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THEME

PROBLÉMES D'EXISTENCE GLOBALE DE SOLUTIONS POUR DES ÉQUATIONS D'ÉVOLUTION NON LINÉAIRES

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Abstract

The aim of this thesis is to study the problems of global existence of solutions for non linear evolution equations.

We first consider the coupled Gierer-Meinhardt systems with homogeneous Neumann boundary conditions. By using the technique of Lyapunov function we prove global existence of solutions. Under suitable conditions, we contribute to the study of the asymptotic behaviour of solutions. The basic idea of this result is a Lyapunov function which is non increasing function. These results are valid for any positive initial data in $C(\bar{\Omega})$, without any differentiability conditions. Moreover, we show that under reasonable conditions on the exponents of the non linear terms the solutions for considered system blow up in finite time.

The second part is devoted to study the uniform boundedness and so global existence of solutions for a Gierer-Meinhardt model of three substances described by reaction-diffusion equations with homogeneous Neumann boundary conditions. The proof of this result is based on a suitable Lyapunov functional and from which a result on the asymptotic behaviour of the solutions is established.

In the third and the last part, we investigate the local existence and uniqueness of mild solution for some hyper-viscous Hamilton-Jacobi equations. Under suitable conditions, the local existence of weak and strong solution, and the uniqueness of strong solution are also studied for considered problem. Moreover, we show the blow up in finite time of weak solution for some fractional Hamilton Jacobi-type equations

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CHAPTER 1

Introduction

It is well known that, the global existence of solutions for the non linear evolution equations is one of the fundamental question from the mathematical point of view. Inspired by this question, we first study in this thesis the large-time behaviour and blow up of solutions for Gierer-Meinhardt systems. Then we investigate the boundedness and large-time behaviour of solutions for a Gierer-Meinhardt system with three equations. Finally, we prove the local well-possedness for some fractional Hamilton-Jacobi-type equations.

In the second chapter, we improve the proof of global existence of solutions for coupled Gierer-Meinhardt systems with homogeneous Neumann boundary conditions. Our technique is based on a Lyapunov functional argument which yields the uniform boundedness of solutions. Under suitable conditions, we contribute to the study of the behaviour of the solutions. Moreover, we show that under reasonable conditions on the exponents of the non linear terms the solutions for considered system blow up in finite time. These results are valid for any positive initial data in $C(\bar{\Omega})$, without any differentiability conditions.

The third chapter is generalization of the previous chapter. We first treat the uniform boundedness of the solutions for Gierer-Meinhardt systems of three equations with homogeneous Neumann boundary conditions. Our technique is based on a Lyapunov functional, and by this method we also deal under suitable conditions the long-time behaviour of solutions as the time goes to $+\infty$ for considered system. These results are valid for any positive continuous initial data on $\bar{\Omega}$.

In the forth chapter, under suitable conditions on s, and the exponents β and α of the non linear term, we treat the local existence of weak solution for some fractional Hamilton-Jacobi-type equations . Moreover, we show the blow up in finite time of weak solution for considered problem.

Finally, we study the short-time existence and uniqueness of mild solutions for the same family of hyperviscous Hamilton-Jacobi equations which been studied by Bellout, Benachour and Titi [5], with conditions on the exponents β of the non linear term. We also investigate the local existence and uniqueness of strong solutions for some fractional Hamilton-Jacobi-type equations perturbed by the fractional s Laplacien, and the non linearity is of polynomial growth.

1. The Gierer-Meinhardt model

In the first part, we consider a general Gierer-Meinhardt system with the constant of relaxation time τ.

$$\begin{cases} \frac{\partial u}{\partial t} - a_1 \Delta u = -b_1 u + f(u,v) & \text{in } \mathbb{R}^+ \times \Omega, \\ \tau \frac{\partial v}{\partial t} - a_2 \Delta v = -b_2 v + g(u,v) & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{1.1}$$

where

$$\left[\begin{array}{l} f(u,v) = \rho_1(x,u,v) \frac{u^{p_1}}{v^{q_1}} + \sigma_1(x), \\ g(u,v) = \rho_2(x,u,v) \frac{u^{p_2}}{v^{q_2}} + \sigma_2(x), \end{array} \right. \tag{1.2}$$

with boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{1.3}$$

and initial data

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x) \qquad \text{in } \Omega. \tag{1.4}$$

Here Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \eta}$ is the outward normal derivative to $\partial\Omega$. The initial data are assumed to be positive and continuous on $\bar{\Omega}$.
The constants $\tau,\ p_i,\ q_i,\ a_i$ and b_i, $i = 1,\ 2$ are real numbers such that

$$\tau > 0,\ p_i \geq 0,\ q_i \geq 0,\ a_i > 0 \quad \text{and} \quad b_i > 0.$$

We assume that $\sigma_1,\ \sigma_2$ are positive functions in $C(\bar{\Omega})$, and $\rho_1,\ \rho_2$ are positive bounded functions in $C^1(\bar{\Omega} \times \mathbb{R}^2_+)$.

In the second part, we consider the Gierer-Meinhardt type system of three equations

$$\begin{cases} \frac{\partial u}{\partial t} - a_1 \Delta u = -b_1 u + f(u,v,w), & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial v}{\partial t} - a_2 \Delta v = -b_2 v + g(u,v,w), & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial w}{\partial t} - a_3 \Delta w = -b_3 w + h(u,v,w), & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{1.5}$$

where

$$\left[\begin{array}{l} f(u,v,w) = \rho_1(x,u,v,w) \frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)} + \sigma_1(x), \\ g(u,v,w) = \rho_2(x,u,v,w) \frac{u^{p_2}}{v^{q_2} w^{r_2}} + \sigma_2(x), \\ h(u,v,w) = \rho_3(x,u,v,w) \frac{u^{p_3}}{v^{q_3} w^{r_3}} + \sigma_3(x), \end{array} \right. \tag{1.6}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{1.7}$$

and initial data

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x) \text{ and } \quad w(0,x) = \varphi_3(x), \qquad \text{in } \Omega. \tag{1.8}$$

Here Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outer normal $\eta(x)$. The constants $c,\ p_i,\ q_i,\ r_i,\ a_i$ and b_i, $i = 1,\ 2,\ 3$ are real numbers such that

$$c,\ p_i,\ q_i,\ r_i \geq 0 \quad \text{and} \quad a_i,\ b_i > 0.$$

The initial data are assumed to be positive and continuous functions on $\bar{\Omega}$. For $i = 1,\ 2,\ 3$, we assume that σ_i are positive functions in $C(\bar{\Omega})$, and ρ_i are positive bounded functions in $C^1(\bar{\Omega} \times \mathbb{R}^3_+)$.

The Gierer-Meinhardt equations are included in the class of reaction-diffusion system. This model was formulated by Alfred Gierer[1] and Hans Meinhardt[2] in 1972, (see [13]). It describes the morphogenesis

[1] **Alfred Gierer** (April 15, 1929 in Berlin) is a German physicist , professor and director (emeritus) at the Max Planck Institute for Developmental Biology in Tübingen , which refers to biology (including exploration of the tobacco mosaic virus TMV), Biophysics and history philosophy of science has specialized.

[2] **Hans Meinhardt** (23 December 1938) is a German scientist . He studied physics in Cologne and Heidelberg and a PhD 1966th He then worked for two years at CERN in Geneva on computer simulations .Then he turned to biology to and joined the Max Planck Institute for Developmental Biology in Tübingen .

of organisms, and the pattern formation of tissue in particular. The central question is: if all cells of an organism start out the same, how can it be that they could grow out so differently? Sometimes, starting from almost homogeneous tissue, spatial patterns and different structures are formed, and these patterns could be independent of the total size of the tissue.

In 1952, Alan Turning[3] [49] already published a paper on morphogenesis which showed that a system of coupled reaction-diffusion equations can be used to describe differentiation and spatial patterns in biological systems. Gierer-Meinhardt used Turning's conclusions to describe biological pattern differential more thoroughly. They constructed a model consisting of two partial differential equations of reaction-diffusion type. It describes the concentration of two different kinds of substances, called the activator and inhibitor system.

An example of a model that Gierer-Meinhardt found for the activator u and the inhibitor v is

$$\begin{cases} \frac{\partial u}{\partial t} = D_u \Delta u + \rho_u \left(\frac{u^2}{v} - u \right), \\ \\ \frac{\partial v}{\partial t} = D_v \Delta v + \rho_v \left(u^2 - v \right). \end{cases} \tag{1.9}$$

Here Δ is the Laplace operator which depends on the space dimension. In a two dimensional system $\Delta = \frac{\partial^2}{\partial x^2}$. D_u and D_v are the diffusion rates of the activator u and the inhibitor v respectively, and ρ_u, ρ_v are the corresponding cross-reaction coefficients. to prevent activator from infinite growth the inhibitor should show down the increase of u. This means that the diffusion on v should be faster than the diffusion of u, i.e. $D_v \gg D_u$. For further explanation, see [26].

Another model of biological pattern formation, which proposed by Gierer and Meinhardt [13] to explain transplantation experiments on hydra mathematically are well-known. They can be expressed in terms of systems of reaction-diffusion equations of the form

$$\begin{cases} \frac{\partial u}{\partial t} = a_1 \Delta u - \mu_1 u + \frac{u^p}{v^q} + \sigma, & \text{in } \mathbb{R}^+ \times \Omega, \\ \\ \frac{\partial v}{\partial t} = a_2 \Delta v - \mu_2 v + \frac{u^r}{v^s}, & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{1.10}$$

on a bounded $\Omega \subset \mathbb{R}^N$, with the homogeneous Neumann boundary conditions and positive initial data. $a_1,\ a_2,\ \mu_1, \mu_2,\ \sigma$ are positive constants, and $p,\ q,\ r,\ s$ are non negative constants satisfying the basic relations

$$\frac{p-1}{r} < \frac{q}{s+1}.$$

Here $u = (u_1, u_2)$ is the unknown; $u_1,\ u_2$ represent concentrations of two substances, called activator and inhibitor. (For biological meanings of these parameters, and numerical treatment of the above equations, see [13]).

The following system that was obtained in [13] is an activator-substrate system. This is based on another way to stop the growth of the activator. The inhibitor could also be achieved by the depletion of a substance v that is required for the auto catalysis. In this case, the system is called an activator-substrate system. In its simplest form it looks like

$$\begin{cases} \frac{\partial u}{\partial t} = D_u \Delta u + \rho_u (u^2 v - u), \\ \\ \frac{\partial v}{\partial t} = D_v \Delta v + \rho_v (1 - u^2 v). \end{cases}$$

[3] **Alan Turning :** *Born*: 23 June 1912, London, England. *Died*: 7 June 1954 (aged 41), Cheshire, England. ***Fields***: Mathematics, crypt analysis, computer science, Biology. ***Institutions***: University of Manchester, Government Code and Cypher School, National Physical Laboratory, University of Cambridge. ***Thesis***: Systems of Logic based on Ordinals (1938). ***Notable awards***: Smith's Prize (1936), OBE, FRS[3].

The parameters have the same meaning as in (1.9). Here v is supposed to be antagonist. Again, the inhibition caused by the substrate is only effective if $D_v \gg D_u$.

2. Applications of the Gierer-Meinhardt model

In biological structures, polygonal patterns are very common. Think about a giraffe's coat or the veins in the wings of a dragonfly.

Figure 1: Polygonal patterns: Giraffe's skin

Figure 2: Polygonal patterns: Veins in wings of a dragonfly

The main difference between the morphogenesis of the coat of a giraffe and the veins in the wings of a dragonfly, is that the patterns on a giraffe, once formed does not change. This is not the case for the dragonfly. The veins in its wings are not produced in a single step at a particular moment of the development. Therefore, the models that describe the formation of both these patterns are different.

For the dragonfly, it is assumed that at an early stage of its development, a simple pattern is laid down. The main veins are already formed, but the smaller branches are developed later, in order to strengthen the growing wings. The model describing this behaviour is a combination of an activator-substrate at first, which should then be replaced by an activator-inhibitor model. The first model (activator-substrate) is given by

$$\begin{cases} \frac{\partial u}{\partial t} = D_u \Delta u + \rho_u \left(\frac{u^2 v}{1+\kappa_u w^2} - u \right) + \sigma_u, \\ \frac{\partial v}{\partial t} = D_v \Delta v - \rho_v \left(\frac{u^2 v}{1+\kappa_u w^2} \right) + \sigma_v. \end{cases}$$

This describes how a pattern of activation mound is produced. In terms of the dragonfly, it denotes how the main veins are formed. This first pattern triggers another system, which is an activator-inhibitor system

$$\begin{cases} \frac{\partial w}{\partial t} = D_w \Delta w + \rho_w \left(\frac{v}{1+\kappa_w u w^2} \left(\frac{w^2}{h} - \sigma_w \right) - w \right), \\ \frac{\partial h}{\partial t} = D_h \Delta h - \rho_h \left(w^2 - h \right). \end{cases}$$

In this system, the concentration of u determines the saturation value of the activator w. when u has a high concentration, the (w, h) system is turned off. At that moment, w has a low concentration. On the other hand, when u had a low concentration, the (w, h) system is triggered and it will form a pattern. This formation is enhanced by v, the substrate, because of the square term in the first equation of the second system. This will have most impact when v is big, and when the concentration of the substrate is high, the concentration of u must be low. Therefore, the action of h ensures that the stripe like patterns w forms become sharp.

It turns out that this model works very well in describing patterns. In general, activator-inhibitor and activator substrate systems are used to describe the formation of biological patterns.

3. Large-time behaviour of solution for Gierer-Meinhardt systems

3.1. Global existence of solutions for Gierer-Meinhardt systems. The global existence of solutions of (1.10) is one of the interesting question from the mathematical point of view. However, the presence of v^q, v^s in the denominators in non linear terms in (1.10) makes the mathematical analysis of (1.10) difficult.

The global existence of solutions of (1.10) is known only for $N = 3,\ p = 2,\ q = 1,\ r = 2,\ s = 0$ (Rothe [39] in 1984), which is a special case of activator model with the different source (in the terminology of (1.10)). The Rothe's method cannot be applied (at least directly) to the general $p,\ q,\ r,\ s$. Wu and Li [51] obtained the same results for (1.10) so long as u, v^{-1} and σ are suitably small.

It is desirable to consider the $p,\ q,\ r,\ s$ originally proposed by Gierer-Meinhardt. Li, et al [30] showed that the solutions of this problem are bounded all the time for each pair of initial values in $L^\infty(\Omega)$ if

$$\frac{p-1}{r} < \min\left(1, \frac{q}{s+1}\right). \tag{1.11}$$

In 1987, Masuda and Takahashi [33] considered the generalized Gierer-Meinhardt system

$$\frac{\partial u_i}{\partial t} = a_i \Delta u_i - \mu_i u_i + g_i(x, u_1, u_2), \qquad \text{in } \mathbb{R}^+ \times \Omega \quad (i = 1, 2)\,, \tag{1.12}$$

where $a_i,\ \mu_i,\ i = 1, 2$ are positive constants, and

$$\left[\begin{array}{l} g_1(x, u_1, u_2) = \rho_1(x, u_1, u_2)\frac{u_1^p}{u_2^q} + \sigma_1(x), \\ \\ g_2(x, u_1, u_2) = \rho_2(x, u_1, u_2)\frac{u_1^r}{u_2^s} + \sigma_2(x), \end{array}\right. \tag{1.13}$$

with $\sigma_1(.)$ (resp. $\sigma_2(.)$) is a positive (resp. non-negative) C^1 function on $\bar{\Omega}$, and ρ_1 (resp. ρ_2) is a non negative (resp. positive) bounded and C^1 function on $\bar{\Omega} \times \mathbb{R}_+^2$.
They extended the result of global existence of solutions for (1.12)–(1.13) of Li, et al [30] to

$$\frac{p-1}{r} < \frac{2}{N+2}, \tag{1.14}$$

and

$$\left\{\begin{array}{l} \varphi_1,\ \varphi_2 \in W^{2,l}(\Omega),\ l > \max\{N, 2\}\,, \\ \\ \frac{\partial \varphi_1}{\partial \eta} = \frac{\partial \varphi_2}{\partial \eta} = 0 \quad \text{on } \partial\Omega \qquad \text{and} \quad \varphi_1 \geq 0,\ \varphi_2 > 0 \quad \text{in } \bar{\Omega}. \end{array}\right. \tag{1.15}$$

In 2006, Jiang [22] obtained the same results of Masuda and Takahashi [33] by another method such that (1.11) and (1.15) are satisfied.
Abdelmalek et al ([2], in 2012) considered the following Gierer-Meinhardt system of three equations

$$\left\{\begin{array}{ll} \frac{\partial u}{\partial t} - a_1 \Delta u = -b_1 u + \frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)} + \sigma, & \text{in } \mathbb{R}^+ \times \Omega, \\ \\ \frac{\partial v}{\partial t} - a_2 \Delta v = -b_2 v + \frac{u^{p_2}}{v^{q_2} w^{r_2}}, & \text{in } \mathbb{R}^+ \times \Omega, \\ \\ \frac{\partial w}{\partial t} - a_3 \Delta w = -b_3 w + \frac{u^{p_3}}{v^{q_3} w^{r_3}}, & \text{in } \mathbb{R}^+ \times \Omega, \end{array}\right. \tag{1.16}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{1.17}$$

and the initial data

$$\begin{aligned} u(0,x) &= \varphi_1(x) > 0, \\ v(0,x) &= \varphi_2(x) > 0, \\ w(0,x) &= \varphi_3(x) > 0 \end{aligned} \tag{1.18}$$

in Ω, and $\varphi_i \in C(\bar{\Omega})$ for all $i = 1, 2, 3$.
Under the following condition

$$0 < p_1 - 1 < \max\left\{ p_2 \min\left(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1 \right), \ p_3 \min\left(\frac{r_1}{r_3+1}, \frac{q_1}{q_3}, 1 \right) \right\}, \tag{1.19}$$

and by using a suitable Lyapunov functional, they studied the global existence of solutions for the system (1.16)–(1.18). Their method gave only the result of global existence of solutions, and they did not make any attempts to obtain the results about the uniform boundedness of solutions on $(0, +\infty)$.

In 2011, Abdelmalek et al [1] concerned with the existence of global solutions to a reaction-diffusion system with m components generalizing the activator-inhibitor system

$$\begin{cases} \partial_t u_1 - a_1 \Delta u_1 = f_1(u) = \sigma - b_1 u_1 + \dfrac{u_1^{p_{11}}}{\prod\limits_{j=2}^{m} u_j^{p_{1j}}}, & \\ & x \in \Omega,\ t > 0, \\ \partial_t u_i - a_i \Delta u_i = f_i(u) = -b_i u_i + \dfrac{u_1^{p_{i1}}}{\prod\limits_{j=2}^{m} u_j^{p_{ij}}}, & i = 2, ..., m, \end{cases}$$

supplemented with Neumann boundary conditions

$$\frac{\partial u_i}{\partial \eta} = 0, \qquad \text{on } \partial\Omega \times t > 0,\ i = 1, ..., m,$$

and the positive initial data

$$u_i(x, 0) = \varphi_i(x) \qquad \text{on } \Omega,\ i = 1, ..., m.$$

Here $u = (u_1, ..., u_m)$, Ω is an open bounded domain of class C^1 in $\mathbb{R}^N$, with boundary $\partial\Omega$, and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$.
They made the following hypotheses. The indexes p_{ij} are non negative for all $i,\ j = 1, ..., m$, with $\sigma > 0$

$$0 < p_{11} - 1 < \max_{k=2,...,m} \left\{ p_{k1} \min\left\{ 1, \frac{p_{1k}}{p_{kk}}, \frac{p_{1j}}{p_{kj}},\ j = 2, ..., m,\ j \neq k \right\} \right\}.$$

The existence of global solutions was obtained via a judicious Lyapunov functional that generalizes the one introduced by Masuda and Takahashi [33].

Our first result is the following theorems, which show global existence and uniformly bounded of solutions for Gierer-Meinhardt systems.

Theorem 3.1. *If*

$$0 < p_1 - 1 < p_2 \min\left(\frac{q_1}{q_2+1}, 1 \right),$$

then all solutions of (1.1)–(1.4) with positive initial data in $C(\bar{\Omega})$ are global and uniformly bounded on $(0, +\infty) \times \bar{\Omega}$.

Theorem 3.2. *If*

$$0 < p_1 - 1 < \max\left\{p_2 \min\left(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1\right), \; p_3 \min\left(\frac{r_1}{r_3+1}, \frac{q_1}{q_3}, 1\right)\right\}, \tag{1.20}$$

then all solutions of (1.5)–(1.8) *with positive initial data in* $C(\bar{\Omega})$ *are global and uniformly bounded on* $(0,+\infty)\times\bar{\Omega}$.

3.2. Asymptotic behaviour of solutions for Gierer-Meinhardt systems. Wu and Li [51] supposed that the activator and the inhibitor fill a bounded domain Ω in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and that there is no flux through the boundary. They considered the following activator-inhibitor system proposed by Gierer-Meinhardt

$$\begin{cases} \frac{\partial A}{\partial t} = \varepsilon^2 \Delta A - A + \frac{A^p}{H^q}, \\ \tau \frac{\partial H}{\partial t} = D\Delta H - H + \frac{A^r}{H^s}, \end{cases} \tag{1.21}$$

for $x \in \Omega$ and $t > 0$, subject to the boundary condition and the initial data

$$\frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \qquad \text{for } x \in \partial\Omega, \; t > 0, \tag{1.22}$$

$$A(x,0) = A_0(x), \; H(x,0) = H_0(x) \;\; \text{for } x \in \Omega, \tag{1.23}$$

where $\varepsilon, \; D, \; \tau$ are positive constants, and the exponents $p > 1, \; q > 0, \; r > 0, \; s \geq 0$ satisfy

$$0 < \frac{p-1}{r} < \frac{q}{s+1}.$$

They proved that if $\tau > \frac{q}{p-1}$, then there are solutions of (1.21)–(1.23) such that

$$(u(x,t), v(x,t)) \longrightarrow (0,0)$$

uniformly on $\bar{\Omega}$ as $t \to +\infty$.

