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Contents

Abstract

The problem of stability for systems governed by partial differential equations (coupled wave equations, coupled Euler-Bernoulli equations, transmission wave equations) with delay terms in the boundary or internal feedback is considered. Under some assumptions exponential stability is established. The results are obtained by introducing an appropriate energy function and by proving a suitable observability inequality.

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Introduction

Time delay exists in many practical systems such as engineering systems (see Abdel-Rohman [**2**], [**3**], Agrawal and Yang [**4**],[**5**], Phohomsiri et al [**35**]), biological systems (Batzel et al [**6**]), etc... It may be a source of instability. In fact, it is by now well known that certain infinite-dimensional second order systems are destabilized by small time delay in the damping (Datko et al [**9**], Datko [**8**]). On the other hand, it may have a stabilizing effect and it could improve the system performances, (see Abdallah et al [**1**], Chiasson and Abdallah [**7**], Niculescu et al [**31**], Kwon et al [**14**],...).
Thus, the stability analysis of time delay system is an important subject for investigations from both theoretical and practical point of view. The stability analysis of control systems governed by ordinary differential equations with constant or time-varying delays has been studied by many researchers (Kolmanovskii and Myshkis [**13**], Niculescu [**30**], Richard [**37**], Gu et al [**11**],...). Two methods, one is based on Lyapunov-Razumikhin functional and the other is based on Lyapunov-Krasovskii functional , are widely used in order to find a delay independent or a delay dependent stability conditions.
Stability of partial differential equations with delay has also attracted the attention of many authors. Datko et al [**9**] analyzed the effect of time delay in boundary feedback stabilization of the one-dimensional wave equation. They showed that an almost arbitrary small time delay destabilize the system which is exponentially stable in the absence of delay. In Datko [**8**], the author presented two examples of hyperbolic partial differential equations which are stabilized by boundary feedback controls and then destabilized by small time delays in these controls. Li and Liu [**20**] proved that stabilization of parabolic systems is robust with respect to small time delays in their feedbacks, however stabilization of infinite-dimensional conservative systems is not. Xu et al [**39**] established sufficient conditions ensuring the stability of one dimensional wave equation with a constant time delay term in the boundary feedback controller using spectral methods. More precisely, they split the controller into two parts: one has no delay and the other has a time delay. They showed that if the constant gain of the delayed damping term is smaller (larger) than the undelayed one then the system is exponential stable (unstable). When the two constant gains are equal, they proved that the system is asymptotically stable for some time delays. This result have been extended to the multidimensional wave equation with a delay term in the boundary or internal feedbacks by Nicaise and Pignotti [**28**]. Similarly to (Xu et al [**39**]), they established an exponential stability result in the case where the constant gain of the delayed term is smaller than the undelayed one. This result is obtained by introducing an appropriate energy function and by using a suitable observability estimate. In the other cases, they constructed a sequence of time delays for which instability occurs. Nicaise and Rebiai [**29**] considered the multidimensional Schrödinger equation with a delay term in the boundary or internal feedbacks. Adopting the approach of (Nicaise and Pignotti [**28**]), they established stability and instability results.

Shang et al [**38**] investigated the stability of one dimensional Euler Bernoulli beam with input delay in the boundary control by using spectral analysis and Lyapunov method.

The purpose of this thesis is to study the stability and stabilization of some distributed parameter systems with time delays. We begin with compactly coupled wave equations with delay terms in the boundary or internal feedbacks. In the second chapter, the system

of transmission of the wave equation with a delay term in the boundary feedback is considered, whereas chapter three treats the transmission wave equation where both boundary and internal feedbacks contain a delay term. Coupled Euler-Bernoulli equations with delay terms in the boundary feedback controller are studied in chapter four. Finally chapter five is devoted to coupled Euler-Bernoulli equations with distributed controllers containing a delay term. Under some assumptions exponential stability is established.
The results are obtained by introducing an appropriate energy function and by proving a suitable observability estimate.

CHAPTER 1

Stability and instability of compactly coupled wave equations with delay terms in the feedbacks

1.1. Stability of compactly coupled wave equations with delay terms in the boundary feedbacks

1.1. Introduction. In this section, we study a stability problem for compactly coupled wave equations with delay terms in the boundary feedbacks.
Let Ω be an open bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that, $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$.
Furthermore we assume that there exists a scalar function $\Phi \in C^2(\overline{\Omega})$ such that
$(H.1)$ Φ is strictly convex in $\overline{\Omega}$; that is, there exists $\lambda > 0$ such that

$$H(x)\Theta.\Theta \geq \lambda\,|\Theta|^2 \qquad \forall\, x \in \overline{\Omega},\, \Theta \in \mathbb{R}^n,$$

where H is the Hessian matrix of Φ.
$(H.2)$ $h(x).\nu(x) \leq 0$ on Γ_1, where $h(x) = \nabla\Phi(x)$ and ν is the unit normal on Γ pointing towards the exterior of Ω.
In Ω, we consider the following coupled system of two wave equations with delay terms in the boundary conditions:

$$u_{tt}(x,t) - \Delta u(x,t) + l(u(x,t) - v(x,t)) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{1.1}$$

$$v_{tt}(x,t) - \Delta v(x,t) + l(v(x,t) - u(x,t)) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.3}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x) \quad \text{in } \Omega, \tag{1.4}$$

$$u(x,t) = v(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,+\infty), \tag{1.5}$$

$$\frac{\partial u(x,t)}{\partial \nu} = -\alpha_1 u_t(x,t) - \alpha_2 u_t(x,t-\tau) \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{1.6}$$

$$\frac{\partial v(x,t)}{\partial \nu} = -\beta_1 v_t(x,t) - \beta_2 v_t(x,t-\tau) \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{1.7}$$

$$u_t(x,t-\tau) = g(x,t-\tau) \quad \text{on } \Gamma_2 \times (0,\tau), \tag{1.8}$$

$$v_t(x,t-\tau) = h(x,t-\tau) \quad \text{on } \Gamma_2 \times (0,\tau), \tag{1.9}$$

where $l, \alpha_1, \alpha_2, \beta_1$ and β_2 are positive constants, $\tau > 0$ is the time delay, u_0, u_1, v_0, v_1, g and h are the initial data, $\frac{\partial}{\partial \nu}$ is the normal derivative.
In the one-dimensional case, u and v may represent the displacements of two vibrating objects measured from their equilibrium positions, the coupling terms $\pm l(u-v)$ are the distributed springs linking the two vibrating objects. In the absence of delay (*i.e.* $\alpha_2 = \beta_2 = 0$), the solution (u,v) of (1.1)-(1.9)decays exponentially in the energy space $H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)$ (Najafi et al [**27**], Komornik and Rao [**15**]).
Stability problems for the wave equation with a delay term in the feedback has been studied by Xu et al [**39**] in the one-dimensional case and by Nicaise and Pignotti [**28**] in the multidimensional case.
The subject of this section is to investigate the uniform exponential stability of the system $(1.1) - (1.9)$ in the case where the boundary damping coefficients $\alpha_1, \alpha_2, \beta_1$ and β_2 are strictly positive.

1.2. Main result. We assume as in [**28**] that

$$\alpha_1 > \alpha_2,\ \beta_1 > \beta_2. \tag{1.10}$$

and define the energy of a solution of (1.1) – (1.9) by

$$\begin{aligned} E(t) =& \frac{1}{2}\int_\Omega \left[|\nabla u(x,t)|^2 + u_t^2(x,t) + |\nabla v(x,t)|^2 + v_t^2(x,t) + l(u(x,t)-v(x,t))^2\right] dx \\ &+ \frac{1}{2}\int_{\Gamma_2}\int_0^1 \left[\mu u_t^2(x,t-\tau\rho) + \xi v_t^2(x,t-\tau\rho)\right] d\rho\, d\Gamma, \end{aligned} \tag{1.11}$$

where μ and ξ are positive constants satisfying

$$\tau\alpha_2 < \mu < \tau(2\alpha_1 - \alpha_2),$$

and

$$\tau\beta_2 < \xi < \tau(2\beta_1 - \beta_2).$$

We show that if $\{\Omega, \Gamma_1, \Gamma_2\}$ satisfies $(H.1)$ and $(H.2)$, then there is an exponential decay rate for $E(t)$. The proof of this result is based on Carleman estimates for a system of coupled non-conservative hyperbolic systems established by Lasiecka and Triggiani in [**16**] and compactness-uniqueness argument.
The main result of this section can be stated as follows.

Theorem 1.1. *Assume* $(H.1)$, $(H.2)$, (1.10) *and* (1.11). *Then the coupled wave equations system* (1.1) – (1.9) *is uniformly exponentially stable, i.e., there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq M e^{-\omega t} E(0).$$

Theorem 1.1 is proved in Subsection 1.4. In Subsection 1.3, we study the well-posedness of the system (1.1) – (1.9) by using semigroup theory.

1.3. Well-posedness. Inspired from [**28**] and [**29**], we introduce the auxiliary variables

$$y(x,\rho,t) = u_t(x,t-\tau\rho),\ z(x,\rho,t) = v_t(x,t-\tau\rho) \qquad x \in \Gamma_2,\ \rho \in (0,1),\ t > 0.$$

Note that y and z verify the following equations on Γ_2 for $0 < \rho < 1$ and $t > 0$

$$\begin{cases} y_t(x,\rho,t) + \tau^{-1} y_\rho(x,\rho,t) = 0, \\ z_t(x,\rho,t) + \tau^{-1} z_\rho(x,\rho,t) = 0, \\ y(x,0,t) = u_t(x,t), z(x,0,t) = v_t(x,t), \\ y(x,\rho,0) = g(x,-\tau\rho), z(x,\rho,0) = h(x,-\tau\rho). \end{cases}$$

Then, the system (1.1) – (1.9) is equivalent to

$$u_{tt}(x,t) - \Delta u(x,t) + l(u(x,t) - v(x,t)) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{1.12}$$

$$y_t(x,\rho,t) + \tau^{-1} y_\rho(x,\rho,t) = 0 \quad \text{on } \Gamma_2 \times (0,1) \times (0,+\infty), \tag{1.13}$$

$$v_{tt}(x,t) - \Delta v(x,t) + l(v(x,t) - u(x,t)) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{1.14}$$

$$z_t(x,\rho,t) + \tau^{-1} z_\rho(x,\rho,t) = 0 \quad \text{on } \Gamma_2 \times (0,1) \times (0,+\infty), \tag{1.15}$$

$$u(x,t) = v(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,+\infty), \tag{1.16}$$

$$\frac{\partial u(x,t)}{\partial \nu} = -\alpha_1 u_t(x,t) - \alpha_2 y(x,1,t) \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{1.17}$$

$$\frac{\partial v(x,t)}{\partial \nu} = -\beta_1 v_t(x,t) - \beta_2 z(x,1,t) \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{1.18}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.19}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x) \quad \text{in } \Omega, \tag{1.20}$$

$$y(x,0,t) = u_t(x,t), z(x,0,t) = v_t(x,t) \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{1.21}$$

$$y(x,\rho,0) = g(x,-\tau\rho), z(x,\rho,0) = h(x,-\tau\rho) \quad \text{on } \Gamma_2 \times (0,1). \tag{1.22}$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2; L^2(0,1)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2; L^2(0,1)),$$

where

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}.$$

We equip $\mathcal{H}$ with the inner product

$$\left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \phi \\ \chi \\ \psi \end{pmatrix} ; \begin{pmatrix} \widetilde{\zeta} \\ \widetilde{\eta} \\ \widetilde{\theta} \\ \widetilde{\phi} \\ \widetilde{\chi} \\ \widetilde{\psi} \end{pmatrix} \right\rangle = \int_\Omega (\nabla\zeta(x).\nabla\widetilde{\zeta}(x) + \eta(x)\widetilde{\eta}(x))\, dx$$

$$+ \mu \int_{\Gamma_2} \int_0^1 \theta(x,\rho)\widetilde{\theta}(x,\rho)\, d\rho\, d\Gamma + \int_\Omega (\nabla\phi(x).\nabla\widetilde{\phi}(x) + \chi(x)\widetilde{\chi}(x))\, dx$$

$$+ \xi \int_{\Gamma_2} \int_0^1 \psi(x,\rho)\widetilde{\psi}(x,\rho)\, d\rho\, d\Gamma + l \int_\Omega (\zeta(x) - \phi(x))(\widetilde{\zeta}(x) - \widetilde{\phi}(x))\, dx.$$

Define in $\mathcal{H}$ a linear operator $\mathcal{A}$ by

$$\begin{aligned} D(\mathcal{A}) = &\{(\zeta,\eta,\theta,\phi,\chi,\psi)^T \in (E(\Delta, L^2(\Omega)) \cap H^1_{\Gamma_1}(\Omega)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2; H^1(0,1)) \times \\ &(E(\Delta, L^2(\Omega)) \cap H^1_{\Gamma_1}(\Omega)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2; H^1(0,1))\,;\ \frac{\partial\zeta}{\partial\nu} = -\alpha_1\eta - \alpha_2\theta(.,1), \\ &\eta = \theta(.,0) \text{ on } \Gamma_2\,;\ \frac{\partial\phi}{\partial\nu} = -\beta_1\chi - \beta_2\psi(.,1)\,,\ \chi = \psi(.,0) \text{ on } \Gamma_2\}, \end{aligned} \tag{1.23}$$

where

$$E(\Delta, L^2(\Omega)) = \left\{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\right\}.$$

$$A(\zeta,\eta,\theta,\phi,\chi,\psi)^T = (\eta, \Delta\zeta + l\phi - l\zeta, -\tau^{-1}\theta_\rho, \chi, \Delta\phi - l\phi + l\zeta, -\tau^{-1}\psi_\rho)^T. \tag{1.24}$$

Then we can rewrite (1.12) – (1.22) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \dfrac{dW}{dt}(t) = \mathcal{A}W(t), \\ W(0) = W_0 \end{cases} \tag{1.25}$$

where

$$W(t) = (u(x,t), u_t(x,t), y(x,\rho,t), v(x,t), v_t(x,t), z(x,\rho,t))^T,$$
$$W_0 = (u_0, u_1, g(.,-.\tau), v_0, v_1, h(.,-.\tau))^T.$$

We verify that $\mathcal{A}$ generates a strongly continuous semigroup in $\mathcal{H}$ and consequently we have

THEOREM 1.2. *For every $W_0 \in \mathcal{H}$, the problem* (1.25) *has a unique solution W whose regularity depends on the initial datum W_0 as follows:*

$$\begin{aligned} W(.) &\in C([0,+\infty);\mathcal{H}) \text{ if } W_0 \in \mathcal{H}, \\ W(.) &\in C^1([0,+\infty);\mathcal{H}) \cap C([0,+\infty); D(\mathcal{A})) \text{ if } W_0 \in D(\mathcal{A}). \end{aligned}$$

PROOF. We will show that the operator $\mathcal{A}$ defined by (1.24) with the condition (1.10) generates a strongly continuous semigroup in $\mathcal{H}$ by using Lumer-Philips Theorem (see for instance [**34**], Theorem I.4.3).
First, we prove that the operator $\mathcal{A}$ is dissipative.
Let,

$$W = (\zeta,\eta,\theta,\phi,\chi,\psi)^T \in D(A). Then$$

$$\langle AW, W\rangle = \left\langle \begin{pmatrix} \eta \\ \Delta\zeta + l\phi - l\zeta \\ -\tau^{-1}\theta_\rho \\ \chi \\ \Delta\phi - l\phi + l\zeta \\ -\tau^{-1}\psi_\rho \end{pmatrix}, \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \phi \\ \chi \\ \psi \end{pmatrix} \right\rangle$$

$$\begin{aligned} &= \int_\Omega [\nabla\eta(x).\nabla\zeta(x) + (\Delta\zeta(x) + l\phi(x) - l\zeta(x))\eta(x)]\, dx - \tau^{-1}\mu \int_{\Gamma_2}\int_0^1 \theta_\rho(x,\rho)\theta(x,\rho)\, d\rho\, d\Gamma \\ &+ \int_\Omega [\nabla\chi(x).\nabla\phi(x) + (\Delta\phi(x) - l\phi(x) + l\zeta(x))\chi(x)]\, dx - \tau^{-1}\xi \int_{\Gamma_2}\int_0^1 \psi_\rho(x,\rho)\psi(x,\rho)\, d\rho\, d\Gamma \\ &+ l\int_\Omega (\eta(x) - \chi(x))(\zeta(x) - \phi(x))\, dx. \end{aligned}$$

Applying Green's Theorem, we get

$$\begin{aligned} \langle AW, W\rangle =& \int_{\Gamma_2} \eta(x)\frac{\partial\zeta(x)}{\partial\nu}\, d\Gamma - \tau^{-1}\mu \int_{\Gamma_2}\int_0^1 \theta_\rho(x,\rho)\theta(x,\rho)\, d\rho\, d\Gamma \\ &+ \int_{\Gamma_2} \chi(x)\frac{\partial\phi(x)}{\partial\nu}\, d\Gamma - \tau^{-1}\xi \int_{\Gamma_2}\int_0^1 \psi_\rho(x,\rho)\psi(x,\rho)\, d\rho\, d\Gamma. \end{aligned} \tag{1.26}$$

Integrating by parts in ρ, we obtain

$$\int_{\Gamma_2}\int_0^1 \theta_\rho(x,\rho)\theta(x,\rho)\, d\rho\, d\Gamma = \frac{1}{2}\int_{\Gamma_2} [\theta^2(x,1) - \theta^2(x,0)]\, d\Gamma, \tag{1.27}$$

and

$$\int_{\Gamma_2}\int_0^1 \psi_\rho(x,\rho)\psi(x,\rho)\, d\rho\, d\Gamma = \frac{1}{2}\int_{\Gamma_2} [\psi^2(x,1) - \psi^2(x,0)]\, d\Gamma. \tag{1.28}$$

Inserting (1.23), (1.27) and (1.28) into (1.26), we find

$$\begin{aligned} \langle AW, W\rangle =& -\alpha_1 \int_{\Gamma_2} \eta^2(x)\, d\Gamma - \alpha_2 \int_{\Gamma_2} \theta(x,1)\eta(x)\, d\Gamma - \frac{\tau^{-1}\mu}{2}\int_{\Gamma_2} \theta^2(x,1)\, d\Gamma \\ &+ \frac{\tau^{-1}\mu}{2}\int_{\Gamma_2} \eta^2(x)\, d\Gamma - \beta_1 \int_{\Gamma_2} \chi^2(x)\, d\Gamma - \beta_2 \int_{\Gamma_2} \psi(x,1)\chi(x)\, d\Gamma \\ &- \frac{\tau^{-1}\xi}{2}\int_{\Gamma_2} \psi^2(x,1)\, d\Gamma + \frac{\tau^{-1}\xi}{2}\int_{\Gamma_2} \chi^2(x)\, d\Gamma. \end{aligned}$$

Therefore, by Cauchy-Schwarz's inequality we have

$$\begin{aligned}\langle AW, W\rangle \leq & (-\alpha_1 + \frac{\alpha_2}{2} + \frac{\tau^{-1}\mu}{2})\int_{\Gamma_2} \eta^2(x)\, d\Gamma + (\frac{\alpha_2}{2} - \frac{\tau^{-1}\mu}{2})\int_{\Gamma_2} \theta^2(x,1)\, d\Gamma \\ & + (-\beta_1 + \frac{\beta_2}{2} + \frac{\tau^{-1}\xi}{2})\int_{\Gamma_2} \chi^2(x)\, d\Gamma + (\frac{\beta_2}{2} - \frac{\tau^{-1}\xi}{2})\int_{\Gamma_2} \psi^2(x,1)\, d\Gamma.\end{aligned}$$

From (1.10), we conclude that $(AW, W) \leq 0$, thus A is dissipative.
Now we show that $\lambda I - A$ is onto for a fixed $\lambda > 0$.
Let $(f, g, h, k, m, p)^T \in \mathcal{H}$, we seek a $W = (\zeta, \eta, \theta, \phi, \chi, \psi)^T \in D(A)$ solution of

$$(\lambda I - A)W = (f, g, h, k, m, p)^T,$$

or equivalently

$$\lambda\zeta - \eta = f, \tag{1.29}$$

$$\lambda\eta - \Delta\zeta + l\zeta - l\phi = g, \tag{1.30}$$

$$\lambda\theta + \tau^{-1}\frac{\partial\theta}{\partial\rho} = h, \tag{1.31}$$

$$\lambda\phi - \chi = k, \tag{1.32}$$

$$\lambda\chi - \Delta\phi + l\phi - l\zeta = m, \tag{1.33}$$

$$\lambda\psi + \tau^{-1}\frac{\partial\psi}{\partial\rho} = p. \tag{1.34}$$

Suppose that we have found ζ and ϕ with the appropriate regularity, then

$$\eta = \lambda\zeta - f, \tag{1.35}$$

$$\chi = \lambda\phi - k. \tag{1.36}$$

Consequently we can determine θ from (1.31) with (1.23) and ψ from (1.34) with (1.23). In fact, θ is the unique solution for $x \in \Gamma_2$ of the initial value problem

$$\begin{cases} \theta_\rho(x,\rho) = -\tau\lambda\theta(x,\rho) + \tau h(x,\rho),\ \rho \in (0,1), \\ \theta(x,0) = \eta(x), \end{cases}$$

and ψ is the unique solution of the initial value problem :

$$\begin{cases} \psi_\rho(x,\rho) = -\tau\lambda\psi(x,\rho) + \tau p(x,\rho),\ x \in \Gamma_2,\ \rho \in (0,1), \\ \psi(x,0) = \chi(x). \end{cases}$$

Therefore

$$\theta(x,\rho) = \eta(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho h(x,\sigma)e^{\lambda\tau\sigma}d\sigma,\ x \in \Gamma_2,\ \rho \in (0,1),$$

and

$$\psi(x,\rho) = \chi(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho p(x,\sigma)e^{\lambda\tau\sigma}d\sigma,\ x \in \Gamma_2,\ \rho \in (0,1),$$

and in particular

$$\theta(x,1) = \lambda e^{-\lambda\tau}\zeta(x) + z_0(x), \tag{1.37}$$

and

$$\psi(x,1) = \lambda e^{-\lambda\tau}\phi(x) + z_1(x), \tag{1.38}$$

with z_0 and z_1 defined by

$$\begin{aligned} z_0(x) &= -f(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\tau\sigma}d\sigma,\ \ x \in \Gamma_2, \\ z_1(x) &= -k(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 p(x,\sigma)e^{\lambda\tau\sigma}d\sigma,\ \ x \in \Gamma_2. \end{aligned}$$

From (1.35), (1.30), (1.36) and (1.33) the functions ζ and ϕ are verify

$$(1.39)\qquad \begin{cases} \lambda^2\zeta - \Delta\zeta + l\zeta - l\phi = g + \lambda f, \\ \lambda^2\phi - \Delta\phi + l\phi - l\zeta = m + \lambda k. \end{cases}$$

Problem (1.39) can be reformulated as

$$\begin{aligned} &\int_\Omega (\lambda^2\zeta - \Delta\zeta + l\zeta - l\phi) w_1\, dx + \int_\Omega (\lambda^2\phi - \Delta\phi + l\phi - l\zeta) w_2\, dx \\ (1.40)\qquad &= \int_\Omega (g + \lambda f) w_1\, dx + \int_\Omega (m + \lambda k) w_2\, dx, \qquad (w_1, w_2) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega). \end{aligned}$$

Using Green's Theorem and recalling (1.37) and (1.38), we rewrite the left hand side of (1.40) as

$$\begin{aligned} &\int_\Omega (\lambda^2\zeta - \Delta\zeta + l\zeta - l\phi) w_1\, dx + \int_\Omega (\lambda^2\phi - \Delta\phi + l\phi - l\zeta) w_2\, dx \\ &= \int_\Omega (\lambda^2\zeta w_1 + \nabla\zeta.\nabla w_1)\, dx + \int_{\Gamma_2} \Big[\alpha_1(\lambda\zeta - f) + \alpha_2(\lambda e^{-\lambda\tau}\zeta + z_0)\Big]\, w_1\, d\Gamma \\ &+ \int_\Omega (\lambda^2\phi w_2 + \nabla\phi.\nabla w_2)\, dx + \int_{\Gamma_2} \Big[\beta_1(\lambda\phi - k) + \beta_2(\lambda e^{-\lambda\tau}\phi + z_1)\Big]\, w_2\, d\Gamma \\ &+ \int_\Omega (l\zeta - l\phi)(w_1 - w_2)\, dx. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_\Omega (\lambda^2\zeta w_1 + \nabla\zeta.\nabla w_1)\, dx + \int_{\Gamma_2} \lambda(\alpha_1 + \alpha_2 e^{-\lambda\tau})\zeta w_1\, d\Gamma + \int_\Omega (\lambda^2\phi w_2 + \nabla\phi.\nabla w_2)\, dx \\ &+ \int_{\Gamma_2} \lambda(\beta_1 + \beta_2 e^{-\lambda\tau})\phi w_2\, d\Gamma + \int_\Omega l(\zeta - \phi)(w_1 - w_2)\, dx \\ &= \int_\Omega (g + \lambda f) w_1\, dx + \int_\Omega (m + \lambda k) w_2\, dx + \alpha_1 \int_{\Gamma_2} f w_1\, d\Gamma + \beta_1 \int_{\Gamma_2} k w_2\, d\Gamma \\ &- \alpha_2 \int_{\Gamma_2} z_0 w_1\, d\Gamma - \beta_2 \int_{\Gamma_2} z_1 w_2\, d\Gamma \qquad \forall (w_1, w_2) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega). \end{aligned}$$

(1.41)

Since the left-hand side of (1.41) is coercive and continuous on $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, and the right-hand side defines a continuous linear form on $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, the Lax-Milgram's Theorem guarantees the existence and uniqueness of a solution $(\zeta, \phi) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ of (1.41).
If we consider $(w_1, w_2) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ in (1.41), then (ζ, ϕ) is solution in $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)$ of

$$(1.42)\qquad \begin{cases} \lambda^2\zeta - \Delta\zeta + l\zeta - l\phi = g + \lambda f, \\ \lambda^2\phi - \Delta\phi + l\phi - l\zeta = m + \lambda k. \end{cases}$$

Thus $(\zeta, \phi) \in E(\Delta, L^2(\Omega)) \times E(\Delta, L^2(\Omega))$.
We obtain from (1.41) after using Green's Theorem and recalling (1.42)

$$\begin{aligned} &\int_{\Gamma_2} \lambda(\alpha_1 + \alpha_2 e^{-\lambda\tau})\zeta w_1\, d\Gamma + \int_{\Gamma_2} \lambda(\beta_1 + \beta_2 e^{-\lambda\tau})\phi w_2\, d\Gamma + \int_{\Gamma_2} \frac{\partial\zeta}{\partial\nu} w_1 d\Gamma + \int_{\Gamma_2} \frac{\partial\phi}{\partial\nu} w_2 d\Gamma \\ &= \alpha_1 \int_{\Gamma_2} f w_1\, d\Gamma - \alpha_2 \int_{\Gamma_2} z_0(x) w_1\, d\Gamma + \beta_1 \int_{\Gamma_2} k w_2\, d\Gamma - \beta_2 \int_{\Gamma_2} z_1(x) w_2\, d\Gamma. \end{aligned}$$

Hence

$$\frac{\partial \zeta}{\partial \nu} = -\alpha_1 \eta - \alpha_2 \theta(.,1),$$
$$\frac{\partial \phi}{\partial \nu} = -\beta_1 \chi - \beta_2 \psi(.,1).$$

So, we have found $(\zeta, \eta, \theta, \phi, \chi, \psi)^T \in D(A)$ which verifies $(1.29) - (1.34)$. Thus, A is the generator of a C_0- semigroup of contractions on $\mathcal{H}$. ☐

1.4. Proof of the main result. We prove the Theorem 1.1 for smooth initial data. The general case follows by a density argument.
First, we prove that the energy function $E(t)$ defined by (1.11) is decreasing.

PROPOSITION 1.1. *The energy corresponding to any regular solution of problem* $(1.1) - (1.9)$, *is decreasing and there exists a positive constant k such that,*

$$\frac{d}{dt}E(t) \leq -k \int_{\Gamma_2} \left\{ u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau) \right\} d\Gamma, \tag{1.43}$$

where

$$k = \min \left\{ \alpha_1 - \frac{\alpha_2}{2} - \frac{\mu}{2\tau}, \frac{\mu}{2\tau} - \frac{\alpha_2}{2}, \beta_1 - \frac{\beta_2}{2} - \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{\beta_2}{2} \right\}.$$

PROOF. Differentiating E(t) defined by (1.11) with respect to time, we obtain

$$\begin{aligned}
\frac{d}{dt}E(t) =& \int_\Omega \left[\nabla u_t . \nabla u + u_{tt} u_t + \nabla v_t . \nabla v + v_{tt} v_t + l(u-v)(u_t - v_t) \right] dx \\
&+ \int_{\Gamma_2} \int_0^1 \left[\mu u_{tt}(x, t-\tau\rho) u_t(x, t-\tau\rho) + \xi v_{tt}(x,t-\tau\rho) v_t(x,t-\tau\rho) \right] d\rho \, d\Gamma \\
=& \int_\Omega \left[\nabla u_t . \nabla u + [\Delta u - l(u-v)] u_t + \nabla v_t . \nabla v + [\Delta v - l(v-u)] v_t + l(u-v)(u_t - v_t) \right] dx \\
&+ \int_{\Gamma_2} \int_0^1 \left[\mu u_{tt}(x, t-\tau\rho) u_t(x, t-\tau\rho) + \xi v_{tt}(x,t-\tau\rho) v_t(x,t-\tau\rho) \right] d\rho \, d\Gamma .
\end{aligned}$$

Applying Green's Theorem and recalling the boundary condition (1.5) -(1.7), we get

$$\begin{aligned}
\frac{d}{dt}E(t) =& -\alpha_1 \int_{\Gamma_2} u_t^2(x,t) \, d\Gamma - \alpha_2 \int_{\Gamma_2} u_t(x,t) u_t(x,t-\tau) \, d\Gamma - \beta_2 \int_{\Gamma_2} v_t(x,t) v_t(x,t-\tau) \, d\Gamma \\
&- \beta_1 \int_{\Gamma_2} v_t^2(x,t) \, d\Gamma + \int_{\Gamma_2} \int_0^1 \{ \mu u_{tt}(x,t-\tau\rho) u_t(x,t-\tau\rho) + \xi v_{tt}(x,t-\tau\rho) v_t(x,t-\tau\rho) \} d\rho \, d\Gamma .
\end{aligned} \tag{1.44}$$

Now, we have

$$u_t(x, t-\tau\rho) = -\tau^{-1} u_\rho(x, t-\tau\rho),$$
$$v_t(x, t-\tau\rho) = -\tau^{-1} v_\rho(x, t-\tau\rho),$$

which lead to

$$u_{tt}(x, t-\tau\rho) = \tau^{-2} u_{\rho\rho}(x, t-\tau\rho),$$
$$v_{tt}(x, t-\tau\rho) = \tau^{-2} v_{\rho\rho}(x, t-\tau\rho).$$

Therefore

$$\begin{aligned}
&\int_{\Gamma_2} \int_0^1 \{ \mu u_{tt}(x,t-\tau\rho) u_t(x,t-\tau\rho) + \xi v_{tt}(x,t-\tau\rho) v_t(x,t-\tau\rho) \} \, d\rho \, d\Gamma \\
&= -\tau^{-3} \int_{\Gamma_2} \int_0^1 \{ \mu u_{\rho\rho}(x,t-\tau\rho) u_\rho(x,t-\tau\rho) + \xi v_{\rho\rho}(x,t-\tau\rho) v_\rho(x,t-\tau\rho) \} \, d\rho \, d\Gamma \\
&= -\frac{\tau^{-3}\mu}{2} \int_{\Gamma_2} \{ u_\rho^2(x,t-\tau) - u_\rho^2(x,t) \} \, d\Gamma - \frac{\tau^{-3}\xi}{2} \int_{\Gamma_2} \{ v_\rho^2(x,t-\tau) - v_\rho^2(x,t) \} \, d\Gamma ,
\end{aligned}$$

that is,

$$\int_{\Gamma_2}\int_0^1 \{\mu u_{tt}(x,t-\tau\rho)u_t(x,t-\tau\rho)+\xi v_{tt}(x,t-\tau\rho)v_t(x,t-\tau\rho)\}\, d\rho\, d\Gamma =$$

$$\frac{\tau^{-1}\mu}{2}\int_{\Gamma_2}\{u_t^2(x,t)-u_t^2(x,t-\tau)\}\,d\Gamma+\frac{\tau^{-1}\xi}{2}\int_{\Gamma_2}\{v_t^2(x,t)-v_t^2(x,t-\tau)\}\,d\Gamma. \tag{1.45}$$

Introducing (1.45) into (1.44), we get

$$\begin{aligned}\frac{d}{dt}E(t)=&-\alpha_1\int_{\Gamma_2}u_t^2(x,t)\,d\Gamma-\beta_1\int_{\Gamma_2}v_t^2(x,t)\,d\Gamma-\alpha_2\int_{\Gamma_2}u_t(x,t)u_t(x,t-\tau)\,d\Gamma\\ &-\beta_2\int_{\Gamma_2}v_t(x,t)v_t(x,t-\tau)\,d\Gamma+\frac{\tau^{-1}\mu}{2}\int_{\Gamma_2}\{u_t^2(x,t)-u_t^2(x,t-\tau)\}\,d\Gamma\\ &+\frac{\tau^{-1}\xi}{2}\int_{\Gamma_2}\{v_t^2(x,t)-v_t^2(x,t-\tau)\}\,d\Gamma.\end{aligned} \tag{1.46}$$

From (1.46), applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\frac{d}{dt}E(t)\le&(-\alpha_1+\frac{\alpha_2}{2}+\frac{\mu\tau^{-1}}{2})\int_{\Gamma_2}u_t^2(x,t)\,d\Gamma+(\frac{\alpha_2}{2}-\frac{\mu\tau^{-1}}{2})\int_{\Gamma_2}u_t^2(x,t-\tau)\,d\Gamma\\ &+(-\beta_1+\frac{\beta_2}{2}+\frac{\xi\tau^{-1}}{2})\int_{\Gamma_2}v_t^2(x,t)\,d\Gamma+(\frac{\beta_2}{2}-\frac{\xi\tau^{-1}}{2})\int_{\Gamma_2}v_t^2(x,t-\tau)\,d\Gamma,\end{aligned}$$

which implies

$$\frac{d}{dt}E(t)\le -k\int_{\Gamma_2}\{u_t^2(x,t)+u_t^2(x,t-\tau)+v_t^2(x,t)+v_t^2(x,t-\tau)\}\,d\Gamma,$$

with k positive constant verifies,

$$k=\min\left\{\alpha_1-\frac{\alpha_2}{2}-\frac{\mu}{2\tau},\frac{\mu}{2\tau}-\frac{\alpha_2}{2},\beta_1-\frac{\beta_2}{2}-\frac{\xi}{2\tau},\frac{\xi}{2\tau}-\frac{\beta_2}{2}\right\}.$$

□

Now, we give an observability inequality which we will use it to prove the exponential decay of the energy $E(t)$.

PROPOSITION 1.2. *For any regular solution of problem* (1.1) – (1.9), *there exists a positive constant C depending on T such that*

$$E(0)\le C\int_0^T\int_{\Gamma_2}\left\{u_t^2(x,t)+u_t^2(x,t-\tau)+v_t^2(x,t)+v_t^2(x,t-\tau)\right\}\,d\Gamma\,dt. \tag{1.47}$$

PROOF. We rewrite

$$E(t)=\mathcal{E}(t)+E_d(t),$$

where

$$\mathcal{E}(t)=\frac{1}{2}\int_\Omega\left\{|\nabla u(x,t)|^2+u_t^2(x,t)+|\nabla v(x,t)|^2+v_t^2(x,t)+l(u(x,t)-v(x,t))^2\right\}dx,$$

and

$$E_d(t)=\frac{1}{2}\int_{\Gamma_2}\int_0^1\left\{\mu u_t^2(x,t-\tau\rho)+\xi v_t^2(x,t-\tau\rho)\right\}\,d\rho\,d\Gamma.$$

$E_d(t)$ can be rewritten via a change of variable as

$$E_d(t)=\frac{1}{2\tau}\int_t^{t+\tau}\int_{\Gamma_2}\left\{\mu u_t^2(x,s-\tau)+\xi v_t^2(x,s-\tau)\right\}\,d\Gamma\,ds.$$

From the above equality, we obtain

$$E_d(t)\le C\int_0^T\int_{\Gamma_2}\left\{u_t^2(x,s-\tau)+v_t^2(x,s-\tau)\right\}\,d\Gamma\,ds. \tag{1.48}$$

for $0 \leq t + \tau \leq T$. Here and throughout the rest of the section C is some positive constant different at different occurrences.
We have from ([**16**],[**19**] and [**25**], see the Appendix B), for T sufficiently large and for any $\epsilon > 0$

$$\mathcal{E}(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u}{\partial \nu}(x,t) \right|^2 + u_t^2(x,t) + \left| \frac{\partial v}{\partial \nu}(x,t) \right|^2 + v_t^2(x,t) \right\} d\Gamma \, dt$$
$$+ C \left\{ \|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} \right\},$$

Inserting the boundary conditions (1.6) and (1.7) into the above inequality, we have

$$\begin{aligned}\mathcal{E}(0) \leq & C \int_0^T \int_{\Gamma_2} \{ u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau) \} \, d\Gamma dt \\ & + C \left\{ \|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} \right\}. \end{aligned} \tag{1.49}$$

Combining (1.48) with (1.49), we obtain

$$\begin{aligned}E(0) \leq & C \int_0^T \int_{\Gamma_2} \{ u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau) \} \, d\Gamma \, dt \\ & + C \left\{ \|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} \right\}. \end{aligned} \tag{1.50}$$

To obtain the desired estimate (1.47) we need to absorb the lower order terms $\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}$ and $\|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}$ on the right-hand side of (1.50). We do this by a compactness-uniqueness argument.
Suppose that (1.47) is not true. Then, there exists a sequence (u_n, v_n) of solution of problem (1.1)-(1.9) with,

$$u_n(x,0) = u_n^0(x), u_{nt}(x,0) = u_n^1(x), u_n(x,t-\tau) = g_n^0(x,t-\tau)$$
$$v_n(x,0) = v_n^0(x), v_{nt}(x,0) = v_n^1(x), v_n(x,t-\tau) = h_n^0(x,t-\tau).$$

Such that

$$E_n(0) > n \int_0^T \int_{\Gamma_2} \{ u_{nt}^2(x,t) + u_{nt}^2(x,t-\tau) + v_{nt}^2(x,t) + v_{nt}^2(x,t-\tau) \} \, d\Gamma \, dt \tag{1.51}$$

where $E_n(0)$ is the energy corresponding to (u_n, v_n) at the time 0.
From (1.50),

$$\begin{aligned}E_n(0) \leq C & \int_0^T \int_{\Gamma_2} \{ u_{nt}^2(x,t) + u_{nt}^2(x,t-\tau) + v_{nt}^2(x,t) + v_{nt}^2(x,t-\tau) \} \, d\Gamma \, dt \\ & + C \{ \|u_n\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} + \|v_n\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} \} \end{aligned} \tag{1.52}$$

From (1.51) together with (1.52), yields

$$\begin{aligned} & n \int_0^T \int_{\Gamma_2} \{ u_{nt}^2(x,t) + u_{nt}^2(x,t-\tau) + v_{nt}^2(x,t) + v_{nt}^2(x,t-\tau) \} \, d\Gamma \, dt \\ & < C \int_0^T \int_{\Gamma_2} \{ u_{nt}^2(x,t) + u_{nt}^2(x,t-\tau) + v_{nt}^2(x,t) + v_{nt}^2(x,t-\tau) \} \, d\Gamma \, dt \\ & + C \{ \|u_n\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} + \|v_n\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} \} \end{aligned}$$

That is

$$\begin{aligned} (n - C) & \int_0^T \int_{\Gamma_2} \{ u_{nt}^2(x,t) + u_{nt}^2(x,t-\tau) + v_{nt}^2(x,t) + v_{nt}^2(x,t-\tau) \} \, d\Gamma \, dt \\ & < C \{ \|u_n\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} + \|v_n\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} \}, \end{aligned} \tag{1.53}$$

Renormalizing, we obtain a sequence (u_n, v_n) of solution of problem (1.1) – (1.9) verifying

$$\|u_n\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} + \|v_n\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} = 1, \tag{1.54}$$

and

$$\int_0^T \int_{\Gamma_2} \{u_{nt}^2(x,t) + u_{nt}^2(x,t-\tau) + v_{nt}^2(x,t) + v_{nt}^2(x,t-\tau)\}\, d\Gamma\, dt < \frac{C}{n-C} \quad for\, all \quad n > C, \tag{1.55}$$

From (1.52), (1.54) and (1.55), we deduce that the sequence $(u_n^0, u_n^1, g_n^0, v_n^0, v_n^1, h_n^0)$ is bounded in $\mathcal{H}$. Then there is a subsequence still denoted by $(u_n^0, u_n^1, g_n^0, v_n^0, v_n^1, h_n^0)$ that converges weakly to $(u^0, u^1, g^0, v^0, v^1, h^0) \in \mathcal{H}$ Let (u, v) be the solution of problem (1.1) – (1.9) with initial condition $(u^0, u^1, g^0, v^0, v^1, h^0)$. We have from Theorem 1.2

$$(u, v) \in C([0, +\infty); H^1_{\Gamma_1}(\Omega)) \times C([0, +\infty); H^1_{\Gamma_1}(\Omega)).$$

Then

$$(u_n, v_n) \longrightarrow (u, v) \text{ weakly in } L^2(0, T; H^1_{\Gamma_1}(\Omega)) \times L^2(0, T; H^1_{\Gamma_1}(\Omega)).$$

Since $H^1_{\Gamma_1}(\Omega)$ is compactly embedded in $H^{\frac{1}{2}+\varepsilon}(\Omega)$, there exist a subsequence which for simplicity of notation, we still denote by (u_n, v_n) such that,

$$(u_n, v_n) \longrightarrow (u, v) \text{ strongly in } L^2(0, T; H^{\frac{1}{2}+\varepsilon}(\Omega)) \times L^2(0, T; H^{\frac{1}{2}+\varepsilon}(\Omega)).$$

So, (1.54) leads to

$$\|u\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))} = 1. \tag{1.56}$$

and from (1.55), we have

$$\int_0^T \int_{\Gamma_2} \left\{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\right\} d\Gamma\, dt = 0.$$

Then

$$u_t(x,t) = v_t(x,t) = 0 \quad \text{on} \quad \Gamma_2 \times (0,T),$$

and

$$\frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_2 \times (0,T).$$

setting $\varphi := u_t, \psi := v_t$, thus (φ, ψ) satisfies

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) + l(\varphi(x,t) - \psi(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ \psi_{tt}(x,t) - \Delta\psi(x,t) + l(\psi(x,t) - \varphi(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ \varphi(x,t) = \psi(x,t) = 0 & \text{on} \quad \Gamma \times (0,T), \\ \dfrac{\partial\varphi(x,t)}{\partial\nu} = \dfrac{\partial\psi(x,t)}{\partial\nu} = 0 & \text{on} \quad \Gamma_2 \times (0,T). \end{cases} \tag{1.57}$$

Problem (1.57) implies to

$$\begin{cases} (\varphi+\psi)_{tt}(x,t) - \Delta(\varphi+\psi)(x,t) = 0 & \text{in} \quad \Omega \times (0,T), \\ (\varphi+\psi)(x,t) = 0 & \text{on} \quad \Gamma \times (0,T), \\ \dfrac{\partial(\varphi+\psi)(x,t)}{\partial\nu} = 0 & \text{on} \quad \Gamma_2 \times (0,T). \end{cases}$$

Therefore, from Holmgren's uniqueness Theorem (see **[21]**, p.92, Chap.I, Theorem 8.2), we conclude that

$$\varphi(x,t) + \psi(x,t) = 0.$$

So, we can rewrite the problem (1.57) as

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) + 2l\varphi(x,t) = 0 & \text{in} \quad \Omega \times (0,T), \\ \varphi(x,t) = 0 & \text{on} \quad \Gamma \times (0,T), \\ \dfrac{\partial \varphi(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma_2 \times (0,T). \end{cases}$$

We conclude from [**19**] and [**40**] that

$$\varphi(x,t) = \psi(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T).$$

This implies

$$u(x,t) = u(x), v(x,t) = v(x).$$

Thus (u,v) verifies

$$\begin{cases} -\Delta u(x) + l(u(x) - v(x)) = 0 & \text{in} \quad \Omega, \\ -\Delta v(x) + l(v(x) - u(x)) = 0 & \text{in} \quad \Omega, \\ u(x) = v(x) = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial u(x)}{\partial \nu} = \dfrac{\partial v(x)}{\partial \nu} = 0 & \text{on} \quad \Gamma_2. \end{cases}$$

The solution of the above problem is $(u,v) = (0,0)$, which contradicts (1.56). Then, the desired inequality (1.47) is proved. □

Now, we show the exponential decay of the energy
From (1.43), we have

$$E(T) - E(0) \leq -k \int_0^T \int_{\Gamma_2} \left\{ u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau) \right\} d\Gamma\, dt,$$

and the observability inequality (1.47) leads to

$$\begin{aligned} E(T) \leq E(0) &\leq C \int_0^T \int_{\Gamma_2} \left\{ u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau) \right\} d\Gamma\, dt \\ &\leq Ck^{-1}(E(0) - E(T)), \end{aligned}$$

so

$$E(T) \leq \frac{Ck^{-1}}{1 + Ck^{-1}} E(0). \tag{1.58}$$

Since we have $0 < C/(k+C) < 1$, the desired conclusion follows now from (1.58).

