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Par

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THEME

Stabilisation uniforme de quelques équations aux dérivées partielles à coefficients variables.

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Uniform decay rates of certain classes of Partial Differential Equations with variable coefficients .

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Dédicace

À mes parents, mes frères et ma sœur.

À mon mari, mes enfants: Dhia Eddine, Kawther et Romaissa.

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General introduction.

GENERAL INTRODUCTION.

Understanding completely the dynamics of evolution processes is too difficult. However, for many relevant systems an asymptotic process occurs and so the uniform decay rate of the energy is considered. In this thesis, we investigate the energy decay rates of certain classes of partial differential equations with variables coefficients.

Literature.

Considerable research has been done in the last three decades on the problem of the uniform decay rates of certain classes of partial differential equations by inserting distributed or boundary feedback term directly into the equation. This is the so called direct stabilization.

When the feedback term depends on the velocity in a linear way, we can mention the first works in this direction: under various geometrical conditions on the boundary, G. Chen in [$CheI79, CheII79, Che81$] has proved the exponential decay rates of the energy for the wave equations in a bounded domain, subject to Neumann feedback acting on a part of the boundary, by employing a result of R. Datko in [$Dat70$] later extending by A. Pazy in [$Paz83$]. In [$Lag88, KZ90$], a different point of view was taken and the Neumann boundary stabilization of the wave equation was obtained by constructing perturbed energy functionals. Similar results for other models were also obtained, we can see, for instance, J. L. Lagnese [$LagII83$] for elastodynamic equations and E. Machtyngier and E. Zuazua [$Mac90, MZ94$] for Schrödinger equations.

In the case of a nonlinear velocity feedback law, several authors derived the decay rate estimates when the feedback has a polynomial behavior in zero, see for example [$Zua90$] by adapting an extension of the perturbed energy method, [$Kom94$] by using some generalization of the R. Datko's integral inequalities. We can see also [$Nak96$] for the locally distributed damping wave equation and [$CMS96$] for the Neumann boundary damped Schrödinger equations. In [$MarI99, MarII99$], P. Martinez have obtained explicit decay rate estimates for the energy of the wave equations when the nonlinear (distributed and Neumann boundary) feedback term is not supposed to have a polynomial behavior in zero, in particular when it is weaker than that at zero. The proof is based on the construction of a special weight function (that depend on the behavior of the feedback in zero) and on some integral inequalities. I. Lasiecka and D. Tataru studied in [$LT93$] the more general case of a semilinear wave equation, where no growth assumptions at the origin are imposed on the nonlinear Neumann boundary feedback, they have proved that the energy decays as fast as the solution of some associated differential equation (see also [$Las99$] for similar decay rate estimates for other systems). Let us further mention that W. Liu and E. Zuazua in [$LZ99$] have also considered the decay rate for a nonlinear globally distributed damping for the wave equation with no growth assumption at the origin, they have used an equivalent energy which is bounded by the solution of a differential equation that tends to zero as time goes to infinity. Similar technique was used in [$LZ01$] to obtain explicit decay rate for the higher dimensional system of thermoelasticity with nonlinear boundary feedback. An important observation that the behavior of the nonlinearity of the feedback at the origin has a direct influence on the energy decay rates of systems.

Noting that in the case where the feedback control is in the Dirichlet boundary condition, the problem of stabilization is more difficult than the corresponding Neumann problem. It was first introduced and solved by I. Lasiecka and R. Triggiani in $[LT87]$ for the wave equation and in $[LT92]$ for the Schrödinger equation, where they have used a lifting arguments in the topology of the solution.

Over the last decade, the problem of the uniform decay rates of systems which include first order terms have been considered in $[CS97, CDS98, GueI03$, $GueII03, CG05]$. In all these works, uniform decay rate estimates are obtained under strong hypothesis on the first order terms. The inclusion of these terms produce serious additional difficulties since we do not have any information about their influence on the energy of the solution, specially, about the signal of the derivative of the energy. In $[GueI03]$, A. Guesmia have proved some integral inequalities to obtain an estimate on behavior at infinity of a positive and non necessarily decreasing function. This extend some integral inequalities due to V. Komornick in $[Kom94]$ concerning decreasing functions.

Concerning the variable problems, the earlier attempt on this subject was made by J. L. Lagnese in $[LagI83]$ through the classical analysis and the exponential decay rate for the wave equation was obtained, see also $[Wyl94]$ for similar results by a new use of some theorems in semigroup theory. We can mention the pseudo differential method of D. Tataru $[Tat95]$ which is one of the useful tools in handling variable coefficients problems. Recently, Riemann geometric methods have emerged as a powerful tool to obtain continuous observability inequalities and direct stabilization for various classes of partial differential equations with variable coefficients (see [$LTY97$, $LTY99$, $Yao99$, $YaoI00$, $YaoII00$]). It permits to reduce the original variable coefficients principal part problem in a bounded domain to a problem on an appropriate Riemann manifolds (determined by the coefficients of the principal part) where the principal part is the Laplacian. This approach was first introduced into the boundary control problem by P. F. Yao in [$Yao99$] for the exact controllability of real valued wave without the first order term. Then it has extended in [$LTY99$] to handle the first order term by using Carleman estimates, subject to the existence of a pseudo convex function. Similar results was obtained by employing this method, see [$YaoI00$] for the observability estimates of Euler Bernoulli equation, [$TY02$, $LTY03$] for the Carleman estimates of wave and plate equations defined on a Riemann manifold, [$NP06$] for the observability estimates of vector-valued Maxwell's systems. Later, using this approach, several papers were devoted to the uniform decay rates of variable systems with internal and Neumann boundary feedback action (see [$FF04$, $GY06$]). We note that all these results were obtained under some geometric assumptions derived by P. F. Yao in [$Yao99$] in terms of the Riemannian geometry method for exact controllability of wave equation with variable coefficients, however, it is important here to mention the work of S. Feng and D. Feng in [$FF02$], where they have proved that these geometric conditions are equivalent to the analytical conditions given by A. Wyler in [$Wyl94$] for exponential decay rate of wave equation with variable coefficients.

For other comparisons of different methods on variable coefficients we refer to $[LTY97, LTY99]$.

For systems of two coupled equations and damped by two feedback operators, important progress has been obtained on the uniform decay rate estimates. We can mention the following works: $[Gue99]$ for the nonlinear wave-Petrovsky equations, $[ACF00]$ for the nonlinear wave-heat equations, $[Aas99, GM04]$ for the nonlinear wave-wave equations, $[Ham06]$ for the Schrödinger-Schrödinger equations with lower order terms.

Another question was considered in the literature: the problem of indirect stabilization of two coupled equations via zero order terms, this problem was first studied by D. L. Russell in $[Rus93]$ who introduced the terminology of indirect damping, since the first equation can be regarded as a stabilizer for the second one. Recently, A. Aassila $[Aas01]$ proved, using the spectral theory, that solutions of two coupled wave equations damped by one feedback is strongly stable and is not uniformly exponentially stable when the domain is an interval. F. Alabau in $[Ala02]$ has studied the indirect boundary stabilization of system of two coupled hyperbolic equations and damped by one feedback. She has proved that this system is not exponentially stable. However, it must be stable in a weaker sense then the exponential one, more precisely, she has show that the feedback of the first equation is sufficient to stabilize polynomially the total system. Moreover, she has established a polynomial decay lemma for a nonincreasing functional satisfying a generalized integral inequalities. In $[ACK02]$, the authors have studied the problem of indirect stabilization of two coupled second order evolution equation via zero order terms by a feedback acting in the whole domain. This result has been extended in [Bey01], using the piece-wise multiplier to several cases, wave-wave, Petrowsky-Petrowsky coupling, for one locally distributed action. We note that, in all these works, the coupling coefficient is considered as a positive constant witch is sufficiently small.

Main contribution.

The aims of our thesis are multiple and may be summarized as follows.

First goal.

It is well know that the study of the exponential decay rate of the energy of the wave equation with constant coefficients, by adapting the method based on the perturbation of the energy developed by V. Komornick and E. Zuazua, require careful treatment when the lower order term appears in some multiplier estimates (see remark 3.2 in [KZ90]). In this case, we have some difficulty to obtain directly the exponential decay rate of the perturbed energy. Our first goal is to overcome this kind of difficulty in the context of the second order hyperbolic equation with variable coefficients and linear zero order term. Here we use the compactness uniqueness argument and some result of I. Lasiecka and D. Tataru in [LT93] . For details see chapter 2.

Second goal.

We consider a general Riemannian wave equation with linear first order term and no growth conditions at the origin are imposed on the Neumann nonlinear feedback. The second goal of this thesis is to prove that the energy of this system decays faster than the solution of some associated differential equation. This goal is accomplished in a few steps. First, we obtain some energy identities by the multiplier method, here we introduce a new geometric multiplier to handle the first order term, then we combine the idea in $[FF04]$ with the one in $[LT93]$ to absorb the lower order term with respect to the energy. Finally, we derive estimates describing the energy decay rates for the solutions by a very general method introduced by I. Lasiecka and D. Tataru in $[LT93]$. This result improves upon the existing literature where exponential and polynomial decay rates of the wave equations with constant coefficients were derived only with smallness conditions imposed on the linear first order term. For details see chapter 3.

Third goal.

Recently, the Riemann geometric approach was introduced, in the context of real systems with variable coefficients, to study the problems of controllability and direct stabilization (see for example $[Yao99, GY06]$). Our third goal is to prove that we can apply this approach to the complex systems with variable coefficients. Here (see section 1.3), we construct a suitable Riemannian metric on $\mathbb{C}^n$, well adapted to such system. For details see chapter 4.

Fourth goal.

It is well know that system of two coupled real wave equations with constant coefficients and damped by two linear feedbacks has an exponential energy decay rate (see for example $[GM04]$). When the feedback appears only in one of the equation where as no effect term is applied to the second, F. Alabau, P. Cannarsa and V. Komornick have proved in $[ACK02, Ala02]$ that this system fails to has an exponential decay rate. In $[Ala02]$, the author has established a polynomial decay lemma for a nonincreasing functional satisfying a generalized integral inequality. Using this result, they have proved the polynomial decay rate estimate of the energy of the sufficiently smooth solution. In this thesis, we combine this idea with the Riemannian geometry method in order to obtain similar result for system of two coupled complex Schrödinger equations with variable coefficients and damped by one Neumann boundary feedback. For details see chapter 5.

When the system considered in chapter 5 is damped by one Dirichlet boundary feedback, we present a successful combination of the Riemann geometric approach, the ideas of I. Lasiecka and R. Triggiani in $[LT87, LT92]$ used to obtain the exponential decay rate of the energy of one equation with Dirichlet boundary feedback and the generalized integral inequalities in $[ACK02, Ala02]$, to prove the polynomial energy decay rates for smooth solutions. For details see chapter 6.

Organization of the thesis.

We now lay out the organization of the present thesis.

This thesis is divided in **6 chapters**.

In chapter 1, we present some notations and results from the Riemannian geometry on $\mathbb{R}^n$. Next, we introduce a new Riemannian metric on $\mathbb{C}^n$ and prove some formulas with which we are working. It is important to mention that this construction was first given in $[HR08]$. In section 1.4, we recall some notions and results on the semigroup theory. Finally, we give some abstract stabilization inequalities: Datko's integral inequalities, I. Lasiecka's and D. Tatau's inequalities and the generalized integral inequalities of F. Alabau. These inequalities are useful to derive different uniform decay rate estimates in this thesis.

In chapter 2, we give a number of examples where the geometric assumption is verified. Next, by using the classical semigroup theory, we give the result of existence, uniqueness and regularity of solution of the second order hyperbolic equations with variable coefficients and zero order term. Finally, we prove the main result of this chapter by the compactness uniqueness argument and some results of I. Lasiecka and D. Tataru in [LT93].

In chapter 3, we consider the Riemannian wave equation with linear first order term and unspecified behavior of the nonlinear feedback f. By employing the Faedo-Galerkin method (see [DL85]), we prove the existence, uniqueness and regularity of solutions. Next, we use the multiplier method (see [Lio88]) to obtain some energy identities. Combining the idea in [FF04] with the one in [LT93], we absorb the lower order term. In section 3.5, we complete the proof of the main result. The last section of this chapter is devoted to the general second order hyperbolic equation with polynomial growth at the origin of f, where the decay rate of the energy is discussed.

Chapter 4 is devoted to a system of two coupled Schrödinger equations with variable coefficients and damped by two Neumann boundary feedback. First, we use the theory of semigroups to prove the existence, uniqueness and regularity of solution. Finally, we use Datko's integral inequalities to prove the exponential decay rate of the energy.

Chapter 5 is about the polynomial decay rate of a coupled system of two

Schrödinger equations with variable coefficients and damped by one end only Neumann boundary feedback.

In the case of one end only Dirichlet control feedback u, we consider, in **chapter 6**, the system of two coupled Schrödinger equations with variable coefficients. We discuss the existence, uniqueness and regularity of solution under a suitable choice of the function u. Section 6.4 is devoted to the proof of the polynomial decay rate estimate.

In the end of this thesis, we collect a few questions and open problems connected with statements of our results.

It is important to mention that the multiplier method (or energy method) developed for example in [$Lio88$, $Kom94$], applied systematically in this thesis, is remarkably elementary and efficient. Where, by multiplying the equation by suitable multipliers, we can obtain different identities which play an important role in deriving the uniform decay rate of the energy.

INTRODUCTION GENERALE.

Comprendre complètement la dynamique des processus évolutifs est très difficile. C'est pourquoi pour la majorité des systèmes on considère le comportement asymptotique à l'infinie de l'énergie. Dans cette thèse, on étudie la stabilisation uniforme de certaines classes des équations aux dérivées partielles à coefficients variables.

Littérature.

Des recherches considérables on été faite dans les trois dernière décénnies sur les problèmes de la stabilisation uniforme de certaines classes des équations aux dérivées partielles par l'insertion directe des termes feedbacks distribués ou frontières. Ce qu'on appelle la stabilisation directe.

Quand le terme feedback dépend de la vitesse de façons linéaire, on peut citer les premiers travaux dans cette direction: sous des différentes conditions géométriques sur la frontière, G. Chen dans [CheI79, CheII79, Che81] a démontré la stabilisation exponentielle de l'énergie de l'équation des ondes dans un domaine borné et soumis à un feedback de type Neumann sur une partie de la frontière, où il a adopté un résultat de R. Datko dans [Dat70] généralisé après par A. Pazy dans [Paz83]. Dans [Lag88, KZ90], la stabilisation frontière de type Neumann de l'équation des ondes a été obtenu en utilisant une méthode différente, cette méthode est basée sur la construction des fonctions de l'énergie perturbée. Des résultats similaires sur d'autre modèles ont été aussi obtenus: J. L. Lagnese [LagII83] pour les équations élastodynamique, E. Machtyngier et E. Zuazua [Mac90, MZ94] pour l'équation de Schrödinger.

Dans le cas d'un feedback non linéaire, de différentes estimations de la stabilisation de l'énergie ont été obtenus quand ce feedback a un comportement polynomial à l'origine: voir par exemple [Zua90] par une extension de la méthode de l'énergie perturbée, [Kom94] par une généralisation des inégalités d'intégrales de R. Datko. On peut mentionner aussi les travaux de Nakao dans [Nak96] pour les équations des ondes avec une action distribuée locale et [CMS96] pour les équations de Schrödinger avec une action frontière de type Neumann. Dans [MarI99, MarII99], P. Martinez a obtenu une estimation explicite de la stabilisation de l'énergie quand le terme feedback non linéaire (distribué ou frontière de type Neumann) n'a pas un comportement polynomial à l'origine, en particulier, quand il a un comportement plus faible. Sa démonstration est basée sur la construction de fonctions poids (dépendent du comportement du feedback à l'origine) et de certaines inégalités intégrales. I. Lasiecka et D. Tataru ont étudié dans [LT93] un cas plus général d'une équation semi linéaire où aucune hypothèse à l'origine n'a été supposée sur le feedback frontière non linéaire de type Neumann, ils ont démontré que l'énergie décroit plus rapidement que la solution d'une équation différentielle associée (voir aussi [Las99] pour des résultats similaires pour d'autre systèmes). On peut aussi mentionner que W. Liu et E. Zuazua dans [LZ99] ont considéré la stabilisation uniforme de l'équation des ondes avec une action frontière distribuée non linéaire sans aucune hypothèse sur son comportement à l'origine, ils ont utilisé une énergie équivalente bornée par une solution d'une équation différentielle qui tend vers zéro quand le temps tend vers l'infinie. Des techniques analogues ont été utilisées dans [LZ01] pour obtenir des estimations explicites de la stabilisation d'un système thermoélastique avec un feedback frontière non linéaire. D'après ce qui a été dit, il est important d'observer que le comportement à l'origine du feedback non linéaire a une influence directe sur les estimations de la stabilisation uniforme des systèmes.

Il est à noter que dans le cas d'un contrôle feedback frontière de type Dirichlet, le problème de la stabilisation est plus difficile que celui d'une action frontière de type Neumann, dans ce cas les solutions ne sont pas assez régulière. Ce problème a été introduit et résolu par I. Lasiecka et R. Triggiani dans [LT87] pour l'équation des ondes et dans [LT92] pour l'équation de Schrödinger, où ils ont utilisé un argument de lavage de la topologie de la solution basé sur un changement de variable qui transforme le problème initial à un problème dont les solutions sont assez régulière.

Durant la dernière décennie, le problème de la stabilisation uniforme des systèmes qui contiennent des termes d'ordre un a été considéré dans [CS97, CDS98, GueI03, GueII03, CG05]. Dans tout ces travaux, les estimations de la stabilisation uniforme ont été obtenus sous des hypothèses forte sur les termes d'ordre un. Il est à noter que l'inclusion de ces termes produit des difficultés additionnelles car on n'a aucune information de leur influence sur l'énergie, spécialement, le signe de la dérivée de l'énergie. Dans [GueI03], A. Guesmia a démonté quelques inégalités intégrales utiles pour avoir des estimations sur le comportement à l'infinie des fonctions positives non nécessairement décroissantes. Ceci généralise les résultats de V. Komornick dans [Kom94] sur les fonctions décroissantes.

Concernant Les problèmes variables, la première tentative dans cette direction a été faite par J. L. Lagnese dans [LagI83] en utilisant l'analyse classique et la stabilisation exponentielle de l'équation des ondes a été obtenue, voir aussi [Wyl94] pour des résultats similaires par une première utilisation d'un théorème dans la théorie des semi groups. On peut aussi mentionner la méthode du pseudo différentielle due à D. Tataru [Tat95] qui représente l'une des outils éfficace pour traiter les problèmes à coefficients variables. Récemment, la méthode de la géométrie Riemannienne a été classée comme un outil trés riche pour obtenir des estimations d'observabilité et de stabilisation directe pour certaines classes des équations aux dérivées partielles (voir par exemple [LTY97, LTY99, Yao99, YaoI00, YaoII00]). Elle permet de transformer le problème initial avec une partie principale à coefficients variables posé sur un domaine borné à un problème posé sur une variété Riemanniènne (définie par les coefficients de la partie principale) où la partie principale deviendra le Laplacien. Cette Approche a été premièrement introduite dans la théorie du contrôle par P. F. Yao dans [Yao99], où il a obtenu des résultats de la contrôlabilité exacte de l'équation des ondes sans l'inclusion des termes d'ordre un. Ensuite, elle a été généralisée dans [LTY99] pour traiter les termes d'ordre un en utilisant les estimations de Carleman, basées sur la construction des fonctions pseudo convexes. Des résultats similaires ont été obtenues en adoptant la méthode de la géométrie Riemannienne, voir [YaoI00] pour les estimations d'observabilité de l'équation d'Euler Bernoulli, [TY02,LTY03] pour les estimations de Carleman de l'équation des ondes et les équations des plaques définie sur des variétés Riemannienne, [NP06] pour les estimations d'observabilité des systèmes de Maxwell. En adoptant cette méthode, plusieurs papiers ont considéré la stabilisation uniforme des systèmes à coefficients variables avec un feedback interne et frontière de type Neumann (voir par exemple [FF04, GY06]). On note ici que tout ces résultats ont été obtenus sous des conditions géométriques introduite par P. F. Yao dans [Yao99] en terme de la méthode de la géométrie Riemannienne pour avoir la contrôlabilité exacte de l'équation des ondes avec des coefficients variables, pour cela, il est important de citer le travail de S. Feng et D. Feng dans [FF02], où ils ont démontré que ces conditions géométriques sont équivalentes aux conditions analytiques introduites par A. Wyler dans [Wyl94] pour avoir la stabilisation exponentielle de l'équation des ondes avec des coefficients variables. Pour une comparaison détaillé des différentes méthodes utilisées pour traiter les problèmes variables on refera à [LTY97, LTY99].

Pour les systèmes couplés de deux équations et soumis à deux fonctions feedbacks, une progression importante à été touchée sur les estimations de la stabilisation uniforme. On peut citer les travaux suivants: [Gue99] sur les équations non linéaire ondes-Petrovski, [ACF00] sur les équations non linéaire ondes-chaleur, [Aas99,GM04] sur les équations non linéaire ondes-ondes, [Ham06] sur un système Schrödinger-Schrödinger avec des termes d'ordre un dans chaque équation.

Une autre question a été considéré dans la littérature: le problème de la stabilisation indirecte de deux équations couplées via des termes d'ordre zéro, ce problème a été premièrement étudié par D. L. Russell dans [Rus93] où il a introduit cette terminologie, puisque la première équation peut être considèré comme un stabilisateur de la deuxième. Récemment, A. Aassila [Aas01] a démontré, en utilisant la théorie spectral, que les solutions d'un système de deux équations couplées et soumis à un seul feedback sont faiblement stable, en plus elles ne sont jamais exponentiellement stable si le domaine est un intervalle. F. Alabau dans [Ala02] a étudié la stabilisation indirecte frontière d'un système couplé de deux équations hyperboliques soumise à une seule action de type Neumann. Elle a démontré que ce système n'est jamais exponentiellement stable et dans le cas où il y'a une stabilité elle doit être faible qu'une fonction exponentielle, plus précisément, elle a démontré que l'action sur la première équation est suffisante pour avoir la stabilisation polynomiale de tout le système. Elle a démontré un résultat de décroissance polynomiale des fonctions décroissantes qui vérifient des inégalités intégrales généralisées. Dans [ACK02], les auteurs ont étudié le problème de la stabilisation indirecte de deux équations évolutives d'ordre deux couplées via zéro ordre terme et soumises à un seul feedback agissant sur tout le domaine. En utilisant les multiplicateurs par morceau, ce résultat a été généralisé dans le cas des systèmes ondes-ondes et Petrovski-Petrovski avec une seule action distribuée locale (voir [Bey01]). On note ici que dans tous ces travaux les coefficients de couplage sont des constantes suffisamment petites.

Contribution de la thèse

Les objectifs de cette thèse sont multiples, on peut les résumer comme suit.

Premier objectif

Il est bien connu que l'étude de la stabilisation exponentielle de l'équation des ondes avec des coefficients constants, en adaptant la méthode de l'énergie perturbée développé par V. Komornick et E. Zuazua, est difficile quand il y'a un terme d'ordre inférieure dans une certaine estimation (voir remarque 3.2 dans [KZ90]). Dans ce cas, on a une certaine difficulté pour obtenir la décroissance exponentielle de l'énergie perturbée. Notre premier objectif est de démontrer ce résultat dans le context des équations hyperbolique d'ordre deux avec des coefficients variables et un terme d'ordre zéro. Pour cela, on va utiliser l'argument de l'unicité et de la compacité et des résultats de I. Lasiecka et D. Tataru dans [LT93]. Pour plus de détails voir chapitre 2.

Deuxième objectif

On considère une équation des ondes Riemannienne générale avec un terme linéaire d'ordre un et aucune condition sur le comportement à l'origine du feedback frontière non linéaire de type Neumann. Notre deuxième objectif dans cette thèse est de montrer que l'énergie de ce système décroit plus rapidement que la solution d'une équation différentielle associée. On démontre ce résultat en plusieurs étapes: Premièrement, on obtient quelques identités d'énergie par la méthode des multiplicateurs, ici on introduit un nouveau multiplicateur géométrique pour traiter le terme d'ordre un, puis on combine l'idée dans [FF04] avec celle dans [LT93] pour absorber les termes d'ordre inférieur. Finalement, en adoptant une méthode générale due à I. Lasiecka et D. Tataru dans [LT93], on montre la stabilisation uniforme de l'énergie. Ce résultat amiliore les résultats, existant dans la littérature, où la stabilisation exponentielle et polynomiale de l'équation des ondes ont été obtenues seulement avec une condition de petitesse sur le terme linéaire d'ordre un. Pour plus de détails voir chapitre 3.

Troisième objectif

Récemment, la méthode de la géométrie Riemannienne a été introduite, dans le contexte des équations réelles avec des coefficients variables, pour étudier les problèmes de la stabilisation directe et de la contrôlabilité exacte (voir par exemple [Yao99, GY06]). Notre troisième objectif est de montrer qu'on peut appliquer cette approche sur les systèmes complexes avec des coefficients variables. Ici (voir section 1.3), on a construit une métrique convenable sur $\mathbb{C}^n$ bien adapter a ce type de système. Pour plus de détails voir chapitre 4.

Quatrième objectif

Il est bien connu que les systèmes couplés de deux équations des ondes avec des coefficients constants et soumis à deux feedbacks linéaires sont exponentiellement stable (voir par exemple [GM04]). Quand la fonction feedback agit seulement sur une seule équation et aucune action sur l'autre, F. Alabau, P. Cannarsa et V. Komornick ont démontré dans [ACK02, Ala02] que ce système n'est jamais exponentiellement stable. Dans [Ala02], l'auteur a établit un résultat de la décroissance polynomiale pour les fonctions décroissantes qui vérifient des inégalités intégrales généralisées. En utilisant ce résultat, ils ont démontré des estimations polynomiales de l'énergie des solutions régulières. Dans notre thèse, on combine cette idée avec la méthode de la géométrie Riemannienne pour obtenir des résultats similaires pour un système couplé de deux équations de Schrödinger avec des coefficients variables et soumis à une seule action de type Neumann. On note ici que le coéfficient de couplage est une fonction de norme suffisamment petite. Pour plus de détails voir chapitre 5.

Quand le système considéré dans le chapitre 5 est soumis à une seule action de type Dirichlet, on présente une combinaison de la méthode de la géométrie Riemannienne avec l'idée de I. Lasiecka et R. Triggiani dans [LT87, LT92] utilisée pour obtenir la décroissance exponentielle d'une équation des ondes avec un feedback frontière de type Dirichlet, et les inégalités intégrales généralisées dans [ACK02, Ala02], pour démontrer la stabilisation polynomiale de l'énergie des solutions régulières. Pour plus de détails voir chapitre 6.

Organisation de la thèse

Cette thèse est organisée comme suit.

Dans le chapitre 1, on présente quelques notations et résultats sur la géométrie Riemannienne sur $\mathbb{R}^n$. Ensuite, on introduit une nouvelle métrique sur $\mathbb{C}^n$ et on montre quelques formules qu'on aura besoin dans cette thèse. Il est important de signaler ici que cette partie a été introduite premièrement dans [HR08]. Dans la section 1.4, on rappelle quelques notions et résultats sur la théorie du semi groupe. Finalement, on donne quelques inégalités abstraite de la stabilisation uniforme: Inégalités intégrales de R. Datko, inégalités de I. Lasiecka et D. Tatau et les inégalités intégrales généralisées de F. Alabau. Toutes ces inégalités sont utiles pour obtenir de différentes estimations de la stabilisation uniforme dans cette thèse.

Dans le chapitre 2, on donne quelques exemples où la condition géométrique est vérifie. Ensuite, en utilisant la théorie du semi groupe, on montre l'existence, l'unicité et la régularité de la solution d'une équation hyperbolique d'ordre deux avec des coefficients variables et un terme d'ordre zero. Finalement, on montre le résultat principal de ce chapitre en utilisant l'argument de l'unicité et de la compacité et un résultat de I. Lasiecka et D. Tataru dans [LT93].

Dans le chapitre 3, on étudie une équation des ondes Riemannienne avec un terme linéaire d'ordre un et un comportement à l'origine non spécifier de la fonction feedback non linéaire f. En utilisant la méthode de Faedo-Galerkin (voir [DL85]), on montre l'existence, l'unicité et la régularité de la solution. Ensuite, on utilise la méthode des multiplicateurs (voir [Lio88]) pour obtenir quelques identités d'énergie. En combinant l'idée dans [FF04] avec celle dans [LT93], on peut absorber le terme d'ordre inferieur. Dans la section 3.5, on termine la démonstration du résultat principal de ce chapitre. La dernière section de ce chapitre concerne la stabilisation uniforme de l'énergie d'une équation hyperbolique générale avec un comportement polynomiale à l'origine de la fonction f.

Chapitre 4 concerne un système couplé de deux équations de Schrödinger avec des coefficients variables et soumis à deux feedbacks de type Neumann. Au début, on utilise la théorie du semi groupe pour démontrer l'existence, l'unicité et la régularité de la solution. Puis, on utilise les inégalités intégrales de R. Datko pour démontrer la stabilisation exponentielle de l'énergie.

Chapitre 5 concerne la stabilisation polynomiale de l'énergie d'un système couplé de deux équations de Schrödinger avec des coefficients variables et soumis à une action frontière de type Neumann.

Dans le cas d'une seule fonction control u de type Dirichlet, on considère dans le **chapitre 6**, un système couplé de deux équations de Schrödinger avec des coefficients variables. On démontre l'existence, l'unicité et la régularité de la solution sous un choix convenable de la fonction u. Section 6.4 concerne la démonstration de la stabilisation polynomiale de l'énergie.

On termine notre thèse par une conclusion et quelques questions et problèmes ouverts liés au contenu de cette thèse.

Il est important de mentionner que la méthode des multiplicateurs (où la méthode de l'énergie), développée par exemple dans [Lio88, Kom94], utilisée systématiquement dans cette thèse est élémentaire et très efficace. Où, en multipliant les équations par des multiplicateurs convenables, on obtient de différentes identités très importantes pour avoir de différentes etimations de la stabilisation uniforme de l'énergie.

List of Notations.

Let us collect some notations which shall use in this thesis.

$\mathbb{N}$	Set of the positive integers.
$\mathbb{R}$	Set of the real numbers.
$\mathbb{C}$	Set of the complex numbers.
T	Positive constant.
	Open bounded domain of $\mathbb{R}^n$, $n \in \mathbb{N}^*$.
Γ	The sufficiently smooth boundary of .
$\{\Gamma_0, \Gamma_1\}$	Partition of Γ such that $\Gamma_0 \neq \varnothing$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \varnothing$.

$Q =]0, T[\times$.

$\Sigma =]0, T[\times \Gamma$.

$\Sigma_0 =]0, T[\times \Gamma_0$.

$\Sigma_1 =]0, T[\times \Gamma_1$.

