) Let $z = \begin{pmatrix} 0 \\ h_1 \end{pmatrix}$ and $z' = \begin{pmatrix} 0 \\ f_1 \end{pmatrix}$ such that $h_1,\ f_1 \in L^2\left(-b,\ 0;\ \mathbb{R}^N\right)$.

Then

$$\left\langle \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \begin{pmatrix} 0 \\ h_1 \end{pmatrix}, \begin{pmatrix} 0 \\ f_1 \end{pmatrix} \right\rangle_H = \int_0^{+\infty} \left\langle \Phi^1(t)h_1, Q\Phi^1(t)f_1 \right\rangle dt.$$

In view of (A.5)

$$\left\langle \begin{pmatrix} P_{01}h_1 \\ \\ P_{11}h_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \\ f_1 \end{pmatrix} \right\rangle_H = \int_0^{+\infty} \left\langle \int_{-b}^0 \Phi^1(t,\alpha)h_1(\alpha)d\alpha, Q\int_{-b}^0 \Phi^1(t,\beta)f_1(\beta)d\beta \right\rangle dt$$

which implies that

$$\langle P_{11}h_1,\ f_1\rangle_{L^2(-b,\ 0;\ \mathbb{R}^N)} = \int_{-b}^0 \left\langle \int_{-b}^0 \left(\int_0^{+\infty} \Phi^1(t,\beta)^* Q^* \Phi^1(t,\alpha)dt \right) h_1(\alpha)d\alpha,\ f_1(\beta) \right\rangle d\beta$$

hence

$$\begin{aligned} & \int_{-b}^0 \langle (P_{11}h_1)(s),\ f_1(s)\rangle\, ds \\ = \ & \int_{-b}^0 \left\langle \int_{-b}^0 \left(\int_0^{+\infty} \Phi^1(t,\beta)^* \widetilde{Q} \Phi^1(t,\alpha)dt \right) h_1(\alpha)d\alpha,\ f_1(\beta) \right\rangle d\beta \end{aligned}$$

using (A.12) we obtain that

$$\begin{aligned} & \int_{-b}^0 \left\langle \int_{-b}^0 P_{11}(s,\ \alpha)h_1(\alpha)d\alpha,\ f_1(s) \right\rangle ds \\ = \ & \int_{-b}^0 \left\langle \int_{-b}^0 \left(\int_0^{+\infty} \Phi^1(t,\beta)^* Q \Phi^1(t,\alpha)dt \right) h_1(\alpha)d\alpha,\ f_1(\beta) \right\rangle d\beta. \end{aligned}$$

which yields for all h_1 and f_1 in $L^2\left(-b,\ 0;\ \mathbb{R}^N\right)$

$$\begin{aligned} & \int_{-b}^0 \left\langle \int_{-b}^0 P_{11}(s,\ \alpha)h_1(\alpha)d\alpha,\ f_1(s) \right\rangle ds \\ = \ & \int_{-b}^0 \left\langle \int_{-b}^0 \left(\int_0^{+\infty} \Phi^1(t,\beta)^* Q^* \Phi^1(t,\alpha)dt \right) h_1(\alpha)d\alpha,\ f_1(\beta) \right\rangle d\beta. \end{aligned}$$

we deduce that

$$P_{11}(\beta,\ \alpha) = \int_0^{+\infty} \Phi^1(t,\beta)^* Q^* \Phi^1(t,\alpha)dt$$

Which implies that

$$P_{11}(\beta,\ \alpha)^* = \int_0^{+\infty} \Phi^1(t,\alpha)^* Q \Phi^1(t,\beta)dt.$$

Now we use (A.17) to express $P_{11}(\alpha,\,\beta)$ in terms of Φ^0.

$$
\begin{aligned}
P_{11}(\alpha,\beta) &= \int\limits_0^{+\infty}\left[\sum_{i=1}^{k}\begin{cases} A_i^*\Phi^0(t-\alpha+\theta_i)^*, & t\geqslant \alpha-\theta_i\geqslant 0\\ 0 & ,\ \text{otherwise}\end{cases}\right]\\
&\quad\times\widetilde{Q}\left[\sum_{j=1}^{k}\begin{cases}\Phi^0(t-\beta+\theta_j)A_j, & t\geqslant \beta-\theta_j\geqslant 0\\ 0 & ,\ \text{otherwise}\end{cases}\right]dt
\end{aligned}
$$

$$
=\sum_{i=1}^{k}\sum_{j=1}^{k}\int\limits_0^{+\infty}\begin{cases} A_i^*\Phi^0(t-\alpha+\theta_i)^*Q\Phi^0(t-\beta+\theta_j)A_j, & t\geqslant\alpha-\theta_i\geqslant 0,\ t\geqslant\beta-\theta_j\geqslant 0\\ 0 & ,\ \text{otherwise}\end{cases}\ dt
$$

$$
=\sum_{i=1}^{k}\sum_{j=1}^{k}\left[\begin{cases}\int_{\alpha-\theta_i}^{+\infty}dtA_i^*\Phi^0(t-\alpha+\theta_i)^*Q\Phi^0(t-\beta+\theta_j)A_j, & \alpha-\theta_i\geqslant\beta-\theta_j\geqslant 0\\ \int_{\beta-\theta j}^{+\infty}dtA_i^*\Phi^0(t-\alpha+\theta_i)^*Q\Phi^0(t-\beta+\theta_j)A_j, & \beta-\theta_j\geqslant\alpha-\theta_i\geqslant 0\\ 0 & ,\ \text{otherwise}\end{cases}\right]
$$

$$
=\sum_{i=1}^{k}\begin{cases}\sum_{j=1}^{k}\begin{cases}\begin{cases}\int_{\alpha-\theta_i}^{+\infty}dtA_i^*\Phi^0(t-\alpha+\theta_i)^*\\ Q\Phi^0(t-\beta+\theta_j)A_j, & \beta-\theta_j-\alpha+\theta_i\leqslant 0\\ \int_{\beta-\theta_j}^{+\infty}dt, & \beta-\theta_j-\alpha+\theta_i>0\end{cases} & ,\ \beta\geqslant\theta_j\\ 0 & ,\ \beta<\theta_j\end{cases} & ,\ \alpha\geq\theta_i\\ 0 & ,\ \alpha<\theta_i\end{cases}
$$

Given α, $P_{11}(\alpha,\beta)$ has jumps at $\beta=\theta_j$, $j=1...,k-1$, of height

$$
\sum_{i=1}^{k}\begin{cases}\int_{\alpha-\theta_i}^{+\infty}dtA_i^*\Phi^0(t-\alpha+\theta_i)^*Q\Phi^0(t)A_j, & \alpha\geqslant\theta_i\\ 0 & ,\ \alpha<\theta_i\end{cases}\quad = P_{10}(\alpha)A_j
$$