In 2008, Suzuki and Takagi ([43], [44]) considered the behaviour of a solution of the following system as $t \to +\infty$

$$\begin{cases} \frac{\partial A}{\partial t} = \varepsilon^2 \Delta A - A + \frac{A^p}{H^q} & \text{in } \mathbb{R}^+ \times \Omega, \\ \tau \frac{\partial H}{\partial t} = D\Delta H - H + \frac{A^r}{H^s} + \sigma_h(x) & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{1.24}$$

with the boundary conditions

$$\frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \;\; \text{in } \mathbb{R}^+ \times \partial\Omega, \tag{1.25}$$

and the initial data

$$A(x,0) = A_0(x), \; H(x,0) = H_0(x) \;\; \text{in } \Omega. \tag{1.26}$$

For the initial data, they assumed that

$$A_0, \; H_0 \in C^{2+\beta}(\bar{\Omega}), \; A_0(x) > 0, \; H_0(x) > 0 \;\; \text{on } \bar{\Omega} \quad \text{and}$$

$$\left.\frac{\partial A_0}{\partial \nu}\right|_{\partial\Omega} = \left.\frac{\partial H_0}{\partial \nu}\right|_{\partial\Omega} = 0, \tag{1.27}$$

where $0 < \beta < 1$. The term $\sigma_h(x)$ is called a basic production term. The exponents satisfy the following condition

$$p - 1 < r \min\left(\frac{q}{s+1}, 1\right).$$

If $\tau > \frac{q}{p-1}$, $\sigma_h(x) \geq 0$, on $\bar{\Omega}$ and

$$\left(\min_{x\bar{\Omega}} H_0(x)\right)^q > \frac{p-1}{p-1-\frac{q}{\tau}} \left(\max_{x\in\bar{\Omega}} A_0(x)\right)^{p-1},$$

then they proved that the solution $(A(t,x), H(t,x))$ of (1.24)–(1.26) satisfies

$$0 < \max_{x\in\bar{\Omega}} A(t,x) \leq Ce^{-t}, \ \max_{x\in\bar{\Omega}} \left(H(t,x) - z(x)\right) \leq Ce^{-\frac{t}{\tau}}$$

in which C is a positive constant depending $(A_0(t,x), H_0(t,x))$, and $z(x)$ is a solution of the problem

$$\begin{cases} D\Delta z - z + \sigma_h(x) = 0 & \text{for } x \in \Omega, \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Our main contributions of this thesis about the asymptotic behaviour of solutions for Gierer-Meinhardt systems are the following

Theorem 3.3. *Assume that*

$$0 < p_1 - 1 < p_2 \min\left(\frac{q_1}{q_2+1}, 1\right),$$

and let (u,v) be the solution of (1.1)–(1.4) in $(0,+\infty)$.
Suppose that $\sigma_1 \equiv 0$ and

$$b_1 > \frac{\tau^{-1}\beta b_2 + K}{2}, \tag{1.28}$$

where

$$K = \frac{\bar{\rho}_1\alpha\left(\frac{\rho_2\tau^{-1}\beta}{\bar{\rho}_1\alpha}\right)^{\frac{1-p_1}{p_2+1-p_1}}}{m_2^{[q_1p_2-(p_1-1)(1+q_2)](p_2+1-p_1)^{-1}}}.$$

Then

$$\lim_{t\longrightarrow\infty} ||u(t,.)||_\infty = \lim_{t\longrightarrow\infty} \left|\left|v(t,.) - \frac{\sigma_2}{b_2}\right|\right|_\infty = 0.$$

Theorem 3.4. *Assume that*

$$0 < p_1 - 1 < \max\left\{p_2 \min\left(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1\right), \ p_3 \min\left(\frac{r_1}{r_3+1}, \frac{q_1}{q_3}, 1\right)\right\}, \tag{1.29}$$

and let (u,v,w) be the solution of (1.5)–(1.8) in $(0,+\infty)$. Suppose that $\sigma_1 \equiv 0$, and

$$b_1 > \frac{\beta b_2 + \gamma b_3 + K}{2}, \tag{1.30}$$

where

$$K = \frac{\alpha\bar{\rho}_1\left(\frac{\beta\rho_2}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1}{p_2-p_1+1}}}{m_2^{[q_1p_2-(q_2+1)(p_1-1)](p_2-p_1+1)^{-1}} m_3^{[r_1p_2-r_2(p_1-1)](p_2-p_1+1)^{-1}}},$$

or

$$K = \frac{\alpha\bar{\rho}_1\left(\frac{\gamma\rho_3}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1}{p_3-p_1+1}}}{m_2^{[q_1p_3-q_3(p_1-1)](p_3-p_1+1)^{-1}} m_3^{[r_1p_3-(r_3+1)(p_1-1)](p_3-p_1+1)^{-1}}}.$$

Then

$$||u(t,.)||_\infty \longrightarrow 0 \qquad \text{as } t \to +\infty,$$

$$\left|\left|v(t,.) - \frac{\sigma_2}{b_2}\right|\right|_\infty \longrightarrow 0 \qquad \text{as } t \to +\infty.$$

$$\left\|w(t,.)-\frac{\sigma_3}{b_3}\right\|_\infty \longrightarrow 0 \qquad as\ t\to+\infty.$$

4. Blow-up of solutions for Gierer-Meinhardt systems

Li et al [30] proved the existence of blow-up solutions of the problem (1.10) for three cases. In the first case, they supposed that

$$p-1>r \qquad \text{and} \qquad rq<(s+1)(p-1).$$

Then, for some initial values, the solutions of the problem (1.10) blow-up in finite time. In the second and the third case, they supposed that the initial data are constants, then the problem (1.10) are transformed into the following ordinary differential equations

$$\begin{cases} u' = -\mu u + \frac{u^p}{v^q} + \sigma, \\ v' = -\nu v + \frac{u^r}{v^s}, \\ u(0)=u_0, \qquad v(0)=v_0. \end{cases} \tag{1.31}$$

If

$$p-1-r>0 \quad \text{and} \quad q-s-1>0$$

and if

$$r+1-p>0 \quad \text{and} \quad rq>(p-1)(q+1),$$

then, for some initial data u_0, v_0 the solutions of (1.31) blow up in finite time.

Pavol and Philippe [37] considered the system

$$\begin{cases} u_t - a\Delta u = -\mu_1 u + \frac{u^p}{v^q} + \sigma, & x\in\Omega,\ t>0, \\ v_t - b\Delta v = -\mu_2 v + \frac{u^r}{v^s}, & x\in\Omega,\ t>0, \\ u_\nu = v_\nu = 0 & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x), & x\in\Omega, \\ v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.32}$$

where $p>1,\ q,\ r,\ s\geq 0,\ a,\ b>0,\ \mu_1,\ \mu_2,\ \sigma\geq 0$ and $u_0,\ v_0\in C(\bar\Omega)$ with $u_0,\ v_0>0$. They proved that if

$$\frac{p-1}{r} > \min\left(\frac{q}{s+1},1\right), \qquad \frac{p-1}{r}\neq 1$$

then there exist space-independent initial data (i.e. solutions of the corresponding ODE system without diffusion) such that the solution $(u,v)=(u(t),v(t))$ of problem (1.32) satisfies $T_{max}<\infty$.

Our main contribution of this thesis about the blow-up of solutions for Gierer-Meinhardt systems is the following

Theorem 4.1. *Suppose that $p_i,\ q_i,\ i=1,2$ satisfy the following condition*

$$p_1-1>p_2\max\left(\frac{q_1}{q_2+1},1\right).$$

Then for some initial data such that φ_1 sufficiently large the solutions of (1.1)–(1.4) blow up in finite time.

5. Hamilton-Jacobi-type equations

In the third and the last part, we consider the fractional Hamilton-Jacobi-type equation

$$\begin{cases} u_t + (-\Delta)^s u = F(u, |\nabla u|) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.33}$$

with periodic boundary conditions, where $\Omega = (0, L)^N$, $L > 0$, $s \geq 2$, and $|\nabla u| = (\nabla u, \nabla u)^{\frac{1}{2}}$.

The Kuramoto-Sivashinsky equation have been studied with various space and time scaling. It can describe instabilities of dissipative trapped ion modes in plasma, instabilities in laminar flame front, phase dynamics in reaction-diffusion systems and fluctuations in fluid films on tilted supports, oscillatory chemical reactions, flow of a thin viscous film along a wall. Moreover, it describes the long-wavelength dynamics at the large length and time scales.

The Kuramoto-Sivashinsky equation in one space dimension, in "derivative" form

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \qquad x \in \left[-\frac{L}{2}, \frac{L}{2}\right], \tag{1.34}$$

or the integral form

$$\phi_t + \phi_{xxxx} + \phi_{xx} + \frac{1}{2}\phi_x^2 = 0, \tag{1.35}$$

where $u = \phi_x$. u_{xx} term carries an instability at large scales, u_{xxxx} term makes damping at small scales, and the non linear term uu_x (the same term as in one-dimensional Navier-Stokes, Burgers equation) is crucial for the global stability of the solution and transport energy between large and small scales.
In the one dimensional case equation (1.34) or (1.35) were studied by several authors both analytically and computationally (see [7]–[8], [9], [10], [15], [21], [23], [25], [27], [35], [36], [45], [46], and references therein).

The Kuramoto-Sivashinsky equation (KSE) in two dimension or higher is given as follows

$$\phi_t + \Delta^2\phi + \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0, \tag{1.36}$$

subject to the appropriate initial and boundary conditions, is an amplitude that arises when studying the propagation of instabilities in hydrodynamics and combustion theory.

Specifically, it appears in hydrodynamics as a model for the flow of thin soap films flowing down an inclined surface, and in combustion theory as a model for the propagation of flame fronts ([28], [41]). To avoid dealing with the average of the solution to this equation, most authors consider, instead, the system of equations for the evolution of $u = \nabla\phi$

$$u_t + \Delta^2 u + \Delta u + \frac{1}{2}\nabla|u|^2 = 0, \tag{1.37}$$

which is also called the KSE.

The question of global regularity of (1.36) or (1.37) in the two-dimensional, or hight, case is one of the major challenging problems in non linear analysis of partial differential equations.
Since $u = \nabla\phi$, equation (1.37) can be written as

$$u_t + \Delta^2 u + \Delta u + (u.\nabla)u = 0, \tag{1.38}$$

in which the non linearity takes a more familiar advection form. Let us assume that it is not difficult to prove the short-time well-posedness for all regular initial data, for any spatial dimension, subject to appropriate boundary conditions, such as periodic conditions, such as periodic boundary conditions.

(see also the work of [40] for global well-posedness for 'small' but not 'too-small' initial data in two-dimensional thin domains, subject to periodic boundary conditions.) However, the major challenge is to show the global well-posedness for (1.37) or (1.38) in the two and higher-dimensional cases. It is clear that the main obstacle in this challenging problem is not due to the destabilizing linear term Δu. In fact, one can equally consider the system

$$u_t + \Delta^2 u + (u.\nabla)u = 0 \tag{1.39}$$

or the equation

$$\phi_t + \Delta^2\phi + \frac{1}{2}|\nabla\phi|^2 = 0. \tag{1.40}$$

Now, equation (1.38) and (1.40) are more familiar. These are hyper-viscous versions of the Burgers-Hopf system of equations

$$u_t - \Delta u + (u.\nabla)u = 0 \tag{1.41}$$

or its scalar version

$$\phi_t - \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0. \tag{1.42}$$

Using the maximum principle for $|u(x,t)|^2$ one can easily show the global regularity for (1.41) in one , two and three dimensions, subject to periodic or homogeneous Dirichlet boundary conditions [29]. Similarly, using the Cole-Hopf transformation $v = e^{-\frac{\phi}{2}} - 1$, one can convert equation (1.42) into the heat equation in the variable v and hence conclude the global regularity in the cases of the Cauchy problem, periodic boundary conditions or homogeneous Dirichlet boundary conditions (see [29] and reference therein). However, it is clear that the maximum principle does not apply to equation (1.39) and the Cole-Hopf transformation does not apply to (1.40); hence, the global regularity for (1.39) or (1.40) in two and three dimensions is still an open question.

Souplet [42] considered the following generalization of the viscous Hamilton-Jacobi equation

$$\begin{cases} u_t - \Delta u = |\nabla u|^p & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.43}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.
They proved under optimal assumption of the growth of $|\nabla u|$ then gradient blow-up occurs for suitable large initial data.

Bellout et al [5] considered the following hyper-viscous Hamilton-Jacobi-type boundary value problem

$$\begin{cases} u_t + \Delta^2 u = |\nabla u|^p & \text{in } \mathbb{R}^+ \times \Omega, \\ u = \Delta u = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.44}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

Under certain constraints on the exponent p, they employed the Galarkin approximation procedure to establish the short-time existence of weak and strong solutions. Moreover, they showed the uniqueness of strong solutions. The uniqueness of weak solutions remains an open question. They proved that certain solutions to the problem (1.44) blow-up in finite time, provided $p > 2$. They also studied global existence for radial initial data in a radially symmetric domain that excludes a neighbourhood of the origin.

However, there is an essential difference in the structure of the formation of singularities in problems (1.43) and (1.44). First, we observe that regardless of the value of p, $p \geq 0$, problem (1.43) satisfies a maximum principle, and hence the $L^\infty(\Omega)$ norm of the solutions exist. Thus, the solutions to (1.43) that blow-up in finite time must develop their singularities in one of their spatial derivatives, while the $L^\infty(\Omega)$ norm remains finite.

On the contrary, for problem (1.44), Bellout et al [5] showed that at the blow-time, the $L^2(\Omega)$ norm of the solution, and therefore the $L^\infty(\Omega)$ norm of the solution must tend to infinity. This is a consequence of the fact that Bellout et al [5] obtained a lower bound on the existence time which depends only on the L^2 norm of the initial data u_0. Notice that, in the case of problem (1.44) we lost the maximum principle. The question of global existence for problem (1.44), in the case $p = 2$, is still open.

Our first result of this part is the following theorem, which gives the local existence and uniqueness of mild solution for the same family of hyper-viscous Hamilton-Jacobi equations which been studied by Bellout et al [5].

Theorem 5.1. *Given $u_0 \in L^2(\Omega)$ and*

$$\beta \geq 1 \qquad \text{for } N \leq 6 \qquad \text{and} \qquad 1 \leq \beta < \frac{N}{N-6} \qquad \text{for } N \geq 7, \tag{1.45}$$

there exist a maximal time $T_{max} > 0$ and a unique mild solution u to the problem (1.44).

Local existence and uniqueness. We assume that, there exist positive constants C_1, C_2 and C_3 independent of u, such that

$$|F(u, |\nabla u|)| \leq C_1 |u|^\alpha |\nabla u|^\beta, \tag{1.46}$$

$$\begin{cases} |\partial_1 F(u, |\nabla u|)| \leq C_2 |u|^{\alpha-1} |\nabla u|^\beta, \\ |\partial_2 F(u, |\nabla u|)| \leq C_3 |u|^\alpha |\nabla u|^{\beta-1}, \end{cases} \tag{1.47}$$

where α and β satisfy the following conditions

$$\begin{cases} \beta \geq 1, \ \alpha \geq 1, \ \text{and} \\ s > \frac{\beta(N+2)+N(\alpha-1)}{4}. \end{cases} \tag{1.48}$$

Theorem 5.2. *Under assumptions (1.46)–(1.48) and for any $u_0 \in L^2(\Omega)$, the problem (1.33) has at least a maximal weak solution.*

Theorem 5.3. *Assume (1.46) and (1.47) are satisfied.*

1. *Let $u_0 \in L^2(\Omega)$.*

   i) *If*

$$\beta \geq 1, \ \alpha \geq 1 \qquad \text{and} \tag{1.49}$$

$$\begin{cases} s \geq \frac{\beta(N+2)+2N(\alpha-1)}{2} & \text{for } N < 2s, \\ s > \frac{\beta(N+2)+N(\alpha-1)}{4} & \text{for } N \geq 2s, \end{cases}$$

then every weak solution to the problem (1.33) is a strong solution.

ii) *Assume* (1.49) *is satisfied.*
If

$$s \geq \frac{\beta(N+2)+2N(\alpha-1)}{2} \quad \text{for } N < 2(s-1),$$

and

$$\begin{cases} s > \frac{\beta(N+2)+N(\alpha-1)}{2(\beta+\alpha)}, & \\ & \text{for } N \geq 2s \\ \text{and } \alpha + s \leq 2, & \end{cases}$$

then the problem (1.33) *has a unique strong solution.*

2. *If* $u_0 \in H^s(\Omega)$ *and* (1.48) *is satisfied, then every weak solution of* (1.33) *is a strong solution. Furthermore, in this case* $u \in L^\infty((0,T); H^s(\Omega))$.
3. *For any* $u_0 \in L^2(\Omega)$, *we assume* (1.48) *is satisfied, every weak solution of* (1.33) *instantaneously becomes a strong solution. That is for any* $\tau > 0$, *we have* $\frac{\partial u}{\partial t} \in L^2((\tau, T); (H^s(\Omega))')$.

Finite time blow up. We have the following result

Theorem 5.4. *Let* $u_0 \in L^2(\Omega)$ *satisfy* $\displaystyle\int_\Omega u_0 \phi(x)dx > M = M(\Omega, \beta) > 0$ *sufficiently large. Assume* $F(u, |\nabla u|) = |\nabla u|^\beta$ *and*

$$2 < \beta < \frac{4s+N}{N+2}, \qquad \text{for} \qquad N < 4(s-1)$$

then, the problem (1.33) *cannot admit a globally defined weak solution, Indeed, there exists* $T^\sharp = T^\sharp(M) < \infty$ *such that* u *satisfies*

$$\lim_{t \to T^\sharp} \|u(.,t)\|_{L^2} = \infty \qquad \text{and} \qquad \lim_{t \to T^\sharp} \|u(.,t)\|_\infty = \infty. \tag{1.50}$$

CHAPTER 2

Large-time behaviour and blow up of solutions for Gierer-Meinhardt systems

1. Introduction

In this paper we improve the result of Masuda and Takahashi [33] and Jiang's method [22] to obtain the global existence of solutions for Gierer-Meinhardt systems. Our technique is based on a Lyapunov functional and by this method we show also the uniform boundedness of solutions. In particular, the results in [22] and in [33] are valid for initial data which are in the Sobolev space $W^{2,l}(\Omega)$, $l > \max\{N, 2\}$, and in the present paper the uniform boundedness of u and v is valid for positive initial data which are only continuous on $\bar{\Omega}$, without any differentiability conditions.

We consider a general Gierer-Meinhardt system for fraction reaction, more exactly, in the same direction of Masuda and Takahashi [33] but with the constant of relaxation time τ, it is a system of reaction-diffusion equations of the form

$$\begin{cases} \frac{\partial u}{\partial t} - a_1 \Delta u = -b_1 u + f(u,v), & \text{in } \mathbb{R}^+ \times \Omega, \\ \tau \frac{\partial v}{\partial t} - a_2 \Delta v = -b_2 v + g(u,v), & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{2.1}$$

where

$$\left[\begin{array}{l} f(u,v) = \rho_1(x,u,v) \frac{u^{p_1}}{v^{q_1}} + \sigma_1(x), \\ g(u,v) = \rho_2(x,u,v) \frac{u^{p_2}}{v^{q_2}} + \sigma_2(x), \end{array} \right. \tag{2.2}$$

with the boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{2.3}$$

and the initial data

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x) \qquad \text{in } \Omega. \tag{2.4}$$

Here Ω is an open-bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \eta}$ is the outward normal derivative to $\partial\Omega$. The initial data are assumed to be positive and continuous on $\bar{\Omega}$.
The constants $\tau,\ p_i,\ q_i,\ a_i$ and b_i, $i = 1,\ 2$ are real numbers such that

$$\tau > 0,\ p_i \geq 0,\ q_i \geq 0,\ a_i > 0 \quad \textit{and} \quad b_i > 0.$$

We assume that $\sigma_1,\ \sigma_2$ are positive functions in $C(\bar{\Omega})$, and $\rho_1,\ \rho_2$ are positive bounded functions in $C^1(\bar{\Omega} \times \mathbb{R}_+^2)$.

In 1972, following the ingenious idea of Turing [49], Gierer and Meinhardt [13] proposed a mathematical model for pattern formations of spatial tissue structure of hydra in morphogenesis, a biological phenomenon discovered by Trembley in 1744 [47]. It can be expressed in the following system

$$\begin{cases} \frac{\partial u}{\partial t} = a_1 \Delta u - \mu_1 u + \frac{u^p}{v^q} + \sigma, & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial v}{\partial t} = a_2 \Delta v - \mu_2 v + \frac{u^r}{v^s}, & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{2.5}$$

on a bounded $\Omega \subset \mathbb{R}^N$, with the homogeneous Neumann boundary conditions and positive initial data. $a_1,\ a_2,\ \mu_1, \mu_2$ and σ are positive constants, and $p,\ q,\ r,\ s$ are non negative constants satisfying the basic relations

$$\frac{p-1}{r} < \frac{q}{s+1}.$$

In 1984, Rothe [39] proved the global existence of solutions for the problem (2.5) with special cases $N = 3,\ p = 2,\ q = 1,\ r = 2$ and $s = 0$. The Rothe's method can not be applied (at least directly) to the general $p,\ q,\ r,\ s$.

Wu and Li [51] obtained the same results for the problem (2.5) so long as $u,\ v^{-1}$ and σ are suitably small.

Li et al [30] showed that the solutions of this problem are bounded all the time for each pair of initial values in $L^\infty(\Omega)$ if

$$\frac{p-1}{r} < \min\left\{1, \frac{q}{s+1}\right\}. \tag{2.6}$$

Masuda and Takahashi [33] (1987) considered the generalized Gierer-Meinhardt system

$$\frac{\partial u_i}{\partial t} = a_i \Delta u_i - \mu_i u_i + g_i(x, u_1, u_2), \qquad \text{in } \mathbb{R}^+ \times \Omega \quad (i = 1, 2)\,, \tag{2.7}$$

where $a_i,\ \mu_i,\ i = 1, 2$ are positive constants, and

$$\left[\begin{array}{l} g_1(x, u_1, u_2) = \rho_1(x, u_1, u_2)\frac{u_1^p}{u_2^q} + \sigma_1(x), \\ \\ g_2(x, u_1, u_2) = \rho_2(x, u_1, u_2)\frac{u_1^r}{u_2^s} + \sigma_2(x), \end{array}\right. \tag{2.8}$$

with $\sigma_1(.)$ (resp. $\sigma_2(.)$) is a positive (resp. non-negative) C^1 function on $\bar{\Omega}$, and ρ_1 (resp. ρ_2) is a non negative (resp. positive) bounded and C^1 function on $\bar{\Omega} \times \mathbb{R}^2_+$.
They extended the result of global existence of solutions for (2.7)–(2.8) of Li, Chen and Qin [30] to

$$\frac{p-1}{r} < \frac{2}{N+2}, \tag{2.9}$$

and

$$\left\{\begin{array}{l} \varphi_1,\ \varphi_2 \in W^{2,l}(\Omega),\ l > \max\{N, 2\}\,, \\ \\ \frac{\partial \varphi_1}{\partial \eta} = \frac{\partial \varphi_2}{\partial \eta} = 0 \quad \text{on } \partial\Omega \qquad \text{and} \quad \varphi_1 \geq 0,\ \varphi_2 > 0 \quad \text{in } \bar{\Omega}. \end{array}\right. \tag{2.10}$$

In 2006, Jiang [22] obtained the same results of Masuda and Takahashi [33] by another method such that (2.6) and (2.10) are satisfied.

The asymptotic behaviour of the solutions for (2.1)–(2.4) was studied by Wu and Li [51], and they proved that if $\sigma_1 \equiv \sigma_2 \equiv 0$ and $\tau > \frac{q}{p-1}$, then $(u(t,x), v(t,x)) \longrightarrow (0,0)$ uniformly on $\bar{\Omega}$ as $t \to +\infty$. Under suitable conditions on τ and on the initial data, Suzuki and Takagi ([43], [44]) also studied the behaviour of the solutions for (2.1)–(2.4).

The existence of blow up of solutions for (2.5) has been shown by Li, Chen and Qin [30] under the following condition

$$p - 1 > r \qquad \text{and} \qquad rq < (s+1)(p-1).$$

For the ordinary differential equations of the form

$$\begin{cases} u' = -\mu_1 u + \frac{u^p}{v^q} + \sigma, \\ v' = -\mu_2 v + \frac{u^r}{v^s}, \\ u(0) = \varphi_1, \qquad v(0) = \varphi_2, \end{cases} \tag{2.11}$$

they established a result of blow up of the solutions under the following conditions

$$\begin{aligned} p-1-r > 0 \qquad &\text{and} \qquad q-s-1>0, \\ r+1-p>0 \qquad &\text{and} \qquad rq < (p-1)(s+1). \end{aligned}$$

In this work, by using the technique of Lyapunov function we give a simple proof of global existence and bounded solutions for all positive time for a general Gierer-Meinhardt system with the constant of relaxation time τ. Under suitable conditions on the coefficients b_1 and b_2 we contribute to the study of the behaviour of the solutions for $\sigma_1(x) = 0$ and $\sigma_2(x) = \sigma_2 \geq 0$ on $\bar{\Omega}$.
Moreover, under suitable conditions on the exponents of the non linear term we show the existence of blow up solutions for the system (2.1)–(2.4). Our results are valid when $\sigma_1 \equiv \sigma_2 \equiv 0$.

2. Notations and preliminary results

2.1. Local existence of solutions. The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\bar{\Omega})$ are respectively denoted by

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p dx, \ \ 1 \leq p < +\infty,$$

$$\|u\|_\infty = ess \sup_{x\in\Omega} |u(x)|,$$

$$\|u\|_{C(\bar{\Omega})} = \max_{x\in\bar{\Omega}} |u(x)|.$$

For $i = 1, 2$ we set

$$\begin{aligned} \underline{\varphi_i} &= \min_{x\in\bar{\Omega}} \varphi_i(x), & \bar{\varphi_i} &= \max_{x\in\bar{\Omega}} \varphi_i(x), \\ \underline{\rho_i} &= \min_{x\in\bar{\Omega}, \xi\in\mathbb{R}_+^2} \rho_i(x,\xi), & \bar{\rho_i} &= \max_{x\in\bar{\Omega}, \xi\in\mathbb{R}_+^2} \rho_i(x,\xi), \\ \underline{\sigma_i} &= \min_{x\in\bar{\Omega}} \sigma_i(x), & \bar{\sigma_i} &= \max_{x\in\bar{\Omega}} \sigma_i(x). \end{aligned}$$

Local existence and uniqueness of solutions to problem (2.1)–(2.4) follow from the basic existence theory for parabolic semi-linear equations (see Friedman [11] and Pazy [38]). All solutions are classical on $(0,T)\times\Omega$, $T < T_{max}$, where $T_{max}(\|u_0\|_\infty, \|v_0\|_\infty)$ denotes the eventual blowing-up time in $L^\infty(\Omega)$.