C Generic positive constant independent of the initial data and it may change from line to line.

$H^1_{\Gamma_0}(\ ) = \left\{u \in H^1(\ ) : u = 0 \text{ on } \Gamma_0\right\}$.

Chapter 1

Riemannian geometry, semigroup theory and some abstract stabilization inequalities.

1.1 *Introduction.*

In this chapter, we shall recall some notions and results that will be needed in this thesis. It is important to mention that section 1.3 was first treated in $[HR08]$.

In the beginning of this chapter, we define a Riemannian metric g on $\mathbb{R}^n$ and give some additional material associated with them. For the general background on Riemannian geometry, we refer to references $[Lee97, Aub98]$ and bibliography cited therein. In order to handle the case of complex systems with variable coefficients in this thesis, we need to define a suitable Riemannian geometry on $\mathbb{C}^n$, well adapted to such systems. For this end, we use the Riemannian geometry on $\mathbb{R}^n$ to construct a suitable Riemannian metric on $\mathbb{C}^n$. Then, we prove some formulas with which we are working. In the next section, we give some notions and results in the semigroup theory that will be needed in the sequel to study the problem of existence, uniqueness and regularity of solution of certain systems considered in this thesis. Finally, we present some abstract stabilization inequalities that will be useful in what follows for obtaining decay rate estimates.

1.2 *Riemannian geometry on* $\mathbb{R}^n$*.*

1.2.1 Riemannian metric on $\mathbb{R}^n$.

Let $A = A(x) = (a_{ij}(x))_{i,j=1,\dots,n}$ be an $n \times n$ matrix with real coefficients, $a_{ij} = a_{ji}$ are C^∞ functions in $\mathbb{R}$ and for some positive constant a_0

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\zeta_i\zeta_j \geq a_0 \sum_{i=1}^{n} \zeta_i^2, \tag{1.1}$$

for all $x \in \mathbb{R}^n$ and for all $\zeta = (\zeta_1, \zeta_2, ..., \zeta_n)^\tau \in \mathbb{R}^n$.

Denote

$$G = G(x) = (g_{ij}(x))_{i,j=1,...,n} = A^{-1},$$

for $x \in \mathbb{R}^n$.

Let $\mathbb{R}^n$ have the usual topology and $x = (x_1, x_2, ..., x_n)$ be the natural coordinates system. For each $x \in \mathbb{R}^n$, we define the inner product and norm on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$g(X, Y) = \langle X, Y \rangle_g = \sum_{i,j=1}^{n} g_{ij}\alpha_i\beta_j \text{ and } \|X\|_g = \left(\langle X, X \rangle_g\right)^{\frac{1}{2}},$$

for all $X = \sum\limits_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$, $Y = \sum\limits_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x$.

Remark 1.1 *It is easily checked from (1.1) that $(\mathbb{R}^n, g)$ is a Riemannian manifold with the Riemannian metric g.*

Definition 1.2 *We define the gradient $\nabla_g f$ of a real valued function $f \in C^1\left(\overline{\ }\right)$ in the Riemannian metric g, via Riesz representation theorem, by*

$$X(f) = \langle \nabla_g f, X \rangle_g,$$

where X is any vector field on the manifold $(\mathbb{R}^n, g)$.

Moreover, if h is a vector field on $(\mathbb{R}^n, g)$, we define the divergence of h in the Riemannian metric g by

$$div_g h = \sum_{i=1}^{n} \left(D_{\frac{\partial}{\partial x_i}} h\right)_i,$$

where $D_{\frac{\partial}{\partial x_i}}h$ *is the covariant derivative of* h *in direction of* $\frac{\partial}{\partial x_i}$ *defined below, and* $\left(D_{\frac{\partial}{\partial x_i}}h\right)_i$ *is his ith coordinate.*

Remark 1.3 *If*

$$a_{ij}(x) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

Then $g_{ij}(x) = \delta_{ij}$ *and* g *is the Euc79 588 Tm□ (u)5j□ 1 0 0 1 27*

1. $D_X Y$ *is linear over* $C^\infty(\mathbb{R}^n)$ *in* X :

$$D_{f_1 X_1 + f_2 X_2} Y = f_1 D_{X_1} Y + f_2 D_{X_2} Y,$$

for all vector fields X_1, X_2, Y *and for all* $f_1, f_2 \in C^\infty(\mathbb{R}^n)$.

2. $D_X Y$ *is linear over* $\mathbb{R}$ *in* Y :

$$D_X(aY_1 + bY_2) = aD_X Y_1 + bD_X Y_2,$$

for all vector fields X, Y_1, Y_2 *and for all* $a, b \in \mathbb{R}$.

3. D *satisfy*

$$D_X(fY) = fD_X(Y) + X(f)Y,$$

for all vector fields X, Y *and for all* $f \in C^\infty(\mathbb{R}^n)$.

$D_X Y$ *is called the covariant derivative of* Y *in direction of* X.

Theorem 1.5 ([*Lee97*]) *There exist a unique linear connection* D *on* $\mathbb{R}^n$ *such that*

1. D *is compatible with* g

$$D_X \langle Y, Z\rangle_g = \langle D_X Y, Z\rangle_g + \langle Y, D_X Z\rangle_g,$$

for all vector fields X, Y *and* Z.

2. D *is symmetric*

$$D_X Y - D_Y X = [X, Y],$$

for all vector fields X *and* Y, *where* $[X, Y]$ *is the Lie bracket of the vector fields* X *and* Y.

This connection is called the Levi Cevita connection or the Riemannian connection and it is given by

$$\begin{aligned}\langle D_X Y, Z\rangle_g &= \frac{1}{2}\left(X\langle Y, Z\rangle_g + \langle X, [Z, Y]\rangle_g\right) + \frac{1}{2}\left(Y\langle Z, X\rangle_g - \langle Y, [X, Y]\rangle_g\right) \\ &\quad -\frac{1}{2}\left(Z\langle X, Y\rangle_g + \langle Z, [Y, X]\rangle_g\right),\end{aligned}$$

for all vector fields X, Y *and* Z.

Definition 1.6 *The total covariant derivative* Dh *(or the covariant differential) of a vector field* h *determines a bilinear form on* $\mathbb{R}^n \times \mathbb{R}^n$ *such that, for each* $x \in \mathbb{R}$,

$$Dh(X, Y) = \langle D_X h, Y\rangle_g,$$

for all vector fields X *and* Y.

Remark 1.7 $Df = \nabla_g f$ *for all real valued function* f.

Definition 1.8 *Let* f *be a real valued function, the Hessian* $D^2 f$ *of* f *with respect to the Riemannian metric* g *is given by*

$$D^2 f(X, Y) = \langle D_Y(Df), X\rangle_g,$$

for all vector fields X *and* Y.

1.2.3 Further relationships.

The following lemma provides some useful identities.

Lemma 1.9 ([*Lee97, Yao99*]) *Let* f *be a real valued function in* $C^1\left(\overline{}\right)$ *and* $h = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x$, *then*

1.

$$\nabla_g f = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(x) \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i} = A \nabla_0 f.$$

2.

$$div_g h = \frac{1}{\sqrt{\det G}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det G} h_i \right).$$

3.

$$h(f) := \langle h, \nabla_g f \rangle_g = h.\nabla_0 f.$$

4.

$$\langle \nabla_g f, \nabla_g (h(f)) \rangle_g = Dh(\nabla_g f, \nabla_g f) + \frac{1}{2} h \left(\|\nabla_g f\|_g^2 \right). \qquad (1.2)$$

We also have

Lemma 1.10 ([$Yao99$]**)** *Let f_1, f_2 be real valued functions in $C^1\left(\overline{}\right)$ and $X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}_x^n$, then*

1.

$$\langle X(x), AY(x) \rangle_g = X(x).Y(x),$$

for all $x \in \mathbb{R}^n$.

2.

$$\langle \nabla_g f_1, \nabla_g f_2 \rangle_g = \nabla_0 f_1.A \nabla_0 f_2,$$

for all $x \in \mathbb{R}^n$.

The relationship between the Riemannian and the Euclidean divergence is given in the first part of the following lemma.

Lemma 1.11 *For all real valued function f in $C^1\left(\overline{\ }\right)$ and $h=\sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i} \in \mathbb{R}_x^n$, we have*

1.

$$div_g h = div_0 h + \frac{1}{\sqrt{\det G}} h.\nabla_0\left(\sqrt{\det G}\right).$$

2.

$$div_g\left(fh\right) = f div_g h + h\left(f\right).$$

Proof.

1.

$$\begin{aligned} div_g h &= \frac{1}{\sqrt{\det G}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det G} h_i\right) \\ &= \sum_{i=1}^{n} \frac{\partial h_i}{\partial x_i} + \frac{1}{\sqrt{\det G}} \sum_{i=1}^{n} h_i \frac{\partial \sqrt{\det G}}{\partial x_i} \\ &= div_0 h + \frac{1}{\sqrt{\det G}} h.\nabla_0\left(\sqrt{\det G}\right). \end{aligned}$$

2. If we apply the first identity we find

$$div_g\left(fh\right) = div_0\left(fh\right) + \frac{1}{\sqrt{\det G}}\left(fh\right).\nabla_0\left(\sqrt{\det G}\right).$$

But we have the following identity in the Euclidean metric

$$div_0\left(fh\right) = h.\nabla_0 f + f div_0 h,$$

then

$$\begin{aligned} div_g\left(fh\right) &= h.\nabla_0 f + f div_0 h + \frac{1}{\sqrt{\det G}}\left(fh\right).\nabla_0\left(\sqrt{\det G}\right) \\ &= f\left(div_0 h + \frac{1}{\sqrt{\det G}} h.\nabla_0\left(\sqrt{\det G}\right)\right) + h.\nabla_0 f \\ &= f div_g h + h\left(f\right). \end{aligned}$$

$\blacksquare$

1.2.4 The integration.

Denote by d the Euclidean volume element on , $d\Gamma$ the Euclidean surface element on Γ and $\upsilon = (\upsilon_1, ..., \upsilon_n)$ the outward unit normal to Γ in the Euclidean metric. Let $d \ \ _g = \sqrt{\det G} d$ be the Riemanian volume element and $d\Gamma_g = \|A\upsilon\|_g \sqrt{\det G} d\Gamma$ be the Riemannian surface element. n is the outward unit normal to Γ in the Riemannian metric g.

Remark 1.12 *If we use identity 2 of lemma 1.10 and the assumption (1.1) on the matrix A we find, for all real valued function f in $H^1(\ )$,*

$$a_0 \|\nabla_0 f\|_0^2 \leq \|\nabla_g f\|_g^2 \leq a_1 \|\nabla_0 f\|_0^2,$$

this imply that

$$a_0 \int \|\nabla_0 f\|_0^2 d \ \ \leq \int \|\nabla_g f\|_g^2 d \ \ \leq a_1 \int \|\nabla_0 f\|_0^2 d \ \ ,$$

for some positive constant a_1.

Moreover, the assumption $\Gamma_0 \neq \varnothing$ implies that, for all $f \in H^1_{\Gamma_0}(\ )$, we have

$$\int |f|^2 d \ \ \leq C_1^2 \int \|\nabla f\|_g^2 d \ \ ,$$

and so

$$\int_{\Gamma_1} |f|^2 d\Gamma \leq C_2^2 \int \|\nabla f\|_g^2 d \ \ ,$$

for some positive constants C_1 and C_2.

We see, now, a few formulas in the Euclidean metric to be invoked in the sequel.

Lemma 1.13 ([$Yao99$]) *For all real valued functions* f_1, f_2 *in* $C^1\left(\overline{\quad}\right)$ *and* $h = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x$, *we have*

1. *Divergence formula in Euclidean metric.*

$$\int div_0 h d \quad = \int_\Gamma h.\upsilon d\Gamma$$

2.

$$\int f_1 h\left(f_2\right) d \quad = \int_\Gamma h.\upsilon f_1 f_2 d\Gamma - \int f_2 div_0\left(f_1 h\right) d \quad . \tag{1.3}$$

We prove, now, the counterpart of lemma 1.13 in the Riemannian metric.

Lemma 1.14 *For all real valued functions* f_1, f_2 *in* $C^1\left(\overline{\quad}\right)$ *and* $h = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x$, *we have*

1. *Divergence formula in the Riemannian metric g.*

$$\int div_g h d \quad _g = \int_\Gamma \langle h, n \rangle_g \, d\Gamma_g.$$

2.

$$\int f_1 h\left(f_2\right) d \quad _g = \int_\Gamma \langle h, n \rangle_g \, f_1 f_2 d\Gamma_g - \int f_2 div_g\left(f_1 h\right) d \quad _g. \tag{1.4}$$

Proof.

1. We have, by lemma 1.11,

$$\begin{aligned} \int div_g h d \quad _g &= \int \sqrt{\det G}\left(div_0 h + \frac{1}{\sqrt{\det G}} h.\nabla_0\left(\sqrt{\det G}\right)\right) d \\ &= \int \sqrt{\det G} div_0 h d \quad + \int h\left(\sqrt{\det G}\right) d \quad . \end{aligned}$$

But

$$\int h\left(\sqrt{\det G}\right) d \quad = \int_\Gamma \sqrt{\det G} h.\upsilon d\Gamma - \int \sqrt{\det G} div_0 hd \quad .$$

Here we have apply lemma 1.13 with $f_1 = 1$ and $f_2 = \sqrt{\det G}$. Then

$$\int div_g hd \quad _g = \int_\Gamma \sqrt{\det G} h.\upsilon d\Gamma = \int_\Gamma \sqrt{\det G} \langle h, A\upsilon\rangle_g \, d\Gamma.$$

But $d\Gamma = \frac{d\Gamma_g}{\|A\upsilon\|_g \sqrt{\det G}}$ then

$$\int div_g hd \quad _g = \int_\Gamma \left\langle h, \frac{A\upsilon}{\|A\upsilon\|_g} \right\rangle_g d\Gamma_g,$$

and $n = \frac{A\upsilon}{\|A\upsilon\|_g}$, then

$$\int div_g hd \quad _g = \int_\Gamma \langle h, n\rangle_g \, d\Gamma_g.$$

2. It's sufficient to see that

$$f_1 h\left(f_2\right) = div_g\left(f_1 f_2 h\right) - f_2 div_g\left(f_1 h\right)$$

integrate over and use the divergence formula in the Riemannian metric.

■

Remark 1.15 *If* $f \in H^1(\quad)$ *then*

$$\begin{aligned} a_0 \int \|\nabla_0 f\|_0^2 \, d \quad _g &\leq \int \|Df\|_g^2 \, d \quad _g = \int \|\nabla_g f\|_g^2 \, d \quad _g \\ &\leq a_1 \int \|\nabla_0 f\|_0^2 \, d \quad _g. \end{aligned}$$

That is

$$a_0 \int \|\nabla_0 f\|_0^2 \, d \quad _g \leq \int \|Df\|_g^2 d \quad _g \leq a_1 \int \|\nabla_0 f\|_0^2 \, d \quad _g.$$

Moreover, if $f \in H^1_{\Gamma_0}(\ )$ *and* ϕ *is a function defined in* $\overline{\ }$ *such that*

$$0 < \phi_* \leq \phi(s) \leq \phi^*, \text{ for all } s \in \overline{\ },$$

for some positive constants ϕ_* *and* ϕ^*.

Then

$$\int \phi\, |f|^2\, d\ \ _g \leq \alpha \int \phi \|Df\|_g^2\, d\ \ _g,$$

and

$$\int_\Gamma \phi |f|^2\, d\Gamma_g \leq \beta \int \phi\, \|Df\|_g^2\, d\ \ _g,$$

for some positive constants α *and* β.

1.2.5 The operator $\mathcal{A}$.

Consider the second order differential operator $\mathcal{A}$ defined by

$$\mathcal{A}y := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial y}{\partial x_j}\right),$$

and the co normal derivative with respect to $\mathcal{A}$ defined by

$$\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} := \sum_{i,j=1}^{n} a_{ij}(x)\,\frac{\partial y}{\partial x_j}\upsilon_i,$$

here $\upsilon_{\mathcal{A}} = A\upsilon$.

We give the Green's formula associated to the operator $\mathcal{A}$.

Lemma 1.16 (First Green's formula $[LTY99]$**)** *For all real valued functions* f_1, f_2 *in* $H^2(\ )$, *we have*

$$\int \mathcal{A}f_1 f_2 d\ \ = \int \langle \nabla_g f_1, \nabla_g f_2 \rangle_g\, d\ \ - \int_\Gamma \frac{\partial f_1}{\partial \upsilon_{\mathcal{A}}} f_2 d\Gamma.$$

1.2.6 The Laplace Beltrami operator.

Consider the Laplace Beltrami operator in the Riemannian metric g given by

$$\Delta_g y = div_g (\nabla_g y) = \frac{1}{\sqrt{\det G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det G} a_{ij}(x) \frac{\partial y}{\partial x_j} \right),$$

for all $y \in H^2(\ \ )$ and $x \in \mathbb{R}^n$.

We give the relationship between the operator $\mathcal{A}$ and the Laplace Beltrami operator Δ_g and the relationship between $\frac{\partial y}{\partial n}$ and $\frac{\partial y}{\partial v_{\mathcal{A}}}$.

Lemma 1.17 *We have*

1.

$$\Delta_g y = \frac{1}{2} \langle Dy, D(\log \det G) \rangle_g - \mathcal{A} y.$$

2.

$$\frac{\partial y}{\partial n} = \frac{1}{\|Av\|_g} \frac{\partial y}{\partial v_{\mathcal{A}}}.$$

Here $\frac{\partial y}{\partial n} := \langle Dy, n \rangle_g$.

Proof.

1. We use lemma 1.11 to obtain

$$\begin{aligned} \Delta_g y &= div_g (\nabla_g y) = div_0 (\nabla_g y) + \frac{1}{\sqrt{\det G}} \nabla_g y . \nabla_0 \left(\sqrt{\det G} \right) \\ &= div_0 (\nabla_g y) + \frac{1}{\sqrt{\det G}} \left\langle \nabla_g y, \nabla_g \left(\sqrt{\det G} \right) \right\rangle_g , \end{aligned}$$

but

$$div_0 (\nabla_g y) = -\mathcal{A} y,$$

and

$$\begin{aligned}
\frac{1}{\sqrt{\det G}}\nabla_g\left(\sqrt{\det G}\right) &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x\right)\frac{1}{\sqrt{\det G}}\frac{\partial\sqrt{\det G}}{\partial x_j}\right)\frac{\partial}{\partial x_i}\\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}\left(x\right)\frac{\partial\log\sqrt{\det G}}{\partial x_j}\right)\frac{\partial}{\partial x_i}\\
&= \nabla_g\left(\log\sqrt{\det G}\right)=\frac{1}{2}\nabla_g\left(\log\det G\right),
\end{aligned}$$

then

$$\Delta_g y=\frac{1}{2}\left\langle Dy, D\left(\log\det G\right)\right\rangle_g-\mathcal{A}y.$$

2. We have

$$\frac{\partial y}{\partial n}=\left\langle Dy,n\right\rangle_g=\left\langle\nabla_g y,\frac{A\upsilon}{\left\|A\upsilon\right\|_g}\right\rangle_g=\frac{1}{\left\|A\upsilon\right\|_g}\left\langle\nabla_g y,A\upsilon\right\rangle_g.$$

So

$$\frac{\partial y}{\partial n}=\frac{1}{\left\|A\upsilon\right\|_g}\nabla_g y.\upsilon=\frac{1}{\left\|A\upsilon\right\|_g}\sum_{i,j=1}^{n}a_{ij}\left(x\right)\frac{\partial y}{\partial x_j}\upsilon_i.$$

That is

$$\frac{\partial y}{\partial n}=\frac{1}{\left\|A\upsilon\right\|_g}\frac{\partial y}{\partial\upsilon_{\mathcal{A}}}.$$

■

Our objective here is to use the Green's formula associated to the operator $\mathcal{A}$ to give a similar formula for the Laplace Beltrami operator.

Lemma 1.18 (Second Green's formula.) *For all real valued function f_1, f_2 in $H^2(\ )$ we have*

$$\int\Delta_g f_1 f_2 d\ \ _g=\int_\Gamma\frac{\partial f_1}{\partial n}f_2 d\Gamma_g-\int\left\langle Df_1,Df_2\right\rangle_g d\ \ _g.$$

Proof. We have

$$
\begin{aligned}
\int \Delta_g f_1 f_2 d \quad {}_g &= \int \left(\frac{1}{2} \langle \nabla_g f_1, \nabla_g (\log \det G) \rangle_g - \mathcal{A} f_1 \right) f_2 \sqrt{\det G} d \\
&= \frac{1}{2} \int \sqrt{\det G} \langle \nabla_g f_1, \nabla_g (\log \det G) \rangle_g f_2 d \\
&\quad - \int \mathcal{A} f_1 \left(\sqrt{\det G} f_2 \right) d \quad ,
\end{aligned}
$$

if we apply the first Green's formula we find

$$
\begin{aligned}
\int \Delta_g f_1 f_2 d \quad {}_g &= \frac{1}{2} \int \sqrt{\det G} \langle \nabla_g f_1, \nabla_g (\log \det G) \rangle_g f_2 d \\
&\quad + \int_\Gamma f_2 \frac{\partial f_1}{\partial \upsilon_{\mathcal{A}}} \sqrt{\det G} d\Gamma - \int \left\langle \nabla_g f_1, \nabla_g \left(\sqrt{\det G} f_2 \right) \right\rangle_g d \quad ,
\end{aligned}
$$

but $\dfrac{\partial f_1}{\partial \upsilon_{\mathcal{A}}} = \|A\upsilon\|_g \dfrac{\partial f_1}{\partial n}$ then

$$
\begin{aligned}
\int \Delta_g f_1 f_2 d \quad {}_g &= \frac{1}{2} \int \sqrt{\det G} \langle \nabla_g f_1, \nabla_g (\log \det G) \rangle_g f_2 d \\
&\quad + \int_\Gamma \frac{\partial f_1}{\partial n} f_2 \left(\|A\upsilon\|_g \sqrt{\det G} d\Gamma \right) \\
&\quad - \int \left\langle \nabla_g f_1, \nabla_g \left(\sqrt{\det G} f_2 \right) \right\rangle_g d \quad ,
\end{aligned}
$$

so

$$
\begin{aligned}
\int \Delta_g f_1 f_2 d \quad {}_g &= \frac{1}{2} \int \sqrt{\det G} \langle \nabla_g f_1, \nabla_g (\log \det G) \rangle_g f_2 d \\
&\quad + \int_\Gamma \frac{\partial f_1}{\partial n} f_2 d\Gamma_g - \int \langle \nabla_g f_1, \nabla_g f_2 \rangle_g d \quad {}_g \\
&\quad - \int \left\langle \nabla_g f_1, \nabla_g \sqrt{\det G} \right\rangle_g f_2 d \quad ,
\end{aligned}
$$

but

$$
\frac{1}{2} \int \sqrt{\det G} \langle \nabla_g f_1, \nabla_g (\log \det G) \rangle_g f_2 d \quad - \int \left\langle \nabla_g f_1, \nabla_g \sqrt{\det G} \right\rangle_g f_2 d \quad = 0,
$$

This complete the proof. ■

1.3 *Riemannian geometry on* $\mathbb{C}^n$.

First, it is important to mention that this section was first considered in $[HR08]$.

Using the inner product $\langle .,. \rangle_g$ on $\mathbb{R}^n$ defined in subsection 1.2.1, we can define an inner product on $\mathbb{C}^n = \mathbb{C}^n_x$ (we take the same symbol g) by for all Z_1, $Z_2 \in \mathbb{C}^n$

$$\begin{aligned}\langle Z_1, Z_2 \rangle_g &= \langle ReZ_1, ReZ_2 \rangle_g + \langle \operatorname{Im} Z_1, \operatorname{Im} Z_2 \rangle_g \\ &\quad -\mathrm{i}\left(\langle ReZ_1, \operatorname{Im} Z_2 \rangle_g - \langle ImZ_1, \operatorname{Re} Z_2 \rangle_g \right),\end{aligned}$$

so the norm is

$$\|Z\|_g^2 = \langle Z, Z \rangle_g = \|ReZ\|_g^2 + \|\operatorname{Im} Z\|_g^2 ,$$

for all $Z \in \mathbb{C}^n$.

Notation.

Let f be a complex valued function and h be a vector field on $(\mathbb{R}^n, g)$. We note

$$h(f) := h(\operatorname{Re} f) + \mathrm{i}h(\operatorname{Im} f),$$

$$\nabla_g f := \nabla_g \operatorname{Re} f + \mathrm{i}\nabla_g \operatorname{Im} f,$$

and

$$div_0 f := div_0 (\operatorname{Re} f) + \mathrm{i}div_0 (\operatorname{Im} f).$$

We give the counterpart of the first Green's formula, identity (1.2) and (1.3) for the complex valued functions.

Lemma 1.19 *Let f_1, f_2 be a complex valued functions in $H^2(\ )$. Then*

1. *Third Green's formula*

$$\int (\mathcal{A}f_1)\,\overline{f_2}d \quad = \int \langle \nabla_g f_1, \nabla_g f_2\rangle_g \, d \quad - \int_\Gamma \frac{\partial f_1}{\partial \upsilon_\mathcal{A}}\overline{f_2}d \quad .$$

2.

$$\int f_1 h\left(\overline{f_2}\right) d \quad = \int_\Gamma h.\upsilon f_1 \overline{f_2} d\Gamma - \int \overline{f_2} div_0\,(f_1 h)\, d \quad . \tag{1.5}$$

Moreover, if f is a complex valued function of $C^1\left(\overline{\ }\right)$ and h is a vector field on $\mathbb{R}^n$, then

$$\begin{aligned} \operatorname{Re}\langle \nabla_g f, \nabla_g (h(f))\rangle_g \quad &= \quad Dh\,(\nabla_g \operatorname{Re} f, \nabla_g \operatorname{Re} f) \\ &\quad +Dh\,(\nabla_g \operatorname{Im} f, \nabla_g \operatorname{Im} f) + \frac{1}{2}h\left(\|\nabla_g f\|_g^2\right), \end{aligned} \tag{1.6}$$

Proof.

1. We have

$$\begin{aligned} \int (\mathcal{A}f_1)\,\overline{f_2}d \quad &= \quad \int \mathcal{A}\,(\operatorname{Re} f_1 + \mathrm{i}\operatorname{Im} f_1)\,(\operatorname{Re} f_2 - \mathrm{i}\operatorname{Im} f_2)\,d \\ &= \quad \int \mathcal{A}\,(\operatorname{Re} f_1)\operatorname{Re} f_2 d \quad + \int \mathcal{A}\,(\operatorname{Im} f_1)\operatorname{Im} f_2 d \\ &\quad +\mathrm{i}\left(\int \mathcal{A}\,(\operatorname{Im} f_1)\operatorname{Re} f_2 d \quad - \int \mathcal{A}\,(\operatorname{Re} f_1)\operatorname{Im} f_2 d \quad\right), \end{aligned}$$

using the first Green's formula concerned with the real valued functions, we find

$$\int \mathcal{A} f_1 \overline{f_2} d \quad = \int \langle \nabla_g \operatorname{Re} f_1, \nabla_g \operatorname{Re} f_2 \rangle_g \, d \quad + \int \langle \nabla_g \operatorname{Im} f_1, \nabla_g \operatorname{Im} f_2 \rangle_g \, d$$

$$+\mathrm{i} \int \left(\langle \nabla_g \operatorname{Im} f_1, \nabla_g \operatorname{Re} f_2 \rangle_g \, d \quad - \langle \nabla_g \operatorname{Re} f_1, \nabla_g \operatorname{Im} f_2 \rangle_g \right) d$$

$$- \int_\Gamma \left(\frac{\partial \operatorname{Re} f_1}{\partial \upsilon_{\mathcal{A}}} \operatorname{Re} f_2 + \frac{\partial \operatorname{Im} f_1}{\partial \upsilon_{\mathcal{A}}} \operatorname{Im} f_2 \right) d\Gamma$$

$$-\mathrm{i} \int_\Gamma \left(\frac{\partial \operatorname{Im} f_1}{\partial \upsilon_{\mathcal{A}}} \operatorname{Re} f_2 - \frac{\partial \operatorname{Re} f_1}{\partial \upsilon_{\mathcal{A}}} \operatorname{Im} f_2 \right) d\Gamma,$$

so

$$\int \mathcal{A} f_1 \overline{f_2} \quad = \int \langle \operatorname{Re}(\nabla_g f_1), \operatorname{Re}(\nabla_g f_2) \rangle_g \, d \quad + \int \langle \operatorname{Im}(\nabla_g f_1), \operatorname{Im}(\nabla_g f_2) \rangle_g \, d$$

$$+\mathrm{i} \int \left(\langle \operatorname{Im}(\nabla_g f_1), \operatorname{Re}(\nabla_g f_2) \rangle_g - \langle \operatorname{Re}(\nabla_g f_1), \operatorname{Im}(\nabla_g f_2) \rangle_g \right) d$$

$$- \int_\Gamma \left(\operatorname{Re}\left(\frac{\partial f_1}{\partial \upsilon_{\mathcal{A}}} \right) \operatorname{Re} f_2 - \operatorname{Im}\left(\frac{\partial f_1}{\partial \upsilon_{\mathcal{A}}} \right) \operatorname{Im} \overline{f_2} \right) d\Gamma$$

$$-\mathrm{i} \int_\Gamma \left(\operatorname{Im}\left(\frac{\partial f_1}{\partial \upsilon_{\mathcal{A}}} \right) \operatorname{Re} f_2 + \operatorname{Re}\left(\frac{\partial f_1}{\partial \upsilon_{\mathcal{A}}} \right) \operatorname{Im} \overline{f_2} \right) d\Gamma,$$

which implies the desired formula.