1.2. Stability of compactly coupled wave equations with delay terms in the internal feedbacks

1.1. Introduction. In this section, we study a stability problem of compactly coupled wave equations with delay terms in the internal feedbacks.
Let Ω be an open bounded domain of $\mathbb{R}^n$ with a boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that, $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$.
Furthermore we assume that there exists a scalar function $\Phi \in C^2(\overline{\Omega})$ such that
$(H.1)$ Φ is strictly convex in $\overline{\Omega}$; that is, there exists $\lambda > 0$ such that

$$H(x)\Theta.\Theta \geq \lambda\,|\Theta|^2 \qquad \forall\, x \in \overline{\Omega},\ \Theta \in \mathbb{R}^n,$$

where H is the Hessian matrix of Φ.
$(H.2)$ $h(x).\nu(x) \leq 0$ on Γ_1, where $h(x) = \nabla\Phi(x)$ and ν is the unit normal on Γ pointing towards the exterior of Ω.
We consider in Ω the following coupled system of two wave equations with delay terms occurring in both internal feedback:

$$\begin{aligned}
&u_{tt}(x,t) - \Delta u(x,t) + l(u(x,t) - v(x,t)) && \\
(1.59)\quad &+ a(x)\left(\alpha_1 u_t(x,t) + \alpha_2 u_t(x,t-\tau)\right) = 0 && \text{in } \Omega \times (0,+\infty),\\
&v_{tt}(x,t) - \Delta v(x,t) + l(v(x,t) - u(x,t)) && \\
(1.60)\quad &+ a(x)\left(\beta_1 v_t(x,t) + \beta_2 v_t(x,t-\tau)\right) = 0 && \text{in } \Omega \times (0,+\infty),\\
(1.61)\quad &u(x,0) = u_0(x), u_t(x,0) = u_1(x) && \text{in } \Omega,\\
(1.62)\quad &v(x,0) = v_0(x), v_t(x,0) = v_1(x) && \text{in } \Omega,\\
(1.63)\quad &u(x,t) = v(x,t) = 0 && \text{on } \Gamma_1 \times (0,+\infty),\\
(1.64)\quad &\frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = 0 && \text{on } \Gamma_2 \times (0,+\infty),\\
(1.65)\quad &u_t(x,t-\tau) = g(x,t-\tau) && \text{in } \Omega \times (0,\tau),\\
(1.66)\quad &v_t(x,t-\tau) = h(x,t-\tau) && \text{in } \Omega \times (0,\tau),
\end{aligned}$$

where $l, \alpha_1, \alpha_2, \beta_1$ and β_2 are positive constants, τ is the time delay, u_0, u_1, v_0, v_1, g and h are the initial data, $\frac{\partial}{\partial\nu}$ is the normal derivative, $a(.)$ is a function in $L^\infty(\Omega)$ such that

$$a(x) \geq 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad a(x) > a_0 > 0 \quad \text{a.e. in } \omega\ ,$$

where $\omega \subset \Omega$ is an open neighbourhood of Γ_2.
The subject of this section is to investigate the uniform exponential stability of system $(1.59) - (1.66)$ in the case where the interior damping coefficients $\alpha_1, \alpha_2, \beta_1$ and β_2 are strictly positive.

1.2. Main result. We assume as before that

$$(1.67)\qquad \alpha_1 > \alpha_2,\ \beta_1 > \beta_2,$$

and define the energy of a solution of $(1.59) - (1.66)$ by

$$\begin{aligned}
F(t) =& \frac{1}{2}\int_\Omega \left[|\nabla u(x,t)|^2 + u_t^2(x,t) + |\nabla v(x,t)|^2 + v_t^2(x,t) + l\left(u(x,t) - v(x,t)\right)^2\right] dx\\
(1.68)\qquad &+ \frac{1}{2}\int_\Omega a(x)\int_0^1 \left[\mu u_t^2(x,t-\tau\rho) + \xi v_t^2(x,t-\tau\rho)\right] d\rho\, dx,
\end{aligned}$$

where

$$\tau\alpha_2 < \mu < \tau(2\alpha_1 - \alpha_2),$$

and

$$\tau\beta_2 < \xi < \tau(2\beta_1 - \beta_2).$$

We show that if $\{\Omega, \Gamma_1, \Gamma_2\}$ satisfies $(H.1)$, $(H.2)$, then there is an exponential decay rate for $F(t)$.

The proof of this result as for the problem with boundary feedback by using a suitable observability estimate and using a compactness-uniqueness argument.
The main theorem of this section can be stated as follows.

THEOREM 1.3. *Assume* $(H.1)$, $(H.2)$, (1.67) *and* (1.68). *Then the coupled wave equations system* $(1.59)-(1.66)$ *is uniformly exponentially stable, i.e., there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$F(t) \leq M e^{-\omega t} F(0).$$

Theorem 1.3 is proved in Subsection 1.4. In Subsection 1.3, we study the well-posedness of system $(1.59)-(1.66)$ by using semigroup theory.

1.3. Well-posedness. We set

$$y(x,\rho,t) = u_t(x,t-\tau\rho), z(x,\rho,t) = v_t(x,t-\tau\rho) \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Problem $(1.59)-(1.66)$ is equivalent to

$$\begin{aligned}
&u_{tt}(x,t) - \Delta u(x,t) + l(u(x,t) - v(x,t)) \\
(1.69)\quad & \quad + a(x)(\alpha_1 u_t(x,t) + \alpha_2 y(x,1,t)) = 0 && \text{in } \Omega \times (0,+\infty),\\
(1.70)\quad & y_t(x,\rho,t) + \tau^{-1} y_\rho(x,\rho,t) = 0 && \text{in } \Omega \times (0,1) \times (0,+\infty),\\
& v_{tt}(x,t) - \Delta v(x,t) + l(v(x,t) - u(x,t)) \\
(1.71)\quad & \quad + a(x)(\beta_1 v_t(x,t) + \beta_2 z(x,1,t)) = 0 && \text{in } \Omega \times (0,+\infty),\\
(1.72)\quad & z_t(x,\rho,t) + \tau^{-1} z_\rho(x,\rho,t) = 0 && \text{in } \Omega \times (0,1) \times (0,+\infty),\\
(1.73)\quad & u(x,t) = v(x,t) = 0 && \text{on } \Gamma_1 \times (0,+\infty),\\
(1.74)\quad & \frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = 0 && \text{on } \Gamma_2 \times (0,+\infty),\\
(1.75)\quad & u(x,0) = u_0(x), u_t(x,0) = u_1(x) && \text{in } \Omega,\\
(1.76)\quad & v(x,0) = v_0(x), v_t(x,0) = v_1(x) && \text{in } \Omega,\\
(1.77)\quad & y(x,0,t) = u_t(x,t), z(x,0,t) = v_t(x,t) && \text{in } \Omega \times (0,+\infty),\\
(1.78)\quad & y(x,\rho,0) = g(x,-\tau\rho), z(x,\rho,0) = h(x,-\tau\rho) && \text{in } \Omega \times (0,1).
\end{aligned}$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Omega; L^2(0,1)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Omega; L^2(0,1)),$$

where

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}.$$

We endow $\mathcal{H}$ with the inner product

$$\left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \phi \\ \chi \\ \psi \end{pmatrix} ; \begin{pmatrix} \widetilde{\zeta} \\ \widetilde{\eta} \\ \widetilde{\theta} \\ \widetilde{\phi} \\ \widetilde{\chi} \\ \widetilde{\psi} \end{pmatrix} \right\rangle = \int_\Omega (\nabla\zeta(x).\nabla\widetilde{\zeta}(x) + \eta(x)\widetilde{\eta}(x))\, dx$$

$$+ \mu \int_\Omega a(x) \int_0^1 \theta(x,\rho)\widetilde{\theta}(x,\rho)\, d\rho\, dx + \int_\Omega (\nabla\phi(x).\nabla\widetilde{\phi}(x) + \chi(x)\widetilde{\chi}(x))\, dx$$

$$+ \xi \int_\Omega a(x) \int_0^1 \psi(x,\rho)\widetilde{\psi}(x,\rho)\, d\rho\, dx + l \int_\Omega (\zeta(x) - \phi(x))(\widetilde{\zeta}(x) - \widetilde{\phi}(x))\, dx.$$

Defined in $\mathcal{H}$ a linear operator A by

$$\begin{aligned} D(\mathcal{A}) =&\{(\zeta,\eta,\theta,\phi,\chi,\psi)^T \in \left(H^2(\Omega)\cap H^1_{\Gamma_1}(\Omega)\right)\times H^1_{\Gamma_1}(\Omega)\times L^2(\Omega;H^1(0,1))\times \\ &\left(H^2(\Omega)\cap H^1_{\Gamma_1}(\Omega)\right)\times H^1_{\Gamma_1}(\Omega)\times L^2(\Omega;H^1(0,1));\ \frac{\partial\zeta}{\partial\nu}=\frac{\partial\phi}{\partial\nu}=0 \text{ on } \Gamma_2 \\ &\eta=\theta(.,0), \chi=\psi(.,0) \text{ in } \Omega\}. \end{aligned} \tag{1.79}$$

$$\begin{aligned} A(\zeta,\eta,\theta,\phi,\chi,\psi)^T =& (\eta, \Delta\zeta + l\phi - l\zeta - a(\alpha_1\eta+\alpha_2\theta(.,1)), -\tau^{-1}\theta_\rho, \\ & \chi, \Delta\phi - l\phi + l\zeta - a(\beta_1\chi+\beta_2\psi(.,1)), -\tau^{-1}\psi_\rho)^T. \end{aligned} \tag{1.80}$$

Then we can rewrite (1.69) – (1.78) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \dfrac{d}{dt}W(t) = \mathcal{A}W(t), \\ W(0)=W_0, \end{cases} \tag{1.81}$$

where

$$\begin{aligned} W(t) &= (u(x,t), u_t(x,t), y(x,\rho,t), v(x,t), v_t(x,t), z(x,\rho,t))^T, \\ W_0 &= (u_0, u_1, g(.,-.\tau), v_0, v_1, h(.,-.\tau))^T. \end{aligned}$$

We verify that $\mathcal{A}$ generates a strongly continuous semigroup in $\mathcal{H}$ and consequently we have

THEOREM 1.4. *For every $W_0 \in \mathcal{H}$, problem* (1.81) *has a unique solution W whose regularity depends on the initial datum W_0 as follows:*

$$\begin{aligned} W(.) &\in C([0,+\infty);\mathcal{H}) \text{ if } W_0\in\mathcal{H}, \\ W(.) &\in C^1([0,+\infty);\mathcal{H})\cap C([0,+\infty);D(\mathcal{A})) \text{ if } W_0\in D(\mathcal{A}). \end{aligned}$$

PROOF. We will show that the operator $\mathcal{A}$ defined by (1.80) with the condition (1.67) generates a strongly continuous semigroup in $\mathcal{H}$ by using Lumer-Philips Theorem (see for instance [**34**], Theorem I.4.3).
First, we prove that the operator $\mathcal{A}$ is dissipative.
Let,

$$W = (\zeta,\eta,\theta,\phi,\chi,\psi)^T \in D(A). \quad \text{Then}$$

$$\begin{aligned} \langle AW;W\rangle =& \int_\Omega [\nabla\eta(x).\nabla\zeta(x) + (\Delta\zeta(x) + l\phi(x) - l\zeta(x) - a(x)(\alpha_1\eta(x)+\alpha_2\theta(x,1)))\,\eta(x)]\,dx \\ &+ \int_\Omega [\nabla\chi(x).\nabla\phi(x) + (\Delta\phi(x) - l\phi(x) + l\zeta(x) - a(x)(\beta_1\chi(x)+\beta_2\psi(x,1)))\,\chi(x)]\,dx \\ &+ l\int_\Omega (\eta(x)-\chi(x))(\zeta(x)-\phi(x))\,dx - \tau^{-1}\mu\int_\Omega a(x)\int_0^1 \theta_\rho(x,\rho)\theta(x,\rho)\,d\rho\,dx \\ &- \tau^{-1}\xi\int_\Omega a(x)\int_0^1 \psi_\rho(x,\rho)\psi(x,\rho)\,d\rho\,dx. \end{aligned}$$

By using Green's Theorem, integrating by parts in ρ and recalling (1.79), we get

$$\begin{aligned} \langle AW,W\rangle =& -\alpha_1\int_\Omega a(x)\eta^2(x)\,dx - \alpha_2\int_\Omega a(x)\eta(x)\theta(x,1)\,dx - \beta_1\int_\Omega a(x)\chi^2(x)\,dx \\ &- \beta_2\int_\Omega a(x)\chi(x)\psi(x,1)\,dx - \frac{\tau^{-1}\mu}{2}\int_\Omega a(x)[\theta^2(x,1)-\theta^2(x,0)]\,dx \\ &- \frac{\tau^{-1}\xi}{2}\int_\Omega a(x)[\psi^2(x,1)-\psi^2(x,0)]\,dx, \end{aligned}$$

from which follows using the Cauchy-Schwarz inequality

$$\begin{aligned} \langle AW,W\rangle \le& (-\alpha_1 + \frac{\alpha_2}{2} + \frac{\tau^{-1}\mu}{2})\int_\Omega a(x)\eta^2(x)\,dx + (\frac{\alpha_2}{2} - \frac{\tau^{-1}\mu}{2})\int_\Omega a(x)\theta^2(x,1)\,dx \\ &+ (-\beta_1 + \frac{\beta_2}{2} + \frac{\tau^{-1}\xi}{2})\int_\Omega a(x)\chi^2(x)\,dx + (\frac{\beta_2}{2} - \frac{\tau^{-1}\xi}{2})\int_\Omega a(x)\psi^2(x,1)\,dx. \end{aligned}$$

From (1.67), we conclude that $(AW, W) \leq 0$, thus A is dissipative.
Let's show now that $\lambda I - A$ is onto for a fixed $\lambda > 0$.
Let $(f, g, h, k, m, p)^T \in \mathcal{H}$, we seek a $W = (\zeta, \eta, \theta, \phi, \chi, \psi)^T \in D(A)$ solution of

$$(\lambda I - A)W = (f, g, h, k, m, p).$$

or equivalently

$$\lambda\zeta - \eta = f, \tag{1.82}$$

$$\lambda\eta - \Delta\zeta + l\zeta - l\phi + a(\alpha_1\eta + \alpha_2\theta(., 1)) = g, \tag{1.83}$$

$$\lambda\theta + \tau^{-1}\frac{d\theta}{d\rho} = h, \tag{1.84}$$

$$\lambda\phi - \chi = k, \tag{1.85}$$

$$\lambda\chi - \Delta\phi + l\phi - l\zeta + a(\beta_1\chi + \beta_2\psi(., 1)) = m, \tag{1.86}$$

$$\lambda\psi + \tau^{-1}\frac{d\psi}{d\rho} = p. \tag{1.87}$$

Suppose that we have found ζ and ϕ with the appropriate regularity, then

$$\eta = \lambda\zeta - f, \tag{1.88}$$

$$\chi = \lambda\phi - k. \tag{1.89}$$

Consequently we can determine θ from (1.84) with (1.79) and ψ from (1.87) with (1.79). In fact, θ is the unique solution of the initial value problem :

$$\begin{aligned} &\theta_\rho(x, \rho) = -\tau\lambda\theta(x, \rho) + \tau h(x, \rho),\ x \in \Omega,\ \rho \in (0, 1), \\ &\theta(x, 0) = \eta(x),\ x \in \Omega. \end{aligned}$$

And ψ is the unique solution of the initial value problem

$$\begin{aligned} &\psi_\rho(x, \rho) = -\tau\lambda\psi(x, \rho) + \tau p(x, \rho),\ x \in \Omega,\ \rho \in (0, 1), \\ &\psi(x, 0) = \chi(x),\ \ x \in \Omega. \end{aligned}$$

Therefore

$$\theta(x, \rho) = \eta(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho h(x, \sigma)e^{\lambda\tau\sigma}d\sigma(x),\ x \in \Omega,\ \rho \in (0, 1),$$

and

$$\psi(x, \rho) = \chi(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho p(x, \sigma)e^{\lambda\tau\sigma}d\sigma(x),\ x \in \Omega,\ \rho \in (0, 1),$$

and in particular

$$\begin{aligned} \theta(x, 1) &= \lambda e^{-\lambda\tau}\zeta(x) + z_0(x), \\ \psi(x, 1) &= \lambda e^{-\lambda\tau}\phi(x) + z_1(x). \end{aligned}$$

with z_0 and z_1 defined by

$$\begin{aligned} z_0(x) &= -f(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 h(x, \sigma)e^{\lambda\tau\sigma}d\sigma(x),\ \ x \in \Omega, \\ z_1(x) &= -k(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 p(x, \sigma)e^{\lambda\tau\sigma}d\sigma(x),\ \ x \in \Omega. \end{aligned}$$

From (1.83), (1.86), (1.88) and (1.89) the functions ζ and ϕ verify

$$\begin{cases} \lambda^2\zeta - \Delta\zeta + l\zeta - l\phi + a(\alpha_1\eta + \alpha_2\theta(., 1)) = g + \lambda f, \\ \lambda^2\phi - \Delta\phi + l\phi - l\zeta + a(\beta_1\chi + \beta_2\psi(., 1)) = m + \lambda k. \end{cases} \tag{1.90}$$

Problem (1.90) can be reformulated as

$$\int_\Omega (\lambda^2\zeta(x) - \Delta\zeta(x) + l\zeta(x) - l\phi(x) + a(x)(\alpha_1\eta(x) + \alpha_2\theta(x,1)))w_1(x)\,dx$$
$$+ \int_\Omega (\lambda^2\phi(x) - \Delta\phi(x) + l\phi(x) - l\zeta(x) + a(x)(\beta_1\chi(x) + \beta_2\psi(x,1)))w_2(x)\,dx$$
$$= \int_\Omega (g(x) + \lambda f(x))w_1(x)\,dx + \int_\Omega (m(x) + \lambda k(x))w_2(x)\,dx, \qquad (w_1, w_2) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$$

Using Green's Theorem and recalling (1.79), we rewrite the left-hand side of the last equality as follow

$$\int_\Omega \left(\lambda^2\zeta(x) - \Delta\zeta(x) + l\zeta(x) - l\phi(x) + a(x)\left(\alpha_1\eta(x) + \alpha_2\theta(x,1)\right)\right) w_1(x)\,dx$$
$$+ \int_\Omega \left(\lambda^2\phi(x) - \Delta\phi(x) + l\phi(x) - l\zeta(x) + a(x)\left(\beta_1\chi(x) + \beta_2\psi(x,1)\right)\right) w_2(x)\,dx$$
$$= \int_\Omega (\lambda^2\zeta w_1 + \nabla\zeta.\nabla w_1)\,dx + \int_\Omega a\left(\alpha_1(\lambda\zeta - f) + \alpha_2(\lambda e^{-\lambda\tau}\zeta + z_0)\right) w_1\,dx$$
$$+ \int_\Omega (\lambda^2\phi w_2 + \nabla\phi.\nabla w_2)\,dx + \int_\Omega a\left(\beta_1(\lambda\phi - k) + \beta_2(\lambda e^{-\lambda\tau}\chi + z_1)\right) w_2\,dx$$
$$+ \int_\Omega (l\zeta - l\phi)w_1\,dx + \int_\Omega (l\phi - l\zeta)w_2\,dx.$$

Therefore

$$\int_\Omega (\lambda^2\zeta w_1 + \nabla\zeta.\nabla w_1)\,dx + \int_\Omega \lambda a(\alpha_1 + \alpha_2 e^{-\lambda\tau})\zeta w_1 + \int_\Omega (\lambda^2\phi w_2 + \nabla\phi.\nabla w_2)\,dx$$
$$+ \int_\Omega \lambda a(\beta_1 + \beta_2 e^{-\lambda\tau})\phi w_2\,dx + \int_\Omega l(\zeta - \phi)(w_1 - w_2)\,dx$$
$$= \int_\Omega (g + \lambda f)w_1\,dx + \int_\Omega (m + \lambda k)w_2\,dx + \alpha_1\int_\Omega afw_1\,dx + \beta_1\int_\Omega akw_2\,dx$$
$$- \alpha_2\int_\Omega az_0w_1\,dx - \beta_2\int_\Omega az_1w_2\,dx \qquad \forall(w_1, w_2) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega). \tag{1.91}$$

Since the left-hand side of (1.91) is coercive and continuous on $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, and the right-hand side defines a continuous linear form on $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, the Lax-Milgram's Theorem guarantees the existence and uniqueness of a solution $(\zeta, \phi) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ of (1.91).
If we consider $(w_1, w_2) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ in (1.91), then (ζ, ϕ) is a solution in $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)$ of

$$\begin{cases} \lambda^2\zeta - \Delta\zeta + l\zeta - l\phi + a(\alpha_1\eta + \alpha_2\theta(.,1)) = g + \lambda f, \\ \lambda^2\phi - \Delta\phi + l\phi - l\zeta + a(\beta_1\chi + \beta_2\psi(.,1)) = m + \lambda k. \end{cases} \tag{1.92}$$

Thus $(\Delta\zeta, \Delta\phi) \in L^2(\Omega) \times L^2(\Omega)$.
From (1.91) after using Green's Theorem and recalling (1.92), we obtain

$$\frac{\partial\zeta}{\partial\nu} = \frac{\partial\phi}{\partial\nu} = 0 \qquad \text{on } \Gamma_2.$$

So, we have found $(\zeta, \eta, \theta, \phi, \chi, \psi)^T \in D(A)$ which verifies (1.82) – (1.87).
By Lumer-Phillips Theorem, A generates a C_0- semigroup of contractions on $\mathcal{H}$. □

1.4. Proof of the main result. We prove the Theorem 1.3 for smooth initial data. First, we prove that the energy function $F(t)$ defined by (1.68) is decreasing.

PROPOSITION 1.3. *The energy corresponding to any regular solution of problem* (1.59) – (1.66), *is decreasing and there exists a positive constant C such that,*

$$\frac{d}{dt}F(t) \le -C\int_\Omega a(x)\left\{u_t^2(x,t)+u_t^2(x,t-\tau)+v_t^2(x,t)+v_t^2(x,t-\tau)\right\}dx, \tag{1.93}$$

where

$$C=\min\left\{\alpha_1-\frac{\alpha_2}{2}-\frac{\mu}{2\tau},\frac{\mu}{2\tau}-\frac{\alpha_2}{2},\beta_1-\frac{\beta_2}{2}-\frac{\xi}{2\tau},\frac{\xi}{2\tau}-\frac{\beta_2}{2}\right\}.$$

PROOF. Differentiating F(t) defined by (1.68) with respect to time, applying Green's Theorem and recalling the boundary condition (1.73) and (1.74), we obtain

$$\begin{aligned}\frac{d}{dt}F(t)=&-\alpha_1\int_\Omega a(x)u_t^2(x,t)\,dx-\alpha_2\int_\Omega a(x)u_t(x,t)u_t(x,t-\tau)\,dx\\&-\beta_2\int_\Omega a(x)v_t^2(x,t)\,dx-\beta_2\int_\Omega a(x)v_t(x,t)v_t(x,t-\tau)\,dx\\&+\int_\Omega\int_0^1 a(x)\{\mu u_{tt}(x,t-\tau\rho)u_t(x,t-\tau\rho)+\xi v_{tt}(x,t-\tau\rho)v_t(x,t-\tau\rho)\}d\rho\,dx.\end{aligned}\tag{1.94}$$

Now, we have

$$\begin{aligned}&\int_\Omega\int_0^1 u_t(x,t-\tau\rho)u_{tt}(x,t-\tau\rho)\,d\rho\,dx+\int_\Omega\int_0^1 v_t(x,t-\tau\rho)v_{tt}(x,t-\tau\rho)\,d\rho\,dx=\\&\frac{\tau^{-1}}{2}\int_\Omega\{u_t^2(x,t)-u_t^2(x,t-\tau)\}\,dx+\frac{\tau^{-1}}{2}\int_{\Gamma_2}\{v_t^2(x,t)-v_t^2(x,t-\tau)\}\,dx.\end{aligned}\tag{1.95}$$

Insertion (1.95) into (1.94) and applying Cauchy-Schwarz inequality, yields

$$\begin{aligned}\frac{d}{dt}F(t)\le&(-\alpha_1+\frac{\alpha_2}{2}+\frac{\mu\tau^{-1}}{2})\int_\Omega a(x)u_t^2(x,t)\,dx+(\frac{\alpha_2}{2}-\frac{\mu\tau^{-1}}{2})\int_\Omega a(x)u_t^2(x,t-\tau)\,dx\\&+(-\beta_1+\frac{\beta_2}{2}+\frac{\xi\tau^{-1}}{2})\int_\Omega a(x)v_t^2(x,t)\,dx+(\frac{\beta_2}{2}-\frac{\xi\tau^{-1}}{2})\int_\Omega a(x)v_t^2(x,t-\tau)\,dx,\end{aligned}$$

which implies

$$\frac{d}{dt}F(t)\le -C\int_\Omega a(x)\{u_t^2(x,t)+u_t^2(x,t-\tau)+v_t^2(x,t)+v_t^2(x,t-\tau)\}\,dx,$$

with

$$C=\min\left\{\alpha_1-\frac{\alpha_2}{2}-\frac{\mu}{2\tau},\frac{\mu}{2\tau}-\frac{\alpha_2}{2},\beta_1-\frac{\beta_2}{2}-\frac{\xi}{2\tau},\frac{\xi}{2\tau}-\frac{\beta_2}{2}\right\}.$$

□

Now we give an observability inequality which we will use it to prove the exponential decay of the energy $F(t)$.

PROPOSITION 1.4. *There exists a time* T^* *such that for all* $T > T^*$, *there exists a positive constant C (depending on T) such that*

$$F(0)\le C\int_0^T\int_\Omega a(x)\left\{u_t^2(x,t)+u_t^2(x,t-\tau)+v_t^2(x,t)+v_t^2(x,t-\tau)\right\}dx\,dt, \tag{1.96}$$

for any regular solution of problem (1.59) – (1.66).

PROOF. We rewrite $F(t)=F_s(t)+F_d(t)$,
where

$$F_s(t)=\frac{1}{2}\int_\Omega\left[|\nabla u(x,t)|^2+u_t^2(x,t)+|\nabla v(x,t)|^2+v_t^2(x,t)+l(u(x,t)-v(x,t))^2\right]dx,$$

and

$$F_d(t)=\frac{1}{2}\int_\Omega a(x)\int_0^1\{u_t^2(x,t-\tau\rho)+v_t^2(x,t-\tau\rho)\}\,d\rho\,dx.$$

By a change of variable, we obtain for $T > \tau$

$$F_d(0) \leq C \int_\Omega a(x) \int_0^T \left\{u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\right\} dt\, dx. \tag{1.97}$$

Now, we decompose the solution (u,v) as follows

$$u = y + \widetilde{y},\ v = z + \widetilde{z},$$

where (y,z) solves

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) + l(y(x,t) - z(x,t)) = 0 & \text{in } \Omega \times (0,+\infty), \\ z_{tt}(x,t) - \Delta z(x,t) + l(z(x,t) - y(x,t)) = 0 & \text{in } \Omega \times (0,+\infty), \\ y(x,0) = u_0(x), y_t(x,0) = u_1(x) & \text{in } \Omega, \\ z(x,0) = v_0(x), z_t(x,0) = v_1(x) & \text{in } \Omega, \\ y(x,t) = z(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial y(x,t)}{\partial \nu} = \frac{\partial z(x,t)}{\partial \nu} = 0 & \text{on } \Gamma_2 \times (0,+\infty), \end{cases} \tag{1.98}$$

and $(\widetilde{y},\widetilde{z})$ is the solution of :

$$\begin{cases} \widetilde{y}_{tt} - \Delta\widetilde{y} + l(\widetilde{y} - \widetilde{z}) + a(x)(\alpha_1 u_t(x,t) + \alpha_2 u_t(x,t-\tau)) = 0 & \text{in } \Omega \times (0,+\infty), \\ \widetilde{z}_{tt} - \Delta\widetilde{z} + l(\widetilde{z} - \widetilde{y}) + a(x)(\beta_1 v_t(x,t) + \beta_2 v_t(x,t-\tau)) = 0 & \text{in } \Omega \times (0,+\infty), \\ \widetilde{y}(x,0) = \widetilde{y}_t(x,0) = 0 & \text{in } \Omega, \\ \widetilde{z}(x,0) = \widetilde{z}_t(x,0) = 0 & \text{in } \Omega, \\ \widetilde{y} = \widetilde{z} = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial\widetilde{y}}{\partial\nu} = \frac{\partial\widetilde{z}}{\partial\nu} = 0 & \text{on } \Gamma_2 \times (0,+\infty), \end{cases} \tag{1.99}$$

Denote by $\mathcal{E}(t)$ the standard energy of (1.98), that is

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \{|\nabla y(x,t)|^2 + y_t^2(x,t) + |\nabla z(x,t)|^2 + z_t^2(x,t) + l(y(x,t) - z(x,t))^2\}\, dx,$$

and by $\widetilde{\mathcal{E}}(t)$ the standard energy of (1.99),

$$\widetilde{\mathcal{E}}(t) = \frac{1}{2}\int_\Omega \{|\nabla\widetilde{y}(x,t)|^2 + \widetilde{y}_t^2(x,t) + |\nabla\widetilde{z}(x,t)|^2 + \widetilde{z}_t^2(x,t) + l(\widetilde{y}(x,t) - \widetilde{z}(x,t))^2\}\, dx.$$

Concerning $\mathcal{E}(0)$, we have the following result

PROPOSITION 1.5. *There exists a time T_0 such that for all $T > T_0$, there exists a positive constant C_1(depending on T) for which*

$$\mathcal{E}(0) \leq C_1 \int_0^T \int_\omega \left\{y_t^2(x,t) + z_t^2(x,t)\right\} dx\, dt, \tag{1.100}$$

for any regular solution (y,z) solution of (1.98).

PROOF. We proceed as in Nicaise and Pignotti [**28**].
So, let ω_0, ω_1 be open neighbourhoods of Γ_2 such that

$$\omega \supset \omega_0 \supset \omega_1 \supset \Gamma_2. \tag{1.101}$$

Let φ be a smooth function such that

$$0 \leq \varphi(x) \leq 1, \qquad \varphi \equiv 0 \quad \text{on} \quad \Omega\backslash\omega_0, \qquad \varphi \equiv 1 \quad \text{on} \quad \omega_1. \tag{1.102}$$

Then $(\varphi y, \varphi z)$ verifies,

$$\begin{aligned} &(\varphi y)_{tt} - \Delta(\varphi y) = F(y) + f(z), \\ &(\varphi z)_{tt} - \Delta(\varphi z) = F(z) + f(y), \\ &(\varphi y) = (\varphi z) = 0 && \text{on } \Gamma_1 \times (0,+\infty), \\ &\frac{\partial(\varphi y)}{\partial\nu} = \frac{\partial(\varphi z)}{\partial\nu} = 0 && \text{on } \Gamma_2 \times (0,+\infty), \end{aligned} \tag{1.103}$$

where $F(y) = -y\Delta\varphi - 2\nabla y.\nabla\varphi - ly\varphi$ and $f(y) = ly\varphi$.
We apply to $(\varphi y, \varphi z)$ Proposition 2.2.1. of [**16**]. Let us recall some notation from ([**16**],

[**18**]).
Denote

$$T_0 = 2\left(\frac{\max_{x\in\overline{\Omega}}\Phi(x)}{\lambda}\right)^{\frac{1}{2}}, \tag{1.104}$$

where λ as in assumption $(H.1)$. Define the function $\phi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\phi(x,t) \equiv \Phi(x) - c\left(t - \frac{T}{2}\right)^2, \tag{1.105}$$

where $T > T_0$ is fixed and c is a constant chosen as follows. From (1.104), there exists a constant $\delta > 0$ such that

$$\max_{x\in\overline{\Omega}}\Phi(x) + 4\delta < \lambda T^2.$$

For fixed δ, there is c such that

$$\max_{x\in\overline{\Omega}}\Phi(x) + 4\delta < cT^2, \qquad c \in (0,\lambda). \tag{1.106}$$

Note that

$$\phi(x,0) < -\delta < 0; \phi(x,T) < -\delta < 0, \quad \text{uniformly in } x \in \Omega. \tag{1.107}$$

We remark that ϕ satisfies:

$$\phi_t(x,t) = -2c(t - \frac{T}{2}); \quad \phi_{tt} = -2c; \quad \phi_t(x,0) = cT; \quad \phi_t(x,T) = -cT, \tag{1.108}$$

$$\mathrm{div}(e^{\gamma\phi}h) = e^{\gamma\phi}[\gamma|h|^2 + \mathrm{div}h], \qquad \nabla(e^{\gamma\phi}) = \gamma e^{\gamma\phi}\nabla\phi.$$

We have

$$\begin{aligned}
&\int_0^T\int_\Gamma e^{\gamma\phi}\frac{\partial(\varphi y)}{\partial\nu}\left[\nabla(\varphi y).h - \phi_t(\varphi y)_t\right]d\Gamma\,dt + \int_0^T\int_\Gamma e^{\gamma\phi}\frac{\partial(\varphi z)}{\partial\nu}[\nabla(\varphi z).h - \phi_t(\varphi z)_t]\,d\Gamma\,dt\\
&+\frac{1}{2}\int_0^T\int_\Gamma e^{\gamma\phi}\left[(\varphi y)_t^2 - |\nabla(\varphi y)|^2\right]h.\nu\,d\Gamma\,dt + \frac{1}{2}\int_0^T\int_\Gamma e^{\gamma\phi}\left[(\varphi z)_t^2 - |\nabla(\varphi z)|^2\right]h.\nu\,d\Gamma\,dt\\
&= \int_0^T\int_\Omega e^{\gamma\phi}H\nabla(\varphi y).\nabla(\varphi y)\,dx\,dt + \int_0^T\int_\Omega e^{\gamma\phi}H\nabla(\varphi z).\nabla(\varphi z)\,dx\,dt\\
&+\frac{1}{2}\int_0^T\int_\Omega[(\varphi y)_t^2 - |\nabla(\varphi y)|^2]\,\mathrm{div}(e^{\gamma\phi}h)\,dx\,dt + \frac{1}{2}\int_0^T\int_\Omega\left[(\varphi z)_t^2 - |\nabla(\varphi z)|^2\right]\,\mathrm{div}(e^{\gamma\phi}h)\,dx\,dt\\
&+\frac{1}{2}\int_0^T\int_\Omega\left[(\varphi y)_t^2 + |\nabla(\varphi y)|^2\right]\,(e^{\gamma\phi}\phi_t)_t\,dx\,dt + \frac{1}{2}\int_0^T\int_\Omega\left[(\varphi z)_t^2 + |\nabla(\varphi z)|^2\right]\,(e^{\gamma\phi}\phi_t)_t\,dx\,dt\\
&-2\gamma\int_0^T\int_\Omega e^{\gamma\phi}h.\nabla(\varphi y)\phi_t(y\varphi)_t\,dx\,dt + \gamma\int_0^T\int_\Omega e^{\gamma\phi}\left(h.\nabla(\varphi y)\right)^2\,dx\,dt - \frac{1}{2}\left[\int_\Omega e^{\gamma\phi}\phi_t|\nabla(\varphi y)|^2\,dx\right]_0^T\\
&-2\tau\int_0^T\int_\Omega e^{\gamma\phi}h.\nabla(\varphi z)\phi_t(z\varphi)_t\,dx\,dt + \gamma\int_0^T\int_\Omega e^{\gamma\phi}\left(h.\nabla(\varphi z)\right)^2\,dx\,dt - \frac{1}{2}\left[\int_\Omega e^{\gamma\phi}\phi_t|\nabla(\varphi z)|^2\,dx\right]_0^T\\
&+\left[\int_\Omega e^{\gamma\phi}\left(h.\nabla(\varphi y) - \frac{1}{2}\phi_t(\varphi y)_t\right)(\varphi y)_t\,dx\right]_0^T + \left[\int_\Omega e^{\gamma\phi}\left(h.\nabla(\varphi z) - \frac{1}{2}\phi_t(\varphi z)_t\right)(\varphi z)_t\,dx\right]_0^T\\
&-\int_0^T\int_\Omega[F(y) + f(z)]\,e^{\gamma\phi}\left[\nabla(\varphi y).h - \phi_t(\varphi y)_t\right]\,dx\,dt\\
&-\int_0^T\int_\Omega[F(z) + f(y)]\,e^{\gamma\phi}\left[\nabla(\varphi z).h - \phi_t(\varphi z)_t\right]\,dx\,dt.
\end{aligned} \tag{1.109}$$

Inserting the boundary conditions (1.103) on the left-hand side of (1.109), we obtain

$$\begin{aligned} BT\omega|_{\Sigma} =& \frac{1}{2}\int_{\Sigma_1} e^{\gamma\phi}(\frac{\partial y}{\partial \nu})^2 h.\nu \, d\Sigma + \frac{1}{2}\int_{\Sigma_2} e^{\gamma\phi}(y_t^2 - |\nabla y|^2)h.\nu \, d\Sigma \\ &+ \frac{1}{2}\int_{\Sigma_1} e^{\gamma\phi}(\frac{\partial z}{\partial \nu})^2 h.\nu \, d\Sigma + \frac{1}{2}\int_{\Sigma_2} e^{\gamma\phi}(z_t^2 - |\nabla z|^2)h.\nu \, d\Sigma. \end{aligned}$$

Then

$$\begin{aligned} BT\omega|_{\Gamma\times(0,T)} \leq& c\int_0^T \int_{\omega} e^{\gamma\phi}\{|\nabla(\varphi y)|^2 + |\nabla(\varphi z)|^2\} \, dx \, dt + c\int_0^T \int_{\omega} \{y_t^2 + z_t^2\} \, dx \, dt \\ &+ c\int_0^T \int_{\Omega} \{y^2 + z^2\} \, dt \, dx + e^{-\delta\tau}\mathcal{E}'(0). \end{aligned} \tag{1.110}$$

Now, consider another smooth function ψ such that

$$0 \leq \psi(x) \leq 1, \qquad \psi \equiv 0 \quad \text{on} \quad \Omega\backslash\omega, \qquad \psi \equiv 1 \quad \text{on} \quad \omega_0. \tag{1.111}$$

We have

$$\int_0^T \int_{\Omega} (y_{tt} - \Delta y)\psi y e^{\gamma\phi} \, dx \, dt = \int_0^T \int_{\Omega} l(z-y)\psi y e^{\gamma\phi} \, dx \, dt.$$

Integrating by parts, we obtain

$$\begin{aligned} \int_0^T \int_{\Omega} \psi|\nabla y|^2 e^{\gamma\phi} \, dx \, dt =& -\left[\int_{\Omega} \psi y y_t e^{\gamma\phi} \, dx\right]_0^T + \int_0^T \int_{\Omega} y_t^2 \psi e^{\gamma\phi} \, dx \, dt \\ &+ \int_0^T \int_{\Omega} \gamma y_t \psi y e^{\gamma\phi}\phi_t \, dx \, dt - \int_0^T \int_{\Omega} \nabla y.\nabla(\psi e^{\gamma\phi}) y \, dx \, dt \\ &+ \int_0^T \int_{\Omega} l(z-y)\psi y e^{\gamma\phi} \, dx \, dt \\ =& -\left[\int_{\Omega} \psi y y_t e^{\gamma\phi} \, dx\right]_0^T + \int_0^T \int_{\Omega} y_t^2 \psi e^{\gamma\phi} \, dx \, dt \\ &+ \int_0^T \int_{\Omega} \gamma y_t \psi y e^{\gamma\phi}\phi_t \, dx \, dt - 2\int_0^T \int_{\Omega} e^{\gamma\phi} y\sqrt{\psi}\nabla(\sqrt{\psi}).\nabla y \, dx \, dt \\ &- \int_0^T \int_{\Omega} \psi y \nabla y.\nabla e^{\gamma\phi} \, dx \, dt + \int_0^T \int_{\Omega} l(z-y)\psi y e^{\gamma\phi} \, dx \, dt. \end{aligned}$$

Applying Cauchy-Schwarz inequality together and Poincaré's inequality and recalling the fact that the energy $\mathcal{E}(.)$ is conserved, we obtain

$$\begin{aligned} \int_0^T \int_{\Omega} \psi\{|\nabla y|^2 + |\nabla z|^2\} e^{\gamma\phi} \, dx \, dt \leq& ce^{-\delta\gamma}\mathcal{E}(0) + \frac{1}{2}\int_0^T \int_{\Omega} \psi\{|\nabla y|^2 + |\nabla z|^2\} e^{\gamma\phi} \, dx \, dt \\ &+ c\int_0^T \int_{\omega} \{y_t^2 + z_t^2\} \, dx \, dt + c\int_0^T \int_{\Omega} \{y^2 + z^2\} \, dx \, dt. \end{aligned}$$

Consequently

$$\begin{aligned} \int_0^T \int_{\omega_0} \{|\nabla y|^2 + |\nabla z|^2\} e^{\gamma\phi} \, dx \, dt \leq& ce^{-\delta\tau}\mathcal{E}(0) + c\int_0^T \int_{\omega} \{y_t^2 + z_t^2\} \, dx \, dt \\ &+ c\int_0^T \int_{\Omega} \{y^2 + z^2\} \, dx \, dt. \end{aligned}$$

Inserting the last inequality in (1.110), gives

$$BT\omega|_{\Sigma} \leq c\int_0^T \int_{\omega} \{y_t^2 + z_t^2\} \, dx \, dt + c\int_0^T \int_{\Omega} \{y^2 + z^2\} \, dt \, dx + e^{-\delta\tau}\mathcal{E}(0). \tag{1.112}$$

From Proposition 3.3. of [**16**], we have

$$BT\omega(y,z)|_{\Sigma} = BT\omega(y)|_{\Sigma} + BT\omega(z)|_{\Sigma} + K_{T,\tau,c,\delta}\int_{\Sigma}|\frac{\partial y}{\partial \nu}y_t + \frac{\partial z}{\partial \nu}z_t|\,d\Sigma,$$

and

$$BT\omega(y,z)|_{\Sigma} + const_{T,\tau}\{\|y\|^2_{C([0,T];L^2(\Omega))} + \|z\|^2_{C([0,T];L^2(\Omega))}\} > K\mathcal{E}(0),$$

where $K = (1 - c - \frac{2C_T}{\tau})K_{t_0,t_1}e^{\tau} - C_1Te^{-\delta\tau}(1 + e^{C_TT})$.
Then

$$BT\omega|_{\Sigma} > K\mathcal{E}(0) - const_{T,\tau}\{\|y\|^2_{C([0,T];L^2(\Omega))} + \|z\|^2_{C([0,T];L^2(\Omega))}\}. \tag{1.113}$$

From (1.112) and (1.113) we obtain

$$\mathcal{E}(0) \le C_1\int_0^T\int_{\omega}\{y_t^2(x,t) + z_t^2(x,t)\}\,dx\,dt + C_1\{\|y\|^2_{C(0,T;L^2(\Omega))} + \|z\|^2_{C(0,T;L^2(\Omega))}\}. \tag{1.114}$$

Now, we prove by a compactness-uniqueness argument that there exists a constant $C_1 > 0$ such that

$$\|y\|^2_{C(0,T;L^2(\Omega))} + \|z\|^2_{C(0,T;L^2(\Omega))} \le C_1\int_0^T\int_{\omega}\{y_t^2(x,t) + z_t^2(x,t)\}\,dx\,dt \tag{1.115}$$

Assume that there exists a sequence $(y_n,\ z_n)$ of solutions of problem (1.98) with

$$y_n(x,0) = y_n^0(x), y_{nt}(x,0) = y_n^1(x), \quad x \in \Omega,$$
$$z_n(x,0) = z_n^0(x), z_{nt}(x,0) = z_n^1(x), \quad x \in \Omega.$$

such that

$$\|y_n\|^2_{C(0,T;L^2(\Omega))} + \|z_n\|^2_{C(0,T;L^2(\Omega))} = 1, \quad n = 1,2,...;$$

$$\int_0^T\int_{\omega}\left\{y_{nt}^2(x,t) + z_{nt}^2(x,t)\right\}\,dx\,dt \to 0 \text{ as } n \to +\infty \tag{1.116}$$

Since each solution satisfies (1.114), we deduce from (1.116) that the sequence $(y_n^0, y_n^1, z_n^0, z_n^1)$ is bounded in $H^1_{\Gamma_1}(\Omega)\times L^2(\Omega)\times H^1_{\Gamma_1}(\Omega)\times L^2(\Omega)$. Hence there is a subsequence still denoted by $(y_n^0, y_n^1, z_n^0, z_n^1)$ which converges weakly to (y^0, y^1, z^0, z^1). Let (y,z) be the solution of problem (1.98) corresponding to such initial conditions. We have

$$(y,z) \in C(0,T;H^1_{\Gamma_1}(\Omega)) \times C(0,T;H^1_{\Gamma_1}(\Omega)).$$

It then follows that

$$(y_n,z_n) \longrightarrow (y,z) \text{ weakly in } C(0,T;H^1_{\Gamma_1}(\Omega)) \times C(0,T;H^1_{\Gamma_1}(\Omega)).$$

Since $H^1_{\Gamma_1}(\Omega)$ is compactly embedded in $L^2(\Omega)$, there exist a subsequence which for simplicity of notation, we still denote by (y_n,z_n) such that,

$$(y_n,z_n) \longrightarrow (y,z) \text{ strongly in } C(0,T;L^2(\Omega)) \times C(0,T;L^2(\Omega)).$$

So, (1.116) leads to

$$\|y\|^2_{C(0,T;L^2(\Omega))} + \|z\|^2_{C(0,T;L^2(\Omega))} = 1, \tag{1.117}$$

and

$$\int_0^T\int_{\omega}\{y_t^2(x,t) + z_t^2(x,t)\}dx\,dt = 0.$$

Then

$$y_t(x,t) = z_t(x,t) = 0 \quad \text{in} \quad \omega\times(0,T),$$

which means

$$y_t(x,t) = z_t(x,t) = 0 \quad \text{on} \quad \Gamma_2\times(0,T),$$

setting $\varphi := y_t, \psi := z_t$, thus (φ, ψ) satisfies

$$(1.118) \qquad \begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) + l(\varphi(x,t) - \psi(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ \psi_{tt}(x,t) - \Delta\psi(x,t) + l(\psi(x,t) - \varphi(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ \varphi(x,t) = \psi(x,t) = 0 & \text{on} \quad \Gamma \times (0,T), \\ \dfrac{\partial\varphi}{\partial\nu}(x,t) = \dfrac{\partial\psi}{\partial\nu}(x,t) = 0 & \text{on} \quad \Gamma_2 \times (0,T). \end{cases}$$

We conclude from the previous chapter that the solution of the above problem is $(\varphi, \psi) = (0,0)$ and $(y,z) = (0,0)$, which contradicts (1.117). Then, the desired inequality (1.100) is proved.

