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par

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THÈME

HOMOGÉNÉISATION D'UNE CLASSE DE FONCTIONNELLES INTÉGRALES ET EXISTENCE ET UNICITÉ DES SOLUTIONS DE QUELQUES ÉQUATIONS D'ÉVOLUTION INCOMPRESSIBLES

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Contents

CHAPTER 1

General Introduction

This thesis is composed of five chapters. The first one is devoted to the essential results, a brief outline for each chapter and the basic tools of the proof. In the second chapter we are concerned with the asymptotic behavior problem for nonconvex random integral functionals depending on second gradient. To identify the $\Gamma-$limit, we use the ergodic theorem for discrete subadditive processes and by $\Gamma-$ convergence argument we treat the problem in question. The third chapter deals with the axisymmetric incompressible Navier-Stokes equations. We treat two problems. In the first one we study the global well-posedness for three-dimensional Navier-Stokes, where the initial data is an axisymmetric vector field and belonging to the critical Besov spaces. In the second part we establish the inviscid limit, when the viscosity ν goes to 0 of the solutions $(v_\nu)_\nu$ of Navier-Stokes equations toward the solution v of Euler equations and we evaluate the rate of convergence. The fourth chapter is reserved to investigate the global well-posedeness of another evolution problem which called the Euler-Boussinesq system with fractional dissipation and initial data lying in critical Besov spaces.
Firstly, we treat the commutator term coming from the commutation between the fractional Laplacian and the regularized flows. Secondly, we prove the smoothing effects of the transport-diffusion equation governing the evolution of the temperature. The final chapter deals with the essential background already used in the third and fourth chapter. We recall to the basic results concerned the Littlewood-Paley theory, as well as the Besov spaces and their properties and the famous paraproduct identity.

CHAPTER 2

1. Nonconvex Random Higher Order Integrals and Homogenization

Composites are structures constituted by two or more materials which are finely mixed at microscopic length scales. Despite the high complexity of their microstructure, composites appear essentially as homogeneous at macroscopic length scale. It suggests to give a description of their effective properties as a kind of average made on the respective properties of the constituents.
The *Homogenization Theory* renders possible to define properly such an average, by thinking of a composite as a limit (in a certain sense) of a sequence of structures whose heterogeneities become finer and finer. There is a wide literature on the subject; we refer

• The asymptotic expansions by multi-scales method, it is adapted in particularly to study the linear problems, we refer to A. Bensoussan, J. L. Lions and G. Papanicolaou [**8**], E. Sanchez Palencia [**53**].
• The $H-$convergence due to F. Murat and L. Tartar [**46**] is permit to describe the asymptotic behavior of a sequence of second order elliptic operators in divergence form. This notion appears as a generalization of the $G-$ convergence (see [**16**], chapter 22) introduced independently for symmetric operators.
• The variational methods like Γ-convergence and Mosco-convergence are devoted to describe the asymptotic analysis of families of minimum problems, usually depending on some parameters whose nature may be geometric or constitutive, deriving from a discretization argument, an approximation procedure, see E. De Giorgi[1] [**19**], H. Attouch [**5**] and U. Mosco[2] [**43**].
• The probabilistic methods are introduced in order to treat the heterogeneous random media, see G. C. Papanicolaou and S. R. S. Varadhan [**49**], G. Dal Maso and L. Modica [**17, 18**], K. Messaoudi and G. Michaille [**40**].
• The two-scales convergence due to G. Neguesting [**48**] developed par G. Allaire et M. Briane [**4**] in order to deal with the weak convergence problems.
The physicists and the mechanics rather use the multiple scales method which has the advantage of being easy to implement. The mathematicians prefer in general the others, because they make it possible to show that the homogenized solution is close (in a precise sense) to the real solution and to consider the error made by replacing the real solution by the homogenized solution. The majority of these methods provide moreover the convergence of energies and marry with the approximation techniques, like duality see P. Suquet [**55**], H. Attouch, D. Azé and R. Wetz [**6**], ...
Our aim here is to characterize the behavior as ε tends to zero of the family of functionals defined on $W^{2,p}(O)$ by

$$G_\varepsilon(\omega)(u) = \int_O J\big(\omega, \varepsilon^{-1}x, \nabla^2 u(x)\big)dx.$$

under periodicity in law and (nonconvexity) hypothesis with O is an open bounded subset of $\mathbb{R}^d$.
The term $G_\varepsilon(\omega)$ can be interpreted as the energy under a deformation u of an elastic body whose microstructure is distributed in random way. We seek to approximate in a $\Gamma-$ convergence sense the microscopic behavior of this kind of material by a macroscopic, or average, description. We combine a $\Gamma-$ limit argument with techniques of ergodic theorem.
The density $J : \Omega \times \mathbb{R}^d \times \mathbb{M}^{d\times d} \to [0, +\infty[$ is a function satisfying the following conditions:

- the map $x \mapsto J(\omega, x, \xi)$ is $x-$ measurable and for every $\xi_1, \xi_2 \in \mathbb{M}^{d\times d}$,

$$|J(\omega,x,\xi_1) - J(\omega,x,\xi_2)| \leq L\big(1+|\xi_1|^{p-1}+|\xi_2|^{p-1}\big)|\xi_1-\xi_2| \quad \text{a.e. } (x,\omega) \in \mathbb{R}^d \times \Omega;$$

- there exists two positive constants α, β such that $0 < \alpha \leq \beta$ and for every $\xi \in \mathbb{M}^{d\times d}$ we have:

$$\alpha|\xi|^p \leq J(\omega,x,\xi) \leq \beta\big(1+|\xi|^p\big) \quad \text{a.e. } (x,\omega) \in \mathbb{R}^d \times \Omega.$$

To determine the effective $\Gamma-$ limit we introduce the following set function:

$$\mathscr{I} \ni Q \mapsto \mathscr{M}_Q(\omega, J, \xi) = \inf\Big\{ \int_Q J\big(\omega, x, \nabla^2 u(x)\big)dx : u \in L_\xi + W_0^{2,p}(Q)\Big\},$$

where $\mathscr{I} = \{[a,b[: a,b \in \mathbb{Z}^d\}$ and L_ξ is an affine function.
Then we show that $\mathscr{T} \ni Q \mapsto \mathscr{M}_Q(\omega, J, \xi)$ is a discrete subadditive process with respect to a given dynamical system (see definition 2.2 below), namely we have:

(1) $\dfrac{\mathscr{M}_{1/\varepsilon Q}(.,J,\xi)}{\mathcal{L}_d(1/\varepsilon Q)} = \dfrac{\mathscr{M}_Q(.,J_\varepsilon,\xi)}{\mathcal{L}_d(Q)}$;

(2) $\|\mathscr{M}_Q(.,J,\xi)\|_{L^1\left(\mathscr{G},\sigma(\mathscr{G}),\mathbb{P}_J\right)} \leq \beta\big(1+|\xi|^p\big)\mathcal{L}_d(Q)$ for all $(Q,J,\xi) \in \mathscr{I} \times \mathscr{G} \times \mathbb{M}^{d\times d}$;

(3) $Q \longmapsto \mathscr{M}_Q(.,.,.)$ is subadditive and covariant;

(4) there exists a constant L' such that for all ξ_1, ξ_2 in $\mathbb{M}^{d\times d}$,

$$\left|\frac{\mathscr{M}_Q(.,J,\xi_1)}{\mathcal{L}_d(Q)} - \frac{\mathscr{M}_Q(.,J,\xi_2)}{\mathcal{L}_d(Q)}\right| \leq L'\big(1+|\xi_1|^{p-1}+|\xi_2|^{p-1}\big)|\xi_1-\xi_2|.$$

Afterward, we identify the $\Gamma-$ limit by applying the ergodic theorem for discrete subadditive processes du to M. Ackoglu and U. Krengel [**3**].
Lastly we establish the almost everywhere $\Gamma-$ convergence of $\{G_\varepsilon\}$ toward G_{hom} by checking the lower and upper Γ-limit (see definition 2.18). Our main result reads as follows:

THEOREM 1.1. *Let J be an ergodic and stationary (periodic in law) integrand. Then the corresponding random process $\{G_\varepsilon(\omega)\}$ defined by $G_\varepsilon(\omega) = \big(\rho_\varepsilon G\big)(\omega)$ for every $\omega \in \Omega$, $\Gamma-$ converge almost every where in $W^{2,p}(O)$ when $\varepsilon \to 0$ to the homogenized functional G_{hom} defined by:*

$$G_{\text{hom}}(u) = \int_O J_{\text{hom}}(\nabla^2 u(x))dx,$$

where the integrand J_{hom} is given by the following statement: for every $\xi \in \mathbb{M}^{d\times d}$,

$$J_{\text{hom}}(\xi) = \lim_{\varepsilon\to 0} \frac{1}{\mathcal{L}_d\big(1/\varepsilon Q\big)} \int_\Omega \min\Big\{ \int_{1/\varepsilon Q} J(\omega,x,\nabla^2 u(x))dx : u \in L_\xi + \in W_0^{1,p}\big(1/\varepsilon Q\big)\Big\} d\mathbb{P}(\omega),$$

with $Q = [0,1[^d$ is a unit cube in $\mathbb{R}^d$.

We emphasize that the homogenized integrand J_{hom} inherit the same properties that J,

(ii) $\alpha|\xi|^p \leq J_{\text{hom}}(\xi) \leq \beta\big(1+|\xi|^p\big)$ for all $\xi \in \mathbb{M}^{d\times d}$ with α, β two constants given by growth and coerciveness conditions (II.3) of Chapter 2, Section 3.

CHAPTER 3

2. Inviscid Limit For Axisymmetric Navier-Stokes System

Historically, the mathematical study of fluid dynamics was initiated by L. Euler[3] in his famous work in the middle of the seventeenth century **[23]** where he showed that the velocity v of a perfect incompressible fluid subjected to an external force f obeys to the following system:

$$
\text{(E)} \qquad \begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = f \\ \text{div } v = 0 \\ v_{|t=0} = v^0. \end{cases}
$$

Here, the first equation describes the conservation of the momentum while the second one means that the fluid mass is preserved through the evolution in time.
The unknown p is a real scalar function called the pressure of the fluid and which can be expressed from the velocity and the external force as below

$$
-\Delta p = \text{div } (v\cdot\nabla v) - \text{div } f.
$$

In many cases this model fails to describe the motion of the fluid because we need to take into account the viscous friction between particles which is very crucial for the dynamics of the fluid. This work was successfully carried out by C. Navier[4] **[47]** and G. Stokes[5] **[54]**. They proved that the dissipation process can be mathematically modeled by Laplace-Beltrami operator. More precisely, the evolution of the velocity $v(t,x)$ is given by the equations

$$
(\text{NS}_\nu) \qquad \begin{cases} \partial_t v_\nu + v_\nu\cdot\nabla v_\nu - \nu\Delta v_\nu + \nabla p_\nu = f \\ \text{div } v_\nu = 0 \\ v_{\nu|t=0} = v^0. \end{cases}
$$

These equations are called Navier-Stokes equations and the parameter $\nu > 0$ is the viscosity. One of the most interesting mathematical field is to study existence and uniqueness solutions for this IVP. Questions that one can ask are in which sense we have to understand these equations: some difficulties arise when we deal with less smooth initial data. Does the solutions exist globally in time or there is some blowup solutions in finite time.
Let us recall some significant results obtained in this direction in the last century and we will restrict ourselves to the viscous case with zero force.

[3] Leonhard Euler, Swiss mathematician, 1707-1783. He worked in Saint Petersburg, Russia

In his pioneering work in the last century J. Leray[6] **[36]** was able to construct globally in time solutions in the energy space which are called weak solutions. More precisely he showed that if the initial velocity belongs to the Lebesgue[7] space $L^2(\mathbb{R}^d)$ then the system (NS_ν) admits a solution v_ν in the function space

$$v_\nu \in L^\infty\big(\mathbb{R}_+; L^2(\mathbb{R}^d)\big) \cap L^2\big(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^d)\big).$$

Moreover this solutions satisfies the energy inequality:

$$\forall t \geq 0, \quad \|v(t)\|^2_{L^2(\mathbb{R}^d)} + 2\nu \int_0^t \|\nabla v(s)\|^2_{L^2(\mathbb{R}^d)} ds \leq \|v^0\|^2_{L^2(\mathbb{R}^d)}.$$

The proof is done by using a compactness method and the a priori bound mentioned above. Unfortunately those solutions are not known to be unique due to the lack of regularity. However the uniqueness is proved in space dimension two since the velocity is almost bounded. Afterward, many authors sought additional criteria allowing to ensure the existence and the uniqueness of global solutions. Thus, in a famous article H. Fujita and T. Kato **[8]** developed another kind of solutions. Their result can be stated in the following way: if v^0 lies in $\dot{H}^{\frac{d}{2}-1}$, then there exists a unique maximal solution v_ν belonging to the space

$$v_\nu \in \mathscr{C}\big([0, T^*[; \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)\big) \cap L^2\big([0, T^*[; \dot{H}^{\frac{d}{2}}(\mathbb{R}^d)\big).$$

Moreover, if $T^* < +\infty$, then we have:

$$\lim_{T \to T^*} \|v\|_{L^2\left([0,T];\dot{H}^{\frac{d}{2}}(\mathbb{R}^d)\right)} = +\infty.$$

If the initial data is sufficiently small, that is $\|v^0\|_{\dot{H}^{\frac{d}{2}-1}} \leq C\nu$ where C is an absolute constant then the solution exists globally in time and remains small. In few words the proof is based in the smoothing effect of the heat operator combined with the Duhamel formula

$$v_\nu(t) = S(t)v^0 + \int_0^t S(t-\tau)\mathcal{P}(v_\nu \cdot \nabla v_\nu)(\tau) d\tau,$$

where $S(t) = e^{t\Delta}$ is the heat semi-flow and $\mathcal{P}$ is Leray's projector over free divergence vector fields.

T. Kato **[31]** established a similar result for small data in L^3 in the absence of external forces. In **[51]**, F. Planchon showed that it was possible to generalize the Kato's result when $v^0 \in L^p \cap \dot{\mathscr{B}}^{\frac{3}{2p}-1}_{2p,\infty}$ with $p > \frac{3}{2}$ if moreover $\|v^0\|_{\dot{\mathscr{B}}^{\frac{3}{2p}-1}_{2p,\infty}}$ is small and $f = 0$. Recently, H.

[6]Jean LERAY: French mathematician, 1906-1998. He received the Wolf Prize in 1979, for pioneering work on the development and application of topological methods to the study of differential equations, jointly with André WEIL. He had worked in Nancy, France, in a prisoner of war camp in Austria (1940-

Koch and D. Tataru [**12**] managed to generalize this type of results by working in space BMO^{-1},

$$BMO^{-1} \stackrel{def}{=} \Big\{ v \in \mathscr{S}' : \sup_{x,R} \Big(|B(x,R)|^{-1} \int_{B(x,R)} \int_0^{R^2} |v|^2 dt dx \Big)^{\frac{1}{2}} < \infty \Big\}.$$

For a more details about this subject we refer to the book of P. G. Lemarié-Rieusset [**35**]. Global existence of smooth solutions with large initial data remains till now one of the most open problem in partial differential equations.

Let us recall that the Besov spaces $\mathscr{B}^s_{p,r}$ (resp. Sobolev spaces H^s) is called critical if $s = \frac{d}{p} + 1$, sub-critical if $s < \frac{d}{p} + 1$ and super-critical $s > \frac{d}{p} + 1$, where d is the space dimension.

Now, we give the different blowup criterion

- Let T^* the maximal time of existence for (NS_ν) solutions, then

$$T^* < \infty \Rightarrow \lim_{t \to T^*} \|v(t)\|_{H^s} = +\infty.$$

- In consequence of the H^s energy estimate,

$$T^* < \infty \Rightarrow \int_0^{T^*} \|v(\tau)\|_{L^\infty} d\tau = +\infty.$$

- Beale-Kato-Majda [**7**]: If $v^0 \in H^s$ ($s > \frac{d}{p} + 1$), then

$$T^* < \infty \Rightarrow \int_0^{T_*} \|\omega(t)\|_{L^\infty} dt = +\infty.$$

We emphasize that the blowup B-K-M criterion rest valid for the Besov spaces $\mathscr{B}^r_{p,r}$, with $s > \frac{d}{p} + 1$.

Moreover, this criterion ensures that the development of finite time singularities for Kato's solutions is related to the blowup of the L^∞ norm of the vorticity near the maximal time existence. A direct consequence of this result is the global well-posedness of two-dimensional Navier-Stokes solutions for smooth initial data since the vorticity is only advected and then does not grow.

Often a significant quantity appear in the systems (E) and (NS_ν) who is called the *vorticity* denoted by ω and defined by the curl of the velocity, i.e., $\omega = \operatorname{curl} v$. The coefficients of the antisymmetric matrix ω are given by :

$$\omega_{ij}(v) \stackrel{df}{=} \partial_j v^i - \partial_i v^j.$$

We apply the curl operator to the equation (NS_ν) leading under the assumption $\operatorname{div} v = 0$ to the following equation:

$$\partial_t \omega + (v \cdot \nabla)\omega + (\omega \cdot \nabla)v - \nu\omega = 0.$$

When the value of ω is known, then one can deduce the velocity v by applying

For $d = 2$ and div $= 0$ the vorticity is reduced to a scalar function, namely $\omega = \partial_2 v^1 - \partial_1 v^2$ and consequently the equation which it checks is the following transport-diffusion equation:

$$\partial_t \omega + (v \cdot \nabla)\omega - \nu \Delta \omega = 0.$$

The incompressibility of the fluid leads that the flow ψ defined by

$$\psi(t,x) = x + \int_0^t v(\tau, \psi(\tau, x))d\tau$$

preserve the Lebesgue measure.

In dimension $d = 3$, the vorticity ω can be identified with a vector. Moreover it obeys the following equation:

$$\partial_t \omega + (v \cdot \nabla)\omega - \nu \Delta \omega = (\omega \cdot \nabla)v.$$

The term of the right-hand side, $(\omega \cdot \nabla)v$, is called the vorticity stretching term. This term is the main obstacle to achieve in time regularity of the three-dimensional Navier-Stokes equations. Nevertheless we have interesting cases in dimension three for which we can prove global existence: we assume some geometric properties for initial data. Namely we have the following two cases. The helicoidal initial data which means that the components of the velocity are constant on the helicoids (see [**22**]). The second is the axisymmetric initial data (see [**58**]). Hereafter, we will focus on the last case.

A vector field v is said to be axisymmetric if in cylindrical coordinates $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ it takes the form:

$$\text{(AX)} \qquad v(t,x) = v_r(t,r,z)\vec{e}_r + v_z(t,r,z)\vec{e}_z,$$

where $\vec{e}_r$ stands for the unit (outer) radial vector and $\vec{e}_z$ for the unit vertical vector. The expression (AX) means that the components $v_r, v_\theta = 0$ and v_z does not depends in angular variable θ. An easy computation shows that the vorticity ω is reduced to $\omega(r,\theta,z) = \omega_\theta(r,z)\vec{e}_\theta$ with $\omega_\theta \stackrel{def}{=} \partial_z v_r - \partial_r v_z$, and obeys to the following equation:

$$\partial_t \omega + (v \cdot \nabla)\omega - \nu \Delta \omega = \frac{v_r}{r}\omega.$$

For the study of axisymmetric solutions of the Navier-Stokes without swirl, Ukhovskii and Yudovich [**21**], and independently Ladyzhenskaya[8] [**33**], proved the existence of generalized solutions, uniqueness and the regularity. S. Leonardi, J. Màlek, J. Necăs and M. Pokornỳ [**34**] gave a refined proof, especially for initial data $v^0 \in H^2$. This result was recently improved by H. Abidi [**1**] for $v^0 \in H^{\frac{1}{2}}$ and axisymmetric external forces $f \in L^2_{loc}(\mathbb{R}_+; H^\beta)$, with $\beta > \frac{1}{4}$.

The inviscid (Euler) axisymmetric flows were the subject of M. Ukhovisky and V. Yudovich [**21**], the authors had to assume that the vorticity vanishes rapidly enough near the axis of symmetry, namely $\frac{\omega}{r} \in L^\infty$ to conclude global existence for initial data $v^0 \in H^s$, with $s > \frac{7}{2}$. This result was relaxed by Yanagisawa [**19**] for Kato's solutions,

recently, H. Abidi, T. Hmidi and S. Keraani **[2]** proved an analogous result for critical Besov spaces $\mathscr{B}_{p,1}^{1+\frac{3}{p}}$. To overcome the non validity of B-K-M criterion they use the special geometric structure of the vorticity leading to a special decomposition of the vorticity. This allows them to bound Lipschitz norm of the velocity.
Here, we aim to investigate the global well-posedness problem for the system (NS$_\nu$) with axisymmetric initial data lying in critical Besov spaces $\mathscr{B}_{p,1}^{1+\frac{3}{p}}$ which have a good local theory **[11]**. We agree to call these critical spaces insofar as they are injected into $W^{1,\infty}$ and admitting the same scales. With this spaces the criterion of Beale-Kato-Majda (see **[7]**) does not applicable, therefore it will be necessary to control the Lipschitz norm of the velocity. The statement of the result in question is:

THEOREM 1.2 (Uniform boundedness of the velocity). *Let $p \in [1, +\infty]$ and v_0 be an axisymmetric vector fields in divergence free. Assume that*

(A1) $v_0 \in \mathscr{B}_{p,1}^{1+\frac{3}{p}}$,
(A2) $\frac{\omega_0}{r} \in L^{3,1}$.

Then there exists a unique global solution $v_\nu \in \mathscr{C}\left(\mathbb{R}_+; \mathscr{B}_{p,1}^{1+\frac{3}{p}}\right)$ to Navier-Stokes system, such that

$$\|v_\nu(t)\|_{\mathscr{B}_{p,1}^{1+\frac{3}{p}}} \leq C_0 e^{\exp C_0 t},$$

where C_0 depends only on the initial data and not on the viscosity.

The proof relies on the uniform estimate of the Lipschitz norm of the velocity. For this purpose we use the method developed in **[2]** for the inviscid case. However the situation in the viscous case is more complicate because of the dissipative term. We have especially to check that it doesn't undermine some geometric properties of the vorticity.

REMARK. *For $p \in [1, 3[$ the second condition (A2) is a consequence of the first one (A1). More precisely, we have:*

$$\left\|\frac{\omega}{r}\right\|_{L^{3,1}} \leq \|v\|_{\mathscr{B}_{p,1}^{1+\frac{3}{p}}}.$$

THEOREM 1.3 (Rate convergence). *Let v_ν and v be respectively the solution of Navier-Stokes and Euler systems with the same initial data $v^0 \in \mathscr{B}_{p,1}^{1+\frac{3}{p}}$. Then we have the rate of convergence*

$$\|v_\nu - v\|_{\mathscr{B}^0_{\max(p,3),1}} \leq C_0 e^{\exp C_0 t} (\nu t)^{\frac{1}{2} + \frac{3}{2\max(p,3)}}, \quad p \in [1, \infty].$$

We use for the proof the uniform bounds in Besov spaces combined with smoothing effects on the vorticity.

CHAPTER 4

3. On the Global well-posedness of the Euler-Boussinesq system with fractional dissipation

In this chapter, we are interested in two-dimensional *Euler-Boussinesq* system with partial viscosity given by the following coupled equations:

$$(\mathrm{B}_\alpha) \qquad \begin{cases} \partial_t v + v\cdot\nabla v + \nabla\Pi = \theta e_2 \\ \partial_t\theta + v\cdot\nabla\theta + \kappa|D|^\alpha\theta = 0 \\ \mathrm{div} v = 0 \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0. \end{cases}$$

Here, the unknowns are:

- the velocity $v = (v^1, v^2)$ is two dimensional vector fields;
- the pressure Π is a real scalar function;
- the temperature θ is a scalar function.

The vector e_2 is given by $(0,1)$, α is a real number in $]0,2]$ and $\kappa \geq 0$ is a diffusion molecular. The operator $|D|^\alpha$ is defined in a standard fashion through its Fourier[9] transform

$$\mathscr{F}(|D|^\alpha u)(\xi) = |\xi|^\alpha(\mathscr{F}u)(\xi).$$

The Boussinesq system describes the influence of the convection (or convection-diffusion) phenomenon in a viscous or inviscid fluid. It is used as a toy model for geophysical fluids whenever rotation and stratification play an important role (for more example see the books of [**41**]). In addition to its intrinsic mathematical importance this equation serves as a $2D$ model in geophysical fluid dynamics, for more details about the subject see [**10, 50**] and has lately received significant attention in mathematical fluid dynamics due to its connection to three-dimensional incompressible flows.

Indeed, the vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisfies a transport-diffusion equation with second member $\partial_1\theta$ given as follows

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega = \partial_1\theta, \\ \partial_t\theta + v\cdot\nabla\theta + \kappa|D|^\alpha\theta = 0, \\ \mathrm{div} v = 0, \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0, \end{cases}$$

which, in turn, obeys a transport equation with second member $(\nabla v)\partial_1\theta$. This quantity is a stretching term in the three-dimensional incompressible Euler vorticity equation (see [**30**]).

[9] Jean Baptiste Joseph Fourier, French mathematician, 1768-1830.

The system (B_α) has a certain mathematical analogies with the quasigeostrophic equations that we recall here in the form:

$$(\mathrm{TD}_\alpha) \qquad \partial_t\theta + v\cdot\nabla\theta + \kappa|D|^\alpha\theta = 0,$$

where $v = (-\partial_2|D|^{-1}\theta, \partial_1|D|^{-1}\theta)$ is the Riesz transform of v.
This equation has been intensively investigated and much attention is carried to the problem of global well-posedness. For the sub-critical case $(\alpha > \frac{1}{2})$ the theory seems to be in a satisfactory state. Indeed, global existence and uniqueness for arbitrary initial data are established in various function spaces (see for example [**15, 52**]). However the critical and super-critical cases, corresponding respectively to $\alpha = \frac{1}{2}$ and $\alpha < \frac{1}{2}$, are harder to deal with. In the super-critical case, we have until now only global results for small initial data, see for instance [**11, 15, 60, 61**]. For critical case , Constantin, Córdoba and Wu showed in [**13**] the global existence in Sobolev space H^1 under smallness assumption of L^∞ norm of θ^0. Very recently, Kiselev, Nazarov and Volberg proved in [**22**] the global well-posedness for arbitrary periodic smooth initial data by using an elegant argument of modulus of continuity. In [**1**] H. Abidi et T. Hmidi established the global well-posedness in the critical case when initial data belong to the homogeneous critical Besov space $\dot{B}^0_{\infty,1}(\mathbb{R}^2)$: they removed the periodic condition and weakened the initial regularity.
We focus our attention that the system (B_α) is investigated by numerous authors in various function spaces [**10, 25**] and the references therein. For $\kappa = 0$ the problem of global well-posedness is well understood. In [**5**], Chae proved global well-posedness for initial data $(v^0;\theta^0)$ lying in Sobolev spaces $H^s \times H^s$; with $s > 2$. This result has been recently improved in [**17**] by taking the data in $H^s \times H^s$; with $s > 0$. However they give only a global existence result without uniqueness in the energy space $L^2 \times L^2$. In [**1**] they prove a uniqueness result for data belonging to $L^2 \cap \mathscr{B}^{-1}_{\infty,1} \times \mathscr{B}^0_{2,1}$. More recently Danchin and Paicu [**12**] have established a uniqueness result in the energy space.
Our goal here is to study the global well-posedness of the system (B_α); with $\kappa > 0$. First of all, let us recall that the two-dimensional incompressible Euler system, corresponding to $\theta^0 = 0$; is globally well-posed in the Sobolev space H^s; with $s > 2$. This is due to the advection of the vorticity by the flow: there is no accumulation of the vorticity and thus there is no finite time singularities according to B-K-M criterion [**7**]. In critical spaces like $\mathscr{B}^{\frac{2}{p}+1}_{p,1}$ the situation is more complicate because we do not know if the B-K-M criterion works or not. In [**26**], Vishik proved that Euler system is globally well-posed in these critical Besov spaces. He used for the proof a new logarithmic estimate taking advantage of the particular structure of the vorticity equation in dimension two. For the Euler-Boussinesq system (B_α), Chae has proved in [**5**] the global well-posedness for initial data $v^0;\theta^0$ lying in Sobolev space H^s; with $s > 2$. His method is basically related to Sobolev logarithmic estimate in which the velocity and the temperature are needed to be

The main result in question is given by the following statement:

THEOREM 1.4. *Let* $(\alpha, p) \in]1,2]\times]1,\infty[$, $v^0 \in B_{p,1}^{1+\frac{2}{p}}$ *be a divergence free vector-field of* $\mathbb{R}^2$ *and* $\theta^0 \in B_{p,1}^{-\alpha+1+\frac{2}{p}} \cap L^r$, *with* $\frac{2}{\alpha-1} < r < \infty$. *Then there exists a unique global solution* (v, θ) *for the system* (B_α) *such that*

$$v \in \mathscr{C}\big(\mathbb{R}_+; B_{p,1}^{1+\frac{2}{p}}\big) \quad \text{and} \quad \theta \in L^\infty_{loc}\big(\mathbb{R}_+; B_{p,1}^{-\alpha+1+\frac{2}{p}} \cap L^r\big) \cap L^1_{loc}\big(\mathbb{R}_+; \mathrm{Lip}(\mathbb{R}^2)\big)$$

In order to prove the previous theorem we need to give the following commutator estimate:

PROPOSITION 1.5 (Commutator estimate). *Let* $v \in L^1_{loc}\big(\mathbb{R}_+; \mathrm{Lip}(\mathbb{R}^2)\big)$ *be a divergence free vector filed. For* $q \in \mathbb{Z}$, *let* ψ_q *the flow of the regularized vector field* $\dot{S}_{q-1}v$. *Then for* $f \in \dot{\mathscr{B}}^{\alpha}_{p,1}$ *and* $(q, \alpha, p) \in \mathbb{Z}\times]1,2] \times [1,\infty]$, *there exists constant* $C \stackrel{def}{=} C(\alpha)$ *such that*

$$\||D|^\alpha(f_q \circ \psi_q) - (|D|^\alpha f_q) \circ \psi_q\|_{L^p} \leq Ce^{CV_q(t)} V_q(t)^{1-\frac{\alpha}{2}} 2^{\alpha q} \|f_q\|_{L^p},$$

where $V_q(t) \stackrel{def}{=} \|\nabla \dot{S}_{q-1}v\|_{L^1_t L^\infty}$ *and* $f_q \stackrel{def}{=} \dot{\Delta}_q f$.

The smoothing effect of the temperature play a significant role in the proof of the theorem 1.4, namely we have:

THEOREM 1.6 (Smoothing effect of the temperature). *Let* $(p, r, m) \in [1,\infty]^3$, $s > -1$ *and* v *be a smooth divergence free vector filed of* $\mathbb{R}^2$ *with vorticity* $\omega \stackrel{def}{=} \mathrm{curl} v$. *Let* θ *be a smooth solution of* (TD_α), *then*

(1) *for every* $t \in \mathbb{R}_+$, *we have*

$$\|\theta\|_{\widetilde{L}^\infty_t B^s_{p,r}} + \kappa^{\frac{1}{m}} \|\theta\|_{\widetilde{L}^m_t B^{s+\frac{2}{m}}_{p,r}} \leq Ce^{CV(t)} \Big(\|\theta^0\|_{B^s_{p,r}} + (\kappa t)^{\frac{1}{m}} + \int_0^t e^{-CV(\tau)} \Gamma_s(\tau) d\tau \Big),$$

with,

$$V(t) \stackrel{def}{=} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau, \quad \Gamma_s(t) \stackrel{def}{=} \begin{cases} \|\nabla\theta(t)\|_{L^\infty} \|v(t)\|_{B^s_{p,r}}, & \text{if } s \geq 1 \\ 0, & \text{if } s \in]-1, 1[; \end{cases}$$

(2) *for every* $q \geq -1$, *we have*

$$\kappa^{\frac{1}{m}} 2^{q\frac{\alpha}{m}} \|\Delta_q \theta\|_{L^m_t L^p} \lesssim \|\theta^0\|_{L^p} \Big(1 + (\kappa t)^{\frac{1}{m}} + (q+2)\|\omega\|_{L^1_t L^\infty} + \|\nabla \Delta_{-1} v\|_{L^1_t L^\infty} \Big).$$

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CHAPTER 2

Nonconvex Random Higher Order Integrals and Homogenization

This work is the subject of the following publication:
M. Zerguine and K. Messaoudi: *Nonconvex random higher order integrals and homogenization*. Accepted in Int. J. Sci. Res. Volume 18.
Abstract. *The asymptotic behavior by* $\Gamma-$ *convergence analysis of the family of nonconvex random integral functionals, depend on the second gradient is obtained. To identify the almost sure* $\Gamma-$ *limit, we use the ergodic theorem for discrete sub-additive processes, see* **[1, 18]**. *The main result of our paper generalizes the one studied by* **[15, 16]** *in the convex case.*
Keywords and phrases. Higher integral functional, ergodic theorem, $\Gamma-$ convergence, homogenization.
2000 Mathematics Subject Classification. 49J45, 73B27, 60Fxx.