It follows that $P_{11}(\alpha,\,-b)=P_{10}(\alpha)A_k$. Now given β, $P_{11}(\alpha,\beta)$ has jumps at $\alpha=\theta_i$, $i=1...,k-1$, of height

$$
\sum_{j=1}^{k}\begin{cases}\int_{\beta-\theta_j}^{+\infty}dtA_i^*\Phi^0(t)^*\widetilde{Q}\Phi^0(t-\beta+\theta_j)A_j, & \beta\geqslant\theta_j\\ 0 & ,\ \beta<\theta_j\end{cases}\quad = A_i^*P_{10}(\beta)^*
$$

Moreover,

$$P_{11}(-b,\ \beta) = \sum_{j=1}^{k} \begin{cases} \int_{\beta-\theta_j}^{+\infty} dt A_k^* \Phi^0(t)^* \widetilde{Q} \Phi^0(t-\beta+\theta_j) A_j, & \beta \geqslant \theta_j \\ 0 & , \beta < \theta_j \end{cases}$$

and

$$P_{11}(-b,\ \beta) = A_k^* P_{10}(\beta)^*.$$

We now express $P_{11}(\alpha,\,\beta)$ in terms of $P_{10}(.)$. We have

$$P_{11}(\alpha,\beta) = \sum_{j=1}^{k} \begin{cases} \sum_{i=1}^{k} \begin{cases} \begin{cases} \int_{\alpha-\theta_i}^{+\infty} dt A_i^* \Phi^0(t-\alpha+\theta_i)^* Q \Phi^0(t-\beta+\theta_j) A_j \\ \qquad ,\beta-\theta_j-\alpha+\theta_i \leqslant 0 \\ 0 \qquad , \text{otherwise} \end{cases} & ,\alpha \geqslant \theta_i \\ 0 & , \alpha < \theta_i \end{cases} & ,\beta \geqslant \theta_i \\ 0 & ,\beta < \theta_i \end{cases}$$

$$+ \sum_{i=1}^{k} \begin{cases} \sum_{j=1}^{k} \begin{cases} \begin{cases} 0, & , \beta-\theta_j-\alpha+\theta_i \leqslant 0 \\ \int_0^{+\infty} dt A_i^* \Phi^0(t-\alpha+\theta_i)^* Q & \\ \Phi^0(t-\beta+\theta_j) A_j & , \beta-\theta_j-\alpha+\theta_i > 0 \end{cases} & , \beta \geqslant \theta_j \\ 0 & , \beta < \theta_j \end{cases} & , \alpha \geqslant \theta_i \\ 0 & ,\alpha < \theta_i \end{cases}$$

$$= \sum_{j=1}^{k} \begin{cases} \sum_{i=1}^{k} \begin{cases} \int_{\alpha-\beta+\theta_j-\theta_i}^{+\infty} dt A_i^* \Phi^0(t-\alpha+\beta-\theta_j+\theta_i)^* Q \\ \Phi^0(t) A_j, \quad -\alpha+\beta-\theta_j+\theta_i \leqslant 0,\ \alpha \geqslant \theta_i \\ 0 \qquad , \text{otherwise} \end{cases} & , \beta \geqslant \theta_i \\ 0 & , \beta < \theta_i \end{cases}$$

$$+\sum_{i=1}^{k}\begin{cases}\sum_{j=1}^{k}\begin{cases}\int_{\beta-\alpha+\theta_i-\theta_j}^{+\infty} dt A_i^*\Phi^0(t)^*Q \\ \Phi^0(t-\beta+\alpha-\theta_i+\theta_j), \quad -\beta+\alpha-\theta_i+\theta_j\leqslant 0\ ,\ \beta\geqslant\theta_j \\ 0 & ,\text{ otherwise}\end{cases} & ,\ \alpha\geqslant\theta_i \\ 0 & ,\ \alpha<\theta_i\end{cases}$$

Part2 In this part we derive equations (A.7), (A.10), (A.14).

1. Our start point is the Lyapunov equation (A.3).

$$\langle Az,\ Pz'\rangle_H+\langle Pz,\ Az'\rangle_H+\left\langle z,\ \widetilde{Q}z'\right\rangle_H=0,\ z,\ z'\in D(A)$$

Let $z,\ z'\in D(A)$ such that $z=\begin{pmatrix} h_0 \\ h\end{pmatrix}$, $z'=\begin{pmatrix} f_0 \\ f\end{pmatrix}$ with $h_0=h\,(0)$ and $f_0=f\,(0)$. We have

$$\begin{aligned}
\langle Az,\ Pz'\rangle_H &= \left\langle\begin{pmatrix} A_0h_0+\sum\limits_{i=1}^{k}A_ih(\theta_i) \\ \frac{dh}{d\theta}\end{pmatrix},\ \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11}\end{bmatrix}\begin{pmatrix} f_0 \\ f\end{pmatrix}\right\rangle_H \\
&= \left\langle A_0h_0+\sum_{i=1}^{k}A_ih(\theta_i),\ P_{00}f_0+P_{01}f\right\rangle \\
&\quad +\int_{-b}^{0}\left\langle\frac{dh}{d\theta}(\theta),\ \left(P_{10}f_0+P_{11}f\right)(\theta)\right\rangle d\theta \\
&= \langle A_0h_0,\ P_{00}f_0\rangle+\langle A_0h_0,\ P_{01}f\rangle+\left\langle\sum_{i=1}^{k}A_ih(\theta_i),\ P_{00}f_0\right\rangle \\
&\quad +\left\langle\sum_{i=1}^{k}A_ih(\theta_i),\ P_{01}f\right\rangle+\int_{-b}^{0}\left\langle\frac{dh}{d\theta}(\theta),\ \left(P_{10}f_0\right)(\theta)\right\rangle d\theta \\
&\quad +\int_{-b}^{0}\left\langle\frac{dh}{d\theta}(\theta),\ \left(P_{11}f\right)(\theta)\right\rangle d\theta
\end{aligned}$$

Similarly we obtain for the second term of the Lyapunov equation ($A.3$).

$$\begin{aligned}\langle Pz,\ Az'\rangle_H &= \langle P_{00}h_0,\ A_0 f_0\rangle + \left\langle P_{00}h_0,\ \sum_{i=1}^{k} A_i f(\theta_i)\right\rangle + \langle P_{01}h,\ A_0 f_0\rangle \\ &\quad + \left\langle P_{01}h,\ \sum_{i=1}^{k} A_i f(\theta_i)\right\rangle + \int_{-b}^{0} \left\langle (P_{10}h_0)(\theta),\ \frac{df}{d\theta}(\theta)\right\rangle d\theta \\ &\quad + \int_{-b}^{0} \left\langle (P_{11}h)(\theta),\ \frac{df}{d\theta}(\theta)\right\rangle d\theta\end{aligned}$$