□

Completion of the proof of Proposition 1.4.
We have

$$F(0) = F_s(0) + F_d(0) = \mathcal{E}(0) + F_d(0),$$

If we take $T > T^* := \max\{T_0, \tau\}$, we get from (1.97) and (1.100)

$$\begin{aligned} F(0) \leq & C_1 \int_0^T \int_\omega \{y_t^2(x,t) + z_t^2(x,t)\} dx dt + C \int_0^T \int_\Omega a(x)\{u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\, dx\, dt \\ & \leq C_1 \int_0^T \int_\Omega a(x)\{y_t^2(x,t) + z_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\, dx\, dt \\ & \leq C_1 \int_0^T \int_\Omega a(x)\left\{u_t^2(x,t) + v_t^2(x,t) + \widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\right\}\, dx\, dt. \end{aligned}$$

It remains to estimate the term $\int_0^T \int_\Omega a(x)\left\{\widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t)\right\}\, dx\, dt$
We differentiate the energy function $\widetilde{\mathcal{E}}(t)$ with respect to t, we obtain

$$\begin{aligned} \frac{d}{dt}\widetilde{\mathcal{E}}(t) = - \int_\Omega & a(x)\{\alpha_1 \widetilde{y}_t(x,t) u_t(x,t) + \alpha_2 \widetilde{y}_t(x,t) u_t(x,t-\tau) \\ & + \beta_1 \widetilde{z}_t(x,t) v_t(x,t) + \beta^2 \widetilde{z}_t(x,t) v_t(x,t-\tau)\}\, dx, \end{aligned}$$

from which we get after using Chauchy-Schwarz inequality

$$\begin{aligned} \frac{d}{dt}\widetilde{\mathcal{E}}(t) \leq & C \int_\Omega a(x)\{u_t^2(x,t) + v_t^2(x,t) + \widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\, dx \\ & + \int_\Omega \{\widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t)\}\, dx. \end{aligned}$$

From the definition of $\widetilde{\mathcal{E}}(t)$, we obtain

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) \leq \widetilde{\mathcal{E}}(t) + C \int_\Omega a(x)\{u_t^2(x,t) + v_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\, dx.$$

Multiplying the last inequality by (e^{-t}) and integrating over $(0,t)$, we get

$$\widetilde{\mathcal{E}}(t) \leq C e^t \int_0^t \int_\Omega a(x)\{u_t^2(x,t) + v_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\, dx\, dt.$$

We conclude for $t \in (0,T)$, that is

$$\widetilde{\mathcal{E}}(t) \leq C \int_0^T \int_\Omega a(x)\{u_t^2(x,t) + v_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\, dx\, dt,$$

which gives

$$\int_0^T \int_\Omega \{\widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t)\}\, dx\, dt \leq C \int_0^T \int_\Omega a(x)\{u_t^2(x,t) + v_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\, dx\, dt.$$

Consequently we have

$$F(0) \leq C_1 \int_0^T \int_\Omega a(x) \left\{ u_t^2(x,t) + v_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau) \right\} \, dx \, dt. \tag{1.119}$$

□

From (1.93), we have

$$F(T) - F(0) \leq -C \int_0^T \int_\Omega a(x) \left\{ u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v(x,t-\tau) \right\} \, dx \, dt,$$

which together with (1.119) leads to

$$F(T) \leq \frac{C_1 C^{-1}}{1 + C_1 C^{-1}} F(0). \tag{1.120}$$

Since we have $0 < C_1/(C + C_1) < 1$, the desired conclusion follows now from (1.120).

1.3. Instability

In this section we show that when $\alpha_2 \geq \alpha_1$ and $\beta_2 \geq \beta_1$ with $\dfrac{\alpha_1}{\alpha_2} = \dfrac{\beta_1}{\beta_2}$, the system (1.1) – (1.9) loses the property of stability for some arbitrary small time delay.
We proceed as in [**28**], We seek a solution of (1.1) – (1.9) in the form

$$u(x,t) = e^{\lambda t}\varphi(x), \quad v(x,t) = e^{\lambda t}\psi(x), \qquad \lambda \in \mathbb{C},$$

where

$$\lambda = ia \qquad a \in \mathbb{R}.$$

Then (φ, ψ) is a solution of the problem

$$(1.121) \qquad \begin{cases} -\Delta\varphi(x) - a^2\varphi(x) + l(\varphi(x) - \psi(x)) = 0 & \text{in } \Omega, \\ -\Delta\psi(x) - a^2\psi(x) + l(\psi(x) - \varphi(x)) = 0 & \text{in } \Omega, \\ \varphi(x) = \psi(x) = 0 & \text{on } \Gamma_1, \\ \frac{\partial\varphi(x)}{\partial\nu} = -ia(\alpha_1 + \alpha_2 e^{-ia\tau})\varphi & \text{on } \Gamma_2, \\ \frac{\partial\psi(x)}{\partial\nu} = -ia(\beta_1 + \beta_2 e^{-ia\tau})\psi & \text{on } \Gamma_2, \end{cases}$$

which can be reformulated, in a variational form as

$$\begin{aligned} &- a^2 \int_\Omega \varphi(x)v(x)\,dx + \int_\Omega \nabla\varphi(x).\nabla v(x)\,dx + ia(\alpha_1 + \alpha_2 e^{-ia\tau}) \int_{\Gamma_2} \varphi(x)v(x)\,d\Gamma \\ &- a^2 \int_\Omega \psi(x)w(x)\,dx + \int_\Omega \nabla\psi(x).\nabla w(x)\,dx + ia(\beta_1 + \beta_2 e^{-ia\tau}) \int_{\Gamma_2} \psi(x)w(x)\,d\Gamma \\ (1.122) \quad &+ \int_\Omega l(\varphi(x) - \psi(x))(v(x) - w(x))\,dx = 0 \qquad \forall (v,w) \text{ in } H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega) \end{aligned}$$

Assume that

$$(1.123) \qquad \cos(a\tau) = \frac{-\alpha_1}{\alpha_2} = \frac{-\beta_1}{\beta_2}.$$

Then

$$\alpha_2 \sin(a\tau) = \sqrt{\alpha_2^2 - \alpha_1^2}\,; \quad \beta_2 \sin(a\tau) = \sqrt{\beta_2^2 - \beta_1^2}\,.$$

Under these assumptions (1.122) is equivalent to

$$\begin{aligned} &- a^2 \int_\Omega \varphi(x)v(x)\,dx + \int_\Omega \nabla\varphi(x).\nabla v(x)\,dx + a\sqrt{\alpha_2^2 - \alpha_1^2} \int_{\Gamma_2} \varphi(x)v(x)\,d\Gamma - a^2 \int_\Omega \psi(x)w(x)\,dx \\ &(1.124) \\ &+ \int_\Omega \nabla\psi(x).\nabla w(x)\,dx + a\sqrt{\beta_2^2 - \beta_1^2} \int_{\Gamma_2} \psi(x)w(x)\,d\Gamma + \int_\Omega l(\varphi(x) - \psi(x))(v(x) - w(x))\,dx = 0 \end{aligned}$$

In particular for $v(x) = \varphi(x)$ and $w(x) = \psi(x)$, (1.124) becomes

$$\begin{aligned} &- a^2 \int_\Omega \varphi^2(x)\,dx + \int_\Omega |\nabla\varphi(x)|^2\,dx + a\sqrt{\alpha_2^2 - \alpha_1^2} \int_{\Gamma_2} \varphi^2(x)\,d\Gamma - a^2 \int_\Omega \psi^2(x)\,dx \\ (1.125) \quad &+ \int_\Omega |\nabla\psi(x)|^2\,dx + a\sqrt{\beta_2^2 - \beta_1^2} \int_{\Gamma_2} \psi^2(x)\,d\Gamma + l\int_\Omega (\varphi(x) - \psi(x))^2\,dx = 0 \end{aligned}$$

We assume

$$(1.126) \qquad \|\varphi\|_2^2 + \|\psi\|_2^2 = 1,$$

then (1.125) can be rewritten as

$$(1.127) \qquad a^2 - a\sqrt{\alpha_2^2 - \alpha_1^2}q_0(\varphi) - a\sqrt{\beta_2^2 - \beta_1^2}q_0(\psi) - q_1(\varphi,\psi) = 0,$$

where

(1.128)

$$q_0(\varphi) := \int_{\Gamma_2} \varphi^2(x)\,d\Gamma\,;\ q_1(\varphi,\psi) := \int_\Omega |\nabla\varphi(x)|^2\,dx + \int_\Omega |\nabla\psi(x)|^2\,dx + l\int_\Omega (\varphi(x) - \psi(x))^2\,dx.$$

We distinguish two cases :
Case 1 : $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$.
(1.127) becomes

$$a^2 = q_1(\varphi, \psi). \tag{1.129}$$

Define

$$a^2 = \min_{\substack{(w_1,w_2)\in H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)\\ \|w_1\|_2^2+\|w_2\|_2^2=1}} q_1(w_1, w_2). \tag{1.130}$$

If (φ, ψ) verifies

$$q_1(\varphi, \psi) = \min_{\substack{(w_1,w_2)\in H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)\\ \|w_1\|_2^2+\|w_2\|_2^2=1}} q_1(w_1, w_2). \tag{1.131}$$

Then (φ, ψ) is a solution of (1.124), and consequently

$$(u(x,t), v(x,t)) = \left(e^{iat}\varphi(x), e^{iat}\psi(x)\right)$$

is a solution of (1.1)-(1.9), whose energy is constant. In fact

$$E(t) = 2a^2 + \frac{a^2}{2}\int_{\Gamma_2} \left\{\mu|\varphi(x)|^2 + \xi|\psi(x)|^2\right\} d\Gamma$$

Assumption (1.123) implies that (φ, ψ) is a solution of

$$\begin{cases} -\Delta\varphi(x) - a^2\varphi(x) + l(\varphi(x) - \psi(x)) = 0 & \text{in } \Omega, \\ -\Delta\psi(x) - a^2\psi(x) + l(\psi(x) - \varphi(x)) = 0 & \text{in } \Omega, \\ \varphi(x) = \psi(x) = 0 & \text{on } \Gamma_1, \\ \frac{\partial\varphi(x)}{\partial\nu} = \frac{\partial\psi(x)}{\partial\nu} = 0 & \text{on } \Gamma_2, \end{cases}$$

which on turn implies that $\varphi + \psi$ is a solution of

$$\begin{cases} -\Delta(\varphi + \psi)(x) - a^2(\varphi + \psi)(x) = 0 & \text{in } \Omega, \\ (\varphi + \psi)(x) = 0 & \text{on } \Gamma_1, \\ \frac{\partial(\varphi+\psi)}{\partial\nu}(x) = 0 & \text{on } \Gamma_2. \end{cases}$$

This is an eigenvalue problem for the Laplacian with Dirichlet-Neumann boundary condition. Therefore, a takes an infinite number of values $a_0, a_1, a_2, ...$ defined by

$$a_n^2 = \lambda_n, \qquad n \in \mathbb{R},$$

where λ_n are the eigenvalues of the Laplace operator with Dirichlet-Neumann boundary condition.
It is known that λ_n are positive and $\lim_{n\to+\infty} \lambda_n = +\infty$.
Now assumption (1.123) holds if

$$a_n\tau = (2p + 1)\pi \qquad p \in \mathbb{N}.$$

So, we have obtained a sequence of time delays

$$\tau_{n,p} = \frac{(2p + 1)}{a_n}\pi \qquad n,\, p \in \mathbb{N},$$

which may be arbitrarily small or large and for which the corresponding solution of the problem is not asymptotically stable.
Case (2) : $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$.
We have

$$a^2 - a\sqrt{\alpha_2^2 - \alpha_1^2}q_0(\varphi) - a\sqrt{\beta_2^2 - \beta_1^2}q_0(\psi) - q_1(\varphi, \psi) = 0,$$

then

$$a = \frac{1}{2}\left(\sqrt{\alpha_2^2 - \alpha_1^2}q_0(\varphi) + \sqrt{\beta_2^2 - \beta_1^2}q_0(\psi) \pm \sqrt{\Delta(\varphi, \psi)}\right),$$

where

$$\Delta(w_1,w_2)=\left(\sqrt{\alpha_2^2-\alpha_1^2}q_0(w_1)+\sqrt{\beta_2^2-\beta_1^2}q_0(w_2)\right)^2+4q_1(w_1,w_2).$$

Define

$$a=\frac{1}{2}\min_{(w_1,w_2)\in H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)}\sqrt{\alpha_2^2-\alpha_1^2}q_0(w_1)+\sqrt{\beta_2^2-\beta_1^2}q_0(w_2)\pm\sqrt{\Delta(w_1,w_2)}. \tag{1.132}$$

We show that if (φ,ψ) verifies

$$\begin{aligned}\sqrt{\alpha_2^2-\alpha_1^2}q_0(\varphi)+&\sqrt{\beta_2^2-\beta_1^2}q_0(\psi)+\sqrt{\Delta(\varphi,\psi)}\\ &=\min_{(w_1,w_2)\in H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)}\sqrt{\alpha_2^2-\alpha_1^2}q_0(w_1)+\sqrt{\beta_2^2-\beta_1^2}q_0(w_2)+\sqrt{\Delta(w_1,w_2)}.\end{aligned} \tag{1.133}$$

then (φ,ψ) is solution of (1.121) with a defined by (1.132).
Take for $\varepsilon\in\mathbb{R}$,

$$\begin{aligned}w_1&=\varphi+\varepsilon v_1\quad\text{with}\quad v_1\in H^1_{\Gamma_1}(\Omega)\quad\text{such that}\quad\int_\Omega\varphi v_1=0,\\ w_2&=\psi+\varepsilon v_2\quad\text{with}\quad v_2\in H^1_{\Gamma_1}(\Omega)\quad\text{such that}\quad\int_\Omega\varphi v_2=0.\end{aligned} \tag{1.134}$$

Then

$$\|w_1\|_2^2+\|w_2\|_2^2=1+\varepsilon^2\left(\|v_1\|_2^2+\|v_2\|_2^2\right). \tag{1.135}$$

Let

$$\begin{aligned}g(\varepsilon)=&\frac{1}{1+\varepsilon^2\left(\|v_1\|_2^2+\|v_2\|_2^2\right)}(\sqrt{\alpha_2^2-\alpha_1^2}q_0(\varphi+\varepsilon v_1)+\sqrt{\beta_2^2-\beta_1^2}q_0(\psi+\varepsilon v_2)\\ &+\sqrt{\left(\sqrt{\alpha_2^2-\alpha_1^2}q_0(\varphi+\varepsilon v_1)+\sqrt{\beta_2^2-\beta_1^2}q_0(\psi+\varepsilon v_2)\right)^2+4q_1(\varphi+\varepsilon v_1,\psi+\varepsilon v_2)}\,),\end{aligned} \tag{1.136}$$

From (1.133), we get

$$g(\varepsilon)\geq g(0)=\sqrt{\alpha_2^2-\alpha_1^2}q_0(\varphi)+\sqrt{\beta_2^2-\beta_1^2}q_0(\psi)+\sqrt{\left(\sqrt{\alpha_2^2-\alpha_1^2}q_0(\varphi)+\sqrt{\beta_2^2-\beta_1^2}q_0(\psi)\right)^2+4q_1(\varphi,\psi)},$$

then, we have

$$\frac{dg(\varepsilon)}{d\varepsilon}|_{\varepsilon=0}=0,$$

which gives

$$\begin{aligned}&\int_\Omega\nabla\varphi.\nabla v_1\,dx+\int_\Omega\nabla\psi.\nabla v_2\,dx+a\sqrt{\alpha_2^2-\alpha_1^2}\int_{\Gamma_2}\varphi v_1\,d\Gamma+a\sqrt{\beta_2^2-\beta_1^2}\int_{\Gamma_2}\psi v_2\,d\Gamma\\ &+\int_\Omega l(\varphi-\psi)(v_1-v_2)\,dx=0.\end{aligned} \tag{1.137}$$

Any function $(\tilde{v_1},\tilde{v_2})$ in $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$ can be decomposed

$$\begin{aligned}\tilde{v_1}&=\gamma\varphi+v_1,\quad\gamma\in\mathbb{R},\,v_1\in H^1_{\Gamma_1}(\Omega)\quad\text{with}\quad\int_\Omega\varphi v_1=0,\\ \tilde{v_2}&=\gamma\psi+v_2,\quad\gamma\in\mathbb{R},\,v_2\in H^1_{\Gamma_1}(\Omega)\quad\text{with}\quad\int_\Omega\varphi v_2=0.\end{aligned}$$

(1.137) and (1.125) yield (φ,ψ) that satisfies (1.124) with a defined by (1.132), so we have found a sequence of delays defined by

$$a\tau=\arccos\left(\frac{-\alpha_1}{\alpha_2}\right)+2p\pi,\quad p\in\mathbb{N}$$

for which the solution of problem is not asymptotically stable.

CHAPTER 2

Stability of the transmission wave equation with a delay term in the boundary feedback

2.1. Introduction

We investigate in this chapter the problem of exponential stability for the system of transmission of the wave equation with a delay term in the boundary feedback. Let Ω be an open bounded domain of $\mathbb{R}^n$ with a boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. Let Γ_0 with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \overline{\Gamma_0} \cap \overline{\Gamma_2} = \emptyset$ be a regular hypersurface of class C^2 which separates Ω into two domains Ω_1 and Ω_2 such that $\Gamma_1 \subset \partial\Omega_1$ and $\Gamma_2 \subset \partial\Omega_2$.
Furthermore, we assume that there exists a real vector field $h \in (C^2(\overline{\Omega}))^n$ such that:
$(H.1)$ The Jacobian matrix J of h satisfies

$$\int_\Omega J(x)\zeta(x).\zeta(x)d\Omega \geq \alpha \int_\Omega |\zeta(x)|^2 \, d\Omega,$$

for some constant $\alpha > 0$ and for all $\zeta \in L^2(\Omega;\mathbb{R}^n)$;
$(H.2)$ $h(x).\nu(x) \leq 0$ on Γ_1;

$(H.3)$ $h(x).\nu(x) \geq 0$ on Γ_0.
where ν is the unit normal on Γ or Γ_0 pointing towards the exterior of Ω or Ω_1.

Let $a_1, a_2 > 0$ be given. Consider the system of transmission of the wave equation with a delay term in the boundary conditions:

(2.1) $\quad y_{tt}(x,t) - \operatorname{div}(a(x)\nabla y(x,t)) = 0 \qquad$ in $\Omega \times (0,+\infty)$,

(2.2) $\quad y(x,0) = y^0(x), y_t(x,0) = y^1(x) \qquad$ in Ω,

(2.3) $\quad y_1(x,t) = 0 \qquad$ on $\Gamma_1 \times (0,+\infty)$,

(2.4) $\quad \dfrac{\partial y_2(x,t)}{\partial \nu} = -\mu_1 y_{2,t}(x,t) - \mu_2 y_{2,t}(x,t-\tau) \qquad$ on $\Gamma_2 \times (0,+\infty)$,

(2.5) $\quad y_1(x,t) = y_2(x,t), \qquad$ on $\Gamma_0 \times (0,+\infty)$,

(2.6) $\quad a_1 \dfrac{\partial y_1(x,t)}{\partial \nu} = a_2 \dfrac{\partial y_2(x,t)}{\partial \nu} \qquad$ on $\Gamma_0 \times (0,+\infty)$,

(2.7) $\quad y_{2,t}(x,t-\tau) = f_0(x,t-\tau) \qquad$ on $\Gamma_2 \times (0,\tau)$.

where:

- (2.8) $$a(x) = \begin{cases} a_1, & x \in \Omega_1, \\ a_2, & x \in \Omega_2. \end{cases}$$
- $y(x,t) = \begin{cases} y_1(x,t), & (x,t) \in \Omega_1 \times (0,+\infty), \\ y_2(x,t), & (x,t) \in \Omega_2 \times (0,+\infty). \end{cases}$
- $\frac{\partial .}{\partial \nu}$ is the normal derivative.
- μ_1 and μ_2 are positive real numbers.
- τ is the time delay.
- y^0, y^1 and f_0 are the initial data which belong to suitable spaces.

From the physical point of view, the transmission problem $(2.1)-(2.7)$ describes the wave propagation from one medium into another different medium, for instance, from air into

glass (see [**22**]).
In the absence of delay, that is $\mu_2 = 0$, Liu and Williams [**23**] have shown that the solution of $(2.1) - (2.6)$ decays exponentially to zero in the energy space $H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)$ provided that

$$a_1 > a_2, \tag{2.9}$$

and $\{\Omega, \Gamma_0, \Gamma_1, \Gamma_2\}$ satisfies $(H.1)$, $(H.2)$, $(H.3)$, and $h(x).\nu(x) \geq \gamma > 0$ on Γ_2.
The purpose of this chapter is to investigate the stability of problem $(2.1) - (2.7)$ in the case where both μ_1 and μ_2 are different from zero. To this end, assume as in [**28**] that

$$\mu_1 > \mu_2. \tag{2.10}$$

and define the energy of a solution of $(2.1) - (2.7)$ by

$$E(t) = \frac{1}{2}\int_\Omega \left[y_t^2(x,t) + a(x)\,|\nabla(y(x,t)|^2\right] dx + \frac{\xi}{2}\int_{\Gamma_2}\int_0^1 y_{2,t}^2(x, t-\tau\rho)\, d\rho\, d\Gamma, \tag{2.11}$$

where

$$a_2\tau\mu_2 < \xi < a_2\tau(2\mu_1 - \mu_2), \tag{2.12}$$

2.2. Main result

We show that if in addition to (2.9) and (2.10), $\{\Omega, \Gamma_0, \Gamma_1, \Gamma_2\}$ satisfies $(H.1), (H.2)$ and $(H.3)$, then there is an exponential decay rate for $E(t)$. The proof of this result combines multipliers techniques and compactness-uniqueness arguments.
The main result of this chapter can be stated as follows.

THEOREM 2.1. *Assume* $(H.1)$, $(H.2)$, $(H.3)$, (2.9) *and* (2.10). *Then there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq M e^{-\omega t} E(0).$$

Theorem 2.1 is proved in Section 2.4. In Section 2.3, we investigate the well-posedness of system $(2.1) - (2.7)$ using semigroup theory.
This chapter is an expanded and revised version of the conference paper by Rebiai [**36**]

2.3. Well-posedness

Inspired from [**28**] and [**29**], we introduce the auxiliary variable $z(x,\rho,t) = y_{2,t}(x, t-\tau\rho)$.
With this new unknown, problem $(2.1) - (2.7)$ is equivalent to

$$y_{tt}(x,t) - div(a(x)\nabla y(x,t)) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{2.13}$$

$$z_t(x,\rho,t) + \tau^{-1} z_\rho(x,\rho,t) = 0 \quad \text{on } \Gamma_2 \times (0,1) \times (0,+\infty), \tag{2.14}$$

$$y(x,0) = y^0(x), y_t(x,0) = y^1(x) \quad \text{in } \Omega, \tag{2.15}$$

$$y_1(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,+\infty), \tag{2.16}$$

$$\frac{\partial y_2(x,t)}{\partial \nu} = -\mu_1 y_{2,t}(x,t) - \mu_2 z(x,1,t) \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{2.17}$$

$$y_1(x,t) = y_2(x,t) \quad \text{on } \Gamma_0 \times (0,+\infty), \tag{2.18}$$

$$a_1\frac{\partial y_1(x,t)}{\partial \nu} = a_2\frac{\partial y_2(x,t)}{\partial \nu} \quad \text{on } \Gamma_0 \times (0,+\infty), \tag{2.19}$$

$$z(x,0,t) = y_{2,t}(x,t) \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{2.20}$$

$$z(x,\rho,0) = f_0(x,-\tau\rho) \quad \text{on } \Gamma_2 \times (0,1). \tag{2.21}$$

Now, we endow the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2; L^2(0,1))$$

with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}; \begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{z} \end{pmatrix} \right\rangle = \int_\Omega (a(x)\nabla u(x)\nabla \overline{u}(x) + v(x)\overline{v}(x))\, dx + \xi \int_{\Gamma_2}\int_0^1 z(x,\rho)\overline{z}(x,\rho) d\rho\, d\Gamma,$$

and define a linear operator in $\mathcal{H}$ by

$$D(A) = \{(u,v,z)^T \in H^2(\Omega_1,\Omega_2,\Gamma_1) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2; H^1(0,1));$$

$$\frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(.,1), v = z(.,0) \text{ on } \Gamma_2\}, \tag{2.22}$$

$$A\,(u,v,z)^T = \left(v, div(a(x)\nabla u), -\tau^{-1} z_\rho\right)^T. \tag{2.23}$$

The spaces used for the definition of $\mathcal{H}$ and $D(A)$ are

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\},$$
$$H^2(\Omega_1,\Omega_2,\Gamma_1) = \{u_i \in H^2(\Omega_i) : u = 0 \text{ on } \Gamma_1,\ u_1 = u_2 \text{ and } a_1\frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} \text{ on } \Gamma_0\}.$$

Then we can rewrite (2.13) – (2.21) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}Y(t) = AY(t), \\ Y(0) = Y_0, \end{cases} \tag{2.24}$$

where

$$Y(t) = (y, y_t, z)^T \text{ and } Y_0 = (y_0, y_1, f_0(., -.\tau))^T.$$

Proposition 2.1. *The operator A defined by* (2.22), (2.23) *and* (2.10) *generates a strongly continuous semigroup on $\mathcal{H}$. Thus, for every $Y_0 \in \mathcal{H}$, problem* (2.24) *has a unique solution Y whose regularity depends on the initial datum Y_0 as follows:*

$$Y(.) \in C([0,+\infty);\mathcal{H}) \text{ if } Y_0 \in \mathcal{H},$$
$$Y(.) \in C([0,+\infty);D(A)) \cap C^1([0,+\infty);\mathcal{H}) \text{ if } Y_0 \in D(A).$$

Proof. Let $Y = \begin{pmatrix} u \\ v \\ z \end{pmatrix} \in D(A)$. Then

$$\langle AY, Y\rangle = \int_\Omega a(x)\nabla u(x).\nabla v(x)\, dx + \int_\Omega v(x)\mathrm{div}(a(x)\nabla u(x))\, dx$$
$$- \frac{\xi}{\tau}\int_{\Gamma_2}\int_0^1 z_\rho(x,\rho) z(x,\rho)\, d\rho\, d\Gamma. \tag{2.25}$$

Applying Green's Theorem and recalling (2.22), we obtain

$$\int_\Omega \mathrm{div}(a(x)\nabla u(x))v(x)\, dx = a_1\int_{\Gamma_1} v(x)\frac{\partial u(x)}{\partial \nu}\, d\Gamma - a_1\int_{\Omega_1}\nabla u(x).\nabla v(x)\, dx$$
$$+ a_2\int_{\Gamma_2} v(x)\frac{\partial u(x)}{\partial \nu}\, d\Gamma - a_2\int_{\Omega_2}\nabla u(x).\nabla v(x)\, dx$$
$$= a_2\int_{\Gamma_2} v(x)\{-\mu_1 v(x) - \mu_2 z(x,1)\}\, d\Gamma - \int_\Omega a(x)\nabla u(x).\nabla v(x)\, dx. \tag{2.26}$$

Integrating by parts in ρ, we get

$$\int_{\Gamma_2}\int_0^1 z_\rho(x,\rho) z(x,\rho)\, d\rho\, d\Gamma = \frac{1}{2}\int_{\Gamma_2}\{z^2(x,1) - z^2(x,0)\}\, d\Gamma. \tag{2.27}$$

Inserting (2.22), (2.26) and (2.27) into (2.25) results in

$$\begin{aligned}\langle AY, Y\rangle = & - a_2\mu_1 \int_{\Gamma_2} v^2(x)\, d\Gamma - a_2\mu_2 \int_{\Gamma_2} v(x)z(x,1)\, d\Gamma \\ & - \frac{\xi}{2\tau}\int_{\Gamma_2} z^2(x,1)\, d\Gamma + \frac{\xi}{2\tau}\int_{\Gamma_2} v^2(x)\, d\Gamma,\end{aligned}$$

from which follows using the Cauchy inequality

$$\langle AY, Y\rangle \leq -(a_2\mu_1 - \frac{a_2\mu_2}{2} - \frac{\xi}{2\tau})\int_{\Gamma_2} v^2(x)\, d\Gamma - (\frac{\xi}{2\tau} - \frac{a_2\mu_2}{2})\int_{\Gamma_2} z^2(x,1)\, d\Gamma. \tag{2.28}$$

(2.28) together with (2.12) implies that

$$\langle AY, Y\rangle \leq 0.$$

Thus A is dissipative.
Now we show that for a fixed $\lambda > 0$ and $(g, h, k)^T \in \mathcal{H}$, there exists $Y = (u, v, z)^T \in D(A)$ such that

$$(\lambda I - A)Y = (g, h, k)^T$$

or equivalently

$$\lambda u - v = g, \tag{2.29}$$

$$\lambda v - \operatorname{div}(a(x)\nabla u) = h, \tag{2.30}$$

$$\lambda z + \frac{1}{\tau} z_\rho = k. \tag{2.31}$$

Suppose that we have found u with the appropriate regularity, then we can determine z. Indeed, from (2.22) and (2.31) we have

$$\begin{cases} z_\rho(x,\rho) = -\lambda\tau z(x,\rho) + \tau k(x,\rho), \\ z(x,0) = v(x). \end{cases}$$

The unique solution of the above initial value problem is

$$z(x,\rho) = e^{-\lambda\tau\rho}v(x) + \tau e^{-\lambda\tau\rho}\int_0^\rho e^{\lambda\tau s}k(x,s)\, ds,$$

and in particular

$$z(x,1) = \lambda e^{-\lambda\tau}u(x) + z_0(x), \quad x \in \Gamma_2,$$

where

$$z_0(x) = -e^{-\lambda\tau}g(x) + \tau e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}k(x,s)\, ds.$$

By (2.29) and (2.30), the function u satisfies

$$\lambda^2 u - \operatorname{div}(a(x)\nabla u) = h + \lambda g. \tag{2.32}$$

Problem (2.32) can be reformulated as

$$\int_\Omega (\lambda^2 u - \operatorname{div}(a(x)\nabla u)w\, dx = \int_\Omega (h + \lambda g)w\, dx, \quad w \in H^1_{\Gamma_1}(\Omega). \tag{2.33}$$

Using Green's Theorem and recalling (2.22), we express the left-hand side of (2.33) as follows

$$\begin{aligned}\int_\Omega (\lambda^2 u - \operatorname{div}(a(x)\nabla u)w\, dx = & \int_\Omega (\lambda^2 uw + a(x)\nabla u.\nabla w)\, dx + a_2\int_{\Gamma_2}\{\mu_1(\lambda u - g)w \\ & + \mu_2(\lambda e^{-\lambda\tau}u(x) + z_0(x))w\}\, d\Gamma.\end{aligned}$$

Therefore (2.33), can be rewritten as

$$\begin{aligned}&\int_\Omega (\lambda^2 uw + a(x)\nabla u.\nabla w)\, dx + a_2\int_{\Gamma_2}(\mu_1 + \mu_2 e^{-\lambda\tau})\lambda uw\, d\Gamma = \int_\Omega (h + \lambda g)w\, d\Gamma \\ & + a_2\mu_1\int_{\Gamma_2} gw\, d\Gamma - a_2\mu_2\int_{\Gamma_2} z_0 w d\Gamma, \qquad \forall w \in H^1_{\Gamma_1}(\Omega).\end{aligned} \tag{2.34}$$

Since the left-hand side of (2.34) is coercive on $H^1_{\Gamma_1}(\Omega)$, the Lax-Milgram Theorem guarantees the existence and uniqueness of the solution $u \in H^1_{\Gamma_1}(\Omega)$ of (2.32). If we consider $w \in \mathcal{D}(\Omega)$ in (2.33), then y is a solution in $\mathcal{D}'(\Omega)$ of

$$\lambda^2 u - \operatorname{div}(a(x)\nabla u) = h + \lambda g, \tag{2.35}$$

and thus $\operatorname{div}(a(x)\nabla u) \in L^2(\Omega)$.
Combining (2.34) together with (2.35), we obtain after using Green's Theorem

$$a_2 \int_{\Gamma_2} (\mu_1 + \mu_2 e^{-\lambda\tau})\lambda u w \, d\Gamma + a_2 \int_{\Gamma_2} \frac{\partial u}{\partial \nu} w \, d\Gamma = a_2\mu_1 \int_{\Gamma_2} g w \, d\Gamma - a_2\mu_2 \int_{\Gamma_2} z_0 w \, d\Gamma,$$

which implies that

$$\frac{\partial u(x)}{\partial \nu} = -\mu_1 v(x) - \mu_2 z(x,1).$$

So, we have found $(u,v,z)^T \in D(A)$ which satisfies (2.29) – (2.31). Thus, by the Lumer-Phillips Theorem (see for instance [**34**], Theorem 1.4.3), A generates a strongly continuous semigroup of contractions on $\mathcal{H}$. □

2.4. Proof of the main result

We prove Theorem 2.1 for smooth initial data. The general case follows by a standard density argument.

We proceed in several steps.
Step 1.
We differentiate the energy defined by (2.11) and apply Green's Theorem. We obtain

$$\frac{d}{dt}E(t) = a_2 \int_{\Gamma_2} y_{2,t}(x,t)\frac{\partial y_2(x,t)}{\partial \nu}\, d\Gamma + \xi \int_{\Gamma_2}\int_0^1 y_{2,t}(x,t-\tau\rho) y_{2,tt}(x,t-\tau\rho)\, d\rho \, d\Gamma, \tag{2.36}$$

after using the boundary condition (2.3) and the transmission condition (2.5), (2.6).
Now, it follows from

$$y_t(x,t-\tau\rho) = -\frac{1}{\tau} y_\rho(x,t-\tau\rho),$$

and

$$y_{tt}(x,t-\tau\rho) = \frac{1}{\tau^2} y_{\rho\rho}(x,t-\tau\rho),$$

that

$$\begin{aligned}\int_{\Gamma_2}\int_0^1 y_{2,t}(x,t-\tau\rho) y_{2,tt}(x,t-\tau\rho)\, d\rho \, d\Gamma &= -\frac{1}{\tau^3}\int_{\Gamma_2}\int_0^1 y_{2,\rho}(x,t-\tau\rho) y_{2,\rho\rho}(x,t-\tau\rho)\, d\rho \, d\Gamma \\ &= -\frac{1}{2\tau^3}\int_{\Gamma_2}\int_0^1 \frac{d}{d\rho}\{y^2_{2,\rho}(x,t-\tau\rho)\}\, d\rho\, d\Gamma \\ &= -\frac{1}{2\tau^3}\int_{\Gamma_2} \{y^2_{2,\rho}(x,t-\tau) - y^2_{2,\rho}(x,t)\}\, d\Gamma,\end{aligned}$$

that is

$$\int_{\Gamma_2}\int_0^1 y_{2,t}(x,t-\tau\rho) y_{2,tt}(x,t-\tau\rho)\, d\rho \, d\Gamma = \frac{1}{2\tau}\int_{\Gamma_2}\{y^2_{2,t}(x,t) - y^2_{2,t}(x,t-\tau)\}\, d\Gamma. \tag{2.37}$$

Substituting (2.4) and (2.37) into (2.36), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) =& -\mu_1 a_2 \int_{\Gamma_2} y^2_{2,t}(x,t)\, d\Gamma - \mu_2 a_2 \int_{\Gamma_2} y_{2,t}(x,t) y_{2,t}(x,t-\tau)\, d\Gamma + \frac{\xi}{2\tau}\int_{\Gamma_2} y^2_{2,t}(x,t)\, d\Gamma \\ & - \frac{\xi}{2\tau}\int_{\Gamma_2} y^2_{2,t}(x,t-\tau)\, d\Gamma,\end{aligned}$$

from which we get after using the Cauchy inequality

$$\frac{d}{dt}E(t) \le -k \int_{\Gamma_2} \{y^2_{2,t}(x,t) + y^2_{2,t}(x,t-\tau)\}\, d\Gamma, \tag{2.38}$$

where

$$k = \min\{a_2\mu_1 - \frac{a_2\mu_2}{2} - \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{a_2\mu_2}{2}\}.$$

Step 2.
Set

$$E(t) = \mathcal{E}(t) + E_d(t),$$

where

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \{a(x)\,|\nabla y(x,t)|^2 + y_t^2(x,t)\}\,dx,$$

and

$$E_d(t) = \frac{\xi}{2}\int_{\Gamma_2}\int_0^1 y_{2,t}^2(x, t-\tau\rho)\,d\rho\,d\Gamma.$$

$E_d(t)$ can be rewritten via a change of variable as

$$E_d(t) = \frac{\xi}{2\tau}\int_t^{t+\tau}\int_{\Gamma_2} y_{2,t}^2(x, s-\tau)\,d\Gamma\,ds. \tag{2.39}$$

From (2.39), we obtain

$$E_d(t) \le C\int_0^T\int_{\Gamma_2} y_{2,t}^2(x, s-\tau)\,d\Gamma\,ds, \tag{2.40}$$

for $0 \le t+\tau \le T$. Here and throughout the rest of the chapter C is a positive constant independent of T different at different occurrences.
Step 3.

We multiply both sides of (2.1) by $2h(x).\nabla y(x,t) + (\text{div}h(x) - \alpha)y(x,t)$ and integrate over $\Omega \times (0,T)$;

$$2\int_0^T\int_\Omega y_{tt}(x,t)h(x).\nabla y(x,t)\,dx\,dt + \int_0^T\int_\Omega y_{tt}(x,t)(\text{div}h(x)-\alpha)y(x,t)\,dx\,dt -$$
(2.41)
$$2\int_0^T\int_\Omega \text{div}h(a(x)\nabla y(x,t))h(x).\nabla y(x,t)\,dx\,dt - \int_0^T\int_\Omega \text{div}h(a(x)\nabla y(x,t))(\text{div}h(x)-\alpha)y(x,t)\,dx\,dt = 0.$$

We compute each term of (2.41) separately.