1. Introduction

In this chapter we deal with an homogenization problem for integral functionals of the form:

$$G_\varepsilon(\omega)(u) = \int_O J\big(\omega, \varepsilon^{-1}x, \nabla^2 u(x)\big)dx, \tag{1.1}$$

where u is a real function of the space $W^{2,p}(O)$, O bounded domain in $\mathbb{R}^d$, $\varepsilon > 0$ and assuming that the density function $\Omega \times \mathbb{R}^d \times \mathbb{M}^{d\times d} \ni (\omega, x, \xi) \mapsto J(\omega, x, \xi)$ is *statistically stationary* or *periodic in law* with respect to the spatial variable $x \in \mathbb{R}^d$.
Functionals (1.1) can be interpreted as the stored strain energy of an elastic material and heterogeneous material, u being a deformation or displacement field. The set O represents a reference configuration. In any case, the medium under consideration is composed of several materials, which are randomly distributed at the microscopic scale given by the ε.
In the theory of homogenization only the most simple properties of stationary random fields are used. The notion of stationary random field is formulated in such general terms as to cover various subjects whose is not probabilistic, e.g., periodic or almost-periodic density.
The question is how to describe approximately the macroscopic behavior of the material, especially the limit problem (homogenization) of $\{G_\varepsilon\}$ when $\varepsilon \to 0$.
The present work is an attempt to develop general techniques for the asymptotic

In physical terms, homogenization means that the heterogeneous medium behaves as an ideal homogeneous one at the macroscopic scale. We seek also to identify the homogenized density J_{hom}, by using the argument of discrete subadittive processes. Let us give a brief account of our major results.

- **Deterministic case**. Functionals of the type (1.1) have been studied in the $\Gamma-$convergence sense by many authors within the Sobolev and BV settings. In a Sobolev setting and for functionals of the form:

$$\int_O J\big(\varepsilon^{-1}x, \nabla^2 u(x)\big)dx,$$

  where $x \mapsto J(x,\xi)$ is $[0,1]^d-$periodic and $\xi \mapsto J(x,\xi)$ is convex, we refer for example to H. Attouch **[2]**. For vector-valued u and nonconvex J, the result is extended by A. Braides **[5]** and S. Müller **[44]**. For more similar results we refer **[3, 5, 6, 7, 17]**.
- **Random case**. For the random case the functional (1.1) has the form:

$$\int_O J\big(\omega, \varepsilon^{-1}x, \nabla^2 u(x)\big)dx,$$

  where $\omega \in \Omega$, with $(\Omega, \Sigma, \mathbb{P})$ is a given probability space and $(\omega, x) \mapsto J(\omega, x, \xi)$ is a realization of a random field, i.e., periodic in law. The functional is treated by **[9, 10]** in convex case. We can even obtain the same result in **[22]** with new easy approach. In the nonconvex case the result is investigated by **[40]**. More similar results are established in various cases, we refer for instance **[15, 16, 17, 19, 21, 24]**.

We start by assuming that for each $x \in \mathbb{R}^d$ a random variable Y_x is given. Then the family of random variables Y_x define a random process on $\mathbb{R}^d$ that is called a *random field* and noted by $Y = (Y_x)_{x\in\mathbb{R}^d}$.

DEFINITION 2.1. *A random field is said to be stationary (periodic in law), if for finite set consisting of points $x^1, x^2, ..., x^l \in \mathbb{R}^d$, and any $h \in \mathbb{R}^d$, the distribution of random vector*

$$Y_{x^1+h}, Y_{x^2+h}, ..., Y_{x^l+h} \tag{1.2}$$

does not depend on $h \in \mathbb{R}^d$.

Assume that the random field $Y = (Y_x)_{x\in\mathbb{R}^d}$ is defined on the same probability space $(\Omega, \Sigma, \mathbb{P})$ by $Y_x(\omega) = Y(x,\omega),\ \omega \in \Omega$. Then we can claim the field Y to be stationary, if it can be represented in the form

$$Y(x,\omega) = Z(\tau_x\omega), \tag{1.3}$$

where $Z(\omega)$ is a fixed random variable and $\tau = (\tau_x)_{\in\mathbb{R}^d} : \Omega \to \Omega$ is a group of transformation which preserves the measure $\mathbb{P}$ on Ω in the sense of definition 2.2. In a certain sense, a converse statement is true: subjecting a given stationary field

2. Probabilistic Description of Non-Homogeneous Media

In this section we collect some ingredients concerning the random medium by starting with the concept of dynamical system corresponding to such medium, some particular examples like : the periodic, quasi periodic medium. Afterward we give the ergodic theorem for discrete subadditive processes which play a crucial step to determine the $\Gamma-$limit.

2.1. Subadditive processes and ergodic theorem. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, with $\mathbb{P}$ is countably additive non-negative normalized measure.

DEFINITION 2.2. *A dynamical system with $d-$ dimensional time, or simply a dynamical system is defined as a group of mappings $\tau_x : \Omega \to \Omega,\ x \in \mathbb{R}^d$, which satisfy the following conditions:*

(1) *the group property: $\tau_0 = \mathbb{I},\ \mathbb{I}$ is the identity map;*

$$\tau_x \circ \tau_y = \tau_{x+y} \quad \textit{for all} \ \ x, y \in \mathbb{R}^d;$$

(2) *the mappings $\tau_x : \Omega \to \Omega$ preserve the measure $\mathbb{P}$ on Ω, i.e., for every $x \in \mathbb{R}^d$, and every $\mathbb{P}-$measurable set $E \in \Sigma$, we have*

$$\tau_x E \in \Sigma, \quad \mathbb{P} \circ \tau_x(E) = P(E);$$

(3) *for any measurable function $f(\omega)$ on Ω, the function $f(\tau_x\omega)$ defined on $\Omega \times \mathbb{R}^d$ is also measurable (where $\mathbb{R}^d$ is endowed with the Borel measure).*

DEFINITION 2.3. *Let $f(\omega)$ be a measurable function on Ω and $\tau = (\tau_x)_{x\in\mathbb{R}^d}$ a dynamical system defined on Ω. Then we have :*

(i) *the function $f(\tau_x\omega)$ for $x \in \mathbb{R}^d$ is said to be a realization of function f;*
(ii) *f is invariant with respect to $\tau = (\tau_x)_{x\in\mathbb{R}^d}$ if $f(\tau_x\omega) = f(\omega)$ for any $x \in \mathbb{R}^d$ almost everywhere in Ω.*

DEFINITION 2.4. *A dynamical system $\tau = (\tau_x)_{x\in\mathbb{R}^d}$ defined on Ω is said to be ergodic, if every invariant function is constant everywhere in Ω. In this situation we shall also say that the measure $\mathbb{P}$ is ergodic with respect to $\tau = (\tau_x)_{x\in\mathbb{R}^d}$.*

REMARK. *There is analogously definition of the concept of ergodicity given by: a dynamical system $\tau = (\tau_x)_{x\in\mathbb{R}^d}$ defined on Ω is called ergodic if all $\tau-$invariant sets E ($\tau_x E = E$ for every $x \in \mathbb{R}^d$) have the property $\mathbb{P}(E) = 0$ or $\mathbb{P}(E^c) = 0$.*

A random medium is modeled by a probability space $(\Omega, \Sigma, \mathbb{P})$ where Ω is the set of all the possible realization. Σ is a $\sigma-$algebra on Ω, and the probability $\mathbb{P}$ is a non-negative measure on (Ω, Σ) such that $\mathbb{P}(\Omega) = 1$. We shall always assume that Σ is $\mathbb{P}-$complete. In the rest of the context we restraint ourselves to the dynamical system defined on the group $\mathbb{Z}^d$. For this let $\mathscr{I} = \{[a, b[: a, b \in \mathbb{Z}^d\}$ and consider the set map

DEFINITION 2.5. *Let $(\tau_z)_{z\in\mathbb{Z}^d}$ be a dynamical system defined on $(\Omega,\Sigma,\mathbb{P})$. A set map $Q \mapsto \mathscr{M}_Q$ is called a subadditive process with respect to $(\tau_z)_{z\in\mathbb{Z}^d}$ if the following conditions are fulfilled:*

(i) $\mathscr{M}_{Q_1\cup Q_2} \le \mathscr{M}_{Q_1} + \mathscr{M}_{Q_2}$ *for every $Q_1, Q_2 \in \mathscr{I}$ be such that $Q_1 \cap Q_2 = \emptyset$;*
(ii) $\mathscr{M}_{z+Q}(.) = \mathscr{M}_Q(\tau_z.)$ *for every $(z,Q) \in \mathbb{Z}^d \times \mathscr{I}$;*
(iii)
$$\gamma(\mathscr{M}) = \inf\left\{ \int_\Omega \frac{\mathscr{M}_Q(\omega)}{\mathcal{L}_d(Q)} d\mathbb{P}(\omega) : Q \in \mathscr{I}, \mathcal{L}_d(Q) \neq 0 \right\} > -\infty.$$

In order to give the ergodic theorem we need to introduce the following definition of regular families

DEFINITION 2.6. *Let $\{Q_\vartheta\}$ be a family of sets in $\mathscr{I}$, where η ranges overs a subsets of the positive rational numbers. Then $\{Q_\vartheta\}$ is called regular (with the constant $C < \infty$) if there exists another family $\{Q'_\vartheta\}$ of sets in $\mathscr{I}$ such that:*

(i) $Q_\vartheta \subset Q'_\vartheta$ *for all $\vartheta > 0$;*
(ii) $Q'_{\vartheta_1} \subset Q'_{\vartheta_2}$ *whenever $\vartheta_1 < \vartheta_2$;*
(iii) $0 < \mathcal{L}_d(Q'_\vartheta) \le C\mathcal{L}_d(Q_\vartheta)$ *for all $\vartheta > 0$.*

According to M. Ackoglu and U. Krengel **[1]** and recently C. Licht and G. Michaille **[19]**, we have the following theorem:

THEOREM 2.7. *Let $(\tau_z)_{z\in\mathbb{Z}^d}$ be a dynamical system defined on $(\Omega,\Sigma,\mathbb{P})$ and a subadditive precess $Q \mapsto \mathscr{M}_Q$ with respect to $(\tau_z)_{z\in\mathbb{Z}^d}$. We assume that there exists $f \in L^1(\Omega,\Sigma,\mathbb{P})$ such that $\|\mathscr{M}_Q(.)\|_{L^1(\Omega,\Sigma,\mathbb{P})} \le f$. Let $\{Q_\vartheta\}$ be a regular sequence of $\mathscr{I}$ satisfying $\lim_{\vartheta\to 0}\rho(Q_\vartheta) = +\infty$. Then almost surely*

$$\lim_{\vartheta\to 0} \frac{\mathscr{M}_{Q_\vartheta}(\omega)}{\mathcal{L}_d(Q_\vartheta)} \quad \textit{exists.}$$

In particular we have:

$$\lim_{\varepsilon\to 0} \frac{\mathscr{M}_{1/\varepsilon Q}(\omega)}{\mathcal{L}_d(1/\varepsilon Q)} = \inf_{\varepsilon>0} \mathbb{E}_{|\Lambda}\left\{ \frac{\mathscr{M}_{1/\varepsilon Q}(\omega)}{\mathcal{L}_d(1/\varepsilon Q)} \right\},$$

where $\mathbb{E}_{|\Lambda}$ is the conditional expectation operator to the $\sigma-$field $\Lambda \stackrel{def}{=} \{E \in \Sigma : \tau_z E = E\ \forall z \in \mathbb{Z}^d\}$. Moreover, if $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic

$$\lim_{\varepsilon\to 0} \frac{\mathscr{M}_{1/\varepsilon Q}(\omega)}{\mathcal{L}_d(1/\varepsilon Q)} = \inf\left\{ \int_\Omega \frac{\mathscr{M}_Q(\omega)}{\mathcal{L}_d(Q)} d\mathbb{P}(\omega) : Q \in \mathscr{I}, \mathcal{L}_d(Q) \neq 0 \right\} = \gamma(\mathscr{M}).$$

REMARK. *Recently C. Licht and G. Michaille* **[19]** *have been extending the result of the above theorem to the family $\mathscr{B}_b(\mathbb{R}^d)$ of all bounded Borel convex sets of $\mathbb{R}^d$.*

2.2. Examples. To illustrate the above notions we give some example and particular cases:

• **Shift map.** Let $(E, \mathscr{E})$ be a measurable space and $\mathbb{I} \neq \emptyset$ an arbitrary index set. The *unilateral* sequence $(\omega_0, \omega_1, \omega_2, \dots)$, and in the case $\mathbb{J} = \mathbb{Z}$ with the *bilateral* sequence $(\dots, \omega_{-1}, \omega_{-2}, \omega_0, \omega_1, \omega_2, \dots)$. The shift $\theta : \Omega \to \Omega$ is the transformation defined by:

$$X_k(\theta\omega) = X_{k+1}(\omega).$$

The shift in $\Omega = E^{\mathbb{Z}}$ is often called *bilateral shift*. It is bijective and both θ and θ^{-1} are measurable with respect to $\Sigma = \mathscr{F}^{\mathbb{Z}}$. The shift in $\Omega^+ = E^{\mathbb{Z}^+}$ is measurable with respect to $\Sigma^+ = \mathscr{F}^{\mathbb{Z}^+}$ and surjective but not invertible. It is called *unilateral shift*.
If E is a topological space, the unilateral shift is continuous in the product topology, and the bilateral shift is even a homeomorphism.
The simplest way to define a $\theta-$invariant measure in Ω is to take a product measure. It is easy to see that $\mathbb{P} = \bigotimes_{j\in\mathbb{Z}} \mathbb{P}_j$ is $\theta-$invariant iff all $\mathbb{P}_j$ are identical. The automorphism θ in $(\Omega, \Sigma, \mathbb{P})$ is then called a (bilateral) *Bernoulli shift*. Unilateral Bernoulli shifts are defined in the same way in Ω^+ with $\mathbb{P} = \bigotimes_{j\in\mathbb{Z}_+} \mathbb{P}_j$ and identical $\mathbb{P}_j$.
• **Periodic case**. Let $\Omega = \mathscr{U}$ the unit cube $\mathscr{U} = \{\omega \in \mathbb{R}^d : 0 \le \omega_j \le 1,\ j = 1, \dots, d\}$. The relation $\tau_x\omega = \omega + x \pmod 1$ defines a dynamical system on Ω. The Lebesgue measure is invariant and ergodic with respect to this system; a realization of measurable function f has the form $f(\omega + x)$.
• **Quasi-periodic case**. Let $\Omega = \mathscr{U}$ the unit cube in $\mathbb{R}^m$, and let $\mathbb{P}$ denote the Lebesgue measure on $\mathscr{U}$. For $x \in \mathbb{R}^d$ set $\tau_x\omega = \omega + \gamma x \pmod 1$, where $\gamma = \{\gamma_{ij}\}$ is an $d \times m$ matrix. Obviously, the mapping τ_x preserves measure $\mathbb{P}$ on Ω. The property of ergodicity will be present if $\gamma_{ij}\kappa_j \neq 0$ for any vector $\kappa \neq 0$ with integer components. The realization have the form $f(\omega + \gamma x)$. It should be mentioned that quasiperiodic functions form is a special class of almost-periodic functions.
We recall that a function $f \in L^2_{loc}(\mathbb{R}^d)$ is almost-periodic if there is a sequence of trigonometric polynomials converging to f with respect to the norm

$$\|f\| = \left(\limsup \frac{1}{t^d \mathcal{L}_d(B(0,1))} \int_{|x|\le t} |f(x)|^2 dx \right)^{1/2}.$$

3. A general Homogenization Theorem

3.1. Assumptions on the integrands. We denote by $\mathscr{O}$ the family of all bounded subsets in $\mathbb{R}^d$. Let us consider a family $\mathscr{G}$ of all $\hat{f} : \mathbb{R}^d \times \mathbb{M}^{d\times d} \to \mathbb{R}$ having the following properties:

(II.1) $x \mapsto \hat{f}(x, \xi)$ is measurable for all $\xi \in \mathbb{M}^{d\times d}$;
(II.2) $\xi \mapsto \hat{f}(x, \xi)$ satisfies the following: there exists a positive constant $L > 0$ such that for every $\xi_1, \xi_2 \in \mathbb{M}^{d\times d}$ and a.e. $x \in \mathbb{R}^d$ we have:

$$|\hat{f}(x, \xi_1) - \hat{f}(x, \xi_2)| \le L\big(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}\big)|\xi_1 - \xi_2|;$$

$\mathscr{G}$ will be equipped of the trace $\sigma-$field denoted by $\sigma(\mathscr{G})$, of the product $\sigma-$field of $\mathbb{R}^{\mathbb{R}^d \times \mathbb{M}^{d\times d}}$. The energy density for a random medium is a map:

$$\begin{aligned} J: \ & \Omega \to \mathscr{G} \\ & \omega \mapsto J(\omega, ., .) \end{aligned}$$

defined by

$$J(\omega, x, \xi) = \hat{J}(\tau_x \omega, \xi), \quad \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{M}^{d\times d}$$

and satisfies the stationary condition (1.2) or (1.3).
An example of such density is given as follows: let g and h be two homogeneous stored energy density belonging to $\mathscr{G}$. Consider an infinite composite medium consisting of a matrix with identical spherical particles randomly embedded within the matrix. Then a realization $\omega \in \Omega$ is identified to the set $\omega = \{x_n : n \in \mathbb{N}\}$ of the centers x_n of the particles.
Let $N(\omega, O)$ denote the number of centers that fall in the domain $O \in \mathscr{O}$, i.e.,

$$N(\omega, O) = \sum_{y \in \omega} \delta_y(O),$$

where δ_y stands the Dirac[1] mass with support $\{y\}$. Σ is defined as the smallest $\sigma-$algebra containing the subsets of Ω of the form:

$$\{\omega \in \Omega : N(b, O_1) = k_1, \ldots, N(\omega, O_n) = k_n\},$$

$O_1, \ldots, O_n$ are a collection of n disjoint domains in $\mathscr{O}$ and $k_1, \ldots, k_n$ are a collection of n positive integers. The probability measure $\mathbb{P}$ is uniquely defined on Σ by its values on these subsets. The translation operator acts on Ω as follows:

$$\forall x \in \mathbb{R}^3 \quad \forall \omega \in \Omega : \ \tau_x \omega = \{x_n + x : n \in \mathbb{N}\}.$$

In order to guarantee the statistical homogeneity of the composite, $\mathbb{P}$ must be obeys the Poisson distribution:

$$\mathbb{P}\big(N(\omega, O_1) = k_1, \ldots, N(\omega, O_n) = k_n\big) = \mathbb{P}\big(N(\omega, O_1) = k_1\big) \times \cdots \times \mathbb{P}\big(N(\omega, O_n) = k_n\big)$$

with

$$\mathbb{P}\big(N(\omega, O) = k\big) = \frac{(a\mathcal{L}_3(O))}{k!} \exp\big(-a\mathcal{L}_3(O)\big),$$

where $a > 0$ is a constant and $\mathcal{L}_3(O)$ is the measure of O.
For $r > 0$, we define the random non homogeneous stored energy density by:

$$J(\omega, x, \xi) = g(\xi) + (h(\xi) - g(\xi)) \min\{1, N(., B(x, r))\},$$

i.e.,

$$J(\omega, x, \xi) = \begin{cases} g(\xi) & \text{if } x \in \cup_{y\in\omega} B(y, r), \\ h(\xi) & \text{otherwise.} \end{cases}$$

The function J is a model for the energy density of such composite material, $(B(y,r))$ being the rescaled random inclusions with a probability expectation

For a given random medium, let $J \in \mathscr{G}$ be a density and define the integral functional G from Ω into $\overline{\mathbb{R}}^{L^p(\mathbb{R}^d)\times\mathscr{O}}$:

$$G(\omega)(u,O) = \begin{cases} \int_O J(\omega,x,\nabla^2 u(x))dx, & \text{if } u\mid_O \in W^{2,p}(O), \\ +\infty & \text{otherwise,} \end{cases}$$

where $(u,O) \in L^p(\mathbb{R}^d) \times \mathscr{O}$.
For every $(\varepsilon,z) \in \mathbb{R}_+^* \times \mathbb{Z}^d$, we define the operators τ_z and ρ_ε, respectively, of translation and dilatation

$$\begin{cases} \tau_z u(x) = u(x-z) & \tau_z O = \{x \in \mathbb{R}^d : x - z \in O\}, \\ \rho_\varepsilon u(x) = u(\varepsilon^{-1}x) & \rho_\varepsilon O = \{x \in \mathbb{R}^d : \varepsilon x \in O\}. \end{cases}$$

Moreover, if G is random process then the functional $\rho_\varepsilon G$ is defined by:

(3.1) $$(\rho_\varepsilon G)(\omega)(u,O) = \varepsilon^d G(\omega)(\rho_\varepsilon u, \rho_\varepsilon O).$$

for every $u \in L^p(\mathbb{R}^d), O \in \mathscr{O}$.
If J denotes the integrand of G then (3.1) becomes

(3.2) $$(\rho_\varepsilon G)(\omega)(u,O) = \int_O J(\omega,\varepsilon^{-1}x,\nabla^2 u(x))dx.$$

for every $u \in W^{2,p}(O), O \in \mathscr{O}$.
Let us introduce the small parameter ε being the size of inhomogeneity. Then in view of (3.2) the random processes $\{G_\varepsilon(\omega)\} = (\rho_\varepsilon G)(\omega)$ is defined by:

(3.3) $$G_\varepsilon(\omega)(u,O) = \begin{cases} \int_O J(\omega,\varepsilon^{-1}x,\nabla^2 u(x))dx & \text{if } u\mid_O \in W^{2,p}(O), \\ +\infty & \text{otherwise.} \end{cases}$$

Then we easily check the following proposition:

PROPOSITION 2.8. *For $(\omega,O,u) \in \Omega \times \mathscr{O} \times W^{2,p}(O)$, the two following mappings $G(\omega)(u,O), (\rho_\varepsilon)(\omega)(u,O)$ are random variables.*

3.2. Identification and properties of $\Gamma-$limit. Let $J : \Omega \times \mathbb{R}^d \times \mathbb{M}^{d\times d} \to \mathbb{R}$ be such that the conditions (II.1)-(II.3) are hold, i.e., $J \in \mathscr{G}$. The Dirichlet problem for the variational functional associated with J plays a central role in the calculus of variations: for each $(O,\omega) \in \mathscr{O} \times \Omega$, we introduce:

(3.4) $$\mathscr{M}_O(\omega,J,\varphi) = \inf\Big\{ \int_O J(\omega,x,\nabla^2 u(x))dx : u \in \varphi + W_0^{2,p}(O) \Big\},$$

where $\varphi \in W^{2,p}_{loc}(\mathbb{R}^d)$ is fixed. We have thus defined a set function:

$$\mathscr{M}_{(.)}(\omega,J,\xi) : \mathscr{O} \to [0,\infty[,$$

which satisfies for every $O \in \mathscr{O}$,

$$|\xi|^p \mathcal{L}_d(O) \le \mathscr{M}_O(\omega,J,\xi) \le (1+|\xi|^p)\mathcal{L}_d(O).$$

The function $O \mapsto \mathscr{M}_O$ have been used to identify densities in $\Gamma-$convergence sense. In

$Q \in \mathscr{I}$ and φ is usually an affine function, i.e., $\varphi(x) = \xi x$.
Now, let us consider the group $(\tau_z)_{z\in\mathbb{Z}^d}$ which acts on $\mathscr{G}$ in the following way: for every $(\omega, x, \xi) \in \Omega \times O \times \mathbb{M}^{d\times d}$

$$(\tau_z J)(\omega, x, \xi) = J(\omega, x+z, \xi) \quad \forall z \in \mathbb{Z}^d. \tag{3.5}$$

According to the following mapping, where $\mathbb{P}_J \stackrel{def}{=} \mathbb{P} \circ J^{-1}$,

$$(\Omega, \Sigma, \mathbb{P}) \to \big(\mathscr{G}, \sigma(\mathscr{G}), \mathbb{P}_J\big), \quad \omega \mapsto J(\omega, ., .),$$

the triplet $\big(\mathscr{G}, \sigma(\mathscr{G}), \mathbb{P}_J\big)$ is a probability space, and it is easily seen

PROPOSITION 2.9. *$\big(\mathscr{G}, \sigma(\mathscr{G}), \mathbb{P}_J\big)$ is a dynamical system. Moreover, if J satisfies (3.5) then $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic.*

Let us give some properties of $\mathscr{O} \ni O \mapsto \mathscr{M}_O$:

PROPOSITION 2.10. (P1) $\dfrac{\mathscr{M}_{1/\varepsilon O}(., J, \xi)}{\mathcal{L}_d(1/\varepsilon O)} = \dfrac{\mathscr{M}_O(., J_\varepsilon, \xi)}{\mathcal{L}_d(O)}$;

(P2) *there exists a constant L', such that for every ξ_1, ξ_2 in $\mathbb{M}^{d\times d}$*

$$\left| \frac{\mathscr{M}_O(., J, \xi_1)}{\mathcal{L}_d(O)} - \frac{\mathscr{M}_O(., J, \xi_2)}{\mathcal{L}_d(O)} \right| \leq L'\big(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}\big)|\xi_1 - \xi_2|;$$

(P3) $\|\mathscr{M}_O(., J, \xi)\|_{L^1\left(\mathscr{G}, \sigma(\mathscr{G}), \mathbb{P}_J\right)} \leq \beta\,(1 + |\xi|^p)\,\mathcal{L}_d(O)$ *for* $(O, J, \xi) \in \mathscr{O} \times \mathscr{G} \times \mathbb{M}^{d\times d}$;

(P4) $O \longmapsto \mathscr{M}_O(., J, \xi)$ *is subadditive and covariant with respect to (3.5).*

PROOF. A straightforward computation yields (P1) and (P3). For (P2) put:

$$H(\xi) = \frac{\mathscr{M}_O(\omega, J, \xi)}{\mathcal{L}_d(O)}.$$

Let $\eta > 0$ and $u_\eta \in W_0^{2,p}(O)$ be such that

$$H(\xi) \geq \frac{1}{\mathcal{L}_d(O)} \Big[\int_O J(\omega, \varepsilon^{-1}x, \nabla^2 u_\eta(x) + \xi)dx - \eta \Big].$$

The Lipshitzian assumption (II.2) on J and Hölder's[2] inequality leading to

(3.6)

$$\begin{aligned}
&H(\xi_1) - H(\xi_2) \\
&\leq \frac{1}{\mathcal{L}_d(O)} \int_O \Big[J(\omega, \varepsilon^{-1}x, \nabla^2 u_\eta(x) + \xi_1) - J(\omega, \varepsilon^{-1}x, \nabla^2 u_\eta(x) + \xi_2) \Big] dx + \eta \\
&\leq \frac{L}{\mathcal{L}_d(O)} \int_O \big(1 + |\nabla^2 u_\eta(x) + \xi_1|^{p-1} + |\nabla^2 u_\eta(x) + \xi_2|^{p-1}\big)^{p-1/p} |\xi_1 - \xi_2| dx + \frac{\eta}{\mathcal{L}_d(O)} \\
&\leq M\Big[\frac{L}{\mathcal{L}_d(O)} \int_O \big(1 + |\xi_1|^p + |\xi_2|^p + |\nabla^2 u_\eta(x) + \xi_2|^p\big) dx \Big]^{p-1/p} |\xi_1 - \xi_2| + \frac{\eta}{\mathcal{L}_d(O)},
\end{aligned}$$

On the other hand, the coerciveness condition (II.3) on J yields

$$\text{(3.7)} \qquad \begin{aligned} \frac{1}{\mathcal{L}_d(O)}\int_O |\nabla^2 u_\eta(t)+\xi_2|^p dx &\leq \frac{1}{\alpha\mathcal{L}_d(O)}\int_O J(\omega,\varepsilon^{-1}x,\nabla^2 u_\eta(x)+\xi_2)dx \\ &\leq \frac{1}{\alpha}\Big[H(\xi_2)+\frac{\eta}{\mathcal{L}_d(O)}\Big] \\ &\leq \frac{\beta}{\alpha}(1+|\xi_2|^p)+\frac{\eta}{\alpha\mathcal{L}_d(O)}. \end{aligned}$$

Inserting (3.7) in (3.6) and letting η go to 0 one has

$$H(\xi_1)-H(\xi_2)\leq L'(1+|\xi_1|^{p-1}+|\xi_2|^{p-1})|\xi_1-\xi_2|,$$

where $L^{'}$ depends only on p,α,β and L. We conclude the proof by interchanging the roles of ξ_1 and ξ_2.
Let us move now to (P4). Let $z\in\mathbb{Z}^d$ then we have

$$\begin{aligned} \mathscr{M}_{\tau_z O}(\omega,J,\xi) &= \inf\Big\{\int_{\tau_z O} J(\omega,x,\nabla^2 u(x))dx : u\in L_\xi+W_0^{2,p}(\tau_z O)\Big\} \\ &= \inf\Big\{\int_O J(\omega,x+z,\nabla^2 u(x+z)+\xi)dx : \tau_{-z}u\in W_0^{2,p}(O)\Big\}. \end{aligned}$$

Setting $v_z(x)=\tau_{-z}u(x)=u(x+z)$. We easily check that $\nabla^2 v_z(x)=\nabla^2 u(x+z)$ and from (3.5) we get

$$\begin{aligned} \mathscr{M}_{\tau_z O}(\omega,J,\xi) &= \inf\Big\{\int_O J(\omega,x+z,\nabla^2 v_z(x)+\xi)dx : v_z\in W_0^{2,p}(O)\Big\} \\ &= \inf\Big\{\int_O (\tau_z J)(\omega,x,\nabla^2 v_z(x)+\xi)dx : v_z\in W_0^{2,p}(O)\Big\}. \end{aligned}$$

This implies

$$\mathscr{M}_{z+O}(\omega,J,\xi)=\mathscr{M}_O(\tau_z\omega,J,\xi)$$

which give $\mathscr{M}_{z+O}=\mathscr{M}_O\circ\tau_z$. The proof is completed. □

REMARK. *The properties of above proposition remains true if we replace the family $\mathscr{O}$ of all bounded subsets O in $\mathbb{R}^d$ by the family $\mathscr{I}$ defined in paragraph 2. 1, Section 2.*

Then the main result of this chapter reads as follows:

THEOREM 2.11. *Let $J\in\mathscr{G}$ be such that the stationary assumption (3.5) is verified. Then the corresponding random process $\{G_\varepsilon(\omega)\}$ defined by (3.3), $\Gamma-$ converge almost every where in $W^{2,p}(O)$ when $\varepsilon\to 0$ to the homogenized functional G_{hom},*

$$G_{\text{hom}}(u,O)=\int_O J_{\text{hom}}(\nabla^2 u(x))dx,$$

where its integrand J_{hom} takes the form:

$$J_{\text{hom}}(\xi) = \lim \frac{\mathscr{M}_{1/\varepsilon Q}(\omega,J,\xi)}{\cdots} \qquad \forall\xi\in\mathbb{M}^{d\times d}$$

COROLLARY 2.12 (definition of J_{hom}). *There exists a subset $\Omega^{'} \subset \Omega$, with $\mathbb{P}(\Omega^{'}) = 1$ and a function J_{hom} given for every $\xi \in \mathbb{M}^{d\times d}$ by:*

$$
\begin{aligned}
J_{\text{hom}}(\xi) &= \lim_{\varepsilon\to 0} \frac{\mathscr{M}_{1/\varepsilon Q}(\omega, J, \xi)}{\mathcal{L}_d(1/\varepsilon Q)} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\mathcal{L}_d(1/\varepsilon Q)} \int_\Omega \inf\Big\{ \int_{1/\varepsilon Q} J(\omega, x, \xi + \nabla^2 u(x))dx : u \in W_0^{2,p}(1/\varepsilon Q) \Big\} d\mathbb{P}(\omega).
\end{aligned}
$$

The next proposition give the main properties of J_{hom}, namely we have:

PROPOSITION 2.13. *The homogenized integrand J_{hom} given by Corollary 2.12 having the following properties:*

(i) $|J_{\text{hom}}(\xi_1) - J_{\text{hom}}(\xi_2)| \leq L^{'}\big(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}\big)|\xi_1 - \xi_2|$ *for all* $\xi_1, \xi_2 \in \mathbb{M}^{d\times d}$ *with* $L^{'}$ *is defined in (P2) of Proposition 2.10;*

(ii) $\alpha|\xi|^p \leq J_{\text{hom}}(\xi) \leq \beta\big(1 + |\xi|^p\big)$ *for all* $\xi \in \mathbb{M}^{d\times d}$ *with* $\alpha < \beta$ *is given by (II.2) condition.*

REMARK. *According to above proposition, the homogenized density J_{hom} belonging to the family $\mathscr{G}$ defined in paragraph 3.1 of Section 3.*

PROOF OF COROLLARY 2.13. For (i). In view of Proposition 2.10, we have for every $\varepsilon > 0$,

$$
\left| \frac{\mathscr{M}_{1/\varepsilon Q}(., J, \xi_1)}{\mathcal{L}_d(1/\varepsilon Q)} - \frac{\mathscr{M}_{1/\varepsilon Q}(., J, \xi_2)}{\mathcal{L}_d(1/\varepsilon Q)} \right| \leq L'(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1})|\xi_1 - \xi_2|.
$$

Going to the limit on ε, according to Corollary 2.12 we get the result.
Concerning (ii). Let Q be any cube in $\mathscr{I}$ then for every $u \in W_0^{2,p}(Q)$ we have

$$
\mathscr{M}_{1/\varepsilon Q}(\omega, J, \xi) \leq \int_{1/\varepsilon Q} J\big(\omega, x, \xi + \nabla^2 u(x)\big)dx \quad \forall \varepsilon > 0.
$$

If we take $u = 0$, it follows from (II.2) the following

$$
\mathscr{M}_{1/\varepsilon Q}(\omega, J, \xi) \leq \int_Q J\big(\omega, x, \xi + \nabla^2 u(x)\big)dx \leq \beta(1 + |\xi|^p)\mathcal{L}_d(1/\varepsilon Q).
$$

We divide the both members by $\mathcal{L}_d(1/\varepsilon Q)$ and letting $\varepsilon \to 0$ we shall have for every $\xi \in \mathbb{M}^{d\times d}$,

$$
J_{\text{hom}}(\xi) \leq \beta(1 + |\xi|^p).
$$

For the lower bound estimate, using the fact J satisfying (II.2) we get

$$
\frac{\mathscr{M}_{1/\varepsilon Q}(\omega, J, \xi)}{\mathcal{L}_d(1/\varepsilon Q)} \geq \alpha \inf\Big\{ \frac{1}{\mathcal{L}_d(1/\varepsilon Q)} \int_{1/\varepsilon Q} |\xi + \nabla^2 u(x)|dx : u \in W_0^{2,p}(1/\varepsilon Q) \Big\}.
$$

Since the elementary function $r \mapsto |r|^p$ is convex, we obtain

$$
\frac{1}{\mathcal{L}_d(1/\varepsilon Q)} \int_{1/\varepsilon Q} |\xi + \nabla^2 u(x)|dx \geq |\xi|^p.
$$

Let ε go to 0 it follows for every $\xi \in \mathbb{M}^{d\times d}$

4. Proof of the main result

Before stating the proof of the theorem 2.11, we need to introduce the notion of $\Gamma-$convergence[3] of a sequence of functions defined on a topological space, and compare this definition with the classical notion of convergence of sets in the sense of Kuratowski. For a large literature of this subject we consult **[2, 8, 5, 11, 17]**.