2. We have

$$\int f_1 h\left(\overline{f_2}\right) d \quad = \int (\operatorname{Re} f_1 + \mathrm{i} \operatorname{Im} f_1)\left(h\left(\operatorname{Re} f_2\right) - \mathrm{i} h\left(\operatorname{Im} f_2\right)\right) d$$

$$= \int \operatorname{Re} f_1 h(\operatorname{Re} f_2) \, d \quad + \int \operatorname{Im} f_1 h\left(\operatorname{Im} f_2\right) d$$

$$+\mathrm{i} \left(\int \operatorname{Im} f_1 h(\operatorname{Re} f_2) \, d \quad - \int \operatorname{Re} f_1 h\left(\operatorname{Im} f_2\right) d \right).$$

The formula (1.3) concerned with the real valued function gives

$$\begin{aligned}\int f_1 h\left(\overline{f_2}\right) d \quad &= \int_\Gamma h.\upsilon \operatorname{Re} f_1 \operatorname{Re} f_2 d\Gamma - \int \operatorname{Re} f_2 div_0 \left(\operatorname{Re} f_1 h\right) d \\ &-\mathrm{i}\int_\Gamma h.\upsilon \operatorname{Re} f_1 \operatorname{Im} f_2 d\Gamma + \mathrm{i}\int \operatorname{Im} f_2 div_0 \left(\operatorname{Re} f_1 h\right) d \\ &+\mathrm{i}\int_\Gamma h.\upsilon \operatorname{Im} f_1 \operatorname{Re} f_2 d\Gamma - \mathrm{i}\int \operatorname{Re} f_2 div_0 \left(\operatorname{Im} f_1 h\right) d \\ &+\int_\Gamma h.\upsilon \operatorname{Im} f_1 \operatorname{Im} f_2 d\Gamma - \int \operatorname{Im} f_2 div_0 \left(\operatorname{Im} f_1 h\right) d \quad ,\end{aligned}$$

so

$$\int f_1 h\left(\overline{f_2}\right) d \quad = \int_\Gamma h.\upsilon f_1 \overline{f_2} d\Gamma - \int \overline{f_2} div_0 \left(f_1 h\right) d \quad .$$

2. It is sufficient to see that

$$\operatorname{Re} \langle \nabla_g f, \nabla_g (h(f))\rangle_g = \langle \nabla_g \operatorname{Re} f, \nabla_g (h(\operatorname{Re} f))\rangle_g + \langle \nabla_g \operatorname{Im} f, \nabla_g (h(\operatorname{Im} f))\rangle_g ,$$

so, (1.6) is obtained by (1.2).

■

Remark 1.20 *If f is a complex valued function in $H^1(\ )$ then*

$$\begin{aligned} & a_0 \int \left(\|\operatorname{Re} \nabla_0 f\|_0^2 + \|\operatorname{Im} \nabla_0 f\|_0^2\right) d \\ \leq \; & \int \|\nabla_g f\|_g^2 d \quad = \int \|\operatorname{Re} \nabla_g f\|_g^2 d \quad + \int \|\operatorname{Im} \nabla_g f\|_g^2 d \\ \leq \; & a_1 \int \left(\|\operatorname{Re} \nabla_0 f\|_0^2 + \|\operatorname{Im} \nabla_0 f\|_0^2\right) d \quad ,\end{aligned}$$

so

$$a_0 \int \|\nabla_0 f\|_0^2 d \quad \leq \int \|\nabla_g f\|_g^2 d \quad \leq a_1 \int \|\nabla_0 f\|_0^2 d \quad .$$

Moreover, if $f \in H^1_{\Gamma_0}(\ )$ *then*

$$\int |f|^2 \, d \quad \leq C_1^2 \int \|\nabla_g f\|_g^2 \, d \quad ,$$

and

$$\int_{\Gamma_1} |f|^2 \, d\Gamma \leq C_2^2 \int \|\nabla_g f\|_g^2 \, d \quad .$$

1.4 *The theory of semigroups of linear operators.*

The problem of existence, uniqueness and regularity of solution is a critical preliminary step in studying questions related to uniform decay rate estimate. For this reason, we present, in this section, some notions and results in the theory of semigroups of bounded operators. This theory is one of the useful tools in the resolution of certain classes of partial differential equations. We will use this method in chapter 2, chapter 4, chapter 5 and chapter 6 of this thesis.

It is mentioned that all definitions and results are standard and classical in the literature (see $[Paz83, DL85, Bre87]$).

Let H be a Hilbert space, $(.,.)_H$ and $\|.\|_H$ are the inner product and norm on H.

Definition 1.21 *The family* $(T(t))_{t\geq 0}$ *of bounded linear operators from* H *into* H *is a* C_0-*semigroup on* H *if*

1. $T(0) = I$ *(*I *is the identity operator on* H*).*

2. $T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$.

3. $\lim_{t\to 0^+} T(t)\, Y_0 = Y_0$, *for all* $Y_0 \in H$.

Definition 1.22 *Let* $(T(t))_{t\geq 0}$ *be a* C_0-*semigroup on* H. *The operator* $\mathfrak{A}$ *defined by*

$$D(\mathfrak{A}) = \left\{ x \in H : \lim_{t\to 0^+} \frac{T(t)\, x - x}{t} \text{ exists} \right\}$$

and

$$\mathfrak{A}x = \lim_{t\to 0^+} \frac{T(t)\, x - x}{t} \text{ for all } x \in D(\mathfrak{A}),$$

is the infinitesimal generator of the C_0-*semigroup* $(T(t))_{t\geq 0}$.

In this case we note

$$T(t) = \exp(t\mathfrak{A}),$$

for all $t \geq 0$.

Definition 1.23 *Let* $\mathfrak{A}$ *be a linear operator on a Hilbert space* H *and* $D(\mathfrak{A})$ *its domain.*

$\mathfrak{A}$ *is a dissipative operator on* H *if*

$$\operatorname{Re}(\mathfrak{A}x, x)_H \leq 0,$$

for all $x \in D(\mathfrak{A})$.

If moreover

$$(I - \mathfrak{A})\, D(\mathfrak{A}) = H,$$

then $\mathfrak{A}$ *is a maximal dissipative operator on* H.

Theorem 1.24 ([DL85]) *Let* $\mathfrak{A}$ *be a linear operator with dense domain* $D(\mathfrak{A})$ *in* H.

The following properties are equivalent.

1. $\mathfrak{A}$ *is the infinitesimal generater of a* C_0-*semigroup of contraction.*

2. $\mathfrak{A}$ *is a maximal dissipative operator on* H.

Cauchy's problem.

Given $Y_0 \in H$, the Cauchy problem for $\mathfrak{A}$ with initial data Y_0 consists to finding a solution Y of the system

$$\begin{cases} Y_t = \mathfrak{A}Y, \\ Y(0) = Y_0. \end{cases} \tag{1.7}$$

Then, we have

Theorem 1.25 ($[DL85, Bre87]$) *If* $\mathfrak{A}$ *is maximal dissipative operator on* H. *Then*

1. *For each* $Y_0 \in H$ *the system (1.7) has a unique solution*

$$Y \in C([0, +\infty); H).$$

2. *Let* $N \geq 1$. *For each* $Y_0 \in D\left(\mathfrak{A}^N\right)$ *the system (1.7) has a unique solution*

$$Y \in C^{N-j}([0, +\infty); D\left(\mathfrak{A}^j\right)),$$

for $j = 0, ..., N$.

Here

$$D\left(\mathfrak{A}^0\right) = H,\ D\left(\mathfrak{A}^1\right) = D\left(\mathfrak{A}\right)$$

and

$$D\left(\mathfrak{A}^N\right) = \left\{Y_0 \in D\left(\mathfrak{A}^{N-1}\right) : \mathfrak{A}Y_0 \in D\left(\mathfrak{A}^{N-1}\right)\right\},$$

for $N \geq 2$.

1.5 *Some abstract stabilization inequalities.*

Let H be a Hilbert space, $(.,.)_H$ and $\|.\|_H$ are the inner product and norm on H.

Consider a function $E : [0,+\infty[\times H \rightarrow [0,+\infty[$. If $Y_0 \in H$, then we put $E(t) = E(t, Y_0)$ for all $t \geq 0$.

We give the notions of uniform, exponential and polynomial decay rate of the function E.

Definition 1.26 ($[LagI83, Las99]$) *E decays to zero at an uniform rate (or simply E decays at an uniform rate) if there exist $T_0 > 0$, a constant M which is independent of t, and a function f such that*

$$\lim_{t \to +\infty} f(t) = 0 \text{ and } E(t) \leq ME(0) f(t),$$

for all $t \geq T_0$ and $Y_0 \in H$.

In this case, f is called the decay rate of the function E.

Remark 1.27 *In the application, E represents the energy of the system.*

Remark 1.28 *If $f(t) = e^{-\omega t}$ for some positive constant $\omega > 0$ then we have an exponential decay rate of E. Moreover, if $f(t) = t^{-n}$ for some $n \in \mathbb{N}^*$ then we have a polynomial decay rate of E.*

1.5.1 Datko's integral inequalities.

We give now the extend of a well know theorem of A. M. Liapunov. This result permit us to obtain the exponential decay rate of the energy of certain partial differential equations.

Theorem 1.29 ([$Dat70$]) *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on H.*

A necessary and sufficient condition that a C_0-semigroups $(T(t))_{t\geq 0}$ defined on H satisfy, for some positive constants c and ω, the condition

$$\|T(t)\| \leq \sqrt{c}e^{-\frac{\omega}{2}t},$$

for all $t \geq 0$, is that, for each Y_0 in H, the integral

$$\int_0^\infty \|T(t)\,Y_0\|_H^2\,dt$$

be convergent.

In this case, $(T(t))_{t\geq 0}$ is said to be an exponential stable semigroup.

Remark 1.30 *Let $Y_0 \in H$. We consider $E(t) = E(t, Y_0) = \|T(t)\,Y_0\|_H^2$ for all $t \geq 0$.*

If $\int_0^\infty E(t)\,dt$ is convergent for all $Y_0 \in H$, then E has an exponential decay rate. Indeed, we have

$$\int_0^\infty \|T(t)\,Y_0\|_H^2\,dt = \int_0^\infty E(t)\,dt$$

is convergent.

Then

$$\begin{aligned} E(t) &= \|T(t)\,Y_0\|_H^2\,dt \leq \|T(t)\|^2\,\|Y_0\|_H^2 = \|T(t)\|^2\,\|T(0)\,Y_0\|_H^2 \\ &= \|T(t)\|^2\,E(0) \leq cE(0)\,e^{-\omega t}. \end{aligned}$$

That is

$$E(t) \leq cE(0)\,e^{-\omega t},$$

for all $t \geq 0$.

1.5.2 Lasiecka's and Tataru's inequalities.

We present here a very general method due to I Lasiecka and D. Tataru which specifies the estimates which are necessary to obtain an uniform decay rate of the energy of certain partial differential equations, under appropriate conditions imposed on the nonlinear damping.

Let p be a given nonlinear function which is positive, strictly increasing of class C^1 and zero at the origin, and $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function.

Theorem 1.31 ([$LT93$]) *If for all T and S such that $T - S \geq T_0$, for some $T_0 > 0$, we have*

$$p\left(E\left(T\right)\right) + E\left(T\right) \leq E\left(S\right).$$

Then E decays faster than the solution of an appropriate ordinary differential equation. That is

$$E\left(t\right) \leq S\left(\frac{t}{T_0} - 1\right) \text{ for } t > T_0,$$

where the function S is a solution of the following system

$$\begin{cases} S_t\left(t\right) + S\left(t\right) - \left(I + p\right)^{-1}\left(S\left(t\right)\right) = 0, \\ S\left(0\right) = E\left(0\right). \end{cases} \tag{1.8}$$

Moreover, if $p(s) > 0$ for $s > 0$, then

$$\lim_{t \to \infty} E\left(t\right) = 0$$

The proof of this theorem is based on the following result.

Lemma 1.32 ([$LT93$]) *If (s_m) is a sequence of positive numbers such that*

$$p\left(s_{m+1}\right) + s_{m+1} \leq s_m,$$

then

$$s_m \leq S(m),$$

where $S(t)$ is solution of system (1.8) with $S(0) = s_0$.

Remark 1.33 *In the application, the function p depends on the parameters in the equation, in particular, on the behavior of nonlinear damping at the origin.*

1.5.3 Alabau's generalized integral inequalities.

We give now a generalized integral inequalities due to F. Alabau that will be useful in what follows for obtaining polynomial decay rate for the energy of the smooth solutions of coupled equations when only one of the equations is stabilized.

Let $\mathfrak{A}$ be the infinitesimal generator of a C_0-semigroup $(\exp(t\mathfrak{A}))_{t\geq 0}$ on H and $D(\mathfrak{A})$ its domain, and let $Y(t) = \exp(t\mathfrak{A})Y_0$, for all $Y_0 \in H$.

Theorem 1.34 ([$ACK02$]) *Let $E : H \to [0, +\infty[$ be a decreasing and continuous function.*

If there exist a positive integer K and a nonnegative constant C such that

$$\int_0^T E(Y(t)) \leq C \sum_{l=0}^{K} E\left(Y^{(l)}(0)\right),$$

for all $T \geq 0$ and for all $Y_0 \in D\left(\mathfrak{A}^K\right)$.

Then for every positive integer N, we have, for a certain constant C_N depending on N,

$$E(Y(t)) \leq C_N \left(\sum_{l=0}^{KN} E\left(Y^{(l)}(0)\right)\right) t^{-N},$$

for all $t > 0$ *and* $Y_0 \in D\left(\mathfrak{A}^{KN}\right)$.

Moreover, we have

$$\lim_{t\to\infty} E\left(Y\left(t\right)\right) = 0,$$

for all $Y_0 \in H$.

Remark 1.35 *In the application, we take*

$$E\left(Y\left(t\right)\right) = \left\|Y\left(t\right)\right\|_H^2,$$

for all $t \geq 0$ *and* $Y_0 \in H$.

Chapter 2

Exponential decay rate of the second order hyperbolic equation with zero order term by the energy perturbed method.

2.1 *Introduction.*

In this chapter, we consider the system

$$\begin{cases} y_{tt} + \mathcal{A}y + qy = 0 \text{ in } Q, \\ y = 0 \text{ on } \Sigma_0, \\ \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} + by_t = 0 \text{ on } \Sigma_1, \\ y(0) = y_0,\, y_t(0) = y_1 \text{ in } \quad . \end{cases} \tag{2.1}$$

Here $b : \Gamma_1 \to \mathbb{R}$ and $q : \quad \to \mathbb{R}$ are two positive bounded functions, that is there exists four positive constants b_*, b^*, q_*, q^* such that

$$0 < b_* \leq b(s) \leq b^*, \tag{2.2}$$

for all $s \in \Gamma_1$, and

$$0 < q_* \leq q(s) \leq q^*$$

for all $s \in \quad$.

We define the energy of the system (2.1) by

$$E(t) = \frac{1}{2} \int \left(|y_t|^2 + q\,|y|^2 + \|\nabla_g y\|_g^2 \right) d \quad ,$$

for all $t \geq 0$.

For the constant case when $\mathcal{A} = -\Delta$, and $q = 0$, V. Komornick and E. Zuazua in $[KZ90]$ have shown that the energy E decays exponentially, that is for some positive constants c and ω, we have

$$E(t) \leq cE(0)\, e^{-\omega t}, \tag{2.3}$$

for all $t \geq T_0$.

They have used a method based on the construction of energy functionals E^{ε} defined, for all $\varepsilon > 0$, by

$$E^{\varepsilon}(t) = E(t) + \varepsilon\rho(t),$$

with

$$\rho(t) = \int y_t My d \quad ,$$

and

$$My = 2h(y) + (div_0 h - m_0)\, y,$$

for some given vector h and positive constant m_0.

So, with a convenable choice of ε, (2.3) is obtained from the exponential decay rate of the energy perturbed E^{ε}. Indeed, assume that there exist two positive constants $\acute{c}, \acute{\omega} > 0$ such that

$$E^{\varepsilon}(t) \leq \acute{c} E^{\varepsilon}(0)\, e^{-\acute{\omega} t}, \tag{2.4}$$

for all $t \geq T_0$.

It is easy to see, from the definition of E^{ε}, that

$$(1 - \varepsilon\theta)\, E(t) \leq E^{\varepsilon}(t) \leq (1 + \varepsilon\theta)\, E(t), \tag{2.5}$$

where θ is a constant verifying

$$|\rho(t)| = \left| \int y_t My d \quad \right| \leq \theta E(t).$$

Combining (2.4) and (2.5) with ε sufficiently small, we find

$$
\begin{aligned}
E(t) &\leq \frac{1}{1-\varepsilon\theta}E^{\varepsilon}(t) \leq \frac{\acute{c}}{1-\varepsilon\theta}E^{\varepsilon}(0)\, e^{-\acute{\omega}t} \\
&\leq \frac{\acute{c}(1+\varepsilon\theta)}{1-\varepsilon\theta}E(0)\, e^{-\acute{\omega}t}.
\end{aligned}
$$

so, (2.3) is verified with

$$
c = \frac{\acute{c}(1+\varepsilon\theta)}{1-\varepsilon\theta} \text{ and } \omega = \acute{\omega}.
$$

When $q \neq 0$, there is some difficulty to obtain the exponential decay rate of the energy perturbed E^{ε}, since we have a lower order term with respect to the energy in some multiplier estimate (see remark 3.2 in $[KZ90]$). The purpose of this chapter is to overcome this kind of difficulty. Here we use the Riemannian geometry method to handle the case of variable coefficients then we use the compactness uniqueness argument to absorb the lower order term with respect to the energy. Finally, we employ some results of I. Lasiecka and D. Tataru.

This chapter is organized as follows. In section 2.2, we give some examples, where the geometric assumption is illustrated. In section 2.3, we discuss the well posedeness of the system (2.1). Section 2.4 is devoted to the proof of (2.4) for smooth solutions. Finally, we deduce the exponential decay rate of E for general solutions.

2.2 *Geometric assumptions and examples.*

Assume that there exist a function $d : \overline{} \to \mathbb{R}^{+}$ of class C^3 verifying

$$
\inf \|\nabla_g d\|_g > 0, \tag{2.6}
$$

and, for some constant $m_0 > 0$, we have

$$D^2 d(X, X) \geq m_0 \|X\|_g^2 , \tag{2.7}$$

for all $X \in \mathbb{R}_x^n$.

Moreover, if $h = \nabla_g d$, we take

$$\Gamma_0 = \{x \in \Gamma : h.\upsilon \leq 0\} , \tag{2.8}$$

and

$$\Gamma_1 = \{x \in \Gamma : h.\upsilon \geq h_0\} , \tag{2.9}$$

for some constant $h_0 > 0$.

Let us give some examples of functions d satisfying (2.6) and (2.7).

Example 2.1 *For the classical Euclidean metric where $a_{ij} = \delta_{ij}$, we may take*

$$d = \frac{1}{2} \|x - x_0\|_0^2 ,$$

with x_0 outside $\overline{\ }$.

Example 2.2 *If $A = (a_{ij})$ is positive definite, symmetric and constant matrix. Put*

$$d(x_1, x_2, ..., x_n) = \sum_{i,j=1}^{n} g_{ij} \left(x_i - x_i^0\right) \left(x_j - x_j^0\right) ,$$

where $x^0 = \left(x_1^0, x_2^0, ..., x_n^0\right)$ outside $\overline{\ }$.

Then (2.7) is verified (see example 3.1 in [Yao99]*). Moreover, we have for some positive constant c*

$$\begin{aligned} \inf \|\nabla_g d\|_g^2 &= \inf \left(\sum_{i,j=1}^{n} g_{ij} \left(x_i - x_i^0\right) \left(x_j - x_j^0\right) \right) \\ &= \inf \left(\left(x - x^0\right)^T G \left(x - x^0\right) \right) \geq c \inf \left\|x - x^0\right\|_0^2 > 0. \end{aligned}$$

Then (2.6) is verified.

Example 2.3 ([*Ham*08]) *In* [*FF*02] *the authors have proved that the geometric condition (2.7), derived in term of the Riemannian geometry method (see remark 2.4 below), is equivalent with the following analytical condition given by A. Wyler in* [*Wyl*94] *for the boundary stabilization of wave equations with variable coefficients*

$$\begin{cases} (p_{ij}) \text{ is uniformly positive definite matrix in } \quad \text{, where} \\ p_{ij} = p_{ji} = \sum_{k=1}^{k=n} a_{ik} \frac{\partial h_j}{\partial x_k} + \sum_{k=1}^{k=n} a_{jk} \frac{\partial h_i}{\partial x_k} - \nabla_0 a_{ij}.h. \end{cases}$$

Here $h = Dd$.

If (a_{ij}) *is the matrix defined by*

$$a_{ij}(x_1, x_2, ..., x_n) = \begin{cases} 0 : i \neq j, \\ f_i(x_i) : i = j, \end{cases}$$

where, for all $i = 1, ..., n$, $f_i : \mathbb{R} \to \mathbb{R}$ *is a function of class* C^3 *satisfying the condition*

$$\inf f_i > 0.$$

Let $d : \overline{} \to \mathbb{R}^+$ *be a function of class* C^3 *verifying*

$$\inf \sum_{i=1}^{n} \left(\frac{\partial d}{\partial x_i} \right)^2 > 0,$$

$$\inf \left(\frac{\partial f_i}{\partial x_i} \frac{\partial d}{\partial x_i} + 2 f_i \frac{\partial^2 d}{\partial x_i^2} \right) > 0, \text{ for all } i,$$

$$\frac{\partial^2 d}{\partial x_i \partial x_j} = 0, \qquad i \neq j.$$

If

$$h = \nabla_g d = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i},$$

where $h_i = f_i \frac{\partial d}{\partial x_i}$.

Then the matrix (p_{ij}) *defined by*

$$
\begin{aligned}
p_{ij} &= p_{ji} = \sum_{k=1}^{k=n} a_{ik}\frac{\partial h_j}{\partial x_k} + \sum_{k=1}^{k=n} a_{jk}\frac{\partial h_i}{\partial x_k} - \nabla_0 a_{ij}.h \\
&= \left\{ \begin{array}{ll} f_i\left(\frac{\partial f_i}{\partial x_i}\frac{\partial d}{\partial x_i} + 2f_i\frac{\partial^2 d}{\partial x_i^2}\right) : i = j, & \\ 2f_i f_j \frac{\partial^2 d}{\partial x_i \partial x_j} = 0, & i \neq j, \end{array} \right.
\end{aligned}
$$

is an uniformly positive definite matrix in . *So the geometric condition (2.7) is verified. On the other hand, we have*

$$inf \, \|\nabla_g d\|_g^2 = inf \sum_{i=1}^{n} f_i \left(\frac{\partial d}{\partial x_i}\right)^2 > 0.$$

Thus, (2.6) is verified.

As an example of such a function and matrix we may take

$$d(x_1, x_2, ..., x_n) = \frac{1}{2}\sum_{i=1}^{n}\left(x_i - x_i^0\right)^2,$$

where $x^0 = \left(x_1^0, x_2^0, ..., x_n^0\right)$ *outside* $\overline{\ }$, *and*

$$f_i(x_i) = \left(x_i - x_i^0\right)^{2\lambda},$$

where $\lambda \in \mathbb{N}^*$.

Remark 2.4 *Geometric condition (2.7) is used in [LTY99] to obtain the exact controllability for second order hyperbolic equation with variable coefficients principal part.*

Remark 2.5 *Assumptions (2.6) and (2.7) are needed to have the uniqueness result.*

If z is the solution of the system

$$\begin{cases} z_{tt} + \mathcal{A}z + qz = 0 \ \text{in } Q, \\ z = 0 \ \text{on } \Sigma, \\ \frac{\partial z}{\partial v_{\mathcal{A}}} = 0 \ \text{on } \Sigma_1, \end{cases} \tag{2.10}$$

then $z = 0$.

Indeed, by lemma 1.17, the system (2.10) is equivalent to

$$\begin{cases} z_{tt} - \Delta_g z + \frac{1}{2} \langle D(\log \det G), Dz \rangle_g + qz = 0 \ \text{in } Q, \\ z = 0 \ \text{on } \Sigma_0, \\ \frac{\partial z}{\partial n} = 0 \ \text{on } \Sigma_1. \end{cases}$$

From Theorem 8.1 in [TY02], we have

$$z = 0,$$

for all $T > T_0$, where T_0 is sufficiently large.

This uniqueness result is needed to absorb the lower order term from certain multiplier estimates below.

Remark 2.6 *Since*

$$q(s) \geq q_* > 0,$$

for all $s \in$, then we can take $\Gamma_0 = \varnothing$.

2.3 *Existence, uniqueness and regularity of solution.*

Let us introduce the linear operators $\mathbb{A}, \mathbb{B}, \mathbb{F} : V = H^1_{\Gamma_0}(\ \ ) \to V^{'}$ by

$$(\mathbb{A}u, w)_{V',V} = \int \langle \nabla_g u, \nabla_g w \rangle_g \, d\ \ ,$$

$$(\mathbb{B}u, w)_{V',V} = \int_{\Gamma_1} buw d\Gamma,$$

and

$$(\mathbb{F}u, w)_{V',V} = \int quw d\ \ .$$

Here, we have equipped $H^1_{\Gamma_0}(\ \ )$ by the following inner product

$$(u, w)_{H^1_{\Gamma_0}(\ )} = \int \left(quw + \langle \nabla_g u, \nabla_g w \rangle_g \right) d\ \ ,$$

for all $u, w \in H^1_{\Gamma_0}(\ \ )$.

If $Y = (y, z) = (y, y_t)$, we define an operator $\mathfrak{A}$ as

$$\mathfrak{A}(y, z) = (z, -\mathbb{A}y - \mathbb{B}z - \mathbb{F}y),$$

with domain

$$D(\mathfrak{A}) = \left\{ \begin{array}{c} (u, w) \in \left(H^2(\ \ ) \cap H^1_{\Gamma_0}(\ \ ) \right) \times H^1_{\Gamma_0}(\ \ ) : \\ \frac{\partial u}{\partial \upsilon_{\mathcal{A}}} = -bw \text{ on } \Gamma_1 \end{array} \right\},$$

then we may interprets the system (2.1) in the following form

$$\left\{ \begin{array}{c} Y_t = \mathfrak{A}Y, \\ Y(0) = Y_0 = (y_0, y_1). \end{array} \right. \quad (2.11)$$

Theorem 2.7 *For every*

$$Y_0 = (y_0, y_1) \in H^1_{\Gamma_0}(\ ) \times L^2(\ ),$$

the system (2.11) (or equivalently the system (2.1)) has a unique solution such that

$$Y = (y, y_t) \in C([0, +\infty[; H^1_{\Gamma_0}(\ ) \times L^2(\ )).$$

If

$$Y_0 = (y_0, y_1) \in D(\mathfrak{A}),$$

the system (2.1) has a unique solution such that

$$(y, y_t) \in C([0, +\infty[; D(\mathfrak{A})) \cap C^1([0, +\infty[; H^1_{\Gamma_0}(\ ) \times L^2(\ )).$$

Proof. It is sufficient to show that $\mathfrak{A}$ is a maximal dissipative operator on $H^1_{\Gamma_0}(\ ) \times L^2(\ )$.

For all $Y = (u, w) \in D(\mathfrak{A})$, we have

$$\begin{aligned}(\mathfrak{A}Y, Y)_{H^1_{\Gamma_0}(\ )\times L^2(\ )} &= ((w, -\mathbb{A}u - \mathbb{B}w - \mathbb{F}u), (u, w))_{H^1_{\Gamma_0}(\ )\times L^2(\ )} \\ &= (w, u)_{H^1_{\Gamma_0}(\ )} - (\mathbb{A}u + \mathbb{B}w + \mathbb{F}u, w)_{L^2(\ )},\end{aligned}$$

using the definitions of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{F}$, we obtain

$$(\mathfrak{A}Y, Y)_{H^1_{\Gamma_0}(\ )\times L^2(\ )} = -\int_{\Gamma_1} b\,|w|^2\, d\Gamma \leq 0.$$

Now, we show that

$$(I - \mathfrak{A})\, D(\mathfrak{A}) = H^1_{\Gamma_0}(\ ) \times L^2(\ ). \tag{2.12}$$

Let $(v_1, v_2) \in H^1_{\Gamma_0}(\ ) \times L^2(\ )$. The system

$$\begin{cases} u_1 - u_2 = v_1 \\ u_2 + \mathbb{A}u_1 + \mathbb{B}u_2 + \mathbb{F}u_1 = v_2 \end{cases}$$

is equivalent to

$$u_2 = u_1 - v_1 \tag{2.13}$$

$$u_1 + \mathbb{A}u_1 + \mathbb{B}u_1 + \mathbb{F}u_1 = v_1 + v_2 + \mathbb{B}v_1. \tag{2.14}$$

But, by the elliptic theory, we can find the solution u_1 of (2.14). Next, by replacing in (2.13) we find u_2. So, (2.12) is obtained. ■

2.4 *Exponential decay rate of* E^ε *for* $(y_0, y_1) \in D(\mathfrak{A})$.

In this section, we prove (2.4) in the case where (y_0, y_1) belongs to $D(\mathfrak{A})$. Thus, we will assume that solution y of (2.1) satisfies $(y, y_t) \in C([0, +\infty[\,; D(\mathfrak{A})) \cap C^1([0, +\infty[\,; H^1_{\Gamma_0}(\ ) \times L^2(\ ))$. This justify all computations that follow.

Using the multiplier method developed for example in $[Lio88, Kom94]$ we can show that the energy E of the system (2.1) is a decreasing function.

Lemma 2.8 *We have*

$$E_t(t) = \frac{dE}{dt} = -\int_{\Gamma_1} b\,|y_t|^2\,d\Gamma \leq 0, \tag{2.15}$$

for all $t \geq 0$.

Proof. If we multiply both sides the first equation of the system (2.1) by y_t, integrate over , use first Green's formula and the identity

$$\int y_{tt} y_t d \quad + \int q y y_t d \quad + \int \langle \nabla_g y, \nabla_g y_t \rangle_g \, d \quad = E_t(t),$$

we find the desired result. ■

Remark 2.9 *(2.15) is called the energy dissipation law.*

The proof of theorem 2.12 below involves two lemmas.

Lemma 2.10 *Let T and S be two positive constants such that $T - S > T_0$, for some sufficiently large positive constant T_0, we have*

1.

$$\frac{m_0 \varepsilon T_0}{2(1+\varepsilon\theta)} E^{\varepsilon}(T) \leq E^{\varepsilon}(S) + \varepsilon C \left(\int_S^T \int_{\Gamma_1} b |y_t|^2 \, d\Sigma + \int_S^T \int |y|^2 \, dQ \right), \tag{2.16}$$

for ε sufficiently small.

Here $dQ = d \quad dt$ and $d\Sigma = d\Gamma \, dt$.

2. *Moreover,*

$$E^{\varepsilon}(S) \leq C \left(\int_S^T \int_{\Gamma_1} b |y_t|^2 \, d\Sigma + \int_S^T \int |y|^2 \, dQ \right). \tag{2.17}$$

Proof.