- Term $2\int_0^T\int_\Omega y_{tt}(x,t)h(x).\nabla y(x,t)\,dx\,dt$
  Integration by parts in t yields

$$2\int_0^T\int_\Omega y_{tt}(x,t)h(x).\nabla y(x,t)\,dx\,dt = 2\left[\int_\Omega y_t(x,t)h(x).\nabla y(x,t)\,dx\right]_0^T -$$
$$2\int_0^T\int_\Omega y_t(x,t)h(x).\nabla y_t(x,t)\,dx\,dt =$$
$$2\left[\int_\Omega y_t(x,t)h(x).\nabla y(x,t)\,dx\right]_0^T - \int_0^T\int_\Omega h(x).\nabla(y_t^2(x,t))\,dx\,dt. \tag{2.42}$$

  Applying Green's theorem to the second integral on the right-hand side of (2.42), we obtain

$$2\int_0^T\int_\Omega y_{tt}(x,t)h(x).\nabla y(x,t)\,dx\,dt = 2\left[\int_\Omega y_t(x,t)h(x).\nabla y(x,t)\,dx\right]_0^T$$
$$-\int_0^T\int_\Gamma y_t^2(x,t)h(x).\nu(x)\,d\Gamma\,dt + \int_0^T\int_\Omega y_t^2(x,t)\text{div}h(x)\,dx\,dt. \tag{2.43}$$

- Term $\int_0^T \int_\Omega y_{tt}(x,t)(\text{div}h(x)-\alpha)y(x,t)\,dx\,dt$
  Using again integration by parts with respect to t, we obtain

$$\int_0^T \int_\Omega y_{tt}(x,t)(\text{div}h(x)-\alpha)y(x,t)dxdt = \left[\int_\Omega y_t(x,t)(\text{div}h(x)-\alpha)y(x,t)\,dx\right]_0^T - \int_0^T \int_\Omega y_t^2(x,t)(\text{div}h(x)-\alpha)\,dx\,dt. \tag{2.44}$$

- Term $\int_0^T \int_\Omega \text{div}(a(x)\nabla y(x,t))h(x).\nabla y(x,t)\,dx\,dt$
  We have by (2.8),

$$2\int_0^T \int_\Omega \text{div}(a(x)\nabla y(x,t))h(x).\nabla y(x,t)\,dx\,dt = 2a_1\int_0^T \int_{\Omega_1} \Delta y_1(x,t)h(x).\nabla y_1(x,t)\,dx\,dt + 2a_2\int_0^T \int_{\Omega_2} \Delta y_2(x,t)h(x).\nabla y_2(x,t)\,dx\,dt. \tag{2.45}$$

From Green's Theorem, we obtain for the first integral on the right-hand side of (2.45),

$$2a_1\int_0^T \int_{\Omega_1} \Delta y_1(x,t)h(x).\nabla y_1(x,t)\,dx\,dt = 2a_1\int_0^T \int_{\Gamma_1} \frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)\,d\Gamma\,dt + 2a_1\int_0^T \int_{\Gamma_0} \frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)d\Gamma dt - 2a_1\int_0^T \int_{\Omega_1} \nabla y_1(x,t).\nabla(h(x).\nabla y_1(x,t))\,dx\,dt. \tag{2.46}$$

Applying the identity

$$\nabla w(x).\nabla(h(x).\nabla w(x)) = J(x)\nabla w(x).\nabla w(x) + \frac{1}{2}h(x).\nabla(|\nabla w(x)|^2)$$

to the last integral on the right hand side of (2.46), we find

$$2a_1\int_0^T \int_{\Omega_1} \Delta y_1(x,t)h(x).\nabla y_1(x,t)\,dx\,dt = 2a_1\int_0^T \int_{\Gamma_1} \frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)\,d\Gamma\,dt + 2a_1\int_0^T \int_{\Gamma_0} \frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)\,d\Gamma\,dt - 2a_1\int_0^T \int_{\Omega_1} J(x)\nabla y_1(x,t).\nabla y_1(x,t))\,dx\,dt - a_1\int_0^T \int_{\Omega_1} h(x).\nabla(|\nabla y_1(x,t)|^2).$$

Another use of Green's Theorem yields

$$2a_1\int_0^T \int_{\Omega_1} \Delta y_1(x,t)h(x).\nabla y_1(x,t)\,dx\,dt = 2a_1\int_0^T \int_{\Gamma_1} \frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)\,d\Gamma\,dt + 2a_1\int_0^T \int_{\Gamma_0} \frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)\,d\Gamma\,dt - 2a_1\int_0^T \int_{\Omega_1} J(x)\nabla y_1(x,t).\nabla y_1(x,t))\,dx\,dt - a_1\int_0^T \int_{\Gamma_1} |\nabla y_1(x,t)|^2\,h(x).\nu(x)\,d\Gamma\,dt - a_1\int_0^T \int_{\Gamma_0} |\nabla y_1(x,t)|^2\,h(x).\nu(x)\,d\Gamma\,dt + a_1\int_0^T \int_{\Omega_1} |\nabla y_1(x,t)|^2\,divh(x)\,dx\,dt. \tag{2.47}$$

For the second on the right-hand side of (2.45), we proceed as above to find

$$
\begin{aligned}
&2a_2 \int_0^T \int_{\Omega_2} \Delta y_2(x,t) h(x).\nabla y_2(x,t)\,dx\,dt = 2a_2 \int_0^T \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial \nu} h(x).\nabla y_2(x,t)\,d\Gamma\,dt - \\
&2a_2 \int_0^T \int_{\Gamma_0} \frac{\partial y_2(x,t)}{\partial \nu} h(x).\nabla y_2(x,t)\,d\Gamma\,dt - 2a_2 \int_0^T \int_{\Omega_2} J(x)\nabla y_2(x,t).\nabla y_2(x,t))\,dx\,dt - \\
&a_2 \int_0^T \int_{\Gamma_2} |\nabla y_2(x,t)|^2 h(x).\nu(x)\,d\Gamma\,dt + a_2 \int_0^T \int_{\Gamma_0} |\nabla y_2(x,t)|^2 h(x).\nu(x)\,d\Gamma\,dt + \\
&a_2 \int_0^T \int_{\Omega_2} |\nabla y_2(x,t)|^2 \operatorname{div} h(x)\,dx\,dt.
\end{aligned}
\tag{2.48}
$$

Substitution of (2.47) and (2.48) into (2.45) yields

$$
\begin{aligned}
&2 \int_0^T \int_{\Omega} \operatorname{div}(a(x)\nabla y(x,t)) h(x).\nabla y(x,t)\,dx\,dt = 2a_1 \int_0^T \int_{\Gamma_1} \frac{\partial y_1(x,t)}{\partial \nu} h(x).\nabla y_1(x,t)\,d\Gamma\,dt + \\
&2a_1 \int_0^T \int_{\Gamma_0} \frac{\partial y_1(x,t)}{\partial \nu} h(x).\nabla y_1(x,t)\,d\Gamma\,dt - 2a_1 \int_0^T \int_{\Omega_1} J(x)\nabla y_1(x,t).\nabla y_1(x,t))\,dx\,dt - \\
&a_1 \int_0^T \int_{\Gamma_1} |\nabla y_1(x,t)|^2 h(x).\nu(x)\,d\Gamma\,dt - a_1 \int_0^T \int_{\Gamma_0} |\nabla y_1(x,t)|^2 h(x).\nu(x)\,d\Gamma\,dt + \\
&a_1 \int_0^T \int_{\Omega_1} |\nabla y_1(x,t)|^2 \operatorname{div} h(x)\,dx\,dt + 2a_2 \int_0^T \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial \nu} h(x).\nabla y_2(x,t)\,d\Gamma\,dt - \\
&2a_2 \int_0^T \int_{\Gamma_0} \frac{\partial y_2(x,t)}{\partial \nu} h(x).\nabla y_2(x,t)\,d\Gamma\,dt - 2a_2 \int_0^T \int_{\Omega_2} J(x)\nabla y_2(x,t).\nabla y_2(x,t))\,dx\,dt - \\
&a_2 \int_0^T \int_{\Gamma_2} |\nabla y_2(x,t)|^2 h(x).\nu(x)\,d\Gamma\,dt + a_2 \int_0^T \int_{\Gamma_0} |\nabla y_2(x,t)|^2 h(x).\nu(x)\,d\Gamma\,dt + \\
&a_2 \int_0^T \int_{\Omega_2} |\nabla y_2(x,t)|^2 \operatorname{div} h(x)\,dx\,dt.
\end{aligned}
\tag{2.49}
$$

We conclude from the boundary conditions (2.3) and (2.5) that

$$
\nabla y_1(x,t) = \frac{\partial y_1(x,t)}{\partial \nu} \nu(x), \qquad \text{on } \Gamma_1 \times (0,T),
\tag{2.50}
$$

and

$$
\nabla (y_2(x,t) - y_1(x,t)) = \frac{\partial (y_2(x,t) - y_1(x,t))}{\partial \nu} \nu(x), \quad \text{on } \Gamma_0 \times (0,T)
$$

then

$$
\begin{aligned}
|\nabla y_2(x,t)|^2 &= |\nabla y_1(x,t)|^2 + 2\left(\frac{\partial y_2}{\partial \nu}(x,t) - \frac{\partial y_1}{\partial \nu}(x,t)\right)\frac{\partial y_1}{\partial \nu}(x,t) + \left(\frac{\partial y_2}{\partial \nu}(x,t) - \frac{\partial y_1}{\partial \nu}(x,t)\right)^2 \\
&= |\nabla y_1(x,t)|^2 + \left(\frac{\partial y_2}{\partial \nu}(x,t)\right)^2 - \left(\frac{\partial y_1}{\partial \nu}(x,t)\right)^2, \quad \text{on } \Gamma_0 \times (0,T)
\end{aligned}
$$

so on $\Gamma_0 \times (0,T)$,

$$\begin{aligned}
& 2a_1 \frac{\partial y_1}{\partial \nu}(x,t)h(x).\nabla y_1(x,t) - 2a_2 \frac{\partial y_2}{\partial \nu}(x,t)h(x).\nabla y_2(x,t) \\
& - a_1 \left|\nabla y_1(x,t)\right|^2 h(x).\nu(x) + a_2 \left|\nabla y_2(x,t)\right|^2 h(x).\nu(x) \qquad (2.51) \\
& = 2a_1 \frac{\partial y_1}{\partial \nu}(x,t)h(x).\nabla y_1(x,t) - a_1 \left|\nabla y_1(x,t)\right|^2 h(x).\nu(x) \\
& - 2a_2 \frac{\partial y_2}{\partial \nu}(x,t)\left(\nabla y_1(x,t) + \left(\frac{\partial y_2}{\partial \nu}(x,t) - \frac{\partial y_1}{\partial \nu}(x,t)\right)\nu(x)\right).h(x) \\
& + a_2 \left(\left|\nabla y_1(x,t)\right|^2 + \left(\frac{\partial y_2}{\partial \nu}(x,t)\right)^2 - \left(\frac{\partial y_1}{\partial \nu}(x,t)\right)^2\right) h(x).\nu(x) \\
& = -2a_1(\frac{a_1}{a_2} - 1)\left(\frac{\partial y_1}{\partial \nu}(x,t)\right)^2 h(x).\nu(x) + (a_2 - a_1)\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x) \\
& + (\frac{a_1^2}{a_2} - a_2)\left(\frac{\partial y_1}{\partial \nu}(x,t)\right)^2 h(x).\nu(x) \\
& = (a_2 - a_1)\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x) - \frac{(a_2-a_1)^2}{a_2}\left(\frac{\partial y_1}{\partial \nu}(x,t)\right)^2 h(x).\nu(x) \qquad (2.52)
\end{aligned}$$

Insertion of (2.50) and (2.52) into (2.49) results in

$$\begin{aligned}
& 2\int_0^T \int_\Omega \operatorname{div}(a(x)\nabla y(x,t))h(x).\nabla y(x,t)\,dx\,dt = a_1 \int_0^T \int_{\Gamma_1} \left(\frac{\partial y_1}{\partial \nu}(x,t)\right)^2 h(x).\nu(x)\,d\Gamma\,dt - \\
& (a_1 - a_2)\int_0^T \int_{\Gamma_0} \left|\nabla y_1(x,t)\right|^2 h(x).\nu(x)\,d\Gamma\,dt - \frac{(a_2-a_1)^2}{a_2}\int_0^T \int_{\Gamma_0} \left(\frac{\partial y_1}{\partial \nu}(x,t)\right)^2 h(x).\nu(x)\,d\Gamma\,dt + \\
& 2a_2 \int_0^T \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial \nu} h(x).\nabla y_2(x,t)\,d\Gamma\,dt - a_2 \int_0^T \int_{\Gamma_2} \left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)\,d\Gamma\,dt - \qquad (2.53) \\
& 2\int_0^T \int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)\,dx\,dt + \int_0^T \int_\Omega a(x)\left|\nabla y(x,t)\right|^2 \operatorname{div}h(x)\,dx\,dt.
\end{aligned}$$

- Term $\int_0^T \int_\Omega \operatorname{div}(a(x)\nabla y(x,t))(\operatorname{div}h(x) - \alpha)y(x,t)\,dx\,dt$

  From (2.8), we may write

$$\begin{aligned}
\int_0^T \int_\Omega \operatorname{div}(a(x)\nabla y(x,t))(\operatorname{div}h(x) - \alpha)y(x,t)\,dx\,dt = & a_1 \int_0^T \int_{\Omega_1} \Delta y_1(x,t)(\operatorname{div}h(x) - \alpha)y_1(x,t)\,dx\,dt \\
& + a_2 \int_0^T \int_{\Omega_2} \Delta y_2(x,t)(\operatorname{div}h(x) - \alpha)y_2(x,t)\,dx\,dt.
\end{aligned}$$

It follows from Green's Theorem that

$$\begin{aligned}
& \int_0^T \int_\Omega \operatorname{div}(a(x)\nabla y(x,t))(\operatorname{div}h(x) - \alpha)y(x,t)\,dx\,dt = a_1 \int_0^T \int_{\Gamma_1} \frac{\partial y_1(x,t)}{\partial \nu}(\operatorname{div}h(x) - \alpha)y_1(x,t)\,d\Gamma\,dt + \\
& a_1 \int_0^T \int_{\Gamma_0} \frac{\partial y_1(x,t)}{\partial \nu}(\operatorname{div}h(x) - \alpha)y_1(x,t)\,d\Gamma\,dt - a_1 \int_0^T \int_{\Omega_1} \left|\nabla y_1(x,t)\right|^2 (\operatorname{div}h(x) - \alpha)\,dx\,dt - \\
& a_1 \int_0^T \int_{\Omega_1} y_1(x,t)\nabla y_1(x,t).\nabla(\operatorname{div}h(x) - \alpha)\,dx\,dt + a_2 \int_0^T \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial \nu}(\operatorname{div}h(x) - \alpha)y_2(x,t)\,d\Gamma\,dt - \\
& a_2 \int_0^T \int_{\Gamma_0} \frac{\partial y_2(x,t)}{\partial \nu}(\operatorname{div}h(x) - \alpha)y_2(x,t)\,d\Gamma\,dt - a_2 \int_0^T \int_{\Omega_2} \left|\nabla y_2(x,t)\right|^2 (\operatorname{div}h(x) - \alpha)\,dx\,dt - \\
& a_2 \int_0^T \int_{\Omega_2} y_2(x,t)\nabla y_2(x,t).\nabla(\operatorname{div}h(x) - \alpha)\,dx\,dt.
\end{aligned}$$

Thus from (2.3), (2.5) and (2.6), we conclude that

$$\int_0^T\int_\Omega \operatorname{div}(a(x)\nabla y(x,t))(\operatorname{div}h(x)-\alpha)y(x,t)\,dx\,dt = a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(\operatorname{div}h(x)-\alpha)y_2(x,t)\,d\Gamma\,dt-$$

(2.54)

$$\int_0^T\int_\Omega a(x)\left|\nabla y(x,t)\right|^2(\operatorname{div}h(x)-\alpha)\,dx\,dt-\int_0^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(divh(x)-\alpha)\,dx\,dt.$$

From (2.41), (2.43), (2.44), (2.53) and (2.54). We obtain

$$2\int_0^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)\,dx\,dt+\alpha\int_0^T\int_\Omega\{y_t^2(x,t)-a(x)\left|\nabla y(x,t)\right|^2\}\,dx\,dt=$$

$$-\left|\left[\int_\Omega\{2y_t(x,t)h(x).\nabla y(x,t)+(\operatorname{div}h(x)-\alpha)y_t(x,t)y(x,t)\}\,dx\right]_0^{\mathrm{T}}\right|-$$

$$\int_0^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(\operatorname{div}h(x)-\alpha)\,dx\,dt+a_1\int_0^T\int_{\Gamma_1}(\frac{\partial y_1(x,t)}{\partial\nu})^2h(x).\nu(x)\,d\Gamma\,dt-$$

$$(a_1-a_2)\int_0^T\int_{\Gamma_0}\left|\nabla y_1(x,t)\right|^2h(x).\nu(x)\,d\Gamma\,dt-\frac{(a_1-a_2)^2}{a_2}\int_0^T\int_{\Gamma_0}(\frac{\partial y_1(x,t)}{\partial\nu})^2h(x).\nu(x)\,d\Gamma\,dt+$$

$$\int_0^T\int_{\Gamma_2}y_{2,t}^2(x,t)h(x).\nu(x)\,d\Gamma\,dt+2a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)\,d\Gamma\,dt-$$

(2.55)

$$a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2h(x).\nu(x)\,d\Gamma\,dt+a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(\operatorname{div}h(x)-\alpha)y_2(x,t)\,d\Gamma\,dt,$$

after using the boundary conditions (2.3), (2.5) and (2.6).
It follows from (2.9) and Assumption $(H3)$ that

$$2\int_0^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)\,dx\,dt+\alpha\int_0^T\int_\Omega\{y_t^2(x,t)-a(x)\left|\nabla y(x,t)\right|^2\}\,dx\,dt\le$$

$$-\left|\left[\int_\Omega\{2y_t(x,t)h(x).\nabla y(x,t)+(\operatorname{div}h(x)-\alpha)y_t(x,t)y(x,t)\}\,dx\right]_0^{\mathrm{T}}\right|-$$

$$\int_0^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(\operatorname{div}h(x)-\alpha)\,dx\,dt+a_1\int_0^T\int_{\Gamma_1}(\frac{\partial y_1(x,t)}{\partial\nu})^2h(x).\nu(x)\,d\Gamma\,dt+$$

$$\int_0^T\int_{\Gamma_2}y_{2,t}^2(x,t)h(x).\nu(x)\,d\Gamma\,dt+2a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)\,d\Gamma\,dt-$$

(2.56)

$$a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2h(x).\nu(x)\,d\Gamma\,dt+a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(\operatorname{div}h(x)-\alpha)y_2(x,t)\,d\Gamma\,dt.$$

We now estimate both sides of (2.56). From $(H1)$, we have

$$\begin{aligned}\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)\,dx &= \int_\Omega J(x)(\sqrt{a(x)}\nabla y(x,t)).(\sqrt{a(x)}\nabla y(x,t))\,dx\\ &\ge \alpha\int_\Omega a(x)\left|\nabla y(x,t)\right|^2\,dx.\end{aligned}$$

Hence

(2.57)

$$2\int_0^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)\,dx\,dt+\alpha\int_0^T\int_\Omega\{y_t^2(x,t)-a(x)\left|\nabla y(x,t)\right|^2\}\,dx\,dt\ge 2\alpha\mathcal{E}(t).$$

For the terms on the right-hand side (RHS) of (2.56), we have by the Cauchy inequality

$$\left|\left[\int_\Omega \{2y_t(x,t)h(x).\nabla y(x,t) + (\text{div}h(x)-\alpha)y_t(x,t)y(x,t)\}\,dx\right]_0^T\right| \le C(\mathcal{E}(T)+\mathcal{E}(0)). \tag{2.58}$$

$$\begin{aligned}\left|\int_0^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(\text{div}h(x)-\alpha)\,dx\,dt\right| \le{}& \frac{\eta}{2}\int_0^T\int_\Omega a(x)\left|\nabla y(x,t)\right|^2\,dx\,dt\\ &+\frac{C}{\eta}\int_0^T\int_\Omega y^2(x,t)\,dx\,dt.\end{aligned} \tag{2.59}$$

$$\begin{aligned}2a_2\left|\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)\,d\Gamma\,dt\right| \le{}& C\int_0^T\int_{\Gamma_2}\left(\frac{\partial y_2}{\partial\nu}(x,t)\right)^2\,d\Gamma\,dt\\ &+a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2\,d\Gamma\,dt.\end{aligned} \tag{2.60}$$

and from the Poincaré inequality combined with the trace inequality in $H^1(\Omega)$, we obtain

$$\begin{aligned}\left|\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(\text{div}h(x)-\alpha)y_2(x,t)\,d\Gamma\,dt\right| \le{}& \frac{C}{\eta}\int_0^T\int_{\Gamma_2}\left(\frac{\partial y_2}{\partial\nu}(x,t)\right)^2\,d\Gamma\,dt\\ &+\eta C\int_0^T\int_\Omega a(x)\left|\nabla y(x,t)\right|^2\,dx\,dt.\end{aligned} \tag{2.61}$$

In (2.59) and (2.60), η is a positive constant that will be fixed later.
We also have

$$\int_0^T\int_{\Gamma_2} y_{2,t}^2(x,t)h(x).\nu(x)\,d\Gamma\,dt \le C\int_0^T\int_{\Gamma_2} y_{2,t}^2(x,t)\,d\Gamma\,dt, \tag{2.62}$$

$$a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)\,d\Gamma\,dt \le C\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2\,d\Gamma\,dt. \tag{2.63}$$

Inserting (2.57) – (2.63) into (2.56) and recalling Assumption (H2), we obtain

$$\begin{aligned}RHS \text{ of } (2.55) \le{}& C\{\mathcal{E}(T)+\mathcal{E}(0)\}+\eta(\frac{1}{2}+C)\int_0^T\int_\Omega a(x)\left|\nabla y(x,t)\right|^2\,dx\,dt\\ &+C\int_0^T\int_\Omega y^2(x,t)\,dx\,dt + C\int_0^T\int_{\Gamma_2}\left|y_{2,t}(x,t)\right|^2\,d\Gamma\,dt\\ &+C\int_0^T\int_{\Gamma_2}(\frac{\partial y_2(x,t)}{\partial\nu})^2\,d\Gamma\,dt + C\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2\,d\Gamma\,dt.\end{aligned} \tag{2.64}$$

(2.55) together with (2.57) and (2.64) leads to

$$\begin{aligned}(\alpha-\eta(\frac{1}{2}+C))\int_0^T\int_\Omega a(x)\left|\nabla y(x,t)\right|^2\,dx\,dt \le{}& C\{\mathcal{E}(T)+\mathcal{E}(0)\}+C\int_0^T\int_\Omega y^2(x,t)\,dx\,dt\\ &+C\int_0^T\int_{\Gamma_2}\{y_{2,t}^2(x,t)+\left(\frac{\partial y_2}{\partial\nu}(x,t)\right)^2\}\,d\Gamma\,dt\\ &+C\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2\,d\Gamma\,dt.\end{aligned}$$

We choose η sufficiently small to make $\alpha-\eta(\frac{1}{2}+C)>0$, we obtain

$$\begin{aligned}\int_0^T\mathcal{E}(t)\,dt \le{}& C\{\mathcal{E}(T)+\mathcal{E}(0)\}+C\int_0^T\int_{\Gamma_2}\{y_{2,t}^2(x,t)+\left(\frac{\partial y_2}{\partial\nu}(x,t)\right)^2\}\,d\Gamma\,dt\\ &+C\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2\,d\Gamma\,dt + C\int_0^T\int_\Omega y^2(x,t)\,dx\,dt.\end{aligned} \tag{2.65}$$

Using the fact that

$$\|\nabla y\|^2_{L^2(\Gamma_2)} = \left\|\frac{\partial y}{\partial \nu}\right\|^2_{L^2(\Gamma_2)} + \|\nabla_\sigma y\|^2_{L^2(\Gamma_2)}.$$

where $\nabla_\sigma y$ is the tangential gradient of y, (2.65) becomes

$$\int_0^T \mathcal{E}(t)\,dt \leq C\{\mathcal{E}(T)+\mathcal{E}(0)\} + C\int_0^T\int_{\Gamma_2}\{y^2_{2,t}(x,t) + \left(\frac{\partial y_2}{\partial \nu}(x,t)\right)^2\}\,d\Gamma\,dt$$
$$+ C\int_0^T\int_{\Gamma_2}|\nabla_\sigma y_2(x,t)|^2\,d\Gamma\,dt + C\int_0^T\int_\Omega y^2(x,t)\,dx\,dt. \tag{2.66}$$

Step 4.

For fixed $\epsilon > 0$ small we apply estimate (2.66) over the interval $(\epsilon, T-\epsilon)$ rather than $(0,T)$. We obtain

$$\int_\epsilon^{T-\epsilon} \mathcal{E}(t)\,dt \leq C\{\mathcal{E}(T-\epsilon)+\mathcal{E}(\epsilon)\} + C\int_\epsilon^{T-\epsilon}\int_{\Gamma_2}\{y^2_{2,t}(x,t) + \left(\frac{\partial y_2}{\partial \nu}(x,t)\right)^2\}\,d\Gamma\,dt$$
$$+ C\int_\epsilon^{T-\epsilon}\int_{\Gamma_2}|\nabla_\sigma y_2(x,t)|^2\,d\Gamma\,dt + C\int_\epsilon^{T-\epsilon}\int_\Omega y^2(x,t)\,dx\,dt. \tag{2.67}$$

We eliminate the tangential gradient from (2.67) by using the following estimate due to Lasiecka and Triggiani (Lemma 7.2 in [**16**])

$$\int_\epsilon^{T-\epsilon}\int_{\Gamma_2}|\nabla_\sigma y_2(x,t)|^2\,d\Gamma\,dt \leq C(\epsilon,\delta,T)\{\int_0^T\int_{\Gamma_2}\{\left(\frac{\partial y_2(x,t)}{\partial \nu}\right)^2 + y^2_{2,t}(x,t)\}\,d\Gamma\,dt$$
$$+ \|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\},$$

where δ is an arbitrarily small positive constants and $C(\epsilon,\delta,T)$ denotes a positive constant that depends on ϵ,δ and T. We obtain

$$\int_\epsilon^{T-\epsilon} \mathcal{E}(t)\,dt \leq C(\mathcal{E}(T-\epsilon)+\mathcal{E}(\epsilon)) + C(\epsilon,\delta,T)\int_0^T\int_{\Gamma_2}\{\left(\frac{\partial y_2}{\partial \nu}(x,t)\right)^2 + y^2_{2,t}(x,t)\}\,d\Gamma\,dt$$
$$+ C(\epsilon,\delta,T)\,\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}, \tag{2.68}$$

since the $H^{1/2+\delta}-$norm dominates the L^2-norm.

Step 5.

We differentiate $\mathcal{E}(t)$ and apply Green's Theorem to obtain

$$\frac{d}{dt}\mathcal{E}(t) = a_2\int_{\Gamma_2} y_{2,t}(x,t)\frac{\partial y_2(x,t)}{\partial \nu}\,d\Gamma\,dt. \tag{2.69}$$

Integration of both sides of (2.69) from ϵ to $T-\epsilon$, yields

$$\mathcal{E}(\epsilon) = \mathcal{E}(T-\epsilon) - a_2\int_\epsilon^{T-\epsilon}\int_{\Gamma_2} y_{2,t}(x,t)\frac{\partial y_2(x,t)}{\partial \nu}\,d\Gamma\,dt.$$

Application of the Cauchy inequality gives

$$\mathcal{E}(\epsilon) \leq \mathcal{E}(T-\epsilon) + \frac{a_2}{2}\int_0^T\int_{\Gamma_2}\{(\frac{\partial y_2(x,t)}{\partial \nu})^2 + y^2_{2,t}(x,t)\}\,d\Gamma\,dt. \tag{2.70}$$

Insertion of (2.70) into (2.68) results in

$$\int_\epsilon^{T-\epsilon} \mathcal{E}(t)\,dt \leq C\mathcal{E}(T-\epsilon) + C(\epsilon,\delta,T)\int_0^T\int_{\Gamma_2}\{(\frac{\partial y_2(x,t)}{\partial \nu})^2 + y^2_{2,t}(x,t)\}\,d\Gamma\,dt.$$
$$+ C(\epsilon,\delta,T)\,\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}. \tag{2.71}$$

Step 6.
Since $E(t)$ is non-increasing and $E(t) = \mathcal{E}(t) + E_d(t)$, then (2.71) together with (2.40) implies that

$$(T-2\epsilon)E(T-\epsilon) \leq C\mathcal{E}(T-\epsilon) + C(\epsilon,\delta,T)\int_0^T\int_{\Gamma_2}\{(\frac{\partial y_2(x,t)}{\partial \nu})^2 + y_{2,t}^2(x,t)\}\,d\Gamma\,dt +$$

(2.72)
$$C(\epsilon,\delta,T)\,\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))} + TC\int_0^T\int_{\Gamma_2} y_{2,t}^2(x,t-\tau)d\Gamma\,dt,$$

for T large enough. Thus invoking again the identity $E(t) = \mathcal{E}(t) + E_d(t)$ and recalling the boundary condition (2.4), we obtain from (2.72)

$$(T-2\epsilon-C)E(T-\epsilon) \leq C(\epsilon,\delta,T)\int_0^T\int_{\Gamma_2}\{y_{2,t}^2(x,t) + y_{2,t}^2(x,t-\tau)\}d\Gamma dt$$

(2.73)
$$+\, C(\epsilon,\delta,T)\,\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\,.$$

We deduce from (2.73) that for T sufficiently large
(2.74)

$$E(T) \leq C(\epsilon,\delta,T)\int_0^T\int_{\Gamma_2}\{y_{2,t}^2(x,t) + y_{2,t}^2(x,t-\tau)\}d\Gamma dt + C(\epsilon,\delta,T)\,\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))}\,.$$

Step 7.
We prove by a compactness-uniqueness argument that there exists a constant $C>0$ such that

(2.75)
$$\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))} \leq C\int_0^T\int_{\Gamma_2}\{y_{2,t}^2(x,t) + y_{2,t}^2(x,t-\tau)\}\,d\Gamma\,dt.$$

Assume that there exists a sequence y_n of solutions of problem (2.1) – (2.7) with

$$\begin{aligned} y_n(x,0) &= y_n^0(x), y_{nt}(x,0) = y_n^1(x), \quad x\in\Omega,\\ y_n(x,t-\tau) &= f_{n0}(x,t-\tau), \quad x\in\Omega, t\in(0,\tau). \end{aligned}$$

such that
(2.76)

$$\|y_n\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega))} = 1, \quad n=1,2,...; \quad \int_0^T\int_{\Gamma_2}\{y_{2n,t}^2(x,t)+y_{2n,t}^2(x,t-\tau)\}d\Gamma dt \to 0 \text{ as } n\to+\infty$$

Since each solution satisfies (2.74), we deduce from (2.38) and (2.76) that the sequence $Y_n^0 = (y_n^0, y_n^1, f_{n0})$ is bounded in H. Hence there is a subsequence still denoted by Y_n^0 which converges weakly to some $Y^0 = (y^0, y^1, f_0)$. Let y be the solution of problem (2.1) – (2.7) corresponding to such initial conditions. We have from Proposition (2.1)

$$y \in C(0,T;H^1_{\Gamma_1}(\Omega)) \cap C^1(0,T;L^2(\Omega)).$$

Then

$$y_n \to y \text{ in } L^\infty(0,T;H^1_{\Gamma_1}(\Omega)) \qquad \text{weak-star.}$$

This fact along with the compactness $H^1_{\Gamma_1}(\Omega) \to H^{1/2+\delta}(\Omega)$ implies that there exists a subsequence still denoted by y_n such that $y_n \to y$ strongly in $L^\infty(0,T;H^{1/2+\delta}(\Omega))$. Then we have from (2.76)

(2.77)
$$\|y\|_{L^2(0,T;H^{1/2+\delta}(\Omega))} = 1,$$

and

$$\int_0^T\int_{\Gamma_2}\{y_{2,t}^2(x,t) + y_{2,t}^2(x,t-\tau)\}\,d\Gamma\,dt = 0.$$

Thus y satisfies

$$y_t(x,t) = 0 \qquad \text{on } \Gamma_2\times(0,T),$$

and

$$\frac{\partial y(x,t)}{\partial\nu} = 0 \qquad \text{on } \Gamma_2\times(0,T).$$

Let $u(x,t) = y(x,t)$. Then

$$
(2.78) \qquad \begin{cases} u_{tt}(x,t) - \mathrm{div}(a(x)\nabla u(x,t)) = 0 & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \Gamma \times (0,T), \\ \frac{\partial u(x,t)}{\partial \nu} = 0 & (x,t) \in \Gamma_2 \times (0,T), \\ u_1(x,t) = u_2(x,t) & (x,t) \in \Gamma_0 \times (0,T), \\ a_1 \frac{\partial u_1(x,t)}{\partial \nu} = a_2 \frac{\partial u_2(x,t)}{\partial \nu} & (x,t) \in \Gamma_0 \times (0,T). \end{cases}
$$

The solution of (2.78) can be written as

$$
(2.79) \qquad u(x,t) = \begin{cases} u_2(x,t) & (x,t) \in \Omega \times (0,T), \\ u_1(x,t) & (x,t) \in \Omega_1 \times (0,T), \end{cases}
$$

where u_2 and u_1 satisfy respectively

$$
(2.80) \qquad \begin{cases} u_{2,tt}(x,t) - a_2 \Delta u_2(x,t) = 0 & (x,t) \in \Omega \times (0,T), \\ u_2(x,t) = 0 & (x,t) \in \Gamma \times (0,T), \\ \frac{\partial u_2(x,t)}{\partial \nu} = 0 & (x,t) \in \Gamma_2 \times (0,T), \end{cases}
$$

and

$$
(2.81) \qquad \begin{cases} u_{1,tt}(x,t) - a_1 \Delta u_1(x,t) = 0 & (x,t) \in \Omega \times (0,T), \\ u_1(x,t) = 0 & (x,t) \in \Gamma_1 \times (0,T), \\ u_1(x,t) = u_2(x,t) & (x,t) \in \Gamma_0 \times (0,T), \\ a_1 \frac{\partial u_1(x,t)}{\partial \nu} = a_2 \frac{\partial u_2(x,t)}{\partial \nu} & (x,t) \in \Gamma_0 \times (0,T). \end{cases}
$$

From Holmgren's uniqueness theorem applied to problem (2.80), we obtain

$$
u_2(x,t) = 0, \quad (x,t) \in \Omega \times (0,T),
$$

and hence

$$
u_1(x,t) = \frac{\partial u_1(x,t)}{\partial \nu} = 0, \quad (x,t) \in \Gamma_0 \times (0,T).
$$

We have again from Holmgren's uniqueness theorem applied this time to problem (2.81)

$$
u_1(x,t) = 0, \; (x,t) \in \Omega_1 \times (0,T).
$$

(2.79) together with (2.80) and (2.81) implies that

$$
u(x,t) = 0, \qquad (x,t) \in \Omega \times (0,T),
$$

and consequently

$$
y(x,t) = y(x).
$$

Thus y verifies

$$
(2.82) \qquad \begin{cases} -\mathrm{div}(a(x)\nabla y(x)) = 0 & x \in \Omega, \\ y_1(x) = 0 & x \in \Gamma_1, \\ \frac{\partial y_2(x)}{\partial \nu} = 0 & x \in \Gamma_2, \\ y_1(x) = y_2(x) & x \in \Gamma_0, \\ a_1 \frac{\partial y_1(x)}{\partial \nu} = a_2 \frac{\partial y_2(x)}{\partial \nu} & x \in \Gamma_0, \end{cases}
$$

and so $y(x) = 0$ for $x \in \Omega$, and this contradicts (2.77).

Step 8.

The estimate (2.38) together with (2.74) and (2.75) yields

$$
(2.83) \qquad E(T) \leq \frac{C}{k+C} E(0)
$$

The desired conclusion follows now from (2.83) since $0 < \frac{C}{k+C} < 1$ (see [**10**], page 299, Proposition 1.7).

CHAPTER 3

Stability of the transmission wave equation with delay terms in the boundary and internal feedbacks

3.1. Introduction

In this chapter we investigate stability of the transmission wave equation with a delay terms in the boundary and internal feedbacks.
Let Ω be an open bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega = \Gamma$ which consists of two parts Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 \cup \Gamma_2$, with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$ and $\Gamma_1 \neq \emptyset$. Let Γ_0 be a smooth hypersurface which separates Ω into two subdomains Ω_1 and Ω_2 such that $\Gamma_1 \subset \partial\Omega_1$ and $\Gamma_2 \subset \partial\Omega_2$.
Let ω be a subdomain of Ω such that $\Gamma_0 \subset \omega$ and $\omega_i = \omega \cap \Omega_i$ with i=1,2.
Furthermore, we assume that there exists a real vector field $h \in (C^2(\overline{\Omega}))^n$ such that:
$(H.1)$ The Jacobian matrix J of h satisfies

$$\int_\Omega J(x)\zeta(x).\zeta(x)d\Omega \geq \alpha \int_\Omega |\zeta(x)|^2 \, d\Omega,$$

for some constant $\alpha > 0$ and for all $\zeta \in L^2(\Omega;\mathbb{R}^n)$;
$(H.2)$ $h(x).\nu(x) \leq 0$ on Γ_1;
where ν is the unit normal on Γ or Γ_0 pointing towards the exterior of Ω or Ω_1.
In Ω, we consider the problem of transmission of the wave equation with time delay terms in both internal and boundary feedbacks

$$\begin{aligned}
&(3.1) && y_{tt}(x,t) - div(a(x)\nabla y(x,t)) \\
& && + \chi_w(x)\{\alpha_1 y_t(x,t) + \alpha_2 y_t(x,t-\tau)\} = 0 \quad \text{in } \Omega \times (0,+\infty),\\
&(3.2) && y(x,0) = y^0(x), y_t(x,0) = y^1(x) && \text{in } \Omega,\\
&(3.3) && y_1(x,t) = 0 && \text{on } \Gamma_1 \times (0,+\infty),\\
&(3.4) && \frac{\partial y_2}{\partial \nu}(x,t) = -\beta_1 y_{2,t}(x,t) - \beta_2 y_{2,t}(x,t-\tau) && \text{on } \Gamma_2 \times (0,+\infty),\\
&(3.5) && y_1(x,t) = y_2(x,t) && \text{on } \Gamma_0 \times (0,+\infty),\\
&(3.6) && a_1\frac{\partial y_1}{\partial \nu}(x,t) = a_2\frac{\partial y_2}{\partial \nu}(x,t) && \text{on } \Gamma_0 \times (0,+\infty),\\
&(3.7) && y_t(x,t-\tau) = f^0(x,t-\tau) && \text{in } \omega_2 \times (0,\tau),\\
&(3.8) && y_{2,t}(x,t-\tau) = g^0(x,t-\tau) && \text{on } \Gamma \times (0,\tau),
\end{aligned}$$

where:

- $a(x) = \begin{cases} a_1, & x \in \Omega_1, \\ a_2, & x \in \Omega_2. \end{cases}$
- $y(x,t) = \begin{cases} y_1(x,t), (x,t) \in \Omega_1 \times (0,+\infty), \\ y_2(x,t), (x,t) \in \Omega_2 \times (0,+\infty). \end{cases}$
- $f^0 = \begin{cases} f_1^0(x,t-\tau), & \text{in } \omega_1 \times (0,\tau), \\ f_2^0(x,t-\tau), & \text{in } \omega_2 \times (0,\tau). \end{cases}$
- $\dfrac{\partial}{\partial \nu}$ is the normal derivative.
- τ is the time delay.
- y^0, y^1, f^0 and g^0 are the initial data which belong to suitable spaces.

It is well known that in the absence of delay (i.e. $\alpha_2 = \beta_2 = 0$), the solution of problem (3.1) – (3.8) decays exponentially to zero (see [**22**]).
In this chapter we investigate the uniform exponential stability of the system in the case where the internal and boundary damping coefficients are strictly positive such that the condition $a_1 > a_2$, contrary to the previous chapter, is not needed to establish the exponential stability.

3.2. Main result

Assume that

$$\alpha_1 > \alpha_2, \beta_1 > \beta_2, \tag{3.9}$$

and define the energy of a solution of (3.1) – (3.8) by

$$\begin{aligned} E(t) =& \frac{1}{2}\int_\Omega \left[a(x)\,|\nabla y(x,t)|^2 + y_t^2(x,t)\right]\,dx + \frac{\mu}{2}\int_\Omega \chi_\omega(x)\int_0^1 y_t^2(x,t-\tau\rho)\,d\rho\,dx \\ &+ \frac{\xi}{2}\int_{\Gamma_2}\int_0^1 y_{2,t}^2(x,t-\tau\rho)\,d\rho\,d\Gamma \end{aligned} \tag{3.10}$$

where

$$\tau\alpha_2 \le \mu \le \tau(2\alpha_1 - \alpha_2), \tag{3.11}$$

$$a_2\tau\beta_2 \le \xi \le a_2\tau(2\beta_1 - \beta_2). \tag{3.12}$$

We show that if in addition to (3.9), $\{\Omega, \Gamma_0, \Gamma_1, \Gamma_2\}$ satisfies $(H.1)$ and $(H.2)$, then there is an exponential decay rate for $E(t)$. The proof of this result combines multiplier technique and compactness-uniqueness arguments.
The main result of this chapter can be stated as follows.

THEOREM 3.1. *Assume* $(H.1)$, $(H.2)$ *and* (3.9). *Then there exist constants* $M \ge 1$ *and* $\omega > 0$ *such that*

$$E(t) \le Me^{-\omega t}E(0).$$

Theorem 3.1 is proved in Section 3.4. In Section 3.3 , we study the well-posedness of system (3.1) – (3.8) using semigroup theory.