4.1. Γ-Convergence. Let X be a topological space. The family of neighborhood of x in X will be denoted by $\mathscr{V}(x)$. Let $\{G_\varepsilon\}_\varepsilon$ be a sequence of functions from X into $\overline{\mathbb{R}}$.

DEFINITION 2.14. *The $\Gamma-$lower limit and the $\Gamma-$upper limit of the sequence $\{G_\varepsilon\}$ are the functions from X into $\overline{\mathbb{R}}$ defined by*

$$\begin{aligned}\big(\Gamma-\liminf_{\varepsilon\to 0} G_\varepsilon\big)(x) &= \sup_{V\in\mathscr{V}(x)} \liminf_{\varepsilon\to 0} \inf_{y\in V} G_\varepsilon(y),\\ \big(\Gamma-\limsup_{\varepsilon\to 0} G_\varepsilon\big)(x) &= \sup_{V\in\mathscr{V}(x)} \limsup_{\varepsilon\to 0} \inf_{y\in V} G_\varepsilon(y).\end{aligned}$$

If there exists a function $G_{\text{hom}} : X \longrightarrow \overline{\mathbb{R}}$ such that

$$\Gamma-\liminf_{\varepsilon\to 0} G_\varepsilon = \Gamma-\limsup_{\varepsilon\to 0} G_\varepsilon = G_{\text{hom}},$$

then we write $G_{\text{hom}} = \Gamma-\lim_{\varepsilon\to 0} G_\varepsilon$ and we say that $\{G_\varepsilon\}$ $\Gamma-$converges to G_{hom} in X.

REMARK 1. *It is clear that*

- $\Gamma-\liminf_{\varepsilon\to 0} G_\varepsilon \le \Gamma-\limsup_{\varepsilon\to 0} G_\varepsilon$, *hence $\{G_\varepsilon\}$ $\Gamma-$converges to G_{hom} if and only if*

$$\Gamma-\liminf_{\varepsilon\to 0} G_\varepsilon \le G_{\text{hom}} \le \Gamma-\limsup_{\varepsilon\to 0} G_\varepsilon.$$

- *If the functions $\{G_\varepsilon\}$ are independent of ε, i.e., there exists $G : X \longrightarrow \overline{\mathbb{R}}$ such that $G_\varepsilon = G$ for every $x \in X$ and every $\varepsilon > 0$, then*

$$\Gamma-\lim_{\varepsilon\to 0} G_\varepsilon = \overline{G},$$

where $\overline{G}$ is the regularized lower semicontinuous of G.

EXAMPLES. *In these two examples we take $X = \mathbb{R}$.*

- *If $G_\varepsilon(x) = (1/\varepsilon)xe^{-2(x/\varepsilon)^2}$, then $\{G_\varepsilon\}$ $\Gamma-$converges in $\mathbb{R}$ to the function*

$$G_{\text{hom}}(x) = \begin{cases} -\frac{1}{2}e^{-\frac{1}{2}} & \text{if } x = 0,\\ 0 & \text{if } x \ne 0,\end{cases}$$

whereas $\{G_\varepsilon\}$ converges pointwise to 0.

- *If $G_\varepsilon(x) = \arctan(x/\varepsilon)$, then $\{G_\varepsilon\}$ $\Gamma-$converges in $\mathbb{R}$ to the function*

$$G_{\text{hom}}(x) = \begin{cases} -\frac{\pi}{2} & \text{if } x \le 0,\\ \frac{\pi}{2} & \text{if } x \ge 0,\end{cases}$$

whereas $\{G_\varepsilon\}$ converges pointwise to the function

$$G(x) = \begin{cases} -\frac{\pi}{2} & \textit{if } x \leq 0, \\ 0 & \textit{if } x = 0, \\ \frac{\pi}{2} & \textit{if } x \geq 0. \end{cases}$$

We now illustrate the relationships between $\Gamma-$convergence and topological set convergence in the sense of Kuratowski. Let $\{E_\varepsilon\}$ be a sequence of subsets of the topological space X. Then we have:

DEFINITION 2.15. *The $K-$lower limit and $K-$upper limit of the sequence $\{E_\varepsilon\}$ are sets of X defined by*

$$\begin{aligned} K - \liminf_{\varepsilon \to 0} E_\varepsilon &= \{x \in X : \forall V \in \mathscr{V}(x)\ \exists \kappa > 0 : V \cap E_\varepsilon \neq \emptyset\ \forall \varepsilon \leq \kappa\}, \\ K - \limsup_{\varepsilon \to 0} E_\varepsilon &= \{x \in X : \forall V \in \mathscr{V}(x)\ \forall \kappa > 0\ \exists \varepsilon \leq \kappa : V \cap E_\varepsilon \neq \emptyset\}. \end{aligned}$$

REMARK. *It is clear that*

- $K - \liminf_{\varepsilon \to 0} E_\varepsilon \subset K - \limsup_{\varepsilon \to 0} E_\varepsilon$*, hence $\{E_\varepsilon\}$ $K-$converges to E if and only if*
$$K - \liminf_{\varepsilon \to 0} E_\varepsilon \subset E \subset E - \limsup_{\varepsilon \to 0} E_\varepsilon.$$
- *If E is a subset of X and $E_\varepsilon = E$ for every $\varepsilon > 0$, then $\{E_\varepsilon\}$ $K-$converges to $\overline{E}$, the closure of E in X (using just the definition).*

EXAMPLE. *Let $X = \mathbb{R}^2$. If $E_\varepsilon = \{(\varepsilon, y) : 0 < y < 1\}$, then $\{E_\varepsilon\}$ $K-$ converges to* $E = \{(0, y) : 0 \leq y \leq 1\}$.

We recall that, for every $E \subset X$, $\mathbf{I}_E$[4] denotes the indicator function of E. The following proposition shows that the $K-$convergence of sets is equivalent to the $\Gamma-$convergence of the corresponding indicator functions.

PROPOSITION 2.16 (**[8, 17]**). *Let $\{E_\varepsilon\}$ be a sequence of subsets of X, and let*

$$E_1 = K - \liminf_{\varepsilon \to 0} E_\varepsilon, \quad E_2 = K - \limsup_{\varepsilon \to 0} E_\varepsilon.$$

Then

$$\mathbf{I}_{E_1} = \Gamma - \limsup_{\varepsilon \to 0} \mathbf{I}_{E_\varepsilon}, \quad \mathbf{I}_{E_2} = \Gamma - \liminf_{\varepsilon \to 0} \mathbf{I}_{E_\varepsilon}.$$

In particular $\{E_\varepsilon\}$ $K-$converges to E in X if and only if $(\mathbf{I}_{E_\varepsilon})$ $\Gamma-$converges to $\mathbf{I}_E$ in X.

The following theorem shows the connection between $\Gamma-$convergence of functions and $K-$convergence of their epigraphs[5]. This is the reason why $\Gamma-$convergence is sometimes called epi $-$convergence.

THEOREM 2.17. *Let $\{G_\varepsilon\}$ be a sequence of functions from X into $\overline{\mathbb{R}}$, and let*

$$G_1 = \Gamma - \liminf_{\varepsilon \to 0} G_\varepsilon, \quad G_2 = \Gamma - \limsup_{\varepsilon \to 0} G_\varepsilon.$$

Then

where the K−limits are taken in the product topology of $X \times \mathbb{R}$. *In particular* $\{G_\varepsilon\}$ $\Gamma-$*converges to G in X if and only if* $\{\text{epi } G_\varepsilon\}$ $K-$*converges in* $X \times \mathbb{R}$.

The following proposition provides a characterization of $\Gamma - \liminf$ and $\Gamma - \limsup$ in terms of sequences, when X satisfies the first axiom of countability (for example a metric space), i.e., the neighborhood system of every point of X has a countable base.

DEFINITION 2.18. *Assume that X satisfies the first axiom of countability. Then* $\{G_\varepsilon\}$ $\Gamma-$ *converges to* G_{hom} *in X if and only if the two following sentences hold:*

(a) Lower bound: *for every* $x \in X$ *and for every sequence* (x_ε) *converging to x in X,*

$$G_{\text{hom}}(x) \leq \liminf_{\varepsilon \to 0} G_\varepsilon(x_\varepsilon);$$

(b) Upper bound: *for every* $x \in X$ *there exists a sequence* (x_ε) *converging to x in X such that*

$$G_{\text{hom}}(x) \geq \limsup_{\varepsilon \to 0} G_\varepsilon(x_\varepsilon).$$

Then we write

$$G_{\text{hom}}(x) = \Gamma - \lim_{\varepsilon \to 0} G_\varepsilon(x).$$

Other significant properties of the $\Gamma-$convergence are given by the following proposition:

PROPOSITION 2.19. *Let* $\{G_\varepsilon\}$ *be a sequence* $\Gamma-$*converges to* G_{hom} *and* $G : X \to \mathbb{R}$ *be a continuous function. Then*

(i) Stability by continuous perturbation:

$$\Gamma - \lim_{\varepsilon \to 0}(G_\varepsilon + G) = \big(\Gamma - \lim_{\varepsilon \to 0} G_\varepsilon\big) + G;$$

(ii) Regularity: G_{hom} *is lower semicontinuous on X;*

(iii) Convergence of minimum: *Let* (x_ε) *be sequence of minimizers of* $\{G_\varepsilon\}$ *in X, i.e.,*

$$G_\varepsilon(x_\varepsilon) \leq \inf_{y \in X} G_\varepsilon(y) + \varepsilon.$$

If x is cluster point of (x_ε), *then x is a minimizer of* G_{hom} *in X, and*

$$\min_{x \in X} G_{\text{hom}}(x) = \lim_{\varepsilon \to 0} \inf_{x \in X} G_\varepsilon(x) \quad .$$

4.2. Proof of the lower bound.

LEMMA 2.20. *For every sequence* $(u_\varepsilon)_{\varepsilon>0}$ *in* $W^{2,p}(O)$ *such that* $u_\varepsilon \to u$ *in* $W^{2,p}(O)$. *Then we have*

$$\liminf_{\varepsilon \to 0} G_\varepsilon(\omega)(u_\varepsilon, O) \geq G_{\text{hom}}(u, O).$$

PROOF. We divide the proof in two steps.

Step 1. We denote by $\mathcal{M}_R(O)$ the set of all Radon[6] measures on O and set

$$\mathcal{M}_R^+(O) = \{\lambda \in \mathcal{M}_R(O) : \lambda \geq 0\}$$

For every $\varepsilon > 0$ define

$$\lambda_\varepsilon \stackrel{def}{=} J\big(\omega, \varepsilon^{-1}x, \nabla^2 u(.)\big),$$

namely, for all $O \in \mathscr{O}$

$$\lambda_\varepsilon(O) = \int_O J\big(\omega, \varepsilon^{-1}x, \nabla^2 u(x)\big)dx.$$

By growth condition (II.3) the sequence $\{\lambda_\varepsilon\}$ is uniformly bounded in $\mathcal{M}_R^+(O)$. Then up to a subsequence there exists $\lambda \in \mathcal{M}_R^+(O)$ such that

$$\lambda_\varepsilon \rightharpoonup \lambda \text{ weakly in } \mathcal{M}_R(O).$$

Namely we have

$$\forall \varphi \in \mathcal{C}_0(O) : \int_O \varphi \lambda_\varepsilon \longrightarrow \int_O \varphi \lambda.$$

Let $\lambda_{\text{hom}} \in \mathcal{M}_R^+(O)$ be defined by

$$\lambda_{\text{hom}}(O) = \int_O J_{\text{hom}}\big(\nabla^2 u(x)\big)dx.$$

The idea is to compare the limit measure λ with λ_{hom}. Thanks to the Aleksandrov theorem, see **[12]** we have:

$$\lambda(O) \le \liminf_{\varepsilon \to 0} \lambda_\varepsilon(O).$$

Then it is enough to prove that

$$\lambda_{\text{hom}}(O) \le \lambda(O).$$

For this aim using the Lebesgue's decomposition theorem we find that

$$\lambda = \lambda_a \mathcal{L}_d + \lambda_e, \quad \lambda_a \ll \mathcal{L}_d \text{ and } \lambda_e \perp \mathcal{L}_d,$$

where λ_a and λ_e are absolutely continuous and singular parts of λ with respect to the Lebesgue measure. From Besicovitch differentiation theorem of measure, there exists $\phi \in L^1(O, \mathbb{R}_+)$ such that $\lambda_a = \phi \mathcal{L}_d$, and for a.e. $x_0 \in O$ we have

$$\phi(x_0) = \lim_{\delta \to 0} \frac{\lambda_a\big(B_\delta(x_0)\big)}{\mathcal{L}_d\big(B_\delta(x_0)\big)} = \lim_{\delta \to 0} \frac{\lambda\big(B_\delta(x_0)\big)}{\mathcal{L}_d\big(B_\delta(x_0)\big)}. \tag{4.1}$$

Here $B_\delta(x_0)$ is the open ball in $\mathbb{R}^d$, centered in x_0, with radius $\delta > 0$. For every $\delta \in [0, \delta_0[\setminus D$, where D is a countable set. In the sequel, we will take δ such that $\lambda\left(\partial B_\delta(x_0)\right) = 0$, according to the Aleksandrov[7] theorem, we have in particular

$$\lambda\big(B_\delta(x_0)\big) = \lim_{\varepsilon \to 0} \lambda_\varepsilon\big(B_\delta(x_0)\big).$$

Consequently (4.1) becomes

$$\phi(x_0) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon\big(B_\delta(x_0)\big)}{\mathcal{L}_d\big(B_\delta(x_0)\big)}.$$

Then it is enough to show that

Indeed: if $\mathcal{L}_d-$ a.e. $x \in O$

$$J_{\text{hom}}\big(\nabla^2 u(x)\big) \leq \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon\big(B_\delta(x)\big)}{\mathcal{L}_d\big(B_\delta(x)\big)}.$$

Then $\mathcal{L}_d-$ a.e. $x \in O, J_{\text{hom}}\big(\nabla^2 u(x)\big) \leq \phi(x)$ consequently

$$\int_O J_{\text{hom}}\big(\nabla^2 u(x)\big)dx \leq \int_O \phi(x)dx,$$

i.e.,

$$\lambda_{\text{hom}}(O) \leq \lambda_a(O) \leq \lambda_a(O) + \lambda_s(O) = \lambda(O).$$

We prove then

$$J_{\text{hom}}\big(\nabla^2 u(x_0)\big) \leq \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon\left(B_\delta(x_0)\right)}{\mathcal{L}_d\left(B_\delta(x_0)\right)}.$$

We assume that $u_\varepsilon = u_0$ in $\partial B_\delta(x_0)$ with

$$u_0(x) = u(x_0) + \nabla^2 u(x_0)\frac{(x-x_0)^T(x-x_0)}{2}.$$

Therefore in view of (3.4) we have

$$\begin{aligned}
\text{(4.2)}\qquad & \frac{1}{\mathcal{L}_d(B_\rho(x_0))}\int_{B_\rho(x_0)} J\big(\omega,\varepsilon^{-1}x,\nabla^2 u_\varepsilon(x)\big)dx \\
= \;& \frac{1}{\mathcal{L}_d(B_\rho(x_0))}\int_{B_\rho(x_0)} J\big(\omega,\varepsilon^{-1}x,\nabla^2(u_\varepsilon-u_0)(x)+\nabla^2 u_0(x)\big)dx \\
\geq \;& \frac{1}{\mathcal{L}_d(B_\rho(x_0))}\inf\Big\{\int_{B_\rho(x_0)} J\big(\omega,\varepsilon^{-1}x,\nabla^2 v(x)+\nabla^2 u_0(x)\big) : v \in W_0^{2,p}(B_\rho(x_0))\Big\}dx.
\end{aligned}$$

We apply the Corollary 2.12 we thus find

$$\begin{aligned}
\text{(4.3)}\qquad J_{\text{hom}}\big(\nabla^2 u(x_0)\big) &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\mathscr{M}_{1/\varepsilon B_\delta(x_0)}(\omega,J,\nabla^2 u(x_0))}{\mathcal{L}_d\big(1/\varepsilon B_\delta(x_0)\big)} \\
&\leq \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\mathcal{L}_d\left(B_\delta\left(x_0\right)\right)}\int_{B_\delta(x_0)} J\big(\omega,\varepsilon^{-1}x,\nabla^2 u_\varepsilon(x)\big)dx \\
&= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon\left(B_\delta\left(x_0\right)\right)}{\mathcal{L}_d\left(B_\delta\left(x_0\right)\right)} = \phi(x_0).
\end{aligned}$$

This achieved the result.

Step 2. Following E. De Giorgi it is suffices to modify the sequence u_ε defined in step 1, by a function of $W^{2,p}\big(\partial B_\delta(x_0)\big)$, which coincides with u_0 on $\partial B_\delta(x_0)$ in the trace sense (for more details of this techniques, see [**13**]).

We fix $0 < r < 1$ and we set

$$B_i = B_{r\rho\delta + i\frac{(\rho\delta - r\rho\delta)}{s}}(x_0) \text{ for } i = 0,\ldots,s.$$

We obviously remark

$$B_{r\rho\delta}(x_0) = B_0 \subset B_1 \subset \cdots \subset B_s = B_{\rho\delta}(x_0).$$

Define

$$\begin{aligned} u_{\varepsilon,i}(x) &\overset{def}{=} u_\varepsilon(x)\psi_i(x) + (1-\psi_i(x))\Big(u(x_0) + \nabla^2 u(x_0)\frac{(x-x_0)_l\,(x-x_0)_k}{2}\Big) \\ &= u_0(x) + \psi_i(x)\big(u_\varepsilon(x) - u_0(x)\big). \end{aligned}$$

Then we have $u_\varepsilon \in W^{2,p}\big(B_{\rho\delta}(x_0)\big), u_\varepsilon = u_0$ on a neighborhood of $\partial B_{\rho\delta}(x_0)$,

$$\nabla^2 u_{\varepsilon,i}(x) = \begin{cases} \nabla^2 u_\varepsilon(x) & \text{in } B_{i-1}, \\ \nabla^2 u(x_0) & \text{in } B_{\rho\delta}(x_0)\backslash B_i \end{cases}$$

and

$$\begin{aligned} \nabla^2 u_{\varepsilon,i}(x) &= \nabla^2 u(x_0) + \nabla^2\psi_i(x)(u_\varepsilon(x) - u_0(x)) \\ &\quad +2\nabla\psi_i(x)\otimes\nabla(u_\varepsilon(x) - u_0(x)) \quad \text{in } B_i\backslash B_{i-1}. \end{aligned}$$

On the other hand, for $i = 1,\dots s$ we have

$$\begin{aligned} \mathscr{M}_{B_{\rho\delta}(x_0)}\big(\omega, J_\varepsilon, \nabla^2 u(x_0)\big) &\le \int_{B_{i-1}} J\big(\omega, \varepsilon^{-1}x, \nabla^2 u_\varepsilon(x)\big)dx + \int_{B_i\backslash B_{i-1}} J\big(\omega, \varepsilon^{-1}x, \nabla^2 v_{\varepsilon,i}(x)\big)dx \\ &\quad + \int_{B_{\rho\delta}(x_0)\backslash B_i} J\big(\omega, \varepsilon^{-1}x, \nabla^2 u(x_0)\big)dx \\ &\overset{def}{=} \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3. \end{aligned}$$

For $\mathbb{I}_1$, we obviously check

$$\mathbb{I}_1 \le \int_{B_{\rho\delta}(x_0)} J\big(\omega, \varepsilon^{-1}x, \nabla^2 u_\varepsilon(x)\big)dx.$$

Concerning $\mathbb{I}_2$, according to growth condition (II.3) we obtain

$$\begin{aligned} \mathbb{I}_2 &\le \beta\big(1 + |\nabla^2 u(x_0)|^p\big)\mathcal{L}_d(B_{\rho\delta}(x_0))(1-r^d) \\ &\quad + \int_{B_i\backslash B_{i-1}} |\nabla^2\psi_i(x)|^p |u_\varepsilon(x) - u_0(x)|^p dx \\ &\quad +2^p \int_{B_i\backslash B_{i-1}} |\nabla\psi_i(x)|^p |\nabla u_\varepsilon(x) - \nabla u_0(x)|^p dx. \end{aligned}$$

Then we deduce

$$\begin{aligned} \mathbb{I}_2 &\le \beta\big(1 + |\nabla^2 u(x_0)|^p\big)\mathcal{L}_d\big(B_{\rho\delta}(x_0)\big)(1-r^d) \\ &\quad + \frac{s^p}{\rho^{2p}\delta^{2p}(1-r)^{2p}} \int_{B_i\backslash B_{i-1}} |u_\varepsilon(x) - u_0(x)|^p dx \\ &\quad + \frac{2^p s^p}{\rho^p\delta^p(1-r)^p} \int_{B_i\backslash B_{i-1}} |\nabla u_\varepsilon(x) - \nabla u_0(x)|^p dx. \end{aligned}$$

Let us move to the proof of the member $\mathbb{I}_3$. Using once again the growth condition (II.3) we get

$$\mathbb{I}_3 \le \int J\big(\omega, \varepsilon^{-1}x, \nabla^2 u(x_0)\big)dx$$

Putting together $\mathbb{I}_1, \mathbb{I}_2$ and $\mathbb{I}_3$ it follows

$$
\begin{aligned}
\mathscr{M}_{B_{\rho\delta}(x_0)}\big(\omega, J_\varepsilon, \nabla^2 u(x_0)\big) \quad \leq \quad & \int_{B_{\rho\delta}(x_0)} J\big(\omega, \varepsilon^{-1}x, \nabla^2 u_\varepsilon(x)\big)dx \\
& +\beta\big(1+|\nabla^2 u(x_0)|^p\big)\mathcal{L}_d\big(B_{\rho\delta}(x_0)\big)(1-r^d) \\
& +\frac{s^p}{\rho^{2p}\delta^{2p}(1-r)^{2p}}\int_{B_i\setminus B_{i-1}} |u_\varepsilon(x)-u_0(x)|^p dx \\
& +\frac{2^p s^p}{\rho^p\delta^p(1-r)^p}\int_{B_i\setminus B_{i-1}} |\nabla u_\varepsilon(x)-\nabla u_0(x)|^p dx.
\end{aligned}
$$

We divide by $\mathcal{L}_d(B_{\rho\delta}(x_0))$, then the above estimates becomes

$$
\begin{aligned}
& \frac{\mathscr{M}_{B_{\rho\delta}(x_0)}\big(\omega, J_\varepsilon, \nabla^2 u(x_0)\big)}{\mathcal{L}_d\big(B_{\rho\delta}(x_0)\big)} \\
& \leq \frac{1}{\mathcal{L}_d(B_{\rho\delta}(x_0))}\int_{B_{\rho\delta}(x_0)} J\big(\omega, \varepsilon^{-1}x, \nabla^2 u_\varepsilon(x)\big)dx + \beta\big(1+|\nabla^2 u(x_0)|^p\big)(1-r^d) \\
& +\frac{s^p}{\rho^{2p}\delta^{2p}(1-r)^{2p}}\frac{1}{\mathcal{L}_d(B_{\rho\delta}(x_0))}\int_{B_i\setminus B_{i-1}} |u_\varepsilon(x)-u_0(x)|^p dx \\
& +\frac{2^p s^p}{\rho^p\delta^p(1-r)^p}\frac{1}{\mathcal{L}_d\big(B_{\rho\delta}(x_0)\big)}\int_{B_i\setminus B_{i-1}} |\nabla u_\varepsilon(x)-\nabla u_0(x)|^p dx.
\end{aligned}
$$

Going to the limit on ε and δ, then combining (P1) of Proposition 2.10, Corollary 2.12 and [Theorem 3.4.2, p. 129] in W. P. Ziemer [**25**] or [Lemma 4.2.1, p. 428] in H. Attouch, G. Buttazzo and G. Michaille [**3**] , i.e., for a.e. $x \in O$,

$$
\begin{aligned}
& \lim_{\delta\to 0}\frac{1}{\mathcal{L}_d\big(B_{\rho\delta}(x_0)\big)}\int_{B_{\rho\delta}(x_0)} |u(x)-u_0(x)|^p dx = 0, \\
& \lim_{\delta\to 0}\frac{1}{\mathcal{L}_d\big(B_{\rho\delta}(x_0)\big)}\int_{B_{\rho\delta}(x_0)} |\nabla u(x)-\nabla u_0(x)|^p\, dx = 0.
\end{aligned}
$$

We shall have

$$
\begin{aligned}
J_{\hom}\big(\nabla^2 u(x_0)\big) \quad \leq \quad & \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\rho^d}\frac{1}{\mathcal{L}_d\big(B_\delta(x_0)\big)}\int_{B_{\rho\delta}(x_0)} J\big(\omega, \varepsilon^{-1}x, \nabla^2 u_\varepsilon(x)\big)dx \\
& +\beta\big(1+|\nabla^2 u(x_0)|^p\big)(1-r^d).
\end{aligned}
$$

Finally, letting $\rho \to 1$ and $r \to 1$ we get

$$
\begin{aligned}
J_{\hom}\big(\nabla^2 u(x_0)\big) \quad & \leq \quad \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\mathcal{L}_d\big(B_\delta(x_0)\big)}\int_{B_\delta(x_0)} J\big(\omega, \varepsilon^{-1}x, \nabla^2 u_\varepsilon(x)\big)dx \\
& = \quad \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon\big(B_\delta(x_0)\big)}{\mathcal{L}_d\big(B_\delta(x_0)\big)},
\end{aligned}
$$

which ends the proof. □

4.3. Proof of the upper bound.

LEMMA 2.21. *For every $u \in W^{2,p}(O)$ there exists a sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ in $W^{2,p}(O)$ such that*

PROOF. We proceed by steps.