1. We can see that

$$E_t^{\varepsilon}(t) = E_t(t) + \varepsilon \rho_t(t),$$

where

$$\rho_t(t) = \frac{d\rho}{dt} \text{ and } E_t^{\varepsilon}(t) = \frac{dE^{\varepsilon}}{dt}.$$

But

$$\begin{aligned}
\rho_t(t) \quad = \quad & \int y_{tt} M y d \quad + \int y_t M y_t d \quad = - \int \mathcal{A} y M y d \\
& - \int q y M y d \quad + 2 \int y_t h(y_t) d \qquad\qquad (2.18) \\
& + \int (div_0 h - m_0) |y_t|^2 d \quad .
\end{aligned}$$

If we use the first Green's formula we find

$$\begin{aligned}
- \int \mathcal{A} y M y d \quad = \quad & \int_{\Gamma} \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} M y d\Gamma - 2 \int \langle \nabla_g y, \nabla_g (h(y)) \rangle_g d \\
& - \int \langle \nabla_g y, \nabla_g (div_0 h) \rangle_g y d \\
& - \int (div_0 h - m_0) \|\nabla_g y\|_g^2 d \quad ,
\end{aligned}$$

identity (1.2) gives

$$\begin{aligned}
- \int \mathcal{A} y M y \quad = \quad & \int_{\Gamma} \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} M y d\Gamma - 2 \int D h(\nabla_g y, \nabla_g y) d \\
& - \int h\left(\|\nabla_g y\|_g^2\right) d \quad - \int \langle \nabla_g y, \nabla_g (div_0 h) \rangle_g y d \\
& - \int (div_0 h - m_0) \|\nabla_g y\|_g^2 d \quad .
\end{aligned}$$

If we apply (1.3) with $f_1 = 1$ and $f_2 = \|\nabla_g y\|_g^2$ we obtain

$$\begin{aligned}
& - \int \mathcal{A} y M y d \quad = \int_{\Gamma} \left(\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} M y - h.\upsilon \|\nabla_g y\|_g^2 \right) d\Gamma \\
& - 2 \int D h(\nabla_g y, \nabla_g y) d \quad - \int \langle \nabla_g y, \nabla_g (div_0 h) \rangle_g y d \qquad (2.19) \\
& + m_0 \int \|\nabla_g y\|_g^2 d \quad .
\end{aligned}$$

On the other hand, we have

$$2\int y_t h(y_t)\,d \quad = \quad 2\int y_t \langle h, \nabla_g y_t\rangle_g\, d \quad = \int \left\langle h, \nabla_g\left(|y_t|^2\right)\right\rangle_g d$$
$$= \quad \int h\left(|y_t|^2\right) d \quad ,$$

we apply (1.3) with $f_1 = 1$ and $f_2 = |y_t|^2$ to find

$$2\int y_t h(y_t)\,d \quad = \int_\Gamma h.\upsilon\,|y_t|^2\,d\Gamma - \int div_0 h\,|y_t|^2\,d \quad . \tag{2.20}$$

If we replace (2.19) and (2.20) in (2.18), we find

$$\begin{aligned}
\rho_t(t) \quad &= \quad \int_\Gamma \left(\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} My + h.\upsilon\left(|y_t|^2 - \|\nabla_g y\|_g^2\right)\right) d\Gamma \\
&\quad -2\int Dh\left(\nabla_g y, \nabla_g y\right) d \qquad (2.21)\\
&\quad -\int \left\langle \nabla_g y, \nabla_g\left(div_0 h\right)\right\rangle_g y\,d \\
&\quad -\int qyMyd \quad - m_0\int \left(|y_t|^2 - \|\nabla_g y\|_g^2\right) d \\
&= \quad I_{\Gamma_0} + I_{\Gamma_1} + I \quad ,
\end{aligned}$$

where

$$I_{\Gamma_0} = \int_{\Gamma_0}\left(2\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} h(y) - h.\upsilon\,\|\nabla_g y\|_g^2\right) d\Gamma,$$

$$I_{\Gamma_1} = \int_{\Gamma_1}\left(\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} My + h.\upsilon\left(|y_t|^2 - \|\nabla_g y\|_g^2\right)\right) d\Gamma,$$

and

$$\begin{aligned}
I \quad &= \quad -2\int Dh\left(\nabla_g y, \nabla_g y\right) d \quad - \int \left\langle \nabla_g y, \nabla_g\left(div_0 h\right)\right\rangle_g y\,d \\
&\quad -\int qyMyd \quad - m_0\int \left(|y_t|^2 - \|\nabla_g y\|_g^2\right) d \quad .
\end{aligned}$$

Since $y = 0$ on Γ_0 then (see [Yao99])

$$\|\nabla_g y\|_g^2 = \frac{1}{\|\upsilon_{\mathcal{A}}\|_g^2} \left|\frac{\partial y}{\partial \upsilon_{\mathcal{A}}}\right|^2 \text{ and } h(y) = \frac{h.\upsilon}{\|\upsilon_{\mathcal{A}}\|_g^2} \frac{\partial y}{\partial \upsilon_{\mathcal{A}}},$$

this imply that

$$\begin{aligned} I_{\Gamma_0} &= \int_{\Gamma_0} \left(2\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} h(y) - h.\upsilon \|\nabla_g y\|_g^2\right) d\Gamma \\ &= \int_{\Gamma_0} \frac{h.\upsilon}{\|\upsilon_{\mathcal{A}}\|_g^2} \left|\frac{\partial y}{\partial \upsilon_{\mathcal{A}}}\right|^2 d\Gamma \leq 0. \end{aligned} \tag{2.22}$$

On the other hand,

$$\begin{aligned} I_{\Gamma_1} &= \int_{\Gamma_1} \left(\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} My + h.\upsilon \left(|y_t|^2 - \|\nabla_g y\|_g^2\right)\right) d\Gamma \\ &\leq C \int_{\Gamma_1} b |y_t|^2 d\Gamma + \sup_{\Gamma_1} (h.\upsilon) \int_{\Gamma_1} |y_t|^2 d\Gamma \\ &\quad + \left(\eta \sup_{\Gamma_1} \|h\|_g^2 - h_0\right) \int_{\Gamma_1} \|\nabla_g y\|_g^2 d\Gamma + \eta C_2^2 E(t), \end{aligned}$$

so

$$\begin{aligned} I_{\Gamma_1} &\leq C \int_{\Gamma_1} b |y_t|^2 d\Gamma + \eta C_2^2 E(t) \\ &\quad + \left(\eta \sup_{\Gamma_1} \|h\|_g^2 - h_0\right) \int_{\Gamma_1} \|\nabla_g y\|_g^2 d\Gamma, \end{aligned} \tag{2.23}$$

and, using (2.7) to find

$$\begin{aligned} I &= -2 \int Dh(\nabla_g y, \nabla_g y)\, d \quad - \int \langle \nabla_g y, \nabla_g (div_0 h) \rangle_g y d \\ &\quad -m_0 \int |q|^2 d \quad + m_0 \int |q|^2 d \quad - \int qyMyd \\ &\quad -m_0 \int \left(|y_t|^2 - \|\nabla_g y\|_g^2\right) d \\ &\leq 2(\eta C - m_0) E(t) + C \int |y|^2 d \quad . \end{aligned} \tag{2.24}$$

If we replace (2.22), (2.23) an (2.24) in (2.21) and taking η sufficiently small, we obtain

$$\rho_t(t) \leq -m_0 E(t) + C\left(\int_{\Gamma_1} b\,|y_t|^2\, d\Gamma + \int |y|^2\, d \quad\right).$$

Thus

$$E_t^\varepsilon(t) \leq E_t(t) - m_0\varepsilon E(t) + \varepsilon C\left(\int_{\Gamma_1} b\,|y_t|^2\, d\Gamma + \int |y|^2\, d \quad\right).$$

If we integrate the last inequality over $]S,T[$, use the decreasing of E and (2.5), we find

$$\begin{aligned} -E^\varepsilon(S) \quad &\leq \quad E^\varepsilon(T) - E^\varepsilon(S) \leq \frac{1}{1-\varepsilon\theta}E^\varepsilon(T) - \frac{m_0\varepsilon T_0}{1+\varepsilon\theta}E^\varepsilon(T) \\ &\quad + \varepsilon C\left(\int_S^T\int_{\Gamma_1} b\,|y_t|^2\, d\Sigma + \int_S^T\int |y|^2\, dQ\right). \end{aligned}$$

Therefore

$$\left(\frac{m_0\varepsilon T_0}{1+\varepsilon\theta} - \frac{1}{1-\varepsilon\theta}\right)E^\varepsilon(T) \leq E^\varepsilon(S) + \varepsilon C\left(\int_S^T\int_{\Gamma_1} b\,|y_t|^2\, d\Sigma + \int_S^T\int |y|^2\, dQ\right),$$

with T_0 sufficiently large, we find

$$\frac{m_0\varepsilon T_0}{2(1+\varepsilon\theta)} \leq \frac{m_0\varepsilon T_0}{1+\varepsilon\theta} - \frac{1}{1-\varepsilon\theta},$$

so

$$\frac{m_0\varepsilon T_0}{2(1+\varepsilon\theta)}E^\varepsilon(T) \leq E^\varepsilon(S) + \varepsilon C\left(\int_S^T\int_{\Gamma_1} b\,|y_t|^2\, d\Sigma + \int_S^T\int |y|^2\, dQ\right),$$

this represents the first estimate.

2. Concerning the second estimate.

First, we have from the first estimate

$$\begin{aligned} E^{\varepsilon}(T) &\leq \frac{2(1+\varepsilon\theta)}{m_0\varepsilon T_0}E^{\varepsilon}(S) \\ &+\frac{2(1+\varepsilon\theta)C}{m_0T_0}\left(\int_S^T\int_{\Gamma_1} b|y_t|^2\,d\Sigma+\int_S^T\int |y|^2\,dQ\right), \end{aligned}$$

with T_0 sufficiently large we have

$$\frac{2(1+\varepsilon\theta)C}{m_0T_0}\leq 1.$$

Then, we obtain

$$E^{\varepsilon}(T)\leq\frac{2(1+\varepsilon\theta)}{m_0\varepsilon T_0}E^{\varepsilon}(S)+\int_S^T\int_{\Gamma_1} b|y_t|^2\,d\Sigma+\int_S^T\int |y|^2\,dQ. \tag{2.25}$$

On the other hand, we have

$$\begin{aligned} E(S) &= E(T)-\int_S^T E_t(t)\,dt=E(T)+\int_S^T\int_{\Gamma_1} b|y_t|^2\,d\Gamma \\ &\leq \frac{1}{1-\varepsilon\theta}E^{\varepsilon}(T)+\int_S^T\int_{\Gamma_1} b|y_t|^2. \end{aligned}$$

Then

$$\begin{aligned} E^{\varepsilon}(S) &\leq (1+\varepsilon\theta)E(S) \\ &\leq \frac{1+\varepsilon\theta}{1-\varepsilon\theta}E^{\varepsilon}(T)+(1+\varepsilon\theta)\int_S^T\int_{\Gamma_1} b|y_t|^2, \end{aligned}$$

by (2.25), we find

$$E^{\varepsilon}(S)\leq\frac{2(1+\varepsilon\theta)^2}{m_0\varepsilon(1-\varepsilon\theta)T_0}E^{\varepsilon}(S)+C\left(\int_S^T\int_{\Gamma_1} b|y_t|^2\,d\Sigma+\int_S^T\int |y|^2\,dQ\right),$$

with T_0 sufficiently large, we find

$$E^{\varepsilon}(S)\leq C\left(\int_S^T\int_{\Gamma_1} b|y_t|^2\,d\Sigma+\int_S^T\int |y|^2\,dQ\right).$$

■

To absorb the lower order term $\int_S^T \int |y|^2 \, dQ$ from the estimate in (2.16) we use the compactness uniqueness argument.

Lemma 2.11 *For $T - S > T_0$, where T_0 is sufficiently large, we have*

$$\int_S^T \int |y|^2 \, dQ \le C \int_S^T \int_{\Gamma_1} b \, |y_t|^2 \, d\Sigma. \tag{2.26}$$

Proof. It is sufficient to prove (see $[FF04]$) that, for some T_0 large enough, we have

$$\int_0^{T_0} \int |y|^2 \, dQ \le C \int_0^{T_0} \int_{\Gamma_1} b \, |y_t|^2 \, d\Sigma.$$

We argue by contradiction. There exists a sequence of solutions (y_k) of system (2.1) such that

$$\lim_{k \to \infty} \int_0^{T_0} \int_{\Gamma_1} b \, |y_{kt}|^2 \, d\Sigma = 0, \tag{2.27}$$

and

$$\int_0^{T_0} \int |y_k|^2 \, dQ = 1 \text{ for all } k. \tag{2.28}$$

Let, for all k,

$$E_k^{\varepsilon}(t) = E_k(t) + \varepsilon \rho_k(t),$$

where E_k represents the energy of y_k and

$$\rho_k(t) = \int y_{kt} M y_k d \quad .$$

If we apply (2.17) with $E^{\varepsilon}(t) = E_k^{\varepsilon}(t)$, $S = 0$ and $T = T_0$, we obtain by (2.27), (2.28) and (2.5) that $(E_k(0))$ is bounded and therefore there exists a subsequence (y_k) such that

$$y_k \to y \text{ weakly}^* \text{ in } L^{\infty}\left(0, T_0; H^1_{\Gamma_0}(\ )\right),$$

and

$$y_k \to y \text{ weakly in } L^2\left(]0,T_0[\times\Gamma\right).$$

Using (2.27) and passing to the limit, we obtain

$$\begin{aligned} y_{tt}+\mathcal{A}y+qy &= 0 \text{ in }]0,T_0[\times \quad , \\ y &= 0 \text{ on }]0,T_0[\times\Gamma_0, \\ \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} &= -by_t = 0 \text{ on }]0,T_0[\times\Gamma_1. \end{aligned}$$

Hence we find

If

$$y=0,$$

then

$$\underset{k\to\infty}{lim}\int_0^{T_0}\int |y_k|^2\,dQ=0.$$

This contradicts (2.28).

If

$$y\neq 0,$$

then $z=y_t$ is solution of

$$\begin{aligned} z_{tt}+\mathcal{A}z+qz &= 0 \text{ in }]0,T_0[\times \quad , \\ z &= 0 \text{ on }]0,T_0[\times\Gamma, \\ \frac{\partial z}{\partial \upsilon_{\mathcal{A}}} &= 0 \text{ on }]0,T_0[\times\Gamma_1, \end{aligned}$$

then, for T_0 sufficiently large, $z = 0$. So, y is a solution of

$$\begin{aligned} \mathcal{A}y + qy &= 0 \text{ in }]0, T_0[\times \quad , \\ y &= 0 \text{ on }]0, T_0[\times \Gamma_0, \\ \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} &= 0 \text{ on }]0, T_0[\times \Gamma_1. \end{aligned}$$

If we multiply the first equation by y, integrate over Q and use the first Green's formula we find

$$\int_0^{T_0} \int \|\nabla_g y\|_g^2 \, dQ + \int_0^{T_0} \int q |y|^2 \, dQ = 0,$$

thus

$$y = 0.$$

■

We give, now, the proof of the following result.

Theorem 2.12 ([*Ham3*]) *For any initial data* $(y_0, y_1) \in D(\mathfrak{A})$, *(2.4) is verified.*

Proof. If we insert (2.26) in (2.16) we find

$$\frac{m_0 \varepsilon T_0}{2(1+\varepsilon\theta)} E^{\varepsilon}(T) \leq E^{\varepsilon}(S) + C \int_S^T \int_{\Gamma_1} b |y_t|^2 \, d\Sigma.$$

But

$$\begin{aligned} \int_S^T \int_{\Gamma_1} b |y_t|^2 \, d\Sigma &= -\int_S^T E_t(t) \, dt = E(S) - E(T) \\ &\leq \frac{1}{1-\varepsilon\theta} E^{\varepsilon}(S). \end{aligned}$$

Then

$$\frac{m_0 \varepsilon T_0}{2(1+\varepsilon\theta)} E^{\varepsilon}(T) \leq C E^{\varepsilon}(S).$$

If we choose T_0 sufficiently large we find $0 < r < 1$ such that

$$\frac{1}{r}E^{\varepsilon}(T) \leq E^{\varepsilon}(S), \tag{2.29}$$

for all $T - S \geq T_0$.

First, we can see that

$$E^{\varepsilon}(T) \leq E^{\varepsilon}(S), \tag{2.30}$$

for all $T - S \geq T_0$.

On the other hand, if we apply (2.29) repeatedly on the intervals $[mT_0, (m+1)T_0]$, $m = 0, 1, \ldots$, we get

$$\begin{aligned} & \frac{1-r}{r}E^{\varepsilon}((m+1)T_0) + E^{\varepsilon}((m+1)T_0) \\ = & \frac{1}{r}E^{\varepsilon}((m+1)T_0) \leq E^{\varepsilon}(mT_0). \end{aligned}$$

Put $p(s) = \frac{1-r}{r}s$ and $s_m = E^{\varepsilon}(mT_0)$, to find

$$p(s_{m+1}) + s_{m+1} \leq s_m.$$

Using lemma 1.32 to obtain

$$s_m \leq S(m),$$

for all m.

Where $S(t)$ is the solution of the system

$$\begin{cases} S_t(t) + (1-r)S(t) = 0, \\ S(0) = E^{\varepsilon}(0). \end{cases}$$

Here we have used

$$\begin{aligned} & S(t) - (I+p)^{-1}(S(t)) \\ = & S(t) - \left(I + \frac{1-r}{r}I\right)^{-1}(S(t)) = (1-r)S(t). \end{aligned}$$

The last system have the solution

$$S(t) = e^{-(1-r)t} E^{\varepsilon}(0).$$

So

$$E^{\varepsilon}(mT_0) \leq e^{-(1-r)m} E^{\varepsilon}(0),$$

for all m.

Let $t \geq T_0$, then

$$t = T_0 + mT_0 + \tau,$$

where $0 \leq \tau \leq T_0$.

This imply that

$$t = mT_0 + \tau^*,$$

where $T_0 \leq \tau^* = T_0 + \tau \leq 2T_0$.

Then by (2.30)

$$E^{\varepsilon}(t) = E^{\varepsilon}(mT_0 + \tau^*) \leq E^{\varepsilon}(mT_0).$$

So

$$\begin{aligned} E^{\varepsilon}(t) &\leq E^{\varepsilon}(mT_0) \leq e^{-(1-r)\frac{t-\tau^*}{T_0}} E^{\varepsilon}(0) \\ &\leq e^{-(1-r)\frac{t-2T_0}{T_0}} E^{\varepsilon}(0). \end{aligned}$$

Thus

$$E^{\varepsilon}(t) \leq c E^{\varepsilon}(0) e^{-\omega t},$$

for $t \geq T_0$.

Where

$$c' = e^{2(1-r)} \text{ and } \omega' = \frac{1-r}{T_0}.$$

■

Remark 2.13 *Since (2.4) is verified in* $D(\mathfrak{A})$ *then we have (2.3) in* $D(\mathfrak{A})$*. That is*

$$E(t) \leq cE(0)e^{-\omega t},$$

for all $t \geq T_0$ *and* $(y_0, y_1) \in D(\mathfrak{A})$*.*

2.5 *Exponential decay rate of* E *for initial data in* $H^1_{\Gamma_0}(\ ) \times L^2(\ )$*.*

In this section, we prove (2.3) for all $(y_0, y_1) \in H^1_{\Gamma_0}(\ ) \times L^2(\ )$.

Theorem 2.14 *If* $(y_0, y_1) \in H^1_{\Gamma_0}(\ ) \times L^2(\ )$*, then (2.3) is verified.*

Proof. Let y be the solution of system (2.1) with initial condition $(y_0, y_1) \in H^1_{\Gamma_0}(\ ) \times L^2(\ )$. Since $D(\mathfrak{A})$ is dense in $H^1_{\Gamma_0}(\ ) \times L^2(\ )$ then there exist a sequence $(y_{k0}, y_{k1}) \in D(\mathfrak{A})$ which is convergent in the energy space to (y_0, y_1). That is

$$\lim_{k\to\infty} \|(y_{k0}, y_{k1}) - (y_0, y_1)\|_{H^1_{\Gamma_0}(\ ) \times L^2(\ )} = 0,$$

but

$$\begin{aligned}
&\left\|(y_{k0},y_{k1})-(y_0,y_1)\right\|^2_{H^1_{\Gamma_0}(\ )\times L^2(\ )}\\
=\ &\left\|y_{k0}-y_0\right\|^2_{H^1_{\Gamma_0}(\ )}+\left\|y_{k1}-y_1\right\|^2_{L^2(\ )} \qquad (2.31)\\
=\ &\int \left(\left|y_{k1}-y_1\right|^2+q\left|y_{k0}-y_0\right|^2+\left\|\nabla_g\left(y_{k0}-y_0\right)\right\|^2_g\right)d\ \ ,
\end{aligned}$$

then

$$\lim_{k\to\infty}\int\left(\left|y_{k1}-y_1\right|^2+q\left|y_{k0}-y_0\right|^2+\left\|\nabla_g\left(y_{k0}-y_0\right)\right\|^2_g\right)d\ \ =0. \qquad (2.32)$$

Let y_k be the solution of (2.1) with initial condition (y_{k0},y_{k1}), for all $k\in\mathbb{N}$. Then

$$E_k(t)\le cE_k(0)\,e^{-\omega t}, \qquad (2.33)$$

for all $t\ge T_0$.

First, we have (see theorem 7.1 in $[Kom94]$)

$$0\le\mathcal{E}(t)\le\mathcal{E}(0)\ \text{ for all } t\ge 0, \qquad (2.34)$$

here $\mathcal{E}(t)$ is the energy of y_k-y, that is

$$\begin{aligned}
\mathcal{E}(t)\ &=\ \frac{1}{2}\left\|(y_k-y,\,y_{kt}-y_t)\right\|^2_{H^1_{\Gamma_0}(\ )\times L^2(\ )}\\
&=\ \frac{1}{2}\int\left(\left|y_{kt}-y_t\right|^2+q\left|y_k-y\right|^2+\left\|\nabla_g\left(y_k-y\right)\right\|^2_g\right)d\ \ .
\end{aligned}$$

This imply that

$$\begin{aligned}
\mathcal{E}(0)\ &=\ \frac{1}{2}\int\left(\left|y_{kt}(0)-y_t(0)\right|^2+q\left|y_k(0)-y(0)\right|^2+\left\|\nabla_g\left(y_k-y\right)(0)\right\|^2_g\right)d\\
&=\ \frac{1}{2}\int\left(\left|y_{k1}-y_1\right|^2+q\left|y_{k0}-y_0\right|^2+\left\|\nabla_g\left(y_{k0}-y_0\right)\right\|^2_g\right)d\ \ ,
\end{aligned}$$

and from (2.32) we obtain

$$\lim_{k\to\infty} \mathcal{E}(0) = 0.$$

By (2.34), we find

$$\lim_{k\to\infty} \mathcal{E}(t) = 0,$$

so

$$\begin{aligned}\lim_{k\to\infty} \|(y_k, y_{kt}) - (y, y_t)\|^2_{H^1_{\Gamma_0}(\ )\times L^2(\ )} &= \lim_{k\to\infty} \|(y_k - y, y_{kt} - y_t)\|^2_{H^1_{\Gamma_0}(\ )\times L^2(\ )} \\ &= \lim_{k\to\infty} \mathcal{E}(t) = 0.\end{aligned}$$

So

$$E_k(t) = \|(y_k, y_{kt})\|_{H^1_{\Gamma_0}(\ )\times L^2(\ )} \underset{k\to\infty}{\to} E(t) = \|(y, y_t)\|_{H^1_{\Gamma_0}(\ )\times L^2(\ )}.$$

Finally, letting $k \to \infty$ in (2.33), to find

$$E(t) \leq cE(0)\, e^{-\omega t}.$$

■

Remark 2.15 *We can treat exactly in the same way the situation of the polynomial growth of the nonlinear feedback near the origin. In this case, we show that we have an exponential or polynomial decay rate of the perturbed energy functional defined for all* $\varepsilon > 0$ *by*

$$E^{\varepsilon}(t) = E(t) + \varepsilon\rho(t)\,(E(t))^{\frac{\gamma-1}{2}},$$

where γ *depends on the behavior of nonlinear damping at the origin.*

Chapter 3

Uniform decay rate of the Riemannian wave equation without smallness condition on the linear first order term.

3.1 *Introduction.*

Consider the following Riemannian wave equation with linear first order term

$$\begin{cases} y_{tt} - \Delta_g y - \langle D\varphi, Dy \rangle_g = 0 \text{ in } Q, \\ \qquad y = 0 \text{ on } \Sigma_0, \\ \frac{\partial y}{\partial n} + b y_t + f(y_t) = 0 \text{ on } \Sigma_1, \\ y(0) = y_0,\ y_t(0) = y_1 \text{ in } \quad . \end{cases} \tag{3.1}$$

The following assumptions are made.

$(H1)$ $\varphi \in W^{2,\infty}(\quad)$ such that for some positive constants φ_* and φ^* we have

$$\varphi_* \leq \varphi(s) \leq \varphi^*,$$

for all $s \in \overline{\ \ }$.

$(H2)$ There exists two positive constants b_* and b^* such that

$$0 < b_* \leq b(s) \leq b^*$$

for all $s \in \Gamma_1$.

$(H3)$ $f : \mathbb{R} \to \mathbb{R}$ is an increasing function of class C^1 such that for some positive constants c_1 and c_2 we have

$$c_1 |s|^2 \leq f(s)\, s \leq c_2 |s|^2 \ \text{ for } |s| \geq 1,$$

and

$$f(s)\, s > 0 \text{ for all } s \neq 0.$$

$(H4)$ There exists a function $d : \overline{\ \ } \to \mathbb{R}^+$ of class C^3 verifying

$$\inf \|Dd\|_g > 0, \tag{3.2}$$

and

$$D^2 d\,(X, X) \geq m_0 \, \|X\|_g^2 \,, \text{ for all } X \in \mathbb{R}^n_x. \tag{3.3}$$

for some constant $m_0 > 0$.

Moreover, we take

$$\Gamma_0 = \left\{ x \in \Gamma : \langle Dd, n \rangle_g \leq 0 \right\}, \tag{3.4}$$

and

$$\Gamma_1 = \left\{ x \in \Gamma : \langle Dd, n \rangle_g \geq h_0 \right\} \tag{3.5}$$

for some constant $h_0 > 0$.

Remark 3.1 *Assumption $(H1)$ is on the linear first order term. The assumption $(H4)$ is a geometric condition on , while the assumptions $(H2)$ and $(H3)$ are on the feedback.*

Remark 3.2 *We note here that no growth conditions at the origin are imposed on the nonlinear feedback f. But, by virtue of Assumption $(H3)$, we can always (See $[LT93]$) construct a concave, strictly increasing function $h : \mathbb{R}_+ \to \mathbb{R}$ such that $h(0) = 0$ and*

$$h\left(f(s)\, s\right) \geq |s|^2 + |f(s)|^2 \;\; \textit{for} \;\; |s| \leq 1.$$

When $a_{ij}(x) = \delta_{ij}$ and f has a polynomial decay at the origin, the problem of energy decay rate of system (3.1) has studied by A. Guesmia , assuming that $\|\varphi\|_{L^\infty} := \sup_{x \in \overline{\ }} \|\nabla_0 \varphi(x)\|_0$ is sufficiently small (see theorem 2.4 in $[GueII03]$). We note that when the damping is applied in all , the internal stabilization can be obtained without any condition of smallness on $\|\varphi\|_{L^\infty}$ (see [$Mes00$] and theorem 2.2 in [$GueII03$]).

The main goal of this paper is to show that, without any condition of smallness on $\|\varphi\|_\infty := \sup_{x\in} \|D\varphi(x)\|_g$, the energy of the solution decays faster than the solution of some associated differential equation. This result generalizes the corresponding case of a second order hyperbolic equation and a linear growth of f at the origin which was proved recently in [$Ham08$].

To obtain our result, we use the energy (multiplier) method, where we introduce a new geometric multiplier $Dd(\varphi)y$ to handle the linear first order term. In order to absorb the lower order term with respect to the energy, we combine the idea in [$FF04$] with the one of I. Lasiecka and D. Tataru in [$LT93$]. Finally, we conclude by employing the Lasiecka's and Tataru's abstract stabilization inequalities.

As it is well know, the presence of the first order term provides the nondissipation for the usual energy. For this reason, we consider an equivalent energy E of the system (3.1) defined, for all $t \geq 0$, by

$$E(t) = \frac{1}{2}\int e^{\varphi}\left(|y_t|^2 + \|Dy\|_g^2\right) d \quad _g,$$

and we shall see in lemma 3.6 below that it is a decreasing function.

We state, now, the main result of this chapter.

Theorem 3.3 ([$Ham2$]) *There exist $T_0 > 0$ such that*

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right) \text{ for all } t > T_0,$$

where $S(t)$ is the solution of the following differential equation

$$\begin{cases} S_t(t) + q(S(t)) = 0, \\ S(0) = E(0). \end{cases}$$

Here, for all $s > 0$, $q(s) = s - (I+p)^{-1}(s)$, with $p(s) = e^{\varphi_*}\left(\sigma I + \widetilde{h}\right)^{-1}(Ks)$, $\widetilde{h}(s) = h\left(\frac{s}{mes\Sigma_1}\right)$, $K = \frac{1}{Ce^{\varphi^*}mes\Sigma_1}$ and $\sigma = \frac{c_1^{-1}+c_2}{e^{\varphi^*}mes\Sigma_1}$.

Remark 3.4 *It is important to emphasize, that the function $S(t)$ does not depend on a profile of the initial condition, but only on the $E(0)$.*

This chapter is organized as follows. In section 3.2, we will use the Faedo-Galerkin method to prove the existence, uniqueness and regularity of solution of system (3.1). In section 3.3, we will give two energy inequalities that we will use in the proof of the main result. Section 3.4 is devoted to the absorbtion of the lower order term. In section 3.5, we complete the proof of the main result. Finally, we study the case of second order hyperbolic equation with polynomial behavior at the origin of the nonlinear feedback f.

3.2 *Existence, uniqueness and regularity of solution.*

In this section, we discuss the existence, uniqueness and regularity of the solution of the system (3.1).

Theorem 3.5 *If*

$$(y_0, y_1) \in H^1_{\Gamma_0}(\ \ ) \times L^2(\ \ )$$

then the system (3.1) has a unique solution

$$y \in C([0,+\infty[; H^1_{\Gamma_0}(\ \ )) \cap C^1([0,+\infty[; L^2(\ \ )).$$

Moreover, if

$$(y_0, y_1) \in H^1_{\Gamma_0}(\ \ ) \cap H^2(\ \ ) \times H^1_{\Gamma_0}(\ \ )$$

such that

$$\frac{\partial y_0}{\partial n} + by_1 + f(y_1) = 0 \ \text{on} \ \Gamma_1,$$

then system (3.1) has a unique solution y verifying

$$y \in L^\infty([0,+\infty[; H^1_{\Gamma_0}(\ \ ) \cap H^2(\ \ )), \ y_t \in L^\infty([0,+\infty[; H^1_{\Gamma_0}(\ \ )),$$

and

$$y_{tt} \in L^\infty([0,+\infty[; L^2(\ \ )).$$

Proof. It will be done by the Faedo-Galerkin method (see $[DL85]$) and some techniques in $[CS97, CDS98, CCSP01]$ and the references therein.

Change of variable.