3.3. Well-posedness

Set

$$u(x,\rho,t) = y_t(x,t-\tau\rho), \quad x \in \omega,\ \rho \in (0,1),$$
$$v(x,\rho,t) = y_{2,t}(x,t-\tau\rho), \quad x \in \Gamma_2,\ \rho \in (0,1).$$

With these new unknowns the problem (3.1) – (3.8) equivalent to

$$
\begin{aligned}
&y_{tt}(x,t) - \operatorname{div}(a(x)\nabla y(x,t)) \\
(3.13)\quad & + \chi_w(x)\{\alpha_1 y_t(x,t) + \alpha_2 u(x,1,t)\} = 0 && \text{in } \Omega \times (0,+\infty),\\
(3.14)\quad & u_t(x,\rho,t) + \tau^{-1}u_\rho(x,\rho,t) = 0 && \text{in } \omega \times (0,1) \times (0,+\infty),\\
(3.15)\quad & v_t(x,\rho,t) + \tau^{-1}v_\rho(x,\rho,t) = 0 && \text{on } \Gamma_2 \times (0,1) \times (0,+\infty),\\
(3.16)\quad & y(x,0) = y^0(x), y_t(x,0) = y^1(x) && \text{in } \Omega,\\
(3.17)\quad & y_1(x,t) = 0 && \text{on } \Gamma_1 \times (0,+\infty),\\
(3.18)\quad & \frac{\partial y_2}{\partial \nu}(x,t) = -\beta_1 y_t(x,t) - \beta_2 v(x,1,t) && \text{on } \Gamma_2 \times (0,+\infty),\\
(3.19)\quad & y_1(x,t) = y_2(x,t) && \text{on } \Gamma_0 \times (0,+\infty),\\
(3.20)\quad & a_1\frac{\partial y_1}{\partial \nu}(x,t) = a_2\frac{\partial y_2}{\partial \nu}(x,t) && \text{on } \Gamma_0 \times (0,+\infty),\\
(3.21)\quad & u(x,0,t) = y_t(x,t) && \text{in } \omega \times (0,\tau),\\
(3.22)\quad & v(x,0,t) = y_{2,t}(x,t) && \text{on } \Gamma_2 \times (0,\tau),\\
(3.23)\quad & u(x,\rho,0) = f^0(x,t-\tau) && \text{in } \omega \times (0,\tau),\\
(3.24)\quad & v(x,\rho,0) = g^0(x,t-\tau) && \text{on } \Gamma_2 \times (0,\tau).
\end{aligned}
$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Omega; L^2(0,1)) \times L^2(\Gamma_2; L^2(0,1)),$$

where

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\},$$

endowed with the inner product

$$
\left\langle \begin{pmatrix} \zeta \\ \eta \\ \varphi \\ \psi \end{pmatrix}; \begin{pmatrix} \overline{\zeta} \\ \overline{\eta} \\ \overline{\varphi} \\ \overline{\psi} \end{pmatrix} \right\rangle = \int_\Omega (a(x)\nabla\zeta(x).\nabla\overline{\zeta}(x) + \eta(x)\overline{\eta}(x))\,dx + \mu\int_\Omega \chi_w(x)\int_0^1 \varphi(x,\rho)\overline{\varphi}(x,\rho)\,d\rho\,dx
$$
$$
+\,\xi\int_{\Gamma_2}\int_0^1 \psi(x,\rho)\overline{\psi}(x,\rho)\,d\rho\,d\Gamma.
$$

Define in $\mathcal{H}$ a linear operator $\mathcal{A}$ by

$$
\begin{aligned}
&D(\mathcal{A}) = \{(\zeta,\eta,\varphi,\psi)^T \in H^2(\Omega,\Gamma_0) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega; H^1(0,1)) \times L^2(\Gamma_2; H^1(0,1))\\
(3.25)\quad &\qquad \frac{\partial\zeta}{\partial\nu} = -\beta_1\eta - \beta_2\psi(.,1), \eta = \varphi(.,0) \text{ in } \omega,\ \eta = \psi(.,0) \text{ on } \Gamma_2\},\\
(3.26)\quad &\qquad A(\zeta,\eta,\varphi,\psi)^T = (\eta, \operatorname{div}(a(x)\nabla\zeta) - \chi_w(x)\{\alpha_1\eta + \alpha_2\varphi(.,1)\}, -\tau^{-1}\varphi_\rho, -\tau^{-1}\psi_\rho)^T,
\end{aligned}
$$

where

$H^2(\Omega,\Gamma_0) = \{u_i \in H^2(\Omega_i) : \quad u = 0 \quad \text{on } \Gamma_1, \quad u_1 = u_2 \quad \text{and} \quad a_1\frac{\partial u_1}{\partial\nu} = a_2\frac{\partial u_2}{\partial\nu} \text{ on } \Gamma_0\}.$

Then we can rewrite (3.13) – (3.24) as an abstract Cauchy problem in $\mathcal{H}$

$$
(3.27)\qquad \begin{cases} \dfrac{dU}{dt}(t) = \mathcal{A}U(t); \\ U(0) = U_0. \end{cases}
$$

where

$$
\begin{aligned}
&U(t) = (y(x,t), y_t(x,t), u(x,\rho,\tau), v(x,\rho,t))^T,\\
&\text{and } U_0 = (y^0, y^1, f^0(.,-.\tau), g^0(.,-.\tau))^T.
\end{aligned}
$$

We verify that $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{H}$ and consequently we have

Proposition 3.1. *For every $U_0 \in \mathcal{H}$, problem* (3.27) *has a unique solution U whose regularity depends on the initial datum U_0 as follows:*

$$\begin{aligned} U(.) &\in C([0,+\infty);\mathcal{H}) \text{ if } U_0 \in \mathcal{H},\\ U(.) &\in C^1([0,+\infty);\mathcal{H}) \cap C([0,+\infty);D(\mathcal{A})) \text{ if } U_0 \in D(\mathcal{A}). \end{aligned}$$

Proof. Let $(\zeta,\eta,\varphi,\psi)^T \in D(A)$. Then

$$\langle AU;U\rangle = \int_\Omega \left[a(x)\nabla\eta(x).\nabla\zeta(x) + \operatorname{div}(a(x)\nabla\zeta(x))\eta(x) - \chi_w(x)\{\alpha_1\eta^2(x) + \alpha_2\varphi(x,1)\eta(x)\}\right] dx$$

(3.28)

$$-\tau^{-1}\mu\int_\Omega \chi_\omega(x)\int_0^1 \varphi_\rho(x,\rho)\varphi(x,\rho)\,d\rho\,d\Gamma - \tau^{-1}\xi\int_{\Gamma_2}\int_0^1 \psi_\rho(x,\rho)\psi(x,\rho)\,d\rho\,d\Gamma.$$

Applying Green's Theorem and recalling (3.25), we get

(3.29)

$$\int_\Omega \operatorname{div}(a(x)\nabla\zeta(x))\eta(x)\,dx = -\,a_2\beta_1\int_{\Gamma_2}\eta^2(x)\,d\Gamma - a_2\beta_2\int_{\Gamma_2}\psi(x,1)\eta(x)\,d\Gamma - \int_\Omega a(x)\nabla\eta(x).\nabla\zeta(x)\,dx.$$

Integrating by parts with respect to ρ, we obtain

$$\tau^{-1}\mu\int_\Omega \chi_\omega(x)\int_0^1 \varphi_\rho(x,\rho)\varphi(x,\rho)\,d\rho\,d\Gamma + \tau^{-1}\xi\int_{\Gamma_2}\int_0^1 \psi_\rho(x,\rho)\psi(x,\rho)\,d\rho\,d\Gamma =$$

$$\frac{\tau^{-1}\mu}{2}\int_\omega [\varphi^2(x,1) - \eta^2(x)]\,dx + \frac{\xi\tau^{-1}}{2}\int_{\Gamma_2}[\psi^2(x,1) - \eta^2(x)]\,d\Gamma. \tag{3.30}$$

Inserting (3.29), (3.30) into (3.28) and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\langle AU,U\rangle \le{}& (-\alpha_1 + \frac{\alpha_2}{2} + \frac{\tau^{-1}\mu}{2})\int_\omega \eta^2(x)\,dx + (\frac{\alpha_2}{2} - \frac{\tau^{-1}\mu}{2})\int_\omega \varphi^2(x,1)\,dx\\ &(-\beta_1 a_2 + \frac{\beta_2 a_2}{2} + \frac{\tau^{-1}\xi}{2})\int_{\Gamma_2}\eta^2(x)\,d\Gamma + (\frac{\beta_2 a_2}{2} - \frac{\tau^{-1}\xi}{2})\int_{\Gamma_2}\psi^2(x,1)\,d\Gamma\end{aligned}$$

From (3.11) and (3.12), we conclude that $\langle AU,U\rangle \le 0$. Thus A is dissipative.
Now, we show that $\lambda I - A$ is onto for a fixed $\lambda > 0$, that is for $(f,g,h,p)^T \in \mathcal{H}$, there exists $U = (\zeta,\eta,\varphi,\psi)^T \in D(A)$ solution of

$$(\lambda I - A)U = (f,g,h,p)$$

or equivalently

$$\lambda\zeta - \eta = f, \tag{3.31}$$

$$\lambda\eta - \operatorname{div}(a\nabla\zeta) - \chi_w(x)\{\alpha_1\eta + \alpha_2\varphi(.,1)\} = g, \tag{3.32}$$

$$\lambda\varphi + \tau^{-1}\varphi_\rho = h, \tag{3.33}$$

$$\lambda\psi + \tau^{-1}\psi_\rho = p. \tag{3.34}$$

Suppose that we have found ζ with the appropriate regularity, then

$$\eta = \lambda\zeta - f, \tag{3.35}$$

consequently we can find φ from (3.33) with (3.25) and ψ from (3.34) with (3.25).
In fact, φ is the unique solution of the initial value problem

$$\begin{aligned}&\varphi_\rho(x,\rho) = -\tau\lambda\varphi(x,\rho) + \tau h(x,\rho),\ x \in \Omega,\ \rho \in (0,1),\\ &\varphi(x,0) = \eta(x),\ x \in \Omega.\end{aligned}$$

given by

$$\varphi(x,\rho) = \eta(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho h(x,\sigma)e^{\lambda\tau\sigma}d\sigma,\ x \in \Omega,\ \rho \in (0,1),$$

and in particular

$$\varphi(x,1)=\lambda e^{-\lambda\tau}\zeta(x)+z_0(x), \tag{3.36}$$

with z_0 defined by

$$z_0(x)=-f(x)e^{-\lambda\tau}+\tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma,$$

and ψ is the unique solution of the initial value problem

$$\begin{aligned}&\psi_\rho(x,\rho)=-\tau\lambda\psi(x,\rho)+\tau p(x,\rho),\ x\in\Gamma_2,\ \rho\in(0,1),\\&\psi(x,0)=\eta(x),\ x\in\Gamma_2.\end{aligned}$$

given by

$$\psi(x,\rho)=\eta(x)e^{-\lambda\tau\rho}+\tau e^{-\lambda\tau\rho}\int_0^1 p(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma,$$

and in particular

$$\psi(x,1)=\lambda e^{-\lambda\tau}\zeta(x)+z_1(x), \tag{3.37}$$

with z_1 defined by

$$z_1(x)=-f(x)e^{-\lambda\tau}+\tau e^{-\lambda\tau}\int_0^1 p(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma(x),\ x\in\Gamma_2,$$

From (3.32) and (3.35), the function ζ verifies

$$\lambda^2\zeta(x)-\operatorname{div}(a(x)\nabla u(x))-\chi_w(x)\{\alpha_1\eta(x)+\alpha_2\varphi(x,1)\}=g(x)+\lambda f(x), \tag{3.38}$$

Problem (3.38) can be reformulated as

$$\begin{aligned}&\int_\Omega\left[\lambda^2\zeta(x)-\operatorname{div}(a(x)\nabla u)-\chi_w(x)\{\alpha_1\eta(x)+\alpha_2\varphi(x,1)\}\right]w(x)\,dx\\&=\int_\Omega(g(x)+\lambda f(x))w(x)\,dx\quad\forall w\in H^1_{\Gamma_1}(\Omega).\end{aligned}$$

We rewrite the left-hand side of last equality after using Green's Theorem and recalling (3.35), (3.36) and (3.37)

$$\begin{aligned}&\int_\Omega(\lambda^2\zeta(x)-\operatorname{div}(a(x)\nabla u)-\chi_w(x)\{\alpha_1\eta(x)+\alpha_2\varphi(x,1)\})w(x)\,dx\\&=\int_\Omega(\lambda^2\zeta(x)w(x)+a(x)\nabla\zeta(x)\nabla w(x))\,dx\\&-\alpha_1\int_\omega(\lambda\zeta(x)-f(x))w(x)\,dx-\alpha_2\int_\omega(\lambda e^{-\lambda\tau}\zeta(x)+z_0(x))w(x)\,dx\\&+a_2\int_{\Gamma_2}\left[\beta_1(\lambda\zeta(x)-f(x))w(x)+\beta_2(\lambda e^{-\lambda\tau}\zeta(x)+z_1(x))w(x)\right]d\Gamma.\end{aligned}$$

Therefore

(3.39)

$$\begin{aligned}&\int_\Omega(\lambda^2\zeta(x)w(x)+a(x)\nabla\zeta(x).\nabla w(x))\,dx-\int_\omega\lambda(\alpha_1+\alpha_2e^{-\lambda\tau})\zeta(x)\,dx\\&+\int_{\Gamma_2}a_2\lambda(\beta_1+\beta_2e^{-\lambda\tau})\zeta(x)w(x)\,d\Gamma=\int_\Omega(g(x)+\lambda f(x))w(x)\,dx-\alpha_1\int_\omega f(x)w(x)\,dx\end{aligned}$$

(3.40)

$$+\alpha_2\int_\omega z_0(x)w(x)\,dx-a_2\beta_1\int_{\Gamma_2}f(x)w(x)\,d\Gamma-a_2\beta_2\int_{\Gamma_2}z_1(x)w(x)\,d\Gamma$$

Since the left-hand side of (3.40) is coercive and continuous on $H^1_{\Gamma_1}(\Omega)$, and the right-hand side defines a continuous linear form on $H^1_{\Gamma_1}(\Omega)$, the Lax-Milgram Theorem guarantees the

existence and uniqueness of a solution $u \in H^1_{\Gamma_1}(\Omega)$ of (3.40).
If we consider $w \in \mathcal{D}(\Omega)$ in (3.40), then ζ is a solution in $\mathcal{D}'(\Omega)$ of

$$\lambda^2\zeta(x) - \operatorname{div}(a(x)\nabla\zeta(x)) - \chi_w(x)\{\alpha_1\eta(x) + \alpha_2\varphi(x,1)\} = g(x) + \lambda f(x) \tag{3.41}$$

and thus $\operatorname{div}(a(x)\nabla\zeta) \in L^2(\Omega)$.
Combining (3.40) together with (3.41), we obtain after using Green's Theorem

$$a_2\int_{\Gamma_2}(\beta_1 + \beta_2 e^{-\lambda\tau})\lambda\zeta w\,d\Gamma + a_2\int_{\Gamma_2}\frac{\partial\zeta}{\partial\nu}w\,d\Gamma = -a_2\beta_1\int_{\Gamma_2} fw\,d\Gamma - a_2\beta_2\int_{\Gamma_2} z_1 w\,d\Gamma$$

which implies that

$$\frac{\partial\zeta}{\partial\nu} = -\beta_1\eta - \beta_2\psi(.,1)$$

So, we have found $(\zeta,\eta,\varphi,\psi)^T \in D(A)$ which satisfies (3.31) – (3.34). Thus, by the Lumer-Phillips Theorem, A is the generator of a C_0- semigroup of contractions on $\mathcal{H}$. □

3.4. Proof of the main result

We prove Theorem 3.1 for smooth solution. The general case follows by density argument. We proceed in several steps.

Step 1.

We first show that the energy $E(t)$ defined by (3.10) is decreasing.
We differentiate $E(t)$ with respect to time and recall the boundary condition in (3.3)-(3.6), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) =& -\alpha_1\int_\omega y_t^2(x,t)\,dx - \alpha_2\int_\omega y_t(x,t)y_t(x,t-\tau)\,dx - a_2\beta_1\int_{\Gamma_2} y_{2,t}^2(x,t)\,d\Gamma\\ &- a_2\beta_2\int_{\Gamma_2} y_{2,t}(x,t)y_{2,t}(x,t-\tau)\,d\Gamma + \mu\int_\omega\int_0^1 y_{tt}(x,t-\tau\rho)y_t(x,t-\tau\rho)\,d\rho\,dx\\ &+ \xi\int_{\Gamma_2}\int_0^1 y_{2,tt}(x,t-\tau\rho)y_{2,t}(x,t-\tau\rho)\,d\rho\,d\Gamma.\end{aligned} \tag{3.42}$$

Now, we have

$$\int_\omega\int_0^1 y_{tt}(x,t-\tau\rho)y_t(x,t-\tau\rho)\,d\rho\,dx = \frac{\tau^{-1}}{2}\int_\omega\left\{y_t^2(x,t) - y_t^2(x,t-\tau)\right\}dx. \tag{3.43}$$

and

$$\int_{\Gamma_2}\int_0^1 y_{2,tt}(x,t-\tau\rho)y_{2,t}(x,t-\tau\rho)\,d\rho\,d\Gamma = \frac{\tau^{-1}}{2}\int_{\Gamma_2}\left\{y_{2,t}^2(x,t) - y_{2,t}^2(x,t-\tau)\right\}d\Gamma, \tag{3.44}$$

Applying Cauchy-Schwarz inequality after inserting (3.43) and (3.44) into (3.42), we get

$$\begin{aligned}\frac{d}{dt}E(t) \le& (-\alpha_1 + \frac{\alpha_2}{2} + \frac{\mu\tau^{-1}}{2})\int_\omega y_t^2(x,t)\,dx + (\frac{\alpha_2}{2} - \frac{\mu\tau^{-1}}{2})\int_\omega y_t^2(x,t-\tau)\,dx\\ &+ (-a_2\beta_1 + \frac{a_2\beta_2}{2} + \frac{\xi\tau^{-1}}{2})\int_{\Gamma_2} y_{2,t}^2(x,t)\,d\Gamma + (\frac{a_2\beta_2}{2} - \frac{\xi\tau^{-1}}{2})\int_{\Gamma_2} y_{2,t}(x,t-\tau)\,d\Gamma,\end{aligned}$$

which implies

$$\frac{d}{dt}E(t) \le -C_1\left\{\int_\omega\{y_t^2(x,t) + y_t^2(x,t-\tau)\}\,dx + \int_{\Gamma_2}\{y_{2,t}^2(x,t) + y_{2,t}^2(x,t-\tau)\}\,d\Gamma\right\}, \tag{3.45}$$

where

$$C_1 = \min\left\{(\alpha_1 - \frac{\alpha_2}{2} - \frac{\tau^{-1}\mu}{2}), (-\frac{\alpha_2}{2} + \frac{\tau^{-1}\mu}{2}), (a_2\beta_1 - \frac{a_2\beta_2}{2} - \frac{\xi\tau^{-1}}{2}), (-\frac{a_2\beta_2}{2} + \frac{\xi\tau^{-1}}{2})\right\}.$$

Step 2.

We rewrite

$$E(t) = \mathcal{E}(t) + E_d(t) + E_b(t),$$

where

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \{a(x)\,|\nabla y(x,t)|^2 + y_t^2(x,t)\}\,dx,$$

$$E_d(t) = \frac{\mu}{2}\int_\Omega \chi_\omega(x)\int_0^1 y_t^2(x,t-\tau\rho)\,d\rho\,dx,$$

and

$$E_b(t) = \frac{\xi}{2}\int_{\Gamma_2}\int_0^1 y_{2,t}^2(x,t-\tau\rho)\,d\rho\,d\Gamma.$$

With a change of variable we can rewrite

$$E_d(t) = \frac{\mu}{2\tau}\int_t^{t+\tau}\int_\Omega \chi_\omega(x) y_t^2(x,s-\tau)\,dx\,ds,$$

and

$$E_b(t) = \frac{\xi}{2\tau}\int_t^{t+\tau}\int_{\Gamma_2} y_{2,t}^2(x,s-\tau)\,d\Gamma\,ds.$$

from which we deduce

(3.46)
$$E_d(t) \le C_2\int_S^T\int_\omega y_t^2(x,t-\tau)\,dx\,dt,$$

and

(3.47)
$$E_b(t) \le C_3\int_S^T\int_{\Gamma_2} y_{2,t}^2(x,s-\tau)\,d\Gamma\,dt.$$

for $0 < S \le t \le T$ and T large enough. **Step 3.**
Concerning $\mathcal{E}(.)$, we multiply both sides of (3.1) by $2h.\nabla y + (\mathrm{div}h - \alpha)y$ and integrate over $\Omega \times (S,T)$, we obtain

(3.48)
$$\int_S^T\int_\Omega (y_{tt} - \mathrm{div}(a\nabla y) + \chi_\omega\{\alpha_1 y_t(x,t) + \alpha_2 y_t(x,t-\tau)\})\,(2h.\nabla y + (\mathrm{div}h-\alpha)y)\;dx\,dt = 0.$$

From the previous chapter, we have

$$\begin{aligned}
&\int_S^T\int_\Omega (y_{tt}(x,t) - \mathrm{div}(a(x)\nabla y(x,t)))\,(2h.\nabla y + (\mathrm{div}h-\alpha)y)\;dx\,dt\\
&= -2\int_S^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)\,dx\,dt - \alpha\int_S^T\int_\Omega \{y_t^2(x,t) - a(x)\,|\nabla y(x,t)|^2\}\,dx\,dt\\
&- \left|\left[\int_\Omega \{2y_t(x,t)h(x).\nabla y(x,t) + (\mathrm{div}h(x)-\alpha)y_t(x,t)y(x,t)\}\,dx\right]_S^{\mathrm{T}}\right|\\
&- \int_S^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(\mathrm{div}h(x)-\alpha)\,dx\,dt\\
&+ a_1\int_S^T\int_{\Gamma_1}\{2a_1\frac{\partial y_1}{\partial\nu}(x,t)h(x).\nabla y_1(x,t) - a_1\,|\nabla y_1(x,t)|^2\,h.\nu\,\}\,d\Gamma\,dt\\
&+ 2a_1\int_S^T\int_{\Gamma_0}\frac{\partial y_1}{\partial\nu}(x,t)h.\nabla y_1\,d\Gamma\,dt - 2a_2\int_S^T\int_{\Gamma_0}\frac{\partial y_2}{\partial\nu}(x,t)h.\nabla y_2\,d\Gamma\,dt\\
&- a_1\int_S^T\int_{\Gamma_0}|\nabla y_1(x,t)|^2\,h.\nu\,d\Gamma\,dt + a_2\int_S^T\int_{\Gamma_0}|\nabla y_2(x,t)|^2\,h.\nu\,d\Gamma\,dt\\
&+ \int_S^T\int_{\Gamma_2} y_{2,t}^2(x,t)h.\nu\,d\Gamma\,dt + 2a_2\int_S^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)\,d\Gamma\,dt -
\end{aligned}$$

(3.49)
$$a_2\int_S^T\int_{\Gamma_2}|\nabla y_2(x,t)|^2\,h.\nu\,d\Gamma\,dt + a_2\int_S^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(\mathrm{div}h(x)-\alpha)y_2(x,t)\,d\Gamma\,dt.$$

Insertion (3.49) into (3.48) yields

$$\begin{aligned}
&2\int_S^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)\,dx\,dt+\alpha\int_S^T\int_\Omega\{y_t^2(x,t)-a(x)\left|\nabla y(x,t)\right|^2\}\,dx\,dt\\
&=-\left|\left[\int_\Omega\{2y_t(x,t)h(x).\nabla y(x,t)+(\mathrm{div}h(x)-\alpha)y_t(x,t)y(x,t)\}\,dx\right]_S^{\mathrm{T}}\right|\\
&-\int_S^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(\mathrm{div}h(x)-\alpha)\,dx\,dt\\
&+a_1\int_S^T\int_{\Gamma_1}\{2a_1\frac{\partial y_1}{\partial\nu}(x,t)h(x).\nabla y_1(x,t)-a_1\left|\nabla y_1(x,t)\right|^2h.\nu\,\}\,d\Gamma\,dt\\
&+2a_1\int_S^T\int_{\Gamma_0}\frac{\partial y_1}{\partial\nu}(x,t)h.\nabla y_1\,d\Gamma\,dt-2a_2\int_S^T\int_{\Gamma_0}\frac{\partial y_2}{\partial\nu}(x,t)h.\nabla y_2\,d\Gamma\,dt\\
&-a_1\int_S^T\int_{\Gamma_0}\left|\nabla y_1(x,t)\right|^2h.\nu\,d\Gamma\,dt+a_2\int_S^T\int_{\Gamma_0}\left|\nabla y_2(x,t)\right|^2h.\nu\,d\Gamma\,dt\\
&+\int_S^T\int_{\Gamma_2}y_{2,t}^2(x,t)h.\nu\,d\Gamma\,dt+2a_2\int_S^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)\,d\Gamma\,dt-\\
&a_2\int_S^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2h.\nu\,d\Gamma\,dt+a_2\int_S^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(\mathrm{div}h(x)-\alpha)y_2(x,t)\,d\Gamma\,dt
\end{aligned}$$

(3.50)

$$+\int_S^T\int_\omega\{\alpha_1y_t(x,t)+\alpha_2y_t(x,t-\tau)\}\left(2h.\nabla y+(\mathrm{div}h-\alpha)y\right)\,dx\,dt.$$

Now, we estimate both sides of (3.50). From (H.1), we have

$$\begin{aligned}
\int_\Omega a(x)J(x)\nabla y.\nabla y\,dx&=\int_\Omega J(x)(\sqrt{a(x)}\nabla y).(\sqrt{a(x)}\nabla y)\,dx\\
&\geq\alpha\int_\Omega a(x)\left|\nabla y(x,t)\right|^2\,dx,
\end{aligned}\tag{3.51}$$

then, for the term on the left-hand side of (3.50) we have

$$2\int_S^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)\,dx\,dt+\alpha\int_S^T\int_\Omega\{y_t^2(x,t)-a(x)\left|\nabla y(x,t)\right|^2\}\,dx\,dt\geq2\alpha\mathcal{E}(t).\tag{3.52}$$

For the terms on the right-hand side of (3.50), we have by the Cauchy Schwarz inequality

$$\left|\left[\int_\Omega\{2y_t(x,t)h(x).\nabla y(x,t)+(\mathrm{div}h(x)-\alpha)y_t(x,t)y(x,t)\}\,dx\right]_S^{\mathrm{T}}\right|\leq C\left(\mathcal{E}(T)+\mathcal{E}(S)\right)\tag{3.53}$$

$$\begin{aligned}
\left|\int_S^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(\mathrm{div}h(x)-\alpha)\,dx\,dt\right|\leq&\frac{\eta}{2}\int_S^T\int_\Omega a(x)\left|\nabla y(x,t)\right|^2\,dx\,dt\\
&+\frac{C}{\eta}\int_S^T\int_\Omega y^2(x,t)\,dx\,dt,
\end{aligned}\tag{3.54}$$

$$\begin{aligned}
2a_2\left|\int_S^T\int_{\Gamma_2}\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2(x,t)\,d\Gamma\,dt\right|\leq&C\int_S^T\int_{\Gamma_2}\left(\frac{\partial y_2}{\partial\nu}(x,t)\right)^2\,d\Gamma\,dt\\
&+a_2\int_S^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2\,d\Gamma\,dt,
\end{aligned}\tag{3.55}$$

$$\left|\int_S^T\int_\omega\{\alpha_1 y_t(x,t)+\alpha_2 y_t(x,t-\tau)\}\,(2h.\nabla y+(\operatorname{div}h-\alpha)y)\;dx\,dt\right|\le C\eta\int_S^T\int_\omega a(x)\,|\nabla y(x,t)|^2\;d\Gamma\,dt$$
(3.56)
$$+\frac{C}{\eta}\int_S^T\int_\omega\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\,dx\,dt+C\int_S^T\int_\Omega y^2(x,t)\,dx\,dt.$$

By using the fact that $y_1(x,t)=0$ on $\Gamma_1\times(0,T)$ and $h.\nu\le 0$ on Γ_1, we get

$$\int_S^T\int_{\Gamma_1}\{2a_1\frac{\partial y_1}{\partial\nu}(x,t)h(x).\nabla y_1(x,t)-a_1\,|\nabla y_1(x,t)|^2\,h.\nu\,\}\,d\Gamma\,dt=\int_S^T\int_{\Gamma_1}a_1\left(\frac{\partial y_1}{\partial\nu}(x,t)\right)^2h.\nu\,d\Gamma\,dt \le 0$$

and from the Poincaré inequality combined with the trace inequality in $H^1(\Omega)$, we obtain

$$\left|\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(\operatorname{div}h(x)-\alpha)y_2(x,t)\,d\Gamma\,dt\right|\le\frac{C}{\eta}\int_0^T\int_{\Gamma_2}\left(\frac{\partial y_2}{\partial\nu}(x,t)\right)^2\;d\Gamma\,dt$$
$$+\eta C\int_0^T\int_\Omega a(x)\,|\nabla y(x,t)|^2\;dx\,dt. \tag{3.57}$$

In (3.54), (3.56) and (3.57), η is a positive constant that will be fixed later.
We have also

$$\left|\int_S^T\int_{\Gamma_2}y_{2,t}^2(x,t)h.\nu\,d\Gamma\,dt\right|\le C\int_S^T\int_{\Gamma_2}y_{2,t}^2(x,t)\,d\Gamma\,dt. \tag{3.58}$$

$$\left|a_2\int_S^T\int_{\Gamma_2}|\nabla y_2(x,t)|^2\,h.\nu\,d\Gamma\,dt\right|\le C\int_S^T\int_{\Gamma_2}|\nabla y_2(x,t)|^2\;d\Gamma\,dt. \tag{3.59}$$

Inserting (3.53) – (3.59) into the right-hand side of (3.50) and using (3.52) leads to

$$\begin{aligned}(\alpha-\eta(\frac{1}{2}+2C))\int_S^T\int_\Omega a(x)\,|\nabla y(x,t)|^2\;dx\,dt&\le C(\mathcal{E}(S)+\mathcal{E}(T))+C\int_S^T\int_\omega\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\,dx\,dt\\&+\int_S^T\int_{\Gamma_2}\{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\,d\Gamma\,dt+C\int_S^T\int_\Omega y^2(x,t)\,dx\,dt\\&+\int_S^T\int_{\Gamma_0}\{2a_1\frac{\partial y_1}{\partial\nu}(x,t)h(x).\nabla y_1(x,t)-a_1\,|\nabla y_1(x,t)|^2\,h.\nu\\&-2a_2\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2(x,t)+a_2\,|\nabla y_2(x,t)|^2\,h.\nu\}\,d\Gamma\,dt\\&+\int_S^T\int_{\Gamma_2}\{2a_2\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2(x,t)-a_2\,|\nabla y_2(x,t)|^2\,h.\nu\}\,d\Gamma\,dt.\end{aligned}$$

We choose η sufficiently small to make $\alpha-\eta(\frac{1}{2}+2C)>0$, we obtain

$$\begin{aligned}\int_S^T\mathcal{E}(t)dt&\le C(\mathcal{E}(S)+\mathcal{E}(T))+C\int_S^T\int_\omega\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\,dx\,dt\\&+\int_S^T\int_{\Gamma_2}\{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\,d\Gamma\,dt+C\int_S^T\int_\Omega y^2(x,t)\,dx\,dt\\&+\int_S^T\int_{\Gamma_0}\{2a_1\frac{\partial y_1}{\partial\nu}(x,t)h(x).\nabla y_1(x,t)-a_1\,|\nabla y_1(x,t)|^2\,h.\nu\\&-2a_2\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2(x,t)+a_2\,|\nabla y_2(x,t)|^2\,h.\nu\}\,d\Gamma\,dt\\&+C\int_S^T\int_{\Gamma_2}\{2a_2\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2(x,t)-a_2\,|\nabla y_2(x,t)|^2\,h.\nu\}\,d\Gamma\,dt.\end{aligned} \tag{3.60}$$

Step 4.
Choose open subsets $\omega^{'}$ and $\omega^{''}$ of Ω and the vector field m on Ω such that

$$\begin{cases} m = h \quad \text{on} \quad \Gamma_0, \\ \text{supp} m \subset \omega^{'}, \\ \omega^{'} \subset \overline{\omega^{'}} \subset \omega^{''} \subset \overline{\omega^{''}} \subset \omega. \end{cases}$$

and then multiply both sides of (3.1) by $2m(x).\nabla y(x,t)$ and integrate by parts over $(S,T)\times\Omega$. We obtain from (3.50)

$$\begin{aligned} &2a_1\int_S^T\int_{\Gamma_0}\frac{\partial y_1}{\partial\nu}(x,t)h(x).\nabla y_1\,d\Gamma\,dt - 2a_2\int_S^T\int_{\Gamma_0}\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2\,d\Gamma\,dt \\ &- a_1\int_S^T\int_{\Gamma_0}|\nabla y_1(x,t)|^2\,h(x).\nu\,d\Gamma\,dt + a_2\int_S^T\int_{\Gamma_0}|\nabla y_2(x,t)|^2\,h(x).\nu\,d\Gamma\,dt \\ &= -2\int_S^T\int_\Omega a(x)M(x)\nabla y(x,t).\nabla y(x,t)\,dx\,dt - \alpha\int_S^T\int_\Omega\{y_t^2(x,t) - a(x)\,|\nabla y(x,t)|^2\}\,dx\,dt \\ &- \left|\left[\int_\Omega\{2y_t(x,t)m(x).\nabla y(x,t) + (\text{div}m(x)-\alpha)y_t(x,t)y(x,t)\}\,dx\right]_S^{\mathrm{T}}\right| \\ &- \int_S^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(\text{div}m(x)-\alpha)\,dx\,dt \end{aligned}$$

(3.61)

$$+\int_S^T\int_\omega\{\alpha_1 y_t(x,t) + \alpha_2 y_t(x,t-\tau)\}\,(2m(x).\nabla y + (\text{div}m(x)-\alpha)y)\;dx\,dt,$$

where M is a Jacobian matrix of m. Therefore

$$\begin{aligned} &\int_S^T\int_{\Gamma_0}\{2a_1\frac{\partial y_1}{\partial\nu}(x,t)h(x).\nabla y_1(x,t) - a_1\,|\nabla y_1(x,t)|^2\,h.\nu - 2a_2\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2(x,t) \\ &+ a_2\,|\nabla y_2(x,t)|^2\,h.\nu\}\,d\Gamma\,dt \le C(\mathcal{E}(S)+\mathcal{E}(T)) + C\int_s^T\int_{\omega'} a(x)\,|\nabla y(x,t)|^2\;dx\,dt \end{aligned}$$

(3.62)

$$+\,C\int_S^T\int_{\omega'}\{y_{2,t}^2(x,t) + y_{2,t}^2(x,t-\tau)\}\,dx\,dt.$$

Step 5.
Let θ be a smooth function defined on Ω such that

$$\begin{cases} \theta(x) = 1, \quad x\in\omega^{'}, \\ \theta(x) = 0, \quad x\in\Omega\backslash\overline{\omega''}, \\ \max\limits_{x\in\omega''}\dfrac{|\nabla\theta(x)|^2}{\theta} \le C. \end{cases}$$

Multiplying both sides of (3.1) by $\theta(x)y(x,t)$ and integrating by parts over $(S,T)\times\Omega$, we obtain

(3.63)

$$\int_S^T\int_\Omega (y_{tt}(x,t) - \text{div}(a(x)\nabla y(x,t)) + \chi_\omega(x)\{\alpha_1 y_t(x,t) + \alpha_2 y_t(x,t-\tau)\})\,(\theta(x)y(x,t))\;dx\,dt = 0.$$

Integrating by parts, we get

(3.64)

$$\int_S^T\int_\Omega y_{tt}(x,t)\theta(x)y(x,t)\,dx\,dt = \left[\int_\Omega y_t(x,t)\theta(x)y(x,t)\,dx\right]_S^T - \int_S^T\int_\Omega\theta(x)y_t^2(x,t)\,dx.$$

$$\int_S^T \int_\Omega \operatorname{div}(a(x)\nabla y(x,t))\theta(x)y(x,t)\,dx\,dt = -\int_S^T \int_\Omega a(x)y(x,t)\nabla\theta(x).\nabla y(x,t)\,dx\,dt$$
$$-\int_S^T \int_\Omega a(x)\theta(x)\left|\nabla y(x,t)\right|^2\,dx\,dt. \tag{3.65}$$

Insertion (3.64) and (3.65) into (3.63) yields

$$\begin{aligned}\int_S^T \int_\Omega a(x)\theta(x)\left|\nabla y(x,t)\right|^2\,dx\,dt = &-\left[\int_\Omega y_t(x,t)\theta(x)y(x,t)\,dx\right]_S^T + \int_S^T \int_\Omega \theta(x)y_t^2(x,t)\\ &-\int_S^T \int_\Omega a(x)y(x,t)\nabla\theta(x).\nabla y(x,t)\,dx\,dt\\ &-\int_S^T \int_\Omega \chi_\omega(x)\{\alpha_1 y_t(x,t)+\alpha_2 y_t(x,t-\tau)\}\theta(x)y(x,t)\,dx\,dt.\end{aligned}$$

We have by Cauchy schwarz inequality

$$\begin{aligned}\int_S^T \int_\Omega a(x)\theta(x)\left|\nabla y(x,t)\right|^2\,dx\,dt \le &C(\mathcal{E}(S)+\mathcal{E}(T)) + C\int_S^T \int_{\omega''} \theta(x)\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\\ &+\int_S^T \int_\Omega a(x)y(x,t)\left(\frac{1}{\sqrt{2\theta(x)}}\nabla\theta(x)\right).\left(\sqrt{2\theta(x)}\nabla y(x,t)\right)dx\,dt\\ &+\int_S^T \int_{\omega''} y^2(x,t)\,dx\,dt.\end{aligned}$$

Application of Cauchy schwarz inequality again gives

$$\begin{aligned}\int_S^T \int_\Omega a(x)\theta(x)\left|\nabla y(x,t)\right|^2\,dx\,dt \le &C(\mathcal{E}(S)+\mathcal{E}(T)) + C\int_S^T \int_{\omega''} \theta(x)\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\\ &+\int_S^T \int_{\omega''} y^2(x,t)\,dx\,dt.\end{aligned} \tag{3.66}$$

From (3.62) we deduce

$$\begin{aligned}&\int_S^T \int_{\Gamma_0} \{2a_1\frac{\partial y_1}{\partial\nu}(x,t)h(x).\nabla y_1(x,t) - a_1\left|\nabla y_1(x,t)\right|^2 h.\nu - 2a_2\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2(x,t)\\ &+a_2\left|\nabla y_2(x,t)\right|^2 h.\nu\}\,d\Gamma\,dt \le C(\mathcal{E}(S)+\mathcal{E}(T)) + C\int_s^T \int_\omega y^2(x,t)\,dx\,dt\\ &+C\int_S^T \int_\omega \{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\,dx\,dt.\end{aligned}$$

Step 6.
We rewrite the estimate established in Step 3 over $[\epsilon, T-\epsilon]$ rather than $[S,T]$ and apply the results of Steps 4 and 5, we obtain

$$\begin{aligned}\int_\epsilon^{T-\epsilon} \mathcal{E}(t)dt \le &C(\mathcal{E}(\epsilon)+\mathcal{E}(T-\epsilon)) + C\int_\epsilon^{T-\epsilon} \int_\omega \{y_t^2(x,t)+y_t^2(x,t-\tau)\}\,dx\,dt\\ &+\int_\epsilon^{T-\epsilon} \int_{\Gamma_2} \{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\,d\Gamma\,dt + C\int_\epsilon^{T-\epsilon} \int_\Omega y^2(x,t)\,dx\,dt\\ &+C\int_\epsilon^{T-\epsilon} \int_{\Gamma_2} \{2a_2\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2(x,t) - a_2\left|\nabla y_2(x,t)\right|^2 h.\nu\}\,d\Gamma\,dt.\end{aligned}$$

Step 7.
Noting that

$$\|\nabla y\|_{L^2(\Gamma_2)}^2 = \left\|\frac{\partial y}{\partial\nu}\right\|_{L^2(\Gamma_2)}^2 + \|\nabla_\sigma y\|_{L^2(\Gamma_2)}^2,$$

where $\nabla_\sigma y_2$ is the tangential gradient of y, and using the following estimate due to Lasiecka and Triggiani [**16**]:

$$\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_2}|\nabla_\sigma y(x,t)|^2\, d\sigma_2\, dt \le C(T,\epsilon,\delta)\{\int_0^T\int_{\Gamma_2}\{\left|\frac{\partial y_2}{\partial\nu}(x,t)\right|^2 + y_{2,t}^2(x,t)\}\, d\Gamma_2\, dt + \|y\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))}\},$$

where ϵ and δ are arbitrary positive constants, and $C(T,\epsilon,\delta)$ is a positive constant which depends on T, ϵ and δ, then we can estimate the last term of the inequality in Step 6 as follows

$$\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2}\{2a_2\frac{\partial y_2}{\partial\nu}(x,t)h(x).\nabla y_2(x,t) - a_2\,|\nabla y_2(x,t)|^2\,.h.\nu\,\}\, d\Gamma\, dt$$
$$\le C(T,\epsilon,\delta)\left[\int_0^T\int_{\Gamma_2}\{\left|\frac{\partial y_2}{\partial\nu}(x,t)\right|^2 + y_{2,t}^2(x,t)\}\, d\Gamma\, dt + \|y\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))}\right],$$

from which, we obtain

$$\int_{\epsilon}^{T-\epsilon}\mathcal{E}(t)dt \le C(\mathcal{E}(\epsilon)+\mathcal{E}(T-\epsilon)) + C(T,\epsilon,\delta)\int_{\epsilon}^{T-\epsilon}\int_{\omega}\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\, dx\, dt$$
(3.67)
$$+ C(T,\epsilon,\delta)\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2}\{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\, d\Gamma\, dt + C(T,\epsilon,\delta)\|y\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))},$$

since the $H^{\frac{1}{2}+\delta}$-norm dominates the L^2-norm.

Step 8.