The third term is

$$\left\langle z,\ \widetilde{Q}z'\right\rangle_H = \left\langle \begin{pmatrix} h_0 \\ h \end{pmatrix}, \begin{pmatrix} Qf_0 \\ 0 \end{pmatrix}\right\rangle = \langle h_0,\ Qf_0\rangle$$

The Lyapunov equation (A.3) is then equivalent to

$$\begin{aligned}&\langle A_0 h_0,\ P_{00} f_0\rangle + \langle A_0 h_0,\ P_{10} f\rangle + \left\langle \sum_{i=1}^{k} A_i h(\theta_i),\ P_{10} f\right\rangle \\ &+ \left\langle \sum_{i=1}^{k} A_i h(\theta_i),\ P_{00} f_0\right\rangle + \left\langle P_{00}h_0,\ \sum_{i=1}^{k} A_i f(\theta_i)\right\rangle \\ &+ \int_{-b}^{0} \left\langle \frac{dh}{d\theta}(\theta),\ (P_{10}f_0)(\theta)\right\rangle d\theta + \int_{-b}^{0} \left\langle \frac{dh}{d\theta}(\theta),\ (P_{11}f)(\theta)\right\rangle d\theta \\ &\int_{-b}^{0} \left\langle (P_{11}h)(\theta),\ \frac{df}{d\theta}(\theta)\right\rangle d\theta + \int_{-b}^{0} \left\langle (P_{10}h_0)(\theta),\ \frac{df}{d\theta}(\theta)\right\rangle d\theta \\ &+ \left\langle P_{01}h,\ \sum_{i=1}^{k} A_i f(\theta_i)\right\rangle + \langle P_{00}h_0,\ A_0 f\rangle \\ &+ \langle P_{01}h,\ A_0 f_0\rangle + \langle h_0,\ Qf_0\rangle = 0\end{aligned} \tag{A.18}$$

Using (A.11) and (A.12), equation (A.18) yields

$$
\begin{aligned}
&\langle A_0 h_0,\ P_{00} f_0\rangle + \left\langle \sum_{i=1}^{k} A_i h(\theta_i),\ P_{00} f_0 \right\rangle \\
&+ \left\langle A_0 h_0,\ \int_{-b}^{0} P_{10}^{*}(\alpha) f(\alpha) d\alpha \right\rangle + \left\langle \sum_{i=1}^{k} A_i h(\theta_i),\ \int_{-b}^{0} P_{10}^{*}(\alpha) f(\alpha) d\alpha \right\rangle \\
&+ \int_{-b}^{0} \left\langle \frac{dh}{d\alpha}(\alpha),\ P_{10}(\alpha) f_0 \right\rangle d\alpha + \int_{-b}^{0} \left\langle \frac{dh}{d\alpha}(\alpha),\ \int_{-b}^{0} P_{11}(\alpha,\ \beta) f(\beta) d\beta \right\rangle d\alpha \\
&+ \langle P_{00} h_0,\ A_0 f_0\rangle \ + \left\langle P_{00} h_0,\ \sum_{i=1}^{k} A_i f(\theta_i\ ) \right\rangle \\
&+ \left\langle \int_{-b}^{0} P_{10}^{*}(\alpha) h(\alpha) d\alpha,\ A_{00} f_0 \right\rangle + \left\langle \int_{-b}^{0} P_{10}^{*}(\alpha) h(\alpha) d\alpha,\ \sum_{i=1}^{k} A_i f(\theta_i) \right\rangle \\
&+ \int_{-b}^{0} \left\langle P_{10}(\theta) h_0,\ \frac{df}{d\theta}(\theta) \right\rangle d\theta \\
&+ \int_{-b}^{0} \left\langle \int_{-b}^{0} P_{11}\left(\beta,\ \alpha\right) h(\alpha) d\alpha,\ \frac{df}{d\beta}(\beta) \right\rangle d\beta + \langle h_0,\ Q f_0\rangle = 0 \qquad \text{(A.19)}
\end{aligned}
$$

Let

$$
h_n(\theta) = \begin{cases} h_0\left(1 + n\frac{\theta}{b}\right), & -\frac{b}{n} \le \theta \le 0. \\ 0, & \text{otherwise,} \end{cases}
$$

where n is chosen in such a way such that $n > b\theta_1^{-1}$. Then

$$
h_n(0) \underset{n\to+\infty}{\longrightarrow} h_0 \text{ and } h_n \longrightarrow 0 \text{ in } L^2(-b,\ 0;\mathbb{R}^N).
$$

Let f_i be chosen in such a way that

$$
\text{supp } f_i \subset (\theta_i,\ \theta_{i-1}) \cup (\theta_1, 0]\,.
$$

Let $h = h_n$ and $f = f_i$ in (A.19):

$$\begin{aligned}
&\langle A_0 h_0, P_{00} f_i(0)\rangle + \left\langle A_0 h_0, \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) P_{10}^*(\alpha) f_i(\alpha) d\alpha \right\rangle \\
&+\int_{\theta_1}^{0} \left\langle \frac{dh_n}{d\alpha}(\alpha), P_{10}(\alpha) f_i(0) \right\rangle d\alpha \\
&+\int_{\theta_1}^{0} \left\langle \frac{dh_n}{d\alpha}(\alpha), \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) P_{11}(\alpha, \beta) f_i(\beta) d\beta \right\rangle d\alpha \\
&+ \langle P_{00} h_0, A_0 f_i(0)\rangle + \left\langle \int_{\theta_1}^{0} P_{10}^*(\alpha) h_n(\alpha) d\alpha, A_0 f_i(0) \right\rangle \\
&+ \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) \left\langle P_{10}(\theta) h_0, \frac{df_i}{d\theta}(\theta) \right\rangle d\theta \\
&+ \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) \left\langle \int_{\theta_1}^{0} P_{11}(\beta,\ \alpha)\, h_n(\alpha) d\alpha, \frac{df}{d\beta}(\beta) \right\rangle d\beta + \langle h_0, Q f_i(0)\rangle = 0
\end{aligned} \tag{A.20}$$

Since $\alpha \to P_{01}(\alpha)$, $\alpha \to P_{11}(\alpha,\ \beta)$ and $\alpha \to P_{11}(\beta,\ \alpha)$ are absolutely continuous in $(\theta_i,\ \theta_{i-1})$ and $(\theta_1,\ 0)$ we can now integrate by parts. Equation