Step 1. We assume that u is a square function, i.e.,

$$u(x) = L_\xi(x) = \xi_{lk}\frac{x_l x_k}{2},$$

with the second gradient is ξ_{lk}.
Let $\gamma > 0$, we consider $(Q_i)_{i\in I(\gamma)}$ and $(Q_i)_{i\in K(\gamma)}$ two families of open disjoints cubes with side γ and lattice in $\mathbb{R}^d$ spanned by $]0,\gamma[$ such that

$$\bigcup_{i\in I(\gamma)} Q_i \subset O \subset \bigcup_{i\in I(\gamma)\cup K(\gamma)} Q_i,$$

with

$$\mathcal{L}_d\Big(\bigcup_{i\in I(\gamma)} Q_i\Big) = \sigma(\gamma) \text{ and } \lim_{\gamma\to 0}\sigma(\gamma) = 0.$$

In view of Corollary 2.12 we have

$$\begin{aligned} G_{\text{hom}}(u,O) \geq G_{\text{hom}}\Big(u, \bigcup_{i\in I(\gamma)} Q_i\Big) &= \sum_{i\in I(\gamma)} \mathcal{L}_d(Q_i) J_{\text{hom}}(\xi) \\ &= \lim_{\varepsilon\to 0}\sum_{i\in I(\gamma)} \mathscr{M}_{Q_i}(\omega, J_\varepsilon, \xi). \end{aligned} \tag{4.4}$$

The construction of the sequence $(u_\varepsilon(\omega))_{\varepsilon>0}$ follows from $\gamma-$minimizers of $\mathscr{M}_{Q_i}(\omega, J_\varepsilon, \xi)$. Let $v_{i,\varepsilon,\gamma}$ in $W^{2,p}(Q_i)$ such that

$$G_\varepsilon(\omega)(v_{i,\varepsilon,\gamma} + L_\xi, Q_i) \leq \mathscr{M}_{Q_i}(\omega, J_\varepsilon, \xi) + \frac{\gamma}{\operatorname{card}\big(I(\gamma)\cup K(\gamma)\big)}. \tag{4.5}$$

Setting $v_{\varepsilon,\gamma} = v_{i,\varepsilon,\gamma}$ in Q_i and $u_{\varepsilon,\gamma} = v_{\varepsilon,\gamma} + L_\xi$. In view of (4.4) and upper growth conditions (II.3) we get

$$\begin{aligned} G_{\text{hom}}(u,O) &\geq \limsup_{\varepsilon\to 0} G_\varepsilon(\omega)\Big(u_{\varepsilon,\gamma}, \bigcup_{i\in I(\gamma)} Q_i\Big) - \gamma \\ &\geq \limsup_{\varepsilon\to 0} G_\varepsilon(\omega)(u_{\varepsilon,\gamma}, O) - \beta\big(1+|\xi|^p\big)\sigma(\gamma) - 2\gamma. \end{aligned}$$

Letting γ tend to 0 we find

$$G_{\text{hom}}(u,O) \geq \limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0} G_\varepsilon(\omega)(u_{\varepsilon,\gamma}, O). \tag{4.6}$$

On the other hand, we apply (II.3)

$$\begin{aligned} \|\nabla^2 v_{i,\varepsilon,\gamma} + \xi\|_{2,p}^p &\leq \frac{1}{\alpha} G_\varepsilon(\omega)(v_{i,\varepsilon,\gamma} + L_\xi, Q_i) \\ &\leq \frac{1}{\alpha}\mathscr{M}_{Q_i}(\omega, J_\varepsilon, \xi) + \frac{\gamma}{\alpha \operatorname{card}\big(I(\gamma)\cup K(\gamma)\big)} \end{aligned} \tag{4.7}$$

Using the Poincaré inequality two times for $v_{i,\varepsilon}$ and for $\nabla v_{i,\varepsilon}$ in $W^{2,p}_0(Q_i)$, there exists $C_1, C_2 > 0$ such that

$$\|v_{i,\varepsilon,\gamma}\|^p_{0,p} \leq C_1\gamma^p \|\nabla v_{i,\varepsilon,\gamma}\|^p_{0,p} \quad \text{and} \quad \|\nabla v_{i,\varepsilon,\gamma}\|^p_{0,p} \leq C_2\gamma^p \|\nabla^2 v_{i,\varepsilon,\gamma}\|^p_{0,p}.$$

Then we deduce

$$\|v_{i,\varepsilon,\gamma}\|^p_{0,p} \leq C_1 C_2 \gamma^{2p} \|\nabla^2 v_{i,\varepsilon,\gamma}\|^p_{2,p}$$

Combining the above estimates with (4.7) we thus get

$$\|v_{i,\varepsilon,\gamma}\|^p_{0,p} \ \leq \ C\gamma^{2p}\Big(\mathcal{L}_d(Q_i) + \frac{\gamma}{\operatorname{card}\big(I(\gamma)\cup K(\gamma)\big)}\Big). \tag{4.8}$$

By definition of $u_{\varepsilon,\gamma}$ we have $\|u_{\varepsilon,\gamma} - L_\xi\|^p_{0,p} = \|v_{\varepsilon,\gamma}\|^p_{0,p}$ and from (4.6), (4.7) one has

$$\begin{aligned} \|u_{\varepsilon,\gamma} - L_\xi\|^p_{0,p} \ &\leq \ C \sum_{i\in I(\gamma)\cup K(\gamma)} \left[\gamma^{2p}\Big(\mathcal{L}_d(Q_i) + \frac{\gamma}{\operatorname{card}(I(\gamma)\cup K(\gamma))}\Big)\right] \\ &\leq \ C\gamma^{2p}(\mathcal{L}_d(\hat{O}) + \gamma), \end{aligned} \tag{4.9}$$

where C is a constant that depends only on p, α, β, ξ and $\hat{O}$ is any bounded set containing O. From (4.6), (4.7) and diagonalization argument there exists a map $\varepsilon \longmapsto \gamma(\varepsilon)$, which $\gamma(\varepsilon)$ tends to 0 when ε tends to 0. Setting $u_\varepsilon = u_{\varepsilon,\gamma(\varepsilon)}$. Clearly we have $\lim_{\varepsilon\to 0} u_\varepsilon = L_\xi$ and

$$G_{\text{hom}}(u, O) \geq \limsup_{\varepsilon\to 0} G_\varepsilon(\omega)(u_\varepsilon, O).$$

Step 2. We assume that $u \in W^{2,p}(O)$. By continuity of G_{hom} on $W^{2,p}(O)$ it is enough to prove the Lemma 2.21 for u is piecewise affine continuous function and applying the previous step. More precisely, let $(O_i)_{i\in I}$ be a finite open partition of O such that $u(x) = L_{\xi^i}(x) + w^i$ in O_i, with $\xi^i \in \mathbb{M}^{d\times d}$ and $w^i \in \mathbb{R}^d$.
Using the first step there exists $v_{i,\varepsilon} \in W^{2,p}(O_i)$ depending on ω such that $\lim_{\varepsilon\to 0} v_{i,\varepsilon} = u$ and

$$G_{\text{hom}}(u, O_i) \geq \limsup_{\varepsilon\to 0} G_\varepsilon(\omega)(v_{i,\varepsilon}, O_i).$$

By the same manner as in step 1 we construct a sequence $(u_{i,\varepsilon})$ depending on ω such that

$$\begin{cases} \lim_{\varepsilon\to 0} u_{i,\varepsilon} = u, \quad u_{i,\varepsilon} = u \text{ in } \partial O_i \\ G_{\text{hom}}(u, O_i) \geq \limsup_{\varepsilon\to 0} G_\varepsilon(\omega)(u_{i,\varepsilon}, O_i). \end{cases}$$

After summing over i it follows that

$$\begin{cases} \lim_{\varepsilon\to 0} u_\varepsilon = u, \\ G_{\text{hom}}(u, O) \geq \limsup_{\varepsilon\to 0} G_\varepsilon(\omega)(u_\varepsilon, O). \end{cases}$$

REMARK 2. *By minor modifications in the proof, one can easily obtain a similar version of Theorem 2.11 for the case when J depends explicitly on the first gradient ∇u. More precisely, G_ε takes the form:*

$$G_\varepsilon(\omega)(u,O) = \int_O J\big(\omega, \varepsilon^{-1}x, \nabla u(x), \nabla^2 u(x)\big)dx,$$

where the density $J : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M}^{d\times d} \longrightarrow [0,+\infty[$ having the following conditions:

- *$x \longmapsto J(\omega, x, \xi, \zeta)$ is measurable for every $(\omega, \xi, \zeta) \in \Omega \times \mathbb{R}^d \times \mathbb{M}^{d\times d}$;*
- *for every $(\omega, \xi, \zeta_1, \zeta_2) \in \Omega \times \mathbb{R}^d \times \mathbb{M}^{2d\times d}$, and a.e. $x \in \mathbb{R}^d$*

$$|J(\omega,x,\xi,\zeta_1) - J(\omega,x,\xi,\zeta_2)| \le L\big(1 + |\zeta_1|^{p-1} + |\zeta_2|^{p-1}\big)|\zeta_1 - \zeta_2|;$$

- *there exists two positive constants $\alpha, \beta : 0 < \alpha \le \beta < +\infty$ such that for every $(\omega, \xi, \zeta) \in \Omega \times \mathbb{R}^d \times \mathbb{M}^{d\times d}$ and a.e. $x \in \mathbb{R}^d$*

$$\alpha\big(|\xi|^p + |\zeta|^p\big) \le J(\omega,x,\xi,\zeta) \le \beta\big(1 + |\xi|^p + |\zeta|^p\big);$$

- *there exists a positive constant S such that for every $(\omega, \xi_1, \xi_2, \zeta) \in \Omega \times \mathbb{R}^{2d} \times \mathbb{M}^{d\times d}$ and a.e. $x \in \mathbb{R}^d$*

$$\left|\sqrt{J(\omega,x,\xi_1,\zeta)} - \sqrt{J(\omega,x,\xi_2,\zeta)}\right| \le S\|\xi_1 - \xi_2\|.$$

The corresponding $\Gamma-$ limit is still given by (2.12), explicitly we have

$$G_{\text{hom}}(u) = \int_O J_{\text{hom}}(\nabla u(x), \nabla^2 u(x))dx,$$

with

$$J_{\text{hom}}(y,\xi) = \lim_{\varepsilon\to 0} \frac{1}{\mathcal{L}_d\big(1/\varepsilon Q\big)} \int_\Omega \min_{u\in\Lambda_\xi} \Big\{G(\omega)\big(u, \big(1/\varepsilon Q\big)\big)\Big\}dP(\omega)$$

where

$$\Lambda_\xi = \Big\{u; u_i(x) - y_{lk}x_k - \frac{1}{2}\xi_{lkm}x_k x_m \in W^{2,p}(Q_{\frac{1}{\varepsilon}})\Big\}$$

Concerning the expression of the homogenized integrand J_{hom} which is given by a minimum, we computed its value explicitly in various examples, in particular, in the example of a heterogeneous material, which its heterogeneousness is distributed in a random way, endowed with a Poisson process, see **[4]**. We associate to J_{hom} the suitable Euler equation and with a Kolmogorov[8] theorem (central limit) we infer its expression.

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CHAPTER 3

Inviscid limit For Axisymmetric Navier-Stokes System

This work is the subject of the following publication :
T. Hmidi and M. Zerguine: *Inviscid limit for axisymmetric Navier-Stokes system*. Differential and Integral Equations **22** (2009), no. 11-12, 1223-1246.

Abstract. *We are interested in the global well-posedness of the axisymmetric Navier-Stokes system with initital data belonging to the critical Besov spaces* $B^{1+\frac{3}{p}}_{p,1}$. *We obtain uniform estimates of the viscous solutions* (v_ν) *with respect to the viscosity in the spirit of the work* **[2]** *concerning the axisymmetric Euler equations. We provide also a strong convergence result in* L^p *norm of the viscous solutions* (v_ν) *to the Eulerian one* v.
Keywords and phrases. Axisymmetric data, Navier-Stokes system, global existence, dyadic decomposition, paradifferential calculus.
2000 Mathematics Subject Classification. 76D03 (35B33, 35Q35, 76D05).

1. Introduction

In this paper we deal with the incompressible Navier-Stokes system described by:

$$(\mathrm{NS}_\nu)\begin{cases}\partial_t v_\nu + v_\nu\cdot\nabla v_\nu - \nu\Delta v_\nu = -\nabla p_\nu\\ \mathrm{div}\, v_\nu = 0\\ v_{\nu|t=0} = v_0.\end{cases}$$

It models the flow of an homogeneous incompressible viscous fluid of viscosity $\nu > 0$. The velocity v_ν is three-dimensional vector-field, the pressure $p_\nu = p_\nu(t,x)$ is a real scalar. The condition div $v_\nu = 0$ means that the fluid is incompressible.
The mathematical theory of (NS_ν) was started by J. Leray in his pioneering work **[15]**. He proved the global existence of weak solutions in energy space by using a compactness method. Nevertheless, the uniqueness of weak solutions is only known in space dimension two. According to the work of H. Fujita and T. Kato **[8]**, we can prove local well-posedness for initial data lying in the critical Sobolev space $\dot{H}^{\frac{1}{2}}$. More similar results are established in various functional spaces like L^3, $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ and BMO^{-1}. We refer to **[13]** for more details about the subject. The global existence of these solutions for arbitrary initial data is an outstanding open problem in PDEs.
When the viscosity $\nu = 0$, the Navier-Stokes system is reduced to Euler system (E) which has a local theory in a satisfactory state. We will restrict ourselves to some significant result: in **[11]** Kato proved local well-posedness for initial data in H^s, with $s > \frac{5}{2}$. We can

they have in cylindrical coordinates $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ the following structure:

$$v(t,x) = v_r(t,r,z)\vec{e}_r + v_z(t,r,z)\vec{e}_z.$$

For these flows the vorticity ω takes the form $\omega = (\partial_z v^r - \partial_r v^z)\vec{e}_\theta$ and obeys the equation:

$$\partial_t \omega - \nu\Delta\omega + (v\cdot\nabla)\omega = \frac{v_r}{r}\omega.$$

For the axisymmetric Navier-Stokes system, M. Ukhoviskii and V. Yudovich [**21**] proved global well-posedness for initial data $v^0 \in H^1$ such that $\omega_0, \frac{\omega_0}{r} \in L^2 \cap L^\infty$, with uniform bounds on the viscosity. In [**14**], S. Leonardi, J. Màlek, J. Necăs and M. Pokorný proved the global well-posedness for initial data $v^0 \in H^2$. This result was recently improved by H. Abidi [**1**] for $v^0 \in H^{\frac{1}{2}}$ and external axisymmetric forces $f \in L^2_{loc}(\mathbb{R}_+; H^\beta)$, with $\beta > \frac{1}{4}$. In the case of axisymmetric Euler system, M. Ukhoviskii and V. Yudovich [**21**] proved global well-posedness for initial data $v^0 \in H^s$, with $s > \frac{7}{2}$. This result was relaxed by Yanagisawa [**19**] for Kato's solutions, $v^0 \in H^s, s > \frac{5}{2}$. We point out that their proofs are based on B-K-M criterion. More recently, H. Abidi, T. Hmidi and S. Keraani [**2**] proved a similar result for critical Besov spaces $B^{1+\frac{3}{p}}_{p,1}$, with $1 \le p \le \infty$. To overcome the non validity of B-K-M criterion they use the special geometric structure of the vorticity leading to a new decomposition of the vorticity. This allows them to bound Lipschitz norm of the velocity.

In this paper we study the persistence of the Besov regularity $B^{1+\frac{3}{p}}_{p,1}$ for Navier-Stokes solutions uniformly with respect to the viscosity. The inviscid limit problem is also treated. We notice that this problem was studied by Majda for smooth initial data in all dimensions, see [**16**]. In space dimension two we refer to the papers of T. Hmidi and S. Keraani [**17, 18**] where they proved the uniform persistence in critical Besov spaces $B^{1+\frac{2}{p}}_{p,1}, p \in [1,\infty]$.

Here are the main results of this paper:

THEOREM 3.1 (Uniform boundedness of the velocity). *Let $p \in [1,+\infty]$ and v_0 be an axisymmetric divergence free vector-field. Assume that*

(A1) $v_0 \in B^{1+\frac{3}{p}}_{p,1}$,
(A2) $\frac{\omega_0}{r} \in L^{3,1}$.

Then there exists a unique global solution $v_\nu \in \mathscr{C}\left(\mathbb{R}_+; B^{1+\frac{3}{p}}_{p,1}\right)$ to the Navier-Stokes system, such that

$$\|v_\nu(t)\|_{B^{1+\frac{3}{p}}_{p,1}} \le C_0 e^{e^{\exp C_0 t}},$$

where C_0 depends only on the initial data and not on the viscosity.

The proof relies on the uniform estimate of the Lipschitz norm of the velocity. For this

REMARK 3. *For $p \in [1,3[$ the second condition (A2) is a consequence of the first one (A1). More precisely, we have*

$$\Big\|\frac{\omega}{r}\Big\|_{L^{3,1}} \leq C\|v\|_{B^{1+\frac{3}{p}}_{p,1}}.$$

Our second main result deals with the inviscid limit, namely we have:

THEOREM 3.2 (Rate convegence). *Let v_ν and v be respectively the solution of Navier-Stokes and Euler systems with the same initial data $v^0 \in B^{1+\frac{3}{p}}_{p,1}$. Then we have the rate of convergence*

$$\|v_\nu - v\|_{B^0_{\max(p,3),1}} \leq C_0 e^{\exp C_0 t}(\nu t)^{\frac{1}{2}+\frac{3}{2\max(p,3)}}, \quad p \in [1,\infty].$$

We use for the proof the uniform bounds in Besov spaces combined with some smoothing effects on the viscous vorticity.
The paper is organized as follows: section 2 is devoted to some basic tools: we introduce the functional framework needed for the proofs and we recall some useful lemmas. We discuss in section 3 the persistence of some important geometric properties for a vorticity like equation. This part is essential for the proof of the main results. We give in section 4 some a priori estimates and we prove a new decomposition of vorticity which allows us to prove the result of Theorem 1.1. The proof of the inviscid limit is given in section 5. We end this paper by an appendix where we give the proof of a technical lemma.

2. Preliminaries

We recall in this section some functional spaces and tools frequently used in this paper. We begin with the usual Lebesgue space L^p defined as the set of $p-$ integrable functions, endowed with the following norm

$$\|v\|_{L^p} = \Big(\int_{\mathbb{R}^3} |v(x)|^p dx\Big)^{\frac{1}{p}}.$$

We recall now Lorentz spaces.

DEFINITION 3.3. *Let $1 < p < \infty$ and $q \in [1,\infty]$. The Lorentz[1] space $L^{p,q}$ can be defined by the real interpolation theory,*

$$L^{p,q} = [L^1, L^\infty]_{1-\frac{1}{p},q}.$$

The spaces $L^{p,q}$ have the following properties:

1) $L^{p,p} = L^p$,
2) $L^{p,q_0} \hookrightarrow L^{p,q_1}$ for all $1 \leq q_0 \leq q_1 \leq \infty$,
3) $\|uv\|_{p,q} \leq \|u\|_\infty \|v\|_{p,q}$.

Now, we give the Littlewood[2]-Paley operators based on a dyadic partition of the unity, for more details we refer the reader to chapter 5, especially to the proposition 5.1 of paragraph 5.1.2.

PROPOSITION 3.4. *There exists two radial functions $\chi \in \mathscr{D}(\mathbb{R}^3)$ and $\varphi \in \mathscr{D}(\mathbb{R}^3 \backslash \{0\})$ such that*

(i) $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}^3$,

(ii) $\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}^3 \backslash \{0\}$,

(iii) $|q' - q| \geq 2 \Rightarrow \operatorname{supp} \varphi(2^{-q'} \cdot) \cap \operatorname{supp} \varphi(2^{-q} \cdot) = \varnothing$,

(iv) $q \geq 1 \Rightarrow \operatorname{supp} \chi \cap \operatorname{supp} \varphi(2^{-q} \cdot) = \varnothing$.

DEFINITION 3.5. *For every v in $\mathscr{S}'$, we define the Littlewood-Paley operators by*

$$\Delta_{-1}v = \chi(D)v; \quad \forall q \in \mathbb{N} \quad \Delta_q v = \varphi(2^{-q}D)v, \quad S_q v = \sum_{-1 \leq j \leq q-1} \Delta_j v.$$

Consequently

$$\begin{aligned} \Delta_{-1}v(x) = g \star v(x) &= \int_{\mathbb{R}^3} g(y)v(x-y)dy \\ \forall q \in \mathbb{N}, \quad \Delta_q v(x) = 2^{3q}h(2^q.) \star v(x) &= 2^{3q} \int_{\mathbb{R}^3} h(2^q y)v(x-y)dy, \end{aligned}$$

where $\hat{g} \equiv \chi$ and $\hat{h} \equiv \varphi$. Let us notice that the operators Δ_q and S_q maps continuously L^p into itself uniformly on q and p. The homogeneous operators $\dot{\Delta}_q$ and $\dot{S}_q$ are defined by

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q = \varphi(2^q D)u, \quad \dot{S}_q = \sum_{j \leq q-1} \dot{\Delta}_j v.$$

Now, we will recall the definition of the Besov spaces.

DEFINITION 3.6. *Let $s \in \mathbb{R}, p, r \in [1, \infty]$. The inhomogeneous Besov space $B^s_{p,r}$ (resp. the homogeneous Besov space $\dot{B}^s_{p,r}$) is the set of all tempered distributions $v \in \mathscr{S}'$ (resp. $v \in \mathscr{S}'_{|P}$) such that*

$$\|v\|_{B^s_{p,r}} \overset{def}{=} \Big(2^{qs}\|\Delta_q v\|_{L^p}\Big)_{\ell^r} < \infty.$$

$$\Big(resp. \ \|v\|_{\dot{B}^s_{p,r}} \overset{def}{=} \Big(2^{qs}\|\dot{\Delta}_q v\|_{L^p}\Big)_{\ell^r(\mathbb{Z})} < \infty\Big).$$

We have denoted by P the set of polynomials.

Let us recall the Bony decomposition **[2]**. For $u, v \in \mathscr{S}'$. The product of uv is formally defined by

$$uv = T_u v + T_v u + R(u,v),$$

where

$$T_u v \overset{def}{=} \sum_q S_{q-1}\Delta_q v, \quad R(u,v) \overset{def}{=} \sum_q \Delta_q u \widetilde{\Delta}_q v,$$

with $\widetilde{\Delta}_q = \sum_{i=l}^{-1} \Delta_{q+l}$. The terms $T_u v$ and $T_v u$ are called paraproducts and the third term $R(u,v)$ is the remainder. For a detailed explanation of this subject, we consult the section 5. 3 of chapter 5.

We need the following mixed spaces. For $T > 0$ and $q \geq 1$. We define the spaces $L^q_T B^s$

$$\widetilde{L^a_T}B^s_{p,r} = \Big\{v : [0,T] \to \mathscr{S}'; \|v\|_{\widetilde{L^a_T}B^s_{p,r}} \stackrel{def}{=} \big(2^{qs}\|\Delta_q v\|_{L^a_T L^p}\big)_{\ell^r} < \infty\Big\}.$$

We have the following embeddings:

$$\begin{cases} L^a_T B^s_{p,r} \hookrightarrow \widetilde{L^a_T}B^s_{p,r} & \text{if } a \leq r, \\ \widetilde{L^a_T}B^s_{p,r} \hookrightarrow L^a_T B^s_{p,r} & \text{if } a \geq r. \end{cases}$$

In addition, we have the interpolation result: let $T > 0$, $s_1 < s < s_2$ and $\kappa \in (0,1)$ such that $s = \kappa s_1 + (1-\kappa)s_2$. Then we have

$$\|v\|_{\widetilde{L^a_T}B^s_{p,r}} \leq C\|v\|^{\kappa}_{\widetilde{L^a_T}B^{s_1}_{p,\infty}}\|v\|^{1-\kappa}_{\widetilde{L^a_T}B^{s_2}_{p,\infty}}. \tag{2.1}$$

Next, we state the following proposition which deals with the persistence of Besov regularities in a transport-diffusion equation.

PROPOSITION 3.7. *Let v be a smooth divergence free vector-field and f be a smooth solution of the transport-diffusion equation*

$$\begin{cases} \partial_t f - \nu\Delta f + v\cdot\nabla f = g \\ f_{|t=0} = f_0 \end{cases} \tag{TD$_\nu$}$$

where $f_0 \in B^s_{p,r}$, $g \in L^1_{loc}(\mathbb{R}_+; B^s_{p,r})$ and $(s,r,p) \in]-1,1[\times[1,\infty]^2$. Then we have for $t \geq 0$,

$$\|f(t)\|_{B^s_{p,r}} \leq Ce^{CV(t)}\Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|g(\tau)\|_{B^s_{p,r}}d\tau\Big),$$

with

$$V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau$$

and C a constant which depends only on s and not on the viscosity. For the limit case

$$s = -1, r = \infty \text{ and } p \in [1,\infty] \quad \text{or} \quad s = 1, r = 1 \text{ and } p \in [1,\infty]$$

the above estimate remains true despite we change $V(t)$ by $Z(t) \stackrel{def}{=} \|v\|_{L^1_t B^1_{\infty,1}}$. In addition if $f = \operatorname{curl} v$, then the above estimate holds true for all $s \in [1,+\infty[$.

PROOF. We will only restrict ourselves to the proof of the limiting cases $s = \mp 1$. The remainder cases are done for example in **[2]**. First, let $q \in \mathbb{N}\cup\{-1\}$ and define $f_q \stackrel{def}{=} \Delta_q$, $g_q \stackrel{def}{=} \Delta_q g$. Then applying the Littlewood-Paley block to the equation, we get

$$\begin{aligned} \partial_t f_q + (v\cdot\nabla)f_q - \nu\Delta f_q &= g_q + (v\cdot\nabla)f_q - (v\cdot\nabla f_q) \\ &= \Delta_q g - [\Delta_q, v\cdot\nabla]f. \end{aligned}$$

Multiplying the above equation by $|f_q|f_q$ and using Hölder inequalities we shall have:

$$\|f_q\|_{L^p} \leq \|f^0_q\|_{L^p} + \int_0^t \|g_q(\tau)\|_{L^p}d\tau + \int_0^t \Big\|[\Delta_q, v\cdot\nabla]f\Big\|_{L^p}d\tau.$$

According to Bony's decomposition, the commutator is given by:

where $T_v^{'}u$ stands for $T_v u + \mathscr{R}(v,u)$. To treat the first member $\mathscr{M}_q^1$, we write from the definition

$$\mathscr{M}_q^1 = \sum_{k\geq q-3} \Delta_q \partial_j (\Delta_k f \widetilde{\Delta}_k v^j).$$

Therefore in view of Bernstein inequalities of Chapter 5, Lemma 5.1 we get for $s=-1$,

$$\sup_{q\geq -1} 2^{-q} \|\mathscr{M}_q^{-1}\|_{L^p} \lesssim \|f\|_{\mathscr{B}^{-1}_{p,\infty}} \|v\|_{\mathscr{B}^1_{p,\infty}}. \tag{2.2}$$

To estimate $\mathscr{M}_q^2$, we write

$$\mathscr{M}_q^2 = \Delta T_{\partial_j f} v^j = \sum_{|q-k|\leq 4} \Delta_q (S_{k-1}\partial_j f \Delta_k v^j).$$

The Bernstein and Young inequalities leads to

$$\begin{aligned} \sup_q 2^{-q} \|\mathscr{M}_q^2\|_{L^p} &\lesssim \sup_q 2^{-q} \|S_{q-1} f\|_{L^p} 2^q \|\Delta_q v^j\|_{L^p} \\ &\lesssim \|v\|_{\mathscr{B}^1_{\infty,\infty}} \sup_q \sum_{-1\leq m\leq q-2} 2^{m-q} 2^{-m} \|\Delta_m f\|_{L^p} \\ &\lesssim \|f\|_{\mathscr{B}^{-1}_{p,\infty}} \|v\|_{\mathscr{B}^1_{\infty,\infty}}. \end{aligned} \tag{2.3}$$

Concerning the member $\mathscr{M}_q$, we obviously check that can be rewritten as follows

$$\mathscr{M}_q^3 = T_{\Delta_q \partial_j f} v^j = \sum_{k\geq q-2} S_{k+2} \Delta_q \partial_j \Delta_k v^j.$$

We apply once again the Bernstein inequality, we shall have

$$2^{-q} \|\mathscr{M}_q^3\|_{L^\infty} \lesssim 2^{-q} \|\Delta_q\|_{L^p} \sum_{k\geq q-2} 2^{q-k} 2^k \|\Delta_k v\|_{L^\infty},$$

Therefore the convolution inequality yields

$$\sup_{q\geq -1} 2^{-q} \|\mathscr{M}_q^3\|_{L^p} \lesssim \|f\|_{\mathscr{B}^{-1}_{p,\infty}} \|v\|_{\mathscr{B}^1_{p,\infty}}. \tag{2.4}$$

For the last member we write

$$\mathscr{M}_q^4 = [\Delta_q, T_{v^j}]\partial_j f = \sum_{|k-q|\leq 4} [\Delta_q, S_{k-1} v^j] \Delta_k \partial_j f.$$

The following is classical (see for example **[6]**),

$$\begin{aligned} \|[\Delta_q, S_{k-1} v^j]\Delta_k \partial_j f\|_{L^\infty} &\lesssim 2^{-q} \|\nabla S_{k-1} v\|_{L^\infty} \|\partial_j \Delta_k f\|_{L^p} \\ &\lesssim 2^{k-q} \|\nabla v\|_{L^\infty} \|\Delta_k f\|_{L^p}. \end{aligned}$$

This implies

$$\sup_{q\geq -1} 2^{-q} \|\mathscr{M}\|_{L^\infty} \lesssim \|f\|_{\mathscr{B}^s_{p,\infty}} \|\nabla v\|_{L^\infty}. \tag{2.5}$$

Putting together the estimates (2.2), (2.3), (2.4) and (2.5), we get

$$\sup\, 2^{-q} \Big\|[\Delta_q, v\cdot\nabla] f\Big\|_{L^p} \lesssim \|f\|_{\mathscr{B}^{-1}_{p,\infty}} \|v\|_{\mathscr{B}^1_{p,\infty}}$$

To conclude the desired result it suffices to apply the Grönwall's[3] inequality.
Let us now move to the case $s=1$ that will briefly explained. We estimate $\mathscr{M}_q^1$ as follows:

$$\begin{aligned}\sum_q 2^q\|\mathscr{M}_q^1\| &\lesssim \sum_{k\geq q-3} 2^{q-k}2^k\|\Delta_k f\|_{L^p}2^k\|\widetilde{\Delta}_k v^j\|_{L^\infty}\\ &\lesssim \|f\|_{\mathscr{B}^1_{p,1}}\|v\|_{\mathscr{B}^1_{\infty,\infty}}.\end{aligned}$$

For the second member we have

$$\begin{aligned}\sum_q 2^q\|\mathscr{M}_q^2 &\lesssim \sum_q 2^q\|S_{q-1}\partial_j f\|_{L^p}\|\Delta_q v^j\|_{L^\infty}\\ &\lesssim \|\nabla f\|_{L^p}\|v\|_{\mathscr{B}^1_{\infty,1}}\\ &\lesssim \|f\|_{\mathscr{B}^1_{p,1}}\|v\|_{\mathscr{B}^1_{\infty,1}}\end{aligned}$$

The third and the last members are treated in the same way to the first one. The proof is completed □

In the sequel, we denote by C a harmless constant whose value may vary from line to line. The notation $X\lesssim Y$ means that $X\leq CY$ for some constant C.