2.2. Positivity of solutions. We introduce the following lemma.

Lemma 2.1. *If (u, v) is a solution of the problem (2.1)–(2.4), then for all $(t,x) \in (0,T)\times\Omega$, we have*

1. $$\begin{cases} u(t,x) \geq e^{-b_1 t}\underline{\varphi_1} > 0, \\ v(t,x) \geq e^{-\frac{b_2}{\tau}t}\underline{\varphi_2} > 0. \end{cases} \tag{2.12}$$

2.
$$\begin{cases} u(t,x) \geq \min\left(\frac{\sigma_1}{b_1}, \underline{\varphi_1}\right) = m_1, \\ \\ v(t,x) \geq \min\left(\frac{\sigma_2}{b_2}, \underline{\varphi_2}\right) = m_2. \end{cases} \tag{2.13}$$

PROOF. Immediate from the maximum principle. □

3. Boundedness of the solutions

In this section we assume that p_i, q_i, $i = 1, 2$ satisfy the following condition
$$0 < p_1 - 1 < p_2 \min\left(\frac{q_1}{q_2+1}, 1\right). \tag{H.1}$$
For proving the global existence of solutions for the problem (2.1)–(2.4), it suffices to prove that the solutions remains bounded in $(0,T) \times \bar{\Omega}$.

Now, let us define, for any $t \in (0,T)$,
$$L(t) = \int_\Omega \frac{u^\alpha(t,x)}{v^\beta(t,x)} dx, \tag{2.14}$$

where α and β are positive constants satisfying the following conditions
$$\alpha > \max\left(2, \frac{3b_2\tau^{-1}\beta}{b_1}\right) \quad \textit{and} \quad \frac{1}{\beta} > \frac{\left(a_1 + \tau^{-1}a_2\right)^2}{2\tau^{-1}a_1 a_2}. \tag{H.2}$$
One of the main results of this paper is the following.

THEOREM 3.1. *Under the hypotheses (H.1) and (H.2), all solutions of (2.1)–(2.4) with positive initial data in $C(\bar{\Omega})$ are global and uniformly bounded on $(0, +\infty) \times \bar{\Omega}$.*

Before proving this theorem we first need the following lemmas.

LEMMA 3.1. *Suppose that $x > 0$ and $y > 0$, then for each group of indexes p, q, δ, θ, λ satisfies $\lambda < p < \delta$ (not necessarily positive), and any constant $\Lambda > 0$, we have*
$$\frac{x^p}{y^q} \leq \Lambda \frac{x^\delta}{y^\theta} + \Lambda^{-\frac{p-\lambda}{\delta-p}} \frac{x^\lambda}{y^\eta}, \tag{2.15}$$
where $\eta = [q(\delta - \lambda) - \theta(p - \lambda)](\delta - p)^{-1}$.

PROOF. We can write
$$\frac{x^p}{y^q} = \left(x^{\frac{\delta(p-\lambda)}{\delta-\lambda}} y^{-\frac{\theta(p-\lambda)}{\delta-\lambda}}\right)\left(x^{\frac{\lambda(\delta-p)}{\delta-\lambda}} y^{\frac{\theta(p-\lambda)}{\delta-\lambda}-q}\right).$$
By using Young's inequality we get
$$\frac{x^p}{y^q} \leq \varepsilon \frac{x^\delta}{y^\theta} + \varepsilon^{-\frac{p-\lambda}{\delta-p}} \frac{x^\lambda}{y^\eta},$$
where $\eta = [q(\delta - \lambda) - \theta(p - \lambda)](\delta - p)^{-1}$.
Then the Lemma 3.1 is completely proved. □

LEMMA 3.2. *Let (u, v) be a solution to (2.1)–(2.4), then there exists a positive constant C such that for all $t \in (0,T)$ the functional*
$$L(t) = \int_\Omega \frac{u^\alpha(t,x)}{v^\beta(t,x)} dx \tag{2.16}$$

satisfies the inequality

$$\frac{d}{dt}L(t) \leq -\left(\alpha b_1 - 3b_2\tau^{-1}\beta\right) L(t) + C. \tag{2.17}$$

PROOF. Differentiating $L(t)$ we get for any $t \in (0, T)$

$$L'(t) \;=\; I + J,$$

where

$$I \;=\; a_1\alpha\int_\Omega \frac{u^{\alpha-1}}{v^\beta}\Delta u dx - a_2\tau^{-1}\beta\int_\Omega \frac{u^\alpha}{v^{\beta+1}}\Delta v dx,$$

and

$$\begin{aligned} J \;=\; & \left(-\alpha b_1 + b_2\tau^{-1}\beta\right) L(t) + \alpha\int_\Omega \rho_1(x,u,v)\frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}}dx \\ & -\tau^{-1}\beta\int_\Omega \rho_2(x,u,v)\frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}}dx + \alpha\int_\Omega \sigma_1(x)\frac{u^{\alpha-1}}{v^\beta}dx \\ & -\tau^{-1}\beta\int_\Omega \sigma_2(x)\frac{u^\alpha}{v^{\beta+1}}dx. \end{aligned}$$

By simple use of Green's formula, we may write I as follows

$$\begin{aligned} I \;=\; & -a_1\alpha(\alpha-1)\int_\Omega \frac{u^{\alpha-2}}{v^\beta}|\nabla u|^2 dx + \alpha\beta(a_1+\tau^{-1}a_2)\int_\Omega \frac{u^{\alpha-1}}{v^{\beta+1}}\nabla u\nabla v dx \\ & -a_2\tau^{-1}\beta(\beta+1)\int_\Omega \frac{u^\alpha}{v^{\beta+2}}|\nabla v|^2 dx. \end{aligned} \tag{2.18}$$

Using Young's inequality we get

$$\begin{aligned} \left|\alpha\beta(a_1+\tau^{-1}a_2)\int_\Omega \frac{u^{\alpha-1}}{v^{\beta+1}}\nabla u\nabla v dx\right| \;\leq\; & \frac{\alpha^2\beta(a_1+\tau^{-1}a_2)^2}{4\tau^{-1}(\beta+1)a_2}\int_\Omega \frac{u^{\alpha-2}}{v^\beta}|\nabla u|^2 dx \\ & +\tau^{-1}\beta(\beta+1)a_2\int_\Omega \frac{u^\alpha}{v^{\beta+2}}|\nabla v|^2 dx. \end{aligned}$$

It follows that

$$I \leq -a_1\alpha(\alpha-1)\int_\Omega \frac{u^{\alpha-2}}{v^\beta}|\nabla u|^2 dx + \frac{\alpha^2\beta(a_1+\tau^{-1}a_2)^2}{4\tau^{-1}(\beta+1)a_2}\int_\Omega \frac{u^{\alpha-2}}{v^\beta}|\nabla u|^2 dx.$$

From (H.2) we obtain that

$$I \leq 0 \qquad \text{for all } t \in (0,T). \tag{2.19}$$

We intend to estimate J, for this, we have

$$\begin{aligned} J \;\leq\; & \left(-\alpha b_1 + b_2\tau^{-1}\beta\right) L(t) + \bar{\rho}_1\alpha\int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}}dx - \underline{\rho}_2\tau^{-1}\beta\int_\Omega \frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}}dx \\ & +\alpha\bar{\sigma}_1\int_\Omega \frac{u^{\alpha-1}}{v^\beta}dx. \end{aligned} \tag{2.20}$$

Applying Lemma 3.1 with $p = \alpha - 1,\ q = \theta = \beta,\ \delta = \alpha$ and $\lambda = 0$, we get

$$\alpha\bar{\sigma}_1\int_\Omega \frac{u^{\alpha-1}}{v^\beta}dx \leq \tau^{-1}\beta b_2\int_\Omega \frac{u^\alpha}{v^\beta}dx + C_1\int_\Omega \frac{1}{v^\beta}dx, \tag{2.21}$$

where $C_1 = \alpha\bar{\sigma}_1 \left(\frac{\tau^{-1}\beta b_2}{\alpha\bar{\sigma}_1}\right)^{1-\alpha}$.
Next, we choose $\epsilon \in (0,\alpha)$ such that

$$\beta + \frac{\alpha(q_1-1-q_2)}{p_2+1-p_1} + \alpha\frac{q_1p_2-(p_1-1)(1+q_2)}{\epsilon(p_2+1-p_1)} \geq 0. \tag{2.22}$$

Again applying Lemma 3.1 for $p=\alpha-1+p_1,\ q=\beta+q_1,\ \delta=\alpha+p_2,\ \theta=\beta+1+q_2$ and $\lambda=\alpha-\epsilon$, we get

$$\bar{\rho}_1\alpha\int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}}dx \leq \underline{\rho}_2\tau^{-1}\beta\int_\Omega \frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}}dx + C_2\int_\Omega \frac{u^{\alpha-\epsilon}}{v^{\eta_1}}dx, \tag{2.23}$$

where

$$\eta_1 \ = \ \beta + \left[q_1p_2-(p_1-1)(1+q_2)+\epsilon(q_1-q_2-1)\right](p_2+1-p_1)^{-1},$$

and $C_2 = \bar{\rho}_1\alpha\left(\frac{\underline{\rho}_2\tau^{-1}\beta}{\bar{\rho}_1\alpha}\right)^{-\frac{p_1-1+\epsilon}{p_2+1-p_1}}$.
In an analogue way, we have

$$C_2\int_\Omega \frac{u^{\alpha-\epsilon}}{v^{\eta_1}}dx \leq b_2\tau^{-1}\beta\int_\Omega \frac{u^{\alpha}}{v^{\beta}}dx + C_3\int_\Omega \frac{1}{v^{\eta_2}}dx, \tag{2.24}$$

where

$$\eta_2 = \beta + \alpha\left(\frac{q_1-q_2-1+\epsilon^{-1}\left[q_1p_2-(p_1-1)(1+q_2)\right]}{p_2+1-p_1}\right) \geq 0,$$

thinks to (2.22), and $C_3 = C_2\left(\frac{b_2\tau^{-1}\beta}{C_2}\right)^{-\frac{\alpha-\epsilon}{\epsilon}}$.

We deduce immediately from (2.19)–(2.24) the following inequality

$$L'(t) \leq -(\alpha b_1 - 3\tau^{-1}\beta b_2)L(t) + C, \quad \text{for all } \ t\in(0,T), \tag{2.25}$$

where

$$C \ = \ |\Omega|\left(\frac{C_1}{m_2^\beta} + \frac{C_3}{m_2^{\eta_2}}\right).$$

This completes the proof of Lemma 3.2. □

Now we come back to the proof of the main theorem.

Proof of Theorem 3.1. In this proof, we will make use the result established in [5] and [4]. Let (u,v) be the solution of the system (2.1)–(2.4) in $(0,T)$. Multiplying the inequality (2.17) by $e^{(\alpha b_1 - 3b_2\tau^{-1}\beta)t}$ and then integrating over $[0,t]$, we deduce

$$L(t) \leq L(0) + \frac{C}{\alpha b_1 - 3\tau^{-1}b_2\beta} \quad \text{for all } \ t\in(0,T). \tag{2.26}$$

Then by using classical method of the semi group and the fractional powers of operators (see the appendix), and since $(\varphi_1,\varphi_2)\in(C(\bar{\Omega}))^2$, we conclude that

$$u \in L^\infty\left((0,T),L^\infty(\Omega)\right) \quad \textit{and} \quad v \in L^\infty\left((0,T),L^\infty(\Omega)\right).$$

Finally, we deduce that the solutions of the system (2.1)-(2.4) are global and uniformly bounded on $(0,+\infty)\times\bar{\Omega}$, □

Remark 3.1. *It is clear that the results of this section are valid when* $\sigma_1 \equiv \sigma_2 \equiv 0$.

4. Asymptotic behaviour of the solutions

In this section, we treat the asymptotic behaviour of solutions for the following system

$$\begin{cases} \frac{\partial u}{\partial t} - a_1 \Delta u = -b_1 u + f_1(u,v), & \text{in } \mathbb{R}^+ \times \Omega, \\ \tau \frac{\partial v}{\partial t} - a_2 \Delta v = -b_2 v + g_1(u,v), & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{2.27}$$

where

$$\left[\begin{array}{l} f_1(u,v) = \rho_1(x,u,v) \frac{u^{p_1}}{v^{q_1}}, \\ g_1(u,v) = \rho_2(x,u,v) \frac{u^{p_2}}{v^{q_2}} + \sigma_2, \end{array} \right. \tag{2.28}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{2.29}$$

and positive initial data

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x) \qquad \text{in } \Omega. \tag{2.30}$$

Here σ_2 is a non negative constant.

Before stating the results, let us expose some simple facts concluded from the result of the previous section.
From Lemma 3.2, and by using classical method of the semi group and the fractional powers of operators (see [5], [12]) we can find the positive constants M_1 and M_2 which are given explicitly in the appendix such that for any t in $(0,+\infty)$

$$\begin{aligned} \|u(t,.)\|_\infty &\leq M_1, \\ \|v(t,.)\|_\infty &\leq M_2. \end{aligned}$$

Let us consider a similar function as in Lemma 3.2 which will be used to study the asymptotic behaviour of solutions,

$$R(t) = \int_\Omega \frac{u^\alpha(t,x)}{v^\beta(t,x)} dx, \qquad t > 0,$$

where α and β are positive constants satisfying

$$\alpha > \max\left(2, \frac{3b_2\tau^{-1}\beta}{b_1}\right), \ \frac{1}{\beta} > \frac{(a_1 + \tau^{-1}a_2)^2}{2\tau^{-1}a_1 a_2}. \tag{2.31}$$

The main result in this section reads as follows.

Theorem 4.1. *Assume (H.1) holds. Let (u,v) be the solution of (2.27)–(2.30) in $(0,+\infty)$. Suppose that*

$$b_1 > \frac{\tau^{-1}\beta b_2 + K}{2}, \tag{2.32}$$

where

$$K = \frac{\bar{\rho}_1 \alpha \left(\frac{\underline{\rho}_2 \tau^{-1}\beta}{\bar{\rho}_1 \alpha} \right)^{\frac{1-p_1}{p_2+1-p_1}}}{m_2^{[q_1 p_2 - (p_1-1)(1+q_2)](p_2+1-p_1)^{-1}}}.$$

Then

$$\lim_{t\longrightarrow\infty}||u(t,.)||_\infty = \lim_{t\longrightarrow\infty}\left|\left|v(t,.)-\frac{\sigma_2}{b_2}\right|\right|_\infty = 0.$$

PROOF. We begin by proving that R is a non increasing function.
From (2.19) and (2.20), we get for all $t\in(0,+\infty)$

$$R'(t) \;\leq\; \left(-\alpha b_1 + b_2\tau^{-1}\beta\right)R(t) + \bar{\rho_1}\alpha\int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}}dx - \underline{\rho_2}\tau^{-1}\beta\int_\Omega\frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}}dx. \tag{2.33}$$

Now, we apply Lemma 3.1 for $p=\alpha-1+p_1,\ q=\beta+q_1,\ \delta=\alpha+p_2,\ \theta=\beta+1+q_2,\ \lambda=\alpha$, we get

$$\bar{\rho_1}\alpha\int_\Omega\frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}}dx \leq \underline{\rho_2}\tau^{-1}\beta\int_\Omega\frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}}dx + C_6\int_\Omega\frac{u^\alpha}{v^{\eta_3}}dx, \tag{2.34}$$

where

$$\eta_3 \;=\; \beta + \left[q_1p_2-(p_1-1)(1+q_2)\right](p_2+1-p_1)^{-1},$$

and $C_6 = \bar{\rho_1}\alpha\left(\frac{\underline{\rho_2}\tau^{-1}\beta}{\bar{\rho_1}\alpha}\right)^{\frac{1-p_1}{p_2+1-p_1}}$.

We set

$$\gamma = [q_1p_2-(p_1-1)(1+q_2)](p_2+1-p_1)^{-1}.$$

By (H.1) we find that γ is positive, and $\lambda<p<\delta$.
Then we get

$$\bar{\rho_1}\alpha\int_\Omega\frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}}dx - \underline{\rho_2}\tau^{-1}\beta\int_\Omega\frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}}dx \leq C_7R(t), \tag{2.35}$$

with $C_7=\frac{C_6}{m_2^\gamma}$,
We deduce from (2.33)–(2.35) the following inequality

$$R'(t)\leq -(\alpha b_1-\tau^{-1}\beta b_2-K)R(t), \tag{2.36}$$

where

$$K \;=\; \frac{\bar{\rho_1}\alpha\left(\frac{\underline{\rho_2}\tau^{-1}\beta}{\bar{\rho_1}\alpha}\right)^{\frac{1-p_1}{p_2+1-p_1}}}{m_2^{[q_1p_2-(p_1-1)(1+q_2)](p_2+1-p_1)^{-1}}}.$$

Using (2.32) we deduce that $R'(t)\leq 0$ for all positive time.

Now, for all $(t,x)\in(0,+\infty)\times\Omega$, setting

$$w_1(t,x)=u(t,x),$$

and

$$w_2(t,x)=v(t,x)-\frac{\sigma_2}{b_2}.$$

We have for $i=1,2$

$$\tau_i\frac{dw_i}{dt}-a_i\Delta w_i = -b_iw_i+\rho_i(x,u,v)\frac{u^{p_i}}{v^{q_i}} \tag{2.37}$$

such that $\tau_1 = 1,\ \tau_2 = \tau$.
Multiplying (2.37) by $w_i(t,x),\ i = 1,\ 2$ and integrating over $[0,t] \times \Omega$ we get

$$\frac{\tau_i}{2}\int_\Omega w_i^2 dx + a_i \int_0^t \int_\Omega |\nabla w_i|^2 dxds + b_i \int_0^t \int_\Omega w_i^2 dxds = \frac{\tau_i}{2}\int_\Omega w_i^2(0)dx + \int_0^t \int_\Omega w_i \rho_i(x,u,v)\frac{u^{p_i}}{v^{q_i}} dxds.$$

From (2.25) we obtain for $i = 1,2$

$$\int_0^t \int_\Omega w_i \rho_i(x,u,v)\frac{u^{p_i}}{v^{q_i}} dxds \ \leq\ \bar{\rho}_i M_i \frac{M_1^{p_i} M_2^\beta}{m_2^{q_i} m_1^\alpha}\int_0^t \int_\Omega \frac{u^\alpha}{v^\beta} dxds < +\infty.$$

One obviously deduces that for $i = 1,2$

$$w_i(t,.) \in L^2(\Omega), \qquad \int_0^{+\infty}\int_\Omega |\nabla w_i|^2 dxds < +\infty$$

and

$$\int_0^{+\infty}\int_\Omega w_i^2 dxds < +\infty,$$

so that Barbalate's lemma (see [16] Lemma 1.2.2) permits to conclude that

$$\lim_{t\to+\infty} \|w_i(t,.)\|_2 = 0, \qquad i = 1,2.$$

On the other hand, since the orbits $\{w_i(t,.)/t \geq 0,\ i = 1,2\}$ are relatively compact in $C(\bar{\Omega})$ (see [17]), it follows readily that

$$\lim_{t\to+\infty} \|w_i(t,.)\|_\infty = 0, \qquad i = 1,2.$$

Then the Theorem 4.1 is completely proved. □

5. Blow up results

In this section, we will show under suitable conditions on the exponents of the non linear term the solution to the problem (2.1)–(2.4) blows up in finite time.

Theorem 5.1. *Suppose that $p_i,\ q_i,\ i = 1,2$ satisfy the following condition*

$$p_1 - 1 > p_2 \max\left(\frac{q_1}{q_2+1}, 1\right). \tag{2.38}$$

Then for some initial data such that φ_1 sufficiently large the solutions of (2.1)–(2.4) blow up in finite time.

For the proof of the theorem, we need the following lemma.

Lemma 5.1. *Let (u,v) be the solution of the problem (2.1)–(2.4) in $(0,T)$. Then for any $\kappa > 0$, we have*

$$\int_\Omega \frac{1}{u^\kappa}dx + \kappa \underline{\rho_1}\int_0^t e^{s-t}\int_\Omega \frac{u^{p_1-1-\kappa}}{v^{q_1}} dxds \leq \int_\Omega \frac{1}{\varphi_1^\kappa}dx + (1+b_1\kappa)|\Omega| m_1^{-\kappa}. \tag{2.39}$$

Proof. Let $\kappa > 0$, we have

$$\frac{d}{dt}\int_\Omega \frac{1}{u^\kappa}dx \leq b_1\kappa \int_\Omega \frac{1}{u^\kappa}dx - \kappa\underline{\rho_1}\int_\Omega \frac{u^{p_1-1-k}}{v^{q_1}}dx.$$

Multiplying by e^t, and integrating from 0 to t, we obtain

$$\int_\Omega \frac{1}{u^\kappa}dx \leq \int_\Omega \frac{1}{\varphi_1^\kappa}dx + (1+b_1 k)|\Omega| m_1^{-\kappa} - \kappa\underline{\rho_1}\int_0^t e^{s-t}\int_\Omega \frac{u^{p_1-1-\kappa}}{v^{q_1}}dxds.$$

Thus the lemma is proved. □

Proof of Theorem 5.1. For all $t \in (0,T)$, let $W_1(t) = \displaystyle\int_\Omega \frac{v^n(t,x)}{u^m(t,x)}dx$, where

$$n > \max\left(2,\ \frac{(3+b_1 m)\tau}{b_2}\right), \quad \textit{and} \quad \frac{1}{m} > \frac{(a_1+\tau^{-1}a_2)^2}{2\tau^{-1}a_1a_2}. \tag{2.40}$$

We first prove that W_1 is a bounded function on $(0,T)$.
Differentiating $W_1(t)$ and using Green's formula, one obtains

$$W_1'(t) \;=\; H_1 + H_2,$$

where

$$\begin{aligned} H_1 \;&=\; -n(n-1)\tau^{-1}a_2\int_\Omega \frac{v^{n-2}}{u^m}|\nabla v|^2dx - m(m+1)a_1\int_\Omega \frac{v^n}{u^{m+2}}|\nabla u|^2dx \\ &\quad +nm(a_1+\tau^{-1}a_2)\int_\Omega \frac{v^{n-1}}{u^{m+1}}\nabla v\nabla u dx, \end{aligned}$$

and

$$\begin{aligned} H_2 \;&=\; (b_1m - b_2\tau^{-1}n)W_1(t) + n\tau^{-1}\int_\Omega \rho_2(x,u,v)\frac{v^{n-1-q_2}}{u^{m-p_2}}dx \\ &\quad +\tau^{-1}n\int_\Omega \sigma_2(x)\frac{v^{n-1}}{u^m}dx - m\int_\Omega \rho_1(x,u,v)\frac{v^{n-q_1}}{u^{m+1-p_1}}dx \\ &\quad -m\int_\Omega \sigma_1(x)\frac{v^n}{u^{m+1}}dx. \end{aligned}$$

Now we can write

$$H_1 \;=\; -\int_\Omega \left[\frac{v^{n-2}}{u^{m+2}}U^TQU\right]dx,$$

where

$$Q = \begin{pmatrix} \tau^{-1}a_2n(n-1) & -nm\frac{a_1+\tau^{-1}a_2}{2} \\ -nm\frac{a_1+\tau^{-1}a_2}{2} & a_1m(m+1) \end{pmatrix},$$

and

$$U = (u\nabla v \qquad v\nabla u)^T. \tag{2.41}$$

The successive principal minors of Q are positive. Indeed

1. $\Delta_1 = \tau^{-1}a_2n(n-1)$

Using (2.40), we deduce that $\Delta_1 > 0$.