The variational formulation of problem (3.1) is given by

$$\begin{cases} \int y_{tt}\omega d\ \ _g + \int \langle Dy, D\omega\rangle_g d\ \ _g - \int \langle D\varphi, Dy\rangle_g \omega d\ \ _g \\ \qquad = -\int\limits_{\Gamma_1} (by_t + f(y_t))\omega d\Gamma_g, \\ \qquad \text{for all } \omega \in H^1_{\Gamma_0}(\ \ ). \end{cases}$$

To prove the existence of the solution we need to estimate $y_{tt}(0)$, but we have technical difficulties because of the boundary term $\int\limits_{\Gamma_1} f(y_t)\omega d\Gamma_g$. In order to avoid these difficulties, we transform (3.1) into an equivalent problem with initial value equal to zero (see $[CS97, CDS98]$). In fact, the change of variables

$$v\,(x,t) = y\,(x,t) - \phi\,(x,t)\,,$$

where

$$\phi\,(x,t) = y_0\,(x) + t y_1\,(x)\,,$$

leads to

$$\begin{cases} v_{tt} - \Delta_g v - \langle D\varphi, Dv\rangle_g = F \text{ in } Q, \\ v = 0 \text{ on } \Sigma_0, \\ \frac{\partial v}{\partial n} + bv_t + f\,(v_t + \phi_t) = G \text{ on } \Sigma_1, \\ v(0) = 0,\ v_t\,(0) = 0 \text{ in } \quad , \end{cases}$$

where

$$F = \Delta_g \phi + \langle D\varphi, D\phi\rangle_g \text{ and } G = -b\phi_t - \frac{\partial \phi}{\partial n}.$$

First. We consider $(y_0, y_1) \in H^1_{\Gamma_0}(\ ) \cap H^2(\ ) \times H^1_{\Gamma_0}(\ )$ such that $\frac{\partial y_0}{\partial n} + by_1 + f\,(y_1) = 0$ on Γ_1.

(i) Approximate problem.

Let $\{\omega_j\}$ be the basis of $H^1_{\Gamma_0}(\ )$. V_m is the subspace of $H^1_{\Gamma_0}(\ )$ generated by the m first vectors $\omega_1, ..., \omega_m$ and $v^m\,(x,t) = \sum_{j=1}^{j=m} \gamma^{jm}\,(t)\,\omega_j\,(x)$ is the solution of the system

$$\begin{cases} \int v_{tt}^m \omega_j d \quad_g + \int \langle Dv^m, D\omega_j\rangle_g d \quad_g - \int \langle D\varphi, Dv^m\rangle_g \omega_j d \quad_g \\ = -\int\limits_{\Gamma_1} (bv_t^m + f\,(v_t^m + \phi_t))\,\omega_j d\Gamma_g + \int\limits_{\Gamma_1} G\omega_j d\Gamma_g + \int F\omega_j d \quad_g, \\ j = 1, ..., m. \end{cases} \tag{3.6}$$

This system is an ordinary differential equations which has a solution v^m defined on $[0, t_m)$, where $0 < t_m < T$. The extension of the solution to the whole interval $[0, T]$ is a consequence of the first estimate which we are going to obtain below.

(ii) A priori estimates.

First estimate.

By multiplying both sides of (3.6) by γ_t^{jm} and adding from $j = 1$ to $j = m$ we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int \left(|v_t^m|^2 + \|Dv^m\|_g^2 \right) d\;\,_g \\ = & \int \langle D\varphi, Dv^m \rangle_g \, v_t^m d\;\,_g - \int_{\Gamma_1} b \, |v_t^m|^2 \, d\Gamma_g - \int_{\Gamma_1} f\left(v_t^m + \phi_t\right)\left(v_t^m + \phi_t\right) d\Gamma_g \\ & + \int_{\Gamma_1} f\left(v_t^m + \phi_t\right)\phi_t d\Gamma_g + \int_{\Gamma_1} G v_t^m d\Gamma_g + \int F v_t^m d\;\,_g, \end{aligned}$$

then, for all $\eta > 0$,

$$\begin{aligned} & \frac{d}{dt}\int \left(|v_t^m|^2 + \|Dv^m\|_g^2 \right) d\;\,_g \\ \leq & \ (\eta - 2b_*)\int_{\Gamma_1} |v_t^m|^2 \, d\Gamma_g + C\int_{\Gamma_1} |G|^2 \, d\Gamma_g \\ & + \int |F|^2 \, d\;\,_g + 2\int |v_t^m|^2 \, d\;\,_g \\ & + \sup \|D\varphi\|_g^2 \int \|Dv^m\|_g^2 \, d\;\,_g \\ & + \eta \int_{\Gamma_1} |f\left(v_t^m + \phi_t\right)|^2 \, d\Gamma_g + C\int_{\Gamma_1} |\phi_t|^2 \, d\Gamma_g, \end{aligned}$$

but

$$\begin{aligned}
\int\limits_{\Gamma_1} |f(v_t^m+\phi_t)|^2 \, d\Gamma_g &= \int\limits_{|v_t^m+\phi_t|\geq 1} |f(v_t^m+\phi_t)|^2 \, d\Gamma_g \\
&\quad + \int\limits_{|v_t^m+\phi_t|\leq 1} |f(v_t^m+\phi_t)|^2 \, d\Gamma_g \\
&\leq c_2^2 \int\limits_{|v_t^m+\phi_t|\geq 1} |v_t^m+\phi_t|^2 \, d\Gamma_g \\
&\quad + C,
\end{aligned}$$

that is

$$\begin{aligned}
\int\limits_{\Gamma_1} |f(v_t^m+\phi_t)|^2 \, d\Gamma_g &\leq 2c_2^2 \int\limits_{\Gamma_1} |v_t^m|^2 \, d\Gamma_g + 2c_2^2 \int\limits_{\Gamma_1} |\phi_t|^2 \, d\Gamma_g \\
&\quad + C.
\end{aligned}$$

So

$$\begin{aligned}
& \frac{d}{dt} \int \left(|v_t^m|^2 + \|Dv^m\|_g^2 \right) d \quad {}_g \\
& + (2b_* - C\eta) \int\limits_{\Gamma_1} |v_t^m|^2 \, d\Gamma_g \\
\leq \quad & M_1 \int \left(|v_t^m|^2 + \|Dv^m\|_g^2 \right) d \quad {}_g + M_2,
\end{aligned}$$

for some positive constants M_1 and M_2.

Integrating above inequality over $]0,t[$ $(t<T)$, taking η sufficiently small to obtain

$$\begin{aligned}
& \int \left(|v_t^m|^2 + \|Dv^m\|_g^2 \right) d \quad {}_g + b_* \int\limits_0^t \int\limits_{\Gamma_1} |v_t^m|^2 \, d\Gamma_g ds \\
\leq \quad & M_1 \int\limits_0^t \left[\int \left(|v_t^m|^2 + \|Dv^m\|_g^2 \right) d \quad {}_g + b_* \int\limits_0^s \int\limits_{\Gamma_1} |v_t^m|^2 \, d\Gamma_g d\tau \right] ds \\
& + TM_2,
\end{aligned}$$

employing Gronwall's inequality, we find

$$\int \left(|v_t^m|^2 + \|Dv^m\|_g^2 \right) d\;\;_g + b_* \int_0^t \int_{\Gamma_1} |v_t^m|^2 \, d\Gamma_g ds \le L_1, \;\; \forall m \in \mathbb{N},$$

where L_1 is a positive constant independent of $m \in \mathbb{N}^*$. Thus

$$(v^m) \text{ is bounded in } L^\infty \left(0, T; H^1_{\Gamma_0}(\;\;)\right), \tag{3.7}$$

$$(v_t^m) \text{ is bounded in } L^\infty \left(0, T; L^2(\;\;)\right).$$

Second estimate.

Firstly, we are going to estimate $v_{tt}^m(0)$.

Taking $t = 0$ in (3.6) we have

$$\begin{aligned} & \int v_{tt}^m(0)\omega_j d\;\;_g + \int \langle Dv^m(0), D\omega_j \rangle_g d\;\;_g \\ & - \int \langle D\varphi, Dv^m(0) \rangle_g \omega_j d\;\;_g \\ = & - \int_{\Gamma_1} \left(bv_t^m(0) + f\left(v_t^m(0) + \phi_t(0)\right)\right) \omega_j d\Gamma_g \\ & + \int_{\Gamma_1} G(0)\,\omega_j d\Gamma_g + \int F(0)\,\omega_j d\;\;_g, \end{aligned}$$

but

$$G(0) = -by_1 - \frac{\partial y_0}{\partial n}, \;\; \phi_t(0) = y_1,$$

and

$$v^m(0) = v_t^m(0) = 0,$$

then

$$\begin{aligned} \int v_{tt}^m(0)\omega_j d\;\;_g &= -\int_{\Gamma_1} \left(\frac{\partial y_0}{\partial n} + by_1 + f(y_1) \right) \omega_j d\Gamma_g \\ & \quad + \int F(0)\,\omega_j d\;\;_g. \end{aligned}$$

Considering the assumption made on the initial data we obtain

$$\int v_{tt}^m(0)\,\omega_j\,d\quad_g = \int F(0)\,\omega_j d\quad_g.$$

If we multiply both side of this identity by $\gamma_{tt}^{jm}(0)$ and summing over $1 \leq j \leq m$ we arrive at

$$\begin{aligned}\int |v_{tt}^m(0)|^2\,d\quad_g &= \int F(0)\,v_{tt}^m(0)\,d\quad_g \\ &\leq \frac{1}{2}\int |v_{tt}^m(0)|^2\quad_g + \frac{1}{2}\int |F(0)|^2\,d\quad_g,\end{aligned}$$

so

$$\int |v_{tt}^m(0)|^2\,d\quad_g \leq \frac{1}{2}\int |F(0)|^2\,d\quad_g.$$

On the other hand, taking the derivative of (3.6) with respect to t, multiplying both side by γ_{tt}^{jm} and summing over $1 \leq j \leq m$ we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \left(|v_{tt}^m|^2 + \|Dv_t^m\|_g^2\right) d\quad_g \\ =\;& \int \langle D\varphi, Dv_t^m\rangle_g\, v_{tt}^m\, d\quad_g - \int_{\Gamma_1} b\,|v_{tt}^m|^2\, d\Gamma_g \\ &- \int_{\Gamma_1} f'(v_t^m + \phi_t)\,|v_{tt}^m|^2\, d\Gamma_g \\ &+ \int_{\Gamma_1} G' v_{tt}^m d\Gamma_g + \int F' v_{tt}^m\, d\quad_g,\end{aligned}$$

then

$$
\begin{aligned}
& \frac{d}{dt}\int \left(|v_{tt}^m|^2 + \|Dv_t^m\|_g^2\right) d\quad_g \\
& + (2b_* - \eta)\int_{\Gamma_1} |v_{tt}^m|^2 \, d\Gamma_g \\
\leq \;& 2\int |v_{tt}^m|^2 \, d\quad_g + \sup \|D\varphi\|_g^2 \int \|Dv_t^m\|_g^2 \, d\quad_g \\
& + C\int_{\Gamma_1} |G|^2 \, d\Gamma_g + \int |F|^2 \, d\quad_g .
\end{aligned}
$$

for all $\eta > 0$.

Or

$$
\begin{aligned}
& \frac{d}{dt}\int \left(|v_{tt}^m|^2 + \|Dv_t^m\|_g^2\right) d\quad_g \\
& + (2b_* - \eta)\int_{\Gamma_1} |v_{tt}^m|^2 \, d\Gamma_g \\
\leq \;& M_3 \left(\int \left(|v_{tt}^m|^2 + \|Dv_t^m\|_g^2\right) d\quad_g \right) + M_4,
\end{aligned}
$$

for some positive constants M_3 and M_4.

Taking η sufficiently small, integrating above $]0,t[$ and employing Gronwall's inequality, we find

$$
\int \left(|v_{tt}^m|^2 + \|Dv_t^m\|_g^2\right) d\quad_g \leq L_2,
$$

where L_2 is a positive constant independent of $m \in \mathbb{N}^*$. Thus

$$
\begin{aligned}
& (v_t^m) \text{ is bounded in } L^\infty\left(0,T; H^1_{\Gamma_0}(\quad)\right), \qquad (3.8) \\
& (v_{tt}^m) \text{ is bounded in } L^\infty\left(0,T; L^2(\quad)\right).
\end{aligned}
$$

(iii) Limit of the approximate solutions.

(3.7), (3.8) permit us to obtain a subsequence of (v^m), which we denote as the original sequence, such that

$$\begin{aligned}
v^m &\rightarrow v \text{ weakly * in } L^\infty\left(0,T;H^1_{\Gamma_0}(\ )\right), \qquad (3.9)\\
v^m_t &\rightarrow v_t \text{ weakly * in } L^\infty\left(0,T;H^1_{\Gamma_0}(\ )\right),\\
v^m_{tt} &\rightarrow v_{tt} \text{ weakly in } L^2\left(0,T;L^2(\ )\right),\\
v^m_t &\rightarrow v_t \text{ weakly in } L^2\left(0,T;L^2(\Gamma_1)\right),\\
\langle D\varphi, Dv^m\rangle_g &\rightarrow \langle D\varphi, Dv\rangle_g \text{ weakly in } L^2\left(0,T;L^2(\ )\right),\\
f(v^m_t) &\rightarrow f(v_t) \text{ weakly in } L^2\left(0,T;L^2(\Gamma_1)\right).
\end{aligned}$$

After passing to the limit and using standard arguments, we conclude that

$$v_{tt} - \Delta_g v - \langle D\varphi, Dv\rangle_g = F \text{ in } L^2\left(0,T;L^2(\ )\right), \qquad (3.10)$$

and

$$\frac{\partial v}{\partial n} + bv_t + f(v_t) = G \text{ in } L^2\left(0,T;L^2(\ )\right).$$

Uniqueness.

Assume that we have two solutions y and $\widehat{y}$ to the problem (3.1), then $z = y - \widehat{y}$ satisfies

$$\begin{cases}
\int_Q z_{tt}\omega dQ_g + \int_Q \langle Dz, D\omega\rangle_g dQ_g = -\int_{\Sigma_1} (bz_t - f(y_t) + f(\widehat{y}_t))\omega d\Sigma_g \\
\qquad \text{for all } \omega \in H^1_{\Gamma_0}(\ ), \\
\qquad z(0) = z_t(0) = 0,
\end{cases} \qquad (3.11)$$

where $d\Sigma_g = d\Gamma_g dt$ and $dQ_g = d\ _g dt$.

Taking $\omega = 2z_t$ we have

$$\begin{aligned} \frac{d}{dt}\int_Q \left(|z_t|^2 + \|Dz\|_g^2\right) dQ_g &= -2\int_{\Gamma_1} b\,|z_t|^2\, d\Sigma_g \\ +2\int_{\Sigma_1} \left(f\left(\widehat{y}_t\right) - f\left(y_t\right)\right) z_t d\Sigma_g &\leq 0 \end{aligned}$$

so

$$z = 0.$$

Second. Consider now $(y_0, y_1) \in H_0^1(\ \ ) \times L^2(\ \ )$, by standard arguments of density and considering analogous arguments used to prove the first and second estimates, and the uniqueness, we can prove existence and uniqueness of the solution (see for example [$CDS98$, $CCSP01$] for similar results). ■

3.3 *Energy inequalities.*

We show that the system (3.1) is dissipative.

Lemma 3.6 *We have*

$$E_t(t) = \frac{dE}{dt} = -\int_{\Gamma_1} e^{\varphi}\left(by_t + f\left(y_t\right)\right) y_t d\Gamma_g \leq 0,$$

for all $t \geq 0$.

Proof. First, we have

$$\begin{aligned} De^{\varphi} &= \sum_i \left(\sum_j a_{ij}\frac{\partial}{\partial x_j} e^{\varphi}\right)\frac{\partial}{\partial x_i} \\ &= e^{\varphi}\sum_i \left(\sum_j a_{ij}\frac{\partial \varphi}{\partial x_j}\right)\frac{\partial}{\partial x_i} = e^{\varphi} D\varphi. \end{aligned}$$

If we use the second Green's formula we find

$$\begin{aligned}
0 &= \int e^{\varphi}\left(y_{tt}-\Delta_g y-\langle D\varphi, Dy\rangle_g\right) y_t d \quad_g \\
&= \left[\int e^{\varphi} y_{tt} y_t d \quad_g + \int e^{\varphi}\langle Dy, Dy_t\rangle_g d \quad_g\right] \\
&\quad -\int_{\Gamma} e^{\varphi}\frac{\partial y}{\partial n} y_t d\Gamma_g = \frac{dE}{dt} - \int_{\Gamma_1} e^{\varphi}\frac{\partial y}{\partial n} y_t d\Gamma_g,
\end{aligned}$$

then

$$\frac{dE}{dt} = \int_{\Gamma_1} e^{\varphi}\frac{\partial y}{\partial n} y_t d\Gamma_g = -\int_{\Gamma_1} e^{\varphi}\left(by_t + f\left(y_t\right)\right) y_t d\Gamma_g.$$

■

To prove the following inequality, we need to introduce a new geometric multiplier $h(\varphi)$, where $h = Dd$.

Lemma 3.7 *For all $T - S > T_0$, where T_0 is sufficiently large, we have*

$$E(T) \leq C\left(\int_S^T \int_{\Gamma_1} e^{\varphi}\left(\left|f\left(y_t\right)\right|^2 + \left|y_t\right|^2\right) d\Sigma_g + \int_S^T \int e^{\varphi}|y|^2 dQ_g\right).$$

Proof. First, we have by lemma 3.6

$$E(T) \leq E(S) \leq E(T) + C\int_S^T \int_{\Gamma_1} e^{\varphi}\left(\left|f\left(y_t\right)\right|^2 + \left|y_t\right|^2\right) d\Sigma_g. \tag{3.12}$$

If we multiply the first equation by $My = 2h(y) + \left(div_g h - m_0 + h(\varphi)\right) y$, integrate over $]S,T[\times$ we find

$$\begin{aligned}
0 &= \int_S^T \int e^{\varphi}\left(y_{tt}-\Delta_g y-\langle D\varphi, Dy\rangle_g\right) My dQ \qquad (3.13)\\
&= \int_S^T \int e^{\varphi} y_{tt} My dQ + \int_S^T \int e^{\varphi}\left(-\Delta_g y - \langle D\varphi, Dy\rangle_g\right) My dQ.
\end{aligned}$$

The integration by part gives

$$\int_S^T \int e^{\varphi} y_{tt} M y dQ_g = \left. \int e^{\varphi} y_t M y d \quad _g \right|_S^T - \int_S^T \int e^{\varphi} y_t M y_t dQ_g$$

$$= \left. \int e^{\varphi} y_t M y d \quad _g \right|_S^T - 2 \int_S^T \int e^{\varphi} h(y_t) y_t dQ_g$$

$$- \int_S^T \int e^{\varphi} (div_g h - m_0 + h(\varphi)) |y_t|^2 dQ_g.$$

But

$$\int_S^T \int e^{\varphi} y_t h(y_t) dQ_g = \int_S^T \int_{\Gamma_1} e^{\varphi} \langle h, n \rangle_g |y_t|^2 d\Sigma_g - \int_S^T \int y_t div_g (e^{\varphi} h y_t) dQ_g$$

$$= \int_S^T \int_{\Gamma_1} e^{\varphi} \langle h, n \rangle_g |y_t|^2 d\Sigma_g - \int_S^T \int e^{\varphi} y_t h(y_t) dQ_g$$

$$- \int_S^T \int e^{\varphi} (div_g h + h(\varphi)) |y_t|^2 dQ_g,$$

then

$$2 \int_S^T \int e^{\varphi} y_t h(y_t) dQ_g = \int_S^T \int_{\Gamma_1} e^{\varphi} \langle h, n \rangle_g |y_t|^2 d\Sigma_g$$

$$- \int_S^T \int e^{\varphi} (div_g h + h(\varphi)) |y_t|^2 dQ_g,$$

so

$$\int_S^T \int e^{\varphi} y_{tt} M y dQ_g = \left. \int e^{\varphi} y_t M y d \quad _g \right|_S^T \tag{3.14}$$

$$- \int_S^T \int_{\Gamma_1} e^{\varphi} \langle h, n \rangle_g |y_t|^2 d\Sigma_g$$

$$+ m_0 \int_S^T \int e^{\varphi} |y_t|^2 dQ_g.$$

On the other hand, if we use the second Green's formula we obtain

$$\begin{aligned}
&\int_S^T \int e^{\varphi} \left(-\Delta_g y - \langle D\varphi, Dy\rangle_g\right) My dQ_g \\
= &\ -\int_S^T \int_\Gamma e^{\varphi} \frac{\partial y}{\partial n} My d\Sigma_g + \int_S^T \int e^{\varphi} \langle Dy, D\left(My\right)\rangle_g dQ_g \\
= &\ -\int_S^T \int_\Gamma e^{\varphi} \frac{\partial y}{\partial n} My d\Sigma_g + 2\int_S^T \int e^{\varphi} \langle Dy, D\left(h\left(y\right)\right)\rangle_g dQ_g \\
&+ \int_S^T \int e^{\varphi} \langle Dy, D\left(div_g h + h\left(\varphi\right)\right)\rangle_g y dQ_g \\
&+ \int_S^T \int e^{\varphi} \left(div_g h - m_0 + h\left(\varphi\right)\right) \|Dy\|_g^2 dQ_g.
\end{aligned}$$

By identity (1.2)

$$\begin{aligned}
&\int_S^T \int e^{\varphi} \left(-\Delta_g y - \langle D\varphi, Dy\rangle_g\right) My dQ_g \\
= &\ -\int_S^T \int_\Gamma e^{\varphi} \frac{\partial y}{\partial n} My d\Sigma_g + 2\int_S^T \int e^{\varphi} Dh\left(Dy, Dy\right) dQ_g \\
&+ \int_S^T \int e^{\varphi} h\left(\|Dy\|_g^2\right) dQ_g + \int_S^T \int e^{\varphi} \langle Dy, D\left(div_g h + h\left(\varphi\right)\right)\rangle_g y dQ_g \\
&+ \int_S^T \int e^{\varphi} \left(div_g h - m_0 + h\left(\varphi\right)\right) \|Dy\|_g^2 dQ_g.
\end{aligned}$$

Then, we find

$$\int_S^T \int e^{\varphi} \left(-\Delta_g y - \langle D\varphi, Dy \rangle_g \right) MydQ_g \tag{3.15}$$

$$= \quad -\int_S^T \int_\Gamma e^{\varphi} \frac{\partial y}{\partial n} Myd\Sigma_g + \int_S^T \int_\Gamma e^{\varphi} \langle h, n \rangle_g \|Dy\|_g^2 \, d\Sigma_g$$

$$+2\int_S^T \int e^{\varphi} Dh\,(Dy, Dy)\, dQ_g - m_0 \int_S^T \int e^{\varphi} \|Dy\|_g^2 \, dQ_g$$

$$+\int_S^T \int e^{\varphi} \langle Dy, D\,(div_g h + h\,(\varphi)) \rangle_g \, ydQ_g.$$

If we replace (3.14) and (3.15) in (3.13), we find

$$2\int_S^T \int e^{\varphi} Dh\,(Dy, Dy)\, dQ_g + m_0 \int_S^T \int e^{\varphi} \left(|y_t|^2 - \|Dy\|_g^2\right) dQ_g$$

$$= \quad -\int e^{\varphi} y_t Myd \quad {}_g \Big|_S^T + \int_S^T \int_{\Gamma_0} e^{\varphi} \left(2\frac{\partial y}{\partial n} h\,(y) - \langle h, n \rangle_g \|Dy\|_g^2\right) d\Sigma_g$$

$$+\int_S^T \int_{\Gamma_1} e^{\varphi} \left(\frac{\partial y}{\partial n} My + \langle h, n \rangle_g \left(|y_t|^2 - \|Dy\|_g^2\right)\right) d\Sigma_g$$

$$-\int_S^T \int e^{\varphi} \langle Dy, D\,(div_g h + h\,(\varphi)) \rangle_g \, ydQ_g.$$

This imply

$$2m_0\,(T - S)\,E\,(T) \le 2m_0 \int_S^T E\,(t)\,dt \le I \quad + I_{\Sigma_0} + I_{\Sigma_1} + I_Q,$$

where

$$I \quad = -\int e^{\varphi} y_t Myd \quad {}_g \Big|_S^T,$$

$$I_{\Sigma_0} = \int_S^T \int_{\Gamma_0} e^{\varphi} \left(2\frac{\partial y}{\partial n} h\,(y) - \langle h, n \rangle_g \|Dy\|_g^2\right) d\Sigma_g,$$

$$I_{\Sigma_1} = \int_S^T \int_{\Gamma_1} e^{\varphi} \left(\frac{\partial y}{\partial n} My + \langle h, n \rangle_g \left(|y_t|^2 - \|Dy\|_g^2 \right) \right) d\Sigma_g,$$

and

$$I_Q = - \int_S^T \int e^{\varphi} \langle Dy, D(div_g h + h(\varphi)) \rangle_g y dQ_g.$$

But, by (3.12),

$$\begin{aligned} I &= - \int e^{\varphi} y_t My d \quad {}_g \Big|_S^T \leq C(E(S) + E(T)) \\ &\leq CE(T) + C \int_S^T \int_{\Gamma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma_g, \end{aligned}$$

since $y = 0$ on Γ_0 then (see $[LTY01]$)

$$\|Dy\|_g^2 = \left| \frac{\partial y}{\partial n} \right|^2 \text{ and } h(y) = \langle h, Dy \rangle_g = \langle h, n \rangle_g \frac{\partial y}{\partial n},$$

so

$$I_{\Sigma_0} = \int_S^T \int_{\Gamma_0} e^{\varphi} \langle h, n \rangle_g \left| \frac{\partial y}{\partial n} \right|^2 d\Sigma_g \leq 0.$$

$$\begin{aligned} I_{\Sigma_1} &= \int_S^T \int_{\Gamma_1} e^{\varphi} \frac{\partial y}{\partial n} (2h(y) + (div_g h - m_0 + h(\varphi)) y) d\Sigma_g \\ &\quad + \int_S^T \int_{\Gamma_1} e^{\varphi} \langle h, n \rangle_g \left(|y_t|^2 - \|Dy\|_g^2 \right) d\Sigma_g \\ &\leq -2 \int_S^T \int_{\Gamma_1} e^{\varphi} (by_t + f(y_t)) \langle h, Dy \rangle_g d\Sigma_g \\ &\quad - \int_S^T \int_{\Gamma_1} e^{\varphi} (div_g h - m_0 + h(\varphi)) (by_t + f(y_t)) y d\Sigma_g \\ &\quad + \sup_{\Gamma_1} \langle h, n \rangle_g \int_S^T \int_{\Gamma_1} e^{\varphi} |y_t|^2 - h_0 \int_S^T \int_{\Gamma_1} \|Dy\|_g^2. \end{aligned}$$

We have

$$\begin{aligned} I_{\Sigma_1} &\leq C\int_S^T\int_{\Gamma_1} e^{\varphi}\left(|f(y_t)|^2+|y_t|^2\right)d\Sigma_g \\ &+\left(\eta\sup_{\Gamma_1}\|h\|_g^2-h_0\right)\int_S^T\int_{\Gamma_1} e^{\varphi}\|Dy\|_g^2\,d\Sigma_g+\eta\beta\int_S^T E(t)\,dt, \end{aligned}$$

for all $\eta>0$.

So

$$\begin{aligned} I_{\Sigma_1} &\leq C\int_S^T\int_{\Gamma_1} e^{\varphi}\left(|f(y_t)|^2+|y_t|^2\right)d\Sigma_g \\ &+\left(\eta\sup_{\Gamma_1}\|h\|_g^2-h_0\right)\int_S^T\int_{\Gamma_1} e^{\varphi}\|Dy\|_g^2\,d\Sigma_g \\ &+\eta C_2^2(T-S)E(S), \end{aligned}$$

This imply, by (3.12),

$$\begin{aligned} I_{\Sigma_1} &\leq C\int_S^T\int_{\Gamma_1} e^{\varphi}\left(|f(y_t)|^2+|y_t|^2\right)d\Sigma_g \\ &+\left(\eta\sup_{\Gamma_1}\|h\|_g^2-h_0\right)\int_S^T\int_{\Gamma_1} e^{\varphi}\|Dy\|_g^2\,d\Sigma_g \\ &+\eta\beta(T-S)E(T). \end{aligned}$$

Similarly, we find

$$
\begin{aligned}
I_Q &= -\int\limits_S^T \int e^{\varphi} \langle Dy, D(div_g h + h(\varphi)) \rangle_g \, y dQ_g \\
&\leq \eta C \int\limits_S^T E(t)\, dt + C \int\limits_S^T \int e^{\varphi} |y|^2 \, dQ_g \\
&\leq \eta C (T-S) E(T) + C \int\limits_S^T \int\limits_{\Gamma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma_g \\
&\quad + C \int\limits_S^T \int e^{\varphi} |y|^2 \, dQ_g .
\end{aligned}
$$

with η sufficiently small and $T-S \geq T_0$, where T_0 is sufficiently large, we obtain the desired result. ■

Remark 3.8 *The multiplier $h(y) + (div_g h - m_0)\, y$ is the counterparts of the classical multiplier $m(y) + (n-1)\, y$ where $m = x - x_0$ ($x_0 \in \mathbb{R}^n$) and Δ_g is the classical Laplacian in the Euclidean metric.*

3.4 *Absorbtion of the lower order term.*

To absorb the lower order term in lemma 3.7 we combine the idea in $[FF04]$ with the one in $[LT93]$.

Lemma 3.9 *For all $T - S > T_0$, where T_0 sufficiently large, we have*

$$
\int\limits_S^T \int e^{\varphi} |y|^2 \, dQ_g \leq C \int\limits_S^T \int\limits_{\Gamma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma_g .
$$

Proof. It is sufficient to prove (see $[FF04]$) that, for some T_0 large enough, we have

$$\int_0^{T_0}\int e^{\varphi}|y|^2 \leq C\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(|f(y_t)|^2+|y_t|^2\right)d\Sigma_g.$$

We argue by contradiction. Let (y_k) be a sequence of solutions to (3.1) such that

$$\lim_{k\to\infty}\frac{\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(|f(y_{kt})|^2+|y_{kt}|^2\right)d\Sigma_g}{\int_0^{T_0}\int e^{\varphi}|y_k|^2\,dQ_g}=0.$$

If we put

$$C_k=\left(\int_0^{T_0}\int e^{\varphi}|y_k|^2\,dQ_g\right)^{\frac{1}{2}}$$

and

$$\overline{y}_k=\frac{y_k}{C_k},$$

then we can see that $\overline{y}_k$ is solution of

$$\begin{cases}\overline{y}_{ktt}-\Delta_g\overline{y}_k-\langle D\varphi, D\overline{y}_k\rangle_g=0 \text{ in }]0,T_0[\times \quad ,\\ \overline{y}_k=0 \text{ on }]0,T_0[\times\Gamma_0,\\ \frac{\partial\overline{y}_k}{\partial n}+b\overline{y}_k+\frac{1}{C_k}f(y_{kt})=0 \text{ on }]0,T_0[\times\Gamma_1,\end{cases}$$

moreover, we have

$$\int_0^{T_0}\int e^{\varphi}|\overline{y}_k|^2\,dQ_g=1, \tag{3.16}$$

$$\lim_{k\to\infty}\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(\left|\frac{f(y_{kt})}{C_k}\right|^2+|\overline{y}_{kt}|^2\right)d\Sigma_g=0, \tag{3.17}$$

and

$$\overline{E}_k(t):=\frac{1}{2}\left(\int_0^{T_0}\int e^{\varphi}\left(|\overline{y}_{kt}|^2+\|D\overline{y}_k\|_g^2\right)d\quad_g\right)=\frac{E_k(t)}{C_k^2},$$

where E_k represents the energy of y_k and $\overline{E}_k$ the energy of $\overline{y}_k$.