3. Study of a vorticity like equation

In this section we study some geometrical properties of any solution satisfying a vorticity like equation given by:

$$\begin{cases}\partial_t\Gamma-\nu\Delta\Gamma+(v\cdot\nabla)\Gamma=(\Gamma\cdot\nabla)v\\ \Gamma_{|t=0}=\Gamma^0,\end{cases}\tag{3.1}$$

where $\Gamma=(\Gamma_1,\Gamma_2,\Gamma_3)$ is an unknown vector-valued function. Our main result in this section reads as follows:

PROPOSITION 3.8. *Let v be an axisymmetric smooth vector-field with zero divergence and Γ be the solution of* (3.1) *with smooth initial data Γ^0. Then we have the following properties:*

(i) *if* $\operatorname{div}\Gamma_0=0$, *then*

$$\operatorname{div}\Gamma(t)=0,\quad\forall t\in\mathbb{R}_+;$$

(ii) *if* $\Gamma^0=\Gamma^0_\theta(r,z)\,\vec{e}_\theta$, *then*

$$\Gamma(t,x)=\Gamma_\theta(t,r,z)\,\vec{e}_\theta,\quad\forall t\in\mathbb{R}_+;$$

(iii) *under assumption (ii) we have* $\Gamma_1(t,x_1,0,z)=\Gamma_2(t,0,x_2,z)$ *and*

$$\partial_t\Gamma_\theta+(v\cdot\nabla)\Gamma_\theta-\nu\left[\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r-\frac{1}{r^2}\right]\Gamma_\theta=\frac{v^r}{r}\Gamma_\theta.$$

PROOF. **(i)** We apply the divergence operator to the equation (3.1) and by an easy computations, using the incompressibility of v, we get

$$\begin{cases}\partial_t\operatorname{div}\Gamma-\nu\Delta\operatorname{div}\Gamma+(v\cdot\nabla)\operatorname{div}\Gamma=0\end{cases}\tag{3.2}$$

From maximum principle, we obtain

$$\|\mathrm{div}\Gamma\|_{L^\infty} \leq \|\mathrm{div}\Gamma^0\|_{L^\infty}.$$

This gives the desired result.
(ii) Let $(\Gamma_r, \Gamma_\theta, \Gamma_z)$ denote the coordinates of Γ in cylindrical basis. The result will be done in two steps: we show first that the cylindrical components of Γ do not depend on the angular parameter θ. We prove in the second one that the components Γ_r and Γ_z are zero. To establish the first point it is enough to prove that (3.1) is stable under rotation transforms. For this purpose we will check that for every $\alpha \in \mathbb{R}$, the quantity $\Gamma_\alpha(t,x) = \mathcal{R}_\alpha^{-1}\Gamma(t, \mathcal{R}_\alpha x)$ satisfies also (3.2). Here $\mathcal{R}_\alpha$ is a rotation with angle α and axis (oz), i.e.

$$\mathcal{R}_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is well-known that the operator Δ commutes with rotations, i.e.,

$$\Delta(\Gamma(t, \mathcal{R}_\alpha x)) = (\Delta\Gamma)(t, \mathcal{R}_\alpha x).$$

This gives,

$$\Delta\Gamma_\alpha(t,x) = \mathcal{R}_\alpha^{-1}(\Delta\Gamma)(t, \mathcal{R}_\alpha x). \tag{3.3}$$

For the advection term we write

$$(\mathcal{R}_\alpha^{-1}(v\cdot\nabla\Gamma))(t, \mathcal{R}_\alpha x) = (v \cdot \nabla\, \mathcal{R}_\alpha^{-1}\Gamma)(t, \mathcal{R}_\alpha x).$$

On the other hand we have

$$v(t,x)\cdot\nabla\,(\mathcal{R}_\alpha^{-1}\Gamma(t, \mathcal{R}_\alpha x)) = \Big(\mathcal{R}_\alpha v(t, \mathcal{R}_\alpha^{-1}x)\cdot\nabla\mathcal{R}_\alpha^{-1}\Gamma\Big)(t, \mathcal{R}_\alpha(x)).$$

As the velocity is axisymmetric then

$$v(t,x)\cdot\nabla\,\Gamma_\alpha(t,x) = \big(v\cdot\nabla\mathcal{R}_\alpha^{-1}\Gamma\big)(t, \mathcal{R}_\alpha(x)).$$

Combining these estimates we find

$$(\mathcal{R}_\alpha^{-1}(v\cdot\nabla\Gamma))(t, \mathcal{R}_\alpha x) = v(t,x)\cdot\nabla\,\Gamma_\alpha(t,x). \tag{3.4}$$

For the stretching term we write by the same way as before

$$\begin{aligned} (\mathcal{R}_\alpha^{-1}(\Gamma\cdot\nabla v))(t, \mathcal{R}_\alpha x) &= (\Gamma\cdot\nabla\,\mathcal{R}_\alpha^{-1}v)(t, \mathcal{R}_\alpha x) \\ &= \big(\Gamma\cdot\nabla\, v(t, \mathcal{R}_\alpha^{-1}x)\big)(t, \mathcal{R}_\alpha x) \\ &= \Gamma_\alpha\cdot\nabla\, v(t,x). \end{aligned} \tag{3.5}$$

Plugging together the equation (3.1) with the identities (3.3), (3.4) and (3.5) we thus get

$$\partial_t\Gamma_\alpha + v\cdot\nabla\,\Gamma_\alpha - \nu\Delta\Gamma_\alpha = \Gamma_\alpha\cdot\nabla v.$$

Since $\Gamma^0(x) = \Gamma^0(x)$ then by uniqueness we get $\Gamma_\alpha(t,x) = \Gamma(t,x)$. This shows that the

Since v is axisymmetric then we get by straightforward computations

$$\begin{aligned}(v\cdot\nabla\Gamma)\cdot\vec{e}_r &= v_r\partial_r\Gamma_r + v_z\partial_z\Gamma_r\\ &= v\cdot\nabla\Gamma_r,\end{aligned}$$

$$\begin{aligned}(\Gamma\cdot\nabla v)\cdot\vec{e}_r &= \Gamma_r\partial_r v_r + \frac{1}{r}\Gamma_\theta\partial_\theta v_r + \Gamma_z\partial_z v_r\\ &= \Gamma_r\partial_r v_r + \Gamma_z\partial_z v_r.\end{aligned}$$

For the dissipative term we have by definition and by the first step,

$$\begin{aligned}\nu\Delta\Gamma\cdot\vec{e}_r &= \nu\Big[\partial_r^2\Gamma\cdot\vec{e}_r + \frac{1}{r}\partial_r\Gamma_r\cdot\vec{e}_r + \frac{1}{r^2}\partial_\theta^2\Gamma\cdot\vec{e}_r + \partial_z^2\Gamma\cdot\vec{e}_r\Big]\\ &= \nu\Big[\partial_r^2(\Gamma\cdot\vec{e}_r) + \frac{1}{r}\partial_r(\Gamma\cdot\vec{e}_r) + \frac{1}{r^2}\Big(\partial_\theta^2(\Gamma\cdot\vec{e}_r) + (\Gamma\cdot\vec{e}_r)\\ &\qquad -2\partial_\theta\Gamma\cdot\vec{e}_\theta\Big) + \partial_z^2(\Gamma\cdot\vec{e}_r)\Big]\\ &= \nu\Big[\partial_r^2\Gamma_r + \frac{1}{r}\partial_r\Gamma_r + \frac{1}{r^2}\partial_\theta^2\Gamma_r - \frac{1}{r^2}\Gamma_r - \frac{2}{r^2}\partial_\theta\Gamma_\theta + \partial_z^2\Gamma_r\Big]\\ &= \nu\Big[\Delta\Gamma_r - \frac{1}{r^2}\Gamma_r\Big].\end{aligned}$$

It follows that,

$$\begin{cases}\partial_t\Gamma_r + v\cdot\nabla\Gamma_r - \nu\big[\Delta\Gamma_r - \frac{1}{r^2}\Gamma_r\big] = \Gamma_r\partial_r v_r + \Gamma_z\partial_z v_r\\ \Gamma_{r|t=0} = \Gamma_r^0.\end{cases}\tag{3.6}$$

By the same method, we can find that the component Γ_z satisfies the following equation:

$$\begin{cases}\partial_t\Gamma_z + v\cdot\nabla\Gamma_z - \nu\Delta\Gamma_z = \Gamma_r\partial_r v_z + \Gamma_z\partial_z v_z\\ \Gamma_{z|t=0} = 0.\end{cases}\tag{3.7}$$

We multiply (3.6) by $|\Gamma_r|^{p-2}\Gamma_r$, integrating by parts and using the fact that div $v=0$,

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\Gamma_r(t)\|_{L^p}^p &+ \nu(p-1)\int_{\mathbb{R}^3}|\nabla\Gamma_r|^2|\Gamma_r|^{p-2}dx + \nu\int_{\mathbb{R}^3}\frac{|\Gamma_r|^p}{r^2}dx\\ &\le \int_{\mathbb{R}^3}|\Gamma_r|^p\partial_r v_r dx + \int_{\mathbb{R}^3}\Gamma_z|\Gamma_r|^{p-2}\Gamma_r\partial_z v_r dx\\ &\le \Big(\|\Gamma_r\|_{L^p}^p + \|\Gamma_z\|_{L^p}\|\Gamma_r\|_{L^p}^{p-1}\Big)\|\nabla v\|_{L^\infty},\end{aligned}$$

where we have used Hölder inequality. This gives

$$\|\Gamma_r(t)\|_{L^p} \le \|\Gamma_r^0\|_{L^p} + \int_0^t\big(\|\Gamma_r(\tau)\|_{L^p} + \|\Gamma_z(\tau)\|_{L^p}\big)\|\nabla v(\tau)\|_{L^\infty}d\tau.$$

Applying the same argument to (3.7) we get,

$$\|\Gamma_z(t)\|_{L^p} \le \|\Gamma_z^0\|_{L^p} + \int_0^t\big(\|\Gamma_r(\tau)\|_{L^p} + \|\Gamma_z(\tau)\|_{L^p}\big)\|\nabla v(\tau)\|_{L^\infty}d\tau.$$

It suffices now to use Gronwall's lemma leading for every $p\in[2,\infty]$,

(iii) The first statement is a direct consequence of $\Gamma \wedge \vec{e}_\theta = \vec{0}$. Let's give the equation governing the angular component Γ_θ. By easy computations one has

$$\begin{aligned}\Delta\Gamma \cdot \vec{e}_\theta &= \Delta(\Gamma_\theta \vec{e}_\theta) \cdot \vec{e}_\theta\\ &= \Delta\Gamma_\theta - \frac{\Gamma_\theta}{r^2},\end{aligned}$$

and

$$(v \cdot \nabla\Gamma) \cdot \vec{e}_\theta = v \cdot \nabla\Gamma_\theta, \quad (\Gamma \cdot \nabla v) \cdot \vec{e}_\theta = \frac{v_r}{r}\Gamma_\theta.$$

Then taking the angular component in the system (3.1) we get

$$\left\{\begin{array}{l}\partial_t\Gamma_\theta + v \cdot \nabla\Gamma_\theta - \nu\big[\Delta\Gamma_\theta - \frac{\Gamma_\theta}{r^2}\big] = \frac{v_r}{r}\Gamma_\theta.\\ \Gamma_{\theta|t=0} = \Gamma_\theta^0.\end{array}\right. \tag{3.8}$$

This achieves the proof.

□

We need the following properties of the vorticity, see for example **[2]**.

PROPOSITION 3.9. *Assume that v is an axisymmetric vector-field with zero divergence and $\omega = \nabla \wedge v$ its vorticity. Then the following properties hold true:*

(i)′ *the vector ω satisfies*

$$\vec{\omega} \wedge \vec{e}_\theta = \vec{0};$$

in particular, for every (x_1, x_2, z) in $\mathbb{R}^3$ we have

$$\omega_3 = 0, \quad \omega_1(x_1, 0, z) = \omega_2(0, x_2, z) = 0;$$

(ii)′ *for every $q \geq -1$, $\Delta_q v$ is axisymmetric and*

$$\Delta_q\omega \wedge \vec{e}_\theta = \vec{0}.$$

4. Proof of Theorem 1.1

4.1. Some a priori estimates.

In this section we give some elementary estimates.

PROPOSITION 3.10. *Let v be an axisymmetric solution of Navier-Stokes system. Then we have for all $t \in \mathbb{R}_+$*

(i) $\left\|\frac{\omega(t)}{r}\right\|_{L^{3,1}} \leq \left\|\frac{\omega_0}{r}\right\|_{L^{3,1}}$;

(ii) $\left\|\frac{v_r(t)}{r}\right\|_{L^\infty} \leq C\left\|\frac{\omega_0}{r}\right\|_{L^{3,1}}$;

(iii) $\|\omega(t)\|_{L^\infty} \leq C\|\omega_0\|_{L^\infty} e^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}$;

(iv) $\|v(t)\|_{L^\infty} \leq C\big(\|v_0\|_{L^\infty} + \|\omega_0\|_{L^\infty}\big) e^{\exp(Ct\|\frac{\omega_0}{r}\|_{L^{3,1}})}$.

The constant C does not depend on the viscosity.

PROOF. **(i)** We set $\eta = \frac{\omega_\theta}{r}$, then we have

By interpolation we get for $1 < p < \infty$ and $q \in [1, \infty]$,

$$\|\eta(t)\|_{L^{p,q}} \le \|\eta_0\|_{L^{p,q}}.$$

(ii) We use the following inequality due to T. Shirota and T. Yanagisawa **[19]**

$$\frac{|v^r|}{r} \lesssim \frac{1}{|.|^2} \star \Big|\frac{\omega_\theta}{r}\Big|.$$

As $\frac{1}{|.|^2} \in L^{\frac{3}{2},\infty}$, then from the convolution laws $L^{p,q} \star L^{p',q'} \to L^\infty$, we have

$$\Big\|\frac{v_r(t)}{r}\Big\|_{L^\infty} \lesssim \Big\|\frac{\omega(t)}{r}\Big\|_{L^{3,1}}.$$

It suffices now to combine this estimate with (i).
(iii) Since ω satisfies

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega - \nu\Delta\omega = \dfrac{v_r}{r}\omega \\ \omega_{|t=0} = \omega^0. \end{cases}$$

Then maximum principle and the estimate of (ii) yields

$$\begin{aligned} \|\omega(t)\|_{L^\infty} &\le \|\omega^0\|_{L^\infty} + \int_0^t \Big\|\frac{v_r(\tau)}{r}\omega(\tau)\Big\|_{L^\infty} d\tau \\ &\le \|\omega_0\|_{L^\infty} + \Big\|\frac{\omega_0}{r}\Big\|_{L^{3,1}} \int_0^t \|\omega(\tau)\|_{L^\infty} d\tau. \end{aligned}$$

This gives in view of Granwall's inequality

$$\|\omega\|_{L^\infty} \le \|\omega_0\|_{L^\infty} e^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}.$$

The desired estimate is then proved.
(iv) We will use an argument due to P. Serfati **[18]** and applied for Euler case. From homogeneous Littlewood-Paley decomposition,

$$\|v(t)\|_{L^\infty} \le \|\dot S_{-N} v\|_{L^\infty} + \sum_{q\ge -N} \|\dot\Delta_q v\|_{L^\infty},$$

where N is a parameter that will be judiciously chosen later. Using Bernstein's inequality we get,

$$\sum_{q\ge -N} \|\dot\Delta v\|_{L^\infty} \lesssim 2^N \|\omega\|_{L^\infty}. \tag{4.1}$$

Since $\dot S_{-N} v$ satisfies the equation

$$(\partial_t - \nu\Delta)\dot S_{-N} v = -\mathbb{P}\big(v\cdot\nabla v\big),$$

then we get easily

$$\|\dot S_{-N} v\|_{L^\infty} \le \|\dot S_{-N} v_0\|_{L^\infty} + \int_0^t \|\dot S_{-N}\mathbb{P}(v\cdot\nabla v)(\tau) d\tau\|_{L^\infty},$$

where $\mathbb{P}$ is Leray's projector of degree zero and $\dot\Delta_q\mathbb{P}$ maps continuously L^∞ into itself.

We combine (4.1) and (4.2), we get

$$\|v(t)\|_{L^\infty} \lesssim \|v_0\|_{L^\infty} + 2^N\|\omega\|_{L^\infty} + 2^{-N}\int_0^t \|v(\tau)\|^2_{L^\infty} d\tau. \tag{4.3}$$

If we choose N such that

$$2^{2N} \approx 1 + \|\omega\|^{-1}_{L^\infty}\int_0^t \|v(\tau)\|^2_{L^\infty} d\tau,$$

then the estimate (4.3) becomes

$$\|v(t)\|^2_{L^\infty} \lesssim \|v_0\|^2_{L^\infty} + \|\omega(t)\|^2_{L^\infty} + \|\omega(t)\|_{L^\infty}\int_0^t \|v(\tau)\|^2_{L^\infty} d\tau.$$

We apply again the Granwall inequality

$$\|v(t)\|_{L^\infty} \lesssim \big(\|v_0\|_{L^\infty} + \|\omega(t)\|_{L^\infty_t L^\infty}\big) e^{Ct\|\omega\|_{L^\infty_{t,x}}}. \tag{4.4}$$

Inserting the estimate (iii) of Proposition 3.10 into (4.4),

$$\|v(t)\|_{L^\infty} \lesssim \big(\|v_0\|_{L^\infty} + \|\omega_0\|_{L^\infty}\big) e^{\exp Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}.$$

The proof is now completed. □

4.2. Vorticity decomposition and Lipschitz bound. The following result is the main step to bound the Lipschitz norm of the velocity. We will establish a new decomposition of the vorticity based on the special structure of axisymmetric flows. We mention that this result were first proved for Euler case **[2]** and we generalize it here for the viscous case uniformly with respect to the viscosity.

PROPOSITION 3.11. *Let ω be the vorticity of the viscous axisymmetric solution. Then there exists a decomposition $\{\tilde{\omega}_q\}_{q\geq -1}$ of the vorticity ω such that for every $t \in \mathbb{R}_+$,*

- (B1) $\omega(t,x) = \sum_{q\geq -1} \tilde{\omega}_q(t,x)$;
- (B2) $\operatorname{div} \tilde{\omega}_q(t,x) = 0$;
- (B3) $\forall q \geq -1,\ \|\tilde{\omega}_q(t)\|_{L^\infty} \leq \|\Delta_q\omega_0\|_{L^\infty} e^{Ct\|\frac{\omega_0}{r}\|_{L^{3,1}}}$;
- (B4) *there exists a constant $C > 0$ independent on the viscosity such that for*

every $k, q \geq -1$

$$\|\Delta_k\tilde{\omega}_q(t)\|_{L^\infty} \leq C2^{-|k-q|}e^{CZ(t)}\|\Delta_q\omega_0\|_{L^\infty},$$

with $Z(t) \overset{def}{=} \|v\|_{L^1_t B^1_{\infty,1}}$.

PROOF. For $q \geq -1$ we define $\tilde{\omega}_q$ as the solution of the following linear Cauchy problem

$$\begin{cases} \partial_t\tilde{\omega}_q - \nu\Delta\tilde{\omega}_q + (v\cdot\nabla)\tilde{\omega}_q = \tilde{\omega}_q\cdot\nabla v \\ \tilde{\omega}_q|_{t=0} = \Delta_q\omega^0. \end{cases} \tag{4.5}$$

Since div $\Delta_q \omega^0 = 0$, then Proposition 4.12 gives div $\tilde{\omega}_q(t) = 0$. From Proposition 3.9, we have $\Delta_q \omega^0 \wedge \vec{e}_\theta = \vec{0}$. It follows from Proposition 4.12 that this property is preserved in time and

$$\begin{cases} \partial_t \tilde{\omega}_q - \nu \Delta \tilde{\omega}_q + (v \cdot \nabla) \tilde{\omega}_q = \frac{v^r}{r} \tilde{\omega}_q \\ \tilde{\omega}_q|_{t=0} = \Delta_q \omega_0. \end{cases} \tag{4.6}$$

Applying the maximum principle, we obtain

$$\|\tilde{\omega}_q(t)\|_{L^\infty} \leq \|\Delta_q \omega_0\|_{L^\infty} + \int_0^t \left\| \frac{v^r}{r} \right\|_{L^\infty} \|\tilde{\omega}_q\|_{L^\infty} d\tau.$$

Therefore we get from Gronwall's lemma and (ii) of Proposition 3.10,

$$\|\tilde{\omega}_q(t)\|_{L^\infty} \leq \|\Delta_q \omega_0\|_{L^\infty} e^{Ct \|\frac{\omega^0}{r}\|_{L^{3,1}}}.$$

The proof of (B4) is equivalent to

$$\|\Delta_k \tilde{\omega}_q(t)\|_{L^\infty} \leq C 2^{k-q} e^{CZ(t)} \|\Delta_q \omega_0\|_{L^\infty} \tag{4.7}$$

and

$$\|\Delta_k \tilde{\omega}_q(t)\|_{L^\infty} \leq C 2^{q-k} e^{CZ(t)} \|\Delta_q \omega_0\|_{L^\infty}. \tag{4.8}$$

To prove (4.7) we apply Proposition 3.7,

$$e^{-CZ(t)} \|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}} \leq C\Big(\|\Delta_q \omega_0\|_{B^{-1}_{\infty,\infty}} + \int_0^t e^{-CZ(\tau)} \|\tilde{\omega}_q \cdot \nabla v(\tau)\|_{B^{-1}_{\infty,\infty}} d\tau \Big).$$

According to Bony's decomposition,

$$\tilde{\omega}_q \cdot \nabla v = T_{\tilde{\omega}_q} \cdot \nabla v + T_{\nabla v} \cdot \tilde{\omega}_q + R(\tilde{\omega}_q^i, \partial_i v).$$

Then we have

$$\begin{aligned} \|\tilde{\omega}_q \cdot \nabla v\|_{B^{-1}_{\infty,\infty}} &\leq \|T_{\tilde{\omega}_q} \cdot \nabla v\|_{B^{-1}_{\infty,\infty}} + \|T_{\nabla v} \cdot \tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}} + \|R(\tilde{\omega}_q \cdot, \nabla v)\|_{B^{-1}_{\infty,\infty}} \\ &\lesssim \|\nabla v\|_{L^\infty} \|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}} + \|R(\tilde{\omega}_q^i, \partial_i v)\|_{B^{-1}_{\infty,\infty}}. \end{aligned}$$

Using (B2) we obtain

$$\begin{aligned} \|R(\tilde{\omega}_q^i, \partial_i v)\|_{B^{-1}_{\infty,\infty}} &= \|\partial_i R(\tilde{\omega}_q^i, v)\|_{B^{-1}_{\infty,\infty}} \\ &\lesssim \sup_k \sum_{j \geq k-3} \|\Delta_j \tilde{\omega}_q\|_{L^\infty} \|\tilde{\Delta}_j v\|_{L^\infty} \\ &\lesssim \|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}} \|v\|_{B^1_{\infty,1}}. \end{aligned}$$

Consequently,

$$\|\tilde{\omega}_q \cdot \nabla v\|_{B^{-1}_{\infty,\infty}} \lesssim \|v\|_{B^1_{\infty,1}} \|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}}.$$

We finally obtain

$$e^{-CZ(t)} \|\tilde{\omega}_q\|_{B^{-1}_{\infty,\infty}} \lesssim \|\Delta_q \omega_0\|_{B^{-1}_{\infty,\infty}} + \int_0^t \|v(\tau)\|_{B^1_{\infty,1}} e^{-CZ(\tau)} \|\tilde{\omega}_q(\tau)\|_{B^{-1}_{\infty,\infty}} d\tau.$$

It follows that

$$\|\Delta_k\tilde{\omega}_q(t)\|_{L^\infty} \leq C2^{k-q}e^{CZ(t)}\|\Delta_q\omega_0\|_{L^\infty}. \tag{4.9}$$

Let us now move to the estimate (4.8). As $v^\theta = 0$, then

$$\frac{v^r}{r} = \frac{v^1}{x_1} = \frac{v^2}{x_2},$$

where $(v^1, v^2, 0)$ is the components of v in cartesian basis. According to Proposition 4.12 the vector-valued solution $\tilde{\omega}_q$ has two components in cartesian basis $\tilde{\omega}_q^1$ and $\tilde{\omega}_q^2$. We restrict ourselves to the proof of the estimate of the first component. The second one is done by the same way. We have

$$\begin{cases} \partial_t\tilde{\omega}_q^1 - \nu\Delta\tilde{\omega}_q^1 + (v\cdot\nabla)\tilde{\omega}_q^1 = \dfrac{v^2}{x_2}\tilde{\omega}_q^1 \\ \tilde{\omega}_q^1|_{|t=0} = \Delta_q\omega_0^1. \end{cases} \tag{4.10}$$

By Proposition 3.9, we have

$$e^{-CZ(t)}\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}} \lesssim \|\Delta_q\omega_0^1\|_{B^1_{\infty,1}} + \int_0^t e^{-CZ(\tau)}\Big\|v^2\frac{\tilde{\omega}_q^1}{x_2}(\tau)\Big\|_{B^1_{\infty,1}}d\tau. \tag{4.11}$$

For the last term of the right-hand side we write

$$\begin{aligned} \Big\|v^2\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^1_{\infty,1}} &\leq \Big\|T_{\frac{\tilde{\omega}_q^1}{x_2}}v^2\Big\|_{B^1_{\infty,1}} + \Big\|T_{v^2}\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^1_{\infty,1}} + \Big\|R\Big(v^2, \frac{\tilde{\omega}_q^1}{x_2}\Big)\Big\|_{B^1_{\infty,1}} \\ &\overset{def}{=} F_1 + F_2 + F_3. \end{aligned} \tag{4.12}$$

To estimate F_1 we use the definitions of paraproducts and Besov spaces

$$F_1 \lesssim \sum_{k\geq -1} 2^k\Big\|S_{k-1}\Big(\frac{\tilde{\omega}_q^1}{x_2}\Big)\Big\|_{L^\infty}\|\Delta_k v^2\|_{L^\infty} \lesssim \|v\|_{B^1_{\infty,1}}\Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{L^\infty}. \tag{4.13}$$

Similarly we have for (F_3),

$$F_3 \lesssim \sum_{l\geq k-3} 2^k\|\Delta_l v^2\|_{L^\infty}\Big\|\tilde{\Delta}_l\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{L^\infty} \lesssim \|v\|_{B^1_{\infty,1}}\Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{L^\infty}. \tag{4.14}$$

The estimate of F_2 is more subtle,

$$F_2 \lesssim \sum_{l\in\mathbb{N}} 2^l\Big\|S_{l-1}v^2(x)\Delta_l\Big(\frac{\tilde{\omega}_q^1(x)}{x_2}\Big)\Big\|_{L^\infty}. \tag{4.15}$$

It is easily seen that,

$$\begin{aligned} \Big\|S_{l-1}v^2(x)\Delta_l\Big(\frac{\tilde{\omega}_q^1(x)}{x_2}\Big)\Big\|_{L^\infty} &\leq \Big\|S_{l-1}v^2(x)\frac{\Delta_l\tilde{\omega}_q^1(x)}{x_2}\Big\|_{L^\infty} \\ &\quad + \Big\|S_{l-1}v^2(x)\Big[\Delta_l, \frac{1}{x_2}\Big]\tilde{\omega}_q^1\Big\|_{L^\infty}. \end{aligned}$$

By Proposition 3.9, $S_{l}v$ is axisymmetric and $S_{l-1}v^2(0, x_3) = 0$. Then we obtain by

Therefore we get

$$\sum_{l\in\mathbb{N}} 2^l \Big\| S_{l-1}v^2(x)\frac{\Delta_l\tilde{\omega}_q^1(x)}{x_2}\Big\|_{L^\infty} \leq \|\nabla v\|_{L^\infty}\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}}. \tag{4.16}$$

To treat the commutator term we write by definition,

$$\begin{aligned} S_{l-1}v^2(x)\Big[\Delta_l,\frac{1}{x_2}\Big]\tilde{\omega}_q^1 &= \frac{S_{l-1}v^2}{x_2}2^{3l}\int_{\mathbb{R}^3} h(2^l(x-y))(x_2-y_2)\frac{\tilde{\omega}_q^1(y)}{y_2}dy \\ &= 2^{-l}\Big(\frac{S_{l-1}v^2}{x_2}\Big)2^{3l}\tilde{h}(2^l.)\star\Big(\frac{\tilde{\omega}_q^1}{x_2}\Big)(x), \end{aligned}$$

where $\tilde{h}(x) = x_2 h(x)$. The following identity holds true for every $f\in S'(\mathbb{R}^3)$.

$$2^{3l}\tilde{h}(2^l.)\star f = \sum_{|l-k|\leq 1} 2^{3l}\tilde{h}(2^l.)\star\Delta_k f.$$

Indeed, we have $\hat{\tilde{h}} = i\partial_{\xi_2}\hat{h} = i\partial_{\xi_2}\varphi(\xi)$. This implies that $\mathrm{supp}\hat{\tilde{h}}\subset\mathrm{supp}\varphi$, and so

$$2^{3l}\tilde{h}(2^l.)\star\Delta_l f \equiv 0, \text{ for } |l-k|\geq 2.$$

Consequently,

$$\begin{aligned} \sum_{l\in\mathbb{N}} 2^l \Big\|S_{l-1}v^2(x)\Big[\Delta_l,\frac{1}{x_2}\Big]\tilde{\omega}_q^1\Big\|_{L^\infty} &\lesssim \sum_{|l-k|\leq 1}\Big\|\frac{S_{l-1}v^2}{x_2}\Big\|_{L^\infty}\Big\|\Delta_k\Big(\frac{\tilde{\omega}_q^1}{x_2}\Big)\Big\|_{L^\infty} \\ &\lesssim \|\nabla v\|_{L^\infty}\Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^0_{\infty,1}}. \end{aligned} \tag{4.17}$$

Thus it follows from (4.16) and (4.17),

$$F_2 \lesssim \|\nabla v\|_{L^\infty}\Big(\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}} + \Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^0_{\infty,1}}\Big). \tag{4.18}$$

Putting together (4.13), (4.14) and (4.18), we get

$$\Big\|v^2\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^1_{\infty,1}} \lesssim \|\nabla v\|_{B^1_{\infty,1}}\Big(\|\tilde{\omega}_q^1\|_{B^1_{\infty,1}} + \Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^0_{\infty,1}}\Big).$$

Thanks to (4.11) and the above estimate one has

$$\begin{aligned} e^{-CZ(t)}\|\tilde{\omega}_q^1(\tau)\|_{B^1_{\infty,1}} &\lesssim \|\tilde{\omega}_q^1(0)\|_{B^1_{\infty,1}} \\ &+ \int_0^t e^{-CZ(\tau)}\|v(\tau)\|_{B^1_{\infty,1}}\|\tilde{\omega}_q^1(\tau)\|_{B^1_{\infty,1}}d\tau \\ &+ \int_0^t e^{-CZ(\tau)}\|v(\tau)\|_{B^1_{\infty,1}}\Big\|\frac{\tilde{\omega}_q^1(\tau)}{x_2}\Big\|_{B^0_{\infty,1}}d\tau. \end{aligned} \tag{4.19}$$

In order to estimate the quantity $\Big\|\frac{\tilde{\omega}_q^1}{x_2}\Big\|_{B^0_{\infty,1}}$ we will make use of Lemma 3.15 (see appendix): first of all, we have due to Proposition 3.9, $\tilde{\omega}_q^1(x_1,0,z)=0$. Hence we get by

$$\begin{aligned}\Big\|\frac{\tilde\omega_q^1}{x_2}\Big\|_{B^0_{\infty,1}} &\lesssim \int_0^1 \|\partial_y\tilde\omega_q^1\|_{B^0_{\infty,1}}(1-\log\mu)d\mu\\ &\lesssim \|\tilde\omega_q^1\|_{B^1_{\infty,1}}\int_0^1(1-\log\mu)d\mu\\ &\lesssim \|\tilde\omega_q^1\|_{B^1_{\infty,1}}.\end{aligned}$$

Then the estimate (4.19) becomes

$$e^{-CZ(t)}\|\tilde\omega_q^1(\tau)\|_{B^1_{\infty,1}}\lesssim\|\tilde\omega_q^1(0)\|_{B^1_{\infty,1}}+\int_0^t e^{-CZ(\tau)}\|v(\tau)\|_{B^1_{\infty,1}}\|\tilde\omega_q^1(\tau)\|_{B^1_{\infty,1}}d\tau.$$

It follows from Gronwall's lemma that

$$\|\tilde\omega_q^1(\tau)\|_{B^1_{\infty,1}}\leq C\|\tilde\omega_q^1(0)\|_{B^1_{\infty,1}}e^{CZ(t)}.$$

This gives in particular the estimate (4.8)

$$\|\Delta_k\tilde\omega_q(t)\|_{L^\infty}\leq C2^{q-k}e^{CZ(t)}\|\Delta_q\omega_0\|_{L^\infty}.$$

The proof of (B4) is now completed. □

Now, we achieve the proof of Theorem 3.1 by giving the persistence of the initial regularity uniformly on the viscosity.

PROPOSITION 3.12. *Let $p\in[1,\infty]$ and v be the solution of (NS_ν) with initial data $v_0\in B^{1+\frac{3}{p}}_{p,1}$ such that $\frac{\omega_0}{r}\in L^{3,1}$. Then we have:*

(E1) *case $p=+\infty$.*

$$\forall t\geq 0,\quad \|\omega(t)\|_{B^0_{\infty,1}}+\|v(t)\|_{B^1_{\infty,1}}\leq C_0e^{\exp C_0t};$$

(E2) *case $1\leq p<+\infty$.*

$$\forall t\geq 0,\quad \|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}}+\|v(t)\|_{B^{1+\frac{3}{p}}_{p,1}}\leq C_0e^{e^{\exp C_0t}},$$

where the constant C_0 depends on the norm of v_0 but not on the viscosity.

PROOF. To prove (E1), we fix an integer N which will be judiciously choosen later. By virtue of (B1) of Proposition 3.11, we have

$$\begin{aligned}\|\omega(t)\|_{B^0_{\infty,1}} &\leq \sum_j\Big\|\Delta_j\sum_q\tilde\omega_q(t)\Big\|_{L^\infty}\\ &\leq \sum_{|j-q|\geq N}\|\Delta_j\tilde\omega_q(t)\|_{L^\infty}+\sum_{|j-q|<N}\|\Delta_j\tilde\omega_q(t)\|_{L^\infty}\\ &\overset{def}{=} H_1+H_2.\end{aligned}\tag{4.20}$$

To estimate the first member H_1 we apply the last part (B4) of Proposition 3.11

$$H_1\leq 2^{-N}\|\omega_0\|_{B^0_{\infty,1}}e^{CZ(t)}.\tag{4.21}$$

For H_2, we use (B3) of Proposition 3.11

Combining this estimate with (4.21) we obtain

$$H_1 + H_2 \lesssim 2^{-N} e^{CZ(t)} + N e^{C_0 t}.$$

It is enough to take N equals to $[CZ(t)] + 1$,

$$\|\omega(t)\|_{B^0_{\infty,1}} \lesssim \big(Z(t) + 1\big) e^{C_0 t}.$$

On the other hand we have

$$\|v\|_{B^1_{\infty,1}} \lesssim \|v\|_{L^\infty} + \|\omega\|_{B^0_{\infty,1}}.$$

Thus it follows from (iii) and (iv) of Proposition 3.10

$$\|v\|_{B^1_{\infty,1}} \leq C_0 e^{\exp C_0 t} + C_0 e^{C_0 t} \int_0^t \|v(\tau)\|_{B^1_{\infty,1}} d\tau.$$

Hence we get by Gronwall's lemma

$$\|v\|_{B^1_{\infty,1}} \leq C_0 e^{\exp C_0 t},$$

and consequently we have

$$\|\omega\|_{B^0_{\infty,1}} \lesssim C_0 e^{\exp C_0 t}.$$

This gives (E1).