2.

$$\begin{aligned} \Delta_2 \;&=\; \begin{vmatrix} \tau^{-1}a_2n(n-1) & -nm\frac{a_1+\tau^{-1}a_2}{2} \\ -nm\frac{a_1+\tau^{-1}a_2}{2} & a_1m(m+1) \end{vmatrix} \\ &=\; \tau^{-1}a_1a_2n^2m^2\left(\frac{(n-1)}{n}\frac{(m+1)}{m} - \frac{(a_1+\tau^{-1}a_2)^2}{4\tau^{-1}a_1a_2}\right). \end{aligned}$$

Also by using the condition (2.40), one gets $\Delta_2 > 0$.
Consequently Q is positive definite, and we have $H_1 \leq 0$ for all $(t,x) \in (0,T)\times\Omega$.
Concerning H_2, one observe that for all $t \in (0,T)$

$$\begin{aligned} H_2 \;&\leq\; (b_1m - \tau^{-1}b_2n)W_1 + \bar{\rho}_2\tau^{-1}n\int_\Omega \frac{v^{n-1-q_2}}{u^{m-p_2}}dx - \underline{\rho}_1 m\int_\Omega \frac{v^{n-q_1}}{u^{m+1-p_1}}dx \\ &\quad +\tau^{-1}n\bar{\sigma}_2\int_\Omega \frac{v^{n-1}}{u^m}dx. \end{aligned}$$

From Lemma 3.1 with $p=n-1,\ q=\theta=m,\ \delta=n,\ \lambda=0$, we obtain
$$\tau^{-1}n\bar{\sigma}_2\int_\Omega\frac{v^{n-1}}{u^m}dx\leq\int_\Omega\frac{v^n}{u^m}dx+A_1\int_\Omega\frac{1}{u^m}dx,\tag{2.42}$$
where $A_1=\tau^{-1}n\bar{\sigma}_2\left(\frac{\tau}{n\bar{\sigma}_2}\right)^{1-n}$.
Now, we choose ε satisfying $-m<\varepsilon<\min(p_1-p_2-1-m,\,p_2-m)$ such that
$$(q_2+1)(p_1-1)-p_2q_1=(n+q_2+1)(\varepsilon+m).\tag{2.43}$$
Again applying Lemma 3.1 with $p=p_2-m,\ q=q_2+1-n,\ \delta=p_1-1-m,\ \theta=q_1-n$ and $\lambda=\varepsilon$, we get
$$\bar{\rho}_2\tau^{-1}n\int_\Omega\frac{u^{p_2-m}}{v^{q_2+1-n}}dx\leq\underline{\rho}_1m\int_\Omega\frac{u^{p_1-m-1}}{v^{q_1-n}}dx+A_2\int_\Omega\frac{u^\varepsilon}{v^{\eta_4}}dx,\tag{2.44}$$
where
$$\eta_4\ =\ -n+(\varepsilon+m)(n+q_1)(p_1-1-p_2)^{-1},$$
thinks to (2.43), and $A_2=\bar{\rho}_2\tau^{-1}n\left(\frac{\underline{\rho}_1m}{\bar{\rho}_2\tau^{-1}n}\right)^{-\frac{p_2-m-\varepsilon}{p_1-1-p_2}}$.

In the same way, we have
$$A_2\int_\Omega\frac{u^\varepsilon}{v^{\eta_4}}dx\leq A_3\int_\Omega\frac{u^{p_1-1-p_2-m}}{v^{q_1}}dx+\int_\Omega\frac{v^n}{u^m}dx,\tag{2.45}$$
where $A_3=A_2^{\frac{p_1-1-p_2}{\varepsilon+m}}$.
It follows from (2.40), (2.42), (2.44) and (2.45)
$$W_1'(t)\leq -W_1(t)+A_1m_1^{-m}|\Omega|+A_3\int_\Omega\frac{u^{p_1-1-p_2-m}}{v^{q_1}}dx.\tag{2.46}$$
Multiplying (2.46) by e^t and integrating from 0 to t, we get
$$W_1(t)\leq W_1(0)+A_1m_1^{-m}|\Omega|+A_3\int_0^te^{s-t}\int_\Omega\frac{u^{p_1-1-p_2-m}}{v^{q_1}}dxds.\tag{2.47}$$
Now, if we apply Lemma 5.1 with $\kappa=p_2+m$, we will deduce that W_1 is a bounded function on $(0,T)$.

Setting $W_2(t)=\displaystyle\int_\Omega u^\zeta dx$ with $0<\zeta<1$, we get
$$W_2'(t)\ =\ \zeta\int_\Omega u^{\zeta-1}\left(a_1\Delta u-b_1u+\rho_1(x,u,v)\frac{u^{p_1}}{v^{q_1}}+\sigma_1(x)\right)dx.$$
Applying Green's formula we obtain
$$W_2'(t)\ \geq\ -\zeta b_1\int_\Omega u^\zeta dx+\underline{\rho}_1\zeta\int_\Omega\frac{u^{p_1+\zeta-1}}{v^{q_1}}dx.\tag{2.48}$$
Now, we choose $k>2$, such that for n large enough, m satisfies the conditions (2.40),
$$k\zeta=\frac{n(p_1-1+\zeta)-mq_1}{q_1+n}.\tag{2.49}$$
Using Hölder's inequality it yields
$$\left(\int_\Omega u^\zeta dx\right)^k\leq A_4\int_\Omega u^{\zeta k}dx,$$

with $A_4 = C|\Omega|^{k-1}$.
Applying Lemma 3.1 with $p = \zeta k,\ q = 0,\ \delta = p_1 + \zeta - 1,\ \theta = q_1, \lambda = -m$ we obtain

$$\left(\int_\Omega u^\zeta dx\right)^k \leq A_4 \left(\int_\Omega \frac{u^{p_1+\zeta-1}}{v^{q_1}} dx + \int_\Omega \frac{u^{-m}}{v^{\eta_2}} dx\right), \tag{2.50}$$

since (2.49) we get that the condition $\lambda < p < \delta$ is satisfied and

$$\eta_2 \ = \ -n.$$

Then, from (2.48) and (2.50) we have

$$W_2'(t) \geq -\zeta b_1 \int_\Omega u^\zeta dx - \zeta \underline{\rho}_1 \int_\Omega \frac{v^n}{u^m} dx + \zeta \underline{\rho}_1 A_4^{-1} \left(\int_\Omega u^\zeta dx\right)^k.$$

It follows that

$$W_2'(t) \geq \zeta \underline{\rho}_1 \left(A_4^{-1} W_2^k(t) - A_5\right) - \zeta b_1 W_2(t),$$

where

$$A_5 = \int_\Omega \frac{\varphi_2^n}{\varphi_1^m} dx + A_1 m_1^{-m} |\Omega| + \frac{A_3}{\underline{\rho}_1 (p_2 + m)} \left(\int_\Omega \frac{1}{\varphi_1^{p_2+m}} dx + (1 + b_1 p_2 + b_1 m)|\Omega| m_1^{-p_2-m}\right)$$

Then we can choose the initial data φ_1 sufficiently large, such that

$$\zeta \underline{\rho}_1 \left(A_4^{-1} W_2^k(0) - A_5\right) - \zeta b_1 W_2(0) > 0.$$

Thus we have proved that the derivative of $W_2(t)$ is positive and increasing which implies that $W_2(t)$ blows up in finite time (see [37]). $\square$

Boundedness and large-time behaviour of solutions for a Gierer-Meinhardt system with three equations

1. Introduction

In this paper, we consider a Gierer-Meinhardt type system of three equations

$$\begin{cases} \frac{\partial u}{\partial t} - a_1\Delta u = -b_1 u + f(u,v,w), & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial v}{\partial t} - a_2\Delta v = -b_2 v + g(u,v,w), & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial w}{\partial t} - a_3\Delta w = -b_3 w + h(u,v,w), & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{3.1}$$

where

$$\left[\begin{array}{l} f(u,v,w) = \rho_1(x,u,v,w)\frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)} + \sigma_1(x), \\ g(u,v,w) = \rho_2(x,u,v,w)\frac{u^{p_2}}{v^{q_2}w^{r_2}} + \sigma_2(x), \\ h(u,v,w) = \rho_3(x,u,v,w)\frac{u^{p_3}}{v^{q_3}w^{r_3}} + \sigma_3(x), \end{array}\right. \tag{3.2}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{3.3}$$

and initial data

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x) \quad \text{and} \quad w(0,x) = \varphi_3(x), \quad \text{in } \Omega. \tag{3.4}$$

Here Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outer normal $\eta(x)$. The constants $c,\ p_i,\ q_i,\ r_i,\ a_i$ and b_i, $i = 1,\ 2,\ 3$ are real numbers such that

$$c,\ p_i,\ q_i,\ r_i \geq 0, \quad \text{and} \quad a_i, b_i > 0.$$

and

$$0 < p_1 - 1 < \max\left\{p_2 \min\left(\frac{q_1}{q_2+1},\ \frac{r_1}{r_2},\ 1\right),\ p_3 \min\left(\frac{r_1}{r_3+1},\ \frac{q_1}{q_3},\ 1\right)\right\}. \tag{3.5}$$

The initial data are assumed to be positive and continuous functions on $\bar{\Omega}$. For $i = 1,\ 2,\ 3$, we assume that σ_i are positive functions in $C(\bar{\Omega})$, and ρ_i are positive bounded functions in $C^1(\bar{\Omega} \times \mathbb{R}^3_+)$.

In 1972, following the ingenious idea of Turing [49], Gierer and Meinhardt [13] proposed a mathematical model for pattern formations of spatial tissue structure of hydra in morphogenesis, a biological phenomenon discovered by Trembley in 1744 [47]. It can be expressed in the following system

$$\begin{cases} \frac{\partial u}{\partial t} = a_1\Delta u - \mu_1 u + \frac{u^p}{v^q} + \sigma, & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial v}{\partial t} = a_2\Delta v - \mu_2 v + \frac{u^r}{v^s}, & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{3.6}$$

on a bounded $\Omega \subset \mathbb{R}^N$, with the homogeneous Neumann boundary conditions and positive initial data: $a_1,\ a_2,\ \mu_1, \mu_2$ and σ are positive constants, and $p,\ q,\ r,\ s$ are non negative constants satisfying the relation

$$\frac{p-1}{r} < \frac{q}{s+1}.$$

The global existence of solutions to the system (3.6) is proved by Rothe [39] with special cases $N = 3,\ p = 2,\ q = 1,\ r = 2$ and $s = 0$. The Rothe's method can not be applied (at least directly) to general $p,\ q,\ r,\ s$.
Wu and Li [51] obtained the same results for the problem (3.6) so long as $u,\ v^{-1}$ and σ are suitably small.

Li, et al [30] showed that the solutions of this problem are bounded all the time for each pair of initial values in $L^{\infty}(\Omega)$ if

$$\frac{p-1}{r} < \min\left(1, \frac{q}{s+1}\right). \tag{3.7}$$

Masuda and Takahashi [33] considered the generalized Gierer-Meinhardt system

$$\frac{\partial u_i}{\partial t} = a_i \Delta u_i - \mu_i u_i + g_i(x, u_1, u_2), \qquad \text{in } \mathbb{R}^+ \times \Omega \quad (i = 1, 2), \tag{3.8}$$

where $a_i,\ \mu_i,\ i = 1, 2$ are positive constants, and

$$\left[\begin{array}{l} g_1(x, u_1, u_2) = \rho_1(x, u_1, u_2)\frac{u_1^p}{u_2^q} + \sigma_1(x), \\ \\ g_2(x, u_1, u_2) = \rho_2(x, u_1, u_2)\frac{u_1^r}{u_2^s} + \sigma_2(x), \end{array}\right. \tag{3.9}$$

with $\sigma_1(.)$ (resp. $\sigma_2(.)$) is a positive (resp. non-negative) C^1 function on $\bar{\Omega}$, and ρ_1 (resp. ρ_2) is a non negative (resp. positive) bounded and C^1 function on $\bar{\Omega} \times \mathbb{R}_+^2$.
They extended the result of global existence of solutions for (3.8)–(3.9) of Li, Chen and Qin [30] to

$$\frac{p-1}{r} < \frac{2}{N+2}, \tag{3.10}$$

and

$$\left\{\begin{array}{l} \varphi_1,\ \varphi_2 \in W^{2,l}(\Omega),\ l > \max\{N, 2\}, \\ \\ \frac{\partial \varphi_1}{\partial \eta} = \frac{\partial \varphi_2}{\partial \eta} = 0 \quad \text{on } \partial\Omega \qquad \text{and} \quad \varphi_1 \geq 0,\ \varphi_2 > 0 \quad \text{in } \bar{\Omega}. \end{array}\right. \tag{3.11}$$

Jiang [22] obtained the same results as Masuda and Takahashi [33] by another method such that (3.7) and (3.11) are satisfied.

Abdelmalek, et all [2] considered the following Gierer-Meinhardt system of three equations

$$\left\{\begin{array}{ll} \frac{\partial u}{\partial t} - a_1 \Delta u = -b_1 u + \frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)} + \sigma, & \text{in } \mathbb{R}^+ \times \Omega, \\ \\ \frac{\partial v}{\partial t} - a_2 \Delta v = -b_2 v + \frac{u^{p_2}}{v^{q_2} w^{r_2}}, & \text{in } \mathbb{R}^+ \times \Omega, \\ \\ \frac{\partial w}{\partial t} - a_3 \Delta w = -b_3 w + \frac{u^{p_3}}{v^{q_3} w^{r_3}}, & \text{in } \mathbb{R}^+ \times \Omega, \end{array}\right. \tag{3.12}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{3.13}$$

and the initial data

$$\begin{array}{rcl} u(0,x) & = & \varphi_1(x) > 0, \\ v(0,x) & = & \varphi_2(x) > 0, \\ w(0,x) & = & \varphi_3(x) > 0 \end{array} \tag{3.14}$$

in Ω, and $\varphi_i \in C(\bar{\Omega})$ for all $i = 1, 2, 3$.
Under the condition (3.5) and by using a suitable Lyapunov functional, they studied the global existence of solutions for the system (3.12)–(3.14). Their method gave only the result of global existence of solutions, and they did not make any attempt to obtain the results about the uniform boundedness of solutions on $(0, +\infty)$.

For asymptotic behaviour of the solutions, Wu and Li [51] considered the system

$$\begin{cases} \frac{\partial u_1}{\partial t} = a_1 \Delta u_1 - u_1 + \frac{u_1^p}{u_2^q} + \sigma_1(x), & \text{in } \mathbb{R}^+ \times \Omega, \\ \tau \frac{\partial u_2}{\partial t} = a_2 \Delta u_2 - u_2 + \frac{u_1^r}{u_2^s} + \sigma_2(x), & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{3.15}$$

with the constant of relaxation time $\tau > 0$, and they proved that if $\sigma_1 \equiv \sigma_2 \equiv 0$ and $\tau > \frac{q}{p-1}$, then $(u(t,x), v(t,x)) \longrightarrow (0,0)$ uniformly on $\bar{\Omega}$ as $t \to +\infty$.

Under suitable conditions on τ and on the initial data, Suzuki and Takagi ([43], [44]) also studied the behaviour of the solutions for (3.15) with the constant of relaxation time τ.

We first treat the uniform boundedness of the solutions for Gierer-Meinhardt system of three equations by proving that the Lyapunov function argument proposed in [2] can be adapted to our situation. Interestingly, we show that the same Lyapunov function satisfies a differential inequality from which the uniform boundedness of the solutions is deduced for any positive time.

Then under reasonable conditions on the coefficients b_1, b_2 and b_3, and by using the uniform boundedness of the solutions and the Lyapunov function which is non-increasing function, we deal with the long-time behaviour of solutions as the time goes to $+\infty$. In particular we are concerned with $\sigma_1 \equiv 0$, σ_2 and σ_3 are non-negative constants to assure that

$$\lim_{t \to +\infty} ||u(t,.)||_\infty = \lim_{t \to +\infty} \left|\left| v(t,.) - \frac{\sigma_2}{b_2} \right|\right|_\infty = \lim_{t \to +\infty} \left|\left| w(t,.) - \frac{\sigma_3}{b_3} \right|\right|_\infty = 0$$

2. Notations and preliminary results

2.1. Local existence of solutions.

For $i = 1,\ 2,\ 3$ we set

$$\begin{array}{rcll} \underline{\varphi}_i & = & \min\limits_{x \in \bar{\Omega}} \varphi_i(x), & \bar{\varphi}_i = \max\limits_{x \in \bar{\Omega}} \varphi_i(x), \\ \underline{\rho}_i & = & \min\limits_{x \in \bar{\Omega}, \xi \in \mathbb{R}^3_+} \rho_i(x,\xi), & \bar{\rho}_i = \max\limits_{x \in \bar{\Omega}, \xi \in \mathbb{R}^3_+} \rho_i(x,\xi), \\ \underline{\sigma}_i & = & \min\limits_{x \in \bar{\Omega}} \sigma_i(x), & \bar{\sigma}_i = \max\limits_{x \in \bar{\Omega}} \sigma_i(x). \end{array}$$

The basic existence theory for abstract semi linear differential equations directly leads to a local existence result to system (3.1)–(3.4) (see, Henry [20]). All solutions are classical on $(0,T) \times \Omega$, $T < T_{max}$, where $T_{max}(\|\varphi_1\|_\infty, \varphi_2\|_\infty, \|\varphi_3\|_\infty)$ denotes the eventual blowing-up time in $L^\infty(\Omega)$.

2.2. Positivity of solutions.

Lemma 2.1. *If (u, v, w) is a solution of the problem (3.1)–(3.4), then for all $(t, x) \in (0, T_{max}) \times \Omega$, we have*

1.
$$\begin{cases} u(t,x) \geq e^{-b_1 t}\underline{\varphi}_1 > 0, \\ v(t,x) \geq e^{-b_2 t}\underline{\varphi}_2 > 0, \\ w(t,x) \geq e^{-b_3 t}\underline{\varphi}_3 > 0. \end{cases}$$

2.
$$\begin{cases} u(t,x) \geq \min\left(\frac{\underline{\sigma}_1}{b_1}, \underline{\varphi}_1\right) = m_1, \\ v(t,x) \geq \min\left(\frac{\underline{\sigma}_2}{b_2}, \underline{\varphi}_2\right) = m_2, \\ w(t,x) \geq \min\left(\frac{\underline{\sigma}_3}{b_3}, \underline{\varphi}_3\right) = m_3. \end{cases}$$

Proof. Immediate from the maximum principle. □

3. Boundedness of the solutions

For proving the global existence of solutions for the problem (3.1)–(3.4), it suffices to prove that the solutions remains bounded in $(0, T) \times \bar{\Omega}$.

One of the main results of this section is the following

Theorem 3.1. *Assume that (3.5) holds. Let (u, v, w) be a solution to (3.1)–(3.4), and let*

$$L(t) = \int_\Omega \frac{u^\alpha(t,x)}{v^\beta(t,x) w^\gamma(t,x)} dx, \qquad \textit{for all} t \in (0, T), \tag{3.16}$$

where α, β and γ are positive constants satisfying the following conditions

$$\alpha > 2\max\left(1,\ \frac{3b_2 + b_3}{b_1}\right), \quad \frac{1}{\beta} > \frac{(a_1 + a_2)^2}{2a_1 a_2}, \tag{3.17}$$

and

$$\left(\frac{1}{2\beta} - \frac{(a_1+a_2)^2}{4a_1a_2}\right)\left(\frac{1}{2\gamma} - \frac{(a_1+a_3)^2}{4a_1a_3}\right) > \left(\frac{(\alpha-1)(a_2+a_3)}{2\alpha\sqrt{a_2a_3}} - \frac{(a_1+a_2)(a_1+a_3)}{4\sqrt{a_1^2a_2a_3}}\right)^2. \tag{3.18}$$

Then there exists a positive constant C such that for all $t \in (0, T)$

$$\frac{d}{dt}L(t) \leq -(\alpha b_1 - 3b_2\beta - \gamma b_3) L(t) + C. \tag{3.19}$$

Corollary 3.1. *Under the assumptions of Theorem 3.1, all solutions of (3.1)–(3.4) with positive initial data in $C(\bar{\Omega})$ are global and uniformly bounded on $(0, +\infty) \times \bar{\Omega}$.*

Before proving this theorem we first need the following technical lemma.

Lemma 3.1. *Suppose that $x > 0$, $y > 0$ and $z > 0$, then for each group of indices r, p, q, δ, θ, λ and ξ satisfies $\lambda < p < \delta$ (not necessarily positive), and any constant $\Lambda > 0$, we have*

$$\frac{x^p}{y^q z^r} \leq \Lambda \frac{x^\delta}{y^\theta z^\xi} + \Lambda^{-\frac{p-\lambda}{\delta-p}} \frac{x^\lambda}{y^{\eta_1} z^{\eta_2}},$$

where

$$\begin{aligned}\eta_1 &= [q(\delta-\lambda)-\theta(p-\lambda)](\delta-p)^{-1},\\ \eta_2 &= [r(\delta-\lambda)-\xi(p-\lambda)](\delta-p)^{-1}.\end{aligned}$$

Proof. We can write

$$\frac{x^p}{y^q z^r} = \left(x^{\frac{\delta(p-\lambda)}{\delta-\lambda}} y^{-\frac{\theta(p-\lambda)}{\delta-\lambda}} z^{-\frac{\xi(p-\lambda)}{\delta-\lambda}}\right)\left(x^{\frac{\lambda(\delta-p)}{\delta-\lambda}} y^{\frac{\theta(p-\lambda)}{\delta-\lambda}-q} z^{\frac{\xi(p-\lambda)}{\delta-\lambda}-r}\right).$$

By using Young's inequality we get

$$\frac{x^p}{y^q z^r} \le \varepsilon \frac{x^\delta}{y^\theta z^\xi} + \varepsilon^{-\frac{p-\lambda}{\delta-p}} \frac{x^\lambda}{y^{\eta_1} z^{\eta_2}},$$

where

$$\eta_1 = [q(\delta-\lambda)-\theta(p-\lambda)](\delta-p)^{-1},$$
$$\eta_2 = [r(\delta-\lambda)-\xi(p-\lambda)](\delta-p)^{-1}.$$

Then Lemma 3.1 is completely proved. □

Proof of Theorem 3.1. Let (u, v, w) be the solution of system (3.1)–(3.4) in $(0, T)$. Differentiating $L(t)$ respect to t, we get

$$L'(t) = I + J,$$

where

$$I = a_1\alpha \int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma}\Delta u\, dx - a_2\beta \int_\Omega \frac{u^\alpha}{v^{\beta+1} w^\gamma}\Delta v\, dx - a_3\gamma \int_\Omega \frac{u^\alpha}{v^\beta w^{\gamma+1}}\Delta w\, dx,$$

and

$$\begin{aligned} J &= (-\alpha b_1 + \beta b_2 + \gamma b_3)\, L(t) + \alpha \int_\Omega \rho_1(x,u,v,w)\frac{u^{\alpha-1+p_1}}{v^{\beta+q_1} w^{\gamma+r_1}} dx \\ &\quad -\beta \int_\Omega \rho_2(x,u,v,w)\frac{u^{\alpha+p_2}}{v^{\beta+1+q_2} w^{\gamma+r_2}} dx - \gamma \int_\Omega \rho_3(x,u,v,w)\frac{u^{\alpha+p_3}}{v^{\beta+q_3} w^{\gamma+1+r_3}} dx \\ &\quad +\alpha \int_\Omega \sigma_1(x)\frac{u^{\alpha-1}}{v^\beta w^\gamma} dx - \beta \int_\Omega \sigma_2(x)\frac{u^\alpha}{v^{\beta+1} w^\gamma} dx - \gamma \int_\Omega \sigma_3(x)\frac{u^\alpha}{v^\beta w^{\gamma+1}} dx. \end{aligned}$$

Using Green's formula we obtain for all $t \in (0, T)$ (see [2])

$$I \le 0. \tag{3.20}$$

Now let us get an estimate for the term J.
For all $t \in (0, T)$ we have

$$\begin{aligned} J &\le (-\alpha b_1 + \beta b_2 + \gamma b_3)\, L(t) + \alpha\bar{\rho}_1 \int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1} w^{\gamma+r_1}} dx - \beta\underline{\rho}_2 \int_\Omega \frac{u^{\alpha+p_2}}{v^{\beta+1+q_2} w^{\gamma+r_2}} dx \\ &\quad -\underline{\rho}_3\gamma \int_\Omega \frac{u^{\alpha+p_3}}{v^{\beta+q_3} w^{\gamma+1+r_3}} dx + \alpha\bar{\sigma}_1 \int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma} dx. \end{aligned} \tag{3.21}$$

Applying Lemma 3.1 with $p = \alpha - 1,\ q = \theta = \beta,\ r = \gamma,\ \delta = \alpha,\ \xi = \gamma$ and $\lambda = 0$, one gets

$$\alpha\bar{\sigma}_1 \int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma} dx \le \beta b_2 \int_\Omega \frac{u^\alpha}{v^\beta w^\gamma} dx + C_1 \int_\Omega \frac{1}{v^\beta w^\gamma} dx, \tag{3.22}$$

where $C_1 = \alpha\bar{\sigma}_1 \left(\frac{\beta b_2}{\alpha\bar{\sigma}_1}\right)^{1-\alpha}$.
Now, we choose $\epsilon_1 \in (0,\alpha)$ such that

$$\beta + \alpha\frac{q_1p_2 - (p_1-1)(1+q_2)}{\epsilon_1(p_2+1-p_1)} + \alpha\frac{q_1-1-q_2}{p_2+1-p_1} \geq 0,$$

$$\gamma + \alpha\frac{r_1p_2 - r_2(p_1-1)}{\epsilon_1(p_2-p_1+1)} + \alpha\frac{r_1-r_2}{p_2-p_1+1} \geq 0.$$

Again, applying Lemma 3.1 for $p = \alpha-1+p_1,\ q = \beta+q_1,\ r = \gamma+r_1,\ \delta = \alpha+p_2,\ \theta = \beta+1+q_2,\ \xi = \gamma+r_2$ and $\lambda = \alpha - \epsilon_1$, we obtain

$$\alpha\bar{\rho}_1 \int_\Omega \frac{u^{\alpha-1+p_1}}{v^{q_1+\beta}w^{r_1+\gamma}}dx \leq \beta\underline{\rho}_2 \int_\Omega \frac{u^{p_2+\alpha}}{v^{q_2+\beta+1}w^{r_2+\gamma}}dx + C_2\int_\Omega \frac{u^{\alpha-\epsilon_1}}{v^{\eta_1}w^{\eta_2}}dx, \tag{3.23}$$

where

$$\begin{aligned}
\eta_1 &= \beta + \left[q_1p_2 - (q_2+1)(p_1-1) + \epsilon_1(q_1-q_2-1)\right](p_2-p_1+1)^{-1},\\
\eta_2 &= \gamma + \left[r_1p_2 - r_2(p_1-1) + \epsilon_1(r_1-r_2)\right](p_2-p_1+1)^{-1},
\end{aligned}$$

and $C_2 = \alpha\bar{\rho}_1 \left(\frac{\beta\underline{\rho}_2}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1+\epsilon_1}{p_2-p_1+1}}$.