We apply lemma 3.7 with $S=0$ and $T=T_0$ to find

$$\overline{E}_k\left(T_0\right) \leq C\left(\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(\left|\frac{f\left(y_{kt}\right)}{C_k}\right|^2+\left|\overline{y}_{kt}\right|^2\right) d\Sigma_g+1\right). \tag{3.18}$$

On the other hand, by (3.12),

$$\overline{E}_k\left(0\right)=\frac{E_k\left(0\right)}{C_k^2} \leq \overline{E}_k\left(T_0\right)+C\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(\left|\frac{f\left(y_{kt}\right)}{C_k}\right|^2+\left|\overline{y}_{kt}\right|^2\right) d\Sigma_g.$$

If we use (3.18) we find

$$\overline{E}_k\left(0\right) \leq C\left(\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left(\left|\frac{f\left(y_{kt}\right)}{C_k}\right|^2+\left|\overline{y}_{kt}\right|^2\right) d\Sigma_g+1\right).$$

From (3.17), we obtain that $\left(\overline{E}_k\left(0\right)\right)$ is bounded, then there exists a subsequence $\left(\overline{y}_k\right)$ denoted by the same symbol such that

$$\overline{y}_k \rightarrow \overline{y} \text{ weakly * in } L^{\infty}\left(]0,T_0[\times \quad\right),$$

and

$$\overline{y}_k \rightarrow \overline{y} \text{ weakly in } L^2\left(]0,T_0[\times\Gamma\right).$$

We shall consider two cases

Case 1 $\overline{y}=0$.

Then

$$\lim_{k\rightarrow\infty}\int_0^{T_0}\int e^{\varphi}\left|\overline{y}_k\right|^2 dQ_g=0,$$

this contradicts (3.16).

Case 2 $\overline{y}\neq 0$.

First we have from (3.17)

$$\lim_{k\rightarrow\infty}\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left|\overline{y}_{kt}\right|^2 d\Sigma_g=\lim_{k\rightarrow\infty}\int_0^{T_0}\int_{\Gamma_1} e^{\varphi}\left|\frac{f\left(y_{kt}\right)}{C_k}\right|^2 d\Sigma_g=0,$$

then $z = \overline{y_t}$ is solution of

$$\begin{cases} z_{tt} - \Delta_g z - \langle D\varphi, Dz \rangle_g = 0 \text{ in }]0, T_0[\times \quad , \\ z = 0 \text{ on }]0, T_0[\times \Gamma, \\ \frac{\partial z}{\partial n} = 0 \text{ on }]0, T_0[\times \Gamma_1. \end{cases}$$

So $z = 0$, then

$$\begin{cases} -\Delta_g \overline{y} - \langle D\varphi, D\overline{y} \rangle_g = 0 \text{ in }]0, T_0[\times \quad , \\ \overline{y} = 0 \text{ on }]0, T_0[\times \Gamma_0, \\ \frac{\partial \overline{y}}{\partial n} = 0 \text{ on }]0, T_0[\times \Gamma_1. \end{cases}$$

If we multiply the first equation by $e^{\varphi}\overline{y}$, integrate over and use the second Green's formula we find

$$0 < \int e^{\varphi} |\overline{y}|^2 \, d \quad_g \leq C \int e^{\varphi} \|D\overline{y}\|_g^2 \, d \quad_g = 0$$

Contradiction. ■

3.5 *Completion of the proof of main theorem.*

By combining the result of lemma 3.7 with the one of lemma 3.9 we obtain, for any value of T and S such that $T - S \geq T_0$, where T_0 sufficiently large,

$$E(T) \leq C \int_S^T \int_{\Gamma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma_g. \tag{3.19}$$

But

$$
\begin{aligned}
&\int_S^T \int_{\Gamma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma_g \\
= \quad & \int_{|y_t| \geq 1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma_g + \int_{|y_t| \leq 1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma_g \\
\leq \quad & \left(c_1^{-1} + c_2 \right) \int_{|y_t| \geq 1} e^{\varphi} f(y_t) y_t d\Sigma_g + e^{\varphi^*} \int_{|y_t| \leq 1} h \left(f(y_t) y_t \right) d\Sigma_g .
\end{aligned}
$$

If we use Jensen's inequality we find

$$
\begin{aligned}
&\int_S^T \int_{\Gamma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma_g \leq \left(c_1^{-1} + c_2 \right) \int_{|y_t| \geq 1} e^{\varphi} f(y_t) y_t d\Sigma_g \\
&+ e^{\varphi^*} mes\Sigma_1 h \left(\frac{1}{mes\Sigma_1} \int_{|y_t| \leq 1} f(y_t) y_t d\Sigma_g \right),
\end{aligned}
$$

so

$$
\int_S^T \int_{\Gamma_1} e^{\varphi} \left(|f(y_t)|^2 + |y_t|^2 \right) d\Sigma_g \leq \frac{1}{CK} \left(\sigma I + \widetilde{h} \right) \left(\int_S^T \int_{\Gamma_1} f(y_t) y_t d\Sigma_g \right)
$$

where

$$
\widetilde{h}(s) = h \left(\frac{s}{mes\Sigma_1} \right), \; K = \frac{1}{C e^{\varphi^*} mes\Sigma_1} \text{ and } \sigma = \frac{c_1^{-1} + c_2}{e^{\varphi^*} mes\Sigma_1}.
$$

C represents the constant in (3.19).

If we replace in (3.19) we find

$$
K E(T) \leq \left(\sigma I + \widetilde{h} \right) \left(\int_S^T \int_{\Gamma_1} f(y_t) y_t d\Sigma_g \right).
$$

Since $\left(\sigma I+\widetilde{h}\right)$ is invertible for any positive value of a constant σ, we obtain

$$\begin{aligned}\left(cI+\widetilde{h}\right)^{-1}(KE\left(T\right)) &\leq \int_{S}^{T}\int_{\Gamma_1} f\left(y_t\right)y_t d\Sigma_g \\ &\leq e^{-\varphi_*}\int_{S}^{T}\int_{\Gamma_1} e^{\varphi} f\left(y_t\right)y_t d\Sigma_g \\ &\leq e^{-\varphi_*}\left(E\left(S\right)-E\left(T\right)\right).\end{aligned}$$

If we put $p\left(s\right)=e^{\varphi_*}\left(\sigma I+\widetilde{h}\right)^{-1}(Ks)$ then

$$p(E\left(T\right))+E\left(T\right)\leq E\left(S\right)$$

Finally, the result follows from theorem 1.31.

Remark 3.10 *Using the idea in* $[KZ87, Lag88]$, *we can relax the condition (3.5) to*

$$\langle Dd, n\rangle_g \geq 0, \text{ for all } x\in\Gamma_1,$$

at the expense of replacing the boundary feedback in system (3.1) by

$$\frac{\partial y}{\partial n}+\langle Dd, n\rangle_g\left(by_t+f\left(y_t\right)\right)=0 \text{ on } \Gamma_1.$$

Remark 3.11 *The techniques we develop in this chapter can be combined with the method introduced by F. Conrad and B. Rao in* $[CR91]$ *in order to obtain the uniform decay rate of the following problem*

$$\begin{cases} y_{tt}-\Delta_g y-\langle D\varphi, Dy\rangle_g=0 \text{ in } Q, \\ y=0 \text{ on } \Sigma_0, \\ \frac{\partial y}{\partial n}+ay+by_t+f\left(y_t\right)=0 \text{ on } \Sigma_1, \\ y(0)=y_0,\ y_t\left(0\right)=y_1 \text{ in } \quad , \end{cases}$$

here a is a positive bounded function.

In this case we consider the energy defined by

$$E(t) = \frac{1}{2}\int e^{\varphi}\left(|y_t|^2 + \|Dy\|_g^2\right) d \quad_g + \int_{\Gamma_1} e^{\varphi} a\, |y|^2\, d\Gamma_g.$$

3.6 *Application: The case of the second order hyperbolic equation with variable coefficients and a polynomial growth at the origin of the function feedback.*

Consider the second order hyperbolic equations with variable coefficients

$$\begin{cases} y_{tt} + \mathcal{A}y - \langle D\psi, Dy\rangle_g = 0 \text{ in } Q, \\ y = 0 \text{ on } \Sigma_0, \\ \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} + by_t + \xi(y_t) = 0 \text{ on } \Sigma_1, \\ y(0) = y_0,\ y_t(0) = y_1 \text{ in } \quad . \end{cases} \tag{3.20}$$

The following result is a consequence of theorem 3.3.

Theorem 3.12 *If $\xi : \mathbb{R} \to \mathbb{R}$ is an increasing function of class C^1 such that for some positive constants M_5, M_6, M_7 and M_8 we have*

$$M_5 |s|^2 \leq \xi(s)\, s \leq M_6 |s|^2 \text{ for } |s| \geq 1,$$

and

$$\xi(s)\, s > 0 \text{ for all } s \neq 0,$$

and

$$M_7 |s|^\gamma \le |\xi(s)| \le M_8 |s|^{\frac{1}{\gamma}} \ \textit{for}\ |s| \le 1, \tag{3.21}$$

for some $\gamma \ge 1$.

Then

$$E(t) \le ce^{-\omega t} \ \textit{if}\ \gamma = 1,$$

and

$$E(t) \le ct^{\frac{2}{1-\gamma}} \ \textit{if}\ \gamma > 1,$$

where $c, \omega > 0$.

Proof. (3.20) is equivalent to

$$\begin{cases} y_{tt} - \Delta_g y - \langle D\varphi, Dy\rangle_g = 0 \text{ in } Q, \\ y = 0 \text{ on } \Sigma_0, \\ \frac{\partial y}{\partial n} + \|\upsilon_{\mathcal{A}}\|_g^{-1} b y_t + f(y_t) = 0 \text{ on } \Sigma_1, \\ y(0) = y_0,\ y_t(0) = y_1 \text{ in } \quad . \end{cases} \tag{3.22}$$

where $\varphi = \psi - \frac{1}{2}\log\det G$ and $f(s) = \|\upsilon_{\mathcal{A}}\|_g^{-1}\xi(s)$ for all $s \in \mathbb{R}$.

From (3.21) we have

$$M_9 |s|^\gamma \le |f(s)| \le M_{10} |s|^{\frac{1}{\gamma}} \ \text{for}\ |s| \le 1,$$

for some positive constants M_9 and M_{10}.

Then

$$h(s) = \alpha s^m$$

with

$$\alpha = M_9^{\frac{-2}{\gamma+1}} + M_{10}^{\frac{2\gamma}{\gamma+1}} \text{ and } m = \frac{2}{\gamma+1}.$$

Repeating the proof of corollary 2 in [LT93] we find the desired result. ■

Remark 3.13 *Theorem 3.12 removes the assumption of smallness on* $\|\varphi\|_\infty := \sup_{x\in\overline{\ }} \|\nabla_0\varphi(x)\|_0$ *made in* [$GueII$03] *to obtain the uniform stabilization of the wave equation with constant coefficients.*

Chapter 4

Exponential decay rate of coupled system of two Schrödinger equations with variable coefficients and damped by two Neumann boundary feedback.

4.1 *Introduction.*

Consider a coupled system of two complex Schrödinger equations with variable coefficients and damped by two Neumann boundary feedback.

$$\begin{cases} iy_t + \mathcal{A}y + az = 0 \text{ in } Q, \\ iz_t + \mathcal{A}z + ay = 0 \text{ in } Q, \\ y = z = 0 \text{ on } \Sigma_0, \\ \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} + b_1 y_t = 0 \text{ and } \frac{\partial z}{\partial \upsilon_{\mathcal{A}}} + b_2 z_t = 0 \text{ on } \Sigma_1, \\ y(0) = y_0 \text{ and } z(0) = z_0 \text{ in } \quad , \end{cases} \tag{4.1}$$

where

$$0 < a_* \le a(s) \le a^* \text{ for all } s \in \quad ,$$

and, for $l = 1, 2$,

$$0 < b_{l*} \le b_l(s) \le b_l^* \text{ for all } s \in \Gamma_1,$$

for some positive constants a^*, a_*, b_{l*} and b_l^*.

We now set up some geometric conditions which are sufficient to get the energy decay rate estimates of system (4.1).

Assume that there exists a real vector field $h \in \left[C^1\left(\overline{\quad}\right)\right]^n$ on Riemannian manifold $(\mathbb{R}^n, g)$, a constant $m_0 > 0$ such that

$$Dh(X, X) \ge m_0 \|X\|_g^2, \text{ for all } X \in \mathbb{R}_x^n. \tag{4.2}$$

and

$$2m_0 > C_1 C_h, \tag{4.3}$$

where $C_h = \sup \|\nabla_g (div_0 h)\|_g$.

We assume that Γ_0, Γ_1 are taken as

$$\Gamma_0 = \{x \in \Gamma : h.\upsilon \leq 0\},$$

and

$$\Gamma_1 = \{x \in \Gamma : h.\upsilon \geq h_0 > 0\}.$$

Remark 4.1 *We note that assumption (4.3) has been used in* [LT89] *to study the exact controllability of real wave equation with constant coefficients.*

Remark 4.2 *In general, the vector field h is not the covariant differential of a function as in chapter 2 and 3.*

Remark 4.3 *We can replace assumption (4.2) by the following*

$$Dh(X, X) = \varkappa(x) \|X\|_g^2, \text{ for all } X \in \mathbb{R}_x^n, \tag{4.4}$$

where

$$\varkappa_* = \min_{x \in} \varkappa(x) > 0. \tag{4.5}$$

So, this assumption is stronger then assumption (4.2). Indeed, if the vector field h meets conditions (4.4) and (4.5), it will meet condition (4.2) with $m_0 = \varkappa_*$.

Our goal in this chapter is to prove that we can apply the Riemann geometric approach to study the energy decay rates of complex systems (as an example of such systems, we consider the system (4.1)). We note that this approach was first introduced to study the problem of controllability of certain real equations with variable coefficients (see for example [$Yao99$]). Later, several results of energy decay rates for real systems were given by this approach, see for example [$GY06$] for the exponential and polynomial energy decay rates of the nonlinear Euler Bernoulli equations.

4.2 *Existence, uniqueness and regularity of solution.*

We define an operator $\mathfrak{A}$ as

$$\mathfrak{A}(u_1, u_2) = \mathrm{i}\left(\mathcal{A}u_1 + au_2, \mathcal{A}u_2 + au_1\right),$$

with domain

$$D(\mathfrak{A}) = \left\{ \begin{array}{c} (u_1, u_2) \in H^2(\quad) \cap H^1_{\Gamma_0}(\quad) \times H^2(\quad) \cap H^1_{\Gamma_0}(\quad), \\ \left[\frac{\partial u_1}{\partial v_{\mathcal{A}}} + \mathrm{i}b_1\left(\mathcal{A}u_1 + au_2\right)\right]_{\Gamma_1} = 0, \\ \left[\frac{\partial u_2}{\partial v_{\mathcal{A}}} + \mathrm{i}b_2\left(\mathcal{A}u_2 + au_1\right)\right]_{\Gamma_1} = 0. \end{array} \right\}$$

Let $Y = (y, z)$, we may interprets the system (4.1) in the following form

$$\left\{ \begin{array}{l} Y_t = \mathfrak{A}Y, \\ Y(0) = Y_0. \end{array} \right. \tag{4.6}$$

then the solvability of (4.1) is equivalent to the one of (4.6).

We prove the following result.

Theorem 4.4 *If*

$$(y_0, z_0) \in H^1_{\Gamma_0}(\quad) \times H^1_{\Gamma_0}(\quad),$$

then the system (4.6) has a unique solution

$$(y,z) \in C([0,+\infty)\,; H^1_{\Gamma_0}(\ ) \times H^1_{\Gamma_0}(\ )).$$

Moreover, if

$$(y_0, z_0) \in D(\mathfrak{A}),$$

then the system (4.6) has a unique solution

$$(y,z) \in C([0,+\infty)\,; D(\mathfrak{A})) \cap C^1([0,+\infty)\,; H^1_{\Gamma_0}(\ ) \times H^1_{\Gamma_0}(\ )).$$

Proof. It is sufficient to prove that $\mathfrak{A}$ is a maximal dissipative operator on $H^1_{\Gamma_0}(\ ) \times H^1_{\Gamma_0}(\ )$. For this reason, we proceed as in the proof of theorem 2.7.

■

4.3 *Preliminaries results.*

We define the energy E of (4.1) by

$$E(t) = E\left(y\left(t\right), z\left(t\right)\right) = E_1\left(y\left(t\right)\right) + E_2\left(z\left(t\right)\right) + \mathrm{Re}\int a y \bar{z} d\ \ , \tag{4.7}$$

for all $t \geq 0$.

Where

$$E_1\left(t\right) = E_1\left(y\left(t\right)\right) = \frac{1}{2}\int \left\|\nabla_g y\right\|_g^2 d\ \ ,$$

and

$$E_2\left(t\right) = E_2\left(z\left(t\right)\right) = \frac{1}{2}\int \left\|\nabla_g z\right\|_g^2 d\ \ .$$

Here $\bar{\cdot}$ design the conjugate.

Remark 4.5 *It is easy to show that E is equivalent to $E_1 + E_2$ when we take $\|a\|_{L^\infty(\ )}$ sufficiently small.*

We begin by establishing a formula concerning the derivative of the energy of the system (4.1).

Lemma 4.6 *We have*

$$E'(t) = -\int_{\Gamma_1} \left(b_1 |y_t|^2 + b_2 |z_t|^2 \right) d\Gamma, \tag{4.8}$$

for all $t \geq 0$.

Proof. We multiply both side of the first equation of (4.1) by $\overline{y}_t$ and use the third Green's formula, we obtain

$$\mathrm{i} \int |y_t|^2 d \quad - \int_\Gamma \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} \overline{y_t} d\Gamma + \int \langle \nabla_g y, \nabla_g y_t \rangle_g d \quad + \int a z \overline{y_t} d \quad = 0.$$

Taking the real part, we obtain

$$\mathrm{Re} \int \langle \nabla_g y, \nabla_g y_t \rangle_g d \quad = \mathrm{Re} \int_\Gamma \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} \overline{y_t} d\Gamma - \mathrm{Re} \int a z \overline{y_t} d \quad .$$

But

$$\mathrm{Re} \int_\Gamma \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} \overline{y_t} d\Gamma = -\,\mathrm{Re} \int_{\Gamma_1} b_1 |y_t|^2 d\Gamma = - \int_{\Gamma_1} b_1 |y_t|^2 d\Gamma,$$

then

$$\mathrm{Re} \int \langle \nabla_g y, \nabla_g y_t \rangle_g d \quad = - \int_{\Gamma_1} b_1 |y_t|^2 d\Gamma - \mathrm{Re} \int a z \overline{y_t} d \quad . \tag{4.9}$$

We obtain similar identity for z. That is

$$\mathrm{Re} \int \langle \nabla_g z, \nabla_g z_t \rangle_g d \quad = - \int_{\Gamma_1} b_2 |z_t|^2 d\Gamma - \mathrm{Re} \int a y \overline{z_t} d \quad , \tag{4.10}$$

but

$$\operatorname{Re}\int\left(\langle\nabla_g y,\nabla_g y_t\rangle_g+\langle\nabla_g z,\nabla_g z_t\rangle_g+a\left(\overline{z}y_t+\overline{z_t}y\right)\right)d\quad=E^{'}\left(t\right),$$

then (4.9)+(4.10) gives (4.8). ■

We provide, now, an identity which is given by the multiplier method.

Lemma 4.7 *We have*

$$\begin{aligned}&2\int\limits_Q\left(Dh\left(\nabla_g\operatorname{Re}y,\nabla_g\operatorname{Re}y\right)+Dh\left(\nabla_g\operatorname{Im}y,\nabla_g\operatorname{Im}y\right)\right)dQ\\&+2\int\limits_Q\left(Dh\left(\nabla_g ReZ,\nabla_g\operatorname{Re}z\right)+Dh\left(\nabla_g ImZ,\nabla_g ImZ\right)\right)dQ\quad(4.11)\\=\;&I\;+I_{\Sigma_0}+I_{\Sigma_1}+I_Q,\end{aligned}$$

where

$$I\;=\operatorname{Im}\int\left(yh\left(\overline{y}\right)+zh\left(\overline{z}\right)\right)d\;\;\Big|_0^T,$$

$$I_{\Sigma_0}=\int\limits_{\Sigma_0}\frac{h.\upsilon}{\|\upsilon_{\mathcal{A}}\|_g^2}\left(\left|\frac{\partial y}{\partial\upsilon_{\mathcal{A}}}\right|^2+\left|\frac{\partial z}{\partial\upsilon_{\mathcal{A}}}\right|^2\right)d\Sigma,$$

$$\begin{aligned}I_{\Sigma_1}\;=\;&-\operatorname{Im}\int\limits_{\Sigma_1}h.\upsilon\left(y\overline{y_t}+z\overline{z_t}\right)d\Sigma-\int\limits_{\Sigma_1}h.\upsilon\left(\|\nabla_g y\|_g^2+\|\nabla_g z\|_g^2\right)d\Sigma\\&-\operatorname{Re}\int\limits_{\Sigma_1}b_1y_t\left(2h\left(\overline{y}\right)+div_0h\overline{y}\right)d\Sigma-\operatorname{Re}\int\limits_{\Sigma_1}b_2z_t\left(2h\left(\overline{z}\right)+div_0h\overline{z}\right)d\Sigma,\end{aligned}$$

and

$$\begin{aligned}I_Q\;=\;&-\operatorname{Re}\int\limits_Q\langle\nabla_g y,\nabla_g\left(div_0h\right)\rangle_g\,\overline{y}dQ-\operatorname{Re}\int\limits_Q az\left(2h\left(\overline{y}\right)+div_0h\overline{y}\right)dQ\\&-\operatorname{Re}\int\limits_Q\langle\nabla_g z,\nabla_g\left(div_0h\right)\rangle_g\,\overline{z}dQ-\operatorname{Re}\int\limits_Q ay\left(2h\left(\overline{z}\right)+div_0h\overline{z}\right)dQ.\end{aligned}$$

Proof. We multiply the first equation of (4.1) by $h(\overline{y})$ and integrate over Q, we obtain

$$
\begin{aligned}
0 &= \int_Q (\mathrm{i}y_t + \mathcal{A}y + az) h(\overline{y})\, dQ = \mathrm{i}\int_Q y_t h(\overline{y})\, dQ \\
&+ \int_Q \mathcal{A}y h(\overline{y})\, dQ + \int_Q az h(\overline{y})\, dQ.
\end{aligned}
\tag{4.12}
$$

Taking the real part

$$
0 = -\operatorname{Im}\int_Q y_t h(\overline{y})\, dQ + \operatorname{Re}\int_Q \mathcal{A}y h(\overline{y})\, dQ + \operatorname{Re}\int_Q az h(\overline{y})\, dQ. \tag{4.13}
$$

By integration by parts, we have

$$
\int_Q y_t h(\overline{y})\, dQ = \int yh(\overline{y})\, d\,\Big|_0^T - \int_Q yh(\overline{y_t})\, dQ.
$$

So

$$
\begin{aligned}
\int_Q y_t h(\overline{y})\, dQ &= \int yh(\overline{y})\, d\,\Big|_0^T - \int_\Sigma y\overline{y_t}h.\upsilon d\Sigma \\
&+ \int_Q div_0 hy\overline{y_t} dQ + \int_Q \overline{y_t}h(y)\, dQ.
\end{aligned}
$$

Then

$$
2\mathrm{i}\operatorname{Im}\int_Q y_t h(\overline{y})\, dQ = \int yh(\overline{y})\, d\,\Big|_0^T - \int_\Sigma y\overline{y_t}h.\upsilon d\Sigma + \int_Q div_0 hy\overline{y_t} dQ,
$$

so

$$
\begin{aligned}
2\operatorname{Im}\int_Q y_t h(\overline{y})\, dQ &= \operatorname{Im}\int yh(\overline{y})\, d\,\Big|_0^T - \operatorname{Im}\int_\Sigma y\overline{y_t}h.\upsilon d\Sigma \\
&+ \operatorname{Im}\int_Q div_0 hy\overline{y_t} dQ.
\end{aligned}
\tag{4.14}
$$

But

$$
\operatorname{Im}\int_Q div_0 hy\overline{y_t} dQ = \operatorname{Re}\int_Q div_0 hy(-\mathcal{A}\overline{y} - a\overline{z})\, dQ.
$$

Then

$$\operatorname{Im}\int_Q div_0 hy\overline{y_t}dQ = \operatorname{Re}\int_\Sigma \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} div_0 h\overline{y}d\Sigma - \int_Q div_0 h \left\|\nabla_g y\right\|_g^2 dQ$$
$$-\operatorname{Re}\int_Q \left\langle \nabla_g y, \nabla_g (div_0 h)\right\rangle_g \overline{y}dQ - \operatorname{Re}\int_Q aydiv_0 h\overline{z}dQ,$$

we insert in (4.14) to obtain

$$-\operatorname{Im}\int_Q y_t h(\overline{y})\,dQ = -\frac{1}{2}\operatorname{Im}\int yh(\overline{y})\,d\Big|_0^T + \frac{1}{2}\operatorname{Im}\int_{\Sigma_1} y\overline{y_t}h.\upsilon d\Sigma \tag{4.15}$$
$$-\frac{1}{2}\operatorname{Re}\int_{\Sigma_1}\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} div_0 h\overline{y}d\Sigma + \frac{1}{2}\int_Q div_0 h\left\|\nabla_g y\right\|_g^2 dQ$$
$$+\frac{1}{2}\operatorname{Re}\int_Q \left\langle \nabla_g y, \nabla_g (div_0 h)\right\rangle_g \overline{y}dQ + \frac{1}{2}\operatorname{Re}\int_Q ay\,div_0 h\overline{z}dQ.$$

Concerning the term $\operatorname{Re}\int_Q \mathcal{A}yh(\overline{y})\,dQ$.

If we use the third Green´s Formula, we find

$$\operatorname{Re}\int_Q \mathcal{A}yh(\overline{y})\,dQ = -\operatorname{Re}\int_\Sigma \frac{\partial y}{\partial \upsilon_{\mathcal{A}}}h(\overline{y})\,d\Sigma + \int_Q Dh(\nabla_g \operatorname{Re} y, \nabla_g \operatorname{Re} y)\,dQ$$
$$+\int_Q Dh(\nabla_g \operatorname{Im} y, \nabla_g \operatorname{Im} y)\,dQ + \frac{1}{2}\int_Q h\left(\left\|\nabla_g y\right\|_g^2\right)dQ.$$

We apply (1.3) with $f_1 = 1$ and $f_2 = \left\|\nabla_g f\right\|_g^2$ to find

$$\begin{aligned} &\operatorname{Re}\int_Q \mathcal{A}yh(\overline{y})\,dQ \\ =\; & -\operatorname{Re}\int_\Sigma \left(\frac{\partial y}{\partial \upsilon_{\mathcal{A}}}h(\overline{y}) - \frac{1}{2}\left\|\nabla_g y\right\|_g^2 h.\upsilon\right)d\Sigma \\ &+\int_Q \left(Dh(\nabla_g \operatorname{Re} y, \nabla_g \operatorname{Re} y) + Dh(\nabla_g \operatorname{Im} y, \nabla_g \operatorname{Im} y)\right)dQ \\ &-\frac{1}{2}\int_Q div_0 h\left\|\nabla_g y\right\|_g^2 dQ. \end{aligned} \tag{4.16}$$

Finally, we insert (4.15) and (4.16) in (4.13) we obtain

$$
\begin{aligned}
& 2\int_Q \left(Dh\left(\nabla_g \operatorname{Re} y, \nabla_g \operatorname{Re} y\right) + Dh\left(\nabla_g \operatorname{Im} y, \nabla_g \operatorname{Im} y\right)\right) dQ \\
= \ & \operatorname{Im} \int yh\left(\overline{y}\right) d \left|_0^T \right. - \operatorname{Im} \int_{\Sigma_1} y\overline{y_t} h.\upsilon d\Sigma \\
& + \operatorname{Re} \int_\Sigma \left(\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} \left(2h\left(\overline{y}\right) + div_0 h\overline{y}\right) - \left\|\nabla_g y\right\|_g^2 h.\upsilon \right) d\Sigma \qquad (4.17) \\
& - \operatorname{Re} \int_Q az\left(2h\left(\overline{y}\right) + div_0 h\overline{y}\right) dQ - \operatorname{Re} \int_Q \left\langle \nabla_g y, \nabla_g \left(div_0 h\right)\right\rangle_g \overline{y} dQ.
\end{aligned}
$$

Similarly, we obtain for z the following identity

$$
\begin{aligned}
& 2\int_Q \left(Dh\left(\nabla_g ReZ, \nabla_g \operatorname{Re} z\right) + Dh\left(\nabla_g ImZ, \nabla_g ImZ\right)\right) dQ \\
= \ & \operatorname{Im} \int zh\left(\overline{z}\right) d \left|_0^T \right. - \operatorname{Im} \int_{\Sigma_1} z\overline{z_t} h.\upsilon d\Sigma \qquad (4.18) \\
& + \operatorname{Re} \int_\Sigma \left(\frac{\partial z}{\partial \upsilon_{\mathcal{A}}} \left(2h\left(\overline{z}\right) + div_0 h\overline{z}\right) - \left\|\nabla_g z\right\|_g^2 h.\upsilon \right) d\Sigma \\
& - \operatorname{Re} \int_Q ay\left(2h\left(\overline{z}\right) + div_0 h\overline{z}\right) dQ - \operatorname{Re} \int_Q \left\langle \nabla_g z, \nabla_g \left(div_0 h\right)\right\rangle_g \overline{z} dQ.
\end{aligned}
$$

Finally, we deduce (4.11) from (4.17)+(4.18). ■

4.4 *Exponential decay rate of the energy.*

The main result of this chapter is the following.

Theorem 4.8 *The energy E of the system (4.1) decays to zero at an exponential rate.*

Proof. *It is sufficient to prove that the integral $\int_0^\infty E(t)\,dt$ is convergent.*

For this end, we shall prove that there exist a constant C such that

$$\int_0^T E(t) \leq CE(0), \tag{4.19}$$

for all $T > 0$.