(A.20) now reduces to

$$\begin{aligned}
&\langle A_0 h_0,\ P_{00} f_i(0)\rangle + \left\langle A_0 h_0,\ \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) P_{10}^*(\alpha) f_i(\alpha) d\alpha \right\rangle \\
&+ \langle h_0,\ P_{10}(0) f_i(0)\rangle + \left\langle h_0,\ \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) P_{11}(0,\ \alpha)\ f_i(\alpha) d\alpha \right\rangle \\
&- \int_{\theta_1}^{0} \langle h_n(\alpha), P_{10}(\alpha)\ f_i(0)\rangle\, d\alpha + \langle P_{00} h_0,\ A_0 f_i(0)\rangle \\
&- \int_{\theta_1}^{0} \left\langle h_n(\alpha), \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) P_{11}(\alpha,\beta) f_i(\beta) d\beta \right\rangle d\alpha \\
&+ \left\langle \int_{\theta_1}^{0} P_{10}^*(\alpha) h_n(\alpha) d\alpha,\ A_0 f_i(0) \right\rangle + \langle P_{10}(0) h_0,\ f_i(0)\rangle \\
&+ \int_{\theta_1}^{0} \langle P_{11}(0,\alpha) h_n(\alpha), f_i(0)\rangle\, d\alpha - \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) \left\langle \frac{dP_{10}^*}{d\alpha}(\alpha)\, h_n(0), f_i(\alpha) \right\rangle d\alpha \\
&- \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) \left\langle \int_{\theta_1}^{0} \frac{dP_{11}}{d\beta}(\beta,\alpha)\, h_n(\alpha) d\alpha, f_i(\beta) \right\rangle d\beta + \langle h_0, Q f_i(0)\rangle = 0
\end{aligned} \tag{A.21}$$

Given any g in $L^2\left(-b, 0; \mathbb{R}^N\right)$,

$$\left| \int_{-b}^{0} \langle h_n(\theta),\ g(\theta)\rangle\, d\theta \right| \le \|h_n\|_{L^2(-b,0;\ \mathbb{R}^N)} \|g\|_{L^2(-b,0;\ \mathbb{R}^N)}$$

and since $\lim_{n\to+\infty} \|h_n\|_{L^2(-b,0;\ \mathbb{R}^N)} = 0$ it follows that

$$\lim_{n\to+\infty} \int_{-b}^{0} \langle h_n(\theta),\ g(\theta)\rangle\, d\theta = 0$$

Taking the limit of equation (A.21) as n goes to infinity, we obtain

$$\begin{aligned}
&\langle A_0 h_0,\ P_{00} f_i(0)\rangle + \left\langle A_0 h_0,\ \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) P_{10}^*(\alpha) f_i(\alpha) d\alpha \right\rangle \\
&+ \langle h_0,\ P_{10}(0) f_i(0)\rangle + \left\langle h_0,\ \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) P_{11}(0,\ \alpha)\ f_i(\alpha) d\alpha \right\rangle \\
&+ \langle P_{00} h_0,\ A_0 f_i(0)\rangle \quad + \langle P_{10}^*(0) h_0,\ f_i(0)\rangle \\
&- \left(\int_{\theta i}^{\theta_{i-1}} + \int_{\theta_1}^{0}\right) \left\langle \frac{dP_{10}^*}{d\alpha}(\alpha)\, h_0,\ f_i(\alpha) \right\rangle d\alpha + \langle h_0,\ Q f_0\rangle = 0
\end{aligned} \tag{A.22}$$

Let

$$f_i(\theta) = f_m(\theta) = \begin{cases} f_0\left(1 + m\frac{\theta}{b}\right), & \frac{-b}{m} \le \theta \le 0 \\ 0, & \text{otherwise} \end{cases},$$

where m is chosen in such a way that $m > b\theta_1^{-1}$. When we take the limit of equation (A.22) as m goes to infinity we obtain

$$\langle (P_{00}A_0 + P_{10}(0)^* + A_0^* P_{00} + P_{10}(0) + Q)\, h_0,\ f_0 \rangle = 0$$

for all h_0 and f_0 in $\mathbb{R}^N$.

2. To obtain (A.10) in the open interval $(\theta_i,\ \theta_{i-1})$ we choose f_i such that supp $f_i \subset (\theta_i, \theta_{i-1})$. Then equation (A.22) yields

$$\left\langle A_0 h_0, \int_{\theta i}^{\theta_{i-1}} P_{10}^*(\alpha) f_i(\alpha) d\alpha \right\rangle + \left\langle h_0, \int_{\theta i}^{\theta_{i-1}} P_{11}(0,\alpha) f_i(\alpha) d\alpha \right\rangle$$
$$- \int_{\theta i}^{\theta_{i-1}} \left\langle \frac{dP_{10}}{d\alpha}(\alpha)\, h_0, f_i(\alpha) \right\rangle d\alpha = 0$$

which implies that

$$\int_{\theta i}^{\theta_{i-1}} \langle P_{10}(\alpha)^* A_0 h_0, f_i(\alpha) d\alpha \rangle + \int_{\theta i}^{\theta_{i-1}} \langle P_{11}(0,\alpha)^* h_0, f_i(\alpha) d\alpha \rangle$$
$$- \int_{\theta i}^{\theta_{i-1}} \left\langle \frac{dP_{10}}{d\alpha}(\alpha)\, h_0, f_i(\alpha) \right\rangle d\alpha = 0$$

Therefore

$$\int_{\theta_i}^{\theta_i - 1} \left\langle \left(P_{10}(\alpha)\, A_0 + P_{11}(0,\ \alpha)^* - \left(\frac{dP_{10}}{d\alpha}(\alpha) \right)^* \right) h_0,\ f_i(\alpha) \right\rangle d\alpha = 0$$

By density of the set of absolutely continuous maps with support in (θ_i, θ_{i-1}) in $L^2\left(\theta_i, \theta_{i-1}; \mathbb{R}^N\right)$ and the propriety

$$P_{11}(\alpha, \beta)^* = P_{11}(\beta,\ \alpha),$$

the above equation yields for all h_0 in $\mathbb{R}^N$

$$\left(-\frac{dP_{10}}{d\alpha}(\alpha) + P_{10}\,(\alpha)\,A_0 + P_{11}(\alpha,\ 0)\right) h_0 = 0$$

almost everywhere in (θ_i, θ_{i-1}).