For (E2), we apply the Proposition 3.7 to the vorticity equation

$$e^{-CV(\tau)} \|\omega\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega_0\|_{B^{\frac{3}{p}}_{p,1}} + \int_0^t e^{-CV(\tau)} \|\omega \cdot \nabla v(\tau)\|_{B^{\frac{3}{p}}_{p,1}} d\tau. \tag{4.22}$$

We want to prove that

$$\|\omega \cdot \nabla v(\tau)\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega\|_{B^{\frac{3}{p}}_{p,1}} \|\nabla v\|_{L^\infty}. \tag{4.23}$$

According to Bony's decomposition we have

$$\|\omega \cdot \nabla v(\tau)\|_{B^{\frac{3}{p}}_{p,1}} \leq \|T_{\nabla v} \cdot \omega\|_{B^{\frac{3}{p}}_{p,1}} + \|T_\omega \cdot \nabla v\|_{B^{\frac{3}{p}}_{p,1}} + \|R\big(\omega^i, \partial_i v\big)\|_{B^{\frac{3}{p}}_{p,1}}. \tag{4.24}$$

By definition of $\|R\big(\omega^i, \partial_i v\big)\|_{B^{\frac{3}{p}}_{p,1}}$, we have

$$\begin{aligned}
\|R\big(\omega, \nabla v\big)\|_{B^{\frac{3}{p}}_{p,1}} &\leq \sum_{q \in \mathbb{N}} 2^{q\frac{3}{p}} \sum_{j \geq q-3} \|\Delta_j \omega\|_{L^p} \|\Delta_j \nabla v\|_{L^\infty} \\
&\lesssim \|\nabla v\|_{L^\infty} \sum_{j \geq q-3} 2^{(q-j)\frac{3}{p}} 2^{j\frac{3}{p}} \|\Delta_j \omega\|_{L^p} \\
&\lesssim \|\nabla v\|_{L^\infty} \|\omega\|_{B^{\frac{3}{p}}_{p,1}}.
\end{aligned}$$

For the third term we write,

$$\begin{aligned}
\|T_\omega\cdot\nabla v\|_{B^{\frac{3}{p}}_{p,1}} &\lesssim \sum_{q\in\mathbb{N}} 2^{q\frac{3}{p}}\|S_{q-1}\omega\|_{L^\infty}\|\nabla\Delta_q v\|_{L^p}\\
&\lesssim \|\omega\|_{L^\infty}\sum_{q\in\mathbb{N}} 2^{q\frac{3}{p}}\|\Delta_q\omega\|_{L^p}\\
&\lesssim \|\nabla v\|_{L^\infty}\|\omega\|_{B^{\frac{3}{p}}_{p,1}}.
\end{aligned}$$

Thus we get

$$e^{-CV(t)}\|\omega\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega_0\|_{B^{\frac{3}{p}}_{p,1}} + \int_0^t e^{-CV(t)}\|\nabla v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{B^{\frac{3}{p}}_{p,1}}\,d\tau.$$

Using Gronwall's lemma we get,

$$\|\omega\|_{B^{\frac{3}{p}}_{p,1}} \le \|\omega_0\|_{B^{\frac{3}{p}}_{p,1}} e^{CV(t)} \le C_0 e^{e^{\exp C_0 t}}.$$

By definition,

$$\begin{aligned}
\|v(t)\|_{B^{1+\frac{3}{p}}_{p,1}} &\lesssim \|\Delta_{-1}v\|_{B^{\frac{3}{p}}_{p,1}} + \sum_{q\in\mathbb{N}} 2^{q\frac{3}{p}}2^q\|\Delta_q v\|_{L^p}\\
&\lesssim \|v(t)\|_{L^p} + \|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}}.
\end{aligned}$$

It remains to estimate $\|v(t)\|_{L^p}$. Since for $p\in]1,\infty[$, Riesz transforms map continuously L^p into itself, then

$$\begin{aligned}
\|v(t)\|_{L^p} &\le \|v^0\|_{L^p} + \int_0^t \|v(\tau)\cdot\nabla v(\tau)\|_{L^p}d\tau\\
&\lesssim \|v^0\|_{L^p} + \int_0^t \|v(\tau)\|_{L^p}\|\nabla v(\tau)\|_{L^\infty}d\tau.
\end{aligned}$$

Applying again Gronwall's lemma,

$$\begin{aligned}
\|v(t)\|_{L^p} &\lesssim \|v^0\|_{L^p}e^{CV(t)}\\
&\le C_0 e^{e^{\exp C_0 t}}.
\end{aligned}$$

For the case $p=1$ we write

$$\begin{aligned}
\|v(t)\|_{L^1} &\le \|\dot S_0 v(t)\|_{L^1} + \sum_{q\ge 0}\|\dot\Delta_q v(t)\|_{L^1}\\
&\lesssim \|\dot S_0 v(t)\|_{L^1} + \sum_{q\ge 0} 2^{-q}\|\dot\Delta_q \nabla v(t)\|_{L^1}\\
&\lesssim \|\dot S_0 v(t)\|_{L^1} + \|\omega(t)\|_{L^1} \overset{def}{=} Y_1 + Y_2.
\end{aligned}$$

To estimate Y_1 we use the Navier-Stokes equations,

$$Y_1 \le \|\dot S_0 v^0\|_{L^1} + \sum_{q\le -1}\|\dot\Delta_q\mathbb{P}\big((v\cdot\nabla)v(t)\big)\|_{L^1}$$

For Y_2, we have

$$Y_2 \leq \|\omega\|_{B^0_{1,1}} \leq \|\omega\|_{B^3_{1,1}}.$$

Combining Y_1 and Y_2, we get

$$\|v(t)\|_{L^1} \leq C_0 e^{e^{\exp C_0 t}}.$$

This finishes the proof. □

5. The Rate Convergence

Before stating the proof of Theorem 3.2, we need the following lemmata, for more details we refer to T. Hmidi and S. Keraani **[18]**.

LEMMA 3.13. *Let $s \in]-1,1[$, $(p_1, p_2, a) \in [1,\infty]^3$ and $v \in L^1\big(\mathbb{R}_+; Lip(\mathbb{R}^3)\big)$ be a divergence free vector-field. Then, there exists a constant C which depends on s and such that the following holds true: let f to be a smooth solution of (TD_ν), then for all $t \in \mathbb{R}_+$*

$$\nu^{\frac{1}{a}} \|f\|_{\widetilde{L^a_t} B^{s+\frac{2}{a}}_{p_1,p_2}} \leq Ce^{CV(t)}(1+\nu t)^{\frac{1}{a}}\Big(\|f_0\|_{B^s_{p_1,p_2}} + \int_0^t \|g(\tau)\|_{B^s_{p_1,p_2}} d\tau\Big),$$

where $V(t) = \|\nabla v\|_{L^1_t L^\infty}$ and C is a constant that does not depend on the viscosity.

LEMMA 3.14. *Let $v \in B^0_{p,1}$ be a divergence free vector-field and $w \in B^1_{\infty,1}$. Then we have*

(1) $\|v \cdot \nabla w\|_{\dot{B}^0_{p,1}} \lesssim \|v\|_{B^0_{p,1}} \|w\|_{B^1_{\infty,1}}$.

Besides, if $v \equiv w$ then we have

(2) $\|v \cdot \nabla v\|_{\dot{B}^0_{p,1}} \lesssim \|v\|_{L^p} \|v\|_{B^1_{\infty,1}}$.

PROOF OF THEOREM 3.2. We distinguish in the proof three cases: $p < 3$, $p = 3$ and $3 < p$.

• Case $p \in]3,\infty]$. We set $z_\nu = v_\nu - v$ and $P_\nu = p_\nu - p$. We can easily check that z_ν satisfies the system

$$(\widetilde{\mathrm{NS}}_\nu) \qquad \begin{cases} \partial_t z_\nu + (v_\nu \cdot \nabla) z_\nu = \nu \Delta v_\nu - (z_\nu \cdot \nabla) v + \nabla P_\nu \\ z_{|t=0} = 0. \end{cases}$$

By virtue of Proposition 3.7 and Proposition 3.12, we have

$$\begin{aligned} \|z_\nu\|_{B^0_{p,1}} &\leq C_0 e^{e^{\exp C_0 t}} \bigg(\int_0^t \Big(\nu \|\Delta v_\nu(\tau)\|_{B^0_{p,1}} \\ &\qquad + \|z_\nu \cdot \nabla v(\tau)\|_{B^0_{p,1}} + \|\nabla P_\nu(\tau)\|_{B^0_{p,1}} \Big) \bigg) d\tau \\ &\overset{def}{=} C_0 e^{e^{\exp C_0 t}} (X_1 + X_2 + X_3). \end{aligned}$$

Therefore we get from the interpolation inequality (2.1),

$$\begin{aligned} X_1 &\leq C\|\nu\omega_\nu\|_{\widetilde{L_t^1}B^1_{p,1}} \\ &\leq C\|\nu\omega_\nu\|^{\frac12+\frac{3}{2p}}_{\widetilde{L_t^1}B^{\frac3p}_{p,\infty}}\|\nu\omega_\nu\|^{\frac12-\frac{3}{2p}}_{\widetilde{L_t^1}B^{2+\frac3p}_{p,\infty}}. \end{aligned}$$

For the first term of the right-hand side, we apply Hölder inequality and Proposition 3.12,

$$\begin{aligned} \|\nu\omega_\nu\|^{\frac12+\frac{3}{2p}}_{\widetilde{L_t^1}B^{\frac3p}_{p,\infty}} &\leq (\nu t)^{\frac12+\frac{3}{2p}}\|\omega_\nu\|^{\frac12+\frac{3}{2p}}_{\widetilde{L_t^\infty}B^{\frac3p}_{p,\infty}} \\ &\leq C_0(\nu t)^{\frac12+\frac{3}{2p}}e^{e^{\exp C_0 t}}. \end{aligned} \tag{5.1}$$

For the second term we use Lemma 4.7,

$$\|\nu\omega_\nu\|_{\widetilde{L_t^1}B^{2+\frac3p}_{p,\infty}} \leq C(1+\nu t)e^{CV_\nu(t)}\Big(\|\omega^0_\nu\|_{B^{\frac3p}_{p,\infty}}+\|\omega_\nu\cdot\nabla v_\nu\|_{L^1_t B^{\frac3p}_{p,1}}\Big). \tag{5.2}$$

We have the law product

$$\|\omega_\nu\cdot\nabla v_\nu\|_{B^{\frac3p}_{p,1}} \lesssim \|\omega_\nu\|_{B^{\frac3p}_{p,1}}\|v_\nu\|_{B^{1+\frac3p}_{p,1}}.$$

Indeed, for $p<\infty$ this is a direct consequence of the algebra structure of the space $B^{\frac3p}_{p,1}$. But for $p=\infty$, we use the incompressibility of the vorticity (div $\omega_\nu=0$). Thanks to (E2) of Proposition 3.12 and (5.1),

$$\|\nu\omega_\nu\|_{\widetilde{L_t^1}B^{2+\frac3p}_{p,\infty}} \leq C_0 e^{e^{\exp C_0 t}}(1+\nu t). \tag{5.3}$$

Thus we get,

$$X_1 \leq C_0 e^{e^{\exp C_0 t}}(\nu t)^{\frac12+\frac{3}{2p}}(1+\nu t)^{\frac12-\frac{3}{2p}}. \tag{5.4}$$

Concerning X_2, we use (1) of Lemma 3.14 it follows,

$$X_2 \lesssim \|z_\nu\|_{B^0_{p,1}}\|v\|_{B^1_{\infty,1}} \tag{5.5}$$

Applying the differential operator div to $(\widetilde{\mathrm{NS}}_\nu)$, we deduce

$$-\Delta P_\nu = \operatorname{div}(z_\nu\cdot\nabla(v_\nu+v)).$$

Since $\dot B^0_{p,1}\hookrightarrow B^0_{p,1}$ and Riesz transforms act continuously on homogeneous Besov spaces,

$$\|\nabla P_\nu\|_{B^0_{p,1}} \leq C\|z_\nu\cdot\nabla(v_\nu+v)\|_{\dot B^0_{p,1}}.$$

Combining this estimate with Lemma 3.14 and Proposition 3.12, we find

$$\|\nabla P_\nu\|_{B^0_{p,1}} \leq C_0 e^{\exp C_0 t}\|z_\nu\|_{B^0_{p,1}}.$$

Inserting (5.4), (5.5) and (5.6) in (5.1), we get

$$\|z_\nu\|_{B^0_{p,1}} \lesssim C_0 e^{e^{\exp C_0 t}} (\nu t)^{\frac{1}{2}+\frac{3}{2p}} (1+\nu t)^{\frac{1}{2}-\frac{3}{2p}} + C_0 e^{\exp C_0 t} \int_0^t \|z_\nu(\tau)\|_{B^0_{p,1}} d\tau.$$

Gronwall's lemma yields

$$\|z_\nu\|_{B^0_{p,1}} \leq C_0 e^{e^{\exp C_0 t}} (\nu t)^{\frac{1}{2}+\frac{3}{2p}}.$$

• Case $p = 3$. We reproduce the same calculus by changing only the interpolation inequality before (5.1) by

$$\|\nu\omega\|_{\widetilde{L^1} B^1_{3,1}} \leq (\nu t) \|\omega_\nu\|_{\widetilde{L^\infty} B^1_{3,1}}.$$

We get finally

$$\|z_\nu\|_{\widetilde{L^1} B^1_{3,1}} \leq C_0 e^{e^{\exp C_0 t}} (\nu t).$$

• For $1 \leq p < 3$, we use the Besov embedding $B^{1+\frac{3}{p}}_{p,1} \hookrightarrow B^1_{3,1}$. It follows then

$$\|z_\nu\|_{B^0_{3,1}} \leq C_0 e^{e^{\exp C_0 t}} (\nu t).$$

This achieves the proof of the theorem. □

6. Appendix

We have the following result which was proved in **[2]** and for the convenience of the reader we will give here the proof.

LEMMA 3.15. *Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be a function in $B^0_{\infty,1}$ and taking $h_\mu(x_1, x_2, x_3) = h(\mu x_1, x_2, x_3)$ with $\mu \in]0, 1[$. Then, there exists an absolute constant C such the following inequality holds*

$$\|h_\mu\|_{B^0_{\infty,1}} \leq C(1 - \log\mu) \|h\|_{B^0_{\infty,1}}.$$

PROOF. Let μ in $]0, 1[$ and taking $h_{q,\mu} = (\Delta_q h)_\mu$ for $q \geq -1$. It is obvious that $h_\mu = \sum_{q\geq -1} h_{q,\mu}$. By definition we have,

$$\begin{aligned} \|h_\mu\|_{B^0_{\infty,1}} &= \|\Delta_{-1} h_\mu\|_{L^\infty} + \sum_{j\in\mathbb{N}} \|\Delta_j h_\mu\|_{L^\infty} \\ &\leq C\|h\|_{L^\infty} + \sum_{\substack{j\in\mathbb{N} \\ q\geq -1}} \|\Delta_j h_{q,\mu}\|_{L^\infty}. \end{aligned}$$

For $j, q \in \mathbb{N}$, the Fourier transform of $\Delta_j h_{q,\lambda}$ is supported in the set

$$\Big\{|\xi_1| + |\xi'| \approx 2^j \quad \text{and} \quad \mu^{-1}|\xi_1| + |\xi'| \approx 2^q\Big\}$$

Thus we get for an integer n_1

$$\begin{aligned}\|h_\mu\|_{B^0_{\infty,1}} &\lesssim \|h\|_{L^\infty} + \sum_{\substack{q-n_1+\log\mu\le j\\ j\le q+n_1}} \|\Delta_j h_{q,\mu}\|_{L^\infty}\\ &\lesssim \|f\|_{L^\infty} + (n_1-\log\mu)\sum_q \|h_{q,\mu}\|_{L^\infty}\\ &\lesssim \|f\|_{L^\infty} + (n_1-\log\mu)\sum_q \|h_q\|_{L^\infty}\\ &\lesssim (1-\log\mu)\|h\|_{B^0_{\infty,1}}.\end{aligned}$$

□

7. Perspectives

The decomposition of the vorticity argument proposed above allow us to extend to the Navier-Stokes system (resp. Euler sytem) for initial data which satisfies the so-called helical symmetry condition (for more details we refer to A. Dutrifoy [**7**] and recently the paper of B. Ettinger and E.S. Titi [**4**]). Let $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ be a cylindric coordinates in $\mathbb{R}^3$ and v a vector field be such that $v = v_r\vec{e}_r + v_\theta\vec{e}_\theta + v_z\vec{e}_z$. Then, for some positive real number k we have

DEFINITION 3.16. *A vector field v in $\mathbb{R}^3$ is called helical symmetry if*

(i) *the components v_r, v_θ and v_z are constant on the helical $r = r_0, z = z_0 + h\theta$,*
(ii) *at every point of $\mathbb{R}^3$ the vector field v is orthogonal to $\vec{\tau} := r\vec{e}_\theta + h\vec{e}_z$, i.e. $rv_\theta + hv_z = 0$*

An equivalent notion of helical symmetry is given as follows

DEFINITION 3.17. *A vector fields $v : \mathbb{R}^3 \to \mathbb{R}^3$ is helical symmetry, if the following assertion holds*

(i′) *(Helical) $v(\mathcal{H}_\theta x) = \mathcal{R}_\theta v(x)$ for every $\theta \in \mathbb{R}$, where $\mathcal{R}_\theta$ is the $\theta-$ rotation transform defined by*

$$\mathcal{R}_\theta(x) = \begin{pmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

and

$$\mathcal{H}_\theta = \mathcal{R}_\theta(x) + \begin{pmatrix} 0\\ 0\\ hz\end{pmatrix} = \begin{pmatrix} x\cos\theta + x\sin\theta\\ -x\sin\theta + y\cos\theta\\ z + hz\end{pmatrix}.$$

(ii′) *(Symmetry) The vector fields v obeys the following constraint*

$$rv_\theta + hv_z = 0.$$

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CHAPTER 4

On the Global well-posedness of the Euler-Boussinesq system with fractional dissipation

This work is the subject of the following publication :
T. Hmidi and M. Zerguine: *On the global well-posedness of the Euler-Boussinesq system with fractional dissipation*. Article in Press, Physica D.

Abstract. *We study the global well-posedness of the Euler-Boussinesq system with term dissipation $|D|^\alpha$ on the temperature equation. We prove that for $\alpha > 1$ the coupled system has global unique solution for initial data with critical regularities.*
Keywords and phrases. Euler-Boussinesq system, Besov spaces, paradifferential calculus.
2000 Mathematics Subject Classification. 35Q35, 35B65, 76D03.

1. Introduction

In this paper we deal with the two-dimensional *Euler-Boussinesq* system given by

$$(\mathrm{B}_\alpha)\qquad \begin{cases} \partial_t v + v\cdot\nabla v + \nabla \pi = \theta e_2 \\ \partial_t \theta + v\cdot\nabla\theta + \kappa|\mathrm{D}|^\alpha\theta = 0 \\ \mathrm{div} v = 0 \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0. \end{cases}$$

Here, the unknowns are the velocity $v = (v^1, v^2) \in \mathbb{R}^2$, the pressure π and the temperature θ. The vector e_2 is given by $(0,1)$, α is a real number in $]0,2]$ and $\kappa \geq 0$ is called the molecular diffusivity. The fractional Laplacian $|\mathrm{D}|^\alpha$ is defined as follows:

$$|\mathrm{D}|^\alpha f(x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}|\xi|^\alpha \widehat{f}(\xi)d\xi.$$

The fractional Laplacian[1] serves to model many physical phenomena such as overdriven detonations in gases **[9]** or anomalous diffusion in semiconductor growth **[27]**. It is also used in some mathematical models in hydrodynamics, molecular biology and finance mathematics, see **[14, 21, 23]**.
In space dimension two the vorticity is defined by the scalar $\omega = \partial_1 v^2 - \partial_2 v^1$. Thus the system (B_α) can be written under the vorticity-temperature formulation as follows:

$$(1.1)\qquad \begin{cases} \partial_t\omega + v\cdot\nabla\omega = \partial_1\theta \\ \partial_t\theta + v\cdot\nabla\theta + \kappa|\mathrm{D}|^\alpha\theta = 0 \end{cases}$$

In the case of zero diffusivity this system can be seen as an hyperbolic quasi-linear system and thus it is locally well-posed in Sobolev spaces H^s with $s > 2$. Nevertheless the question of whether smooth solutions develop singularities in finite time or not remains till now an outstanding open problem. For $\kappa > 0$ and $\alpha = 2$ the question of global existence is solved recently in a serie of papers **[4, 12, 13, 17]**. In **[4]**, Chae proved the global existence and uniqueness for initial data $(v^0, \theta^0) \in H^s \times H^s$, with $s > 2$. This result has been recently improved in **[17]** by Hmidi and Keraani for initial data $v^0 \in B_{p,1}^{\frac{2}{p}+1}$ and $\theta^0 \in B_{p,1}^{\frac{2}{p}-1} \cap L^r$, with $r \in]2, \infty[$. It seems that the only smoothing effects due to the transport-diffusion equation governing the temperature is sufficient to counterbalance the amplification of the vorticity. More recently, the study of global existence of Yudovich solutions for this system has been done in **[12]**. We mention that in **[13]** Danchin and Paicu proved that if we have only an horizontal viscosity, that is $\partial_{11}^2 \theta$ instead of $\Delta\theta$, then Euler-Boussinesq system admits global unique solution.
In this paper, we aim at solving the question of global existence for less dissipative term $|\mathrm{D}|^\alpha\theta$. As we shall see the difficulty depends on the parameter α and it appears that the system shares some properties with the 2d quasi-geostrophic equation (QG) described by

$$\partial_t\theta + v \cdot \nabla\theta + \kappa|\mathrm{D}|^\alpha\theta = 0, \quad v = (-\partial_2|\mathrm{D}|^{-1}\theta, \partial_1|\mathrm{D}|^{-1}\theta).$$

Indeed, the velocity in the second equation of (B_α) has basically the same regularity as the temperature but it is given through a complex dynamical system. In a similar way to the quasi-geostrophic equation we shall call critical the value $\alpha = 1$. It corresponds to the fact that the likely amplification of the vorticity due to the term $\partial_1\theta$ and the dissipation have the same rate. Thus we expect for the sub-critical case $\alpha > 1$ to have global existence since the dissipation is much stronger than the amplification. This will be the main goal of this paper. We emphasize that our method does not give any answer to the global existence for the critical case. On the other hand the approach developed by Kiselev, Nazarov and Volberg **[22]** to settle global existence for the critical (QG) equation does not work here because the relation between the velocity and the temperature is not local in time. Likewise there is no hope with the method used by Caffarelli and Vasseur **[3]** since we have not sufficient estimates on the velocity like $v \in L^\infty([0, T]; \mathrm{BMO})$. We recall that the BMO space is the set of functions of bounded mean oscillation introduced by John and Nirenberg. Now, we state the main result of this paper:

THEOREM 4.1. *Let* $(\alpha, p) \in]1, 2] \times]1, \infty[$, $v^0 \in B_{p,1}^{1+\frac{2}{p}}$ *be a divergence free vector-field of* $\mathbb{R}^2$ *and* $\theta^0 \in B_{p,1}^{-\alpha+1+\frac{2}{p}} \cap L^r$, *with* $\frac{2}{\alpha-1} < r < \infty$. *Then there exists a unique global solution* (v, θ) *for the system* (B_α) *such that*

$$v \in \mathscr{C}\big(\mathbb{R}_+; B_{p,1}^{1+\frac{2}{p}}\big) \quad \textit{and} \quad \theta \in \mathscr{C}\big(\mathbb{R}_+; B_{p,1}^{-\alpha+1+\frac{2}{p}} \cap L^r\big) \cap L^1_{loc}(\mathbb{R}_+; \mathrm{Lip})$$

REMARK 4. *Notice that the regularity assumption on the velocity is in some sense critical.*

that the incompressible Euler system has global unique solution for initial data lying in theses spaces. It is then legitimate to try to obtain a similar result for our system, which is the subject of this paper.

REMARK 5. *The Besov regularity $B_{p,1}^{-\alpha+1+\frac{2}{p}}$ of the temperature is also optimal with respect to the regularity of the velocity. Indeed, to estimate the quantity $\|v(t)\|_{B_{p,1}^{\frac{2}{p}+1}}$ we need to control $\|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}}$. Now from the maximal smoothing effect of the fractional heat equation the best space of the initial temperature should be $B_{p,1}^{-\alpha+1+\frac{2}{p}}$. It is obvious that when p is sufficiently large then the temperature will not be necessary in any Lebesgue space and thus there is no plain conservation laws. For this reason we need to put the initial temperature in some regular Lebesgue space L^r, with $r > \frac{2}{\alpha-1}$, but we do not know whether we can improve or remove this technical condition.*

The proof of our main theorem relies heavily on some smoothing effects of the transport-diffusion equation governing the evolution of the temperature, see Propositions 4.8 and 4.9. This is the crucial ingredient of the proof and it allows us to control the growth of the vorticity by the quantity $\partial_1\theta$.
Our paper is organized as follows. The second section deals with some basic notions of Littlewood-Paley theory and we recall some useful lemmas. In the third one we are interested in studying a transport-diffusion equation. We prove basically two kinds of estimates: some smoothing effects and a commutator estimate type. The proof of our main result is given in the fourth section.

2. Preliminaries

Throughout this paper, the notation $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$. We denote by C a harmless constant whose value may vary from line to line.
We will gather in this section some definitions and tools frequently used along this paper. We start with the so-called Littlewood-Paley operators which allow us to define the Besov spaces. For the following assertion we can see the Proposition 5.2 given in paragraph 1.2 of chapter 5.

PROPOSITION 4.2. *There exists two radial positive functions $\chi \in \mathscr{D}(\mathbb{R}^2)$ and $\varphi \in \mathscr{D}(\mathbb{R}^2\backslash\{0\})$ such that*

(1) $\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2,$
(2) $\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2\backslash\{0\},$
(3) $|j-q| \geq 2 \Rightarrow supp\ \varphi(2^{-j}\cdot) \cap supp\ \varphi(2^{-q}\cdot) = \varnothing,$
(4) $q \geq 1 \Rightarrow supp\ \chi \cap supp\ \varphi(2^{-q}\cdot) = \varnothing.$

Now we define the the Littlewood-Paley operators as follows: for every tempered

It is easy to see that the operators Δ_q and S_q map continuously L^p into itself uniformly with respect to q and p. We can also define the homogeneous operators $\dot\Delta_q$ and $\dot S_q$

$$\forall q \in \mathbb{Z} \quad \dot\Delta_q = \varphi(2^q D)u \quad \text{and} \quad \dot S_q = \sum_{j \le q-1} \dot\Delta_j v$$

According to **[2]** we can split the product uv into three parts:

$$uv = T_u v + T_v u + R(u, v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad R(u, v) = \sum_q \Delta_q u \tilde\Delta_q v \quad \text{and} \quad \tilde\Delta_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

For $(p, r) \in [1, +\infty]^2$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $B^s_{p,r}$ as the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} \stackrel{def}{=} \Big(2^{qs} \|\Delta_q u\|_{L^p}\Big)_{\ell^r} < +\infty.$$

The homogeneous Besov space $\dot B^s_{p,r}$ is defined as the set of $u \in \mathscr{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot B^s_{p,r}} \stackrel{def}{=} \Big(2^{qs} \|\dot\Delta_q u\|_{L^p}\Big)_{\ell^r(\mathbb{Z})} < +\infty.$$

In the case $(s, p, r) \in]0, 1[\times[1, \infty]^2$ we have an other characterization of the Besov space, (for the proof see **[25]**),

$$C^{-1}\|v\|_{\dot B^s_{p,r}} \le \Big(\int_{\mathbb{R}^d} \frac{\|v(\cdot - x) - v(\cdot)\|^r_{L^p}}{|x|^{sr}} \frac{dx}{|x|^2} \Big)^{\frac{1}{r}} \le C\|v\|_{\dot B^s_{p,r}}. \tag{2.1}$$

Let $T > 0$ and $\rho \ge 1$, we denote by $L^\rho_T B^s_{p,r}$ the space of distributions u such that

$$\|u\|_{L^\rho_T B^s_{p,r}} \stackrel{def}{=} \Big\| \Big(2^{qs} \|\Delta_q u\|_{L^p}\Big)_{\ell^r} \Big\|_{L^\rho_T} < +\infty.$$

We say that u belongs to the space $\widetilde L^\rho_T B^s_{p,r}$ if

$$\|u\|_{\widetilde L^\rho_T B^s_{p,r}} \stackrel{def}{=} \Big(2^{qs} \|\Delta_q u\|_{L^\rho_T L^p}\Big)_{\ell^r} < +\infty.$$

By a direct application of the Minkowski inequality, we have the following links between these spaces. Let $\varepsilon > 0$, then

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde L^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^{s-\varepsilon}_{p,r}, \text{ if } \quad r \ge \rho,$$

$$L^\rho_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde L^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \text{ if } \quad \rho \ge r.$$

LEMMA 4.3. *There exists a constant $C > 0$ such that for $1 \le a \le b \le \infty$ and for every function v and every $q \in \mathbb{Z}$, we have:*

$$\begin{aligned}
\sup_{|\alpha|=k} \|\partial^\alpha S_q v\|_{L^b} &\le C^k 2^{q\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|S_q v\|_{L^a};\\
C^{-k} 2^{qk} \|\dot\Delta_q v\|_{L^a} &\le \sup_{|\alpha|=k} \|\partial^\alpha \dot\Delta_q v\|_{L^a} \le C^k 2^{qk} \|\dot\Delta_q v\|_{L^a}.
\end{aligned}$$

Notice that Bernstein inequalities remain true if we change the derivative ∂^α by the fractional derivative $|\mathrm{D}|^\alpha$. The next proposition deals with some commutator estimates.

PROPOSITION 4.4. *Let u be a smooth function and v be a smooth vector-field of $\mathbb{R}^2$ with zero divergence. Then for every $q \ge -1$, we have*

$$\big\|[\Delta_q, v\cdot\nabla]u\big\|_{L^p} \lesssim \|u\|_{L^p}\Big(\big\|\nabla\Delta_{-1}v\big\|_{L^\infty} + (q+2)\|\mathrm{curl}\, v\|_{L^\infty}\Big).$$

Besides we have for every $s \ge -1$

$$\sum_{q\ge -1} 2^{qs}\|[\Delta_q, v\cdot\nabla]u\|_{L^p} \le \|\nabla v\|_{L^\infty}\|u\|_{B^s_{p,1}} + \|\nabla u\|_{L^\infty}\|v\|_{B^s_{p,1}} 1_{[1,\infty[}(s).$$

The first estimate is proved in **[19]**. However the second one is classical and its proof can be found for example in **[6]**.
Next we recall a logarithmic estimate proven first by Vishik in **[26]** for the particular case of Besov space $B^0_{\infty,1}$. The proof for more general case can be found in **[20]**.

PROPOSITION 4.5. *Let $(p,r) \in [1,\infty]^2$, v be a divergence free vector-field belonging to the space $L^1_{loc}(\mathbb{R}_+; \mathrm{Lip}(\mathbb{R}^2))$ and let a be a smooth solution of the following transport equation,*

$$\begin{cases} \partial_t a + v\cdot\nabla a = f\\ a_{|t=0} = a^0. \end{cases}$$

If the initial data $a^0 \in B^0_{p,r}$, then we have for all $t \in \mathbb{R}_+$

$$\|a\|_{\widetilde L^\infty_t B^0_{p,r}} \lesssim \left(\|a^0\|_{B^0_{p,r}} + \|f\|_{\widetilde L^1_t B^0_{p,r}}\right)\Big(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\Big).$$

Let us now end this section with a classical result about incompressible Euler equation, see for instance **[5]**.

PROPOSITION 4.6. *Let v be a solution of the incompressible Euler system,*

$$\partial_t v + v\cdot\nabla v + \nabla\pi = f, \quad v_{|t=0} = v^0, \quad \mathrm{div} v = 0.$$

Then for $s > -1, (p,r) \in]1,\infty[\times[1,\infty]$ we have

3. Study of a transport-diffusion equation

This section is devoted to some estimates for the following transport-diffusion model

$$(\mathrm{TD}_\alpha)\qquad \begin{cases} \partial_t\theta + v\cdot\nabla\theta + |\mathrm{D}|^\alpha\theta = f \\ \theta_{|t=0} = \theta^0. \end{cases}$$

The first estimate deals with the L^p estimates, see **[11]**.

LEMMA 4.7. *Let $\alpha\in[0,2]$, v be a smooth divergence free vector-field. We assume that θ is a smooth solution of the equation (TD_α). Then for $p\in[1,\infty]$*

$$\|\theta(t)\|_{L^p}\le\|\theta^0\|_{L^p}+\int_0^t\|f(\tau)\|_{L^p}d\tau.$$

Now we intend to discuss some important smoothing effects which are the cornerstone of the proof of Theorem 4.1.

3.1. Smoothing effects. We will discuss here two kinds of smoothing effects. The first one is described by the following proposition.

PROPOSITION 4.8. *Let $\alpha\in[0,2]$, $p\in[2,\infty[$, $\rho\in[1,\infty]$ and v be a smooth divergence free vector-filed of $\mathbb{R}^2$. Let θ be a smooth solution of (TD_α) with a zero force f. Then we have for every $t\ge 0, q\in\mathbb{N}\cup\{-1\}$*

$$2^{q\frac{\alpha}{\rho}}\|\Delta_q\theta\|_{L^\rho_tL^p}\lesssim\|\theta^0\|_{L^p}\Big(1+t+(q+2)\|\mathrm{curl}\,v\|_{L^1_tL^\infty}+\|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big).$$

REMARK 6. *If the velocity belongs to $L^1_t\mathrm{Lip}$ then the previous estimate becomes*

$$2^{q\frac{\alpha}{\rho}}\|\Delta_q\theta\|_{L^\rho_tL^p}\lesssim\|\theta^0\|_{L^p}\big(1+t+\|\nabla v\|_{L^1_tL^\infty}\big).$$

Although we have not a frequency-logarithmic loss in this case, the estimate seems to be not very convenient for our context due to the term $\|\nabla v\|_{L^\infty}$. As we shall see, it is much harder to estimate this quantity rather than the vorticity.