We differentiate $\mathcal{E}(t)$ and apply Green's Theorem to obtain

$$\text{(3.68)}\quad \frac{d}{dt}\mathcal{E}(t) = a_2\int_{\Gamma_2} y_{2,t}(x,t)\frac{\partial y_2(x,t)}{\partial\nu}\, d\Gamma\, dt - \int_{\omega}\{\alpha_1 y_t^2(x,t) + \alpha_2 y_t(x,t)y_t(x,t-\tau)\}\, dx.$$

Integration of both sides of (3.68) from ϵ to $T-\epsilon$, yields

$$\mathcal{E}(\epsilon) = \mathcal{E}(T-\epsilon) - a_2\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2} y_{2,t}(x,t)\frac{\partial y_2(x,t)}{\partial\nu}\, d\Gamma\, dt + \int_{\omega}\{\alpha_1 y_t^2(x,t) + \alpha_2 y_t(x,t)y_t(x,t-\tau)\}\, dx\, dt.$$

Application of the Cauchy inequality gives
(3.69)
$$\mathcal{E}(\epsilon) \le \mathcal{E}(T-\epsilon) + \frac{a_2}{2}\int_0^T\int_{\Gamma_2}\{(\frac{\partial y_2(x,t)}{\partial\nu})^2 + y_{2,t}^2(x,t)\}\, d\Gamma\, dt + C\int_{\omega}\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\, dx\, dt.$$

Insertion of (3.69) into (3.67) results in

$$\int_{\epsilon}^{T-\epsilon}\mathcal{E}(t)dt \le C\mathcal{E}(T-\epsilon) + C(T,\epsilon,\delta)\int_{\epsilon}^{T-\epsilon}\int_{\omega}\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\, dx\, dt$$
$$+ C(T,\epsilon,\delta)\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2}\{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\, d\Gamma\, dt + C(T,\epsilon,\delta)\|y\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))},$$

$$(T-2\epsilon)E(T-\epsilon) \le CE(T-\epsilon)\, dt + C(T,\epsilon,\delta)\int_{\epsilon}^{T-\epsilon}\int_{\omega}\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\, dx\, dt$$
$$+ C(T,\epsilon,\delta)\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2}\{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\, d\Gamma\, dt + C(T,\epsilon,\delta)\|y\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))},$$

which means

$$(T-2\epsilon-C)E(T-\epsilon)\leq C(T,\epsilon,\delta)\int_{\epsilon}^{T-\epsilon}\int_{\omega}\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\,dx\,dt$$

(3.70)

$$+C(T,\epsilon,\delta)\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2}\{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\,d\Gamma\,dt+C(T,\epsilon,\delta)\|y\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))}.$$

We deduce from (3.70) that for T sufficiently large

$$E(T)\leq C(T,\epsilon,\delta)\left\{\int_0^T\int_{\omega}\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\,dx\,dt+\int_0^T\int_{\Gamma_2}\{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\,d\Gamma\,dt\right\}$$

(3.71) $$+C(T,\epsilon,\delta)\|y\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))}.$$

Step 9.
We drop the lower order term on the right hand-side of (3.71) by a compactness-uniqueness argument to obtain

(3.72)

$$E(T)\leq C\left\{\int_0^T\int_{\omega}\{y_t^2(x,t)+y_t^2(x,t-\tau)\}\,dx\,dt+\int_0^T\int_{\Gamma_2}\{y_{2,t}^2(x,t)+y_{2,t}^2(x,t-\tau)\}\,d\Gamma\,dt\right\}.$$

Suppose that (3.72) does not hold. Then, there exists a sequence y_n of solution of problem (3.1)–(3.8) with,

$$\begin{aligned}&y_n(x,0)=y_n^0(x),y_{nt}(x,0)=y_n^1(x),\\&y_{n,t}(x,t-\tau)=f_n^0(x,t-\tau)\quad x\in\omega,\ t\in(0,\tau),\\&y_{2n,t}(x,t-\tau)=g_n^0(x,t-\tau)\quad x\in\Gamma_2,\ t\in(0,\tau),\end{aligned}$$

such that

(3.73)

$$E_n(T)\geq n\left\{\int_0^T\int_{\omega}\{y_{n,t}^2(x,t)+y_{n,t}^2(x,t-\tau)\}\,dx\,dt+\int_0^T\int_{\Gamma_2}\{y_{2n,t}^2(x,t)+y_{2n,t}^2(x,t-\tau)\}\,d\Gamma\,dt\right\},$$

denoting by $E_n(T)$ the energy E related to y_n at the time T.
From (3.71) and (3.45), we have

$$E_n(T)\leq C\left\{\int_0^T\int_{\omega}\{y_{n,t}^2(x,t)+y_{n,t}^2(x,t-\tau)\}\,dx\,dt+\int_0^T\int_{\Gamma_2}\{y_{2n,t}^2(x,t)+y_{2n,t}^2(x,t-\tau)\}\,d\Gamma\,dt\right\}$$

(3.74)

$$+C\|y_n\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))}.$$

Then, from (3.73) and (3.74),

$$\left\{\int_0^T\int_{\omega}\{y_{n,t}^2(x,t)+y_{n,t}^2(x,t-\tau)\}\,dx\,dt+\int_0^T\int_{\Gamma_2}\{y_{2n,t}^2(x,t)+y_{2n,t}^2(x,t-\tau)\}\,d\Gamma\,dt\right\}$$
$$\leq\frac{C}{n-C}\|y_n\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))},\qquad n>C$$

Renormalizing, we obtain a sequence of solution of problem (3.1)–(3.8) verifying

(3.75) $$\|y_n\|^2_{L^2(0;T;H^{\frac{1}{2}+\delta}(\Omega))}=1,\qquad\text{for all } n>C$$

and

(3.76)

$$\left\{\int_0^T\int_{\omega}\{y_{n,t}^2(x,t)+y_{n,t}^2(x,t-\tau)\}\,dx\,dt+\int_0^T\int_{\Gamma_2}\{y_{2n,t}^2(x,t)+y_{2n,t}^2(x,t-\tau)\}\,d\Gamma\,dt\right\}\leq\frac{C}{n-C},\quad n>C$$

From (3.74),(3.75) and (3.76), it follows that the sequence $(y_n^0, y_n^1, f_n^0, g_n^0)$ is bounded in $\mathcal{H}$. Then there is a subsequence still denoted by $(y_n^0, y_n^1, f_n^0, g_n^0))$ which converges weakly to $(y^0, y^1, f^0, g^0) \in \mathcal{H}$. Let y be the solution of problem (3.1)–(3.8) with initial condition (y^0, y^1, f^0, g^0). We have from Proposition 3.1

$$y \in C(0,T; H^1_{\Gamma_1}(\Omega)) \cap C^1(0,T; L^2(\Omega))$$

Then

$$y_n \to y \text{ in } L^\infty(0,T; H^1_{\Gamma_1}(\Omega)) \qquad \text{weak-star}$$

Since the embedding $H^1_{\Gamma_1}(\Omega) \to H^{1/2+\delta}(\Omega)$ is compact, then there exists a subsequence still denoted by y_n such that $y_n \to y$ strongly in $L^\infty(0,T; H^{1/2+\delta}(\Omega))$. Then we have from (3.75)

$$\|y\|_{L^2(0,T;H^{1/2+\delta}(\Omega))} = 1 \tag{3.77}$$

and from (3.76),we have,

$$\int_0^T \int_\omega \{y_t^2(x,t) + y_t^2(x,t-\tau)\}\, dx\, dt + \int_0^T \int_{\Gamma_2} \{y_{2,t}^2(x,t) + y_{2,t}^2(x,t-\tau)\} d\Gamma dt = 0$$

Thus y satisfies

$$\begin{aligned} y_t(x,t) &= y_t(x,t-\tau) = 0 \qquad \text{in } \omega \times (0,T) \\ y_{2,t}(x,t) &= \frac{\partial y_2(x,t)}{\partial \nu} = 0 \qquad \text{on } \Gamma_2 \times (0,T) \end{aligned}$$

Hence, we conclude from the previous chapter that $y(x,t) = 0$ in $\Omega \times (0,T)$. This is in contradiction with (3.77). The desired inequality (3.72) is therefore proved.
We are now ready to end the proof of Theorem 3.1.
End of the proof of Theorem 3.1. We have from (3.45)

$$\frac{d}{dt}E(t) \le -C_1 \left\{ \int_\omega \{y_t^2(x,t) + y_t^2(x,t-\tau)\}\, dx + \int_{\Gamma_2} \{y_{2,t}^2(x,t) + y_{2,t}^2(x,t-\tau)\}\, d\Gamma \right\}.$$

Then

$$\left\{ \int_0^T \int_\omega \{y_t^2(x,t) + y_t^2(x,t-\tau)\}\, dx\, dt + \int_0^T \int_{\Gamma_2} \{y_{2,t}^2(x,t) + y_{2,t}^2(x,t-\tau)\}\, d\Gamma\, dt \right\} \le C_1^{-1}(E(0) - E(T))$$

which together with (3.72) yields

$$E(T) \le \frac{CC_1^{-1}}{1 + CC_1^{-1}} E(0). \tag{3.78}$$

Since $0 < \dfrac{C}{C + C_1} < 1$, the desired conclusion follows now from (3.78).

CHAPTER 4

Stability of coupled Euler-Bernoulli equations with delay terms in the boundary feedbacks

4.1. Introduction

In this chapter, we investigate the problem of exponential stability for a linear system of compactly coupled Euler-Bernoulli equations. Let Ω be an open bounded domain of $\mathbb{R}^n$ with a regular boundary Γ which consists of two non-empty parts Γ_0 and Γ_1 such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.
Furthermore, we assume that there exists $x_0 \in \mathbb{R}^n$ such that:
$(H.1)$ $h(x).\nu(x) \leq 0 \quad on \quad \Gamma_0,$
where $h(x) = x - x_0$ and ν is the unit normal on Γ pointing towards the exterior of Ω.
In Ω, we consider the following coupled system of two Euler-Bernoulli equations with delay terms in the boundary conditions:

$$u_{tt}(x,t) + \Delta^2 u(x,t) + l(u(x,t) - v(x,t)) = 0 \qquad \text{in } \Omega \times (0,+\infty), \tag{4.1}$$

$$v_{tt}(x,t) + \Delta^2 v(x,t) + l(v(x,t) - u(x,t)) = 0 \qquad \text{in } \Omega \times (0,+\infty), \tag{4.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \qquad \text{in } \Omega, \tag{4.3}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x) \qquad \text{in } \Omega, \tag{4.4}$$

$$u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0 \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{4.5}$$

$$\Delta u(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{4.6}$$

$$\frac{\partial \Delta u(x,t)}{\partial \nu} = \alpha_1 u_t(x,t) + \alpha_2 u_t(x,t-\tau) \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{4.7}$$

$$v(x,t) = \frac{\partial v(x,t)}{\partial \nu} = 0 \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{4.8}$$

$$\Delta v(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{4.9}$$

$$\frac{\partial \Delta v(x,t)}{\partial \nu} = \beta_1 v_t(x,t) + \beta_2 v_t(x,t-\tau) \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{4.10}$$

$$u_t(x,t-\tau) = f(x,t-\tau) \qquad \text{on } \Gamma_1 \times (0,\tau), \tag{4.11}$$

$$v_t(x,t-\tau) = g(x,t-\tau) \qquad \text{on } \Gamma_1 \times (0,\tau), \tag{4.12}$$

where t and x represent the time and space variables, respectively. $l, \alpha_1, \alpha_2, \beta_1$ and β_2 are positive constants, τ is the time delay, u_0, u_1, v_0, v_1, f and g are the initial data.
In the one dimensional case, the coupled system $(4.1) - (4.2)$ is known as coupled Euler-Bernoulli beams, u and v represent then the vertical displacements of the beams measured from the horizontal equilibrium positions and the terms $\pm l(u-v)$ are the coupling between the two beams .
The problem of boundary stabilization of coupled Euler-Bernoulli beams has been considered by Najafi et al [**26**] in the case where there is no time delay and by Datko [**8**] and Shang et al [**38**] in the case where there is a time delay term in the boundary conditions.
The subject of this chapter is to investigate the uniform exponential stability of the coupled multidimensional Euler-Bernoulli equation $(4.1) - (4.12)$ in the case where the boundary damping coefficients $\alpha_1, \alpha_2, \beta_1$ and β_2 are strictly positive.

4.2. Main result

We assume

$$\alpha_1 > \alpha_2,\ \beta_1 > \beta_2, \tag{4.13}$$

and define the energy of a solution of (4.1) – (4.12) by

$$\begin{aligned} E(t) =& \frac{1}{2}\int_\Omega \left[|\Delta u(x,t)|^2 + u_t^2(x,t) + |\Delta v(x,t)|^2 + v_t^2(x,t) + l(u(x,t)-v(x,t))^2\right] dx \\ & + \frac{1}{2}\int_{\Gamma_1}\int_0^1 \left[\mu u_t^2(x,t-\tau\rho) + \xi v_t^2(x,t-\tau\rho)\right] d\rho\, d\Gamma, \end{aligned} \tag{4.14}$$

where

$$\tau\alpha_2 < \mu < \tau(2\alpha_1 - \alpha_2), \tag{4.15}$$

and

$$\tau\beta_2 < \xi < \tau(2\beta_1 - \beta_2). \tag{4.16}$$

The main result of this chapter can be stated as follow.

THEOREM 4.1. *Assume* $(H.1)$, (4.13), (4.15) *and* (4.16). *Then the system* (4.1) – (4.12) *is uniformly exponentially stable, i.e., there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq M e^{-\omega t} E(0).$$

Theorem 4.1 is proved in Section 4.4. In Section 4.3, we study the well-posedness of system (4.1)-(4.12) using semigroup theory.

4.3. Well-posedness

We set

$$y(x,\rho,t) = u_t(x,t-\tau\rho), \quad z(x,\rho,t) = v_t(x,t-\tau\rho) \qquad x\in\Gamma_1,\ \rho\in(0,1),\ t>0.$$

Problem (4.1) – (4.12) is equivalent to

$$u_{tt}(x,t) + \Delta^2 u(x,t) + l(u(x,t)-v(x,t)) = 0 \quad \text{in } \Omega\times(0,+\infty), \tag{4.17}$$

$$y_t(x,\rho,t) + \tau^{-1} y_\rho(x,\rho,t) = 0 \quad \text{on } \Gamma_1\times(0,1)\times(0,+\infty), \tag{4.18}$$

$$v_{tt}(x,t) + \Delta^2 v(x,t) + l(v(x,t)-u(x,t)) = 0 \quad \text{in } \Omega\times(0,+\infty), \tag{4.19}$$

$$z_t(x,\rho,t) + \tau^{-1} z_\rho(x,\rho,t) = 0 \quad \text{on } \Gamma_1\times(0,1)\times(0,+\infty), \tag{4.20}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{4.21}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x) \quad \text{in } \Omega, \tag{4.22}$$

$$u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0 \quad \text{on } \Gamma_0\times(0,+\infty), \tag{4.23}$$

$$\Delta u(x,t) = 0 \quad \text{on } \Gamma_1\times(0,+\infty), \tag{4.24}$$

$$\frac{\partial \Delta u(x,t)}{\partial \nu} = \alpha_1 u_t(x,t) + \alpha_2 y(x,1,t) \quad \text{on } \Gamma_1\times(0,+\infty), \tag{4.25}$$

$$v(x,t) = \frac{\partial v(x,t)}{\partial \nu} = 0 \quad \text{on } \Gamma_0\times(0,+\infty), \tag{4.26}$$

$$\Delta v(x,t) = 0 \quad \text{on } \Gamma_1\times(0,+\infty), \tag{4.27}$$

$$\frac{\partial \Delta v(x,t)}{\partial \nu} = \beta_1 v_t(x,t) + \beta_2 z(x,1,t) \quad \text{on } \Gamma_1\times(0,+\infty), \tag{4.28}$$

$$u_t(x,t) = y(x,0,t),\ v_t(x,t) = z(x,0,t) \quad \text{on } \Gamma_1\times(0,+\infty), \tag{4.29}$$

$$y(x,\rho,0) = f(x,-\tau\rho),\ z(x,\rho,0) = g(x,-\tau\rho) \quad \text{on } \Gamma_1\times(0,1). \tag{4.30}$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = H^2_{\Gamma_0}(\Omega)\times L^2(\Omega); L^2(\Gamma_1\times L^2(0,1))\times H^2_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(\Gamma_1; L^2(0,1)),$$

where

$$H^2_{\Gamma_0}(\Omega) = \{u \in H^2(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\}.$$

We equip $\mathcal{H}$ with the inner product

$$\left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \phi \\ \chi \\ \psi \end{pmatrix} ; \begin{pmatrix} \widetilde{\zeta} \\ \widetilde{\eta} \\ \widetilde{\theta} \\ \widetilde{\phi} \\ \widetilde{\chi} \\ \widetilde{\psi} \end{pmatrix} \right\rangle = \int_\Omega (\Delta\zeta(x)\Delta\widetilde{\zeta}(x) + \eta(x)\widetilde{\eta}(x))\, dx +$$

$$\mu \int_{\Gamma_1}\int_0^1 \theta(x,\rho)\widetilde{\theta}(x,\rho)\, d\rho\, d\Gamma + \int_\Omega (\Delta\phi(x)\Delta\widetilde{\phi}(x) + \chi(x)\widetilde{\chi}(x))\, dx +$$

$$\xi \int_{\Gamma_1}\int_0^1 \psi(x,\rho)\widetilde{\psi}(x,\rho)\, d\rho\, d\Gamma + l \int_\Omega (\zeta(x) - \phi(x))(\widetilde{\zeta}(x) - \widetilde{\phi}(x))\, dx.$$

Define in $\mathcal{H}$ a linear operator A by

$$(4.31) \quad A(\zeta,\eta,\theta,\phi,\chi,\psi)^T = (\eta, -\Delta^2\zeta + l\phi - l\zeta, -\tau^{-1}\theta_\rho, \chi, -\Delta^2\phi - l\phi + l\zeta, -\tau^{-1}\psi_\rho)^T,$$

$$D(A) = \{(\zeta,\eta,\theta,\phi,\chi,\psi)^T \in \mathcal{H} : (\eta, -\Delta^2\zeta + l\phi - l\zeta, -\tau^{-1}\theta_\rho, \chi, -\Delta^2\phi - l\phi + l\zeta, -\tau^{-1}\psi_\rho)^T \in \mathcal{H};$$

$$(4.32) \quad \frac{\partial \Delta\zeta}{\partial \nu} = \alpha_1\eta + \alpha_2\theta(.,1),\ \Delta\zeta = 0,\ \frac{\partial\Delta\phi}{\partial\nu} = \beta_1\chi + \beta_2\psi(.,1),\ \Delta\phi = 0,$$

$$\eta = \theta(.,0)\,;\quad \chi = \psi(.,0) \text{ on } \Gamma_1\},$$

then we can rewrite (4.17) – (4.30) as an abstract Cauchy problem in $\mathcal{H}$

$$(4.33) \qquad \begin{cases} \frac{dW}{dt}(t) = \mathcal{A}W(t); \\ W(0) = W_0. \end{cases}$$

where

$$W(t) = (u(x,t), u_t(x,t), y(x,\rho,t), v(x,t), v_t(x,t), z(x,\rho,t))^T,$$

$$\text{and } W_0 = (u_0, u_1, f(., -.\tau), v_0, v_1, g(., -.\tau))^T.$$

We verify that $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{H}$ and consequently we have

THEOREM 4.2. *Assume* (4.13), *for any initial datum* $W_0 \in \mathcal{H}$, *the problem defined by* (4.31) *and* (4.33) *has a unique solution* $W(.) \in C([0,+\infty);\mathcal{H})$
If in addition we assume that $W_0 \in D(\mathcal{A})$, *then the solution is more regular*

$$W(.) \in C^1([0,+\infty);\mathcal{H}) \cap C([0,+\infty); D(\mathcal{A}))\,.$$

PROOF. First, we prove that the operator $\mathcal{A}$ is dissipative, Let

$$W = (\zeta,\eta,\theta,\phi,\chi,\psi)^T \in D(A).\, Then$$

$$\begin{aligned} \langle AW; W\rangle = & \int_\Omega \left[\Delta\eta\Delta\zeta + (-\Delta^2\zeta + l\phi - l\zeta)\eta\right]\, dx - \tau^{-1}\mu\int_{\Gamma_1}\int_0^1 \theta_\rho\theta\, d\rho\, d\Gamma \\ & + \int_\Omega [\Delta\chi\Delta\phi + (-\Delta^2\phi - l\phi + l\zeta)\chi]\, dx - \tau^{-1}\xi\int_{\Gamma_1}\int_0^1 \psi_\rho\psi\, d\rho\, d\Gamma \\ & + l\int_\Omega (\eta - \chi)(\zeta - \phi)\, dx. \end{aligned}$$

Applying Green's Theorem and integrating by part with respect to ρ, we obtain

$$\langle AW;W\rangle = -\int_{\Gamma}\eta(x)\frac{\partial\Delta\zeta(x)}{\partial\nu}\,d\Gamma + \int_{\Gamma}\Delta\zeta(x)\frac{\partial\eta(x)}{\partial\nu}\,d\Gamma - \frac{\tau^{-1}\mu}{2}\int_{\Gamma_1}[\theta^2(x,1)-\theta^2(x,0)]\,d\Gamma$$

$$-\int_{\Gamma}\chi(x)\frac{\partial\Delta\phi(x)}{\partial\nu}\,d\Gamma + \int_{\Gamma}\Delta\varphi(x)\frac{\partial\chi(x)}{\partial\nu}\,d\Gamma - \frac{\tau^{-1}\xi}{2}\int_{\Gamma_1}[\psi^2(x,1)-\psi^2(x,0)]\,d\Gamma, \tag{4.34}$$

Inserting (4.32) in (4.34) and using the Cauchy-Schwarz inequality, we get

$$\langle AW;W\rangle \le (-\alpha_1 + \frac{\alpha_2}{2} + \frac{\tau^{-1}\mu}{2})\int_{\Gamma_1}\eta^2(x)\,d\Gamma + (\frac{\alpha_2}{2} - \frac{\tau^{-1}\mu}{2})\int_{\Gamma_1}\theta^2(x,1)\,d\Gamma$$
$$+(-\beta_1 + \frac{\beta_2}{2} + \frac{\tau^{-1}\xi}{2})\int_{\Gamma_1}\chi^2(x)\,d\Gamma + (\frac{\beta_2}{2} - \frac{\tau^{-1}\xi}{2})\int_{\Gamma_1}\psi^2(x,1)\,d\Gamma.$$

From (4.15) and (4.16), we conclude that $(AW,W)\le 0$. Thus A is dissipative.
Now, we show that $\lambda I - A$ is onto for a fixed $\lambda > 0$ and $(f,g,h,k,m,p)^T \in \mathcal{H}$, there exists $W = (\zeta,\eta,\theta,\phi,\chi,\psi)^T \in D(A)$ solution of

$$(\lambda I - A)W = (f,g,h,k,m,p)^T,$$

or equivalent

$$\lambda\zeta - \eta = f, \tag{4.35}$$
$$\lambda\eta + \Delta^2\zeta + l\zeta - l\phi = g, \tag{4.36}$$
$$\lambda\theta + \tau^{-1}\theta_\rho = h, \tag{4.37}$$
$$\lambda\phi - \chi = k, \tag{4.38}$$
$$\lambda\chi + \Delta^2\phi + l\phi - l\zeta = m, \tag{4.39}$$
$$\lambda\psi + \tau^{-1}\psi_\rho = p. \tag{4.40}$$

Suppose that we have found ζ and ϕ with the appropriate regularity, then

$$\eta = \lambda\zeta - f, \tag{4.41}$$

$$\chi = \lambda\phi - k. \tag{4.42}$$

Consequently we can find θ from (4.37) with (4.32) and ψ from (4.40) with (4.32).
In fact, θ is the unique solution of the initial value problem

$$\theta_\rho(x,\rho) = -\tau\lambda\theta(x,\rho) + \tau h(x,\rho),\ x\in\Gamma_1,\ \rho\in(0,1),$$
$$\theta(x,0) = \eta(x),\ x\in\Gamma_1,$$

given by

$$\theta(x,\rho) = \eta(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho h(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma,\ x\in\Gamma_1,\ \rho\in(0,1),$$

and in particular

$$\theta(x,1) = \lambda e^{-\lambda\tau}\zeta(x) + z_0(x), \tag{4.43}$$

with z_0 defined by

$$z_0(x) = -f(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma,\ \ x\in\Gamma_1,$$

and ψ is the unique solution of the initial value problem

$$\psi_\rho(x,\rho) = -\tau\lambda\psi(x,\rho) + \tau p(x,\rho),\ x\in\Gamma_1,\ \rho\in(0,1),$$
$$\psi(x,0) = \chi(x),\ \ x\in\Gamma_1.$$

Given by

$$\psi(x,\rho) = \chi(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^1 p(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma,\ x\in\Gamma_1,$$

in particular

(4.44) $$\psi(x,1) = \lambda e^{-\lambda\tau}\phi(x) + z_1(x),$$

with z_1 defined by,

$$z_1(x) = -k(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 p(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma,\ \ x\in\Gamma_1,$$

From (4.41), (4.42), (4.36) and (4.39), the function ζ and ϕ verify,

(4.45) $$\begin{cases} \lambda^2\zeta + \Delta^2\zeta + l\zeta - l\phi = g + \lambda f, \\ \lambda^2\phi + \Delta^2\phi + l\phi - l\zeta = m + \lambda k. \end{cases}$$

Problem (4.45) can be reformulated as

$$\int_\Omega(\lambda^2\zeta(x) + \Delta^2\zeta(x) + l\zeta(x) - l\phi(x))w_1(x)\,dx + \int_\Omega(\lambda^2\phi(x) + \Delta^2\phi(x) + l\phi(x) - l\zeta(x))w_2(x)\,dx$$
$$= \int_\Omega(g(x) + \lambda f(x))w_1(x)\,dx + \int_\Omega(m(x) + \lambda k(x))w_2(x)\,dx, \quad (w_1,w_2)\in H^2_{\Gamma_0}(\Omega)\times H^2_{\Gamma_0}(\Omega).$$

We rewrite the left-hand of the last equality after using Green's Theorem and recalling (4.41) – (4.44)

$$\int_\Omega(\lambda^2\zeta(x) + \Delta^2\zeta(x) + l\zeta(x) - l\phi(x))w_1(x)\,dx + \int_\Omega(\lambda^2\phi(x) + \Delta^2\phi(x) + l\phi(x) - l\zeta(x))w_2(x)\,dx$$
$$= \int_\Omega\left(\lambda^2\zeta(x)w_1(x) + \Delta\zeta(x)\Delta w_1(x)\right)\,dx + \int_{\Gamma_1}\Big(\alpha_1(\lambda\zeta(x) - f(x)) + \alpha_2(\lambda e^{-\lambda\tau}\zeta(x) + z_0(x))\Big)\,w_1(x)\,d\Gamma$$
$$+ \int_\Omega\left(\lambda^2\phi(x)w_2(x) + \Delta\phi(x)\Delta w_2(x)\right)\,dx + \int_{\Gamma_1}\Big(\beta_1(\lambda\phi(x) - k(x)) + \beta_2(\lambda e^{-\lambda\tau}\chi(x) + z_1(x))\Big)\,w_2(x)\,d\Gamma$$
$$+ \int_\Omega(l\zeta(x) - l\phi(x))w_1(x)\,dx + \int_\Omega(l\phi(x) - l\zeta(x))w_2(x)\,dx.$$

Therefore

$$\int_\Omega(\lambda^2\zeta(x)w_1(x) + \Delta\zeta(x)\Delta w_1(x))\,dx + \int_{\Gamma_1}\lambda(\alpha_1 + \alpha_2 e^{-\lambda\tau})\zeta(x)w_1(x)\,d\Gamma + \int_\Omega(l\zeta(x) - l\phi(x))w_1(x)\,dx$$
$$+ \int_\Omega(\lambda^2\phi(x)w_2(x) + \Delta\phi(x)\Delta w_2(x))\,dx + \int_{\Gamma_1}\lambda(\beta_1 + \beta_2 e^{-\lambda\tau})\phi(x)w_2(x)\,d\Gamma + \int_\Omega(l\phi(x) - l\zeta(x))w_2(x)\,dx$$
$$= \int_\Omega(g(x) + \lambda f(x))w_1(x)\,dx + \int_\Omega(m(x) + \lambda k(x))w_2(x)\,dx - \alpha_1\int_{\Gamma_1} f(x)w_1(x)\,d\Gamma$$
$$+ \alpha_2\int_{\Gamma_1} z_0(x)w_1(x)\,d\Gamma - \beta_1\int_{\Gamma_1} k(x)w_2(x)\,d\Gamma + \beta_2\int_{\Gamma_1} z_1(x)w_2(x)\,d\Gamma,$$

(4.46)

Since the left-hand side of (4.46) is coercive and continuous on $H^2_{\Gamma_0}(\Omega)\times H^2_{\Gamma_0}(\Omega)$, the Lax-Milgram Lemma guarantees the existence and uniqueness of a solution $(\zeta,\phi)\in H^2_{\Gamma_0}(\Omega)\times H^2_{\Gamma_0}(\Omega)$ of (4.46).
If we consider $(w_1,w_2)\in\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ in (4.46), then (ζ,ϕ) is a solution in $\mathcal{D}'(\Omega)\times\mathcal{D}'(\Omega)$ of

(4.47) $$\begin{cases} \lambda^2\zeta + \Delta^2\zeta + l\zeta - l\phi = g + \lambda f, \\ \lambda^2\phi + \Delta^2\phi + l\phi - l\zeta = m + \lambda k. \end{cases}$$

Thus $(\Delta^2\zeta, \Delta^2\phi) \in L^2(\Omega) \times L^2(\Omega)$.
Combining (4.46) together with (4.47), we obtain after using Green's Theorem

$$\int_{\Gamma_1} \lambda(\alpha_1 + \alpha_2 e^{-\lambda\tau})\zeta(x)w_1(x)\, d\Gamma + \int_{\Gamma_1} \lambda(\beta_1 + \beta_2 e^{-\lambda\tau})\phi(x)w_1(x)\, d\Gamma - \int_{\Gamma_1} \frac{\partial\Delta\zeta(x)}{\partial\nu} w_1(x) d\Gamma$$
$$+ \int_{\Gamma_1} \frac{\partial w_1(x)}{\partial\nu}\Delta\zeta(x)\, d\Gamma - \int_{\Gamma_1} \frac{\partial\Delta\varphi(x)}{\partial\nu} w_2(x)\, d\Gamma + \int_{\Gamma_1} \frac{\partial w_2(x)}{\partial\nu}\Delta\varphi(x)\, d\Gamma = \alpha_1 \int_{\Gamma_1} f(x)w_1(x)\, d\Gamma$$
$$- \alpha_2 \int_{\Gamma_1} z_0(x)w_1(x)\, d\Gamma + \beta_1 \int_{\Gamma_1} k(x)w_2(x)\, d\Gamma - \beta_2 \int_{\Gamma_1} z_1(x)w_2(x)\, d\Gamma,$$

which implies that

$$\begin{aligned} &\frac{\partial\Delta\zeta}{\partial\nu}(x) = \alpha_1\eta(x) + \alpha_2\theta(x,1) \quad \text{on} \quad \Gamma_1 \\ &\frac{\partial\Delta\phi}{\partial\nu}(x) = \beta_1\chi(x) + \beta_2\psi(x,1) \quad \text{on} \quad \Gamma_1 \\ &\Delta\zeta(x) = \Delta\phi(x) = 0 \qquad\qquad \text{on} \quad \Gamma_1. \end{aligned}$$

So, we have found $(\zeta, \eta, \theta, \phi, \chi, \psi)^T \in D(A)$ which verifies (4.35) – (4.40). Thus, by the Lumer-Phillips Theorem, A is the generator of a strongly continuous semigroup of contractions on $\mathcal{H}$. □

4.4. Proof of the main result

Theorem 4.1 will be proved for smooth initial data. First we prove that the energy function $E(t)$ defined by (4.14), (4.15) and (4.16) is decreasing.

PROPOSITION 4.1. *The energy corresponding to any regular solution of problem* (4.1) – (4.12), *is decreasing and there exists a positive constant* K *such that,*

$$\frac{d}{dt}E(t) \le -K \int_{\Gamma_1} \left\{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\right\} d\Gamma \tag{4.48}$$

where

$$K = \min\left\{(\alpha_1 - \frac{\alpha_2}{2} - \frac{\mu}{2\tau}), (\frac{\mu}{2\tau} - \frac{\alpha_2}{2}), \beta_1 - \frac{\beta_2}{2} - \frac{\xi}{2\tau}), (\frac{\xi}{2\tau} - \frac{\beta_2}{2})\right\}.$$

PROOF. Differentiating E defined by (4.14) with respect to time, we obtain

$$\begin{aligned} \frac{d}{dt}E(t) \int_{\Omega} &[\Delta u_t(x,t)\Delta u(x,t) + [-\Delta^2 u(x,t) - l(u(x,t) - v(x,t))]u_t(x,t) + \Delta v_t(x,t)\Delta v(x,t) \\ &+ [-\Delta^2 v(x,t) - l(v(x,t) - u(x,t))]v_t(x,t) + l(u(x,t) - v(x,t))(u_t(x,t) - v_t(x,t))]\, dx \\ &+ \int_{\Gamma_1}\int_0^1 \{\mu u_t(x,t-\tau\rho)u_{tt}(x,t-\tau\rho) + \xi v_t(x,t-\tau\rho)v_{tt}(x,t-\tau\rho)\}\, d\rho\, d\Gamma. \end{aligned}$$

Applying Green's second Theorem, integrating by parts with respect to ρ and recalling the boundary conditions (4.5)-(4.10), we obtain

$$\begin{aligned} \frac{d}{dt}E(t) = &- \alpha_1 \int_{\Gamma_1} u_t^2(x,t)\, d\Gamma - \alpha_2 \int_{\Gamma_1} u_t(x,t)u_t(x,t-\tau)\, d\Gamma \\ &- \beta_1 \int_{\Gamma_1} v_t^2(x,t)\, d\Gamma - \beta_2 \int_{\Gamma_1} v_t(x,t)u_t(x,t-\tau)\, d\Gamma \\ &+ \frac{\tau^{-1}\mu}{2} \int_{\Gamma_1} \{u_t^2(x,t) - u_t^2(x,t-\tau)\}\, d\Gamma + \frac{\tau^{-1}\xi}{2} \int_{\Gamma_1} \{v_t^2(x,t) - v_t^2(x,t-\tau)\}\, d\Gamma. \end{aligned}$$

Applying the Cauchy-Schwarz inequality, we find

$$\begin{aligned}\frac{d}{dt}E(t) \leq & (-\alpha_1 + \frac{\alpha_2}{2} + \frac{\mu\tau^{-1}}{2})\int_{\Gamma_1} u_t^2(x,t)\, d\Gamma + (\frac{\alpha_2}{2} - \frac{\mu\tau^{-1}}{2})\int_{\Gamma_1} u_t^2(x,t-\tau)\, d\Gamma \\ & + (-\beta_1 + \frac{\beta_2}{2} + \frac{\xi\tau^{-1}}{2})\int_{\Gamma_1} v_t^2(x,t)\, d\Gamma + (\frac{\beta_2}{2} - \frac{\xi\tau^{-1}}{2})\int_{\Gamma_1} v_t^2(x,t-\tau)\, d\Gamma,\end{aligned}$$

which implies

$$\frac{d}{dt}E(t) \leq -K\int_{\Gamma_1} \{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\}\, d\Gamma,$$

with

$$K = \min\{(\alpha_1 - \frac{\alpha_2}{2} - \frac{\mu}{2\tau}), (-\frac{\alpha_2}{2} + \frac{\mu}{2\tau}), (\beta_1 - \frac{\beta_2}{2} - \frac{\xi}{2\tau}), (-\frac{\beta_2}{2} + \frac{\xi}{2\tau})\}.$$

Since μ and ξ are such that (4.15) and (4.16). □

We now give an observability estimate which will be used to prove the exponential decay of the energy $E(t)$.

PROPOSITION 4.2. *For any regular solution of problem* (4.1) − (4.12)*, there exists a positive constant* C *(depending on* T*) such that*

$$E(0) \leq C\int_0^T \int_{\Gamma_1} \left\{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\right\}\, d\Gamma\, dt. \tag{4.49}$$

PROOF. To establish (4.49) we follow several steps

Step 1.

We rewrite the energy function E as

$$E(t) = \mathcal{E}(t) + E_d(t),$$

where

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \{|\Delta u(x,t)|^2 + u_t^2(x,t) + |\Delta v(x,t)|^2 + v_t^2(x,t) + l(u(x,t) - v(x,t))^2\}\, dx,$$

and

$$E_d(t) = \frac{1}{2}\int_{\Gamma_1}\int_0^1 \{\mu u_t^2(x,t-\tau\rho) + \xi v_t^2(x,t-\tau\rho)\}\, d\rho\, d\Gamma,$$

$E_d(t)$ can be rewritten via a change of variable as

$$E_d(0) \leq C\int_t^{t+\tau}\int_{\Gamma_1} \{\mu u_t^2(x,s-\tau) + \xi v_t^2(x,s-\tau)\}\, d\Gamma\, ds,$$

then

$$E_d(0) \leq C\int_0^T \int_{\Gamma_1} u_t^2(x,s-\tau) + v_t^2(x,s-\tau)\, d\Gamma\, ds. \tag{4.50}$$

for $\tau + t \leq T$ and T large enough.

Step 2.

Concerning $\mathcal{E}(t)$, We first introduce some notations

$Q = \Omega \times (0,T]$, $\Sigma = \Gamma \times (0,T]$, $\Sigma_0 = \Gamma_0 \times (0,T]$ and $\Sigma_1 = \Gamma_1 \times (0,T]$.

We multiply both side of (4.1),(4.2) by $h.\nabla u$, $h.\nabla v$ respectively and integrate over Q , we obtain

$$\int_Q (u_{tt}(x,t) + \Delta^2 u(x,t))(h.\nabla u)\, dQ = l\int_Q (v(x,t) - u(x,t))(h.\nabla u)\, dQ, \tag{4.51}$$

$$\int_Q (v_{tt}(x,t) + \Delta^2 v(x,t))(h.\nabla v)\, dQ = l\int_Q (u(x,t) - v(x,t))(h.\nabla v)\, dQ. \tag{4.52}$$

We compute each term of left-hand side of (4.51) separately

- Term $\int_Q u_{tt}(x,t)h(x).\nabla u(x,t)\,dQ$

  Integration by parts with respect to t yields

$$\int_Q u_{tt}h.\nabla u\,dQ = \left[\int_\Omega u_t h.\nabla u\,dx\right]_0^T - \int_Q u_t h.\nabla u_t\,dQ.$$

  Green's Theorem gives

$$\int_Q u_{tt}h.\nabla u\,dQ = \left[\int_\Omega u_t h.\nabla u\,dx\right]_0^T - \frac{1}{2}\int_\Sigma u_t^2 h.\nu\,d\Sigma + \frac{1}{2}\int_Q u_t^2 \mathrm{div} h\,dQ. \tag{4.53}$$

- Term $\int_Q \Delta^2 u(x,t)h(x).\nabla u(x,t)\,dQ$.

  Applying Green's Theorem, we obtain

$$\begin{aligned}
\int_Q \Delta^2 u h.\nabla u\,dQ &= \int_\Sigma \frac{\partial \Delta u}{\partial \nu} h.\nabla u\,d\Sigma - \int_Q \nabla\Delta u(x,t).\nabla(h.\nabla u)\,d\Sigma\\
&= \int_\Sigma \frac{\partial \Delta u}{\partial \nu} h.\nabla u\,d\Sigma - \int_\Sigma \Delta u(x,t)(\frac{\partial h_k}{\partial x_j}\frac{\partial u}{\partial x_k} + h_k\frac{\partial^2 u}{\partial x_k \partial x_j})\nu_j\,d\Sigma\\
&+ \int_Q \Delta u(x,t)(\frac{\partial^2 h_k}{\partial^2 x_j}\frac{\partial u}{\partial x_k} + 2\frac{\partial h_k}{\partial x_j}\frac{\partial^2 u}{\partial x_k\partial x_j} + h_k\frac{\partial}{\partial x_j}(\frac{\partial^2 u}{\partial x_k \partial x_j}))\,dQ\\
&= \int_\Sigma \frac{\partial \Delta u}{\partial \nu} h.\nabla u\,d\Sigma - \int_\Sigma \Delta u\frac{\partial u}{\partial x_k}\nu_k\,d\Sigma - \int_\Sigma h_k\Delta u\frac{\partial^2 u}{\partial x_k\partial x_j}\nu_j\,d\Sigma\\
&+ 2\int_Q (\Delta u)^2\,dQ + \int_Q \Delta u h_k\frac{\partial}{\partial x_k}(\Delta u)\,dQ\\
&= \int_\Sigma \frac{\partial \Delta u}{\partial \nu} h.\nabla u\,d\Sigma - \int_\Sigma \Delta u\frac{\partial u}{\partial x_k}\nu_k\,d\Sigma - \int_\Sigma h_k\Delta u\frac{\partial^2 u}{\partial x_k\partial x_j}\nu_j\,d\Sigma\\
&+ 2\int_Q (\Delta u)^2\,dQ + \frac{1}{2}\int_\Sigma (\Delta u)^2 h.\nu\,d\Sigma - \frac{1}{2}\int_Q (\Delta u)^2 \mathrm{div} h\,dQ
\end{aligned} \tag{4.54}$$

  We have from the boundary condition (4.5)

$$\frac{\partial u}{\partial x_k} = \frac{\partial u}{\partial \nu}\nu_k = 0, \quad \text{thus } \nabla u = 0 \ \text{ on}\Gamma_0, \tag{4.55}$$

  and

$$\begin{aligned}
\int_{\Sigma_0} h_k\Delta u\frac{\partial^2 u}{\partial x_k\partial x_j}\nu_j\,d\Sigma &= \int_{\Sigma_0} h_k\Delta u\frac{\partial^2 u}{\partial x_k\partial \nu}\,d\Sigma\\
&= \int_{\Sigma_0} h_k\Delta u\frac{\partial^2 u}{\partial^2 \nu}\nu_k\,d\Sigma\\
&= \int_{\Sigma_0} h_k\Delta u\nu_k\frac{\partial^2 u}{\partial^2 \nu}\sum_{j=1}^n \nu_j^2\,d\Sigma\\
&= \int_{\Sigma_0} h_k\Delta u\nu_k\sum_{j=1}^n \frac{\partial^2 u}{\partial^2 \nu}\nu_j^2\,d\Sigma\\
&= \int_{\Sigma_0} (\Delta u)^2 h.\nu\,d\Sigma.
\end{aligned} \tag{4.56}$$

  Insertion (4.6) together with (4.55) and (4.56) into (4.54) yields

$$\begin{aligned}
\int_Q \Delta^2 u h.\nabla u\,dQ &= \int_{\Sigma_1} \frac{\partial \Delta u}{\partial \nu} h.\nabla u\,d\Sigma - \frac{1}{2}\int_{\Sigma_0} (\Delta u)^2 h.\nu\,d\Sigma\\
&+ 2\int_Q (\Delta u)^2\,dQ - \frac{1}{2}\int_Q (\Delta u)^2 \mathrm{div} h\,dQ.
\end{aligned}$$

So (4.51) becomes

$$(4.57)\quad \begin{aligned}&\left[\int_\Omega u_t h.\nabla u\,dx\right]_0^T - \frac{1}{2}\int_\Sigma u_t^2 h.\nu\,d\Sigma + \frac{n}{2}\int_Q u_t^2\,dQ + \int_{\Sigma_1}\frac{\partial\Delta u}{\partial\nu}h.\nabla u\,d\Sigma\\ &-\frac{1}{2}\int_{\Sigma_0}(\Delta u)^2 h.\nu\,d\Sigma + 2\int_Q(\Delta u)^2\,dQ - \frac{n}{2}\int_Q(\Delta u)^2\,dQ = l\int_Q(v-u)(h.\nabla u)\,dQ.\end{aligned}$$

In similar manner, we obtain

$$(4.58)\quad \begin{aligned}&\left[\int_\Omega v_t h.\nabla v\,dx\right]_0^T - \frac{1}{2}\int_\Sigma v_t^2 h.\nu\,d\Sigma + \frac{n}{2}\int_Q v_t^2\,dQ + \int_{\Sigma_1}\frac{\partial\Delta v}{\partial\nu}h.\nabla v\,d\Sigma\\ &-\frac{1}{2}\int_{\Sigma_0}(\Delta v)^2 h.\nu\,d\Sigma + 2\int_Q(\Delta v)^2\,dQ - \frac{n}{2}\int_Q(\Delta v)^2\,dQ = l\int_Q(u-v)(h.\nabla v)\,dQ.\end{aligned}$$

Summing up (4.57) with (4.57), we get

$$\begin{aligned}&\left[\int_\Omega\{u_t h.\nabla u + v_t h.\nabla v\}\,dx\right]_0^T - \frac{1}{2}\int_{\Sigma_1}\{u_t^2+v_t^2\}h.\nu\,d\Sigma + \frac{n}{2}\int_Q\{u_t^2+v_t^2-(\Delta u)^2-(\Delta v)^2\}\,dQ\\ &+2\int_Q\{(\Delta u)^2+(\Delta v)^2\}\,dQ + \int_{\Sigma_1}\{\frac{\partial\Delta u}{\partial\nu}h.\nabla u + \frac{\partial\Delta v}{\partial\nu}h.\nabla v\}\,d\Sigma - \frac{1}{2}\int_{\Sigma_0}\{(\Delta u)^2+(\Delta v)^2\}h.\nu\,d\Sigma\\ &=\frac{-l}{2}\int_{\Sigma_1}(u-v)^2 h.\nu\,d\Sigma + \frac{ln}{2}\int_Q(u-v)^2\,dQ,\end{aligned}$$

which gives

$$(4.59)\quad \begin{aligned}2\int_0^T\mathcal{E}(t)\,dt =& -\left[\int_\Omega\{u_t h.\nabla u + v_t h.\nabla v\}\,dx\right]_0^T + \frac{1}{2}\int_{\Sigma_1}\{u_t^2+v_t^2\}h.\nu\,d\Sigma - \frac{n}{2}\int_Q\{u_t^2+v_t^2-(\Delta u)^2-(\Delta v)^2\}\,dQ\\ &-\int_Q\{(\Delta u)^2+(\Delta v)^2\}\,dQ - \int_{\Sigma_1}\{\frac{\partial\Delta u}{\partial\nu}h.\nabla u + \frac{\partial\Delta v}{\partial\nu}h.\nabla v\}\,d\Sigma + \frac{1}{2}\int_{\Sigma_0}\{(\Delta u)^2+(\Delta v)^2\}h.\nu\,d\Sigma\\ &+\int_Q\{u_t^2+v_t^2+l(u-v)^2\}\,dQ - \frac{l}{2}\int_{\Sigma_1}(u-v)^2 h.\nu\,d\Sigma + \frac{ln}{2}\int_Q(u-v)^2\,dQ.\end{aligned}$$