3. To obtain (A.14) in the region

$$\{(\alpha,\ \beta) \in [-b,\ 0] \times [-b,\ 0] \,/\, \alpha \in (\theta_i,\ \theta_{i-1})\,,\ \beta \in (\theta_j,\ \theta_{j-1})\}$$

we choose

$$\begin{aligned} h &= h_i,\ \text{supp}\ h_i \subset (\theta_i,\ \theta_{i-1})\,, \\ f &= f_j,\ \text{supp}\ f_j \subset (\theta_j,\ \theta_{j-1}) \end{aligned}$$

and substitute in (A.22) which reduces to the following expression:

$$\begin{aligned} &\int\limits_{\theta_i}^{\theta_{i-1}} \left\langle \frac{dh_i^1}{d\alpha}(\alpha), \int_{\theta_j}^{\theta_{j-1}} P_{11}(\alpha,\beta) f_j^1(\beta) d\beta \right\rangle d\alpha \\ &+ \int\limits_{\theta_j}^{\theta_{j-1}} \left\langle \int_{\theta i}^{\theta_{i-1}} P_{11}\,(\beta,\alpha)\,h_i^1(\alpha) d\alpha, \frac{df_j^1}{d\beta}(\beta) \right\rangle d\beta = 0 \end{aligned} \tag{A.23}$$

The two terms can be integrated by parts:

$$\begin{aligned} &\int\limits_{\theta_i}^{\theta_{i-1}} \left\langle \frac{dh_i^1}{d\alpha}(\alpha), \int_{\theta_j}^{\theta_{j-1}} P_{11}(\alpha,\beta) f_j^1(\beta) d\beta \right\rangle d\alpha \\ &= -\int\limits_{\theta_i}^{\theta_{i-1}} \left\langle h_i^1(\alpha), \int_{\theta_j}^{\theta_{j-1}} \frac{\partial}{\partial\alpha} P_{11}(\alpha,\beta) f_j^1(\beta) d\beta \right\rangle d\alpha \end{aligned}$$

and

$$\begin{aligned} &\int\limits_{\theta_{ij}}^{\theta_{j-1}} \left\langle \int_{\theta i}^{\theta_{i-1}} P_{11}(\beta,\alpha) h_i^1(\alpha) d\alpha, \frac{df_j^1}{d\beta}(\beta) \right\rangle d\beta \\ = &\ -\int\limits_{\theta_j}^{\theta_{j-1}} \int\limits_{\theta_i}^{\theta_{i-1}} \left\langle \frac{\partial}{\partial\beta} P_{11}(\beta,\alpha) h_i^1(\alpha) d\alpha, f_j^1(\beta) \right\rangle d\beta \end{aligned}$$

Equation (A.23) takes the form

$$\int_{\theta_i}^{\theta_{i-1}} \left\langle h_i^1(\alpha), \int_{\theta_j}^{\theta_{j-1}} \frac{\partial}{\partial \alpha} P_{11}(\alpha,\beta) f_j^1(\beta) d\beta \right\rangle d\alpha$$

$$+ \int_{\theta_j}^{\theta_{j-1}} \int_{\theta_i}^{\theta_{i-1}} \left\langle \frac{\partial}{\partial \beta} P_{11}(\beta,\alpha) h_i^1(\alpha) d\alpha, f_j^1(\beta) \right\rangle d\beta = 0$$

which implies that

$$\int_{\theta_i}^{\theta_{i-1}} \left(\int_{\theta_j}^{\theta_{j-1}} \left\langle \left(\frac{\partial}{\partial \alpha} P_{11}(\alpha,\beta) \right)^* h_i^1(\alpha), f_j^1(\beta) \right\rangle d\beta \right) d\alpha$$

$$+ \int_{\theta i}^{\theta_{i-1}} \left(\int_{\theta j}^{\theta_{j-1}} \left\langle \frac{\partial}{\partial \beta} P_{11}(\beta,\alpha) h_i^1(\alpha), f_j^1(\beta) \right\rangle d\alpha \right) d\beta = 0$$

Using the property $P_{11}(\alpha,\ \beta) = P_{11}(\beta,\ \alpha)^*$ it follows that

$$\int_{\theta_i}^{\theta_{i-1}} \left(\int_{\theta_j}^{\theta_{j-1}} \left\langle \left(\left(\frac{\partial}{\partial \alpha} P_{11}(\alpha,\beta) \right)^* + \left(\frac{\partial}{\partial \beta} P_{11}(\alpha,\beta) \right)^* \right) h_i(\alpha), f_j(\beta) \right\rangle d\beta \right) d\alpha = 0$$

Now using the density argument we obtain

$$\left(\frac{\partial}{\partial \alpha} P_{11}(\alpha,\ \beta) \right)^* + \left(\frac{\partial}{\partial \beta} P_{11}(\alpha,\beta) \right)^* = 0$$

from which it follows that

$$\frac{\partial}{\partial \alpha} P_{11}(\alpha,\beta) + \frac{\partial}{\partial \beta} P_{11}(\alpha,\beta) = 0$$

for all most all (α,β) in $(\theta_i,\ \theta_{i-1}) \times (\theta_j,\ \theta_{j-1})$.

Part3.

We now solve equation (A.14) with boundary conditions (A.15).

We let $\eta = \alpha - \beta$ and consider two cases. First let $b \geq \eta \geq 0$; then

$$-b \leq \beta \leq 0 \Longrightarrow \eta - b \leq \alpha \leq 0$$

If we change the variable β to $\eta = \alpha - \beta$, equation (A.14) becomes

$$\frac{d}{d\alpha} P_{11}(\alpha,\ \alpha - \eta) = \sum_{i=1}^{k-1} A_i^* P_{10}\left(\alpha - \eta\right)^* \delta(\alpha - \theta_i) + \sum_{i=1}^{k-1} P_{10}\left(\alpha\right) A_i \delta(\alpha - \eta - \theta_j)$$

This last equation can be integrated from $\eta - b$ to α :

$$\begin{aligned} P_{11}(\alpha,\ \alpha - \eta) &= P_{11}\left(\eta - b,\ -b\right) \\ &+ \sum_{i=1}^{k-1} \begin{cases} A_i^* P_{10}\left(\alpha - \eta\right)^*, & \eta - b \leq \theta_i < \alpha \\ 0, & \text{otherwise} \end{cases} \\ &+ \sum_{j=1}^{k-1} \begin{cases} P_{10}\left(\eta + \theta_j\right) A_j, & \eta - b \leq \eta + \theta_j < \alpha \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Finally for $\alpha \geq \beta$,

$$\begin{aligned} P_{11}\left(\alpha,\ \beta\right) &= P_{10}\left(\alpha - \beta - b\right) A_k \\ &+ \sum_{j=1}^{k-1} \begin{cases} P_{10}\left(\alpha - \beta + \theta_j\right) A_j, & \theta_j < \beta \\ 0 & \text{, otherwise} \end{cases} \\ &+ \sum_{j=1}^{k-1} \begin{cases} A_i^* P_{10}\left(\beta - \alpha + \theta_i\right)^*, & -b \leq \beta - \alpha + \theta_i, \quad \theta_i < \alpha \\ 0 & \text{, otherwise} \end{cases} \\ &= P_{10}\left(\alpha - \beta - b\right) A_k \\ &+ \sum_{j=1}^{k-1} \begin{cases} P_{10}\left(\alpha - \beta + \theta_j\right) A_j, & \theta_j < \beta, \quad -b \leq \alpha - \beta + \theta_j \\ 0 & , \quad \text{otherwise} \end{cases} \\ &+ \sum_{j=1}^{k-1} \begin{cases} A_i^* P_{10}\left(\beta - \alpha + \theta_i\right)^*, & -b \leq \beta - \alpha + \theta_i, \quad \theta_i < \alpha \\ 0 & \text{, otherwise} \end{cases} \end{aligned}$$