Letting $T \to \infty$ *we obtain the estimate*

$$\int_0^\infty E(t) \leq CE(0)\,,$$

this imply that the integral $\int_0^\infty E(t)\,dt$ *is convergent.*

First, we have

$$E(T) - E(0) = \int_0^T E'(t) = -\int_{\Sigma_1} \left(b_1 |y_t|^2 + b_2 |z_t|^2\right) d\Sigma \leq 0,$$

then

$$E(T) \leq E(0)\,, \tag{4.20}$$

and

$$\int_{\Sigma_1} \left(b_1 |y_t|^2 + b_2 |z_t|^2\right) d\Sigma = E(0) - E(T) \leq E(0)\,,$$

thus

$$b_{1*} \int_{\Sigma_1} |y_t|^2 \, d\Sigma \leq \int_{\Sigma_1} b_1 \, |y_t|^2 \, d\Sigma \leq \int_{\Sigma_1} \left(b_1 |y_t|^2 + b_2 |z_t|^2\right) d\Sigma \leq E(0)\,,$$

and

$$b_{2*} \int_{\Sigma_1} |z_t|^2 \, d\Sigma \leq \int_{\Sigma_1} b_2 \, |z_t|^2 \, d\Sigma \leq \int_{\Sigma_1} \left(b_1 |y_t|^2 + b_2 |z_t|^2\right) d\Sigma \leq E(0)\,,$$

so

$$\int_{\Sigma_1} |y_t|^2 \, d\Sigma \leq CE(0) \;\; \textit{and} \;\; \int_{\Sigma_1} |z_t|^2 \, d\Sigma \leq CE(0)\,. \tag{4.21}$$

In order to obtain (4.19) we need to majorette the terms: I *,* I_{Σ_l} $(l = 0, 1)$ *and* I_Q *in (4.11).*

We have

$$I \quad = \operatorname{Im} \int (yh(\overline{y}) + zh(\overline{z}))\, d \quad \Bigg|_0^T \leq C\,(E\,(0) + E\,(T))\,,$$

If we use (4.20) we find

$$I \quad \leq CE(0)\,.$$

We have

$$I_Q \leq C \left(\eta_1 {C_h}^2 + \frac{C_1^2}{\eta_1} + C \left\| a \right\|_{L^\infty(\quad)} \right) \int_0^T E\,(t)\, dt,$$

for all $\eta_1 > 0$.

If we take $\eta_1 = \frac{C_1}{C_h}$ *we find*

$$I_Q \leq C \left(2C_1 C_h + C \left\| a \right\|_{L^\infty(\quad)} \right) \int_0^T E\,(t)\, dt.$$

Concerning the terms I_{Σ_l}*, we have*

$$I_{\Sigma_0} \leq 0.$$

Put

$$I_{\Sigma_1} = I_{\Sigma_1}(y) + I_{\Sigma_1}(z)\,,$$

where

$$\begin{aligned} I_{\Sigma_1}(y) \quad &= \quad -\operatorname{Im} \int_{\Sigma_1} h.\upsilon y \overline{y_t} d\Sigma - \int_{\Sigma_1} h.\upsilon \left\| \nabla_g y \right\|_g^2 d\Sigma \\ &\quad -\operatorname{Re} \int_{\Sigma_1} b_1 y_t \left(2h\,(\overline{y}) + div_0 h \overline{y} \right) d\Sigma, \end{aligned}$$

and

$$\begin{aligned} I_{\Sigma_1}(z) &= -\operatorname{Im}\int_{\Sigma_1} h.\upsilon z\overline{z_t}d\Sigma - \int_{\Sigma_1} h.\upsilon \left\Vert \nabla_g z\right\Vert_g^2 d\Sigma \\ &\quad -\operatorname{Re}\int_{\Sigma_1} b_2 z_t\left(2h\left(\overline{z}\right) + div_0 h\overline{z}\right)d\Sigma, \end{aligned}$$

We start with $I_{\Sigma_1}(y)$.

We have

$$I_{\Sigma_1}(y) \leq C\int_{\Sigma_1} |y_t|^2 d\Sigma + C_2^2\eta_2\int_{Q} \left\Vert \nabla_g y\right\Vert_g^2 dQ + (C\eta_2 - h_0)\int_{\Sigma_1} \left\Vert \nabla_g y\right\Vert_g^2 d\Sigma,$$

for all $\eta_2 > 0$.

If we use (4.21) we find

$$I_{\Sigma_1}(y) \leq CE(0) + C_2^2\eta_2\int_{Q} \left\Vert \nabla_g y\right\Vert_g^2 dQ + (C\eta_2 - h_0)\int_{\Sigma_1} \left\Vert \nabla_g y\right\Vert_g^2 d\Sigma.$$

Similarly for $I_{\Sigma_1}(z)$.

$$I_{\Sigma_1}(z) \leq CE(0) + C_2^2\eta_2\int_{Q} \left\Vert \nabla_g z\right\Vert_g^2 dQ + (C\eta_2 - h_0)\int_{\Sigma_1} \left\Vert \nabla_g z\right\Vert_g^2 d\Sigma.$$

So

$$I_{\Sigma} \leq CE(0) + C\eta_2\int_0^T E(t)\,dt + (C\eta_2 - h_0)\int_{\Sigma_1}\left(\left\Vert \nabla_g y\right\Vert_g^2 + \left\Vert \nabla_g z\right\Vert_g^2\right)d\Sigma.$$

We insert in (4.11) and use (4.2) we obtain

$$\begin{aligned} &\left(m_0 - 2C_1C_h - \left\Vert a\right\Vert_{L^\infty(\ )} - \eta_2\right)C\int_0^T E(t)\,dt \\ &\leq \quad CE(0) + (C\eta_2 - h_0)\int_{\Sigma_1}\left(\left\Vert \nabla_g y\right\Vert_g^2 + \left\Vert \nabla_g z\right\Vert_g^2\right)d\Sigma. \end{aligned}$$

We use (4.3) and choose η_2 *and* $\left\Vert a\right\Vert_{L^\infty(\ )}$ *sufficiently small we find*

$$\int_0^T E(t)\,dt \leq CE(0).$$

■

Chapter 5

Polynomial decay rate of coupled system of two Schrödinger equations with variable coefficients and damped by one Neumann boundary feedback.

5.1 *Introduction.*

In the previous chapter, we have considered a coupled system of two Shrödinger equations with variable coefficients and damped by two Neumann boundary feedback, and an exponential decay rate was obtained. It is interesting to study this system where the Neumann boundary feedback appears only in one of the equation where as no effect term is applied to the second. That is, we consider the following system

$$\begin{cases} iy_t + \mathcal{A}y + az = 0 \text{ in } Q, \\ iz_t + \mathcal{A}z + ay = 0 \text{ in } Q, \\ y = 0 \text{ on } \Sigma_0,\ \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} + by_t = 0 \text{ on } \Sigma_1 \text{ and } z = 0 \text{ on } \Sigma, \\ y(0) = y_0 \text{ and } z(0) = z_0 \text{ in } \quad , \end{cases} \tag{5.1}$$

where a and b are two functions in $L^{\infty}\left(\overline{\ }\right)$ such that for some constants a_*, $b_* > 0$, we have

$$a_* \leq a(s) \text{ for all } s \in \quad ,$$

and

$$b_* \leq b(s) \text{ for all } s \in \Gamma_1.$$

Remark 5.1 *For the geometric assumptions, we kept the same assumptions as in chapter 4.*

In $[ACK02]$, the authors have considered a coupled system of two real wave equations with constant coefficients and damped by one distributed feedback. They have proved that this system fails to have an exponential decay rate. So, if it retains any stability property, then such a property must be weaker than exponential stability. More precisely, they have proved, by using a generalized integral inequalities due to F. Alabau (see theorem 1.34 in this thesis), that such a weaker decay rate is of polynomial type. Similar results were obtained in $[Ala02]$ for a coupled system of two real wave equations with constant coefficients and damped by one Neumann boundary feedback. We note that all these results were obtained with a positive, constant coupling coefficient witch is sufficiently small.

The aim of this chapter, is to prove that we can combine this method with the Riemann geometric approach to prove the polynomial decay rate of the sufficiently smooth solutions of the variable and complex system (5.1). We mention here that the coupling coefficient a is considered as a function with $\|a\|_{L^{\infty}(\ )}$ sufficiently small.

In the next section, we give the existence, uniqueness and regularity of solutions of system (5.1). In section 5.3, we prove the polynomial decay rate of this system.

5.2 *Existence, uniqueness and regularity of solution.*

Let

$$\mathfrak{A} : D(\mathfrak{A}) \subset H^1_{\Gamma_0}(\ ) \times H^1_0(\ ) \rightarrow H^1_{\Gamma_0}(\ ) \times H^1_0(\ )$$

be the operator defined by

$$\mathfrak{A}\left(u_1, u_2\right) = \left(\mathrm{i}\mathcal{A}u_1 + \mathrm{i}au_2, \mathrm{i}\mathcal{A}u_2 + \mathrm{i}au_1\right),$$

where

$$D = D\left(\mathfrak{A}\right) = \left\{ \begin{array}{c} \left(u_1, u_2\right) \in H^1_{\Gamma_0}(\ ) \times H^1_0(\ ) : \left(\mathcal{A}u_1, \mathcal{A}u_2\right) \in H^1_{\Gamma_0}(\ ) \times H^1_0(\ ) \\ \text{and } \frac{\partial u_1}{\partial \upsilon} + \mathrm{i}b\mathcal{A}u_1\Big|_{\Gamma_1} = 0 \end{array} \right\}.$$

Using the idea in $[CMS96]$ we can show that (5.1) is equivalent to

$$\begin{cases} Y_t = \mathfrak{A}Y \\ Y\left(0\right) = \left(y_0, z_0\right), \end{cases} \tag{5.2}$$

where $Y\left(t\right) = \left(y\left(t\right), z\left(t\right)\right).$

As in chapter 2, we prove that $\mathfrak{A}$ is a maximal dissipative operator in the space $H^1_{\Gamma_0}(\ ) \times H^1_0(\ )$. Then we have

Theorem 5.2 *If*

$$\left(y_0, z_0\right) \in H^1_{\Gamma_0}(\ ) \times H^1_0(\ ),$$

the system (5.2) has a unique solution

$$\left(y, z\right) \in C(\left[0, +\infty\right); H^1_{\Gamma_0}(\ ) \times H^1_0(\ )).$$

Moreover, if

$$\left(y_0, z_0\right) \in D\left(\mathfrak{A}^N\right)$$

for $N \geq 1$, *the system (5.2) has a unique solution*

$$\left(y, z\right) \in C^{N-j}(\left[0, +\infty\right); D\left(\mathfrak{A}^j\right))$$

for $j = 0, ..., N$.

5.3 *Polynomial decay rate of the energy.*

Consider the total energy E of the system (5.1) defined by, for all $t \geq 0$,

$$E(t) = E(y(t), z(t)) = E_1(y(t)) + E_2(z(t)) + \mathrm{Re} \int a y \overline{z} d \quad ,$$

where

$$E_1(t) = E_1(y(t)) = \frac{1}{2} \int \|\nabla_g y\|_g^2 \, d \quad ,$$

and

$$E_2(t) = E_2(z(t)) = \frac{1}{2} \int \|\nabla_g z\|_g^2 \, d \quad .$$

The dissipative property of the solution of the system (5.1) is given by the following lemma.

Lemma 5.3 *For all $t \geq 0$,*

$$E^{'}(t) = -\int_{\Gamma_1} b |y_t|^2 \, d\Gamma \leq 0 .$$

Proof. We multiply both side the first equation of (5.1) by $\overline{y}_t$, integrate over , take the real part, use the third Green's formula, finally, we use the boundary condition, we find

$$\mathrm{Re} \int \langle \nabla_g y, \nabla_g y_t \rangle_g \, d \quad + \int_{\Gamma_1} b |y_t|^2 \, d\Gamma + \mathrm{Re} \int a z \overline{y_t} d \quad = 0.$$

We obtain similar identity for z

$$\mathrm{Re} \int \langle \nabla_g z, \nabla_g z_t \rangle_g \, d \quad + \mathrm{Re} \int a y \overline{z_t} d \quad = 0.$$

But

$$E^{'}(t) = \mathrm{Re} \int \langle \nabla_g y, \nabla_g y_t \rangle_g \, d \quad + \mathrm{Re} \int \langle \nabla_g z, \nabla_g z_t \rangle_g \, d \quad + \mathrm{Re} \int a (y\overline{z})_t \, d \quad ,$$

then we find the result. ■

Remark 5.4 *We deduce from lemma 5.3 that*

$$E(T) \leq E(0) \tag{5.3}$$

and

$$\int_{\Sigma_1} |y_t|^2 \, d\Sigma \leq CE(0). \tag{5.4}$$

Our main result is

Theorem 5.5 ([HR08]) *Let $N \geq 1$. For any initial data*

$$(y_0, z_0) \in D(\mathfrak{A}^N),$$

the energy E of the solution of system (5.1) decays polynomially. That is, we have

$$E(y(t), z(t)) \leq \frac{C}{t^N} \sum_{l=0}^{l=N} E\left(y^{(l)}(0), z^{(l)}(0)\right),$$

for all $t > 0$.

Moreover, if

$$(y_0, z_0) \in H^1_{\Gamma_0}(\ ) \times H^1_0(\ )$$

then

$$\lim_{t \to +\infty} E(y(t), z(t)) = 0.$$

Proof. To prove this theorem, we estimate $\int_0^T E_1(t)\, dt$ and $\int_0^T E_2(t)\, dt$ then after summing as these two estimates we conclude applying the theorem 1.34 with $K = 1$.

Step 1. We have to prove an estimate which is useful to estimate the term $\int_0^T E_1(t)\, dt$.

For fixed t, we consider w the solution of the problem

$$\begin{cases} \mathcal{A}w = 0 \text{ in } \quad , \\ \\ w = y \text{ on } \Gamma. \end{cases}$$

Using the elliptic regularity (see theorem 2.2 in $[BM92]$), we can see that

$$\int |w|^2 \, d \quad \leq C \int_{\Gamma_1} |y|^2 \, d\Gamma \leq CE(0).$$

If we use this inequality with the derivatives, integrate over $]0, T[$ and use (5.4) we obtain

$$\int_Q |w_t|^2 \, dQ \leq C \int_{\Sigma_1} |y_t|^2 \, d\Sigma \leq CE(0). \tag{5.5}$$

On the other hand, we have

$$\operatorname{Re} \int \mathcal{A} w \overline{z} d \quad = 0,$$

then

$$\operatorname{Re} \int \langle \nabla_g w, \nabla_g z \rangle_g \, d \quad = 0. \tag{5.6}$$

Multiplying the conjugate of the first equation of (5.1) by $y - w$, integrating over Q and taking the real part

$$\operatorname{Im} \int_Q \overline{z_t} (y - w) \, dQ + \operatorname{Re} \int_Q \mathcal{A} \overline{z} (y - w) \, dQ + \int_Q a |y|^2 \, dQ - \operatorname{Re} \int_Q a \overline{y} w dQ = 0,$$

then, by the third Green's formula, (5.6) and the integration by parts, we find

$$\operatorname{Im} \int \overline{z} (y - w) \Big|_0^T d \quad - \operatorname{Im} \int_Q \overline{z} (y_t - w_t) \, dQ$$
$$+ \operatorname{Re} \int_Q \langle \nabla_g z, \nabla_g y \rangle_g \, dQ + \int_Q a |y|^2 \, dQ - \operatorname{Re} \int_Q a \overline{y} w dQ = 0.$$

If we multiply the first equation of (5.1) by $(-\overline{z})$, integrate over Q, take the real part and summing the result with the last identity, we find

$$\begin{aligned}\int_Q a\,|z|^2\,dQ &= \operatorname{Im}\int \overline{z}\,(y-w)\,d\ \Big|_0^T + \operatorname{Im}\int_Q \overline{z}w_t\,dQ + \int_Q a\,|y|^2\,dQ \\ &\quad -\operatorname{Re}\int_Q a\overline{y}w\,dQ,\end{aligned}$$

but

$$\begin{aligned}\left|\int \overline{z}(y-w)\,d\ \right| &\leq C\left(\int |y|^2\,d\ + \int |z|^2\,d\ + \int |w|^2\,d\ \right) \\ &\leq CE(0)\end{aligned}$$

then, for all $\eta > 0$, we have

$$\begin{aligned}\int_Q a\,|z|^2\,dQ &\leq CE(0) + \frac{\eta}{a_* 2}\int_Q a\,|z|^2\,dQ + \frac{1}{2\eta}\int_Q |w_t|^2\,dQ \\ &\quad + C\,\|a\|_{L^\infty(\ )}\int_0^T E_1(t)\,dt,\end{aligned}$$

then, by (5.5) and if we choose $\eta = a_*$, we find

$$\int_Q a\,|z|^2\,dQ \leq CE(0) + C\,\|a\|_{L^\infty(\ )}\int_0^T E_1(t)\,dt. \tag{5.7}$$

Step 2. We give the estimate of $\int_0^T E_1(t)\,dt$.

Multiplying the first equation of (5.1) by $2h(\overline{y}) + div_0 h\overline{y}$, integrating over Q and taking the real part, we obtain

$$\begin{aligned}0 &= \operatorname{Im}\int_Q y_t\,(2h(\overline{y}) + div_0 h\overline{y})\,dQ - \operatorname{Re}\int_Q \mathcal{A}y\,(2h(\overline{y}) + div_0 h\overline{y})\,dQ \quad (5.8)\\ &\quad -\operatorname{Re}\int_Q a\,z\,(2h(\overline{y}) + div_0\,h\overline{y})\,dQ.\end{aligned}$$

By integration by parts, we have

$$\int_Q y_t h(\overline{y})\, dQ = \int y h(\overline{y})\, d \Big|_0^T - \int_Q y h(\overline{y_t})\, dQ.$$

We apply the identity (1.5) with $f_1 = y$ and $f_2 = y_t$ we obtain

$$\begin{aligned}
\int_Q y_t h(\overline{y})\, dQ &= \int y h(\overline{y})\, d \Big|_0^T - \int_\Sigma h.\upsilon y \overline{y_t} d\Sigma \\
+ \int_Q \overline{y_t} div_0 (hy)\, dQ &= \int y h(\overline{y})\, d \Big|_0^T - \int_\Sigma h.\upsilon y \overline{y_t} d\Sigma \\
&\quad + \int_Q \overline{y_t} div_0 hy dQ + \int_Q \overline{y_t} h(y)\, dQ.
\end{aligned}$$

Then

$$\begin{aligned}
2\,\mathrm{Im} \int_Q y_t h(\overline{y})\, dQ &= \mathrm{Im} \int y h(\overline{y})\, d \Big|_0^T \\
&\quad - \mathrm{Im} \int_\Sigma h.\upsilon y \overline{y_t} d\Sigma - \mathrm{Im} \int_Q y_t div_0 h \overline{y} dQ,
\end{aligned}$$

so

$$\begin{aligned}
\mathrm{Im} \int_Q y_t \left(2h(\overline{y}) + div_0 h \overline{y}\right) dQ &= \mathrm{Im} \int y h(\overline{y})\, d \Big|_0^T \\
&\quad - \mathrm{Im} \int_{\Sigma_1} h.\upsilon y \overline{y_t} d\Sigma.
\end{aligned} \tag{5.9}$$

On the other hand, if we use the third Green's formula, we find

$$\begin{aligned}
&\mathrm{Re} \int_Q \mathcal{A} y \left(2h(\overline{y}) + div_0 h \overline{y}\right) dQ \\
= &- \mathrm{Re} \int_\Sigma \frac{\partial y}{\partial \upsilon_{\mathcal{A}}} \left(2h(\overline{y}) + div_0 h \overline{y}\right) d\Sigma \\
&+2 \int_Q \mathrm{Re} \left\langle \nabla_g y, \nabla_g (h(y)) \right\rangle_g dQ \\
&+ \int_Q \mathrm{Re} \left\langle \nabla_g y, \nabla_g (div_0 hy) \right\rangle_g dQ.
\end{aligned}$$

Indeed, we recall the identity (1.6) we obtain

$$
\begin{aligned}
& \operatorname{Re}\int_Q \mathcal{A}y\left(2h\left(\overline{y}\right)+div_0 h\overline{y}\right)dQ \\
= \quad & -\operatorname{Re}\int_\Sigma \frac{\partial y}{\partial \upsilon_{\mathcal{A}}}\left(2h\left(\overline{y}\right)+div_0 h\overline{y}\right)d\Sigma \\
& +2\int_Q \left(Dh\left(\nabla_g \operatorname{Re} y, \nabla_g \operatorname{Re} y\right)+Dh\left(\nabla_g \operatorname{Im} y, \nabla_g \operatorname{Im} y\right)\right)dQ \\
& +\int_Q h\left(\left\|\nabla_g y\right\|_g^2\right)dQ \\
& +\operatorname{Re}\int_Q \left\langle \nabla_g y, \nabla_g\left(div_0 h\right)\right\rangle_g \overline{y}dQ+\int_Q div_0 h\left\|\nabla_g y\right\|_g^2 dQ,
\end{aligned}
$$

then

$$
\begin{aligned}
& \operatorname{Re}\int_Q \mathcal{A}y\left(2h\left(\overline{y}\right)+div_0 h\overline{y}\right)dQ \\
= \quad & -\operatorname{Re}\int_\Sigma \frac{\partial y}{\partial \upsilon_{\mathcal{A}}}\left(2h\left(\overline{y}\right)+div_0 h\overline{y}\right)d\Sigma \\
& +2\int_Q \left(Dh\left(\nabla_g \operatorname{Re} y, \nabla_g \operatorname{Re} y\right)+Dh\left(\nabla_g \operatorname{Im} y, \nabla_g \operatorname{Im} y\right)\right)dQ \\
& +\int_\Sigma h.\upsilon\left\|\nabla_g y\right\|_g^2 d\Sigma+\operatorname{Re}\int_Q \left\langle \nabla_g y, \nabla_g\left(div_0 h\right)\right\rangle_g \overline{y}dQ.
\end{aligned}
$$

Since, $\operatorname{Re} y=\operatorname{Im} y=0$ on Γ_0, then we have (see [Yao99])

$$
h\left(\operatorname{Re} y\right)=\frac{h.\upsilon}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2}\frac{\partial \operatorname{Re} y}{\partial \upsilon_{\mathcal{A}}} \text{ and } \left\|\nabla_g \operatorname{Re} y\right\|_g^2=\frac{1}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2}\left(\frac{\partial \operatorname{Re} y}{\partial \upsilon_{\mathcal{A}}}\right)^2
$$

and

$$
h\left(\operatorname{Im} y\right)=\frac{h.\upsilon}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2}\frac{\partial \operatorname{Im} y}{\partial \upsilon_{\mathcal{A}}} \text{ and } \left\|\nabla_g \operatorname{Im} y\right\|_g^2=\frac{1}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2}\left(\frac{\partial \operatorname{Im} y}{\partial \upsilon_{\mathcal{A}}}\right)^2.
$$

So

$$
h\left(y\right)=\frac{h.\upsilon}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2}\frac{\partial y}{\partial \upsilon_{\mathcal{A}}} \text{ and } \left\|\nabla_g y\right\|_g^2=\frac{1}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2}\left|\frac{\partial y}{\partial \upsilon_{\mathcal{A}}}\right|^2.
$$

Then

$$
\begin{aligned}
-\operatorname{Re}\int_{Q} \mathcal{A}y\left(2h\left(\overline{y}\right)+div_{0}h\overline{y}\right)dQ &= \int_{\Sigma_0} \frac{h.\upsilon}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2}\left|\frac{\partial y}{\partial \upsilon_{\mathcal{A}}}\right|^2 d\Sigma \\
-\int_{\Sigma_1} h.\upsilon\left\|\nabla_g y\right\|_g^2 d\Sigma+\operatorname{Re}\int_{\Sigma_1}\frac{\partial y}{\partial \upsilon_{\mathcal{A}}}\left(2h\left(\overline{y}\right)+div_0 h\overline{y}\right)d\Sigma \\
-2\int_{Q}\left(Dh\left(\nabla_g \operatorname{Re} y,\nabla_g \operatorname{Re} y\right)+Dh\left(\nabla_g \operatorname{Im} y,\nabla_g \operatorname{Im} y\right)\right)dQ \\
-\operatorname{Re}\int_{Q}\left\langle \nabla_g y,\nabla_g\left(div_0 h\right)\right\rangle_g \overline{y}dQ.
\end{aligned} \tag{5.10}
$$

Finally, we insert (5.9) and (5.10) in (5.8) to obtain

$$
\begin{aligned}
& 2\int_{Q}\left(Dh\left(\nabla_g \operatorname{Re} y,\nabla_g \operatorname{Re} y\right)+Dh\left(\nabla_g \operatorname{Im} y,\nabla_g \operatorname{Im} y\right)\right)dQ \\
= \quad & I \quad + I_{\Sigma_0}+I_{\Sigma_1}+I_Q,
\end{aligned} \tag{5.11}
$$

where

$$
I \quad = \operatorname{Im}\int y h\left(\overline{y}\right) d \quad \Bigg|_0^T,
$$

$$
I_{\Sigma_0} = \int_{\Sigma_0}\frac{h.\upsilon}{\left\|\upsilon_{\mathcal{A}}\left(x\right)\right\|_g^2}\left|\frac{\partial y}{\partial \upsilon_{\mathcal{A}}}\right|^2 d\Sigma,
$$

and

$$
\begin{aligned}
I_{\Sigma_1} \quad = \quad & -\operatorname{Im}\int_{\Sigma_1} h.\upsilon y\overline{y_t}d\Sigma-\int_{\Sigma_1} h.\upsilon\left\|\nabla_g y\right\|_g^2 d\Sigma \\
& -\operatorname{Re}\int_{\Sigma_1} by_t\left(2h\left(\overline{y}\right)+div_0 h\overline{y}\right)d\Sigma,
\end{aligned}
$$

$$
I_Q = -\operatorname{Re}\int_{Q}\left\langle \nabla_g y,\nabla_g\left(div_0 h\right)\right\rangle_g \overline{y}dQ-\operatorname{Re}\int_{Q} az\left(2h\left(\overline{y}\right)+div_0 h\overline{y}\right)dQ.
$$

We can see that by (5.3)

$$
I \quad \leq CE\left(0\right).
$$

We have

$$I_{\Sigma_0} \leq 0.$$

We have for all $\eta_1 > 0$

$$\begin{aligned} I_{\Sigma_1} &\leq \frac{C}{\eta_1} \int\limits_{\Sigma_1} |y_t|^2 + C\eta_1 \int\limits_{Q} \|\nabla_g y\|_g^2 \, dQ + (C\eta_1 - h_0) \int\limits_{\Sigma_1} \|\nabla_g y\|_g^2 \, d\Sigma \\ &\leq CE(0) + C\eta_1 \int\limits_0^T E_1(t)\, dt + (C\eta_1 - h_0) \int\limits_{\Sigma_1} \|\nabla_g y\|_g^2 \, d\Sigma. \end{aligned}$$

We have, for all $\eta_2 > 0$, by (5.7)

$$I_Q \leq CE(0) + \left(\eta_2 C_h{}^2 + \frac{C_1^2}{\eta_2} \right) \int\limits_0^T E_1(t)\, dt + C \|a\|_{L^{\infty}(\ )} \int\limits_0^T E_1(t)\, dt.$$

Replace the majorities of I , I_{Σ_l} $(l = 0, 1)$ and I_Q in (5.11), choose $\eta_2 = \frac{C_1}{C_h}$, η_1 and $\|a\|_{L^{\infty}(\ )}$ sufficiently small, we find

$$\int\limits_0^T E_1(t)\, dt \leq CE(0). \tag{5.12}$$

Step 3. We estimate the term $\int\limits_0^T E_2(t)\, dt$.

First we have by (5.7) and (5.12)

$$\int\limits_Q |z|^2 \, dQ \leq CE(0) = CE(y(0), z(0)). \tag{5.13}$$

If we use this inequality with the derivatives, we obtain

$$\int\limits_Q |z_t|^2 \, dQ \leq CE(y_t(0), z_t(0)). \tag{5.14}$$

On the other hand, if we multiply the second equation of the system (5.1) by $\overline{z}$, integrate over Q, take the real part and we use the third Green's formula, we find

$$\int_Q \|\nabla_g z\|_g^2 \, dQ = \operatorname{Im}\int_Q z_t \overline{z} dQ - \operatorname{Re}\int_Q ay\overline{z} dQ.$$

If we use (5.12), (5.13) and (5.14) we find

$$\int_0^T E_2(t)\, dt \le C\left(E\left(y(0), z(0)\right) + E\left(y_t(0), z_t(0)\right)\right).$$

Step 4. We can now conclude the result of theorem 5.5.

We have, for all $T > 0$

$$\begin{aligned} \int_0^T E\left(y(t), z(t)\right) dt &= \int_0^T E_1\left(y(t)\right) dt + \int_0^T E_2\left(z(t)\right) dt + \operatorname{Re}\int_Q ay\overline{z} dQ \\ &\le C\left(E\left(y(0), z(0)\right) + E\left(y_t(0), z_t(0)\right)\right). \end{aligned}$$

■

Chapter 6

Polynomial decay rate of coupled system of two Schrödinger equations with variable coefficients and damped by one Dirichlet boundary feedback.

6.1 *Introduction.*

We are concerned with the coupled complex valued Schrödinger equations with variable coefficients and forcing term u in the Dirichlet boundary condition. This control function u is acting on one end only (no damping acting on z on Σ_1).

$$\begin{cases} iy_t + \mathcal{A}y + az = 0 \text{ in } Q, \\ iz_t + \mathcal{A}z + ay = 0 \text{ in } Q, \\ y = 0 \text{ on } \Sigma_0,\ y = u \text{ on } \Sigma_1 \text{ and } z = 0 \text{ on } \Sigma, \\ y(0) = y_0 \text{ and } z(0) = z_0 \text{ in } \quad . \end{cases} \tag{6.1}$$

where a is a positive constant (the coupling coefficient).

In this chapter, we combine the Riemannian geometry method to handle the case of variable coefficients, the lifting arguments in the topology of the solutions developed by I. Lasiecka and R. Triggiani in [LT87, LT92] in order to obtain the exponential decay rate of one equation with Dirichlet boundary feedback, and the Alabau's generalized integral inequalities in [Ala02] developed in the context of real systems with constant coefficients, to prove with a suitable choice of the control function u the polynomial energy decay rates for smooth solutions for the system (6.1).

In the next section we give some notation that will be used. Then, we prove that there exist some choice of u which makes the system (6.1) dissipative. Finally, we prove the main result of this chapter.