By differentiating $I = \int_\Omega\{u(x,t)u_t(x,t) + v(x,t)v_t(x,t)\}\,dx$ with respect to t and recalling (4.1), (4.2) after using Green's Theorem we obtain

$$\frac{dI}{dt} = \int_\Omega\{u_t^2+v_t^2-(\Delta u)^2-(\Delta v)^2-l(u-v)^2\}\,dx - \int_{\Gamma_1}\{\frac{\partial\Delta u}{\partial\nu}u + \frac{\partial\Delta v}{\partial\nu}v\}\,d\Gamma.$$

We integrate both sides of the last equality over $(0,T)$, we find

$$(4.60)\quad \int_Q\{u_t^2+v_t^2-(\Delta u)^2-(\Delta v)^2-l(u-v)^2\}\,dx = \int_{\Sigma_1}\{\frac{\partial\Delta u}{\partial\nu}u + \frac{\partial\Delta v}{\partial\nu}v\}\,d\Sigma + \left[\int_\Omega uu_t + vv_t\,dx\right]_0^T.$$

From (4.60), we conclude that

$$(4.61)\quad \begin{aligned}\int_Q\{u_t^2+v_t^2+(\Delta u)^2+(\Delta v)^2+l(u-v)^2\}\,dx =& \int_{\Sigma_1}\{\frac{\partial\Delta u}{\partial\nu}u + \frac{\partial\Delta v}{\partial\nu}v\}\,d\Sigma + \left[\int_\Omega uu_t + vv_t\,dx\right]_0^T\\ &+2\int_Q\{(\Delta u)^2+(\Delta v)^2\}\,dQ + 2l\int_Q(u-v)^2\,dQ.\end{aligned}$$

Insertion (4.60) and (4.61) into (4.59) yields

$$
\begin{aligned}
2\int_0^T \mathcal{E}(t)\,dt = &-\left[\int_\Omega u_t h.\nabla u + v_t h.\nabla v\,dx\right]_0^T + (1-\frac{n}{2})\left[\int_\Omega uu_t + vv_t\,dx\right]_0^T \\
&+ (1-\frac{n}{2})\int_{\Sigma_1}\{\frac{\partial \Delta u}{\partial \nu}u + \frac{\partial \Delta v}{\partial \nu}v\}\,d\Sigma - \int_{\Sigma_1}\{\frac{\partial \Delta u}{\partial \nu}h.\nabla u + \frac{\partial \Delta v}{\partial \nu}h.\nabla v\}\,d\Sigma \\
&+ \frac{1}{2}\int_{\Sigma_1}\{u_t^2 + v_t^2\}h.\nu\,d\Sigma - \frac{l}{2}\int_{\Sigma_1}(u-v)^2 h.\nu\,d\Sigma \\
&+ 2l\int_Q (u-v)^2\,dQ + \frac{1}{2}\int_{\Sigma_0}\{(\Delta u)^2 + (\Delta v)^2\}h.\nu\,d\Sigma.
\end{aligned}
$$

We have from Assumption $(H.1)$

$$
\begin{aligned}
\int_0^T \mathcal{E}(t)\,dt \le &-\frac{1}{2}\left[\int_\Omega u_t h.\nabla u + v_t h.\nabla v\,dx\right]_0^T + \frac{1}{2}(1-\frac{n}{2})\left[\int_\Omega uu_t + vv_t\,dx\right]_0^T \\
&+ \frac{1}{2}(1-\frac{n}{2})\int_{\Sigma_1}\{\frac{\partial \Delta u}{\partial \nu}u + \frac{\partial \Delta v}{\partial \nu}v\}\,d\Sigma - \frac{1}{2}\int_{\Sigma_1}\{\frac{\partial \Delta u}{\partial \nu}h.\nabla u + \frac{\partial \Delta v}{\partial \nu}h.\nabla v\}\,d\Sigma \\
&+ \frac{1}{4}\int_{\Sigma_1}\{u_t^2 + v_t^2\}h.\nu\,d\Sigma - \frac{l}{4}\int_{\Sigma_1}(u-v)^2 h.\nu\,d\Sigma + l\int_Q (u-v)^2\,dQ. \qquad (4.62)
\end{aligned}
$$

Let

$$
\begin{aligned}
&M_h = \max_{\overline{\Omega}}|h|, C_h = \max_{\Gamma_1}|h|, \\
&\int_\Omega u^2\,dx < C_p\int_\Omega |\nabla u|^2\,dx, \qquad u \in H_0^1(\Omega), \quad C_p = \text{Poincaré constant} \\
&\int_\Omega |\nabla u|^2\,dx < C'\int_\Omega |\Delta u|^2\,dx, \qquad \forall u \in H_0^2(\Omega), \qquad (\text{see } [\mathbf{21}], \text{ p.256}).
\end{aligned}
$$

Then, for the tow first terms (I_1) in the right-hand side of (4.62) we have

$$
\begin{aligned}
I_1 = &\frac{1}{2}\left[\int_\Omega u_t h.\nabla u + v_t h.\nabla v\,dx\right]_0^T + \frac{1}{2}(1+\frac{n}{2})\left[\int_\Omega uu_t + vv_t\,dx\right]_0^T \le \\
&(\frac{M_h}{4} + \frac{1}{4}(1+\frac{n}{2}))\{\|u_t(T)\|^2 + \|v_t(T)\|^2 + \|u_t(0)\|^2 + \|v_t(0)\|^2\} \\
&+ (\frac{M_h}{4} + \frac{C_p}{4}(1+\frac{n}{2}))\{\|\nabla u(T)\|^2 + \|\nabla v(T)\|^2\|\nabla u(0)\|^2 + \|\nabla v(0)\|^2)\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
I_1 \le\, &C_1\{\|u_t(T)\|^2 + \|v_t(T)\|^2 + \|\Delta u(T)\|^2 + \|\Delta v(T)\|^2 \\
&+ \|u_t(0)\|^2 + \|v_t(0)\|^2 + \|\Delta u(0)\|^2 + \|\Delta v(0)\|^2\} \\
&\le C_1(\mathcal{E}(T) + \mathcal{E}(0)) \qquad (4.63)
\end{aligned}
$$

For the third and fourth terms (I_2) in the right-hand side of (4.62) we use Cauchy-Schwarz inequality and poincaré inequality we get

$$
\begin{aligned}
I_2 = &\left|\frac{1}{2}(1-\frac{n}{2})\int_{\Sigma_1}\{\frac{\partial \Delta u}{\partial \nu}u + \frac{\partial \Delta v}{\partial \nu}v\}\,d\Sigma\right| + \left|\int_{\Sigma_1}\frac{1}{2}\{\frac{\partial \Delta u}{\partial \nu}h.\nabla u + \frac{\partial \Delta v}{\partial \nu}h.\nabla v\}\,d\Sigma\right| \\
\le &[\frac{C_p}{4}(1+\frac{n}{2}) + \frac{C_h}{4}]\{\|\nabla u\|^2_{L^2(\Sigma_1)} + \|\nabla v\|^2_{L^2(\Sigma_1)}\} + [\frac{1}{4}(1+\frac{n}{2}) + \frac{C_h}{4}]\int_{\Sigma_1}\{\left|\frac{\partial \Delta u}{\partial \nu}\right|^2 + \left|\frac{\partial \Delta v}{\partial \nu}\right|^2\}\,d\Sigma
\end{aligned}
$$

Then

$$
I_2 \le C_2\{\|u\|^2 + \|v\|^2\}_{L^2(0,T,H^1(\Gamma_1))} + C_2\int_{\Sigma_1}\{\left|\frac{\partial \Delta u}{\partial \nu}\right|^2 + \left|\frac{\partial \Delta v}{\partial \nu}\right|^2\}\,d\Sigma. \qquad (4.64)
$$

For the fifth term in the right-hand side of (4.62) we have

$$\int_Q l(u-v)^2\,dx \le 2l\,\{\|u\|^2+\|v\|^2\}_{L^2(0,T,L^2(\Omega))}. \tag{4.65}$$

For the three last terms the right-hand side of (4.62), we obtain

$$\begin{aligned}&\frac{1}{4}\left|\int_{\Sigma_1}\{u_t^2+v_t^2\}h.\nu\,d\Sigma\right|+\frac{l}{4}\left|\int_{\Sigma_1}(u-v)^2h.\nu\,d\Sigma\right|\\ &\le C_3\int_{\Sigma_1}\{u_t^2+v_t^2\}\,d\Sigma+C_3\{\|u\|^2+\|v\|^2\}_{L^2(0,T,H^1(\Gamma_1))}.\end{aligned} \tag{4.66}$$

Insertion (4.63)-(4.66) into (4.62) yields

$$\begin{aligned}\int_0^T\mathcal{E}(t)\,dt &\le C_1(\mathcal{E}(T)+\mathcal{E}(0))+C_2\int_{\Sigma_1}\{\left|\frac{\partial\Delta u}{\partial\nu}\right|^2+\left|\frac{\partial\Delta v}{\partial\nu}\right|^2\}\,d\Sigma\\ &+(C_2+C_3)\{\|u\|^2+\|v\|^2\}_{L^2(0,T,H^1(\Gamma_1))}+2l\,\{\|u\|^2+\|v\|^2\}_{L^2(0,T,L^2(\Omega))}\\ &+C_3\int_{\Sigma_1}\{u_t^2+v_t^2\}\,d\Sigma.\end{aligned}$$

By trace Theorem we have

$$\{\|u\|^2+\|v\|^2\}_{L^2(0,T,H^1(\Gamma_1))}\le C\{\|u\|^2+\|v\|^2\}_{L^2(0,T,H^{\frac{3}{2}}(\Omega))}.$$

Then

$$\begin{aligned}\int_0^T\mathcal{E}(t)\,dt &\le C_1(\mathcal{E}(T)+\mathcal{E}(0))+C_2\int_{\Sigma_1}\{\left|\frac{\partial\Delta u}{\partial\nu}\right|^2+\left|\frac{\partial\Delta v}{\partial\nu}\right|^2\}\,d\Sigma\\ &+C_4\{\|u\|^2+\|v\|^2\}_{L^2(0,T,H^{\frac{3}{2}}(\Omega))}+C_3\int_{\Sigma_1}\{u_t^2+v_t^2\}\,d\Sigma.\end{aligned} \tag{4.67}$$

Now, we differentiate $\mathcal{E}(t)$ with respect to t, and integrate over (0,t] with $t\in[0,T]$, we get

$$\mathcal{E}(t)-\mathcal{E}(0)=-\int_0^t\int_{\Gamma_1}u_t\frac{\partial\Delta u}{\partial\nu}\,d\Gamma\,dt-\int_0^t\int_{\Gamma_1}v_t\frac{\partial\Delta v}{\partial\nu}\,d\Gamma\,dt.$$

For $t\in[0,T]$, we obtain

$$\begin{aligned}\mathcal{E}(t)=&\mathcal{E}(0)-\frac{1}{2}\int_0^t\int_{\Gamma_1}\{(u_t+\frac{\partial\Delta u}{\partial\nu})^2+(v_t+\frac{\partial\Delta v}{\partial\nu})^2\}\,d\Gamma\,dt\\ &+\frac{1}{2}\int_0^t\int_{\Gamma_1}\{u_t^2+(\frac{\partial\Delta u}{\partial\nu})^2+v_t^2+(\frac{\partial\Delta v}{\partial\nu})^2\}\,d\Gamma\,dt.\end{aligned}$$

Then

$$\begin{aligned}\mathcal{E}(t)\ge&\mathcal{E}(0)-\frac{1}{2}\int_0^t\int_{\Gamma_1}\{(u_t+\frac{\partial\Delta u}{\partial\nu})^2+(v_t+\frac{\partial\Delta v}{\partial\nu})^2\}\,d\Gamma\,dt\\ &\ge\mathcal{E}(0)-\frac{1}{2}\int_0^T\int_{\Gamma_1}\{(u_t+\frac{\partial\Delta u}{\partial\nu})^2+(v_t+\frac{\partial\Delta v}{\partial\nu})^2\}\,d\Gamma\,dt.\end{aligned}$$

By integrating the last estimate over (0,T], we get

$$\int_0^T\mathcal{E}(t)\,dt\ge T\mathcal{E}(0)-\frac{T}{2}\int_0^T\int_{\Gamma_1}\{(u_t+\frac{\partial\Delta u}{\partial\nu})^2+(v_t+\frac{\partial\Delta v}{\partial\nu})^2\}\,d\Gamma\,dt,$$

which together with (4.67) gives

$$\begin{aligned}&T\mathcal{E}(0)-\frac{T}{2}\int_0^T\int_{\Gamma_1}\{(u_t+\frac{\partial\Delta u}{\partial\nu})^2+(v_t+\frac{\partial\Delta v}{\partial\nu})^2\}\,d\Gamma\,dt\le C_1(\mathcal{E}(T)+\mathcal{E}(0))\\ &+C_2\int_{\Sigma_1}\{\left|\frac{\partial\Delta u}{\partial\nu}\right|^2+\left|\frac{\partial\Delta v}{\partial\nu}\right|^2\}\,d\Sigma+C_4\{\|u\|^2+\|v\|^2\}_{L^2(0,T,H^{\frac{3}{2}}(\Omega))}+C_3\int_{\Sigma_1}\{u_t^2+v_t^2\}\,d\Sigma.\end{aligned}$$

So

$$(T-C_1)\mathcal{E}(0) \le C_1\mathcal{E}(0) + \frac{C_1}{2}\int_{\Sigma_1}\{u_t^2 + (\frac{\partial \Delta u}{\partial \nu})^2 + v_t^2 + (\frac{\partial \Delta v}{\partial \nu})^2\}\, d\Sigma$$
$$+ C_2\int_{\Sigma_1}\{\left|\frac{\partial \Delta u}{\partial \nu}\right|^2 + \left|\frac{\partial \Delta v}{\partial \nu}\right|^2\}\, d\Sigma + C_4\{\|u\|^2 + \|v\|^2\}_{L^2(0,T,H^{\frac{3}{2}}(\Omega))} + C_3\int_{\Sigma_1}\{u_t^2 + v_t^2\}\, d\Sigma$$
$$+ T\int_{\Sigma_1}\{u_t^2 + (\frac{\partial \Delta u}{\partial \nu})^2 + v_t^2 + (\frac{\partial \Delta v}{\partial \nu})^2\}\, d\Sigma.$$

Then

$$(T-2C_1)\mathcal{E}(0) \le (\frac{C_1}{2} + C_3 + T)\int_{\Sigma_1}\{u_t^2 + v_t^2\}\, d\Sigma + (\frac{C_1}{2} + C_2 + T)\int_{\Sigma_1}\{\left|\frac{\partial \Delta u}{\partial \nu}\right|^2 + \left|\frac{\partial \Delta v}{\partial \nu}\right|^2\}\, d\Sigma$$
$$+ C_4\{\|u\|^2 + \|v\|^2\}_{L^2(0,T,H^{\frac{3}{2}}(\Omega))}.$$

For T assay large and (C_5 depending to T), we conclude that

$$\mathcal{E}(0) \le C_5\int_{\Sigma_1}\{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\}\, d\Sigma$$
$$+ C_5\{\|u\|^2 + \|v\|^2\}_{L^2(0,T,H^{\frac{3}{2}}(\Omega))}. \tag{4.68}$$

From (4.50) and(4.68) we obtain

$$E(0) \le C\int_0^T\int_{\Gamma_1}\{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\}\, d\Gamma\, dt$$
$$+ C\{\|u\|^2 + \|v\|^2\}_{L^2(0,T,H^{\frac{3}{2}}(\Omega))}. \tag{4.69}$$

Step 3.

We prove by a compactness-uniqueness argument that there exists a constant $C > 0$ such that
(4.70)

$$\{\|u\|^2 + \|v\|^2\}_{L^2(0,T,H^{\frac{3}{2}}(\Omega))} \le C\int_0^T\int_{\Gamma_1}\{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\}\, d\Gamma\, dt.$$

Suppose that (4.70) does not hold. Then, there exists a sequence (u_n, v_n) of solution of problem (4.1)-(4.12) with,

$$u_n(x,0) = u_n^0(x), u_{nt}(x,0) = u_n^1(x),$$
$$v_n(x,0) = v_n^0(x), v_{nt}(x,0) = v_n^1(x),$$
$$u_n(x,t-\tau) = f_n^0(x,t-\tau), v_n(x,t-\tau) = g_n^0(x,t-\tau).$$

Such that

$$\{\|u_n\|^2 + \|v_n\|^2\}_{L^2(0,T,H^{\frac{3}{2}}(\Omega))} = 1, \qquad n = 1, 2, ...;$$

$$\int_0^T\int_{\Gamma_1}\{u_{nt}^2(x,t) + u_{nt}^2(x,t-\tau) + v_{nt}^2(x,t) + v_{nt}^2(x,t-\tau)\}\, d\Gamma\, dt \to 0 \,\text{as}\, n \to +\infty, \tag{4.71}$$

Since each solution satisfied (4.69), we deduce from it and (4.71) that the sequence $(u_n^0, u_n^1, f_n^0, v_n^0, v_n^1, g_n^0)$ is bounded in $\mathcal{H} \times \mathcal{H}$. Then there is a subsequence still denoted by $(u_n^0, u_n^1, f_n^0, v_n^0, v_n^1, g_n^0)$ which converges weakly to some $(u^0, u^1, f^0, v^0, v^1, g^0) \in \mathcal{H} \times \mathcal{H}$. Let (u, v) be the solution of problem (4.1)-(4.12) with initial condition $(u^0, u^1, f^0, v^0, v^1, g^0)$. we have from Theorem 4.2

$$(u, v) \in C(0, T; H_0^2(\Omega)) \times C(0, T; H_0^2(\Omega))$$

Then,

$$(u_n, v_n) \longrightarrow (u, v) \text{ weakly in } L^2(0, T, H_0^2(\Omega)) \times L^2(0, T, H_0^2(\Omega)).$$

Since $H_0^2(\Omega)$ is compactly embedded in $H^{\frac{3}{2}}(\Omega)$, there exist a subsequence which for simplicity of notation, we still denote by $\{u_n, v_n\}_n$ such that ,

$$(u_n, v_n) \longrightarrow (u, v) \text{ strongly in } L^2(0,T,H^{\frac{3}{2}}(\Omega)) \times L^2(0,T,H^{\frac{3}{2}}(\Omega)).$$

From (4.71), we get

$$\|u\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Omega))} + \|v\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Omega))} = 1, \tag{4.72}$$

and

$$\int_0^T \int_{\Gamma_1} \{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\}\, d\Gamma\, dt = 0.$$

Then

$$u_t(x,t) = v_t(x,t) = 0 \quad \text{on} \quad \Gamma_1 \times (0,T)$$
$$\frac{\partial \Delta u}{\partial \nu}(x,t) = \frac{\partial \Delta v}{\partial \nu}(x,t) = 0 \quad \text{on} \quad \Gamma_1 \times (0,T)$$

setting $\varphi := u_t, \psi := v_t,$ thus (u,v) satisfies

$$\begin{cases} \varphi_{tt}(x,t) + \Delta^2\varphi(x,t) + l(\varphi(x,t) - \psi(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ \psi_{tt}(x,t) + \Delta^2\psi(x,t) + l(\psi(x,t) - \varphi(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ \varphi(x,t) = \dfrac{\partial \varphi(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma \times (0,T), \\ \psi(x,t) = \dfrac{\partial \psi(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma \times (0,T), \\ \Delta\varphi(x,t) = \Delta\psi(x,t) = 0 & \text{on} \quad \Gamma_1 \times (0,T), \\ \dfrac{\partial \Delta\varphi(x,t)}{\partial \nu} = \dfrac{\partial \Delta\psi(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma_1 \times (0,T). \end{cases} \tag{4.73}$$

Putting $y = \varphi(x,t) + \psi(x,t),$ then the problem (4.73) implies

$$\begin{cases} y_{tt}(x,t) + \Delta^2 y(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T), \\ y(x,t) = \dfrac{\partial y(x,t)}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0,T), \\ \Delta y(x,t) = \dfrac{\partial \Delta y(x,t)}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \times (0,T). \end{cases}$$

which solution is y=0 (see [**21**]. p.276 lemme 3.6), we conclude that

$$\varphi(x,t) = -\psi(x,t)$$

Problem (4.73) becomes

$$\begin{cases} \varphi_{tt}(x,t) + \Delta^2\varphi(x,t) + 2l\varphi(x,t) = 0 & \text{in} \quad \Omega \times (0,T), \\ \varphi(x,t) = \dfrac{\partial \varphi(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma \times (0,T), \\ \Delta\varphi(x,t) = \dfrac{\partial \Delta\varphi(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma_1 \times (0,T), \end{cases}$$

which solution is $\varphi(x,t) = 0$ (see Remark 4 of [**18**]).
Hence

$$u_t(x,t) = v_t(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T).$$

This implies that

$$u(x,t) = u(x),\ v(x,t) = v(x).$$

Thus (u, v) verifies

$$\begin{cases} \Delta^2 u(x) + l(u(x) - v(x)) = 0 & \text{in } \Omega, \\ \Delta^2 v(x) + l(v(x) - u(x)) = 0 & \text{in } \Omega, \\ u(x) = v(x) = \dfrac{\partial u(x)}{\partial \nu} = \dfrac{\partial v(x)}{\partial \nu} = 0 & \text{on } \Gamma, \\ \Delta u(x) = \Delta v(x) = \dfrac{\partial \Delta u(x)}{\partial \nu} = \dfrac{\partial \Delta v(x)}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases}$$

So $(u, v) = (0, 0)$. This is in contradiction with (4.72). The observability inequality (4.49) is therefore proved. □

From (4.48), we have

$$E(T) - E(0) \leq -K \int_0^T \int_{\Gamma_1} \{u_t^2(x,t) + u_t^2(x,t-\tau) + u_t^2(x,t) + v_t^2(x,t-\tau)\}\, d\Gamma\, dt$$

which together with (4.49) leads to

$$E(T) \leq \frac{CK^{-1}}{1 + CK^{-1}} E(0) \tag{4.74}$$

Since we have $0 < C/(K + C) < 1$, the desired conclusion follows now from (4.74).

CHAPTER 5

Stability of coupled Euler-Bernoulli equations with delay terms in the internal feedbacks

5.1. Introduction

The purpose of this chapter is to study the problem of stability for coupled Euler-Bernoulli equations with delay terms in the internal feedbacks.
Let Ω be an open bounded domain of $\mathbb{R}^n$ with regular boundary Γ.
Let $\omega \subset \Omega$ be an open neighbourhood of a subset Γ_0 of Γ defined by

$$\Gamma_0 = \{x \in \Gamma : \quad h(x).\nu(x) > 0\},$$

where ν is the unit normal on Γ towards the exterior of Ω, and $h(x) = x - x_0$, $x_0 \in \mathbb{R}^n$.

In Ω, we consider the following coupled system of two Euler-Bernoulli equations with delay terms in the internal feedbacks :

$$\begin{aligned}
&u_{tt}(x,t) + \Delta^2 u(x,t) + \chi_\omega(x)\{\alpha_1 u_t(x,t) + \alpha_2 u_t(x,t-\tau)\} \\
&= l(v(x,t) - u(x,t)) && \text{in } \Omega \times (0,+\infty), && (5.1)\\
&v_{tt}(x,t) + \Delta^2 v(x,t) + \chi_\omega(x)\{\beta_1 v_t(x,t) + \beta_2 v_t(x,t-\tau)\} \\
&= l(u(x,t) - v(x,t)) && \text{in } \Omega \times (0,+\infty), && (5.2)\\
&u(x,0) = u_0(x), u_t(x,0) = u_1(x) && \text{in } \Omega, && (5.3)\\
&v(x,0) = v_0(x), v_t(x,0) = v_1(x) && \text{in } \Omega, && (5.4)\\
&u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0 && \text{on } \Gamma \times (0,+\infty), && (5.5)\\
&v(x,t) = \frac{\partial v(x,t)}{\partial \nu} = 0 && \text{on } \Gamma \times (0,+\infty), && (5.6)\\
&u_t(x,t-\tau) = f(x,t-\tau) && \text{in } \omega \times (0,\tau), && (5.7)\\
&v_t(x,t-\tau) = g(x,t-\tau) && \text{in } \omega \times (0,\tau), && (5.8)
\end{aligned}$$

where $\chi_\omega(.)$ is the characteristic function of ω, l, α_1, α_2, β_1 and β_2 are positive constants, τ is the time delay, u_0, u_1, v_0, v_1, f and g are the initial data.
In the absence of delay, exponential stability of the system $(5.1)-(5.8)$ has been established by Najafi et al **[26]** for one dimensional domain Ω. In this chapter, we study the exponential stability of the system $(5.1) - (5.8)$ in the case where the interior damping coefficients $\alpha_1, \alpha_2, \beta_1$ and β_2 are strictly positive.

5.2. Main result

Assume

$$\alpha_1 > \alpha_2,\ \beta_1 > \beta_2, \tag{5.9}$$

and define the energy of a solution of $(5.1) - (5.8)$ by

$$\begin{aligned}
E(t) =& \frac{1}{2}\int_\Omega \left[|\Delta u(x,t)|^2 + u_t^2(x,t) + |\Delta v(x,t)|^2 + v_t^2(x,t) + l(u(x,t)-v(x,t))^2\right] dx \\
&+ \frac{1}{2}\int_\Omega \chi_\omega(x)\int_0^1 \left[\mu u_t^2(x,t-\tau\rho) + \xi v_t^2(x,t-\tau\rho)\right] d\rho\, dx,
\end{aligned} \tag{5.10}$$

where

$$\tau\alpha_2 < \mu < \tau(2\alpha_1 - \alpha_2), \tag{5.11}$$

and

$$\tau\beta_2 < \xi < \tau(2\beta_1 - \beta_2). \tag{5.12}$$

The main result of this chapter can be stated as follows.

THEOREM 5.1. *Assume* (5.9), (5.11) *and* (5.12). *Then the system* (5.1) – (5.8) *is uniformly exponentially stable, i.e., there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq Me^{-\omega t}E(0).$$

Theorem 5.1 is proved in Section 5.4. In Section 5.3, we study the well-posedness of system (5.1)-(5.8) using semigroup theory.

5.3. Well-posedness

We introduce the auxiliary variables

$$y(x,\rho,t) = u_t(x, t-\tau\rho),\ z(x,\rho,t) = v_t(x,t-\tau\rho), \qquad x \in \omega,\ \rho \in (0,1),\ t > 0,$$

with these new unknowns, problem (5.1) – (5.8) is equivalent to

$$\begin{aligned}
&(5.13) && u_{tt} + \Delta^2 u = l(v-u) - \chi_\omega(x)\{\alpha_1 u_t(x,t) + \alpha_2 y(x,1,t)\} && \text{in } \Omega \times (0,+\infty),\\
&(5.14) && y_t(x,\rho,t) + \tau^{-1} y_\rho(x,\rho,t) = 0 && \text{in } \omega \times (0,1) \times (0,+\infty),\\
&(5.15) && v_{tt} + \Delta^2 v = l(u-v) - \chi_\omega(x)\{\beta_1 v_t(x,t) + v_2 z(x,1,t)\} && \text{in } \Omega \times (0,+\infty),\\
&(5.16) && z_t(x,\rho,t) + \tau^{-1} z_\rho(x,\rho,t) = 0 && \text{in } \omega \times (0,1) \times (0,+\infty),\\
&(5.17) && u(x,0) = u_0(x), u_t(x,0) = u_1(x) && \text{in } \Omega,\\
&(5.18) && v(x,0) = v_0(x), v_t(x,0) = v_1(x) && \text{in } \Omega,\\
&(5.19) && u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0 && \text{on } \Gamma \times (0,+\infty),\\
&(5.20) && v(x,t) = \frac{\partial v(x,t)}{\partial \nu} = 0 && \text{on } \Gamma \times (0,+\infty),\\
&(5.21) && u_t(x,t) = y(x,0,t) && \text{in } \omega \times (0,+\infty),\\
&(5.22) && v_t(x,t) = z(x,0,t) && \text{in } \omega \times (0,+\infty),\\
&(5.23) && y(x,\rho,0) = f(x,-\tau\rho) && \text{in } \omega \times (0,1),\\
&(5.24) && z(x,\rho,0) = g(x,-\tau\rho) && \text{in } \omega \times (0,1).
\end{aligned}$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = H_0^2(\Omega) \times L^2(\Omega) \times L^2(\omega; L^2(0,1)) \times H_0^2(\Omega) \times L^2(\Omega) \times L^2(\omega; L^2(0,1)),$$

where

$$H_0^2(\Omega) = \{u \in H^2(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma\}.$$

We equip $\mathcal{H}$ with the inner product

$$\left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \phi \\ \vartheta \\ \psi \end{pmatrix} ; \begin{pmatrix} \widetilde{\zeta} \\ \widetilde{\eta} \\ \widetilde{\theta} \\ \widetilde{\phi} \\ \widetilde{\vartheta} \\ \widetilde{\psi} \end{pmatrix} \right\rangle = \int_\Omega (\Delta\zeta(x)\Delta\widetilde{\zeta}(x) + \eta(x)\widetilde{\eta}(x))\, dx +$$

$$\mu \int_\Omega \chi_\omega(x) \int_0^1 \theta(x,\rho)\widetilde{\theta}(x,\rho)\, d\rho\, dx + \int_\Omega (\Delta\phi(x)\Delta\widetilde{\phi}(x) + \vartheta(x)\widetilde{\vartheta}(x))\, dx +$$

$$\xi \int_\Omega \chi_\omega(x) \int_0^1 \psi(x,\rho)\widetilde{\psi}(x,\rho)\, d\rho\, dx + l \int_\Omega (\zeta(x) - \phi(x))(\widetilde{\zeta}(x) - \widetilde{\phi}(x))\, dx,$$

and define a linear operator A in $\mathcal{H}$ by

$$\begin{aligned} D(A) =& \{(\zeta,\eta,\theta,\phi,\vartheta,\psi)^T \in (H^4(\Omega)\cap H_0^2(\Omega)) \times H_0^2(\Omega) \times L^2(\omega; H^1(0,1)) \times \\ & (H^4(\Omega)\cap H_0^2(\Omega)) \times H_0^2(\Omega) \times L^2(\omega; H^1(0,1))\,; \eta = \theta(.,0),\ \vartheta = \psi(.,0) \text{ in } \omega\}, \end{aligned} \tag{5.25}$$

$$\begin{aligned} A(\zeta,\eta,\theta,\phi,\vartheta,\psi)^T = & (\eta, -\Delta^2\zeta + l\phi - l\zeta - \chi_\omega\{\alpha_1\eta + \alpha_2\theta(.,1)\}, -\tau^{-1}\theta_\rho, \\ & \chi, -\Delta^2\phi - l\phi + l\zeta - \chi_\omega\{\beta_1\vartheta + \beta_2\psi(.,1)\}, -\tau^{-1}\psi_\rho)^T. \end{aligned} \tag{5.26}$$

Then we can rewrite (5.13) – (5.24) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \dfrac{dW}{dt}(t) = \mathcal{A}W(t); \\ W(0) = W_0. \end{cases} \tag{5.27}$$

where

$$W(t) = (u(x,t), u_t(x,t), y(x,\rho,t), v(x,t), v_t(x,t), z(x,\rho,t))^T,$$
$$\text{and } W_0 = (u_0, u_1, f(.,-.\tau), v_0, v_1, g(.,-.\tau))^T.$$

We verify that A generates a strongly continuous semigroup on $\mathcal{H}$ and consequently we have

Theorem 5.2. *Assume* (5.9), *then for any initial datum $W_0 \in \mathcal{H}$, the problem defined by* (5.27) *has a unique solution $W(.) \in C([0,+\infty); \mathcal{H})$*
If in addition we assume that $W_0 \in D(A)$, then the solution is more regular

$$W(.) \in C^1([0,+\infty); \mathcal{H}) \cap C([0,+\infty); D(A)).$$

Proof. First, we prove that the operator A is dissipative. Let

$$W = (\zeta,\eta,\theta,\phi,\chi,\psi)^T \in D(A). Then$$

$$\begin{aligned} \langle AW; W\rangle =& \int_\Omega \left[\Delta\eta(x)\Delta\zeta(x) + (-\Delta^2\zeta(x) + l\phi(x) - l\zeta(x))\eta(x) - \chi_\omega(x)\{\alpha_1\eta(x) + \alpha_2\theta(x,1)\}\eta(x)\right] dx \\ &+ \int_\Omega \left[\Delta\vartheta(x)\Delta\phi(x) + (-\Delta^2\phi(x) - l\phi(x) + l\zeta(x))\vartheta(x) - \chi_\omega(x)\{\beta_1\vartheta(x) + \beta_2\psi(x,1)\}\vartheta(x)\right] dx \\ &- \tau^{-1}\mu \int_\Omega \chi_\omega(x) \int_0^1 \theta_\rho(x,\rho)\theta(x,\rho)\, d\rho\, dx - \tau^{-1}\xi \int_\Omega \chi_\omega(x) \int_0^1 \psi_\rho(x,\rho)\psi(x,\rho)\, d\rho\, dx \\ &+ l \int_\Omega (\eta(x) - \vartheta(x))(\zeta(x) - \phi(x))\, dx. \end{aligned}$$

Using Cauchy-Schwarz inequality after applying Green's Theorem and integrating by parts with respect to ρ, we get

$$\langle AW;W\rangle \leq (-\alpha_1+\frac{\alpha_2}{2}+\frac{\tau^{-1}\mu}{2})\int_\omega \eta^2(x)\,dx+(\frac{\alpha_2}{2}-\frac{\tau^{-1}\mu}{2})\int_\omega \theta^2(x,1)\,dx$$
$$+(-\beta_1+\frac{\beta_2}{2}+\frac{\tau^{-1}\xi}{2})\int_\omega \chi^2(x)\,dx+(\frac{\beta_2}{2}-\frac{\tau^{-1}\xi}{2})\int_\omega \psi^2(x,1)\,dx.$$

From (5.11) and (5.12), we conclude that $\langle AW;W\rangle \leq 0$. Thus A is dissipative.
Now, we show that $\lambda I-A$ is onto for a fixed $\lambda>0$, that is for $(f,g,h,k,m,p)^T\in\mathcal{H}$, there exists $W=(\zeta,\eta,\theta,\phi,\chi,\psi)^T\in D(A)$ solution of

$$(\lambda I-A)W=(f,g,h,k,m,p)^T,$$

or equivalently

$$\lambda\zeta-\eta=f, \tag{5.28}$$
$$\lambda\eta+\Delta^2\zeta+l\zeta-l\phi+\chi_\omega\{\alpha_1\eta+\alpha_2\theta(.,1)\}=g, \tag{5.29}$$
$$\lambda\theta+\tau^{-1}\theta_\rho=h, \tag{5.30}$$
$$\lambda\phi-\vartheta=k, \tag{5.31}$$
$$\lambda\chi+\Delta^2\phi+l\phi-l\zeta+\chi_\omega\{\beta_1\vartheta+\beta_2\psi(.,1)\}=m, \tag{5.32}$$
$$\lambda\psi+\tau^{-1}\psi_\rho=p. \tag{5.33}$$

Suppose that we have found ζ and ϕ with the appropriate regularity, then

$$\eta=\lambda\zeta-f, \tag{5.34}$$

$$\vartheta=\lambda\phi-k. \tag{5.35}$$

We have from (5.30) with (5.25),

$$\theta(x,1)=\lambda e^{-\lambda\tau}\zeta(x)+z_0(x), \tag{5.36}$$

with z_0 defined by

$$z_0(x)=-f(x)e^{-\lambda\tau}+\tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma(x), \tag{5.37}$$

and from (5.33) and (5.25), we find

$$\psi(x,1)=\lambda e^{-\lambda\tau}\phi(x)+z_1(x), \tag{5.38}$$

with z_1 defined by,

$$z_1(x)=-k(x)e^{-\lambda\tau}+\tau e^{-\lambda\tau}\int_0^1 p(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma(x).$$

From (5.29), (5.32), (5.36) and (5.38) the functions ζ and ϕ verify

$$\begin{cases}\lambda^2\zeta+\Delta^2\zeta+\lambda\chi_\omega(\alpha_1+\alpha_2e^{-\lambda\tau})\zeta+l\zeta-l\phi=g+(\alpha_1\chi_\omega+\lambda)f+\alpha_2\chi_\omega z_0,\\ \lambda^2\phi+\Delta^2\phi+\lambda\chi_\omega(\beta_1+\beta_2e^{-\lambda\tau})\phi+l\phi-l\zeta=m+(\beta_1\chi_\omega+\lambda)k+\beta_2\chi_\omega z_1.\end{cases} \tag{5.39}$$

Problem (5.39) can be reformulated as

$$\int_\Omega\Big(\lambda^2\zeta(x)+\Delta^2\zeta(x)+\lambda\chi_\omega(x)(\alpha_1+\alpha_2e^{-\lambda\tau})\zeta(x)+l\zeta(x)-l\phi(x)\Big)\,w_1(x)\,dx$$
$$+\int_\Omega\Big(\lambda^2\phi(x)+\Delta^2\phi(x)+\lambda\chi_\omega(x)(\beta_1+\beta_2e^{-\lambda\tau})\phi(x)+l\phi(x)-l\zeta(x)\Big)\,w_2(x)\,dx$$
$$=\int_\Omega(g(x)+(\alpha_1\chi_\omega(x)+\lambda)f(x)+\alpha_2\chi_\omega(x)z_0(x))\,w_1(x)\,dx$$
$$+\int_\Omega(m(x)+(\beta_1\chi_\omega(x)+\lambda)k(x)+\beta_2\chi_\omega(x)z_1(x))\,w_2(x)\,dx,\qquad\forall(w_1,w_2)\in H_0^2(\Omega)\times H_0^2(\Omega).$$

We rewrite the last equality after using Green's Theorem

$$\begin{aligned}
&\int_\Omega(\lambda^2\zeta(x)w_1(x)+\Delta\zeta(x)\Delta w_1(x))\,dx+\int_\Omega\chi_\omega(x)\lambda(\alpha_1+\alpha_2e^{-\lambda\tau})\zeta(x)w_1(x)\,dx\\
&+\int_\Omega(l\zeta(x)-l\phi(x))w_1(x)\,dx+\int_\Omega(\lambda^2\phi(x)w_2(x)+\Delta\phi(x)\Delta w_2(x))\,dx\\
&+\int_\Omega\chi_\omega(x)\lambda(\beta_1+\beta_2e^{-\lambda\tau})\phi(x)w_2(x)\,dx+\int_\Omega(l\phi(x)-l\zeta(x))w_2(x)\,dx \qquad (5.40)\\
&=\int_\Omega(g(x)+(\alpha_1\chi_\omega(x)+\lambda)f(x)+\alpha_2\chi_\omega(x)z_0(x))\,w_1(x)\,dx\\
&+\int_\Omega(m(x)+(\beta_1\chi_\omega(x)+\lambda)k(x)+\beta_2\chi_\omega(x)z_1(x))\,w_2(x)\,dx. \qquad (5.41)
\end{aligned}$$

Since the left-hand side of (5.41) is coercive and continuous on $H_0^2(\Omega)\times H_0^2(\Omega)$, and the right-hand side defines a continuous linear form, the Lax-Milgram Theorem guarantees the existence and uniqueness of a solution $(\zeta,\phi)\in H_0^2(\Omega)\times H_0^2(\Omega)$ of (5.39).
If we consider $(w_1,w_2)\in\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ in (5.41), then (ζ,ϕ) is a solution in $\mathcal{D}'(\Omega)\times\mathcal{D}'(\Omega)$ of

$$\left\{\begin{array}{l}
\lambda^2\zeta+\Delta^2\zeta+\lambda\chi_\omega(\alpha_1+\alpha_2e^{-\lambda\tau})\zeta+l\zeta-l\phi=g+(\alpha_1\chi_\omega+\lambda)f+\alpha_2\chi_\omega z_0,\\
\lambda^2\phi+\Delta^2\phi+\lambda\chi_\omega(\beta_1+\beta_2e^{-\lambda\tau})\phi+l\phi-l\zeta=m+(\beta_1\chi_\omega+\lambda)k+\beta_2\chi_\omega z_1.
\end{array}\right. \qquad (5.42)$$

Thus $(\Delta^2\zeta,\Delta^2\phi)\in L^2(\Omega)\times L^2(\Omega)$.
So, we have found $(\zeta,\eta,\theta,\phi,\vartheta,\psi)^T\in D(A)$ which verifies (5.28) – (5.33). Thus, by the Lumer-Phillips Theorem, A is the generator of a C_0- semigroup of contractions on $\mathcal{H}$. □

5.4. Proof of the main result

We prove Theorem 5.1 for smooth solution. The general case follows by a standard density argument. We first show that the energy function $E(t)$ defined by (5.10), (5.11) and (5.12) is decreasing.

PROPOSITION 5.1. *The energy corresponding to any regular solution of problem* (5.1) – (5.8), *is decreasing and there exists a positive constant* K *such that,*

$$\frac{d}{dt}E(t)\le -K\int_\omega\left\{u_t^2(x,t)+u_t^2(x,t-\tau)+v_t^2(x,t)+v_t^2(x,t-\tau)\right\}\,dx, \qquad (5.43)$$

where

$$K=\min\left\{(\alpha_1-\frac{\alpha_2}{2}-\frac{\mu}{2\tau}),(\frac{\mu}{2\tau}-\frac{\alpha_2}{2}),(\beta_1-\frac{\beta_2}{2}-\frac{\xi}{2\tau}),(\frac{\xi}{2\tau}-\frac{\beta_2}{2})\right\}.$$

PROOF. Differentiating $E(t)$ with respect to time, applying Green's Theorem, recalling the boundary conditions (5.5)-(5.6), and applying Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\frac{d}{dt}E(t)\le&(-\alpha_1+\frac{\alpha_2}{2}+\frac{\mu\tau^{-1}}{2})\int_\Omega\chi_\omega(x)u_t^2(x,t)\,dx+(\frac{\alpha_2}{2}-\frac{\mu\tau^{-1}}{2})\int_\Omega\chi_\omega(x)u_t^2(x,t-\tau)\,dx\\
&+(-\beta_1+\frac{\beta_2}{2}+\frac{\xi\tau^{-1}}{2})\int_\Omega\chi_\omega(x)v_t^2(x,t)\,dx+(\frac{\beta_2}{2}-\frac{\xi\tau^{-1}}{2})\int_\Omega\chi_\omega(x)v_t^2(x,t-\tau)\,dx,
\end{aligned}$$

which implies

$$\frac{d}{dt}E(t)\le -K\int_\omega\{u_t^2(x,t)+u_t^2(x,t-\tau)+v_t^2(x,t)+v_t^2(x,t-\tau)\}\,dx,$$

with

$$K=\min\left\{(\alpha_1-\frac{\alpha_2}{2}-\frac{\mu}{2\tau}),(\frac{\mu}{2\tau}-\frac{\alpha_2}{2}),(\beta_1-\frac{\beta_2}{2}-\frac{\xi}{2\tau}),(\frac{\xi}{2\tau}-\frac{\beta_2}{2})\right\}.$$

□

We now give an observability inequality which we will use it to prove the exponential decay of the energy E.