In an analogue way, we have

$$C_2\int_\Omega \frac{u^{\alpha-\epsilon_1}}{v^{\eta_1\eta_2}}dx \leq b_2\beta\int_\Omega \frac{u^\alpha}{v^\beta w^\gamma}dx + C_3\int_\Omega \frac{1}{v^{\eta_3\eta_4}}dx, \tag{3.24}$$

where

$$\begin{aligned}
\eta_3 &= \beta + \alpha\left[\epsilon_1^{-1}\left(q_1p_2 - (q_2+1)(p_1-1)\right) + q_1 - q_2 - 1\right](p_2-p_1+1)^{-1} \geq 0,\\
\eta_4 &= \gamma + \alpha\left[\epsilon_1^{-1}(r_1p_2 - r_2(p_1-1)) + r_1 - r_2\right](p_2-p_1+1)^{-1} \geq 0,
\end{aligned}$$

and $C_3 = C_2\left(\frac{b_2\beta}{C_2}\right)^{-\frac{\alpha-\epsilon_1}{\epsilon_1}}$.

Or, we choose $\epsilon_2 \in (0,\alpha)$ such that

$$\beta + \alpha\frac{q_1p_3 - q_3(p_1-1)}{\epsilon_2(p_3-p_1+1)} + \alpha\frac{q_1-q_3}{p_3-p_1+1} \geq 0,$$

$$\gamma + \alpha\frac{r_1p_3 - (r_3+1)(p_1-1)}{\epsilon_2(p_3-p_1+1)} + \alpha\frac{r_1-r_2-1}{p_3-p_1+1} \geq 0.$$

Now, applying Lemma 3.1 with $p = p_1+\alpha-1,\ q = q_1+\beta,\ r = r_1+\gamma,\ \delta = p_3+\alpha,\ \theta = q_3+\beta,\ \xi = r_3+\gamma+1$ and $\lambda = \alpha-\epsilon_2$, we find

$$\alpha\bar{\rho}_1 \int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}}dx \leq \gamma\underline{\rho}_3 \int_\Omega \frac{u^{\alpha+p_3}}{v^{\beta+q_3}w^{\gamma+1+r_3}}dx + C_4\int_\Omega \frac{u^{\alpha-\epsilon_2}}{v^{\eta_5}w^{\eta_6}}dx, \tag{3.25}$$

where

$$\begin{aligned}
\eta_5 &= \beta + \left[q_1p_3 - q_3(p_1-1) + \epsilon_2(q_1-q_3)\right](p_3-p_1+1)^{-1},\\
\eta_6 &= \gamma + \left[r_1p_3 - (r_3+1)(p_1-1) + \epsilon_2(r_1-r_3-1)\right](p_3-p_1+1)^{-1},
\end{aligned}$$

and $C_4 = \alpha\bar{\rho}_1 \left(\frac{\gamma\underline{\rho}_3}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1+\epsilon_2}{p_3-p_1+1}}$.
In the same way, we obtain

$$C_4\int_\Omega \frac{u^{\alpha-\epsilon_2}}{v^{\eta_5}w^{\eta_6}}dx \leq b_2\beta\int_\Omega \frac{u^\alpha}{v^\beta w^\gamma}dx + C_5\int_\Omega \frac{1}{v^{\eta_7}w^{\eta_8}}dx, \tag{3.26}$$

where

$$\begin{aligned}\eta_7 &= \beta+\alpha\left[\epsilon_2^{-1}(q_1p_3-q_3(p_1-1))+q_1-q_3\right](p_3-p_1+1)^{-1}\geq 0,\\ \eta_8 &= \gamma+\alpha\left[\epsilon_2^{-1}\left(r_1p_3-(r_3+1)(p_1-1)\right)+r_1-r_3-1\right](p_3-p_1+1)^{-1}\geq 0,\end{aligned}$$

and $C_5 = C_4\left(\frac{b_2\beta}{C_4}\right)^{-\frac{\alpha-\epsilon_2}{\epsilon_2}}$.

From (3.21)–(3.26) there exists a positive constant C such that

$$L'(t)\leq -(b_1\alpha-3\beta b_2-\gamma b_3)L(t)+C, \qquad \forall t\in(0,T).$$

Then Theorem 3.1 is completely proved.

□

PROOF OF COROLLARY 3.1. Since

$$L(t)\leq L(0)+\frac{C}{\alpha b_1-3b_2\beta-\gamma b_3} \quad \textit{for all } t\in(0,T),$$

then there exist non-negative constants C_6, C_7 and C_8 independent of t such that

$$\begin{aligned}\|f(u,v,w)-b_1u\|_N &\leq C_6,\\ \|g(u,v;w)-b_2v\|_N &\leq C_7,\\ \|h(u,v,w)-b_3w\|_N &\leq C_8.\end{aligned}$$

Since $(\varphi_1,\varphi_2,\varphi_3)\in(C(\bar{\Omega}))^3$, we conclude from the L^p-L^q-estimate (see Henry [20], Haraux and Kirane [17]) that

$$u\in L^{\infty}\left((0,T),L^{\infty}(\Omega)\right) \quad v\in L^{\infty}\left((0,T),L^{\infty}(\Omega)\right) \quad \text{and} \quad w\in L^{\infty}\left((0,T),L^{\infty}(\Omega)\right).$$

Finally, we deduce that the solutions of the system (3.1)–(3.4) are global and uniformly bounded on $(0,+\infty)\times\bar{\Omega}$. □

REMARK 3.1. *It is clear that the results of this section are valid when* $\sigma_1\equiv\sigma_2\equiv\sigma_3\equiv 0$.

4. Asymptotic behaviour of the solutions

In this section, we will study the asymptotic behaviour of the solutions for the following system

$$\begin{cases}\frac{\partial u}{\partial t}-a_1\Delta u=-b_1u+f(u,v,w), & \text{in } \mathbb{R}^+\times\Omega,\\ \frac{\partial v}{\partial t}-a_2\Delta v=-b_2v+g(u,v,w), & \text{in } \mathbb{R}^+\times\Omega,\\ \frac{\partial w}{\partial t}-a_3\Delta w=-b_3w+h(u,v,w), & \text{in } \mathbb{R}^+\times\Omega,\end{cases} \tag{3.27}$$

where

$$\left[\begin{array}{l} f(u,v,w)=\rho_1(x,u,v,w)\frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)}+\sigma_1,\\ g(u,v,w)=\rho_2(x,u,v,w)\frac{u^{p_2}}{v^{q_2}w^{r_2}}+\sigma_2,\\ h(u,v,w)=\rho_3(x,u,v,w)\frac{u^{p_3}}{v^{q_3}w^{r_3}}+\sigma_3,\end{array}\right. \tag{3.28}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial\eta}=\frac{\partial v}{\partial\eta}=\frac{\partial w}{\partial\eta}=0 \qquad \text{on } \mathbb{R}^+\times\partial\Omega, \tag{3.29}$$

and initial data

$$u(0,x)=\varphi_1(x),\quad v(0,x)=\varphi_2(x),\quad w(0,x)=\varphi_3(x)\qquad \text{in } \Omega. \tag{3.30}$$

Here $\sigma_1,\ \sigma_2$ and σ_3 are non negative constants.

Before stating the results, let us expose some simple facts concluded from the result of the previous section.
From Theorem 3.1, and by using classical method of a semi group and a power fractional (see [5]) we can find the positive constants $M_1,\ M_2$ and M_3 explicitly (see [34]) such that

$$\begin{aligned}\|u(t,.)\|_\infty &\leq M_1,\\ \|v(t,.)\|_\infty &\leq M_2,\\ \|w(t,.)\|_\infty &\leq M_3.\end{aligned}$$

Let us consider the same function in Theorem 3.1

$$L(t)=\int_\Omega \frac{u^\alpha(t,x)}{v^\beta(t,x)w^\gamma(t,x)}dx,\qquad \forall t\in(0,+\infty),$$

where $\alpha,\ \beta$ and γ are positive constants satisfying the following conditions

$$\alpha>2\max\left(1,\ \frac{3b_2+b_3}{b_1}\right),\quad \frac{1}{\beta}>\frac{(a_1+a_2)^2}{2a_1a_2},$$

and

$$\left(\frac{1}{2\beta}-\frac{(a_1+a_2)^2}{4a_1a_2}\right)\left(\frac{1}{2\gamma}-\frac{(a_1+a_3)^2}{4a_1a_3}\right)>\left(\frac{(\alpha-1)(a_2+a_3)}{2\alpha\sqrt{a_2a_3}}-\frac{(a_1+a_2)(a_1+a_3)}{4\sqrt{a_1^2a_2a_3}}\right)^2.$$

The main result in this section reads as follows.

Theorem 4.1. *Assume* (3.5) *holds. Let* (u,v,w) *be the solution of* (3.27)–(3.30) *in* $(0,+\infty)$. *Suppose that* $\sigma_1=0$, *and*

$$b_1>\frac{\beta b_2+\gamma b_3+K}{2}, \tag{3.31}$$

where

$$K=\frac{\alpha\bar{\rho}_1\left(\frac{\beta\rho_2}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1}{p_2-p_1+1}}}{m_2^{[q_1p_2-(q_2+1)(p_1-1)](p_2-p_1+1)^{-1}}m_3^{[r_1p_2-r_2(p_1-1)](p_2-p_1+1)^{-1}}},$$

or

$$K=\frac{\alpha\bar{\rho}_1\left(\frac{\gamma\rho_3}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1}{p_3-p_1+1}}}{m_2^{[q_1p_3-q_3(p_1-1)](p_3-p_1+1)^{-1}}m_3^{[r_1p_3-(r_3+1)(p_1-1)](p_3-p_1+1)^{-1}}}.$$

Then for all $t\in(0,+\infty)$ *we have*

$$L(t)\leq\int_\Omega\frac{\varphi_1^\alpha(x)}{\varphi_2^\beta(x)\varphi_3^\gamma(x)}dx.$$

Corollary 4.1. *Under the assumptions of Theorem 4.1, for all positive initial data in $C(\bar{\Omega})$ we have*

$$||u(t,.)||_\infty \longrightarrow 0 \qquad as\ t \to +\infty,$$

$$\left|\left|v(t,.) - \frac{\sigma_2}{b_2}\right|\right|_\infty \longrightarrow 0 \qquad as\ t \to +\infty.$$

$$\left|\left|w(t,.) - \frac{\sigma_3}{b_3}\right|\right|_\infty \longrightarrow 0 \qquad as\ t \to +\infty.$$

Proof of Theorem 4.1. From (3.20) and (3.21), we obtain for all $t \in (0,+\infty)$

$$\begin{aligned} L'(t) &\leq -(\alpha b_1 - \beta b_2 - \gamma b_2)L(t) + \alpha\bar{\rho}_1 \int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1} w^{\gamma+r_1}} dx \\ & -\beta\underline{\rho}_2 \int_\Omega \frac{u^{\alpha+p_2}}{v^{\beta+1+q_2} w^{\gamma+r_2}} dx - \gamma\underline{\rho}_3 \int_\Omega \frac{u^{\alpha+p_3}}{v^{\beta+q_3} w^{\gamma+1+r_3}} dx. \end{aligned} \tag{3.32}$$

Now, we apply Lemma 3.1 for $p = \alpha-1+p_1,\ q = \beta+q_1,\ r = \gamma+r_1,\ \delta = \alpha+p_2,\ \theta = \beta+1+q_2,\ \xi = \gamma+r_2$ and $\lambda = \alpha$ we get

$$\alpha\bar{\rho}_1 \int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1} w^{\gamma+r_1}} dx \leq \beta\underline{\rho}_2 \int_\Omega \frac{u^{\alpha+p_2}}{v^{\beta+1+q_2} w^{\gamma+r_2}} dx + A_1 \int_\Omega \frac{u^\alpha}{v^{\eta_9} w^{\eta_{10}}} dx, \tag{3.33}$$

where

$$\begin{aligned} \eta_9 &= \beta + [q_1p_2 - (q_2+1)(p_1-1)]\,(p_2-p_1+1)^{-1} > 0, \\ \eta_{10} &= \gamma + [r_1p_2 - r_2(p_1-1)]\,(p_2-p_1+1)^{-1} > 0, \end{aligned}$$

and $A_1 = \alpha\bar{\rho}_1 \left(\frac{\beta\underline{\rho}_2}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1}{p_2-p_1+1}}$.

Or, applying Lemma 3.1 for $p = \alpha-1+p_1,\ q = \beta+q_1,\ r = \gamma+r_1,\ \delta = \alpha+p_3,\ \theta = \beta+q_3,\ \xi = \gamma+1+r_3$ and $\lambda = \alpha$, we get

$$\alpha\bar{\rho}_1 \int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1} w^{\gamma+r_1}} dx \leq \gamma\underline{\rho}_3 \int_\Omega \frac{u^{\alpha+p_3}}{v^{\beta+q_3} w^{\gamma+1+r_3}} dx + A_2 \int_\Omega \frac{u^\alpha}{v^{\eta_{11}} w^{\eta_{12}}} dx, \tag{3.34}$$

where

$$\begin{aligned} \eta_{11} &= \beta + [q_1p_3 - q_3(p_1-1)](p_3-p_1+1)^{-1} > 0, \\ \eta_{12} &= \gamma + [r_1p_3 - (r_3+1)(p_1-1)](p_3-p_1+1)^{-1} > 0, \end{aligned}$$

and $A_2 = \alpha\bar{\rho}_1 \left(\frac{\gamma\underline{\rho}_3}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1}{p_3-p_1+1}}$.

By combining (3.32) with (3.33) and (3.34) we obtain

$$L'(t) \leq -(\alpha b_1 - \beta b_2 - \gamma b_3 - K)L(t), \qquad \forall t \in (0,+\infty), \tag{3.35}$$

where

$$K = \frac{\alpha\bar{\rho}_1 \left(\frac{\beta\underline{\rho}_2}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1}{p_2-p_1+1}}}{m_2^{[q_1p_2-(q_2+1)(p_1-1)](p_2-p_1+1)^{-1}}\, m_3^{[r_1p_2-r_2(p_1-1)](p_2-p_1+1)^{-1}}},$$

or

$$K = \frac{\alpha\bar{\rho}_1 \left(\frac{\gamma\underline{\rho}_3}{\alpha\bar{\rho}_1}\right)^{-\frac{p_1-1}{p_3-p_1+1}}}{m_2^{[q_1p_3-q_3(p_1-1)](p_3-p_1+1)^{-1}}\, m_3^{[r_1p_3-(r_3+1)(p_1-1)](p_3-p_1+1)^{-1}}}.$$

Using (3.31) we deduce that the function $t \longmapsto L(t)$ is a non-increasing function.

This completes the proof of Theorem 4.1. $\square$

PROOF OF COROLLARY 4.1. Setting for all $(t,x) \in (0,+\infty) \times \Omega$

$$\begin{aligned} h_1(t,x) &= u(t,x), \\ h_2(t,x) &= v(t,x) - \frac{\sigma_2}{b_2}, \end{aligned}$$

and

$$h_3(t,x) = w(t,x) - \frac{\sigma_3}{b_3}.$$

For $i = 1,2,3$ we have

$$\frac{dh_i}{dt} - a_i \Delta h_i = -b_i h_i + \rho_i(x,u,v,w) \frac{u^{p_i}}{v^{q_i} w^{r_i}}. \tag{3.36}$$

Multiplying (3.36) by $h_i(t,x)$, $i = 1,\ 2,\ 3$ and integrating over $[0,t] \times \Omega$ we get

$$\begin{aligned} \frac{1}{2}\int_\Omega h_i^2 dx + a_i \int_0^t \int_\Omega |\nabla h_i|^2 dxds \ &+ \ b_i \int_0^t \int_\Omega h_i^2 dxds = \frac{1}{2}\int_\Omega h_i^2(0)dx \\ &+ \int_0^t \int_\Omega h_i \rho_i(x,u,v) \frac{u^{p_i}}{v^{q_i} w^{r_i}} dxds. \end{aligned}$$

From (3.35), for all $t \in (0,+\infty)$, and for $i = 1,2,\ 3$ we obtain

$$\int_0^t \int_\Omega h_i \rho_i(x,u,v) \frac{u^{p_i}}{v^{q_i} w^{r_i}} dxds \ \leq \ \bar{\rho}_i M_i \frac{M_1^{p_i} M_2^{\beta} M_3^{\gamma}}{m_2^{q_i} m_1^{\alpha} m_3^{r_i}} \int_0^t \int_\Omega \frac{u^\alpha}{v^\beta w^\gamma} dxds < +\infty.$$

One obviously deduces that for $i = 1,2,3$

$$h_i(t,.) \in L^2(\Omega), \qquad \int_0^{+\infty} \int_\Omega |\nabla h_i|^2 dxds < +\infty$$

and

$$\int_0^{+\infty} \int_\Omega h_i^2 dxds < +\infty,$$

so that Barbalate's lemma (see [16] Lemma 1.2.2) permits to conclude that

$$\lim_{t \to +\infty} \|h_i(t,.)\|_2 = 0, \qquad i = 1,2,3.$$

On the other hand, since the orbits $\{h_i(t,.)/t \geq 0,\ i = 1,2,3\}$ are relatively compact in $C(\bar{\Omega})$ (see [17]), it follows readily that

$$\lim_{t \to +\infty} \|h_i(t,.)\|_\infty = 0, \qquad i = 1,2,3.$$

Then Corollary 4.1 is completely proved. $\square$

Local existence and blow up of weak solutions for some fractional Hamilton-Jacobi-type equations

1. Introduction

In this work we deal with the fractional Hamilton-Jacobi-type equation

$$\begin{cases} u_t + (-\Delta)^s u = F(u, |\nabla u|) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4.1}$$

with periodic boundary conditions, where $\Omega = (0, L)^N$, $L > 0$, $s \geq 2$, and $|\nabla u| = (\nabla u, \nabla u)^{\frac{1}{2}}$.

Assume that, there exist positive constants C_1, C_2 and C_3 independent of u, such that

$$|F(u, |\nabla u|)| \leq C_1 |u|^\alpha |\nabla u|^\beta, \tag{4.2}$$

$$\begin{cases} |\partial_1 F(u, |\nabla u|)| \leq C_2 |u|^{\alpha-1} |\nabla u|^\beta, \\ |\partial_2 F(u, |\nabla u|)| \leq C_3 |u|^\alpha |\nabla u|^{\beta-1}, \end{cases} \tag{4.3}$$

where α and β given positive numbers, such that

$$\begin{cases} \beta \geq 1, \ \alpha \geq 1, \ \text{and} \\ s > \frac{\beta(N+2)+N(\alpha-1)}{4}. \end{cases} \tag{4.4}$$

The global existence of solutions for the two and three dimensional Kuramoto-Sivashinsky equations is one of the major open questions in non linear analysis.

The Kuramoto-Sivashinsky equation (KSE) has the following form

$$\phi_t = -\Delta^2 \phi - \Delta \phi - \frac{1}{2} |\nabla \phi|^2. \tag{4.5}$$

We can be written

$$u_t = -\Delta^2 u - \Delta u - \frac{1}{2} \nabla |u|^2, \tag{4.6}$$

or

$$u_t = -\Delta^2 u - \Delta u - (u.\nabla) u, \tag{4.7}$$

with $u = \nabla \phi$, subject to the appropriate initial and boundary conditions.
The KSE has been introduced three decades ago as a model of non linear evolution of linearly unstable interfaces in various contexts such as phase turbulence and flame front propagation in combustion theory.

In one-dimension, it takes the derivative form

$$u_t + u_{xxxx} + u_{xx} + u u_x = 0 \qquad x \in \left[-\frac{L}{2}, \frac{L}{2} \right], \tag{4.8}$$

or the integral form

$$\phi_t + \phi_{xxxx} + \phi_{xx} + \frac{1}{2} \phi_x^2 = 0, \tag{4.9}$$

where $u = \phi_x$. The term u_{xx} in (4.8) is responsible for an instability at large scales; the dissipative u_{xxxx} term provides damping at small scales; and the non-linear term uu_x (which has the same form as that in the Burgers or one-dimensional Navier-Stokes equations) stabilizes by transferring energy between large and small scales. This equation is one of the simplest one dimensional PDE, which was studied by several authors both analytically and computationally. (see [7]–[8], [9], [10], [14], [21], [23], [25], [27], [35], [36], [45], [46], and references therein).

The global regularity of (4.5), (4.6) or (4.7) in the two-dimensional, or higher is one of the major open questions in non linear analysis of partial differential equations. Let us mention that it is not difficult to prove the short-time well-posedness for all regular initial data, or global well-posedness for small initial data, for any equations (4.5), (4.6) or (4.7), at any spatial dimension, subject to appropriate boundary conditions, such as periodic boundary conditions. (See also the work of [40] for global well-posedness for 'small' but not 'too-small' initial data in two-dimensional thin domains, subject to periodic boundary conditions).

These are hyper-viscous versions of the Burgers-Hopf system of equations

$$u_t - \Delta u + (u.\nabla)u = 0 \tag{4.10}$$

or its scalar version

$$\phi_t - \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0. \tag{4.11}$$

Using the maximum principle for $|u(x,t)|^2$ one can easily show the global regularity for (4.10) in one, two and three dimensions, subject to periodic or homogeneous Dirichlet boundary conditions [29]. Similarly, using the Cole-Hopf transformation $v = e^{-\frac{\phi}{2}} - 1$, one can convert equation (4.11) into the heat equation in the variable v and hence conclude the global regularity in the cases of the Cauchy problem, periodic boundary conditions or homogeneous Dirichlet boundary conditions (see [29]).

The major challenge is to show the global well-posedness for (4.5), (4.6) or (4.7) in the two-and higher-dimensional cases. It is clear that the main obstacle in this challenging problem is not due to the destabilizing linear term Δu. In fact, one can equally consider the system

$$u_t + \Delta^2 u + (u.\nabla u)u = 0 \tag{4.12}$$

or the equation

$$\phi_t + \Delta^2\phi + \frac{1}{2}|\nabla\phi|^2 = 0. \tag{4.13}$$

It is clear that the maximum principle does not apply to equation (4.12) and the Cole-Hopf transformation does not apply to (4.13); hence the global regularity for (4.12) or (4.13) in two and three dimensions is still an open question.

In 2002, Souplet [42] considered non linear parabolic equations with gradient dependent non linearities, of the form $u_t - \Delta u = F(u, \nabla u)$ (The viscous Hamilton-Jacobi equations). These equations were studied on smoothly bounded domain of $\mathbb{R}^N$, $N \geq 1$ with arbitrary Direchlet boundary data. Under optimal assumption of growth of F with respect to ∇u then gradient blow-up occurs for suitably large initial data; i.e, ∇u blows up in finite time while u remains uniformly bounded. They also considered some equations where the non linearity is non local with respect to ∇u, and they showed that gradient blow-up usually does not occur in this case.

Bellout, et al [5] (2003) treated the hyper-viscous Hamilton-Jacobi-type problem

$$\begin{cases} u_t + \Delta^2 u = |\nabla u|^p & \text{in } \Omega \times \mathbb{R}^+, \\ u = \Delta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4.14}$$

where Ω is a smooth, bounded, open domain in $\mathbb{R}^n$, p a given positive number.
Under suitable conditions on the exponent p, they proved the local existence of weak and strong solutions to (4.14). They also proved the uniqueness of strong solutions and the blow-up in finite time of certain solutions to this family of equations when $p > 2$. Moreover, they showed the global existence of a radial solution in annulus with Neumann boundary conditions.