PROOF. First, let $q\in\mathbb{N}^*$ and define $\theta_q\overset{def}{=}\Delta_q\theta$. Then applying the Littlewood-Paley operator Δ_q to the equation, we get

$$\partial_t\theta_q+v\cdot\nabla\theta_q+|\mathrm{D}|^\alpha\theta_q=-\big[\Delta_q,v\cdot\nabla\big]\theta. \tag{3.1}$$

Multiplying the above equation by $|\theta_q|^{p-2}\theta_q$ and using Hölder inequalities we get

$$\frac{1}{p}\frac{d}{dt}\|\theta_q\|_{L^p}^p+\int_{\mathbb{R}^2}(|\mathrm{D}|^\alpha\theta_q)|\theta_q|^{p-2}\theta_q dx\le\|\theta_q\|_{L^p}^{p-1}\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}.$$

Now recall from **[8]** the following generalized Bernstein inequality

$$c2^{q\alpha}\|\theta_q\|_{L^p}^p\le\int_{\mathbb{R}^2}(|\mathrm{D}|^\alpha\theta_q)|\theta_q|^{p-2}\theta_q dx,$$

where the constant c depends on p. Inserting this estimate in the previous one yields

$$\frac{1}{p}\frac{d}{dt}\|\theta_q(t)\|_{L^p}^p+c2^{q\alpha}\|\theta_q(t)\|_{L^p}^p\le\|\theta_q(t)\|_{L^p}^{p-1}\|[\Delta_q,v\cdot\nabla]\theta(t)\|_{L^p}$$

Hence

$$\frac{d}{dt}\Big(e^{ct2^{q\alpha}}\|\theta_q(t)\|_{L^p}\Big)\lesssim e^{ct2^{q\alpha}}\|[\Delta_q,v\cdot\nabla]\theta(t)\|_{L^p}.$$

Integrating in time this differential inequality leads to

$$\|\theta_q(t)\|_{L^p}\le\|\theta_q^0\|_{L^p}e^{-ct2^{q\alpha}}+\int_0^t e^{-c(t-\tau)2^{q\alpha}}\|[\Delta_q,v\cdot\nabla]\theta(\tau)\|_{L^p}d\tau.$$

Recall from Proposition 4.4 that

$$\|[\Delta_q,v\cdot\nabla]\theta\|_{L^p}\le\|\theta\|_{L^p}\big((q+2)\|\omega\|_{L^\infty}+\|\nabla\Delta_{-1}v\|_{L^\infty}\big).$$

Integrating once again in time and using the convolution inequalities we find the desired result. □

The second main goal of this section is to establish another kind of smoothing effects for any solution of (TD_α).

PROPOSITION 4.9. *Let $\rho\in[1,\infty],\alpha\in[0,2],\ p\in[1,\infty],s>-1$ and v be a smooth divergence free vector-field of $\mathbb{R}^2$. Let θ be a smooth solution of (TD_α) with a zero force f. Then for $t\ge0$*

$$\|\theta\|_{\widetilde{L}^\rho_t B^{s+\frac{\alpha}{\rho}}_{p,1}}\le Ce^{CV(t)}\Big(\|\theta^0\|_{B^s_{p,1}}(1+t^{\frac{1}{\rho}})+\int_0^t\Gamma_s(\tau)d\tau\Big),$$

with

$$V(t)\overset{def}{=}\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau,\quad \Gamma_s(t)\overset{def}{=}\|\nabla\theta(t)\|_{L^\infty}\|v(t)\|_{B^s_{p,1}}1_{[1,\infty[}(s).$$

REMARK 7. *Using the method described in the proof of Proposition 4.8 one can establish the above proposition for $p\in[2,\infty[$ but the estimates include some constants which blow up when p goes to infinity. On the other hand the method of* **[8]** *does not work for $p\in[1,2[$ due to some composition laws which are not valid in this case although we expect the final result to be true. In order to cover all the value $p\in[1,\infty]$ we will use a different approach based on the Lagrangian coordinates.*

PROOF. The proof will be done in the spirit of **[1, 15]**. Roughly speaking, it consists first in localizing in frequency the evolution equation and second in rewriting the equation in Lagrangian coordinates. This will lead to some technical difficulties, especially, when we have to treat a commutator term coming from the commutation between the fractional Laplacian and the regularized flows.
Let $q\in\mathbb{N}$ and define $\theta_q\overset{def}{=}\Delta_q\theta$. Then localizing in frequency the equation we get

$$\partial_t\theta_q+S_qv\cdot\nabla\theta_q+|\mathrm{D}|^\alpha\theta_q=(S_qv-v)\cdot\nabla\theta_q-\big[\Delta_q,v\cdot\nabla\big]\theta:=\mathscr{R}_q. \tag{3.2}$$

Applying Lemma 4.7 yields

$$\|\theta(t)\|_{L^p}\le\|\theta_q^0\|_{L^p}+\int_0^t\|\mathscr{R}_q(\tau)\|_{L^p}d\tau.$$

Let us recall from Proposition 4.4 the following estimate

$$\sum_q 2^{qs}\|\mathscr{R}_q(\tau)\|_{L^p} \leq \|\nabla v\|_{L^\infty}\|\theta\|_{B^s_{p,1}} + \|\nabla\theta\|_{L^\infty}\|v\|_{B^s_{p,1}} 1_{[1,\infty[}(s). \tag{3.3}$$

Combining together these estimates and using Gronwall's inequality we thus find

$$\|\theta\|_{\widetilde{L}^\infty_t B^s_{p,1}} \leq Ce^{CV(t)}\Big(\|\theta^0\|_{B^s_{p,1}} + \int_0^t e^{-CV(\tau)}\Gamma_s(\tau)d\tau\Big). \tag{3.4}$$

This achieves the proof for the particular case $\rho = +\infty$. Let us now move to the smoothing effect. We define by ψ_q the flow of the regularized velocity $S_q v$, given by the integral equation,

$$\psi_q(t,x) = x + \int_0^t S_q v(\tau, \psi_q(\tau,x))d\tau.$$

Set

$$\bar{\theta}_q(t,x) = \theta_q(t,\psi_q(t,x)) \quad \text{and} \quad \bar{\mathscr{R}}_q(t,x) = \mathscr{R}_q(t,\psi_q(t,x)).$$

It is easily seen that

$$\partial_t\bar{\theta}_q + |\mathrm{D}|^\alpha\bar{\theta}_q = \bar{\mathscr{R}}_q + |\mathrm{D}|^\alpha\big(\theta_q\circ\psi_q\big) - \big(|\mathrm{D}|^\alpha\theta_q\big)\circ\psi_q \overset{def}{=} \mathscr{R}^1_q. \tag{3.5}$$

We will use the following estimate

$$\||\mathrm{D}|^\alpha\big(\theta_q\circ\psi_q\big) - \big(|\mathrm{D}|^\alpha\theta_q\big)\circ\psi_q\|_{L^p} \leq Ce^{CV(t)}V(t)2^{\alpha q}\|\theta_q\|_{L^p}. \tag{3.6}$$

The proof of this estimate is postponed at the end of this section. Now, since the flow ψ_q preserves Lebesgue measure then we get by (3.6)

$$\big\|\mathscr{R}^1_q(t)\big\|_{L^p} \lesssim e^{CV(t)}V(t)2^{\alpha q}\|\theta_q\|_{L^p} + \|\mathscr{R}_q(t)\|_{L^p}. \tag{3.7}$$

At this stage of the proof one can remark that the function $\bar{\theta}_q$ is not necessarily localized in frequency. Thus in order to quantify the smoothing effects we need once again to localize the equation (3.5). Now, let $j \in \mathbb{N}$ then applying the operator Δ_j to the equation (3.5) yields

$$\partial_t\Delta_j\bar{\theta}_q + |\mathrm{D}|^\alpha\Delta_j\bar{\theta}_q = \Delta_j\mathscr{R}^1_q.$$

The frequency description of the smoothing effect of the fractional heat semigroup can be summarized in the following estimate

$$\|e^{-t|\mathrm{D}|^\alpha}\Delta_j f\|_{L^p} \lesssim e^{-ct2^{j\alpha}}\|\Delta_j f\|_{L^p}.$$

The proof of this inequality can be found for example in **[7]** for the case $\alpha = 2$ and **[18]** for $\alpha \in]0,2[$. Combining this estimate with Duhamel formula and (3.7) one obtains

$$\begin{aligned}\|\Delta_j\bar{\theta}_q(t)\|_{L^p} &\lesssim e^{-ct2^{j\alpha}}\|\Delta_j\theta^0_q\|_{L^p}\\ &+ 2^{q\alpha}e^{CV(t)}V(t)\int_0^t e^{-c(t-\tau)2^{j\alpha}}\|\theta_q(\tau)\|_{L^p}d\tau\\ &+ \int_0^t e^{-c(t-\tau)2^{j\alpha}}\big\|\mathscr{R}_q(\tau)\big\|_{L^p}d\tau.\end{aligned} \tag{3.8}$$

By integrating in time and using convolution inequalities, we get for $j \in \mathbb{N}$

Now, let $N \in \mathbb{N}$ be a fixed number that will be chosen later. Since ψ_q preserves Lebesgue measure then we get

$$\begin{aligned} 2^{q(s+\frac{\alpha}{\rho})}\|\theta_q(t)\|_{L^\rho_t L^p} &= 2^{q(s+\frac{\alpha}{\rho})}\|\bar{\theta}_q(t)\|_{L^\rho_t L^p} \\ &\le 2^{q(s+\frac{\alpha}{\rho})}\Big(\sum_{|j-q|\le N}\|\Delta_j\bar{\theta}_q\|_{L^\rho_t L^p} + \sum_{|j-q|>N}\|\Delta_j\bar{\theta}_q\|_{L^\rho_t L^p}\Big) \\ &\stackrel{def}{=} \mathrm{I}_q + \mathrm{II}_q. \end{aligned}$$

If $q \ge N$, then it follows from (3.9),

$$\mathrm{I}_q \lesssim 2^{qs}\|\theta^0_q\|_{L^p} + e^{CV(t)}V(t)2^{\alpha N}2^{q(s+\frac{\alpha}{\rho})}\|\theta_q\|_{L^\rho_t L^p} + 2^{N\frac{\alpha}{\rho}}2^{qs}\|\mathscr{R}_q\|_{L^1_t L^p}. \tag{3.10}$$

To estimate the second term II_q we use the following result due to Vishik **[26]**

$$\|\Delta_j\bar{\theta}_q\|_{L^p} \lesssim 2^{-|q-j|}e^{CV(t)}\|\theta_q\|_{L^p}.$$

Hence

$$\mathrm{II}_q \lesssim 2^{-N}e^{CV(t)}2^{q(s+\frac{\alpha}{\rho})}\|\theta_q\|_{L^\rho_t L^p}. \tag{3.11}$$

For low frequencies, $q \le N$, we have from Hölder's inequality

$$\sum_{q\le N} 2^{q(s+\frac{\alpha}{\rho})}\|\theta_q\|_{L^\rho_t L^p} \lesssim 2^{N\frac{\alpha}{\rho}}t^{\frac{1}{\rho}}\|\theta\|_{\widetilde{L}^\infty_t B^s_{p,1}}.$$

It suffices now to use (3.4), leading to

$$\sum_{q\le N} 2^{q(s+\frac{\alpha}{\rho})}\|\theta_q\|_{L^\rho_t L^p} \lesssim 2^{N\frac{\alpha}{\rho}}t^{\frac{1}{\rho}}e^{CV(t)}\Big(\|\theta^0\|_{B^s_{p,1}} + \int_0^t e^{-CV(\tau)}\Gamma_s(\tau)d\tau\Big). \tag{3.12}$$

Putting together (3.10), (3.11), (3.3) and (3.12), we get

$$\begin{aligned} \|\theta\|_{\widetilde{L}^\rho_t B^{s+\frac{\alpha}{\rho}}_{p,1}} &\le C\|\theta^0\|_{B^s_{p,1}}(1+2^{N\frac{\alpha}{\rho}}t^{\frac{1}{\rho}}e^{CV(t)}) \\ &+ Ce^{CV(t)}\Big(V(t)\,2^{N\alpha}+2^{-N}\Big)\|\theta\|_{\widetilde{L}^\rho_t B^{s+\frac{\alpha}{\rho}}_{p,1}} \\ &+ C2^{N\frac{\alpha}{\rho}}(1+t^{\frac{1}{\rho}})e^{CV(t)}\int_0^t \Gamma_s(\tau)d\tau. \end{aligned}$$

It is easy to check that there exists two absolute constants $N \in \mathbb{N}$ and $C_1 > 0$ such that

$$V(t) \le C_1 \Rightarrow Ce^{CV(t)}\Big(V(t)2^{\alpha N} + 2^{-N}\Big) \le \frac{1}{2}.$$

Indeed, we start with taking t such that $V(t) \le 1$, which is possible since $\lim_{t\to 0} V(t) = 0$. Next, we choose N in order to have $Ce^C 2^{-N} \le \frac{1}{4}$. Now, we take $V(t)$ sufficiently small such that $Ce^{CV(t)}V(t)2^{\alpha N} \le \frac{1}{4}$. This proves the above assertion. Under this assumption $V(t) \le C_1$, we get

$$\|\theta\|_{\widetilde{L}^\rho_t B^{s+\alpha}_{p,1}} \lesssim (1+t^{\frac{1}{\rho}})\Big(\|\theta^0\|_{B^s_{p,1}} + \int_0^t \Gamma_s(\tau)d\tau\Big).$$

This gives the desired result for small time. In order to get the estimate for arbitrary

Then reproducing the same calculation we get

$$\|\theta\|_{\widetilde{L}^{\rho}([t_i,t_{i+1}]B^{s+\alpha}_{p,1})} \lesssim (1+(t_{i+1}-t_i)^{\frac{1}{\rho}})\Big(\|\theta(t_i)\|_{B^s_{p,1}}+\int_{t_i}^{t_{i+1}}\Gamma_s(\tau)d\tau\Big).$$

Since

$$\|\theta\|_{\widetilde{L}^{\rho}([0,t],B^{s+\alpha}_{p,1})} \leq \sum_{i=0}^{K-1}\|\theta\|_{\widetilde{L}^{\rho}([t_i,t_{i+1}]B^{s+\alpha}_{p,1})},$$

then

$$\|\theta\|_{\widetilde{L}^{\rho}([0,t],B^{s+\alpha}_{p,1})} \leq (1+t^{\frac{1}{\rho}})\Big(\sum_{i=0}^{K-1}\|\theta(t_i)\|_{B^s_{p,1}}+\int_0^t\Gamma_s(\tau)d\tau\Big).$$

It suffices now to combine this estimate with (3.4). □

3.2. Proof of (3.6).

PROOF. Note that the case $\alpha\in]0,1]$ was treated in **[1, 17]**. We will use here the method developed in these papers to extend the estimate for $\alpha\in[1,2]$. The case $\alpha=2$ can be done explicitly by Leibniz formula and some estimates of the flow. It is plain that

$$\begin{aligned}|\mathrm{D}|^{\alpha}(f_q\circ\psi_q)-(|\mathrm{D}|^{\alpha}f_q)\circ\psi_q &= |\mathrm{D}|^{\frac{\alpha}{2}}\{(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\circ\psi_q\}-\{|\mathrm{D}|^{\frac{\alpha}{2}}(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\}\circ\psi_q\\ &\quad+|\mathrm{D}|^{\frac{\alpha}{2}}\{|\mathrm{D}|^{\frac{\alpha}{2}}(f_q\circ\psi_q)-(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\circ\psi_q\}\\ &= \mathrm{I}_q+\mathrm{II}_q.\end{aligned}$$

For the term I_q, it suffices to apply Proposition 3.1 of **[18]**, with $\frac{\alpha}{2}$ and $f=|\mathrm{D}|^{\frac{\alpha}{2}}f_q$. Thus we get,

$$\|\mathrm{I}_q\|_{L^p} \lesssim \max\Big(\big|1-\|\nabla\psi_q^{-1}\|_{L^\infty}^{2+\frac{\alpha}{2}}\big|;\big|1-\|\nabla\psi_q\|_{L^\infty}^{-2-\frac{\alpha}{2}}\big|\Big)\|\nabla\psi_q\|_{L^\infty}^{\frac{\alpha}{2}}\|F_q\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}.$$

The flows $\psi_q^1\overset{def}{=}\psi_q$ and ψ_q^{-1} satisfy the classical estimates

$$e^{-CV(t)}\leq\|\nabla\psi_q^{\pm1}\|_{L^\infty}\leq e^{CV(t)}. \tag{3.13}$$

It follows from Bernstein inequality

$$\|\mathrm{I}_q\|_{L^p}\lesssim e^{CV(t)}(e^{CV(t)}-1)2^{\alpha q}\|f_q\|_{L^p}. \tag{3.14}$$

For the second term we use the following representation of the fractional Laplacian

$$|\mathrm{D}|^{\frac{\alpha}{2}}f(x)=C\int_{\mathbb{R}^2}\frac{f(x)-f(y)}{|x-y|^{2+\frac{\alpha}{2}}}dy.$$

Since the flow ψ_q preserves Lebesgue measure then we get easily

$$\begin{aligned}|\mathrm{D}|^{\frac{\alpha}{2}}(f_q\circ\psi_q)(x)-(|\mathrm{D}|^{\frac{\alpha}{2}}f_q)\circ\psi_q(x) &= C\int_{\mathbb{R}^2}\frac{f_q(\psi_q(x))-f_q(\psi_q(y))}{|x-y|^{2+\frac{\alpha}{2}}}\\ &\quad\times\Big(1-\frac{|x-y|^{2+\frac{\alpha}{2}}}{|\psi_q(x)-\psi_q(y)|^{2+\frac{\alpha}{2}}}\Big)dy.\end{aligned}$$

with
$$\bar{\psi}_q(x,h) = 1 - \frac{|h|^{2+\frac{\alpha}{2}}}{|\psi_q(x) - \psi_q(x-h)|^{2+\frac{\alpha}{2}}}.$$
It is not hard to see from Bony's decomposition that for $s > 0$ we have the following law product:
$$\|fg\|_{\dot{B}^s_{p,1}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,1}} + \|f\|_{\dot{B}^s_{\infty,1}}\|g\|_{L^p}.$$
Combining this estimate with the embedding $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$, we find
$$\begin{aligned}
\|\mathrm{II}_q\|_{L^p} &\leq \big\||\mathrm{D}|^{\frac{\alpha}{2}}(f_q \circ \psi_q) - (|\mathrm{D}|^{\frac{\alpha}{2}} f_q)\circ\psi_q\big\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}} \\
&\leq C\|\bar{\psi}_q\|_{L^\infty(\mathbb{R}^4)}\int_{\mathbb{R}^2}|h|^{-2-\frac{\alpha}{2}}\|g_q(\cdot) - g_q(\cdot - h)\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}dh \\
&+ C\sup_{h\in\mathbb{R}^2}\|\bar{\psi}_q(\cdot,h)\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,1}}\int_{\mathbb{R}^2}|h|^{-2-\frac{\alpha}{2}}\|g_q(\cdot) - g_q(\cdot - h)\|_{L^p}dh \\
&= J^1_q + J^2_q.
\end{aligned}$$
To estimate J^1_q we use the mean value Theorem,
$$\frac{1}{\|\nabla\psi\|_{L^\infty}^{2+\frac{\alpha}{2}}} \leq \frac{|h|^{2+\frac{\alpha}{2}}}{|\psi(x) - \psi(x-h)|^{2+\frac{\alpha}{2}}} \leq \|\nabla\psi^{-1}\|_{L^\infty}^{2+\frac{\alpha}{2}}.$$
Therefore we get by the definition of $\bar{\psi}_q$ and the above estimate,
$$\|\bar{\psi}_q\|_{L^\infty} \leq \max\Big(|1 - \|\nabla\psi_q^{-1}\|_{L^\infty}^{2+\frac{\alpha}{2}}|; |1 - \|\nabla\psi_q\|_{L^\infty}^{-2-\frac{\alpha}{2}}|\Big).$$
It follows from (3.13) that
$$\|\bar{\psi}_q\|_{L^\infty} \leq Ce^{CV(t)}(e^{CV(t)} - 1). \tag{3.15}$$
Using the definition of Besov spaces and the commutation of Δ_j with translation operators one finds
$$\begin{aligned}
&\int_{\mathbb{R}^2}|h|^{-2-\frac{\alpha}{2}}\|g_q(\cdot) - g_q(\cdot - h)\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}dh \\
&\qquad\qquad \leq \sum_j 2^{\frac{\alpha}{2}j}\int_{\mathbb{R}^2}|h|^{-\frac{\alpha}{2}}\|\Delta_j g_q(\cdot) - (\Delta_j g_q)(\cdot - h)\|_{L^p}\frac{dh}{|h|^2}.
\end{aligned}$$
The characterization of Besov spaces (2.1) yields
$$\begin{aligned}
\int_{\mathbb{R}^2}|h|^{-2-\frac{\alpha}{2}}\|g_q(\cdot) - g_q(\cdot - h)\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}dh &\leq C\sum_j 2^{\frac{\alpha}{2}j}\|\Delta_j g_q\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}} \\
&\leq C\sum_{|j-k|\leq 1} 2^{j\frac{\alpha}{2}}2^{\frac{\alpha}{2}k}\|\Delta_j\Delta_k g_q\|_{L^p} \\
&\leq C\|g_q\|_{\dot{B}^{\alpha}_{p,1}}.
\end{aligned}$$
Now we use the following interpolation result,
$$\|g_q\|_{\dot{B}^{\alpha}_{p,1}} \lesssim \|g_q\|_{L^p}^{1-\frac{\alpha}{2}}\|\Delta g_q\|_{L^p}^{\frac{\alpha}{2}}.$$

Applying Bernstein inequality and (3.13) we get

$$\|\Delta g_q\|_{L^p} \lesssim e^{CV(t)} 2^{2q} \|f_q\|_{L^p} + 2^q \|f_q\|_{L^p} \|\Delta \psi_q\|_{L^\infty}.$$

The derivative of the flow equation with respect to x and the use of Gronwall and Bernstein inequalities give

$$\begin{aligned} \|\nabla^2 \psi_q(t)\|_{L^\infty} &\lesssim e^{CV(t)} \int_0^t \|\nabla^2 S_q v(\tau)\|_{L^\infty} d\tau \\ &\lesssim e^{CV(t)} V(t) 2^q. \end{aligned} \tag{3.16}$$

Combining both last estimates we obtain

$$\|\Delta g_q\|_{L^p} \lesssim e^{CV(t)} 2^{2q} \|f_q\|_{L^p}. \tag{3.17}$$

Putting together (3.15) and (3.17)

$$\|J_q^1(t)\|_{L^p} \lesssim e^{CV(t)} (e^{CV(t)} - 1) 2^{q\alpha} \|f_q\|_{L^p}.$$

Let us now turn to the second term J_q^2. The integral term can be estimated from (2.1) as follows

$$\int_{\mathbb{R}^2} |h|^{-2-\frac{\alpha}{2}} \|g_q(\cdot) - g_q(\cdot - h)\|_{L^p} dh \lesssim \|g_q\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}}.$$

Using the composition result proven in **[24]**

$$\begin{aligned} \|g_q(t)\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}} &\lesssim \|\nabla \psi_q\|_{L^\infty}^{\frac{\alpha}{2}} \|f_q\|_{\dot{B}^{\frac{\alpha}{2}}_{p,1}} \\ &\lesssim e^{CV(t)} 2^{q\frac{\alpha}{2}} \|f_q\|_{L^p}. \end{aligned} \tag{3.18}$$

In order to estimate $\bar{\psi}_q$ we use the interpolation inequality

$$\|\bar{\psi}_q(\cdot, h)\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,1}} \lesssim \|\bar{\psi}_q(\cdot, h)\|_{L^\infty}^{1-\frac{\alpha}{2}} \|\nabla_x \bar{\psi}_q(\cdot, h)\|_{L^\infty}^{\frac{\alpha}{2}}.$$

This leads in view of (3.15) to

$$\|\bar{\psi}_q(\cdot, h)\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,1}} \leq C e^{CV(t)} \{V(t)\}^{1-\frac{\alpha}{2}} \|\nabla_x \bar{\psi}_q(\cdot, h)\|_{L^\infty}^{\frac{\alpha}{2}}. \tag{3.19}$$

The derivative of $\bar{\psi}_q$ with respect to x yields

$$\begin{aligned} |\nabla_x \bar{\psi}_q(x, h)| &\lesssim \frac{|h|^{3+\frac{\alpha}{2}}}{|\psi_q(x) - \psi_q(x-h)|^{3+\frac{\alpha}{2}}} \frac{|\nabla_x \psi_q(x) - \nabla_x \psi_q(x-h)|}{|h|} \\ &\lesssim \|\nabla \psi_q^{-1}\|_{L^\infty}^{3+\frac{\alpha}{2}} \|\nabla^2 \psi_q\|_{L^\infty}. \end{aligned}$$

Combining (3.13) and (3.16), we obtain

$$\|\nabla_x \bar{\psi}_q(t)\|_{L^\infty(\mathbb{R}^4)} \lesssim e^{CV(t)} V(t)\, 2^q. \tag{3.20}$$

Plugging (3.20) into (3.19) we find

$$\|\bar{\psi}_q(\cdot, h)\|_{\dot{B}^{\frac{\alpha}{2}}_{\infty,1}} \lesssim e^{CV(t)} V(t)\, 2^{q\frac{\alpha}{2}}. \tag{3.21}$$

This achieves the proof of the desired estimate. □

4. Proof of Theorem 4.1

The aim of this section is to prove our main theorem. It will be done in several steps. In the first step we establish some significant *a priori* estimates. In the second one we prove the uniqueness part and the construction of the solution is described in the third step. the last step is devoted to the continuity-in-time of the solution.

4.1. a Priori Estimates. The *a priori* estimates will be described in several propositions. We start with the following one,

PROPOSITION 4.10. *Let $\alpha \in]1,2]$, $(p,r) \in [1,\infty[\times]\frac{2}{\alpha-1},\infty[$ and define $\bar{p} = \max\{p,r\}$. If $\omega^0 \in L^\infty \cap L^p$ and $\theta^0 \in L^r$ then any smooth solution of the Boussinesq system (B_α) satisfies*

(1)
$$\|\theta(t)\|_{L^r} \leq \|\theta^0\|_{L^r};$$

(2)
$$\|\omega(t)\|_{L^\infty\cap L^{\bar{p}}} + \|\nabla\theta\|_{L^1_tL^\infty} \leq C_0 e^{C_0 t}.$$

PROOF. To prove the inequality (1) it is enough to apply Lemma 4.7. Notice that we do not have any restriction on the value of r for this estimate. To establish the second estimate (2), we start with the vorticity equation

$$\partial_t\omega + v\cdot\nabla\omega = \partial_1\theta.$$

It is clear that

$$\|\omega(t)\|_{L^\infty} \leq \|\omega^0\|_{L^\infty} + \|\nabla\theta\|_{L^1_tL^\infty}.$$

Using the classical embedding $B^{1+\frac{2}{r}}_{r,1} \hookrightarrow \mathrm{Lip}(\mathbb{R}^2)$, one can easily obtain

$$\|\omega(t)\|_{L^\infty} \lesssim \|\omega^0\|_{L^\infty} + \|\theta\|_{L^1_tB^{1+\frac{2}{r}}_{r,1}}. \tag{4.1}$$

Combining now Proposition 4.9 with Bernstein inequalities, we get for $\epsilon > 0$

$$\begin{aligned}\|\theta\|_{\widetilde{L}^1_tB^{\alpha-\epsilon}_{r,\infty}} &\lesssim \|\theta^0\|_{L^r}\Big(1+t+\|\omega\|_{L^1_tL^\infty} + \|\Delta_{-1}\nabla v\|_{L^1_tL^\infty}\Big)\\ &\lesssim \|\theta^0\|_{L^r}\Big(1+t+\|\omega\|_{L^1_tL^\infty} + \|\nabla v\|_{L^1_tL^{\bar{p}}}\Big),\end{aligned}$$

with $\bar{p} \stackrel{def}{=} \max\{p,r\}$. Take ϵ such that $1+\frac{2}{r} < \alpha-\epsilon$, then we have $\widetilde{L}^1_tB^{\alpha-\epsilon}_{r,\infty} \hookrightarrow L^1_tB^{1+\frac{2}{r}}_{r,1}$. Thus we find,

$$\|\theta\|_{L^1_tB^{1+\frac{2}{r}}_{r,1}} \lesssim \|\theta^0\|_{L^r}\Big(1+t+\|\omega\|_{L^1_tL^\infty} + \|\nabla v\|_{L^1_tL^{\bar{p}}}\Big).$$

On the other hand, the classical Calderón-Zygmund estimate $\|\nabla v\|_{L^{\bar{p}}} \approx \|\omega\|_{L^{\bar{p}}}$ yields

$$\|\theta\|_{L^1_tB^{1+\frac{2}{r}}_{r,1}} \lesssim \|\theta^0\|_{L^r}\Big(1+t+\|\omega\|_{L^1_tL^\infty} + \|\omega\|_{L^1_tL^{\bar{p}}}\Big). \tag{4.2}$$

The estimate of the $L^{\bar{p}}$ norm of the vorticity can be done similarly to the L^∞ estimate

Set $f(t) \stackrel{def}{=} \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}$, then combining (4.1), (4.2) and (4.3) leads to

$$f(t) \lesssim \|\theta^0\|_{L^r}(1+t+t\|\omega^0\|_{L^\infty\cap L^{\bar{p}}}) + \|\theta^0\|_{L^r}\int_0^t f(\tau)d\tau.$$

It follows from Gronwall's inequality that,

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \leq C_0 e^{C_0 t}, \tag{4.4}$$

where C_0 is a constant depending on the initial data. This gives in view of Besov embeddings

$$\|\nabla\theta\|_{L^1_t L^\infty} \leq C_0 e^{C_0 t}.$$

From (4.1) and (4.2), we deduce

$$\|\omega(t)\|_{L^\infty\cap L^{\bar{p}}} \leq C_0 e^{C_0 t}.$$

This completes the proof of the proposition. □

Next, we will establish the following proposition.

PROPOSITION 4.11. *Under the same assumptions of Proposition 4.10 and if in addition $\omega^0 \in B^0_{\infty,1}$ then we have for every $t \in \mathbb{R}_+$*

$$\|\omega(t)\|_{B^0_{\infty,1}} + \|\nabla v(t)\|_{L^\infty} \leq C_0 e^{\exp C_0 t}.$$

PROOF. Applying Proposition 4.5 to the vorticity equation and using Besov embeddings,

$$\begin{aligned}
\|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}} &\lesssim \big(\|\omega^0\|_{B^0_{\infty,1}} + \|\theta\|_{L^1_t B^1_{\infty,1}}\big)\big(1+\|\nabla v\|_{L^1_t L^\infty}\big)\\
&\lesssim \big(\|\omega^0\|_{B^0_{\infty,1}} + \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}\big)\big(1+\|\nabla v\|_{L^1_t L^\infty}\big).
\end{aligned} \tag{4.5}$$

On the other hand we have

$$\begin{aligned}
\|\nabla v(t)\|_{L^\infty} &\leq \|\nabla\Delta_{-1}v(t)\|_{L^\infty} + \sum_{q\in\mathbb{N}}\|\Delta_q\nabla v(t)\|_{L^\infty}\\
&\lesssim \|\nabla\Delta_{-1}v(t)\|_{L^{\bar{p}}} + \|\omega(t)\|_{B^0_{\infty,1}}\\
&\lesssim \|\omega(t)\|_{L^{\bar{p}}} + \|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}}.
\end{aligned} \tag{4.6}$$

Putting together (4.4), (4.5) and (4.6) and using Gronwall's inequality we deduce

$$\|\nabla v(t)\|_{L^\infty} + \|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}} \leq C_0 e^{\exp C_0 t}. \tag{4.7}$$

Now, the proof is achieved. □

Let us now see how to propagate the initial regularities. The Lipschitz estimate on the velocity will be very crucial.