Notice that in the above expression for $P_{11}\left(\alpha,\ \beta\right)$ all terms but the first are symmetrical. Hence for $\alpha \leq \beta$ we shall obtain the same expression with the exception of the first term which will be equal to

$$A_k^* P_{10} (\alpha - \beta - b)^*$$

But

$$\lim_{\substack{\beta \longrightarrow \alpha \\ \alpha \leq \beta}} P_{10} (\alpha - \beta - b) A_k = P_{10} (-b) A_k = A_k^* P_{00} A_k$$

and

$$\lim_{\substack{\beta \longrightarrow \alpha \\ \beta \leq \alpha}} A_k^* P_{10} (\alpha - \beta - b)^* = A_k^* P_{10} (-b)^* = A_k^* P_{00} A_k$$

imply that this first is continuous at $(\alpha,\ \alpha)$, $-b \leq \alpha < \theta_{k-1.}$ This makes it possible to write the first term as follows:

$$\begin{array}{rcl} P_{10} (\alpha - \beta - b) A_k,\ \alpha & \geq & \beta \\ A_k^* P_{10} (\alpha - \beta - b)^*,\ \alpha & < & \beta \end{array}$$

This yields identity (A.16). ∎

Appendix B

In this part, we give a proof to Lemma 3.7, similar to the one given in [39]. At first we recall some basic definitions from the Graphs Theory.

Let V a set of points which shall be considered to be connected in some fashion. We call V the vertex set and the elements $v \in V$ are vertices.

Definition B.1 *A graph $G = G(V)$ with the vertex set V is a family of associations or pairing*

$$E = (a, b), \ \ a, b \in V \tag{B.1}$$

which indicates which vertices shall be considered to be connected. Each defined couple (B.1) is called an edge of the graph; the vertices a and b are called the endpoints of the edge E.

In the applications a graph is usually interpreted as a network in which the vertices of G are nodes or junctions.

In the definition (B.1) of an edge one may or may not take into account the order in which the two endpoints occur. If the order is immaterial

$$E = (a, b) = (b, a)$$

we say that E is undirected edge. On the other hand, if the order is to be taken into consideration we shall call E a directed edge. A gragh is called undirected when every edge is undirected, while in a directed graph all edges are directed.

Definition B.2 *A sequence of edges in G is any finite or infinite series of edges*

$$S = (...,\ E_0,\ E_1, ..., E_n, ...)$$

such that consecutive edges E_{i-1} and E_i always have a common point.

Definition B.3 *A sequence of edges is a path when no edge appears more than once in it. A non-cyclic path is called a simple path or an arc if none of its vertices is traversed more than once.*

Definition B.4 *A directed graph G is said to be strongly connected if every node of G is connected to every distinct node of G by a direct path in G.*

Let

$$f(\alpha) = \max_{j \in \overline{N}} f_j(\alpha), \qquad f_j(\alpha) = \left\| \sum_{i=1}^{N} \left(\frac{\alpha_i}{\alpha j} \right)^2 H_{ij} \right\|, \ \ \alpha \in (0, \infty)^N .$$

where, for all i, $j \in \overline{N}$, the operator $H_{ij} \in L(U_j)$ is defined by

$$H_{ij} u_j = \theta_j \int_0^{+\infty} D_j^* S^*(t) E_i^* E_i S(t) D_j u_j dt.$$

where the operators $S(t)$, $(E_i, D_i)_{i=1,N}$ and θ_i, $i = 1, ..., N$, are defined as in Chapter 3.

Set

$$f(\alpha) = \max_{j \in \overline{N}} f_j(\alpha), \qquad f_j(\alpha) = \left\| \sum_{i=1}^{N} \left(\frac{\alpha_i}{\alpha j} \right)^2 H_{ij} \right\|, \ \ \alpha \in (0, \infty)^N .$$

Set

$$\hat{\mu} = \inf_{\alpha \in (0,\infty)^N} f(\alpha) = \inf_{\alpha \in (0,\infty)^N} \max_{j \in \overline{N}} \left\| \sum_{i=1}^{N} \left(\frac{\alpha_i}{\alpha j} \right)^2 H_{ij} \right\| \tag{B.2}$$

Let Υ the directed graph with node set $\overline{N} = \{1, ..., N\}$ and set of directed arcs

$$\Theta = \left\{ (i,j) \in \overline{N}^2; \ H_{ij} \neq 0 \right\}$$

Theorem B.5 *Suppose that Υ is strongly connected. Then there exists a subset $J \subset \bar{N}$ and a vector $\hat{\alpha} \in (0, \infty)^N$ satisfying $f(\hat{\alpha}) = \hat{\mu}$, and*

$$\begin{aligned} \left\| \sum_{i=1}^{N} \left(\frac{\hat{\alpha}_i}{\hat{\alpha}_j} \right)^2 H_{ij} \right\| &= \left\| \sum_{i \in J} \left(\frac{\hat{\alpha}_i}{\hat{\alpha}_j} \right)^2 H_{ij} \right\| = \hat{\mu} \quad \text{ if } j \in J. \\ \left\| \sum_{i=1}^{N} \left(\frac{\hat{\alpha}_i}{\hat{\alpha}_j} \right)^2 H_{ij} \right\| &< \hat{\mu} \quad \text{ if } j \in N \backslash J. \end{aligned}$$

In order to prove this theorem we need the following lemmas.

Lemma B.6 *Suppose that $\left(\alpha^k\right)_{k \in \mathbb{N}}$ is a sequence in $S_+^N = \left\{ \alpha \in (0, \infty)^N ; \|\alpha\| = 1 \right\}$ and $\left(f\left(\alpha^k\right)\right)_{k \in \mathbb{N}}$ is bounded. If $j \in \overline{N}$ can be reached from $i \in \overline{N}$ via a direct path in Υ, then*

$$\lim_{k \to +\infty} \alpha_j^k = 0 \Longrightarrow \lim_{k \to +\infty} \alpha_i^k = 0 \tag{B.3}$$

Proof. It suffices to prove (B.3) for the case where $(i,j) \in \Theta$. Since

$$f\left(\alpha^k\right) \geq f_j\left(\alpha^k\right) \geq \left(\frac{\alpha_i^k}{\alpha_j^k} \right)^2 \|H_{ij}\|$$

it follows that

$$\left(\alpha_j^k\right)^2 f\left(\alpha^k\right) \geq \left(\alpha_i^k\right)^2 \|H_{ij}\|$$

from which we obtain the desired result. ■

Applying this lemma we obtain the following existence result.