In the first section, we follow the work of Bellout, et al [5] and prove a general result concerning the local existence of weak solutions for a family of Hamilton-Jacobi equations for fractional Laplacien $(-\Delta)^s$, and the non linearity is of polynomial growth. The uniqueness of such solution is not guaranteed in general. In the second section, we prove the blow up of solutions to certain type of this family of equations.

2. Local existence of weak solutions

In this section, we present one of the main result in this chapter, which asserts the local existence of weak solutions to the problem (4.1), and before that we start by introducing the concept of weak solution.

Definition 2.1. *A weak solution to the problem (4.1) in the interval $(0,T)$ with initial data $u_0 \in L^2(\Omega)$ is a function $u \in L^2((0,T); H^s(\Omega)) \cap L^\infty((0,T); L^2(\Omega))$ for which $\frac{\partial u}{\partial t} \in L^2((0,T); H^{-N-s}(\Omega))$, $F(u, |\nabla u|) \in L^1(\Omega, (0,T))$. The partial differential equation is satisfied in the sense that for any $\phi \in C^\infty(\Omega \times (0,T))$ with compact support in $\Omega \times (0,T)$ the following integral equality holds*

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t} \phi dx d\tau + \int_0^T \int_\Omega ((-\Delta)^{\frac{s}{2}} u)((-\Delta)^{\frac{s}{2}} \phi) dx d\tau = \int_0^T \int_\Omega F(u, |\nabla u|) \phi dx d\tau$$

Theorem 2.1. *Under assumptions (4.2)–(4.4) and for any $u_0 \in L^2(\Omega)$, the problem (4.1) has at least a maximal weak solution.*

The proof of this theorem consists in several steps. We first start by establishing a priori estimates on the solutions.

Lemma 2.1. *Suppose that (4.2), (4.4) hold, and let u be a smooth solution of (4.1). Then, there exists a constant C independent of u such that*

$$\int_\Omega u^2(x,t) dx \le \int_\Omega u_0^2(x) dx + C \int_0^t \left(\int_\Omega u^2(x,\tau) dx \right)^\sigma d\tau, \tag{4.15}$$

where

$$\sigma = 1 + \frac{2s(\beta + \alpha - 1)}{4s - \beta(N+2) - N(\alpha - 1)}.$$

Proof. Multiplying (4.1) by u and integrating by parts we get

$$\frac{1}{2} \int_\Omega u^2(x,t) dx + \int_0^t \|(-\Delta)^{\frac{s}{2}} u(.,\tau)\|_{L^2}^2 d\tau = \frac{1}{2} \int_\Omega u_0^2(x) dx + \int_0^t \int_\Omega F(u, |\nabla u|) u dx d\tau. \tag{4.16}$$

Next, let us get an estimate for

$$\int_0^t \int_\Omega F(u, |\nabla u|) u dx d\tau.$$

By using (4.2) and Hölder's inequality we obtain

$$\int_\Omega F(u,|\nabla u|)u dx \le C\left(\int_\Omega |\nabla u|^{\beta q_1} dx\right)^{\frac{1}{q_1}}\left(\int_\Omega |u|^{(\alpha+1)q_2} dx\right)^{\frac{1}{q_2}},$$

where $\frac{1}{q_1}+\frac{1}{q_2}=1$.
In order to estimate the second term of this inequality, we use the interpolation inequalities and embedding results for Sobolev spaces (see [48]).
We have

$$\begin{aligned} \|\nabla u\|_{L^{\beta q_1}} &\le C\|u\|_{H^{s_1}}, \\ &\le C\|u\|_{L^2}^{1-\theta_1}\|u\|_{H^s}^{\theta_1}, \end{aligned} \tag{4.17}$$

where

$$s_1 = -\frac{N}{\beta q_1}+\frac{N}{2}+1 \quad \text{and} \quad s_1 = s\theta_1 \text{ for some } \theta_1\in(0,1).$$

It follows

$$\theta_1 = -\frac{N}{\beta q_1 s}+\frac{N}{2s}+\frac{1}{s}.$$

In an analogous way, we have

$$\begin{aligned} \|u\|_{L^{(\alpha+1)q_2}} &\le C\|u\|_{H^{s_2}}, \\ &\le C\|u\|_{L^2}^{1-\theta_2}\|u\|_{H^s}^{\theta_2}, \end{aligned} \tag{4.18}$$

with

$$s_2 = -\frac{N}{(\alpha+1)q_2}+\frac{N}{2} \quad \text{and} \quad s_2 = s\theta_2 \text{ for some } \theta_2\in(0,1).$$

It yields that

$$\theta_2 = -\frac{N}{(\alpha+1)q_2 s}+\frac{N}{2s}.$$

From (4.17) and (4.18) we obtain

$$\left|\int_\Omega F(u,|\nabla u|)u dx\right| \le C\|u\|_{L^2}^{\beta(1-\theta_1)+(\alpha+1)(1-\theta_2)}\|u\|_{H^s}^{\beta\theta_1+(\alpha+1)\theta_2}.$$

We would like to have that $\beta\theta_1+(\alpha+1)\theta_2<2$. An elementary calculation shows that this holds whenever (4.4) is satisfied.
By using Young's inequality and since $\|u\|_{H^s}\simeq\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}$ we find

$$\left|\int_\Omega F(u,|\nabla u|)u dx\right| \le C\|u\|_{L^2}^{2\sigma}+\frac{1}{2}\|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^2, \tag{4.19}$$

where

$$\sigma = \frac{\beta(1-\theta_1)+(\alpha+1)(1-\theta_2)}{2}\times\left(1-\frac{\beta\theta_1+(\alpha+1)\theta_2}{2}\right)^{-1}. \tag{4.20}$$

Replacing θ_1 and θ_2 in (4.20) we get

$$\sigma = 1+\frac{2s(\beta+\alpha-1)}{4s-\beta(N+2)-N(\alpha-1)},$$

we observe that $\sigma>1$.
We deduce from (4.16) and (4.19) that

$$\int_\Omega u^2(x,t)dx+\int_0^t\|u(\tau,.)\|_{H^s}^2 d\tau \le \int_\Omega u_0^2(x)dx + C\int_0^t\left(\int_\Omega u^2(x,\tau)dx\right)^\sigma d\tau. \tag{4.21}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let u be a smooth solution to the problem (4.1). Then under the assumptions (4.2) and (4.4), there exist a constant C, independent of u, and a time*

$$T^* = \frac{1}{(\sigma-1)\|u_0\|_{L^2}^{2(\sigma-1)}C},$$

such that for all $t < T^*$,

$$\int_\Omega u^2(x,t)dx \leq \left(\frac{\|u_0\|_{L^2}^{2(\sigma-1)}}{1-(\sigma-1)\|u_0\|_{L^2}^{2(\sigma-1)}Ct}\right)^{\frac{1}{\sigma-1}} < \infty, \tag{4.22}$$

and

$$\int_0^t\int_\Omega ((-\Delta)^{\frac{s}{2}}u)^2dxd\tau \leq \|u_0\|_{L^2}^2 + Ct\left(\frac{\|u_0\|_{L^2}^{2(\sigma-1))}}{1-(\sigma-1)\|u_0\|_{L^2}^{2(\sigma-1)}Ct}\right)^{\frac{\sigma}{\sigma-1}} < \infty. \tag{4.23}$$

Here

$$\sigma - 1 = \frac{2s(\beta+\alpha-1)}{4s-\beta(N+2)-N(\alpha-1)}.$$

Proof. We consider the following problem

$$\begin{cases} \frac{dv}{dt} = C(v(t))^\sigma, \\ v(0) = \displaystyle\int_\Omega u_0^2(x)dx. \end{cases} \tag{4.24}$$

The solution of (4.24) is given by

$$v(t) = \left(\frac{\|u_0\|_{L^2}^{2(\sigma-1)}}{1-(\sigma-1)\|u_0\|_{L^2}^{2(\sigma-1)}Ct}\right)^{\frac{1}{\sigma-1}}.$$

Then from estimate (4.15) of Lemma 2.1 and Gronwall's integral inequality (see [32] p 86) we deduce

$$\int_\Omega u^2(x,t)dx \leq v(t) = \left(\frac{\|u_0\|_{L^2}^{2(\sigma-1)}}{1-(\sigma-1)\|u_0\|_{L^2}^{2(\sigma-1)}Ct}\right)^{\frac{1}{\sigma-1}}. \tag{4.25}$$

The inequality (4.23) follows from the estimates (4.21) and (4.25). □

Now, we will use the Galarkin's method to prove the local existence of solutions for the problem (4.1).

Proof Theorem 2.1. **Step 1** To construct the subspace E_k, we let $w_i, i = 1, 2, 3, ...$ be the eigenfunction of the Laplace operator in $H_0^1(\Omega)$ orthogonalized with respect to the $L^2(\Omega)$ norm, and this set eigenfunction constitutes a basic of $L^2(\Omega)$. We set $E_k = span\{w_1, ..., w_k\}$.

Let us introduce also the projection operator P_k from $L^2(\Omega)$ on E_k defined by

$$P_k v = \sum_{i=1}^k (v, w_i)_{L^2} w_i, \qquad \forall v \in L^2(\Omega). \tag{4.26}$$

From classical results concerning Hilbert spaces, one has that

$$P_k v \longrightarrow v \qquad \text{strongly in } L^2(\Omega),\ \forall v \in L^2(\Omega), \tag{4.27}$$

and furthermore,

$$\|P_k\|_{\mathcal{L}^2(\Omega)} \leq 1. \tag{4.28}$$

Step 2 Since, by assumption $u_0 \in L^2(\Omega)$, if we set $u_k^0 = P_k u_0$ then we have

$$u_k^0 \longrightarrow u_0 \qquad \text{strongly in } L^2(\Omega).$$

Let now introduce, for any $k \in \mathbb{N}^*$, the finite dimensional approximate problem.

$$\begin{cases} \text{Find } u_k = \displaystyle\sum_{i=1}^{k} a_{i,k}(t) w_i(x) \in E_k \text{ such that} \\ \displaystyle\int_\Omega \frac{\partial u_k}{\partial t} w_j dx + \int_\Omega ((-\Delta)^{\frac{s}{2}} u_k)((-\Delta)^{\frac{s}{2}} w_j) dx \\ \qquad\qquad\qquad = \displaystyle\int_\Omega F(u_k, |\nabla u_k|) w_j dx, \qquad \forall j = 1, ..., k \\ u_k(x,0) = u_k^0(x). \end{cases} \tag{4.29}$$

Since $w_1, ..., w_k$ are linearly independent then $a_{i,k}(0) = (u_0, w_i)_{L^2(\Omega)}$.

Consequently, the problem (4.29) is a system of k non-linear ordinary differential equations of the first order with unknowns $a_{1,k},\ a_{2,k}, ..., a_{k,k}$.

This system satisfies the conditions of Picard's theorem. Therefore, it has a unique local solution $a_{i,k}(t),\ i = 1, ..., k$ in some interval about $t = 0$.

Step 3. In this step we will prove that u_k satisfies some a priori estimates.

Let us multiply the jth equation in (4.29) by $a_{j,k}$ and sum over j from 1 to k we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_k^2 dx + \int_\Omega \left((-\Delta)^{\frac{s}{2}} u_k\right)^2 dx = \int_\Omega F(u_k, |\nabla u_k|) u_k dx.$$

By the same steps as in the proof of Lemma 2.1, we deduce that for every k fixed u_k is in $L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^s(\Omega))$ for all

$$T \leq T_k^* = \frac{1}{(\sigma - 1)\|u_k(0,.)\|_{L^2}^{2(\sigma-1)} C}.$$

From (4.28) we have $\|u_k(0,.)\|_{L^2} \leq \|u_0\|_{L^2}$ for all k, it follows that the T_k^* uniformly bounded from below by

$$T^* = \frac{1}{(\sigma - 1)\|u_0\|_{L^2}^{2(\sigma-1)} C}.$$

By the same method as in the proof of (4.22) and (4.23), we can find that for $\tau < T^*$ fixed, u_k is bounded in $L^\infty((0,\tau); L^2(\Omega)) \cap L^2((0,\tau); H^s(\Omega))$ independently of k.

Now we establish an estimate for $\frac{\partial u_k}{\partial t}$.

Let ϕ be a function in $H_0^{s+N}(\Omega)$ and we decompose $\phi = \phi_k + (\phi - \phi_k)$, where ϕ_k is the L^2 projection of ϕ into the space E_k.

By using the orthogonality property of the function w_i, we have

$$\int_\Omega \frac{\partial u_k}{\partial t} \phi dx = \int_\Omega \frac{\partial u_k}{\partial t} \phi_k dx.$$

Since $\phi_k \in E_k$, it follows that

$$\int_\Omega \frac{\partial u_k}{\partial t} \phi dx = -\int_\Omega ((-\Delta)^{\frac{s}{2}} u_k)((-\Delta)^{\frac{s}{2}} \phi_k) dx + \int_\Omega F(u_k, |\nabla u_k|) \phi_k dx.$$

Let us now move to estimate the last term in the equality above.

From (4.2) and by using Hölder's inequality we get

$$\left| \int_\Omega F(u_k, |\nabla u_k|) \phi_k dx \right| \ \leq\ C \|\phi_k\|_{L^\infty} \|\nabla u_k\|_{L^{2\beta q_1}}^{\beta} \|u_k\|_{L^{2\alpha q_2}}^{\alpha},$$

with $\frac{1}{q_1}+\frac{1}{q_2}=1$.
Since the embedding of $H^{N+s}(\Omega)$ into $L^\infty(\Omega)$ is satisfied for any N then we obtain

$$\left|\int_\Omega F(u_k,|\nabla u_k|)\phi_k dx\right| \le C\|\nabla u_k\|^\beta_{L^{2\beta q_1}}\|u_k\|^\alpha_{L^{2\alpha q_2}}\|\phi_k\|_{H^{N+s}}.$$

Now, by using the interpolation inequalities we find

$$\|\nabla u_k\|_{L^{2\beta q_1}} \le C\|u_k\|^{1-\theta_1}_{L^2}\|u_k\|^{\theta_1}_{H^s}, \tag{4.30}$$

with

$$\theta_1 = -\frac{N}{2\beta q_1 s}+\frac{N}{2s}+\frac{1}{s},$$

and

$$\|u_k\|_{L^{2\alpha q_2}} \le C\|u_k\|^{1-\theta_2}_{L^2}\|u_k\|^{\theta_2}_{H^s}, \tag{4.31}$$

with

$$\theta_2 = -\frac{N}{2\alpha q_2 s}+\frac{N}{2s}.$$

Thanks to (4.4) and

$$\begin{cases} \frac{N}{\beta(2+N)} \le q_1 & \text{for} \quad N \le 2(s-1), \\ \frac{N}{\beta(N+2)} \le q_1 \le \frac{N}{\beta(N-2(s-1))} & \text{for} \quad N > 2(s-1), \end{cases}$$

and

$$\begin{cases} \frac{1}{\alpha} \le q_2 & \text{for} \quad N \le 2s, \\ \frac{1}{\alpha} \le q_2 \le \frac{N}{\alpha(N-2s)} & \text{for} \quad N > 2s, \end{cases}$$

we have $\theta_1 \in (0,1)$ and $\theta_2 \in (0,1)$.
We deduce from (4.30) and (4.31) that

$$\left|\int_\Omega F(u_k,|\nabla u_k|)\phi_k dx\right| \le C\|u_k\|^{\beta(1-\theta_1)+\alpha(1-\theta_2)}_{L^2}\|u_k\|^{\beta\theta_1+\alpha\theta_2}_{H^s}\|\phi_k\|_{H^{n+s}}. \tag{4.32}$$

An elementary calculation and since $\beta < \frac{N+8+4(s-2)-N\alpha}{N+2}$ we obtain that

$$\beta\theta_1+\alpha\theta_2 < 2.$$

Moreover, since $u_k \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^s(\Omega))$ then by (4.32) we find

$$\left|\int_\Omega F(u_k,|\nabla u_k|)\phi_k dx\right| \le CG(t)\|\phi_k\|_{H^{N+s}}, \tag{4.33}$$

where G is independent of k. Note that G is in $L^\gamma(0,t)$ with

$$\gamma = \frac{2}{\beta\theta_1+\alpha\theta_2}.$$

By applying Hölder's inequality we obtain

$$\int_\Omega \left((-\Delta)^{\frac{s}{2}}u_k\right)\left((-\Delta)^{\frac{s}{2}}\phi_k\right)dx \le C\|u_k\|_{H^s}\|\phi_k\|_{H^{s+N}}. \tag{4.34}$$

It is well known (see [31]) that thanks to the special choice of the sequence w_i, we have

$$\|\phi_k\|_{H^{N+s}} \le C\|\phi\|_{H^{N+s}}.$$

In light of inequalities (4.33) and (4.34) we reach that $\frac{\partial u_k}{\partial t}$ is uniformly bounded in $L^{\gamma_1}((0,t);H^{-s-n}(\Omega))$, where $\gamma_1 = \min(\gamma,2)$ and $H^{-N-s}(\Omega) = \left(H_0^{N+s}(\Omega)\right)'$.

Step 4. From the weak compactness of the sequence u_k, we can extract a subsequence (still denote by k) and a function u such that for any $t < T^*$

$$u_k \longrightarrow u \quad \text{as } k \to +\infty \qquad \text{in } L^\infty_{weak*}\left((0,t); L^2_{weak}(\Omega)\right), \tag{4.35}$$

$$u_k \longrightarrow u \quad \text{as } k \to +\infty \qquad \text{in } L^2_{weak}((0,t); H^s_{weak}(\Omega)). \tag{4.36}$$

Furthermore by using Aubin's lemma (see [31]) we have that u_k is compact in the strong topology of $L^2((0,t); H^{s-\varepsilon}(\Omega))$ for some $\varepsilon > 0$. Therefore for any

$$\begin{cases} q < \frac{2N}{N-2(s-1)} & \text{when } N > 2(s-1), \\ q < \infty & \text{when } N \leq 2(s-1). \end{cases}$$

there is a subsequence of u_k, still denote u_k, which convergences to u strongly in $L^2((0,t); W^{1,q}(\Omega))$. With the strong convergence of u_k, we are ready to prove that $\forall \psi \in C^\infty(\Omega \times \mathbb{R}^+)$

$$\int_0^t \int_\Omega F(u_k, |\nabla u_k|)\psi dx d\tau \longrightarrow \int_0^t \int_\Omega F(u, |\nabla u|)\psi dx d\tau \qquad \text{as } k \to +\infty.$$

Let $\psi \in C^\infty(\Omega \times \mathbb{R}^+)$, then

$$\begin{aligned} \left| \int_\Omega (F(u_k, |\nabla u_k|) - F(u, |\nabla u|))\psi dx \right| &\leq C\|\psi\|_{L^\infty} \int_\Omega |F(u_k, |\nabla u_k|) - F(u_k, |\nabla u|)| \, dx \\ &+ C\|\psi\|_\infty \int_\Omega |F(u_k, |\nabla u|) - F(u, |\nabla u|)| \, dx, \\ &\leq C\|\psi\|_{L^\infty} \int_\Omega |\partial_2 F(u_k, c_2)||\nabla u_k - \nabla u| dx \\ &+ C\|\psi\|_{L^\infty} \int_\Omega |\partial_1 F(c_1, |\nabla u|)||u_k - u| dx, \end{aligned}$$

with

$$\begin{aligned} c_1 &= \delta_1 u_k - (1-\delta_1)u, \\ c_2 &= \delta_2 |\nabla u_k| - (1-\delta_2)|\nabla u|, \end{aligned}$$

where $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$.
By using the condition (4.3) and Hölder's inequality with $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q} = 1$ we get

$$\begin{aligned} \left| \int_\Omega \left(F(u_k, |\nabla u_k|) - F(u, |\nabla u|)\right) \psi dx \right| &\leq C\|\psi\|_{L^\infty} \Big(\|u_k\|^\alpha_{L^{\alpha q_1}} \|\nabla u_k\|^{\beta-1}_{L^{(\beta-1)q_2}} \\ &+ \|u_k\|^\alpha_{L^{\alpha q_1}} \|\nabla u\|^{\beta-1}_{L^{(\beta-1)q_2}} + \|u\|^{\alpha-1}_{L^{(\alpha-1)q_1}} \|\nabla u\|^\beta_{L^{\beta q_2}} \\ &+ \|u_k\|^{\alpha-1}_{L^{(\alpha-1)q_1}} \|\nabla u\|^\beta_{L^{\beta q_2}} \Big) \|u_k - u\|_{W^{1,q}}. \end{aligned} \tag{4.37}$$

Now, applying the interpolation inequalities [48] we have

$$\|u_k\|_{L^{\alpha q_1}} \leq C\|u_k\|^{1-\theta_1}_{L^2} \|u_k\|^{\theta_1}_{H^s}, \tag{4.38}$$

$$\|u_k\|_{L^{(\alpha-1)q_1}} \leq C\|u_k\|^{1-\theta_2}_{L^2} \|u_k\|^{\theta_2}_{H^s}, \tag{4.39}$$

with

$$\theta_1 = -\frac{N}{\alpha q_1 s} + \frac{N}{2s}, \qquad \theta_2 = -\frac{N}{(\alpha-1)q_1 s} + \frac{N}{2s},$$

and

$$\|\nabla u_k\|_{L^{(\beta-1)q_2}} \leq C\|u_k\|^{1-\theta_3}_{L^2} \|u_k\|^{\theta_3}_{H^s}, \tag{4.40}$$

with
$$\theta_3 = -\frac{N}{(\beta-1)q_2 s} + \frac{N}{2s} + \frac{1}{s}.$$
Combining (4.37) with (4.38)–(4.40) we obtain
$$\begin{aligned}\left|\int_\Omega \left(F(u_k,|\nabla u_k|) - F(u,|\nabla u|)\right)\psi dx\right| \;&\leq\; C\|\psi\|_{L^\infty}\Big(\|u\|^{\alpha-1}_{L^{(\alpha-1)q_1}}\|\nabla u\|^{\beta}_{L^{\beta q_2}} \\ &+\|u_k\|_{L^2}^{\alpha(1-\theta_1)+(\beta-1)(1-\theta_3)}\|u_k\|_{H^s}^{\alpha\theta_1+(\beta-1)\theta_3} + \|u_k\|_{L^2}^{\alpha(1-\theta_1)}\|u_k\|_{H^s}^{\alpha\theta_1}\|\nabla u\|_{L^{(\beta-1)q_2}} \\ &+\|u_k\|_{L^2}^{(\alpha-1)(1-\theta_2)}\|u_k\|_{H^s}^{(\alpha-1)\theta_2}\|\nabla u\|^{\beta}_{L^{\beta q_2}}\Big)\|u_k-u\|_{W^{1,q}}.\end{aligned} \tag{4.41}$$
We choose $q_1>1$ and $q_2>1$ such that
$$\begin{cases} \frac{2}{\alpha}\leq q_1 & \text{for } N\leq 2s,\\ \frac{2}{\alpha}\leq q_1\leq \frac{2N}{(\alpha-1)(N-2s)} & \text{for } N>2s,\end{cases}$$
and
$$\begin{cases} \frac{2N}{(\beta-1)(N+2)}\leq q_2 & \text{for } N\leq 2(s-1),\\ \frac{2N}{(\beta-1)(2+N)}\leq q_2\leq \frac{2N}{(\beta-1)(N-2(s-1))} & \text{for } N>2(s-1).\end{cases}$$
we have $\theta_1\in(0,1)$ and $\theta_2\in(0,1)$.

We can check $\alpha\theta_1+(\beta-1)\theta_2<1$, this is always possible to choose
$$\frac{1}{q_1}+\frac{1}{q_2} > \frac{\alpha N+(\beta-1)(N+2)-2s}{2N}.$$

Passing to the limit in (4.41), as $k\to\infty$ we get for any $\psi\in C^\infty(\Omega\times\mathbb{R}^+)$
$$\left|\int_0^t\int_\Omega \left(F(u_k,|\nabla u_k|)-F(u,|\nabla u|)\right)\psi dx d\tau\right| \longrightarrow 0 \qquad \text{as } k\to\infty,$$
where we made use of the fact that u_k is bounded in $L^\infty((0,t);L^2(\Omega))\cap L^2((0,t);H^s(\Omega))$.
We then deduce that the limit u of the sequence u_k satisfies
$$\int_0^t\int_\Omega \frac{\partial u}{\partial t}\psi dx d\tau + \int_0^t\int_\Omega \left((-\Delta)^{\frac{s}{2}}u\right)\left((-\Delta)^{\frac{s}{2}}\psi\right)dx d\tau = \int_0^t\int_\Omega F(u,|\nabla u|)\psi dx d\tau,$$
for all $\psi\in C^\infty((0,t)\times\Omega)$. □

3. Blow up results

In this section, we want to derive a blow up result by using the technique of Kaplan introduced in [24] for the following problem
$$\begin{cases} u_t + (-\Delta)^s u = |\nabla u|^\beta & \text{in } \Omega\times\mathbb{R}^+,\\ u(x,0)=u_0(x) & \text{on } \Omega,\end{cases} \tag{4.42}$$
with the periodic boundary conditions, where $\Omega=(0,L)^N$ and $L>0$, $s\geq 2$.