Remark 6.1 *Concerning the geometric assumption in this chapter, we take the* *same as in chapter 4, except for the partition* $\{\Gamma_0, \Gamma_1\}$ *where we take it as*

$$\Gamma_0 = \{x \in \Gamma : h.\upsilon \leq 0\} \ \ \textit{and}\, \Gamma_1 \neq \varnothing.$$

6.2 ***Notations.***

Let $\mathbf{A}$ be the positive self adjoint operator on $L^2(\;\;)$ defined by

$$\mathbf{A}f = \mathcal{A}f \text{ and } D(\mathbf{A}) = H^2(\;\;) \cap H_0^1(\;\;).$$

The following space identification are know (with equivalent norms)

$$D\left(\mathbf{A}^{\frac{1}{2}}\right) = H_0^1(\;\;), \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)' = H^{-1}(\;\;),$$

and

$$\|f\|_{D\left(\mathbf{A}^{\frac{1}{2}}\right)} = \left\|\mathbf{A}^{\frac{1}{2}}f\right\|_{L^2(\;\;)}, \|f\|_{\left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)'} = \left\|\mathbf{A}^{-\frac{1}{2}}f\right\|_{L^2(\;\;)}.$$

Let us introduce the operator $D : L^2(\Gamma) \to L^2(\;\;)$ defined by

$$f = Dw \Longleftrightarrow \left(\mathcal{A}f = 0, f_{/\Gamma_0} = 0, f_{/\Gamma_1} = w\right),$$

and his adjoint D^* by

$$(Dw, f)_{L^2(\;\;)} = (w, D^*f)_{L^2(\Gamma)},$$

for all $w \in L^2(\Gamma)$ and $f \in L^2(\;\;)$.

We have (see $[LLT86]$) for all $f \in D(\mathbf{A})$

$$D^*\mathbf{A}f = \begin{cases} 0 \text{ on } \Gamma_0, \\ -\frac{\partial f}{\partial \upsilon_{\mathcal{A}}} \text{ on } \Gamma_1. \end{cases}$$

6.3 *The closed loop system: Choice of the Dirichlet control function* u.

Using the techniques of $[LT87, BT91]$, we put the problem (6.1) into a semigroup frame

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \mathrm{i}\begin{pmatrix} \mathbf{A} & a \\ a & \mathbf{A} \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} \mathrm{i}\mathbf{A}Du \\ 0 \end{pmatrix}. \tag{6.2}$$

If we take $u = F(y) = -\mathrm{i}D^{*}y$ then (6.2) is rewritten as

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \mathfrak{A}_F \begin{pmatrix} y \\ z \end{pmatrix} \tag{6.3}$$

where

$$\mathfrak{A}_F = \mathrm{i}\begin{pmatrix} \mathbf{A} + \mathrm{i}\mathbf{A}DD^{*} & a \\ a & \mathbf{A} \end{pmatrix}$$

with domain

$$D(\mathfrak{A}_F) = \left\{ \begin{array}{l} \begin{pmatrix} y \\ z \end{pmatrix} \in \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)^{'} \times \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)^{'} : \\ \mathfrak{A}_F \begin{pmatrix} y \\ z \end{pmatrix} \in \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)^{'} \times \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)^{'} \end{array} \right\}$$

This choice of u makes the operator $\mathfrak{A}_F$ dissipative on $\left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)^{'} \times \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)^{'}$.

Indeed,

$$\operatorname{Re}\left(\mathfrak{A}_F \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right)_{\left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)^{'} \times \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)^{'}} = -\left\|D^{*}y\right\|^{2}_{L^{2}(\Gamma)} \leq 0.$$

This imply the following result of existence, uniqueness and regularity of solution of system (6.3).

Theorem 6.2 *For all*

$$(y_0, z_0) \in \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)' \times \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)',$$

the system (6.3) has a unique solution

$$(y, z) \in C([0, +\infty) ; \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)' \times \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)').$$

Moreover, for all $N \geq 1$ *and for all*

$$(y_0, z_0) \in D\left(\mathfrak{A}_F^N\right),$$

the system (6.3) has a unique solution

$$(y, z) \in C^{N-j}([0, +\infty) ; D\left(\mathfrak{A}_F^j\right))$$

for $j = 0, ..., N$.

Proof. It sufficient to see that $\mathfrak{A}_F$ with domain $D(\mathfrak{A}_F)$ is a maximal dissipative operator in $\left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)' \times \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)'$. ■

6.4 *Polynomial decay rate of the energy.*

We define the total energy E of (6.3) by

$$\begin{aligned} E(t) &= E(y(t), z(t)) = \frac{1}{2}\|(y, z)\|^2_{\left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)' \times \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)'} \\ &= \frac{1}{2}\left\|\mathbf{A}^{-\frac{1}{2}} y\right\|^2_{L^2(\ )} + \frac{1}{2}\left\|\mathbf{A}^{-\frac{1}{2}} z\right\|^2_{L^2(\ )}. \end{aligned}$$

By the dissipation of the operator $\mathfrak{A}_F$, we can see that E is a decreasing function

$$E_t(t) = -\|D^*y\|^2_{L^2(\Gamma)} \leq 0.$$

We give the main result of this chapter.

Theorem 6.3 ($[Ham1]$**)** *Let $N \geq 1$. For any initial data*

$$(y_0, z_0) \in D\left(\mathfrak{A}_F^N\right),$$

the energy E of the solution of the closed loop dynamics (6.1), with the choice of $u = -\mathrm{i}D^*y$ inserted in the boundary condition, decays polynomially. That is,

$$E\left(y(t), z(t)\right) \leq \frac{C}{t^N} \sum_{l=0}^{l=N} E\left(y^{(l)}(0), z^{(l)}(0)\right),$$

for all $t > 0$.

Moreover, for all

$$(y_0, z_0) \in \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)' \times \left(D\left(\mathbf{A}^{\frac{1}{2}}\right)\right)'$$

$$\lim_{t\to\infty} E\left(y(t), z(t)\right) = 0$$

Proof. Step 1. Change of variable.

Motivated by the techniques of $[LT87, LT92]$ in the context of one equation, we introduce a new variables p and q by setting

$$p = \mathbf{A}^{-1}y \text{ and } q = \mathbf{A}^{-1}z,$$

where $(y_0, z_0) \in D(\mathfrak{A}_F)$.

Then, by (6.3), we obtain the system

$$\begin{cases} \mathrm{i}p_t + \mathcal{A}p + G + aq = 0 \text{ in } Q, \\ \mathrm{i}q_t + \mathcal{A}q + ap = 0 \text{ in } Q, \\ p = q = 0 \text{ on } \Sigma, \\ p(0) = p_0 \text{ and } q(0) = q_0 \text{ in } \quad , \end{cases}$$

where $G = \mathrm{i}DD^*\mathbf{A}p$.

On the other hand, we have

$$E(t) = E_1(t) + E_2(t),$$

where

$$E_1(t) = \frac{1}{2}\left\|\mathbf{A}^{-\frac{1}{2}}y\right\|^2_{L^2(\ )} = \frac{1}{2}\left\|\mathbf{A}^{\frac{1}{2}}p\right\|^2_{L^2(\ )} = \frac{1}{2}\int \|\nabla_g p\|^2_g d \quad ,$$

and

$$E_2(t) = \frac{1}{2}\left\|\mathbf{A}^{-\frac{1}{2}}z\right\|^2_{L^2(\ )} = \frac{1}{2}\left\|\mathbf{A}^{\frac{1}{2}}q\right\|^2_{L^2(\ )} = \frac{1}{2}\int \|\nabla_g q\|^2_g d \quad .$$

We can see that

$$\int_{\Sigma_1} \left|\frac{\partial p}{\partial \upsilon_{\mathcal{A}}}\right|^2 d\Sigma = \|D^*y\|^2_{L^2(\Sigma)} = -\int_0^T \frac{dE(t)}{dt}dt \leq E(0),$$

and

$$\int_Q |G|^2\, dQ = \int_Q |DD^*\mathbf{A}p|^2\, dQ \leq C\int_{\Sigma_1}\left|\frac{\partial p}{\partial \upsilon_{\mathcal{A}}}\right|^2 d\Sigma \leq CE(0).$$

Step 2. In this step we shall estimate the term $\int_0^T E_1(t)\,dt$.

We have

$$\begin{aligned} 0 \quad = \quad & \mathrm{Re}\int_Q (\mathrm{i}p_t + \mathcal{A}p + G + aq)\left(2h(\overline{p}) + div_0 h\overline{p} + \overline{q}\right) dQ \\ & - \mathrm{Re}\int_Q (-\mathrm{i}\overline{q}_t + \mathcal{A}\overline{q} + a\overline{p})\, p dQ, \end{aligned}$$

so

$$
\begin{aligned}
0 \quad = \quad & \operatorname{Re} \mathrm{i} \int\limits_Q \left(p_t \overline{q} + p \overline{q}_t\right) dQ + \operatorname{Re} \int\limits_Q \left(\mathcal{A} p \overline{q} - \mathcal{A} \overline{q} p\right) dQ \\
& + \operatorname{Re} \int\limits_Q \mathcal{A} p \left(2h\left(\overline{p}\right) + div_0 h \overline{p}\right) dQ \\
& - \operatorname{Im} \int\limits_Q p_t \left(2h\left(\overline{p}\right) + div_0 h \overline{p}\right) dQ \\
& + \operatorname{Re} \int\limits_Q G \left(2h\left(\overline{p}\right) + div_0 h \overline{p} + \overline{q}\right) dQ \qquad (6.4) \\
& + \operatorname{Re} \int\limits_Q aq \left(2h\left(\overline{p}\right) + div_0 h \overline{p}\right) dQ \\
& - \int\limits_Q a \left|p\right|^2 dQ + \int\limits_Q a \left|q\right|^2 dQ.
\end{aligned}
$$

But

$$
\operatorname{Re} \mathrm{i} \int\limits_Q \left(p_t \overline{q} + p \overline{q}_t\right) dQ = \operatorname{Re} \mathrm{i} \int\limits_Q \left(p \overline{q}\right)_t dQ = - \operatorname{Im} \int p \overline{q} d \,\Bigg|_0^T, \qquad (6.5)
$$

$$
\operatorname{Re} \int\limits_Q \left(\mathcal{A} p \overline{q} - \mathcal{A} \overline{q} p\right) dQ = 0 \qquad (6.6)
$$

$$
\begin{aligned}
& \operatorname{Re} \int\limits_Q \mathcal{A} p \left(2h\left(\overline{p}\right) + div_0 h \overline{p}\right) dQ \\
= \quad & \operatorname{Re} \int\limits_Q \left\langle \nabla_g p, \nabla_g \left(2h\left(p\right) + div_0 h p\right)\right\rangle_g dQ - 2 \operatorname{Re} \int\limits_\Sigma \frac{\partial \overline{p}}{\partial \upsilon_{\mathcal{A}}} h\left(p\right) d\Sigma \\
= \quad & 2 \int\limits_Q \left(Dh\left(\nabla_g \operatorname{Re} p, \nabla_g \operatorname{Re} p\right) + Dh\left(\nabla_g \operatorname{Im} p, \nabla_g \operatorname{Im} p\right)\right) dQ \\
& + \operatorname{Re} \int\limits_Q \left\langle \nabla_g p, \nabla_g \left(div_0 h\right)\right\rangle_g \overline{p} dQ - 2 \operatorname{Re} \int\limits_\Sigma \frac{\partial \overline{p}}{\partial \upsilon_{\mathcal{A}}} h\left(p\right) d\Sigma \\
& + \int\limits_\Sigma h.\upsilon \left\|\nabla_g p\right\|_g^2 d\Sigma.
\end{aligned}
$$

Since, $\operatorname{Re} p = \operatorname{Im} p = 0$ on Γ, then

$$
h\left(\operatorname{Re} p\right) = \frac{h.\upsilon}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2} \frac{\partial \operatorname{Re} p}{\partial \upsilon_{\mathcal{A}}},
$$

$$\|\nabla_g \operatorname{Re} p\|_g^2 = \frac{1}{\|\upsilon_{\mathcal{A}}\|_g^2}\left(\frac{\partial \operatorname{Re} p}{\partial \upsilon_{\mathcal{A}}}\right)^2,$$

and

$$h(\operatorname{Im} p) = \frac{h.\upsilon}{\|\upsilon_{\mathcal{A}}\|_g^2}\frac{\partial \operatorname{Im} p}{\partial \upsilon_{\mathcal{A}}},$$

$$\|\nabla_g \operatorname{Im} p\|_g^2 = \frac{1}{\|\upsilon_{\mathcal{A}}(x)\|_g^2}\left(\frac{\partial \operatorname{Im} p}{\partial \upsilon_{\mathcal{A}}}\right)^2.$$

So

$$h(p) = \frac{h.\upsilon}{\|\upsilon_{\mathcal{A}}\|_g^2}\frac{\partial p}{\partial \upsilon_{\mathcal{A}}} \text{ and } \|\nabla_g p\|_g^2 = \frac{1}{\|\upsilon_{\mathcal{A}}\|_g^2}\left|\frac{\partial p}{\partial \upsilon_{\mathcal{A}}}\right|^2.$$

So

$$\begin{aligned}
& \operatorname{Re}\int_Q \mathcal{A}p\left(2h(\overline{p}) + div_0 h\overline{p}\right) dQ \\
= \ & 2\int_Q \left(Dh\left(\nabla_g \operatorname{Re} p, \nabla_g \operatorname{Re} p\right) + Dh\left(\nabla_g \operatorname{Im} p, \nabla_g \operatorname{Im} p\right)\right) dQ \\
& + \operatorname{Re}\int_Q \left\langle \nabla_g p, \nabla_g\left(div_0 h\right)\right\rangle_g \overline{p} dQ \qquad (6.7) \\
& - \int_\Sigma \frac{h.\upsilon}{\|\upsilon_{\mathcal{A}}\|_g^2}\left|\frac{\partial p}{\partial \upsilon_{\mathcal{A}}}\right|^2 d\Sigma.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_Q p_t h(\overline{p}) dQ &= \int ph(\overline{p})\, d \,\Big|_0^T - \int_Q ph\left(\overline{p_t}\right) dQ \\
&= \int ph(\overline{p})\, d \,\Big|_0^T - \int_Q ph.\nabla_0 \overline{p_t} dQ \\
&= \int ph(\overline{p})\, d \,\Big|_0^T + \int_Q \overline{p_t} div_0(hp)\, dQ \\
&= \int ph(\overline{p})|_0^T + \int_Q \overline{p_t} div_0 hp dQ + \int_Q \overline{p_t} h(p)\, dQ.
\end{aligned}$$

Then

$$- \operatorname{Im} \int_Q p_t \left(2h\left(\overline{p}\right) + div_0 h \overline{p}\right) dQ = - \operatorname{Im} \int ph\left(\overline{p}\right) d \left. \right|_0^T . \qquad (6.8)$$

Replacing (6.5), (6.6), (6.7) and (6.8) in (6.4), we find

$$\begin{aligned} & 2 \int_Q \left(Dh\left(\nabla_g \operatorname{Re} p, \nabla_g \operatorname{Re} p\right) + Dh\left(\nabla_g \operatorname{Im} p, \nabla_g \operatorname{Im} p\right)\right) dQ \\ = \quad & I \quad + I_\Sigma + I_Q, \end{aligned} \qquad (6.9)$$

where

$$I \ = \operatorname{Im} \int p\left(h\left(\overline{p}\right) + \overline{q}\right) d \left. \right|_0^T \leq CE\left(0\right),$$

$$\begin{aligned} I_\Sigma \quad &= \quad \int_\Sigma \frac{h.\upsilon}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2} \left|\frac{\partial p}{\partial \upsilon_{\mathcal{A}}}\right|^2 d\Sigma \\ &= \quad \int_{\Sigma_0} \frac{h.\upsilon}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2} \left|\frac{\partial p}{\partial \upsilon_{\mathcal{A}}}\right|^2 d\Sigma + \int_{\Sigma_1} \frac{h.\upsilon}{\left\|\upsilon_{\mathcal{A}}\right\|_g^2} \left|\frac{\partial p}{\partial \upsilon_{\mathcal{A}}}\right|^2 d\Sigma \\ &\leq \quad CE\left(0\right), \end{aligned}$$

and

$$\begin{aligned} I_Q \quad = \quad & - \operatorname{Re} \int_Q \left\langle \nabla_g p, \nabla_g \left(div_0 h\right)\right\rangle_g \overline{p} dQ \\ & - \operatorname{Re} \int_Q G\left(2h\left(\overline{p}\right) + div_0 h \overline{p} + \overline{q}\right) dQ \\ & - \operatorname{Re} \int_Q aq\left(2h\left(\overline{p}\right) + div_0 h \overline{p}\right) dQ \\ & + \int_Q a\left|p\right|^2 dQ - \int_Q a\left|q\right|^2 dQ. \end{aligned}$$

We have, for all $\eta_1, \eta_2 > 0$

$$\begin{aligned}
I_Q &\leq \left(\eta_1 {C_h}^2 + \frac{C_1^2}{\eta_1}\right) \int_0^T E_1(t)\,dt + CE(0) \\
&\quad + \eta_2 \left(C \int_0^T E_1(t)\,dt + \int_Q |q|^2 \,dQ \right) \\
&\quad - \frac{a}{2} \int_Q |q|^2 \,dQ + Ca \int_0^T E_1(t)\,dt.
\end{aligned}$$

Replace the majorities of I , I_{Σ_l} $(l = 0, 1)$ and I_Q in (6.9), choose $\eta_1 = \frac{C_1}{C_h}$, η_2 and a sufficiently small, we find

$$\int_0^T E_1(t)\,dt \leq CE(0)\,. \tag{6.10}$$

Step 3. In this step we shall estimate the term $\int_0^T E(t)\,dt$.

We have

$$\begin{aligned}
0 &= \mathrm{Re} \int_Q (-\mathrm{i}\overline{q}_t + \mathcal{A}\overline{q} + a\overline{p})\,p dQ - \mathrm{Re} \int_Q (\mathrm{i}p_t + \mathcal{A}p + G + aq)\,\overline{q} dQ \\
&= \mathrm{Im} \int p\overline{q} d \,\Big|_0^T + \int_Q a|p|^2 \,dQ - \mathrm{Re} \int_Q G\overline{q} dQ - \int_Q a\,|q|^2 \,dQ,
\end{aligned}$$

o

$$\int_Q |q|^2 \,dQ \leq CE(0)\,. \tag{6.11}$$

If we use this inequality with the derivatives, we obtain

$$\int_Q |q_t|^2 \,dQ \leq CE(y_t(0), z_t(0))\,. \tag{6.12}$$

On the other hand, we have

$$\begin{aligned} 0 &= \operatorname{Re}\int\limits_{Q}\left(iq_t+\mathcal{A}q+ap\right)\overline{q}dQ=-\operatorname{Im}\int\limits_{Q}q_t\overline{q}dQ+\int\limits_{Q}\left\|\nabla_g q\right\|_g^2 dQ \\ &\quad +\operatorname{Re}\int\limits_{Q}ap\overline{q}dQ, \end{aligned}$$

then

$$\int\limits_{Q}\left\|\nabla_g q\right\|_g^2 dQ=\operatorname{Im}\int\limits_{Q}q_t\overline{q}dQ-\operatorname{Re}\int\limits_{Q}ap\overline{q}dQ.$$

If we use (6.10), (6.11) and (6.12) we find

$$\int\limits_0^T E_2\left(t\right)dt\leq C\left(E\left(y\left(0\right),z\left(0\right)\right)+E\left(y_t\left(0\right),z_t\left(0\right)\right)\right).$$

Finally, we obtain

$$\begin{aligned} \int\limits_0^T E\left(t\right)dt &= \int\limits_0^T E_1\left(t\right)dt+\int\limits_0^T E_2\left(t\right)dt \\ &\leq C\left(E\left(y\left(0\right),z\left(0\right)\right)+E\left(y_t\left(0\right),z_t\left(0\right)\right)\right). \end{aligned}$$

The desired conclusion follows from theorem 1.34 with $K=1$. ∎

Conclusion and open problems.

CONCLUSION AND OPEN PROBLEMS.

Conclusion.

First, we have considered a second order hyperbolic equation with variable coefficients and linear zero order term and we have directly proved the exponential decay rate of the perturbed energy. To obtain our results we have needed to some geometric assumptions, a number of examples have been illustrated.

The problem of uniform decay rate of the Riemannian wave equation with linear first order term and unspecified behavior at the origin of the nonlinear feedback f has been also studied. We have introduced a new geometric multiplier to prove, by adapting the method introduced by I. Lasiecka and D. Tataru in $[LT93]$, that the energy of the solution decays faster than the solution of some associated differential equation. The decay rate of a general second order hyperbolic equation with polynomial growth at the origin of f has been discussed too. In contrast with the literature (see $[GueII03, CG05]$), our results do not require any assumptions of smallness imposed on the linear first order term.

Next, we have shown that we can apply the Riemann geometric approach to the complex systems with variables coefficients, here we have constructed a suitable Riemannian metric on $\mathbb{C}^n$. We note that this approach was introduced to obtain controllability and direct stabilization of certain real systems with variable coefficients: second order hyperbolic equations $[Yao99, CY04, FF04]$, Euler Bernoulli equation $[YaoI00, GY06]$, Shallow Shells equation $[YaoII00]$, Maxwell's systems $[NP06]$, etc... .

Next, by combining the Riemannian geometry method and the method of F. Alabau developed in the context of two coupled real wave equations with constant coefficients, we have obtained the polynomial decay rate estimate for sufficiently smooth solution of coupled system of two complex Schrödinger equations with variable coefficients and damped by one end only Neumann boundary feedback.

In the case of one end only Dirichlet control function u, we have provided polynomial decay rate estimate for sufficiently smooth solution of system of two coupled complex Shrödinger equations with variable coefficients and a suitable choice of the function u, by using a successful combination of three key ingredients: (1) The Riemann geometric approach to handle the case of the variable coefficients principal part. (2) The ideas of I. Lasiecka and R. Triggiani in $[LT87, LT92]$ used to obtain the exponential decay rate of one equation with Dirichlet boundary feedback. (3) The Alabau's generalized integral inequalities in $[ACK02, Ala02]$.

Open problems.

The following open questions can be made regarding the material presented in this thesis.

1. An immediate question is to see if it is possible to obtain some energy decay rate estimates of systems with nonlinear first order term without any conditions of smallness.

2. Another interesting problem is that: can we obtain uniform decay rate of the energy of the Schrödinger equations with first order terms (linear and

nonlinear). In this case, we shall mention that there is some difficulty to define an equivalent energy which is dissipative as we have do in the case of the wave equation.

3. One would also consider the energy decay rate of two equations (wave-wave, Shrödinger-Schrödinger, etc...) coupled via first order terms. According to our best knowledge, this class of systems is not considered in the literature. The existing one is about systems coupled via zero order terms.

4. In [ALa01], the author has considered the problem of the observability of system of two coupled wave equations with only one of the two components is observed. This is the so called indirect observability. She has proved that the observation of the trace of the normal derivative of the first component of the solution in a part of the boundary allows us to get back a weakened energy of the initial data. It would be interesting to study this problem when the system considered contains first order term in each equations.

5. Another question is to see if there is any relationship between indirect observability and indirect stabilization of coupled systems. Concerning the direct observability and direct stabilization, a complete answer to this question was provided in the literature (see for example [Rus78, Lio88]).

CONCLUSION ET PROBLEMES OUVERTS

Conclusion

Au début, on a considéré une équation hyperbolique d'ordre deux avec des coefficients variables et un terme linéaire d'ordre zéro puis on a démontré la décroissance exponentielle de l'énergie perturbée. Pour obtenir ce résultat on a supposé quelques conditions géométriques, des exemples ont été illustrés.

Le problème de la stabilisation uniforme d'une équation des ondes Riemannienne avec un terme linéaire d'ordre un et un comportement à l'origine non spécifier de la fonction feedback non linéaire f a été aussi étudié. On a introduit un nouveau multiplicateur géométrique pour démontrer que l'énergie décroit plus rapidement que la solution de certaine équation différentielle associée. Ici on a adopté une méthode due à I. Lasiecka et D. Tataru dans [LT93]. La décroissance de l'énergie d'une équation hyperbolique avec un comportement polynomiale à l'origine de la fonction f a été aussi étudiée. En comparant avec ce qui existe dans la littérature (voir [GueII03, CG05]), notre résultat a été obtenu sans aucune condition de petitesse sur le terme linéaire d'ordre un.

Ensuite, on a montré qu'on peut appliquer la méthode de la géométrie Riemannienne sur les systèmes complexes avec des coefficients variables, ici on a construit une métrique Riemannienne convenable sur $\mathbb{C}^n$. On note que cette approche a été introduite pour étudier les problèmes de la contrôlabilité exacte et de la stabilisation directe de certains systèmes réels avec des coefficients variables: équation hyperbolique d'ordre deux [Yao99, CY04, FF04], équation d'Euler Bernoulli [YaoI00, GY06], Equation de Shallow Shell [YaoII00], équation de Maxwell [NP06], etc... .

Puis, en combinant la méthode de la géométrie Riemannienne avec la méthode de F. Alabau développée dans le contexte d'un système couplé de deux équations réelles des ondes avec des coefficients variables, on a obtenu la stabilisation polynomiale de l'énergie de deux équations complexe de Schrödinger couplées avec des coefficients variables et soumises a une seule action frontière de type Neumann.

Dans le cas d'une seule action frontière de type Dirichlet u, on a démontré la stabilisation polynomiale de l'énergie d'un système couplé de deux équations complexe de Schrödinger avec des coefficients variables, en faisant un choix convenable de u, ici on a utilisé une combinaison successive de trois ingrédients: (1) La méthode de la géométrie Riemannienne pour traiter la partie principale à coefficients variables (2) L'idée de I. Lasiecka et R. Triggiani dans [LT87,LT92] utilisée pour obtenir la stabilisation exponentielle d'une seule équation avec un feedback frontière de type Dirichlet. (3) Les inégalités intégrales généralisées de F. Alabau dans [ACK02, Ala02].

A l'issue de notre étude, une remarque importante est que le nombre et le comportement à l'origine de la fonction feedback ont une influence sur les estimations de la stabilisation uniforme, si par exemple:

1. Si le comportement de la fonction feedback est linéaire à l'origine (nombre d'équations égale au nombre d'actions) alors la stabilisation est exponentielle.

2. Si le comportement de la fonction feedback est polynomial à l'origine

(nombre d'équations égale au nombre d'actions) alors la stabilisation est polynomiale.

3. Si les deux équations sont soumises à une seule action feedback linéaire alors la stabilisation n'est jamais exponentielle mais elle est polynomiale.

Problèmes ouverts

Notre étude ouvre la voie à plusieurs questions notamment.

1. Une question immédiate est de voir s'il est possible d'obtenir des estimations de la stabilisation des systèmes avec des termes non linéaire d'ordre un sans aucune hypothèse de petitesse.

2. Un autre problème intéressant est: peut on obtenir des résultats de la stabilisation uniforme de l'équation de Schrödinger avec un terme d'ordre un (linéaire ou non linéaire). Dans le cas linéaire, il est important de signaler qu'on a une certaine difficulté pour définir une énergie dissipative équivalente comme on a fait pour l'équation des ondes.

3. On peut aussi considérer le problème de la stabilisation de deux équations (par exemple ondes-ondes, Schrödinger-Schrödinger, etc...) couplées via un terme d'ordre un. A notre connaissance cette classe de systèmes n'a pas été considérée dans la littérature.

4. Dans [ALa01], l'auteur a considéré le problème de l'observabilité d'un système couplé de deux équations où une seule équation est observé. Ce qu'on l'appelle l'observation indirecte. Elle a démontré que pour un temps assez grand, l'observation de dérivée normale de la première composante de la solution sur une partie du bord, permet de restituer une énergie affaiblie de toute la donnée

initiale. Il est donc intéressant d'étudié ce problème quand le système considéré contient un terme d'ordre un dans chaque équation.

5. Une autre question est de voir s'il y'a une relation entre la stabilisation indirecte et l'observabilité indirecte des systèmes couplés. Concernant l'observabilité directe et la stabilisation directe une comparaison complète a été faite dans la littérature (voir par exemple [Rus78,Lio88]).

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Abstract.

The first goal of this thesis is to prove directly the exponential decay rate of the perturbed energy of the second order hyperbolic equation with variable coefficients and linear zero order term. Our second goal is to remove some conditions of smallness imposed on the linear first order term to obtain uniform decay rate of the wave equation with constant coefficients. The third goal is to prove that we can apply the Riemannian geometry method to study the exponential decay rates of complex systems with variable coefficients. As an example of such systems, we have considered two coupled Schrödinger's equations with variable coefficients and damped by two Neumann boundary feedbacks. When this system is damped by one Neumann boundary feedback, we have obtained the polynomial decay rate estimate for sufficiently smooth solution. Similar result was proved in the case of one Dirichlet boundary feedback.

Key words: Riemannian geometry method, multiplier method, stabilization inequalities.

Résumé.

Notre premier objectif de cette thèse est de montrer directement la décroissance exponentielle de l'énergie perturbée d'une équation hyperbolique d'ordre deux avec un terme linéaire d'ordre zéro. Le deuxième objectif est d'éliminer certaines conditions de petitesse imposées sur le terme linéaire d'ordre un pour obtenir la décroissance uniforme de l'énergie de l'équation des ondes à coefficients constants. Le troisième objectif est de montrer qu'on peut appliquer la méthode de la géométrie Riemannienne pour étudier la décroissance exponentielle de l'énergie des systèmes complexes avec des coefficients variables. Où, on a considéré un système couplé de deux équations de Schrödinger à coefficients variables et soumis à deux actions frontière de type Neumann. Lorsque ce système est soumis à une seule action, on a obtenu une décroissance polynomiale de l'énergie. Des résultats similaires ont été obtenus dans le cas d'une seule action frontière de type Dirichlet.

Mots clès: Méthode de la géométrie Riemannienne, méthode des multiplicateurs, inégalités de stabilisation.

ملخص.

هدفنا الأول هو إثبات مباشرة التناقص الأسي للطاقة لمعادلة ذات معامل خطي من الدرجة صفر. الهدف الثاني هو حذف بعض الشروط على صغر المعامل الخطي من الدرجة الأولي للحصول على تناقص أسي للطاقة لمعادلة الموجة بمعاملات ثابتة. أما الهدف الثالث فهو إثبات أنه يمكن تطبيق طريقة الهندسة الريمانية لدراسة التناقص الأسي للطاقة لمعادلة معقدة ذات معاملات متغيرة حيث اعتبرنا جملة مزدوجة مكونة من معادلتي شرود نجر ذات معاملات متغيرة تحت تأثير قوتين من نوع نيومان علي الحافة. عندما تكون هذه الجملة تحت تأثير قوة واحدة من نوع نيومان علي الحافة حصلنا على تناقص من نوع كثير حدود للطاقة كما حصلنا علي نفس النتيجة في حالة تأثير قوة واحدة من نوع دركلي علي الحافة.

كلمات مفتاحية: طريقة الهندسة الريمانية, طريقة الضرب, متراجحات الاستقرار.