Proposition B.7 *Suppose that Υ is strongly connected. Then there exists $\hat{\alpha} \in S_+^N$ such that*

$$f\left(\hat{\alpha}\right) = \hat{\mu}$$

Proof. Since

$$f(r\alpha) = f(\alpha),\ \alpha \in (0, \infty)^N,\ r > 0$$

it suffices to consider f on S_+^N. Let $\left(\alpha^k\right)_{k\geq 0}$ be a minimizing sequence for f on S_+^N which converge toward some limit $\hat{\alpha}$ in the closure of S_+^N. We have only to show that $\hat{\alpha} \in S_+^N$, i.e., $\hat{\alpha}_i > 0$ for all $i \in \bar{N}$. But if we assume that $\hat{\alpha}_j = 0$ for some $j \in \bar{N}$, then since$\left(f\left(\alpha^k\right)\right)$ is bounded and j can be reached from every $i \in \bar{N}$ via a direct path in Υ, we must have $\hat{\alpha} = 0$ by the previous lemma. On the other hand $\hat{\alpha} \neq 0$ because $\left(\alpha^k\right)_{k\geq 0}$ is a sequence in S_+^N. The contradiction shows that $\hat{\alpha}_i > 0$ for all $i \in \bar{N}$. ■

Lemma B.8 *Suppose H_0, $H_1 \in L^+(U)$, and $r_1 < r_2$ are such that*

$$\|H_0 + r_1 H_1\| = \|H_0 + r_2 H_1\|$$

Then

$$\|H_0 + r H_1\| = \|H_0\|,\ 0 \leq r \leq r_2$$

Proof. Let $v \in U$, $\|v\| = 1$, be such that

$$\|H_0 + r_1 H_1\| = \left\langle \left(H_0 + r_1 H_1\right) v, v\right\rangle$$

Then

$$\langle (H_0 + r_1 H_1)\, v, v\rangle \leq \langle (H_0 + r_2 H_1)\, v, v\rangle \leq \|H_0 + r_2 H_1\| = \langle (H_0 + r_1 H_1)\, v, v\rangle$$

It follows that

$$\langle (H_0 + r_1 H_1)\, v, v\rangle = \langle (H_0 + r_2 H_1)\, v, v\rangle$$

which implies that $\langle H_1 v, v\rangle = 0$. So

$$\|H_0 + r_2 H_1\| = \langle H_0 v, v\rangle \leq \|H_0\|$$

But $r \longrightarrow \|H_0 + rH_1\|$ is increasing and hence the result follows. ■

Proof of Theorem B.5. By proposition B.7 there is a vector $\omega \in (0,\infty)^N$ such that

$$\max_{j\in\overline{N}} \left\| \sum_{i\in\overline{N}} \left(\frac{\omega_i}{\omega_j}\right)^2 H_{ij} \right\| = \hat{\mu}$$

Among all these minimizing vectors we choose one, denoted by $\hat{\omega}$, for which the number of $j \in \overline{N}$ satisfying

$$\left\| \sum_{i\in\overline{N}} \left(\frac{\widehat{\omega}_i}{\widehat{\omega}_j}\right)^2 H_{ij} \right\| = \hat{\mu}$$

is minimal. Let J be the set of these $j \in \overline{N}$. Then

$$\left\| \sum_{i\in\overline{N}} \left(\frac{\widehat{\omega}_i}{\widehat{\omega}_j}\right)^2 H_{ij} \right\| < \hat{\mu}, \qquad j \in \overline{N} \setminus J \tag{B.4}$$

For $r \in [0,1]$, define $\hat{\omega}(r) \in (0,\infty)^N$ by setting $\hat{\omega}_j(r) = \hat{\omega}_j$ if $j \in J$ and $\hat{\omega}_j(r) = r\hat{\omega}_j$ if $j \in \overline{N}\backslash J$. For r sufficiently close to one, say $r \in [\hat{r}, 1]$ with $\hat{r} < r$, the inequalities (B.4) still hold when $\hat{\omega}$ is replaced by $\hat{\omega}(r)$. For these r :

$$\left\| \sum_{i\in\overline{N}} \left(\frac{\hat{\omega}_i(r)}{\hat{\omega}_j(r)}\right)^2 H_{ij} \right\| = \left\| \sum_{i\in J} \left(\frac{\hat{\omega}_i}{\hat{\omega}_j}\right)^2 H_{ij} + r^2 \sum_{i\in\overline{N}\backslash J} \left(\frac{\hat{\omega}_i}{\hat{\omega}_j}\right)^2 H_{ij} \right\| \leq \hat{\mu}, \qquad j \in J$$

But by the minimality assumption on J, none of the above inequalities can be strict for $r \in [\hat{r}, 1]$. Applying Lemma B.8 with $H_0 = \sum\limits_{i \in J} \left(\frac{\hat{\omega}_i}{\hat{\omega}_j}\right)^2 H_{ij}$ and $H_1 = \sum\limits_{i \in \overline{N} \setminus J} \left(\frac{\hat{\omega}_i}{\hat{\omega}_j}\right)^2 H_{ij}$ we conclude that

$$\left\| \sum_{i \in \overline{N}} \left(\frac{\hat{\omega}_i(r)}{\hat{\omega}_j(r)} \right)^2 H_{ij} \right\| = \left\| \sum_{i \in J} \left(\frac{\hat{\omega}_i}{\hat{\omega}_j} \right)^2 H_{ij} \right\| = \hat{\mu}, \qquad r \in [0,1], \quad j \in J \tag{B.5}$$

Setting $\hat{\alpha}_i = \hat{\omega}_i(r)$, $i \in \overline{N}$, we obtain the desired result from (B.4) and (B.5). ■

In the case where Υ is not strongly connected, we introduce the following notations. Let C_k, $k = 1, ..., K$, be the node sets of the strongly connected components of Υ ordered in such a way that for $1 \leq h < k \leq K$ there is no directed arc $(i,j) \in \Theta$ such that $i \in C_k$, $j \in C_h$. Then, for all h, $k \in \overline{K}$

$$h < k \Longrightarrow (i \in C_k \text{ and } j \in C_h \Longrightarrow H_{ij} = 0) \tag{B.6}$$

Since $\overline{N} = \underset{k \in \overline{K}}{\cup} C_k$, it follows from (B.6) that

$$f(\alpha) = \max_{k \in \overline{K}} \max_{j \in C_k} \left\| \sum_{i=1}^{N} \left(\frac{\alpha_i}{\alpha_j} \right)^2 H_{ij} \right\| = \max_{k \in \overline{K}} \max_{j \in C_k} \left\| \sum_{k=1}^{N} \sum_{i \in C_k} \left(\frac{\alpha_i}{\alpha_j} \right)^2 H_{ij} \right\|, \quad \alpha \in (0, \infty)^N.$$

The next theorem shows that problem (B.2) can be solved by restricting our considerations to the strongly connected components of Υ. We denote by $(0,\infty)^{C_k}$ the set $(0,\infty)^{C_k} = \{(\alpha_j),\ \alpha_j \in (0,\ \infty) \text{ for all } j \in C_k\}$.