We start by introducing some notations and recalling some well-know results.
It is well known (see [24]) that under the assumptions we made on Ω, the eigenvalue problem
$$-\Delta\psi=\lambda\psi \qquad \psi\in H_0^1(\Omega) \tag{4.43}$$

has a smallest positive eigenvalue $\lambda = \lambda_1$ and that the associated eigenfunction ϕ does not vanish in Ω. Notice that $\phi \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$. Therefore, we can choose a ϕ such that $\phi > 0$ in Ω and $\int_\Omega \phi dx = 1$. Furthermore, it can be proved (see [3], [4], [42]) that

$$\int_\Omega \phi^{-\alpha} dx = C(\alpha, \Omega) < \infty \qquad \forall \alpha \in (0,1). \tag{4.44}$$

Theorem 3.1. *Let $u_0 \in L^2(\Omega)$ satisfy $\int_\Omega u_0 \phi(x) dx > M = M(\Omega, \beta) > 0$ sufficiently large. Assume*

$$2 < \beta < \frac{4s+N}{N+2}, \qquad \textit{and} \qquad N < 4(s-1)$$

then, the problem (4.42) cannot admit a globally defined weak solution, Indeed, there exists $T^\sharp = T^\sharp(M) < \infty$ such that u satisfies

$$\lim_{t\to T^\sharp} \|u(.,t)\|_{L^2} = \infty \qquad \textit{and} \qquad \lim_{t \to T^\sharp} \|u(.,t)\|_\infty = \infty. \tag{4.45}$$

Proof. Multiplying the equation (4.42) by ϕ and integrating over Ω we obtain

$$\int_\Omega u_t \phi dx + \int_\Omega ((-\Delta)^s u)\phi dx = \int_\Omega |\nabla u|^\beta \phi dx.$$

On the other hand we have

$$\int_\Omega ((-\Delta)^s u)\phi dx = \lambda_1^s \int_\Omega u\phi dx.$$

Therefore, setting $z(t) = \int_\Omega u\phi dx$, we get

$$z'(t) + \lambda_1^s z(t) = \int_\Omega |\nabla u|^\beta \phi dx. \tag{4.46}$$

We will prove that

$$\int_\Omega |\nabla u|^\beta \phi dx \geq C|z(t)|^\beta - C',$$

where C, C' are positive constants.
We have

$$\begin{aligned} |z(t)|^\beta &\leq \|\phi\|^\beta_{L^\infty} \left(\int_\Omega |u| dx \right)^\beta, \\ &\leq C \left(\int_\Omega |u - ⨍ u| dx \right)^\beta + C \left(|\Omega| ⨍ u \right)^\beta, \end{aligned} \tag{4.47}$$

by using Poincaré-Wirtinger's inequality and Hölder's inequality we obtain

$$\left(\int_\Omega |u - ⨍ u| dx \right)^\beta \leq C \left(\int_\Omega |\nabla u|^\beta \phi dx \right) \left(\int_\Omega \phi^{-\frac{\beta'}{\beta}} dx \right)^{\frac{\beta}{\beta'}}, \tag{4.48}$$

where β' is the conjugate of β.
Combining (4.46) with (4.47) and (4.48) we have

$$z'(t) + \lambda_1^s z(t) \geq C|z(t)|^\beta - C'.$$

Then we can choose $\int_\Omega u_0(x)\phi(x) dx > M(\Omega, \beta) > 0$ sufficiently large, such that the problem (4.42) cannot admit a globally defined weak solution. Indeed, there exists $T^\sharp = T^\sharp(M) > 0$ such that either u ceases to exist before $T^\sharp$, or

$$\lim_{t \to T^\sharp} \int_\Omega u(x,t)\phi(x) dx = +\infty.$$

This completes the proof of Theorem 3.1. □

Remark 3.1. *The results of the present chapter are motivated by the work of Bellout, et al* [5], *where they studied the case* $\alpha = 0$, $\beta = p$ *and* $s = 2$. *Our work extends their results in several directions.*

CHAPTER 5

Local existence and uniqueness of mild and strong solutions for some fractional Hamilton-Jacobi-type equation

1. Introduction

It is well known that, the global existence of solutions for the Kuramoto-Sivashinsky equations in two-dimensional, or higher is one of the major open questions in non linearity analysis. Inspired by this question, we introduce a family of fractional Hamilton-Jacobi-type equations perturbed by the fractional s Laplacien, and the non linearity is of polynomial growth. Under suitable conditions on the exponents β of the non linear term, we first study the short-time existence and uniqueness of mild solutions for the same family of hyper-viscous Hamilton-Jacobi equations which been studied by Bellout, et all [5]. Then we establish the local existence and uniqueness of strong solution for considered problem.

2. Local existence of mild solutions and uniqueness

In this section, we prove the local existence and uniqueness of mild solutions to the following hyper-viscous Hamilton-Jacobi equation

$$\begin{cases} u_t + \Delta^2 u = |\nabla u|^\beta & \text{in } \Omega \times \mathbb{R}^+, \\ u = \Delta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 & \text{in } \Omega. \end{cases} \tag{5.1}$$

Here Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and β is a positive number.

Next, we introduce the concept of mild solution on Hilbert space.

Definition 2.1. *Let H be a Hilbert space, A be a maximal monotone operator on H. Suppose $f : [t_0, T] \times D(A^{1-\rho}) \longrightarrow H$ be continuous in t on $[t_0, T]$ and uniformly Lipschitz continuous on H such that $0 < \rho < 1$. Let $u_0 \in H$, the function*

$$u(t) = S(t - t_0)u_0 + \int_{t_0}^{t} S(t-s) f(s, u(s)) ds,$$

uniquely defined, is called a mild solution for the following problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)) & t > t_0, \\ u(t_0) = u_0. \end{cases}$$

Now, the main result in this section is as follows.

Theorem 2.1. *Given $u_0 \in L^2(\Omega)$ and*

$$\beta \geq 1 \qquad \text{for } N \leq 6 \qquad \text{and} \qquad 1 \leq \beta < \frac{N}{N-6} \qquad \text{for } N \geq 7, \tag{5.2}$$

there exist a maximal time $T_{max} > 0$ and a unique mild solution u to the problem (5.1).

Proof. We apply the semi-group theory in $H = L^2(\Omega)$ (see [6], [50], [52]). Consider the unbounded operator $A : D(A) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$\begin{cases} D(A) = \left\{u \in H^4(\Omega) \cap H_0^1(\Omega),\ \Delta u = 0, \text{ on } \partial\Omega \times \mathbb{R}^+\right\}, \\ \\ Au = \Delta^2 u. \end{cases}$$

We already know that A is maximal monotone in $L^2(\Omega)$.

Next, we will show that the non linearity is a locally Lipschitz continuous mapping from $D(A^{1-\rho})$ into $L^2(\Omega)$ for some ρ, $0 < \rho < 1$.
For this, we set $F(u) = |\nabla u|^p$, and let $u \in D(A^{1-\rho})$ then from embedding results for Sobolev spaces (see [48]), we have

$$\|\nabla u\|_{L^q} \leq C\|u\|_{H^s},$$

with

$$\frac{1}{q} = \frac{N - 2(s-1)}{2N} \qquad \text{and} \qquad s = 4(1-\rho).$$

Since $0 < s < 4$ then $F \in L^2(\Omega)$, such that

$$\beta \geq 1 \qquad \text{for } N \leq 6 \qquad \text{and} \qquad 1 \leq \beta < \frac{N}{N-6} \qquad \text{for } N \geq 7. \tag{5.3}$$

On the other hand, let $(u, v) \in (D(A^{1-\rho}))^2$ then we have

$$\|F(u) - F(v)\|_{L^2} \leq C\|\nabla(u-v)\|_{L^{2\beta}}\||\nabla u| + |\nabla v|\|_{L^{2\beta}}^{\beta-1}. \tag{5.4}$$

We will estimate the second term in the inequality (5.4), we obtain

$$\|\nabla(u-v)\|_{L^q} \leq C\|u-v\|_{H^s}, \tag{5.5}$$

with

$$s = 4(1-\rho) \qquad \text{and} \qquad \frac{1}{q} = \frac{N-2(s-1)}{2N}. \tag{5.6}$$

From (5.3)–(5.6) we deduce that F is locally Lipschitz in $L^2(\Omega)$. □

3. Local existence of strong solutions and uniqueness

In this section, we treat the local existence of strong solutions and uniqueness for the same problem in a previous chapter

$$\begin{cases} u_t + (-\Delta)^s u = F(u, |\nabla u|) & \text{in } \Omega \times \mathbb{R}^+, \\ \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{5.7}$$

with periodic boundary conditions, where $\Omega = (0, L)^N$, $L > 0$, $s \geq 2$, and $|\nabla u| = (\nabla u, \nabla u)^{\frac{1}{2}}$.

Assume that, there exist positive constants C_1, C_2 and C_3 independent of u, such that

$$|F(u, |\nabla u|)| \leq C_1|u|^\alpha|\nabla u|^\beta, \tag{5.8}$$

$$\begin{cases} |\partial_1 F(u, |\nabla u|)| \leq C_2|u|^{\alpha-1}|\nabla u|^\beta, \\ \\ |\partial_2 F(u, |\nabla u|)| \leq C_3|u|^\alpha|\nabla u|^{\beta-1}, \end{cases} \tag{5.9}$$

where α and β given positive numbers, such that

$$\begin{cases} \beta \geq 1, \ \alpha \geq 1, \ \text{and} \\ s > \frac{\beta(N+2)+N(\alpha-1)}{4}. \end{cases} \tag{5.10}$$

We first give the following definition.

Definition 3.1. *A strong solution in $\Omega \times (0,T)$ to problem (5.7) with $u_0 \in L^2(\Omega)$ is a weak solution that also satisfies $\frac{\partial u}{\partial t} \in L^2((0,T);(H^s(\Omega))')$.*

Theorem 3.1. *Assume (5.8)–(5.9) are satisfied.*

1. *Let $u_0 \in L^2(\Omega)$.*
   i) *If*

$$\beta \geq 1, \ \alpha \geq 1 \qquad \text{and} \tag{5.11}$$

$$\begin{cases} s \geq \frac{\beta(N+2)+2N(\alpha-1)}{2} & \text{for } N < 2s, \\ s > \frac{\beta(N+2)+N(\alpha-1)}{4} & \text{for } N \geq 2s, \end{cases}$$

then every weak solution to the problem (5.7) is a strong solution.

   ii) *Assume (5.11) is satisfied.*
   If

$$s \geq \frac{\beta(N+2)+2N(\alpha-1)}{2} \quad \text{for } N < 2(s-1),$$

and

$$\begin{cases} s > \frac{\beta(N+2)+N(\alpha-1)}{2(\beta+\alpha)}, & \\ & \text{for } N \geq 2s \\ \text{and } \alpha + s \leq 2, & \end{cases}$$

then the problem (5.7) has a unique strong solution.

2. *If $u_0 \in H^s(\Omega)$ and (5.10) is satisfied, then every weak solution of (4.1) is a strong solution. Furthermore, in this case $u \in L^\infty((0,T);H^s(\Omega))$.*
3. *For any $u_0 \in L^2(\Omega)$, we assume (5.10) is satisfied, every weak solution of (5.7) instantaneously becomes a strong solution. That is for any $\tau > 0$, we have $\frac{\partial u}{\partial t} \in L^2((\tau,T);(H^s(\Omega))')$.*

Proof of Theorem 3.1. 1) Let $u_0 \in L^2(\Omega)$.
i) Since u is a weak solution, then u in $L^2((0,T);H^s(\Omega))$.

Let $v \in L^2((0,T);H^s(\Omega))$ then by applying Hölder's inequality we get

$$\int_\Omega ((-\Delta)^s u)\, v dx \leq C\|u\|_{H^s}\|v\|_{H^s}.$$

It follows that $(-\Delta)^s u \in L^2((0,T);(H^s(\Omega))')$.
Now, we will show that $F(u,|\nabla u|) \in L^2((0,T);(H^s(\Omega))')$, and for this we have to distinguish three cases.
Case 1. For $N > 2s$ we have $H^s(\Omega) \hookrightarrow L^\gamma(\Omega)$ for $1 \leq \gamma < \frac{2N}{N-2s}$.
Now, let $v \in L^2((0,T);H^s(\Omega))$ then by using (5.8) and Hölder's inequality we get

$$\int_\Omega F(u,|\nabla u|) v dx \leq C\|\nabla u\|^\beta_{L^{\beta r_1}}\|u\|^\alpha_{L^{\alpha r_2}}\|v\|_{L^{r_3}}, \tag{5.12}$$

where $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=1$.
By interpolation inequalities we get

$$\|\nabla u\|_{L^{\beta r_1}} \leq C\|u\|_{L^2}^{1-\theta_1}\|u\|_{H^s}^{\theta_1}, \tag{5.13}$$

with

$$\theta_1 = -\frac{N}{\beta r_1 s}+\frac{N}{2s}+\frac{1}{s} \qquad \text{such that } \theta_1 \in (0,1),$$

and

$$\|u\|_{L^{\alpha r_1}} \leq C\|u\|_{L^2}^{1-\theta_2}\|u\|_{H^s}^{\theta_2}, \tag{5.14}$$

where

$$\theta_2 = -\frac{N}{\alpha r_2 s}+\frac{N}{2s} \qquad \text{such that } \theta_2 \in (0,1).$$

By combining (5.12) with (5.13) and (5.14) we obtain

$$\int_\Omega F(u,|\nabla u|)v dx \leq C\|u\|_{L^2}^{\beta(1-\theta_1)+\alpha(1-\theta_2)}\|u\|_{H^s}^{\beta\theta_1+\alpha\theta_2}\|v\|_{H^s}.$$

where we made use the fact that $H^s(\Omega)$ is included in $L^{r_3}(\Omega)$ when $1 \leq r_3 < \frac{2N}{N-2s}$.
Now, setting

$$\frac{1}{r_1}+\frac{1}{r_2}=\beta\left(\frac{N+2}{2N}\right)+\frac{\alpha}{2}-\frac{s}{N},$$

we get $\frac{1}{r_1}+\frac{1}{r_2}<\frac{N+2s}{2N}$ and $\beta\theta_1+\alpha\theta_2=1$.

Since $u \in L^\infty((0,T),L^2(\Omega))\cap L^2((0,T);H^s(\Omega))$ then for $n>2s$ we deduce that $F(u,|\nabla u|) \in L^2((0,T);(H^s(\Omega$
Case 2. For $N<2s$ we have $H^s(\Omega)\hookrightarrow C(\bar{\Omega})$.
Let $v \in L^2((0,T);H^s(\Omega))$. In the light of condition (5.8) and by applying Hölder's inequality we get

$$\int_\Omega F(u,|\nabla u|)v dx \ \leq \ C\|v\|_{H^s}\|\nabla u\|_{L^{\beta r_1}}^{\beta}\|u\|_{L^{\alpha r_2}}^{\alpha},$$

where $\frac{1}{r_1}+\frac{1}{r_2}=1$.
Now, by interpolation inequalities we get

$$\int_\Omega F(u,|\nabla u|)v dx \leq C\|u\|_{L^2}^{\beta(1-\theta_1)+\alpha(1-\theta_2)}\|u\|_{H^s}^{\beta\theta_1+\alpha\theta_2}\|v\|_{H^s},$$

with

$$\theta_1 = -\frac{N}{r_1\beta s}+\frac{N}{2s}+\frac{1}{s} \qquad \text{such that } \theta_1 \in (0,1),$$

and

$$\theta_2 = -\frac{N}{r_2\alpha s}+\frac{N}{2s} \qquad \text{such that } \theta_1 \in (0,1).$$

Since $\beta(N+2)+N(\alpha-1)\leq 2s$ then we have$\beta\theta_1+\alpha\theta_2 \leq 1$.
Then we deduce $F(u,|\nabla u|)\in L^2((0,T);(H^s(\Omega))')$ for $N<2s$.

Case 3. For $N=2s$ we have $H^s(\Omega)\hookrightarrow L^\gamma(\Omega)$ with $\gamma\in[1,+\infty[$.
By the same steps as in the first case we obtain $F(u,|\nabla u|)\in L^2((0,T);(H^s(\Omega))')$ for $N=2s$.

ii) Next, we will prove uniqueness of solution.

Let u_1, u_2 be two strong solutions corresponding to the same initial data u_0.
Taking the difference of equations satisfied by each function we get that

$$\frac{\partial w}{\partial t} + (-\Delta)^s w = F(u_1, |\nabla u_1|) - F(u_2, |\nabla u_2|), \tag{5.15}$$

where $w = u_1 - u_2$.
Since the function w is in $L^2((0,T); H^s(\Omega))$, a well-know lemma from Lions-Magenes [31] implies that the function $\|w(t)\|_{L^2}$ is absolutely continuous and that $\frac{d}{dt}\|w\|_{L^2}^2 = 2\langle \frac{\partial w}{\partial t}, w\rangle_{E'}$.
Therefore, multiplying (5.15) by w, it follows from integrating by part

$$\begin{aligned}
\frac{1}{2}\int_\Omega w^2(x,t)dx + \int_0^t\int_\Omega \left((-\Delta)^{\frac{s}{2}}w\right)^2 dxd\tau \ &\leq \ \int_0^t\int_\Omega |F(u_1,|\nabla u_1|) - F(u_1,|\nabla u_2|)||w|dxd\tau \\
&\quad + \int_0^t\int_\Omega |F(u_1,|\nabla u_2|) - F(u_2,|\nabla u_2|)||w|dxd\tau, \\
&\leq \ \int_0^t\int_\Omega |\partial_2 F(u_1,c_2)||\nabla w||w|dxd\tau \\
&\quad + \int_0^t\int_\Omega |\partial_1 F(c_1,|\nabla u_2|)||w|^2 dxd\tau.
\end{aligned}$$

Moreover, from (5.9) we infer that

$$\begin{aligned}
\frac{1}{2}\int_\Omega w^2(x,t)dx + \int_0^t\int_\Omega \left((-\Delta)^{\frac{s}{2}}w\right)^2 dxd\tau \ &\leq \ C\int_0^t\int_\Omega |u_1|^\alpha |c_2|^{\beta-1}|\nabla w||w|dxd\tau \\
&\quad + C\int_0^t\int_\Omega |c_1|^{\alpha-1}|\nabla u_2|^\beta |w|^2 dxd\tau,
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= \xi_1 u_1 + (1-\xi_1)u_2, \\
c_2 &= \xi_2|\nabla u_1| + (1-\xi_2)|\nabla u_2|,
\end{aligned}$$

such that $0 < \xi_1 < 1$ and $0 < \xi_2 < 1$.

Now, we need to estimate the right-hand side of the last inequality, for this we apply Hölder's inequality we get

$$\int_\Omega |u_1|^\alpha |c_2|^{\beta-1}|\nabla w||w|dx \leq C\|u_1\|_{L^{\alpha r_1}}^\alpha \|c_2\|_{L^{(\beta-1)r_2}}^{\beta-1}\|\nabla w\|_{L^{r_3}}\|w\|_{L^2}, \tag{5.16}$$

where $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{2}$.
Using interpolation inequalities and embedding results for Sobolev spaces, we have

$$\|\nabla w\|_{L^{r_3}} \leq C\|w\|_{L^2}^{1-\theta_3}\|w\|_{H^s}^{\theta_3}, \tag{5.17}$$

with

$$\theta_3 = -\frac{N}{r_3 s} + \frac{N}{2s} + \frac{1}{s}.$$

The constraint $\theta_3 \in (0,1)$ is satisfied whenever we choose r_3 such that for $n > 2(s-1)$

$$\frac{1}{r_3} \geq \frac{N-2(s-1)}{2N} \Longleftrightarrow \frac{1}{r_1} + \frac{1}{r_2} \leq \frac{2(s-1)}{2N}. \tag{5.18}$$

By combining (5.16) with (5.17) and by using Young's inequality we obtain

$$\begin{aligned}
\int_\Omega |u_1|^\alpha |c_2|^{\beta-1}|\nabla w||w|dx \ &\leq \ C\|u_1\|_{L^{\alpha r_1}}^{q\alpha}\left(\|\nabla u_1\|_{L^{(\beta-1)r_2}}^{q(\beta-1)} + \|\nabla u_2\|_{L^{(\beta-1)r_2}}^{q(\beta-1)}\right)\|w\|_{L^2}^2 \\
&\quad + \frac{1}{2}\|w\|_{H^s}^2,
\end{aligned} \tag{5.19}$$

such that $q = \frac{2}{2-\theta_3}$.
Applying again the interpolation inequalities, we get

$$\|u_1\|_{L^{\alpha r_1}} \leq C\|u\|_{L^2}^{1-\theta_1}\|u_1\|_{H^s}^{\theta_1}, \tag{5.20}$$

with

$$\theta_1 = -\frac{N}{\alpha r_1 s} + \frac{N}{2s},$$

and

$$\|\nabla u_1\|_{L^{(\beta-1)r_2}} \leq C\|u_1\|_{L^2}^{1-\theta_2}\|u_1\|_{H^s}^{\theta_2}, \tag{5.21}$$

$$\|\nabla u_2\|_{L^{(\beta-1)r_2}} \leq C\|u_2\|_{L^2}^{1-\theta_2}\|u_2\|_{H^s}^{\theta_2}, \tag{5.22}$$

where

$$\theta_2 = -\frac{N}{(\beta-1)r_2 s} + \frac{N}{2s} + \frac{1}{s}.$$

In order to have $\theta_1 \in (0,1)$ and $\theta_2 \in (0,1)$ we will require that

$$\alpha\left(\frac{1}{2} - \frac{s}{N}\right) \leq \frac{1}{r_1} \leq \frac{\alpha}{2}, \tag{5.23}$$

$$(\beta-1)\left(\frac{1}{2} + \frac{1}{N} - \frac{s}{N}\right) \leq \frac{1}{r_2} \leq (\beta-1)\left(\frac{1}{2} + \frac{1}{N}\right). \tag{5.24}$$

Combining (5.19) with (5.20)–(5.22) we obtain

$$\begin{aligned}\int_\Omega |u_1|^\alpha|c_2|^{\beta-1}|\nabla w||w|dx \leq C\Big(&\|u_1\|_{L^2}^{q\alpha(1-\theta_1)}\|u_1\|_{H^s}^{q\alpha\theta_1}\|u_2\|_{L^2}^{q(\beta-1)(1-\theta_2)}\|u_2\|_{H^s}^{q(\beta-1)\theta_2}\\ &+\|u_1\|_{L^2}^{\delta_1}\|u_1\|_{H^s}^{\delta_2}\Big)\|w\|_{L^2}^2 + \frac{1}{2}\|w\|_{H^s}^2,\end{aligned}$$

where

$$\begin{aligned}\delta_1 &= q\left[\alpha(1-\theta_1) + (\beta-1)(1-\theta_2)\right],\\ \delta_2 &= q\left[\alpha\theta_1 + (\beta-1)\theta_2\right].\end{aligned}$$

we notice that $\delta_2 < 2$ whenever $\beta < \frac{N+8+4(s-2)-N\alpha}{N+2}$.
From (5.18), (5.23) and (5.24) we get

$$\beta(N+2) + 2N(\alpha-1) < 2s \qquad \text{for } N < 2(s-1),$$

and

$$\left\{\begin{array}{l}(\beta-1)(N-2(s-1)) \leq 2(s-1) - \alpha(N-2s),\\ \\ \alpha + s \leq 2.\end{array}\right. \qquad \text{for } N > 2s$$

Now, we estimate the second term of (5.15) by using Hölder's inequality we find

$$\int_\Omega |c_1|^{\alpha-1}|\nabla u_2|^\beta|w|^2dx \leq C\|c_1\|_{L^{(\alpha-1)q_1}}^{\alpha-1}\|\nabla u_2\|_{L^{\beta q_2}}^{\beta}\|w\|_{L^{2q_3}}^2,$$

where $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$.
From interpolation inequalities we get

$$\|w\|_{L^{2q_3}} \leq C\|w\|_{L^2}^{1-\gamma_3}\|w\|_{H^s}^{\gamma_3},$$

with

$$\gamma_3 = -\frac{N}{2q_3 s} + \frac{N}{2s}, \qquad \text{for some } \gamma_3 \in (0,1).$$