PROPOSITION 5.2. *For any regular solution of problem* (5.1) – (5.8)*, there exists a positive constant* C *(depending on* T*) such that*

$$E(0) \leq C \int_0^T \int_\omega \left\{ u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau) \right\} dx\,dt. \tag{5.44}$$

To prove this result, we need to find an observability inequality for the homogeneous coupled Euler-Bernoulli equations

$$\begin{cases} y_{tt}(x,t) + \Delta^2 y(x,t) + l(y(x,t) - z(x,t)) = 0 & \text{in } \Omega \times (0,+\infty), \\ z_{tt}(x,t) + \Delta^2 z(x,t) + l(z(x,t) - y(x,t)) = 0 & \text{in } \Omega \times (0,+\infty), \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x) & \text{in } \Omega, \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x) & \text{in } \Omega, \\ y(x,t) = z(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ \dfrac{\partial y(x,t)}{\partial \nu} = \dfrac{\partial z(x,t)}{\partial \nu} = 0 & \text{on } \Gamma \times (0,+\infty). \end{cases} \tag{5.45}$$

Denote by F the standard energy for (5.45), that is

$$F(t) = \frac{1}{2} \int_\Omega \left\{ |\Delta y(x,t)|^2 + y_t^2(x,t) + |\Delta z(x,t)|^2 + z_t^2(x,t) + l(y(x,t) - z(x,t))^2 \right\} dx.$$

Note that $F(t) = F(0)$.

PROPOSITION 5.3. *For all* $T > 0$*, there exists a positive constant* C *(depending on* T*) for which*

$$F(0) \leq C \int_0^T \int_\omega \left\{ y_t^2(x,t) + z_t^2(x,t) \right\} dx\,dt, \tag{5.46}$$

for any regular solution (y,z) solution of (5.45)

PROOF. We follow several steps to prove the inequality (5.46) .

Step 1.

We multiply both sides of the first two equations in (5.45) by $h.\nabla y$, $h.\nabla z$ respectively and integrate over $\Omega \times (0,T)$, we obtain

$$\int_0^T \int_\Omega (y_{tt}(x,t) + \Delta^2 y(x,t))(h(x).\nabla y(x,t))\,dx\,dt = l \int_0^T \int_\Omega (z(x,t) - y(x,t))(h(x).\nabla y(x,t))\,dx\,dt, \tag{5.47}$$

$$\int_0^T \int_\Omega (z_{tt}(x,t) + \Delta^2 z(x,t))(h(x).\nabla z(x,t))\,dx\,dt = l \int_0^T \int_\Omega (y(x,t) - z(x,t))(h(x).\nabla z(x,t))\,dx\,dt. \tag{5.48}$$

We compute each term on the left-hand side of (5.47) separately

- Term $\int_0^T \int_\Omega y_{tt}(x,t)h(x).\nabla y(x,t)\,dx\,dt$
  Integrating by parts with respect to t and applying Green's Theorem, we get

(5.49)
$$\int_0^T \int_\Omega y_{tt}(x,t)h(x).\nabla y(x,t)\,dx\,dt = \left[\int_\Omega y_t(x,t)h(x).\nabla y(x,t)\,dx\right]_0^T + \frac{1}{2}\int_0^T \int_\Omega y_t^2(x,t)\mathrm{div}h(x)\,dx\,dt.$$

- Term $\int_0^T \int_\Omega \Delta^2 y(x,t)h(x).\nabla y(x,t)\,dx\,dt$.
  Applying Green's Theorem, we obtain

$$\begin{aligned}\int_0^T \int_\Omega \Delta^2 yh.\nabla y\,dx\,dt &= \int_0^T \int_\Gamma \frac{\partial \Delta y}{\partial \nu}h.\nabla y\,d\Gamma\,dt - \int_0^T \int_\Gamma \Delta y\frac{\partial y}{\partial x_k}\nu_k\,d\Gamma\,dt \\ &- \int_0^T \int_\Gamma h_k\Delta y\frac{\partial^2 y}{\partial x_k \partial x_j}\nu_j\,d\Gamma\,dt + 2\int_0^T \int_\Omega (\Delta y)^2\,dx\,dt \\ &+ \frac{1}{2}\int_0^T \int_\Gamma (\Delta y)^2 h.\nu\,d\Gamma\,dt - \frac{1}{2}\int_0^T \int_\Omega (\Delta y)^2\mathrm{div}h\,dx\,dt.\end{aligned} \tag{5.50}$$

We have from the boundary condition $y = \frac{\partial y}{\partial \nu} = 0$ on $\Gamma \times (0,T)$,

$$\frac{\partial y}{\partial x_k} = \frac{\partial y}{\partial \nu}\nu_k = 0, \quad \text{thus} \qquad \nabla y = 0 \ \text{ on}\,\Gamma \times (0,T),$$

and

$$\int_0^T \int_\Gamma h_k\Delta y\frac{\partial^2 y}{\partial x_k \partial x_j}\nu_j\,d\Gamma\,dt = \int_0^T \int_\Gamma (\Delta y)^2 h.\nu\,d\Gamma\,dt.$$

Hence

$$\int_0^T \int_\Omega \Delta^2 yh.\nabla y\,dx\,dt = -\frac{1}{2}\int_0^T \int_\Gamma (\Delta y)^2 h.\nu\,d\Gamma\,dt + 2\int_0^T \int_\Omega (\Delta y)^2\,dx\,dt - \frac{1}{2}\int_0^T \int_\Omega (\Delta y)^2\mathrm{div}h\,dx\,dt.$$

So (5.47) becomes:

$$\begin{aligned}&\left[\int_\Omega y_t h.\nabla y\,dx\right]_0^T + \frac{n}{2}\int_0^T \int_\Omega y_t^2\,dx\,dt - \frac{1}{2}\int_0^T \int_\Gamma (\Delta y)^2 h.\nu\,d\Gamma\,dt + 2\int_0^T \int_\Omega (\Delta y)^2\,dx\,dt \\ &- \frac{n}{2}\int_0^T \int_\Omega (\Delta y)^2\,dx\,dt = l\int_0^T \int_\Omega (z(x,t)-y(x,t))(h.\nabla y)\,dx\,dt.\end{aligned} \tag{5.51}$$

In similar manner, we obtain

$$\begin{aligned}&\left[\int_\Omega z_t h.\nabla z\,dx\right]_0^T + \frac{n}{2}\int_0^T \int_\Omega y_t^2\,dx\,dt - \frac{1}{2}\int_0^T \int_\Gamma (\Delta z)^2 h.\nu\,d\Gamma\,dt + 2\int_0^T \int_\Omega (\Delta z)^2\,dx\,dt \\ &- \frac{n}{2}\int_0^T \int_\Omega (\Delta z)^2\,dx\,dt = l\int_0^T \int_\Omega (y-z)(h.\nabla z)\,dx\,dt.\end{aligned} \tag{5.52}$$

Summing up (5.51) with (5.52), we get

$$\begin{aligned}&\left[\int_\Omega \{y_t h.\nabla y + z_t h.\nabla z\}\,dx\right]_0^T + \frac{n}{2}\int_0^T \int_\Omega \{y_t^2 + z_t^2 - (\Delta y)^2 - (\Delta z)^2 - l(y-z)^2\}\,dx\,dt \\ &- \frac{1}{2}\int_0^T \int_\Gamma \{(\Delta y)^2 + (\Delta z)^2\}h.\nu\,d\Gamma\,dt + 2\int_0^T \int_\Omega \{(\Delta y)^2 + (\Delta z)^2\}\,dx\,dt = 0.\end{aligned} \tag{5.53}$$

Differentiating $I = \displaystyle\int_\Omega \{y(x,t)y_t(x,t) + z(x,t)z_t(x,t)\}\,dx$ with respect to time t and recalling (5.45) after using Green's Theorem, we obtain

$$\frac{dI}{dt} = \int_\Omega \left\{y_t^2 + z_t^2 - (\Delta y)^2 - (\Delta z)^2 - l(y-z)^2\right\}\,dx.$$

We integrate both sides of the last equality over $(0,T)$, we find

$$\int_0^T \int_\Omega \left\{y_t^2 + z_t^2 - (\Delta y)^2 - (\Delta z)^2 - l(y-z)^2\right\}\,dx\,dt = \left[\int_\Omega \{yy_t + zz_t\}\,dx\right]_0^T. \tag{5.54}$$

From (5.54), we conclude

$$\begin{aligned}\int_0^T \int_\Omega \left\{y_t^2 + z_t^2 + (\Delta y)^2 + (\Delta z)^2 + l(y-z)^2\right\}\,dx\,dt &= 2\int_0^T \int_\Omega (\Delta y)^2 + (\Delta z)^2\,dx\,dt \\ &+ 2l\int_0^T \int_\Omega (y-z)^2\,dx\,dt + \left[\int_\Omega \{yy_t + zz_t\}\,dx\right]_0^T.\end{aligned} \tag{5.55}$$

Insertion of (5.54) and (5.55) into (5.53) yields

$$\begin{aligned}2TF(0) = &-\left[\int_\Omega \{y_t h.\nabla y + z_t h.\nabla z\}\,dx\right]_0^T + (1-\frac{n}{2})\left[\int_\Omega \{yy_t + zz_t\}\,dx\right]_0^T \\ &+ \frac{1}{2}\int_0^T \int_\Gamma \{(\Delta y)^2 + (\Delta z)^2\} h.\nu\,d\Gamma\,dt + 2l\int_0^T \int_\Omega (y-z)^2\,dx\,dt.\end{aligned}$$

We have from the definition of Γ_0

$$\begin{aligned}2TF(0) \le &-\left[\int_\Omega \{y_t h.\nabla y + z_t h.\nabla z\}\,dx\right]_0^T + (1-\frac{n}{2})\left[\int_\Omega \{yy_t + zz_t\}\,dx\right]_0^T \\ &+ \frac{1}{2}\int_0^T \int_{\Gamma_0} \{(\Delta y)^2 + (\Delta z)^2\} h.\nu\,d\Gamma\,dt + 2l\int_0^T \int_\Omega (y-z)^2\,dx\,dt.\end{aligned}$$

Let

$$M_h = \max_{\overline{\Omega}} |h|, C_h = \max_{\Gamma_0} |h|,$$

$$\int_\Omega u^2\,dx < C_p \int_\Omega |\nabla u|^2\,dx, \qquad u \in H_0^1(\Omega), \quad C_p = \text{Poincaré constant}$$

$$\int_\Omega |\nabla u|^2 dx < C' \int_\Omega |\Delta u|^2\,dx, \qquad \forall u \in H_0^2(\Omega), \qquad (\text{see } \textbf{[21]}, \text{ p.256}).$$

From Cauchy-Schwarz inequality and Poincaré inequality, we have

$$\begin{aligned}TF(0) \le &(\frac{M_h}{4} + \frac{1}{4}(1+\frac{n}{2}))\{\|y_t(T)\|^2 + \|z_t(T)\|^2 + \|y_t(0)\|^2 + \|z_t(0)\|^2\} \\ &+ (\frac{M_h}{4} + \frac{C_p}{4}(1+\frac{n}{2}))\{\|\nabla y(T)\|^2 + \|\nabla z(T)\|^2 + \|\nabla y(0)\|^2 + \|\nabla z(0)\|^2\} \\ &+ \frac{C_h}{4}\int_0^T \int_{\Gamma_0} \{(\Delta y)^2 + (\Delta z)^2\}\,d\Gamma\,dt + 2lC_p \int_0^T \int_\Omega \{|\nabla y|^2 + |\nabla z|^2\}\,dx\,dt \\ &\le CF(0) + \frac{C_h}{4}\int_0^T \int_{\Gamma_0} \{(\Delta y)^2 + (\Delta z)^2\}\,d\Gamma\,dt.\end{aligned}$$

For T large enough, we get (C depending on T)

$$F(0) \le C\int_0^T \int_{\Gamma_0} \{|\Delta y|^2 + |\Delta z|^2\}\,d\Gamma. \tag{5.56}$$

For fixed $\alpha > 0$ small enough we apply estimate (5.56) over the interval $(\alpha, T-\alpha)$ rather than $(0,T)$. We obtain

$$F(0) \leq C \int_{\alpha}^{T-\alpha} \int_{\Gamma_0} \{|\Delta y|^2 + |\Delta z|^2\} \, d\Gamma. \tag{5.57}$$

Step 2.
To estimate the right-hand side of (5.57), we use a multiplier technique again, by choosing another multiplier $q(x,t) = t(T-t)m(x)$, where $m \in \left(C^2(\overline{\Omega})\right)^n$ is a vector field (see [**21**]) that satisfies

$$m(x).\nu(x) = 1 \text{ on } \Gamma_0, \; m(x).\nu(x) \geq 0 \text{ on } \Gamma, \; \text{supp } m \subset \omega_\varepsilon,$$

such that $\omega_\varepsilon \subset \omega$, where ω_ε is defined by (see [**21**])

$$O_\varepsilon = \bigcup_{x \in \Gamma_0} B(x,\varepsilon), \qquad \omega_\varepsilon = O_\varepsilon \cap \Omega,$$

where $B(x,\varepsilon)$ is the ball of center x and radius ε.
We multiply both sides of first two equations in the problem (5.45) by $(q.\nabla y), (q.\nabla z)$ respectively and integrate over $\Omega \times (0,T)$, we obtain

$$\begin{aligned}
\frac{1}{2}\int_0^T \int_\Gamma q.\nu \left(|\Delta y|^2 + |\Delta z|^2\right) \, d\Gamma \, dt &= \frac{1}{2}\int_0^T \int_\Omega \text{div} q \left(y_t^2 + z_t^2 - |\Delta y|^2 - |\Delta z|^2 - l(y-z)^2\right) \, dx \, dt \\
&+ \int_0^T \int_\Omega \left(\Delta q.\nabla y \Delta y + \Delta q.\nabla z \Delta z\right) \, dx \, dt \\
&+ 2\int_0^T \int_\Omega \Delta y \sum_{j,k=1}^n \partial_j q_k \partial_{jk}^2 y \, dx \, dt \\
&+ 2\int_0^T \int_\Omega \Delta z \sum_{j,k=1}^n \partial_j q_k \partial_{jk}^2 z \, dx \, dt - \int_0^T \int_\Omega \left(y_t q_t.\nabla y + z_t q_t.\nabla z\right) \, dx \, dt.
\end{aligned}$$

By the properties of m, we get

$$\begin{aligned}
\alpha(T-\alpha)\int_\alpha^{T-\alpha} \int_{\Gamma_0} \left(|\Delta y|^2 + |\Delta z|^2\right) \, d\Gamma \, dt &\leq \int_\alpha^{T-\alpha} \int_{\Gamma_0} q.\nu \left(|\Delta y|^2 + |\Delta z|^2\right) \, d\Gamma \, dt \\
&\leq \int_0^T \int_\Gamma q.\nu \left(|\Delta y|^2 + |\Delta z|^2\right) \, d\Gamma \, dt.
\end{aligned}$$

We apply Young's inequality in the previous identity and use the last inequality, we find

$$\begin{aligned}
\int_\alpha^{T-\alpha} \int_{\Gamma_0} \left(|\Delta y|^2 + |\Delta z|^2\right) \, d\Gamma \, dt &\leq C\{\int_0^T \int_{\omega_\varepsilon} \left(y_t^2 + z_t^2 + |\Delta y|^2 + |\Delta z|^2\right) \, dx \, dt + \int_0^T \int_{\omega_\varepsilon} (y^2 + z^2) \, dx \, dt \\
&+ \int_0^T \int_{\omega_\varepsilon} \left(\frac{1}{\eta}|\Delta y|^2 + \eta|\nabla y|^2\right) \, dx \, dt \\
&+ \int_0^T \int_{\omega_\varepsilon} \left(\frac{1}{\eta}|\Delta z|^2 + \eta|\nabla z|^2\right) \, dx \, dt \\
&+ \int_0^T \int_{\omega_\varepsilon} \left(\frac{1}{\eta}(|\Delta y|^2 + |\Delta z|^2) + \eta \sum_{j,k=1}^n (|\partial_{jk}^2 y|^2 + |\partial_{jk}^2 z|^2)\right) \, dx \, dt \\
&+ \int_0^T \int_{\omega_\varepsilon} \left(\frac{1}{\eta}(y_t^2 + z_t^2) + \eta(|\nabla y|^2 + |\nabla z|^2)\right) \, dx \, dt\}
\end{aligned}$$

where η is a positive constant that will be fixed later.
From the last inequality we have for all $\eta \in (0,1)$

$$\begin{aligned}\int_{\alpha}^{T-\alpha}\int_{\Gamma_0}\left(|\Delta y|^2+|\Delta z|^2\right)\,d\Gamma\,dt \leq & \frac{C}{\eta}\int_0^T\int_{\omega_\varepsilon}\left(y_t^2+z_t^2+|\Delta y|^2+|\Delta z|^2\right)\,dx\,dt \\ & +\eta C\int_0^T\left(\|y(.,t)\|^2_{H^2(\Omega)}+\|z(.,t)\|^2_{H^2(\Omega)}\right)\,dt.\end{aligned} \tag{5.58}$$

Using the fact that

$$\|w\|^2_{H^2(\Omega)} \leq C\int_\Omega |\Delta w|^2\,dx \qquad \forall w \in H_0^2(\Omega),$$

then, (5.58) implies

$$\begin{aligned}\int_{\alpha}^{T-\alpha}\int_{\Gamma_0}\left(|\Delta y|^2+|\Delta z|^2\right)\,d\Gamma\,dt \leq & \frac{C}{\eta}\int_0^T\int_{\omega_\varepsilon}\left(y_t^2+z_t^2+|\Delta y|^2+|\Delta z|^2\right)\,dx\,dt \\ & +\eta C\int_0^T\int_\Omega\left(|\Delta y|^2+|\Delta z|^2\right)\,dx\,dt.\end{aligned}$$

From the definition of the energy $F(t)$, the last inequality becomes

(5.59)
$$\int_{\alpha}^{T-\alpha}\int_{\Gamma_0}\left(|\Delta y|^2+|\Delta z|^2\right)\,d\Gamma\,dt \leq \frac{C}{\eta}\int_0^T\int_{\omega_\varepsilon}\left(y_t^2+z_t^2+|\Delta y|^2+|\Delta z|^2\right)\,dx\,dt+\eta CTF(0),$$

for all $\eta \in (0,1)$.
Using (5.57) with (5.59) yields

$$(1-\eta CT)F(0) \leq C\int_0^T\int_{\omega_\varepsilon}\left(y_t^2+z_t^2+|\Delta y|^2+|\Delta z|^2\right)\,dx\,dt.$$

We choose η sufficiently small to make $(1-C\eta T>0)$, we obtain

$$F(0) \leq C\int_0^T\int_{\omega_\varepsilon}\left(y_t^2+z_t^2+|\Delta y|^2+|\Delta z|^2\right)\,dx\,dt. \tag{5.60}$$

Step 3.

We have from (5.60) on $(\alpha, T-\alpha)$ and in the domain $\omega_{\frac{\epsilon}{2}}$

$$F(0)=F(\alpha) \leq C\int_{\alpha}^{T-\alpha}\int_{\omega_{\frac{\epsilon}{2}}}\left(y_t^2+z_t^2+|\Delta y|^2+|\Delta z|^2\right)\,dx\,dt \tag{5.61}$$

Let

$$\begin{aligned}\xi_1(x,t)&=t(T-t)\varphi_\epsilon(x)y(x,t),\\ \xi_2(x,t)&=t(T-t)\varphi_\epsilon(x)z(x,t),\end{aligned}$$

where $\varphi_\epsilon \in W_0^{2,\infty}(O_\epsilon)$ is defined (see **[21]**)

$$\varphi_\epsilon(x)=\begin{cases}1 & \text{on } O_{\frac{\epsilon}{2}},\\ \dfrac{(\epsilon-2d(x))^4}{\epsilon^4} & \text{on} \quad O\epsilon\diagdown O_{\frac{\epsilon}{2}},\\ 0 & \text{elsewhere },\end{cases}$$

where $d(x)$ is the distance from x to $\partial O_{\frac{\epsilon}{2}}$. φ_ϵ satisfies

$$0 \le \varphi_\epsilon \le 1 \quad \text{in } \Omega, \tag{5.62}$$

$$\frac{|\nabla\varphi_\epsilon|^2}{\varphi_\epsilon} \le \frac{C}{\epsilon^4} \quad \text{in } \omega_\epsilon, \tag{5.63}$$

$$\frac{|\Delta\varphi_\epsilon|^2}{\varphi_\epsilon} \le \frac{C}{\epsilon^4} \quad \text{in } \omega_\epsilon. \tag{5.64}$$

Now, we multiply both sides of two first equations of (5.45) by ξ_1, ξ_2 respectively and integrate over $\Omega \times (0,T)$, we obtain

$$\begin{aligned}
&-\int_0^T\int_\Omega y_t\varphi_\epsilon(t(T-t)y_t + (T-2t)y)\,dx\,dt + \int_0^T\int_\Omega \Delta y t(T-t)(\Delta\varphi_\epsilon y + 2\nabla\varphi_\epsilon.\nabla y + \varphi_\epsilon\Delta y)\,dx\,dt\\
&-\int_0^T\int_\Omega z_t\varphi_\epsilon(t(T-t)z_t + (T-2t)z)\,dx\,dt + \int_0^T\int_\Omega \Delta z t(T-t)(\Delta\varphi_\epsilon z + 2\nabla\varphi_\epsilon\nabla z + \varphi_\epsilon\Delta z)\,dx\,dt\\
&+l\int_0^T\int_\Omega (y-z)^2 t(T-t)\varphi_\epsilon\,dx\,dt = 0.
\end{aligned}$$

Since φ_ϵ is zero outside ω_ϵ, we get

$$\begin{aligned}
\int_0^T\int_\Omega \left(|\Delta y|^2 + |\Delta z|^2\right) t(T-t)\varphi_\epsilon\,dx\,dt =& \int_0^T\int_{\omega_\varepsilon} y_t\varphi_\epsilon(t(T-t)y_t + (T-2t)y)\,dx\,dt\\
&-\int_0^T\int_{\omega_\varepsilon} \Delta y t(T-t)(\Delta\varphi_\epsilon y + 2\nabla\varphi_\epsilon.\nabla y)\,dx\,dt\\
&+\int_0^T\int_{\omega_\varepsilon} z_t\varphi_\epsilon(t(T-t)z_t + (T-2t)z)\,dx\,dt\\
&-\int_0^T\int_{\omega_\varepsilon} \Delta z t(T-t)(\Delta\varphi_\epsilon z + 2\nabla\varphi_\epsilon.\nabla z)\,dx\,dt\\
&-l\int_0^T\int_{\omega_\varepsilon} (y-z)^2 t(T-t)\varphi_\epsilon\,dx\,dt,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_0^T\int_\Omega \left(|\Delta y|^2 + |\Delta z|^2\right) t(T-t)\varphi_\epsilon(x)\,dx\,dt \le& C(\int_0^T\int_{\omega_\varepsilon} \left(y_t^2 + z_t^2 + y^2 + z^2\right)\,dx\,dt\\
&+\int_0^T\int_{\omega_\varepsilon} |\Delta y| t(T-t)\sqrt{\varphi_\epsilon}(\frac{|\Delta\varphi_\epsilon|}{\sqrt{\varphi_\epsilon}}|y| + \frac{|\nabla\varphi_\epsilon|}{\sqrt{\varphi_\epsilon}}|\nabla y|)\,dx\,dt\\
&+\int_0^T\int_{\omega_\varepsilon} |\Delta z| t(T-t)\sqrt{\varphi_\epsilon}(\frac{|\Delta\varphi_\epsilon|}{\sqrt{\varphi_\epsilon}}|z| + \frac{|\nabla\varphi_\epsilon|}{\sqrt{\varphi_\epsilon}}|\nabla z|)\,dx\,dt).
\end{aligned}$$

By applying Young's inequality on the right-hand side of the last inequality, we obtain for all $\eta > 0$

$$\begin{aligned}
\int_0^T\int_\Omega \left(|\Delta y|^2 + |\Delta z|^2\right) t(T-t)\varphi_\epsilon(x)\,dx\,dt \le& C(\int_0^T\int_{\omega_\varepsilon} \left(y_t^2 + z_t^2 + y^2 + z^2\right)\,dx\,dt\\
&+\eta\int_0^T\int_{\omega_\varepsilon} t(T-t)\varphi_\epsilon\left(|\Delta y|^2 + |\Delta z|^2\right)\,dx\,dt\\
&+\frac{1}{\eta}\int_0^T\int_{\omega_\varepsilon} t(T-t)(\frac{|\Delta\varphi_\epsilon|^2}{\varphi_\epsilon}|y|^2 + \frac{|\nabla\varphi_\epsilon|^2}{\varphi_\epsilon}|\nabla y|^2\\
&+\frac{|\Delta\varphi_\epsilon|^2}{\varphi_\epsilon}|z|^2 + \frac{|\nabla\varphi_\epsilon|^2}{\varphi_\epsilon}|\nabla z|^2)\,dx\,dt).
\end{aligned}$$

Using (5.63) and (5.64), we find

$$\begin{aligned}\int_0^T \int_\Omega \left(|\Delta y|^2 + |\Delta z|^2\right) t(T-t)\varphi_\epsilon(x)\, dx\, dt \leq & C\Big(\int_0^T \int_{\omega_\varepsilon} \left(y_t^2 + z_t^2 + y^2 + z^2\right)\, dx\, dt \\ & + \eta \int_0^T \int_{\omega_\varepsilon} t(T-t)\varphi_\epsilon \left(|\Delta y|^2 + |\Delta z|^2\right)\, dx\, dt \\ & + \frac{1}{\eta}\int_0^T \int_{\omega_\varepsilon} t(T-t)(y^2 + |\nabla y|^2 + z^2 + |\nabla z|^2)\, dx\, dt\Big).\end{aligned}$$

We choose η sufficiently small to make $(1 - C\eta > 0)$ and recall that φ_ϵ is zero outside ω_ϵ, we obtain

$$\int_0^T \int_\Omega \left(|\Delta y|^2 + |\Delta z|^2\right) t(T-t)\varphi_\epsilon(x)\, dx\, dt \leq C\int_0^T \int_{\omega_\varepsilon} \left(y_t^2 + z_t^2 + y^2 + z^2 + |\nabla y|^2 + |\nabla z|^2\right)\, dx\, dt.$$

We use the fact that $\varphi_\epsilon(x) = 1 \quad \text{on } \omega_{\frac{\epsilon}{2}}$, the last inequality becomes

(5.65)

$$\int_\alpha^{T-\alpha} \int_{\omega_{\frac{\epsilon}{2}}} \left(|\Delta y|^2 + |\Delta z|^2\right)\, dx\, dt \leq C\int_0^T \int_{\omega_\varepsilon} \left(y_t^2 + z_t^2 + y^2 + z^2 + |\nabla y|^2 + |\nabla z|^2\right)\, dx\, dt.$$

Insertion (5.65) into (5.61) yields

$$F(0) \leq C\int_0^T \int_{\omega_\varepsilon} \left(y_t^2 + z_t^2 + y^2 + z^2 + |\nabla y|^2 + |\nabla z|^2\right)\, dx\, dt.$$

Consequently, since $\omega_\epsilon \subset \omega$

$$F(0) \leq C\int_0^T \int_\omega \left(y_t^2 + z_t^2\right)\, dx\, dt + C\{\|y\|^2 + \|z\|^2\}_{C(0,T,H_0^1(\Omega))}. \tag{5.66}$$

Step 4.
We prove by a compactness-uniqueness argument that there exists a constant C such that

$$\{\|y\|^2 + \|z\|^2\}_{C(0,T,H_0^1(\Omega))} \leq C\int_0^T \int_\omega \left(y_t^2 + z_t^2\right)\, dx\, dt. \tag{5.67}$$

Assume that there exists a sequence (y_n, z_n) of solution of problem (5.45) with

$$\begin{aligned} y_n(x,0) &= y_n^0(x), \quad y_{nt}(x,0) = y_n^1(x), \qquad x \in \Omega, \\ z_n(x,0) &= z_n^0(x), \quad z_{nt}(x,0) = z_n^1(x), \qquad x \in \Omega, \end{aligned}$$

such that

(5.68)

$$\{\|y_n\|^2 + \|z_n\|^2\}_{C(0,T,H_0^1(\Omega))} = 1,\ n = 1, 2, ...; \qquad \int_0^T \int_\omega \left(y_{nt}^2(x,t) + z_{nt}^2(x,t)\right)\, dx\, dt \to 0 \,\text{as}\, n \to +\infty.$$

Since each solution satisfies (5.66), we deduce from (5.68) that the sequence $(y_n^0, y_n^1, z_n^0, z_n^0)$ is bounded in $H = H_0^2(\Omega) \times L^2(\Omega) \times H_{\Gamma_0}^2(\Omega) \times L^2(\Omega)$. Hence there is a subsequence still denoted by $(y_n^0, y_n^1, z_n^0, z_n^0)$ which converges weakly to some $(y^0, y^1, z^0, z^1) \in H$. Let (y, z) be the solution of problem (5.45) with initial condition (y^0, z^0). We have

$$(y, z) \in C(0, T; H) \times C(0, T; H).$$

Then

$$(y_n, z_n) \longrightarrow (y, z) \text{ weakly in } L^\infty(0, T; H_0^2(\Omega)) \times L^\infty(0, T; H_0^2(\Omega)).$$

Since $H_0^2(\Omega)$ is compactly embedded in $H_0^1(\Omega)$, there exist a subsequence still denoted by (y_n, z_n) such that ,

$$(y_n, z_n) \longrightarrow (y, z) \text{ strongly in } L^\infty(0, T; H_0^1(\Omega)) \times L^\infty(0, T; H_0^1(\Omega)).$$

Then, we have from (5.68),

$$\{\|y\|^2 + \|z\|^2\}_{C(0,T,H_0^1(\Omega))} = 1. \tag{5.69}$$

and

$$\int_0^T \int_\omega \left(y_t^2(x,t) + z_t^2(x,t)\right)\, dx\, dt = 0.$$

Then

$$\Delta y_t = \Delta z_t = 0 \quad \text{on} \quad \omega \times (0,T).$$

and therefore by taking the trace on Γ_0, we get

$$\Delta y_t = \Delta z_t = 0 \quad \text{on} \quad \Gamma_0 \times (0,T).$$

Let $u := y_t, v := z_t$, then (u, v) satisfies

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) + l(u(x,t) - v(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ v_{tt}(x,t) + \Delta^2 v(x,t) + l(v(x,t) - u(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ \Delta u(x,t) = \Delta v(x,t) = 0 & \text{on} \quad \Gamma_0 \times (0,T), \\ u(x,t) = \dfrac{\partial u(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma \times (0,T), \\ v(x,t) = \dfrac{\partial v(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma \times (0,T). \end{cases} \tag{5.70}$$

The problem (5.70) implies

$$\begin{cases} (u+v)_{tt}(x,t) + \Delta^2 (u+v)(x,t) = 0 & \text{in} \quad \Omega \times (0,T), \\ \Delta (u+v)(x,t) = 0 & \text{on} \quad \Gamma_0 \times (0,T), \\ (u+v)(x,t) = \dfrac{\partial (u+v)(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma \times (0,T). \end{cases}$$

By proposition 2.1 of [**21**], we have

$$u(x,t) + v(x,t) = 0$$

Then, problem (5.70) implies

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) + 2lu(x,t) = 0 & \text{in} \quad \Omega \times (0,T), \\ \Delta u(x,t) = 0 & \text{on} \quad \Gamma_0 \times (0,T), \\ u(x,t) = \dfrac{\partial y(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma \times (0,T). \end{cases}$$

We conclude from ([**24**])

$$u(x,t) = 0, v(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T).$$

This implies that

$$y(x,t) = y(x), z(x,t) = z(x).$$

Thus (y, z) verifies

$$\begin{cases} \Delta^2 y(x) + l(y(x) - z(x)) = 0 \quad \text{in} \quad \Omega, \\ \Delta^2 z(x) + l(z(x) - y(x)) = 0 \quad \text{in} \quad \Omega, \\ \Delta y(x) = \Delta z(x) = 0 \quad \text{on} \quad \Gamma_0, \\ y(x) = \dfrac{\partial y(x)}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \\ z(x) = \dfrac{\partial z(x)}{\partial \nu} = 0 \quad \text{on} \quad \Gamma. \end{cases}$$

We have from ([**24**]) $(y, z) = (0, 0)$. This is in contradiction with (5.69). Then

$$F(0) \leq C \int_0^T \int_\omega \{y_t^2 + z_t^2\} dx\, dt. \tag{5.71}$$

□

Proof of Proposition 5.2
We decompose the solution (u, v) as follows

$$u = y + \widetilde{y},\ v = z + \widetilde{z},$$

where (y, z) is solution of (5.45) with the initial condition

$$\begin{array}{ll} y(x,0) = u_0(x), y_t(x,0) = u_1(x) & \text{in } \Omega, \\ z(x,0) = v_0(x), z_t(x,0) = v_1(x) & \text{in } \Omega. \end{array}$$

and $(\widetilde{y}, \widetilde{z})$ is the solution of :
(5.72)

$$\begin{cases} \widetilde{y}_{tt}(x,t) + \Delta^2 \widetilde{y}(x,t) + l(\widetilde{y}(x,t) - \widetilde{z}(x,t)) + \chi_\omega(x)(\alpha_1 u_t(x,t) + \alpha_2 u_t(x,t-\tau)) = 0 & \text{in } \Omega \times (0, +\infty), \\ \widetilde{z}_{tt}(x,t) + \Delta^2 \widetilde{z}(x,t) + l(\widetilde{z}(x,t) - \widetilde{y}(x,t)) + \chi_\omega(x)(\beta_1 v_t(x,t) + \beta_2 v_t(x,t-\tau)) = 0 & \text{in } \Omega \times (0, +\infty), \\ \widetilde{y}(x,0) = \widetilde{y}_t(x,0) = 0 & \text{in } \Omega, \\ \widetilde{z}(x,0) = \widetilde{z}_t(x,0) = 0 & \text{in } \Omega, \\ \widetilde{y}(x,t) = \widetilde{z}(x,t) = 0 & \text{on } \Gamma \times (0, +\infty), \\ \frac{\partial \widetilde{y}(x,t)}{\partial \nu} = \frac{\partial \widetilde{z}(x,t)}{\partial \nu} = 0 & \text{on } \Gamma \times (0, +\infty). \end{cases}$$

and define the energy function of (5.72) by

$$F_d(t) = \frac{1}{2} \int_\Omega \left[|\Delta \widetilde{y}(x,t)|^2 + \widetilde{y}_t^2(x,t) + |\Delta \widetilde{z}(x,t)|^2 + \widetilde{z}_t^2(x,t) + l(\widetilde{y}(x,t) - \widetilde{z}(x,t))^2 \right] dx.$$

We rewrite the energy E as

$$E(t) = \mathcal{E}(t) + E_d(t),$$

where

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega \left[|\Delta u(x,t)|^2 + u_t^2(x,t) + |\Delta z(x,t)|^2 + v_t^2(x,t) + l(y(x,t) - z(x,t))^2 \right] dx,$$

and

$$F_d(t) = \frac{1}{2} \int_\Omega \chi_\omega(x) \int_0^1 \{\mu u_t^2(x, t-\tau\rho) + \xi v_t(x, t-\tau\rho)\}\, d\rho\, dx.$$

$F_d(t)$ can be rewritten via a change of variable as

$$F_d(t) \leq C \int_0^T \int_\omega \{u_t^2(x, t-\tau) + v_t^2(x, t-\tau)\}\, dx\, dt, \tag{5.73}$$

for T large enough.
We have

$$E(0) = \mathcal{E}(0) + E_d(0) = F(0) + E_d(0).$$

From (5.73) and (5.46), we obtain

$$\begin{aligned} E(0) \leq & C\int_0^T \int_\omega \{y_t^2(x,t) + z_t^2(x,t)\}\,dx\,dt + C\int_0^T \int_\omega \{u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\,dx\,dt \\ \leq & \, C\int_0^T \int_\omega \{u_t^2(x,t) + v_t^2(x,t) + \widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\,dx\,dt. \end{aligned}$$

It remains to estimate the term $\int_0^T \int_\omega \{\widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t)\}\,dx\,dt$.
We differentiate the energy function $F_d(t)$ with respect to t, we obtain

$$\begin{aligned} \frac{d}{dt}F_d(t) = & -\int_\Omega \chi_\omega(x)\{\alpha_1\widetilde{y}_t(x,t)u_t(x,t) + \alpha_2\widetilde{y}_t(x,t)u_t(x,t-\tau) \\ & + \beta_1\widetilde{z}_t(x,t)v_t(x,t) + \beta^2\widetilde{z}_t(x,t)v_t(x,t-\tau)\}\,dx, \end{aligned}$$

from which we get after using Chauchy-schwarz inequality

$$\begin{aligned} \frac{d}{dt}F_d(t) \leq & C\int_\omega \{u_t^2(x,t) + v_t^2(x,t) + \widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\,dx \\ & + \int_\Omega \{\widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t)\}\,dx. \end{aligned}$$

From the definition of F_d, we obtain

$$\frac{d}{dt}F_d(t) \leq F_d(t) + C\int_\omega \{u_t^2(x,t) + v_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\,dx.$$

Multiplying the last inequality by (e^{-t}) and integrating over $(0,t)$, we get

$$F_d(t) \leq Ce^t\int_0^t \int_\omega \{u_t^2(x,t) + v_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\,dx\,dt.$$

We conclude for $t \in (0,T)$, that is

$$F_d(t) \leq C\int_0^T \int_\omega \{u_t^2(x,t) + v_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\,dx\,dt,$$

which gives

$$\int_0^T \int_\omega \{\widetilde{y}_t^2(x,t) + \widetilde{z}_t^2(x,t)\}\,dx\,dt \leq C\int_0^T \int_\omega \{u_t^2(x,t) + v_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t-\tau)\}\,dx\,dt.$$

Consequently we have

$$E(0) \leq C\int_0^T \int_\omega \{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\}\,dx\,dt.$$

Proof of the main result From (5.43), we have

$$E(T) - E(0) \leq -K\int_0^T \int_\omega \left\{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\right\}\,dx\,dt,$$

and (5.44) leads to

$$\begin{aligned} E(0) & \leq C\int_0^T \int_\omega \left\{u_t^2(x,t) + u_t^2(x,t-\tau) + v_t^2(x,t) + v_t^2(x,t-\tau)\right\}\,dx\,dt \\ & \leq CK^{-1}(E(0) - E(T)), \end{aligned}$$

so

$$E(T) \leq \frac{CK^{-1}}{1 + CK^{-1}}E(0). \tag{5.74}$$

The desired conclusion follows now from (5.74) since $0 < \frac{C}{K+C} < 1$.

Conclusion

In this thesis we have studied stability problems for some systems governed by partial differential equations:

- Coupled wave equations.
- Transmission wave equation.
- Coupled Euler-Bernoulli equations.

with time delays in the boundary or internal feedbacks.
The approach we adopted uses:

- An appropriate energy function.
- Observability estimate type for the corresponding homogeneous system whose proof combines either classical or Carleman multiplier techniques and compactness-uniqueness argument.

There are several extensions of the results obtained in this thesis. For example the following questions can be considered for future work :

- Stability of coupled wave or Euler-Bernoulli equations with delay term in one of the boundary feedback without assuming that the constant gain of the delayed term is less than of the undelayed one
- Stability of coupled wave or Euler-Bernoulli equations with time delays in the non linear (boundary or internal) feedbacks.
- Stabilization of wave or Euler-Bernoulli system with time delays in the boundary feedback by an internal feedback.

Appendix A

In this appendix we recall some well known results from the theory of semigroup

Definition 5.1. *A one-parameter family $T(t)$ for $0 \leq t < \infty$ of bounded linear operators on a Banach space X is a C_0-(or strongly continuous) semigroup on X if*

- $T(0) = I$, *(I is the identity operator on X).*
- $T(t+s) = T(t)T(s)$ *for every* $t, s \geq 0$. *(semigroup property)*
- $\lim_{t\to 0} \|T(t)x - x\| = 0$ *for all* $x \in X$.

Theorem 5.3. *Let $T(t)$ be a semigroup. There exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that the following holds:*

$$\|T(t)\| \leq Me^{\omega t}$$

If $\omega = 0$ and $M = 1$, then $T(t)$ is called a C_0-semigroup of contraction.

Theorem 5.4. *(Lumer-Phillips)*
Let A be a linear operator with dense domain $D(A)$ in X.

- *If A is dissipative and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$ of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.*
- *If A is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$, $Re\langle Ax, x^*\rangle \leq 0$.*

Definition 5.2. *A semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called uniformly exponentially stable if there exist constants $C > 0, M \geq 1$ such that*

$$\|T(t)\| \leq Me^{-Ct}$$

for all $t \geq 0$.

Proposition 5.4. *For a strongly continuous semigroup $(T(t))_{t\geq 0}$, the following assertions are equivalent.*

- $(T(t))_{t\geq 0}$ *is uniformly exponentially stable.*
- *There exists $t_0 > 0$ such that $\|T(t_0)\| < 1$.*

Appendix B

Consider the following coupled system of two second-order hyperbolic equations in the unknowns $w(t,x)$ and $z(t,x)$:

$$\begin{cases} w_{tt} &= \Delta w + F_1(w) + P_1(z) \quad \text{in } (0,T] \times \Omega \equiv Q, \\ z_{tt} &= \Delta z + F_2(z) + P_2(w) \quad \text{in} Q, \end{cases}$$

defined on a bounded domain $\Omega \in \mathbb{R}^n$ with smooth boundary Γ, where F_1, F_2, P_1, P_2 are (linear) differential operators of order one in all variables $t, x_1, ..., x_n$, with $L_\infty(Q)$-coefficients, thus satisfying the point wise bounds

$$\begin{aligned} |F_1(w)|^2 + |P_2(w)|^2 &\le c_T[w_t^2 + |\nabla w|^2 + w^2] \qquad \forall t, x \in Q, \\ |F_2(z)|^2 + |P_1(z)|^2 &\le c_T[z_t^2 + |\nabla z|^2 + z^2] \qquad \forall t, x \in Q, \end{aligned}$$

PROPOSITION 5.5. *(Lasiecka and Triggiani* **[16]***) Let w and z be solutions of the above problem in the following class*

$$\begin{cases} w, z \in H^1(Q) = L_2(0,T; H^1(\Omega)) \cap H^1(0,T; L_2(\Omega)) \\ w_t, \frac{\partial w}{\partial \nu}, z_t, \frac{\partial z}{\partial \nu} \in L_2(0,T; L_2(\Gamma)). \end{cases}$$

then the following inequality holds true for τ sufficiently large:

- *there exists a positive constant $k_{\phi,\tau} > 0$ such that*

$$k_{\phi,\tau} E(0) \le \int_0^T \int_\Gamma \left[\left(\frac{\partial w}{\partial \nu}\right)^2 + w_t^2 + \left(\frac{\partial z}{\partial \nu}\right)^2 + z_t^2 \right] d\Gamma\, dt$$
$$+ \mathit{const}_{T,\tau,\epsilon_0} \left\{ \|w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} \right\},$$

  or equivalently,

$$k_{\phi,\tau} [E(0) + E(T)] \le \int_0^T \int_\Gamma \left[\left(\frac{\partial w}{\partial \nu}\right)^2 + w_t^2 + \left(\frac{\partial z}{\partial \nu}\right)^2 + z_t^2 \right] d\Gamma\, dt$$
$$+ \mathit{const}_{T,\tau,\epsilon_0} \left\{ \|w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} \right\},$$

- *If, moreover, w and/or z satisfy the boundary condition*

$$w|_{\Sigma_0} \equiv 0, and/or, respectively, z|_{\Sigma_0} \equiv 0, \qquad \Sigma_0 = (0,T] \times \Gamma_0,$$

  where Γ_0 is the portion of the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ defined by

$$\Gamma_0 = \{x \in \Gamma : \nabla\phi.\nu(x) \le 0\};$$

  then the corresponding integral term for w and/or for z replaces Γ with Γ_1.

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