Theorem B.9 *Let* $\mu_k = \min\limits_{\alpha \in (0,\infty)^{C_k}} \min\limits_{j \in C_k} \left\| \sum\limits_{i \in C_k} \left(\frac{\alpha_i}{\alpha_j}\right)^2 H_{ij} \right\|$, $k \in \overline{K}$. *Then*

$$\hat{\mu} = \max_{k \in \overline{K}} \mu_k$$

Moreover, if $\hat{k}$ *satisfies* $\mu_{\hat{k}} = \max\limits_{k \in \overline{K}} \mu_k$ *there exist a subset* $J \subset C_{\hat{k}}$ *and for every*

$\delta > 0,$ *a vector* $\alpha \in (0,\infty)^N$ *such that*

$$\begin{aligned} \left\|\sum_{i=1}^{N}\left(\frac{\alpha_i}{\alpha_j}\right)^2 H_{ij}\right\| &\leq \hat{\mu}+\delta, \;\; j \in \overline{N}. \\ \left\|\sum_{i\in C_{\hat{k}}}\left(\frac{\alpha_i}{\alpha_j}\right)^2 H_{ij}\right\| &= \left\|\sum_{i\in J}\left(\frac{\alpha_i}{\alpha_j}\right)^2 H_{ij}\right\| = \hat{\mu}, \;\; j \in J. \end{aligned}$$

For the proof we need the following lemmas. For every $\varepsilon > 0$ we define the set

$$X(\varepsilon) = \left\{ \alpha \in (0,\infty)^N;\; \forall k \in \overline{K-1}:\; i \in C_k \wedge j \in C_{k+1} \Longrightarrow \frac{\alpha_i}{\alpha_j} < \varepsilon \right\}$$

Lemma B.10 *Suppose that, for each* $k \in \overline{K},$ *a positive vector* $\omega^k \in (0,\infty)^{C_k}$ *is given. Choose* $r_k > 0$ *for all* $k \in \overline{K}$ *such that*

$$\max_{i\in C_k} r_k(\omega^k)_i < \min_{j\in C_{k+1}} r_{k+1}(\omega^{k+1})_j,\;\; k = 1,...,K-1.$$

and define $\alpha \in (0,\infty)^N$ *by*

$$\alpha_i = r_k \varepsilon^{K-k}(\omega^k)_i, \qquad k \in \overline{K},\;\; i \in C_k.$$

Then

$$\alpha \in X(\varepsilon) \text{ and } \forall i,\; j \in C_k : \frac{\alpha_i}{\alpha_j} = \frac{(\omega^k)_i}{(\omega^k)_j}. \tag{B.7}$$

Lemma B.11 *Given a family of vectors* $\omega^k \in (0,\infty)^{C_k}$, $k \in \overline{K},$ *satisfying*

$$\max_{j\in C_k}\left\|\sum_{i\in C_k}\left(\frac{(\omega^k)_i}{(\omega^k)_j}\right)^2 H_{ij}\right\| = \mu_k, \qquad k \in \overline{K}. \tag{B.8}$$

there exists, for every $\delta > 0,$ *a vector* $\alpha \in (0,\infty)^N$ *such that*

$$f(\alpha) \leq \max_{k\in\overline{K}} \mu_k + \delta \text{ and } \forall k \in \overline{K},\; \forall i,\; j \in C_k : \frac{\alpha_i}{\alpha_j} = \frac{(\omega^k)_i}{(\omega^k)_j}.$$

Proof. Suppose that $\omega^k \in (0,\infty)^{C_k}$, $k \in \overline{K}$, satisfy (B.8), and let $\delta > 0$. Then

$$\max_{k\in\overline{K}} \max_{j\in C_k} \left\| \sum_{i\in C_k} \left(\frac{(\omega^k)_i}{(\omega^k)_j}\right)^2 H_{ij} \right\| = \max_{k\in\overline{K}} \mu_k.$$

For every $\varepsilon > 0$, there exist, by Lemma B.10, a vector $\alpha = \alpha(\varepsilon) \in (0,\infty)^N$ such that $(B.7)$ is satisfied. It follows that

$$\begin{aligned} f(\alpha(\varepsilon)) &= \max_{k\in\overline{K}} \max_{j\in C_k} \left\| \sum_{h=1}^{k} \sum_{i\in C_h} \left(\frac{(\alpha(\varepsilon))_i}{(\alpha(\varepsilon))_j}\right)^2 H_{ij} \right\| \\ &\le \max_{k\in\overline{K}} \max_{j\in C_k} \left\| \sum_{h=1}^{k-1} \sum_{i\in C_h} \left(\frac{(\alpha(\varepsilon))_i}{(\alpha(\varepsilon))_j}\right)^2 H_{ij} \right\| + \max_{k\in\overline{K}} \mu_k \end{aligned}$$

But for all $\varepsilon \in (0,1)$ we have

$$i \in C_h,\ j \in C_k,\ \text{and } h < k \Longrightarrow \frac{\alpha(\varepsilon)_i}{\alpha(\varepsilon)_j} \le \varepsilon$$

Choosing $\varepsilon \in (0,1)$ such that

$$\max_{k\in\overline{K}} \max_{j\in C_k} \left\| \sum_{h=1}^{k-1} \sum_{i\in C_h} \varepsilon^2 H_{ij} \right\| \le \delta$$

we obtain

$$f(\alpha(\varepsilon)) \le \max_{k\in\overline{K}} \mu_k + \delta$$

This concludes the proof. ■

Proof of Theorem B.9. By proposition B.7 there exists a family of vectors ω^k, $k \in \overline{K}$, satisfying (B.8). Hence by Lemma B.11 for $\delta > 0$ there exists $\alpha \in (0,\infty)^N$ such that

$$f(\alpha) \le \max_{k\in\overline{K}} \mu_k + \delta$$

But for every $\alpha \in (0,\infty)^N$,

$$f(\alpha) = \max_{k\in\overline{K}} \max_{j\in C_k} \left\| \sum_{h=1}^{k} \sum_{i\in C_h} \left(\frac{\alpha_i}{\alpha_j}\right)^2 H_{ij} \right\| \ge \max_{k\in\overline{K}} \max_{j\in C_h} \left\| \sum_{i\in C_h} \left(\frac{\alpha_i}{\alpha_j}\right)^2 H_{ij} \right\| \ge \max_{k\in\overline{K}} \mu_k$$

since $\hat{\mu} = \inf_{\alpha \in (0,\infty)^N} f(\alpha)$, we conclude that $\hat{\mu} = \max_{k \in \overline{K}} \mu_k$. The second part of Theorem B.9 follows from this and Theorem B.5. ■

Combining Theorem B.9 and Theorem B.5, we obtain Lemma 3.7.

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