By using Young's inequality with $p = \frac{1}{2-\gamma_3}$ we obtain

$$\begin{aligned}\int_\Omega |c_1|^{\alpha-1}|\nabla u_2|^\beta |w|^2 dx &\leq C\left(\|u_1\|^{p(\alpha-1)}_{L^{(\alpha-1)q_1}} + \|u_2\|^{p(\alpha-1)}_{L^{(\alpha-1)q_1}}\right)\|\nabla u_2\|^{p\beta}_{L^{\beta q_2}}\|w\|^2_{L^2} \\ &+ \frac{1}{2}\|w\|^2_{H^s}. \end{aligned} \tag{5.25}$$

Applying again the interpolation inequalities, we get

$$\|u_2\|_{L^{(\alpha-1)q_1}} \leq C\|u_2\|^{1-\gamma_1}_{L^2}\|u_2\|^{\gamma_1}_{H^s}, \tag{5.26}$$

$$\|u_1\|_{L^{(\alpha-1)q_1}} \leq C\|u_1\|^{1-\gamma_1}_{L^2}\|u_1\|^{\gamma_1}_{H^s}, \tag{5.27}$$

where

$$\gamma_1 = -\frac{N}{(\alpha-1)q_1 s} + \frac{N}{2s}, \qquad \text{for some } \gamma_1 \in (0,1),$$

and

$$\|\nabla u_2\|_{L^{\beta r_2}} \leq C\|u_2\|^{1-\gamma_2}_{L^2}\|u_2\|^{\gamma_2}_{H^s}, \tag{5.28}$$

where

$$\gamma_2 = -\frac{N}{\beta q_2 s} + \frac{N}{2s} + \frac{1}{s}, \qquad \text{for some } \gamma_2 \in (0,1).$$

Combining (5.25) with (5.26)–(5.28) we obtain

$$\begin{aligned}\int_\Omega |c_1|^{\alpha-1}|\nabla u_2|^\beta |w|^2 dx &\leq C\Big(\|u_1\|^{p(\alpha-1)(1-\gamma_1)}_{L^2}\|u_1\|^{p(\alpha-1)\gamma_1}_{H^s}\|u_2\|^{p(\alpha-1)(1-\gamma_1)}_{L^2}\|u_2\|^{p(\alpha-1)\gamma_1}_{H^s} \\ &+ \|u_2\|^{\delta_3}_{L^2}\|u_2\|^{\delta_4}_{L^2}\Big)\|w\|^2_{L^2} + \frac{1}{2}\|w\|^2_{H^s}, \end{aligned} \tag{5.29}$$

where

$$\begin{aligned}\delta_3 &= p[(\alpha-1)(1-\gamma_1) + \beta(1-\gamma_2)], \\ \delta_4 &= p[(\alpha-1)\gamma_1 + \beta\gamma_2].\end{aligned}$$

We recall that we assume that $\beta < \frac{N+8+4(s-2)-N\alpha}{N+2}$, then $\delta_4 < 2$.
We deduce from (5.25) and (5.29)

$$\begin{aligned}\int_\Omega w^2(x,t)dx &\leq C\int_0^t \Big(\|u_1\|^{q\alpha(1-\theta_1)}_{L^2}\|u_1\|^{q\alpha\theta_1}_{H^s}\|u_2\|^{q(\beta-1)(1-\theta_2)}_{L^2}\|u_2\|^{q(\beta-1)\theta_2}_{H^s} \\ &+ \|u_1\|^{p(\alpha-1)(1-\gamma_1)}_{L^2}\|u_1\|^{p(\alpha-1)\gamma_1}_{H^s}\|u_2\|^{p(\alpha-1)(1-\gamma_1)}_{L^2}\|u_2\|^{p(\alpha-1)\gamma_1}_{H^s} \\ &+ \|u_1\|^{\delta_1}_{L^2}\|u_1\|^{\delta_2}_{H^s} + \|u_2\|^{\delta_3}_{L^2}\|u_2\|^{\delta_4}_{L^2}\Big)\|w\|^2_{L^2}d\tau.\end{aligned}$$

Since $\|u_1(t,.)\|_{L^2}$, $\|u_2(t,.)\|_{L^2}$ are bounded in $L^\infty(0,T)$ and $\|u_1(t,.)\|^{\delta_2}_{H^s}$, $\|u_2(t,.)\|^{\delta_4}_{H^s} \in L^1(0,T)$, then from Gronwall's inequality we obtain

$$u_1(t,x) = u_2(t,x) \qquad \text{for all } (t,x) \in (0,T)\times\Omega.$$

2) We multiply (5.7) by $(-\Delta)^s u$ and integrate over $\Omega \times [0,t]$, we obtain

$$\begin{aligned}\int_\Omega \left((-\Delta)^{\frac{s}{2}}u\right)^2 dx + 2\int_0^t\int_\Omega \left((-\Delta)^s u\right)^2 dx d\tau &= \int_\Omega \left((-\Delta)^{\frac{s}{2}}u_0(x)\right)^2 dx \\ &+2\int_0^t\int_\Omega ((-\Delta)^s u)F(u,|\nabla u|)dx d\tau.\end{aligned}$$

We need to estimate the last term in the inequality above.
First, we get from using Young's inequality and (5.8) that

$$\int_\Omega \left((-\Delta)^{\frac{s}{2}}u\right)^2 dx + \int_0^t\int_\Omega \left((-\Delta)^s u\right)^2 dxd\tau \;\le\; \int_\Omega \left((-\Delta)^{\frac{s}{2}}u_0(x)\right)^2 dx$$
$$+C\int_0^t\int_\Omega |u|^{2\alpha}|\nabla u|^{2\beta} dxd\tau.$$

We apply Hölder's inequality we have

$$\int_\Omega |u|^{2\alpha}|\nabla u|^{2\beta}dx \le C\|u\|^{2\alpha}_{L^{2\alpha r_1}}\|\nabla u\|^{2\beta}_{L^{2\beta r_1}}, \tag{5.30}$$

where $\frac{1}{r_1}+\frac{1}{r_2}=1$.
By using Sobolev embedding and interpolation inequalities, we find that

$$\|u\|_{L^{2\beta r_1}} \le C\|u\|^{1-\theta_1}_{L^2}\|u\|^{\theta_1}_{H^{2s}}, \tag{5.31}$$

with

$$\theta_1 = -\frac{N}{4\beta}+\frac{N}{4s}+\frac{1}{2s},$$

and

$$\|u\|_{L^{2\alpha r_2}} \le C\|u\|^{1-\theta_2}_{L^2}\|u\|^{\theta_2}_{H^{2s}}, \tag{5.32}$$

where

$$\theta_2 = -\frac{N}{4\alpha r_2 s}+\frac{N}{4s}.$$

Combining (5.30) with (5.31) and (5.32) we reach

$$\int_\Omega \left((-\Delta)^{\frac{s}{2}}u(t,x)\right)^2 dx + \int_0^t\int_\Omega \left((-\Delta)^s u\right)^2 dxd\tau$$
$$\le C\int_0^t \|u\|^{2\beta(1-\theta_1)+2\alpha(1-\theta_2)}_{L^2}\|u\|^{2\beta\theta_1+2\alpha\theta_2}_{H^{2s}}d\tau + \int_\Omega \left((-\Delta)^{\frac{s}{2}}u_0(x)\right)^2 dx.$$

Again, it is easy to verify that for (5.9) we have $\beta\theta_1+\alpha\theta_2<1$.
By using Young's inequality we get

$$\int_\Omega \left((-\Delta)^{\frac{s}{2}}u(t,x)\right)^2 dx + \frac{1}{2}\int_0^t\int_\Omega ((-\Delta)^s u)^2 dxd\tau$$
$$\le \int_\Omega \left((-\Delta)^{\frac{s}{2}}u_0(x)\right)^2 dx + C\int_0^t \|u\|_{L^2}^{\frac{2\beta(1-\theta_1)+2\alpha(1-\theta_2)}{1-\beta\theta_1-\alpha\theta_2}} d\tau.$$

Since $u_0\in H^s(\Omega)$ then $u\in L^\infty((0,T);H^s(\Omega))\cap L^2((0,T);H^{2s}(\Omega))$.

3) The last case can be easily obtained from the previous parts.

$\square$

Conclusion

This thesis is devoted to the large-time behaviour and blow up of solutions for Gierer-Meinhardt systems, and the local well-posedness for some fractional Hamilton-Jacobi-type equations.

Our first contribution in this work is to study global existence and uniformly boundedness of solutions for Gierer-Meinhardt systems. We also give the result of asymptotic behaviour of solutions for Gierer-Meinhardt systems, and the result of blow up of solutions only for coupled Gierer-Meinhardt systems.

Our second contribution is to show local existence of weak and strong solutions and uniqueness of strong solutions for some fractional Hamilton-Jacobi-type equations. We also prove the local existence and uniqueness of mild solutions for hyper-viscous Hamilton-Jacobi-type problem. The blow up of weak solutions for considered problem is established.

Open problems

In this section we discuss some open problems, which appear to be interesting.

1. In the case of Gierer-Meinhardt systems, there seems to many important problems.
   - In the second chapter, we studied the asymptotic behaviour of solutions only when $\sigma_1 \equiv 0$ and $\sigma_2(x) = \sigma_2 \geq 0$ on $x \in \Omega$. It is important to establish this result for all $\sigma_i,\ i = 1, 2$ are non negative continuous functions on $\bar{\Omega}$. The same thing for the asymptotic behaviour of solutions in the third chapter.
   - The result of blow up of solutions for coupled Gierer-Meinhardt systems is showed assuming that the exponents of the non linear terms satisfy
   $$p_1 - 1 > p_2 \max\left(\frac{q_1}{q_2+1}, 1\right).$$
   But in the following cases
     (a) Case 1:
     $$\frac{p_1 - 1}{p_2} > 1 \qquad \text{and} \qquad \frac{p_1 - 1}{p_2} < \frac{q_1}{q_2+1}$$
     (b) Case 2:
     $$\frac{p_1 - 1}{p_2} < 1 \qquad \text{and} \qquad \frac{p_1 - 1}{p_2} > \frac{q_1}{q_2+1}$$
   We could not obtain any results about blow up of solutions for considered problem in the second chapter. This can be an area for future research.
   - It would be interesting to study blow up of solutions for a Gierer-Meinhardt system with tree equations.
2. In the case of Hamilton-Jacobi-type equations, there seems to many interesting problems. Among them
   - It is well known that, the global regularity of solutions for Kuramoto-Sivashinsky equation in the two-dimensional, or higher is one of the major open questions in non linear analysis of partial differential equations.

- We studied the local existence of weak and strong solutions and uniqueness of strong solutions for some fractional Hamilton-Jacobi-type equations when the non linearity is of polynomial growth. But the non linearity grows faster than a polynomial, nothing seems to be known for instance.

Appendix

In this appendix we introduce the positive constants M_1 and M_2 such that for all $t \in (0,+\infty)$

$$\begin{aligned} \|u(t,.)\|_\infty &\leq M_1, \\ \|v(t,.)\|_\infty &\leq M_2, \end{aligned}$$

are constructed by applying variation of constants and by introducing fractional powers of operators.

This appendix is divided into two subsections.

Preliminary estimates. In this subsection, we recall some classical facts about the semi group formulation and the fractional powers of operators by following [20]. For $p > 1$, let us defined the operator L on $L^p(\Omega)$ by

$$L_p u = d\Delta u \quad \text{for } \ u \in D(L), \quad d > 0,$$

$$\textit{and} \qquad D(L_p) = \left\{ u \in W^{2,p}(\Omega) \ / \ \frac{\partial u}{\partial \eta} = 0 \quad on \ \ \partial\Omega \right\},$$

where $W^{2,p}(\Omega)$ is the usual Sobolev space. It is well known that L generates a compact analytic semi group

$$\mathscr{S}_p = \left\{ e^{tL_p} \ / \ t \geq 0 \right\}$$

of bounded linear operators on $L^p(\Omega)$ and that

$$\|e^{tL_p} u\|_p \leq \|u\|_p, \qquad for \quad t \geq 0, \ u \in L^p(\Omega).$$

It is also well-known fact that for $r > 0$, the fractional powers $(I - L_p)^{-r}$ exist and are injective bounded linear operators on $L^p(\Omega)$ (see, e.g., [38]).
For $0 < r < 1$, let $A_p^r = ((I - L_p)^{-r})^{-1}$ and recall $D(A_p^r)$ is a Banach space with the graph norm $| \, \|u\| \, |_{r,p} = \|A_p^r u\|_p$ and that if $r > s \geq 0$ (where conventionally $L^p(\Omega) = D(A_p^0)$), then $D(A_p^r)$ is a dense space of $D(A_p^s)$ with the inclusion $D(A_p^r) \subset D(A_p^s)$ compact (see, e.g., [38]). Here we will make use of the following two lemmas.

Lemma A. 1. *For the semi group $\mathscr{S}_p$ and the fractional powers A_p^r just considered, one has*

$$\begin{aligned} t > 0, \ u \in L^p(\Omega) &\implies e^{tL_p} u \in D(A_p^r), \\ t > 0, \ u \in L^p(\Omega) &\implies \|A_p^r e^{tL_p} u\|_p \leq K(r,p) t^{-r} \|u\|_p, \\ t > 0, \ u \in L^p(\Omega) &\implies A_p^r e^{tL_p} u = e^{tL_p} A_p^r u, \end{aligned}$$

where $K(r,p)$ is a positive constant independent of t.

Proof. For the proof of this lemma, we refer the reader to Pazy [38] (page 74, Theorem 6.13). □

Lemma A. 2. *Suppose that a fractional power A_p^r (defined above) is such that $r > \frac{N}{2p}$. Then $D(A_p^r) \subset L^\infty(\Omega)$ and*

$$\|u\|_\infty \leq B(r,p) \|A_p^r u\|_p,$$

where $B(r,p) > 0$ is a positive constant.

PROOF. The proof of this lemma can be readily deduced by applying Theorem 1.6.1 exposed in [20] (page 39). $\square$

Construction of the constants M_1 and M_2. For all $(t,x) \in (0,T)\times\Omega$ we set

$$\begin{aligned} F(t)(x) &= -b_1 u(t,x) + \rho_1(x,u(t,x),v(t,x))\frac{u^{p_1}(t,x)}{v^{q_1}(t,x)} + \sigma_1(x), \\ G(t)(x) &= -b_2 v(t,x) + \rho_2(x,u(t,x),v(t,x))\frac{u^{p_2}(t,x)}{v^{q_2}(t,x)} + \sigma_2(x). \end{aligned}$$

Observe that if we choose $\alpha > \max\left\{p_1 N, \frac{\beta p_1}{q_1}\right\}$, then by using Lemma 3.1 and (2.26) we obtain

$$\left\|\frac{u^{p_1}(t)}{v^{q_1}(t)}\right\|_N \le \left(\int_\Omega \frac{\varphi_1^\alpha}{\varphi_2^\beta}dx + \frac{C}{\alpha b_1 - 3\tau^{-1}b_2\beta} + |\Omega| m_2^{\frac{N(\beta p_1 - q_1\alpha)}{\alpha - p_1 N}}\right)^{\frac{1}{N}}. \tag{5.33}$$

Now, for $t\in(0,1)$ we use the proprieties of the semi group $\mathscr{S}_p$ and the inequality (5.33) we get

$$\|u(t,.)\|_\infty \le M_1', \tag{5.34}$$

where

$$M_1' = \bar\varphi_1 + \frac{1}{\sqrt{\pi}}\left[\bar\rho_1\left(\int_\Omega \frac{\varphi_1^\alpha}{\varphi_2^\beta}dx + \frac{C}{\alpha b_1 - 3\tau^{-1}b_2\beta} + |\Omega| m_2^{\frac{N(\beta p_1 - q_1\alpha)}{\alpha - p_1 N}}\right)^{\frac{1}{N}} + |\Omega|^{\frac{1}{N}}\bar\sigma_1\right].$$

On the other hand, by applying variation of constant, one can write for $t_0 \ge 0$ and $0<r<1$

$$\begin{aligned} u(t) &= e^{(t-t_0)L_N}u(t_0) + \int_{t_0}^t e^{(t-s)L_N}F(s)ds, \\ \textit{and} \qquad A_N^r u(t) &= A_N^r e^{(t-t_0)L_N}u(t_0) + \int_{t_0}^t A_N^r e^{(t-s)L_N}F(s)ds. \end{aligned}$$

By Lemma A.1 and the proprieties of the semi group $\mathscr{S}_p$ we obtain for all $t\in(0,T)$

$$\|A_N^r u(t)\|_N \le K(r,N)\left[(t-t_0)^{-r}\|u(t_0)\|_N + \int_{t_0}^t (t-s)^{-r}\left(\bar\rho_1\left\|\frac{u^{p_1}(s)}{v^{q_1}(s)}\right\|_N + \|\sigma_1\|_N\right)ds\right]. \tag{5.35}$$

Note that we have

$$\|u(t_0)\|_N \le |\Omega|^{\frac{1}{N}}\bar\varphi_1 + C_1'\left(\bar\rho_1\left(\int_\Omega \frac{\varphi_1^\alpha}{\varphi_2^\beta}dx + \frac{C}{\alpha b_1 - 3\tau^{-1}b_2\beta} + |\Omega| m_2^{\frac{N(\beta p_1 - q_1\alpha)}{\alpha - p_1 N}}\right)^{\frac{1}{N}} + |\Omega|^{\frac{1}{N}}\bar\sigma_1\right), \tag{5.36}$$

where C_1' is a positive constant independent of t.
From (5.35)–(5.36), we obtain

$$\begin{aligned} \|A_N^r u(t)\|_N \;\le\; & K(r,N)\Bigg[(t-t_0)^{-r}\left(C_1' + \frac{t-t_0}{1-r}\right)\Bigg(\bar\rho_1\left(\int_\Omega \frac{\varphi_1^\alpha}{\varphi_2^\beta}dx + \frac{C}{\alpha b_1 - 3\tau^{-1}b_2\beta} + |\Omega| m_2^{\frac{N(\beta p_1 - q_1\alpha)}{\alpha - p_1 N}}\right)^{\frac{1}{N}} \\ & + |\Omega|^{\frac{1}{N}}\bar\sigma_1\Bigg) + (t-t_0)^{-r}|\Omega|^{\frac{1}{N}}\bar\varphi_1\Bigg]. \end{aligned}$$

Set $t_0 = \lfloor t\rfloor - 1$, where $\lfloor t\rfloor$ denotes the floor of t. We have for $t\ge 1$

$$\begin{aligned} \|A_N^r u(t)\|_N \;\le\; & K(r,N)\Bigg[|\Omega|^{\frac{1}{N}}\bar\varphi_1 + \left(C_1' + \frac{2^{1-r}}{1-r}\right)\Bigg(\bar\rho_1\Bigg(\int_\Omega \frac{\varphi_1^\alpha}{\varphi_2^\beta}dx + \frac{C}{\alpha b_1 - 3\tau^{-1}b_2\beta} \\ & + |\Omega| m_2^{\frac{N(\beta p_1 - q_1\alpha)}{\alpha - p_1 N}}\Bigg)^{\frac{1}{N}} + |\Omega|^{\frac{1}{N}}\bar\sigma_1\Bigg)\Bigg]. \end{aligned}$$

Next, we set $r = \frac{3}{4} > \frac{N}{2N}$, so that by virtue of Lemma A.2 with the positive constant $B(\frac{3}{4}, N) > 0$ introduced therein, one claims that

$$\|u(t,.)\|_\infty \leq M_1''. \tag{5.37}$$

where

$$M_1'' = B(\frac{3}{4}, N)K(\frac{3}{4}, N)\left[|\Omega|^{\frac{1}{N}}\bar{\varphi}_1 + C_2'\left(\bar{\rho}_1\left(\int_\Omega \frac{\varphi_1^\alpha}{\varphi_2^\beta}dx + \frac{C}{\alpha b_1 - 3\tau^{-1}b_2\beta} + |\Omega| m_2^{\frac{N(\beta p_1 - q_1\alpha)}{\alpha - p_1 N}}\right)^{\frac{1}{N}} + |\Omega|^{\frac{1}{N}}\bar{\sigma}_1\right)\right],$$

and C_2' is a positive constant independent of t.

From (5.34) and (5.37) we deduce that $T_{max} = +\infty$ and

$$\|u(t,.)\|_\infty \leq M_1 \qquad \textit{for all} \quad t \in (0, +\infty),$$

such that $M_1 = \max(M_1', M_1'')$.
In an analogue way, we get for all $t \in (0, +\infty)$

$$\|v(t,.)\|_\infty \leq M_2,$$

where $M_2 = \max(M_2', M_2'')$ such that

$$M_2' = \bar{\varphi}_2 + \frac{\tau^{-1}}{\sqrt{\pi}}\left[\bar{\rho}_2\left(\int_\Omega \frac{\varphi_1^\alpha}{\varphi_2^\beta}dx + \frac{C}{\alpha b_1 - 3\tau^{-1}b_2\beta} + |\Omega| m_2^{\frac{N(\beta p_2 - q_2\alpha)}{\alpha - p_2 N}}\right)^{\frac{1}{N}} + |\Omega|^{\frac{1}{N}}\bar{\sigma}_2\right],$$

and

$$M_2'' = B(\frac{3}{4}, N)K(\frac{3}{4}, N)\left[|\Omega|^{\frac{1}{N}}\bar{\varphi}_2 + \tau^{-1}A_1'\left(\bar{\rho}_2\left(\int_\Omega \frac{\varphi_1^\alpha}{\varphi_2^\beta}dx + \frac{C}{\alpha b_1 - 3\tau^{-1}b_2\beta} + |\Omega| m_2^{\frac{N(\beta p_2 - q_2\alpha)}{\alpha - p_2 N}}\right)^{\frac{1}{N}} + |\Omega|^{\frac{1}{N}}\bar{\sigma}_2\right)\right],$$

where A_1' is a positive constant independent of t.

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Abstract

The aim of this thesis is to study the global existence of solutions for some non linear evolution equations.

In the first part, we consider a coupled Gierer-Meinhardt system with homogeneous Neumann boundary conditions. We prove that the solutions are global and are uniformly bounded and under suitable conditions, we contribute to the study of the asymptotic behaviour of these solutions. The basic idea of these results is the judicious Lyapunov functions constructed. Moreover, we will schow that under reasonable conditions on the exponents of the non linear term, these solutions blow up in finite time. These results are valid for any positive continuous initial data in $C(\bar{\Omega})$, without any differentiability conditions.

The second part of this thesis is devoted to study the uniform boundedness and so global existence of solutions for Gierer-Meinhardt model of three substance described by reaction-diffusion equations with homogeneous Neumann boundary conditions. The proofs of these results is based on suitable Lyapunov functionals and from which a result on the asymptotic behaviour of the solutions is established.

In the last part of this thesis, we investigate the local existence and uniqueness of mild solution for some hyper-viscous Hamilton-Jacobi equations. We give some conditions from which these results can be established. Moreover, we show the blow up in finite time of some weak solutions.

Résumé

L'objectif de cette thèse est d'étudier l'existence globale des solutions pour des équations d'évolution non linéaires.

En premiére partie, on considére un système de Gierer-Meinhardt couplé avec des conditions aux limites de Neumann homogènes. On montre que les solutions sont globales et uniformément bornées. Sous des conditions appropriées, nous contribuons à l'étude du comportement asymptotique de ces solutions. L'idée de base de ces résultats est le choix judicieux de fonctionnelles de Lyapunov. Par ailleurs, on montre que sous certaines conditions sur les exposants du terme non linéaire, ces solutions explosent en temps fini. Ces résultats sont valables pour toutes les données initiales positives et continues sur $C(\bar{\Omega})$ avec aucune condition de différentiabilité.

La seconde partie de cette thèse est consacrée à l'étude du bornage uniforme des solutions d'un modéle de Gierer-Meinhardt à trois équations avec des conditions de Neumann homogènes. Par une technique de fonctionnelles de Lyapunov adaptées au système, on établit des résultats sur l'existence globale et sur le comportement à l'infini des solutions.

Dans la derniére partie de cette thèse, on s'intéresse à l'étude de l'existence locale et de l'unicité des solutions pour certaines équations de type Hamilton-Jacobi hyper-visqueux. On montre que sous certaines conditions, certains de ces résultats peuvent être établis et qu'en plus, il est possible d'avoir des cas d'explosion en temps fini de certaines solutions faibles.