PROPOSITION 4.12. *Let $p \in]1,\infty[$, $v^0 \in B^{1+\frac{2}{p}}_{p,1}$ be a divergence free vector-field of $\mathbb{R}^2$ and*

PROOF. From Remark 6 and (4.7) we have

$$\|\theta\|_{\widetilde{L}_t^\rho B_{r,\infty}^{\frac{\alpha}{\rho}}} \leq C_0 e^{\exp C_0 t}.$$

Now we use Besov embedding $B_{r,\infty}^{\frac{\alpha}{\rho}} \hookrightarrow B_{\infty,\infty}^{\frac{\alpha}{\rho}-\frac{2}{r}}$. Since $1+\frac{2}{r}<\frac{\alpha}{\rho}$ then we have the embeddings

$$\widetilde{L}_t^\rho B_{r,\infty}^{\frac{\alpha}{\rho}} \hookrightarrow L_t^\rho B_{\infty,1}^1 \hookrightarrow L_t^\rho \mathrm{Lip}.$$

Hence, it follows that

$$\|\nabla\theta\|_{L_t^\rho L^\infty} \leq C_0 e^{\exp C_0 t}.$$

To establish the second estimate of the proposition we distinguish two cases: the first one is $-\alpha+1+\frac{2}{p}<1$ and the second one is $-\alpha+1+\frac{2}{p}\geq 1$.

• Case $-\alpha+1+\frac{2}{p}<1$. We apply Proposition 4.9 to the temperature equation,

$$\|\theta\|_{\widetilde{L}_t^\infty B_{p,1}^{1+\frac{2}{p}-\alpha}} + \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \lesssim \|\theta^0\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} (1+t) e^{CV(t)}.$$

It suffices now to combine this estimate with the Lipschitz bound of the velocity (4.7).

• Case $-\alpha+1+\frac{2}{p}\geq 1$. Applying once again Proposition 4.9 we get

$$\|\theta\|_{\widetilde{L}_t^\infty B_{p,1}^{1+\frac{2}{p}-\alpha}} + \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \lesssim \|\theta^0\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} \Big(1+t+\|\nabla\theta\|_{L_t^1 L^\infty} \|v\|_{L_t^\infty B_{p,1}^{-\alpha+1+\frac{2}{p}}}\Big) e^{CV(t)}.$$

Hence we obtain from Proposition 4.10 and (4.7)

$$\|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \leq C_0 e^{e^{\exp C_0 t}} \big(1+\|v\|_{L_t^\infty B_{p,1}^{-\alpha+1+\frac{2}{p}}}\big).$$

Applying Proposition 4.6 we get

$$\begin{aligned}
\|v\|_{L_t^\infty B_{p,1}^{-\alpha+1+\frac{2}{p}}} &\lesssim e^{CV(t)} \big(\|v^0\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} + \|\theta\|_{L_t^1 B_{p,1}^{-\alpha+1+\frac{2}{p}}}\big) \\
&\leq C_0 e^{e^{\exp C_0 t}} \big(1+\|\theta\|_{L_t^1 B_{p,1}^{-\alpha+1+\frac{2}{p}}}\big).
\end{aligned}$$

Thus

$$\|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \leq C_0 e^{e^{\exp C_0 t}} \big(1+\|\theta\|_{L_t^1 B_{p,1}^{-\alpha+1+\frac{2}{p}}}\big).$$

Iterating this procedure we get for $n\in\mathbb{N}$

$$\|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \leq C_0 e^{e^{\exp C_0 t}} \big(1+\|\theta\|_{L_t^1 B_{p,1}^{-n\alpha+1+\frac{2}{p}}}\big).$$

To conclude it is enough to choose n such that $-(n+1)\alpha+1+\frac{2}{p}<1$ and then we can apply the first case. Finally we get

$$\|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \leq C_0 e^{e^{\exp C_0 t}}.$$

Applying again Proposition 4.6 we get

$$\begin{aligned}\|v\|_{\widetilde{L}^\infty_t B^{1+\frac{2}{p}}_{p,1}} &\lesssim e^{CV(t)}\big(\|v^0\|_{B^{1+\frac{2}{p}}_{p,1}} + \|\theta\|_{L^1_t B^{1+\frac{2}{p}}_{p,1}}\big)\\ &\le C_0 e^{e^{\exp C_0 t}}.\end{aligned}$$

□

4.2. Uniqueness. We will prove a uniqueness result in the following space

$$\mathcal{X}_T = (L^\infty_T L^p \cap L^1_T \mathrm{Lip}) \times (L^\infty_T L^r \cap L^1_T \mathrm{Lip}), \quad r > 2.$$

Without loss of generality we can suppose that $p \in [r, \infty[$. Indeed, let $\bar{p} = \max\{p, r\}$ then from Besov embedding we have $B^{1+\frac{2}{p}}_{p,1} \hookrightarrow B^{1+\frac{2}{\bar{p}}}_{\bar{p},1} \hookrightarrow L^{\bar{p}}$. Let (v^1, π^1, θ^1) and (v^2, π^2, θ^2) be two solutions of (B_α) belonging to the space $\mathcal{X}_T$ and denote

$$v = v^2 - v^1, \quad \theta = \theta^2 - \theta^1 \quad \text{and} \quad \pi = \pi^2 - \pi^1.$$

Then we have the equations

$$\begin{cases} \partial_t v + v^2 \cdot \nabla v = -\nabla \pi - v \cdot \nabla v^1 + \theta e_2 \\ \partial_t \theta + v^2 \cdot \nabla \theta + |\mathrm{D}|^\alpha \theta = -v \cdot \nabla \theta^1 \\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0. \end{cases}$$

The L^p estimate of the velocity is given by

$$\|v(t)\|_{L^p} \le \|v^0\|_{L^p} + \int_0^t \|v(\tau)\|_{L^p} \|\nabla v^1(\tau)\|_{L^\infty} + \|\nabla \pi(\tau)\|_{L^p} d\tau + \|\theta\|_{L^1_t L^p}.$$

From the incompressibility condition we get

$$\nabla \pi \;=\; \nabla \Delta^{-1} \mathrm{div}\big(-v \cdot \nabla v^1 + \theta e_2\big) - \nabla \Delta^{-1} \mathrm{div}(v^2 \cdot \nabla v).$$

Now due to the identity $\mathrm{div}(v^2 \cdot \nabla v) = \mathrm{div}(v \cdot \nabla v^2)$, one obtains

$$\nabla \pi = \nabla \Delta^{-1} \mathrm{div}\big(-v \cdot \nabla (v^1 + v^2) + \theta e_2\big).$$

Using the continuity of Riesz transform on L^p with $p \in]1, \infty[$ we get

$$\|\nabla \pi\|_{L^p} \lesssim \|v\|_{L^p}\big(\|\nabla v^1\|_{L^\infty} + \|\nabla v^2\|_{L^\infty}\big) + \|\theta\|_{L^p}.$$

Combining this estimate with the L^p estimate of the velocity we get

$$\|v(t)\|_{L^p} \lesssim \|v^0\|_{L^p} + \int_0^t \|v(\tau)\|_{L^p}\big(\|\nabla v^1(\tau)\|_{L^\infty} + \|\nabla v^2(\tau)\|_{L^\infty}\big) d\tau + \|\theta\|_{L^1_t L^p}.$$

Now, we apply Proposition 4.9 with $s = -1 + \epsilon$ and $\epsilon \in]0, 1[$ we get

$$\begin{aligned}\|\theta(t)\|_{L^1_t L^p} &\lesssim \|\theta\|_{L^1_t B^{-1+\epsilon+\alpha}_{p,1}}\\ &\lesssim e^{C\|\nabla v^2\|_{L^1_t L^\infty}}\Big(\|\theta^0\|_{B^{-1+\epsilon}_{p,1}} + \int_0^t \|v \cdot \nabla \theta^1(\tau)\|_{B^{-1+\epsilon}_{p,1}} d\tau\Big)\end{aligned}$$

We have used in the last inequality the embeddings $L^r \hookrightarrow B_{r,1}^{-1+\epsilon+\frac{2}{r}} \hookrightarrow B_{p,1}^{-1+\epsilon}$ valid for ϵ satisfying $-1+\epsilon+\frac{2}{r}<0$. This is possible since $r>2$. Finally we get

$$\begin{aligned}\|v(t)\|_{L^p} \;&\lesssim\; e^{C\|\nabla v^2\|_{L^1_t L^\infty}}\Big(\|v^0\|_{L^p}+\|\theta^0\|_{L^r}\\ &+\; \int_0^t \|v(\tau)\|_{L^p}\big(\|\nabla v^1(\tau)\|_{L^\infty}+\|\nabla v^2(\tau)\|_{L^\infty}+\|\nabla\theta^1\|_{L^\infty}\big)d\tau\Big).\end{aligned}$$

Using Gronwall's inequality we find

$$\|v(t)\|_{L^p} \lesssim e^{C\|(v^1,v^2,\theta^2)\|_{L^1_t \mathrm{Lip}}}\big(\|v^0\|_{L^p}+\|\theta^0\|_{L^r}\big). \tag{4.8}$$

This gives in turn

$$\|\theta\|_{L^1_t L^p} \lesssim e^{C\|(v^1,v^2,\theta^2)\|_{L^1_t \mathrm{Lip}}}\big(\|v^0\|_{L^p}+\|\theta^0\|_{L^r}\big)(1+t). \tag{4.9}$$

This concludes the proof of the uniqueness part.

4.3. Existence. We consider the following system

$$(\mathrm{B}_n)\qquad \begin{cases}\partial_t v_n + v_n\cdot\nabla v_n + \nabla\pi_n = \theta_n e_2\\ \partial_t\theta_n + v_n\cdot\nabla\theta_n + |\mathrm{D}|^\alpha\theta_n = 0\\ \mathrm{div} v_n = 0\\ v_{n|t=0} = S_n v^0, \quad \theta_{n|t=0} = S_n\theta^0.\end{cases}$$

By using the method of **[19]** we can prove that this system has a unique local smooth solution (v_n,θ_n). The global existence of this solution is governed by the quantity $\|\nabla v_n\|_{L^1_T L^\infty}$. Now from the *a priori* estimates the Lipschitz norm can not blow up in finite time and then the solution (v_n,θ_n) is globally defined. Once again from the *a priori* estimates we have

$$\|v_n\|_{\widetilde{L}^\infty_T B_{p,1}^{1+\frac{2}{p}}} + \|\theta_n\|_{\widetilde{L}^\infty_T B_{p,1}^{-\alpha+1+\frac{2}{p}}} + \|\theta_n\|_{L^\infty_T L^r} + \|\nabla\theta_n\|_{L^\rho_T L^\infty} \le \Phi_3(T).$$

Consequently, up to an extraction the sequence (v_n,θ_n) converges weakly to (v,θ) belonging to $\widetilde{L}^\infty_T B_{p,1}^{1+\frac{2}{p}} \times \Big(\widetilde{L}^\infty_T B_{p,1}^{-\alpha+1+\frac{2}{p}} \cap L^\infty_T L^r \cap L^\rho_T \mathrm{Lip}\Big)$.
For $(n,m)\in\mathbb{N}^2$ we set $v_{n,m}=v_n-v_m$ and $\theta_{n,m}=\theta_n-\theta_m$ then according to the estimates (4.8) and (4.9) we get

$$\|v_{n,m}\|_{L^\infty_T L^{\bar p}} + \|\theta_{n,m}\|_{L^1_T L^{\bar p}} \le C_0 e^{e^{\exp C_0 t}}\big(\|S_n v^0 - S_m v^0\|_{L^p} + \|S_n\theta^0 - S_m\theta^0\|_{L^r}\big),$$

with $\bar p = \max\{p,r\}$. This proves that the sequence (v_n,θ_n) converges strongly to (v,θ) in $L^\infty_T L^{\bar p}\times L^1_T L^{\bar p}$. This allows us to pass to the limit in the system (B_n) and then we get that (v,θ) is a solution of the Boussinesq system (B_α).

4.4. Continuity-in-time. Let us first sketch the proof of the continuity in time of the velocity. Let $\epsilon>0, N\in\mathbb{N}^*$ and $T>0$, then for every $0\le\tau\le t\le T$, Since $v \in \widetilde{L}^\infty_T B^{1+\frac{2}{p}}_{p,1}$, then there exists N sufficiently large such that

$$\sum_{q>N} 2^{q(1+\frac{2}{p})} \|\Delta_q v\|_{L^\infty_T L^p} \leq \epsilon.$$

Thus we get

$$\|v(t)-v(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} \lesssim 2^{N(1+\frac{2}{p})}\|v(t)-v(\tau)\|_{L^p} + \epsilon.$$

On the other hand from the equation of the velocity we get

$$v(t,x)-v(\tau,x) = \int_\tau^t \mathcal{P}(v\cdot\nabla v)(t',x)dt' + \int_\tau^t \mathcal{P}(\theta e_2)(t',x)dt',$$

where $\mathcal{P}$ denotes Leray's projector. Since $\mathcal{P}$ acts continuously on L^p for $p \in]1,\infty[$ then

$$\begin{aligned}
\|v(t)-v(\tau)\|_{L^p} &\lesssim \int_\tau^t \|\nabla v(t')\|_{L^\infty}\|v(t')\|_{L^p}dt' + \int_\tau^t \|\theta(t')\|_{L^p}dt' \\
&\lesssim |t-\tau|\|\nabla v\|_{L^\infty_T L^\infty}\|v\|_{L^\infty_T L^p} + \int_\tau^t \|\theta(t')\|_{L^p}dt'.
\end{aligned}$$

Let $\rho > \alpha$, then using Hölder inequality, Besov embeddings and Proposition 4.9 we get

$$\begin{aligned}
\int_\tau^t \|\theta(t')\|_{L^p}dt' &\leq (t-\tau)^{1-\frac{1}{\rho}}\|\theta\|_{\widetilde{L}^\rho_t B^0_{p,1}} \\
&\lesssim e^{CV(t)}(t-\tau)^{1-\frac{1}{\rho}}(1+t^{\frac{1}{\rho}})\|\theta^0\|_{B^{-\frac{\alpha}{\rho}}_{p,1}}.
\end{aligned}$$

Choose ρ such that $-\alpha+1+\frac{2}{p} \geq -\frac{\alpha}{\rho}$, which is possible for ρ close to α. Then

$$\int_\tau^t \|\theta(t')\|_{L^p}dt' \lesssim e^{CV(t)}(t-\tau)^{1-\frac{1}{\rho}}(1+t^{\frac{1}{\rho}})\|\theta^0\|_{B^{-\alpha+1+\frac{2}{p}}_{p,1}}.$$

Finally we obtain

$$\|v(t)-v(\tau)\|_{L^p} \lesssim |t-\tau|\|\nabla v\|_{L^\infty_T L^\infty}\|v\|_{L^\infty_T L^p} + e^{CV(t)}|t-\tau|^{1-\frac{1}{\rho}}(1+t^{\frac{1}{\rho}})\|\theta^0\|_{B^{-\alpha+1+\frac{2}{p}}_{p,1}}.$$

This ensures the continuity in time of the velocity.
Let us now move to the proof of the continuity-in-time of the temperature. We will first prove that $\theta \in \mathscr{C}([0,T], B^{-\alpha+1+\frac{2}{p}}_{p,1})$. Similarly to the velocity we write since $\theta \in \widetilde{L}^\infty_T B^{-\alpha+1+\frac{2}{p}}_{p,1}$ that for large N

$$\|\theta(t)-\theta(\tau)\|_{B^{-\alpha+1+\frac{2}{p}}_{p,1}} \leq \sum_{q\leq N} 2^{q(-\alpha+1+\frac{2}{p})}\|\Delta_q\theta(t)-\Delta_q\theta(\tau)\|_{L^p}$$

From the equation of θ and Bernstein inequality we find for $0 \le \tau \le t$

$$
\begin{aligned}
\|\Delta_q\theta(t) - \Delta_q\theta(\tau)\|_{L^p} &\le \int_\tau^t \|\Delta_q(v\cdot\nabla\theta)(t')\|_{L^p}dt' + 2^{q\alpha}(t-\tau)\|\Delta_q\theta\|_{L^p}\\
&\lesssim \|v\|_{L^\infty_T L^p}\int_\tau^t \|\nabla\theta(t')\|_{L^\infty}dt' + 2^{q(2\alpha-1-\frac{2}{p})}(t-\tau)\|\theta\|_{\widetilde{L}^\infty_T B_{p,1}^{-\alpha+1+\frac{2}{p}}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|\theta(t) - \theta(\tau)\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} &\le 2^{N(1+\frac{2}{p})}\|v\|_{L^\infty_T L^p}\int_\tau^t \|\nabla\theta(t')\|_{L^\infty}dt'\\
&+ 2^{N\alpha}(t-\tau)\|\theta\|_{\widetilde{L}^\infty_T B_{p,1}^{-\alpha+1+\frac{2}{p}}} + \epsilon.
\end{aligned}
$$

This proves that $\theta \in \mathscr{C}([0,T], B_{p,1}^{-\alpha+1+\frac{2}{p}})$.
Let us now prove that $\theta \in \mathscr{C}([0,T], L^r)$. Denote by $S(t) = e^{-t|\mathrm{D}|^\alpha}$ and $f = -v\cdot\nabla\theta$. Then from Duhamel formulae we get

$$
\begin{aligned}
\theta(t,x) - \theta(\tau,x) &= [S(t) - S(\tau)]\theta^0(x) + S(t)\Big(\int_\tau^t S(-t')f(t')dt'\Big)\\
&+ [S(t) - S(\tau)]\Big(\int_0^\tau S(-t')f(t')dt'\Big).
\end{aligned}
$$

To conclude we use the fact that $(S(t))_{t\ge 0}$ is a C_0-semigroup of contractions with positive kernels combined with the estimates

$$
\int_\tau^t \|f(t')\|_{L^r}dt' \le \|v\|_{L^\infty_T L^\infty}\int_\tau^t \|\nabla\theta(t')\|_{L^r}dt'
$$

and

$$
\Big|[S(t) - S(\tau)]\Big(\int_0^\tau S(-t')f(t',x)dt'\Big)\Big| \le [S(t) - S(\tau)]\Big(\int_0^T |S(-t')f(t',x)|dt'\Big).
$$

This integral $\int_0^t \|f(t')\|_{L^p}dt'$ is bounded according to (4.4) and Proposition 4.12.

5. Perspectives

There remain many questions to explore in the Cauchy problem for the systems which we saw but I will limit myself to the critical case $\alpha = \frac{1}{2}$. This case seems a difficult question and I think that it requires a new approach similar to that introduced by Kiselev and *therein* in the case of the critical quasigeostrophic equation.
For the supercritical quasigeostrophic equation one did not establish the global existence apart from the small data yet then it is completely possible to build a family of non small initial data and generating an infinite time of existence. To be done one can be possibly inspired on the work of J.-Y. Chemin et I. Gallagher within the framework of the Navier-Stokes system of **[?]**. Let us mention that their condition of smallness does not

There is also a significant question which still remains in the case of the critical quasigeostrophic equation and which refers to the stability of the global solutions which are built in [**1**]. It is a question in particular of knowing if the flow $\theta^0 \to \theta(t)$ is continuous from critical space $\dot{\mathscr{B}}^0_{\infty,1}$ into $L^\infty_{loc}\big(\mathbb{R}_+;\dot{\mathscr{B}}^0_{\infty,1}\big)$, for more details we refer T. Hmidi [**16**].

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CHAPTER 5

Littlewood-Paley thoeory and Besov Spaces

In this chapter, we introduce a basic tool for the second and third chapter: the Littlewood-Paley decomposition, $v = S_0 f + \sum_{q\geq 1} \Delta_q v$, and the Besov spaces $\mathscr{B}^s_{p,r}$. We will discuss the localization in frequency (Bernstein inequalities), homogeneous decomposition for distributions modulo the polynomials. We study how the product and the composition acts on Besov spaces. Our main references for Besov spaces are the books by Bergh and Löfström [**1**], by Meyer [**5**], and by Peetre [**6**], and P.-G Lemarié-Rieusset [**4**], and Y. Chemin [**2**], and J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier [**3**]. Another classical reference is the book by Triebel [**7**].

1. Localization in Frequency Space

The very basic idea of this theory consists in a localization procedure in the frequency space. The interest of this method is that the derivatives (or more generally the Fourier multipliers) act in a very special way on distributions the Fourier transform of which is supported in a ball or a ring. More precisely, we have the following lemma:

1.1. Bernstein inequalities.

LEMMA 5.1 (of localization). *Let $\mathcal{C}$ be a ring, B a ball. Then there exists a constant $C > 0$ such that for $1 \leq a \leq b \leq \infty$, for any function $v \in L^a$ and every $q \in \mathbb{Z}$, we have:*

$$\begin{aligned}
\operatorname{supp}\hat{v} \subset \lambda B \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha v\|_{L^b} &\leq C^k \lambda^{k+d\left(\frac{1}{a}-\frac{1}{b}\right)} \|v\|_{L^a};\\
\operatorname{supp}\hat{v} \subset \lambda\mathcal{C} \Rightarrow C^{-k}\lambda^k \|v\|_{L^a} &\leq \sup_{|\alpha|=k} \|\partial^\alpha v\|_{L^a} \leq C^k\lambda^k \|v\|_{L^a}.
\end{aligned}$$

PROOF. Let ϕ be a function of $\mathscr{D}(\mathbb{R}^d)$ such that $\phi = 1$ near the ball B. As $\hat{v}(\xi) = \phi(\lambda^{-1}\xi)\hat{v}(\xi)$, we can write, if g denotes the inverse Fourier transform of ϕ,

$$\partial^\alpha v = \partial^\alpha g \star v.$$

where $\star$ denotes the convolution operator, i.e.

$$\partial^\alpha g \star v(x) = \int_{\mathbb{R}^d} (\partial^\alpha g)(x-y) v(y) dy.$$

By Young's[1] inequality, we get

$$\|\partial^\alpha v\|_{L^b} \leq \lambda^{|\alpha|}\lambda^{d(1-1/c)} \|\partial^\alpha g\|_{L^c} \|v\|_{L^a}, \quad 1/c = 1 + 1/b - 1/a.$$

Then using the general convexity inequality

$$\forall (A,B) \in \mathbb{R}_+ \times \mathbb{R}_+,\ \forall \zeta \in]0,1], \quad AB \leq \zeta A^{1/\zeta} + (1-1/\zeta) B^{1/1-\zeta}.$$

Therefore the result follows through

$$\begin{aligned} \|\partial^\alpha g\|_{L^c} &\leq \|\partial^\alpha g\|_{L^\infty} + \|\partial^\alpha g\|_{L^1} \\ &\leq \|(1+|.|^2)^d \partial^\alpha g\|_{L^\infty} \\ &\leq \|(Id-\Delta)^d\big((.)^\alpha \phi\big)\|_{L^1} \\ &\leq C^k. \end{aligned}$$

To prove the second assertion, let us consider a function $\tilde{\phi} \in \mathscr{D}(\mathbb{R}^d \backslash \{0\})$ such that $\tilde{\phi} = 1$ near the ring $\mathcal{C}$. Using the following algebraic identity

$$\begin{aligned} |\xi|^{2k} &= \sum_{|\alpha|=k} \xi^\alpha \xi^\alpha \\ &= \sum_{|\alpha|=k} (i\xi)^\alpha (-i\xi)^\alpha, \end{aligned} \tag{1.1}$$

and stating $g_\alpha \overset{def}{=} \mathscr{F}^{-1}(i\xi)^\alpha |\xi|^{-2k} \tilde{\phi}(\xi)$, we can write, as $\hat{v} = \tilde{\phi}\hat{v}$ that

$$\hat{v} = \sum_{|\alpha|=k} (-i\xi)^\alpha \hat{g}_\alpha \hat{v},$$

which implies that

$$v = \sum_{|\alpha|=k} g_\alpha \star \partial^\alpha v \tag{1.2}$$

and then the result. This proves the whole lemma. □

1.2. Dyadic partition of unity. Now, let us define a dyadic partition of unity. We shall use it along this text.

PROPOSITION 5.2. *Let us define by $\mathcal{C}$ the ring of center* 0*, of small radius* $3/4$ *and great radius* $8/3$*. It exists two radial functions χ and φ the values of which are in the interval* $[0,1]$*, belonging respectively to $\mathscr{D}\big(B(0,4/3)\big)$ and to $\mathscr{D}(\mathcal{C})$ such that*

$$\forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1; \tag{1.3}$$

$$\forall \xi \in \mathbb{R}^d \backslash \{0\}, \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1; \tag{1.4}$$

$$|q' - q| \geq 2 \Rightarrow \text{supp } \varphi(2^{-q'}.) \cap \text{supp } \varphi(2^{-q}.) = \emptyset; \tag{1.5}$$

$$q \geq 1 \Rightarrow \text{supp } \chi \cap \text{supp } \varphi(2^{-q}.) = \emptyset. \tag{1.6}$$

If $\tilde{\mathcal{C}} = B(0,2/3) + \mathcal{C}$, then $\tilde{\mathcal{C}}$ is a ring and we have

$$\forall \xi \in \mathbb{R}^d \backslash \{0\}, \quad \frac{1}{2} \leq \sum_{q\in\mathbb{Z}} \varphi^2(2^{-q}\xi) \leq 1. \tag{1.9}$$

PROOF. Let us choose α in the interval $]1, 4/3[$ let us denote by $\mathcal{C}'$ the ring of small radius α^{-1} and big radius 2α. Let us choose a smooth function θ, radial with value in $[0,1]$, supported in $\mathcal{C}$ with value 1 in the neighborhood of $\mathcal{C}'$. The important point is the following. For any couple of integers (q, q') we have

$$|q - q'| \geq 2 \Rightarrow 2^q\mathcal{C} \cap 2^{q'}\mathcal{C} = \emptyset. \tag{1.10}$$

Indeed, let us assume that $2^q\mathcal{C} \cap 2^{q'}\mathcal{C} \neq \emptyset$ and that $q \geq q'$. It turns out that $2^q \times 3/4 \leq 4 \times 2^{q+1}/3$, which implies $q - q' \leq 1$. Now let us state

$$S(\xi) = \sum_{q\in\mathbb{Z}} \theta(2^{-q}\xi).$$

Thanks to (1.10), this sum is locally finite on the space $\mathbb{R}^d \backslash \{0\}$. Thus the function S is smooth on this space. As α is greater than 1,

$$\bigcup_{q\in\mathbb{Z}} 2^q\mathcal{C}' = \mathbb{R}^d \backslash \{0\}.$$

As the function θ is non negative and has value 1 near $\mathcal{C}'$, it comes from the above covering property that the above function is positive. Then let us state:

$$\varphi = \frac{\theta}{S}. \tag{1.11}$$

Let us check that φ fits. It is obvious that $\varphi \in \mathcal{D}(\mathcal{C})$. The function $1 - \sum_{q\geq 0} \varphi(2^{-q}\xi)$ is smooth thanks to (1.10). As the support of θ is included in $\mathcal{C}$, we have

$$|\xi| \geq \frac{4}{3} \Rightarrow \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1. \tag{1.12}$$

We get identities (1.3) and (1.5) thus stating:

$$\chi(\xi) = 1 - \sum_{q\geq 0} \varphi(2^q\xi), \tag{1.13}$$

Identity (1.6) is a obvious consequence of (1.10) and of (1.12). Now let us prove (1.7) which will be useful in Section 5.2. It is clear that the ring $\tilde{\mathcal{C}}$ is the ring of center 0, of small radius $1/12$ and of big radius $10/3$. Then it turns out that

$$2^q\tilde{\mathcal{C}} \cap 2^{q'}\tilde{\mathcal{C}} \neq \emptyset \Rightarrow \left(\frac{3}{4} \times 2^{q'} \leq 2^q \times \frac{10}{3} \text{ ou } \frac{1}{12} \times 2^q \leq 2^{q'} \times \frac{8}{3}\right)$$

and (1.7) is proved.
Now let us prove (1.8). As χ and φ have their values in $[0,1]$, it is clear that:

$$\chi^2(\xi) + \sum_{q\geq 0} \varphi^2(2^{-q}\xi) \leq 1. \tag{1.14}$$

with

$$\Pi_0(\xi) = \sum_{q\equiv 0(2), q\geq 0} \varphi(2^{-q}\xi), \quad \Pi_1(\xi) = \sum_{q\equiv 1(2), q\geq 0} \varphi(2^{-q}\xi).$$

From this it comes that $1 \leq 3\big(\chi^2(\xi) + \Pi_0^2(\xi) + \Pi_1^2(\xi)\big)$. But thanks to (1.5), we get

$$\Pi_i^2(\xi) = \sum_{q\geq 0, q\equiv i(2)} \varphi^2(2^{-q}\xi).$$

The proposition is proved. □

Now let us to fix the notations that will be used in all the following of this text. We choose two functions χ and φ satisfying the assertions (1.3)-(1.8).

Notations.

- $h = \mathscr{F}^{-1}\varphi$ and $g = \mathscr{F}^{-1}\chi$;
- $\Delta_{-1}v = \chi(D)v = \mathscr{F}^{-1}\big(\chi\hat{v}\big)$;
- $\forall q \in \mathbb{N} \quad \Delta_q v = \varphi(2^{-q}D)v, \quad S_q v = \sum_{-1\leq j\leq q-1} \Delta_j v$;
- $\forall q \leq -2, \quad \Delta_q v = 0$;
- $\Delta_{-1}v(x) = g \star v(x) = \int_{\mathbb{R}^3} g(y)v(x-y)dy$;
- $\forall q \in \mathbb{N},\ \Delta_q v(x) = 2^{dq}h(2^q.) \star v(x) = 2^{dq}\int_{\mathbb{R}^3} h(2^q y)v(x-y)dy$.

REMARK. *Let us notice that the operators Δ_q and S_q maps continuously L^p into itself uniformly on q and p.*

Now let us have a look of the case when we may write:

$$\mathrm{Id} = \sum_q \Delta_q \quad \text{or} \quad \mathrm{Id} = \sum_q \dot{\Delta}_q.$$

This is described by the following proposition:

PROPOSITION 5.3. *Let v be in $\mathscr{S}'(\mathbb{R}^d)$. Then, we have, in the sense of the convergence in the space $\mathscr{S}'(\mathbb{R}^d)$,*

$$v = \lim_{q\to\infty} S_q v.$$

PROOF. Let $\psi \in \mathscr{S}(\mathbb{R}^d)$. We have $\langle v - S_q v, \psi\rangle = \langle v, \psi - S_q\psi\rangle$. Thus it is enough to prove that in the space $\mathscr{S}(\mathbb{R}^d)$, we have

$$\psi = \lim_{q\to\infty} S_q\psi.$$

We shall use the family of semi norms $\mathscr{N}_k$ of $\mathscr{S}$ defined by

$$\mathscr{N}_k(\psi) \stackrel{def}{=} \sup_{|\alpha|\leq k,\ \xi\in\mathbb{R}^d} (1+|\xi|)^k |\partial^\alpha \hat{\psi}(\xi)|.$$

Thanks to Leibniz formula, we have

$$\mathscr{N}_k(\psi - S_q\psi) \leq \sup_{|\alpha|\leq k,\ \xi\in\mathbb{R}^d} \Big\{(1+|\xi|)^k \Big(|1-\chi(2^{-q}\xi)| \times |\partial^\alpha\hat{\psi}(\xi)|$$
$$+ \sum C_\alpha^\beta 2^{-q|\beta|} |(\partial^\beta\chi)(2^{-q}\xi)| \times |\partial^{\alpha-\beta}\hat{\psi}(\xi)|\Big)\Big\}.$$

This achieved the proof of the proposition. □

The following proposition tells us that the condition of convergence in $\mathscr{S}'$ is somehow weak for series, the Fourier transform of which is supported in dyadic rings.

PROPOSITION 5.4. *Let $(u_q)_{q\in\mathbb{N}}$ be a sequence of bounded functions such that the Fourier transform of u_q is supported in $2^q\widetilde{\mathcal{C}}$ where $\widetilde{\mathcal{C}}$ is a given ring. Let us assume that*

$$\|v_q\|_{L^\infty} \leq C2^{qN}.$$

Then the series $(u_q)_{q\in\mathbb{N}}$ is convergent in $\mathscr{S}'$.

PROOF. Let us use the relation (1.2). After rescaling it can be written as

$$v_q = 2^{-qk} \sum_{|\alpha|=k} 2^{dq} g_\alpha(2^q.) \star \partial^\alpha v_q.$$

Then for any test function ϕ in $\mathscr{S}$, let us write that

$$\begin{aligned} \langle u_q, \phi\rangle &= 2^{-qk} \sum_{|\alpha|=k} \langle v_q, 2^{dq} g_\alpha(2^q.) \star \partial^\alpha \phi\rangle \\ &\leq C2^{-dq} \sum_{|\alpha|=k} 2^{qN} \|\partial^\alpha \phi\|_{L^1}. \end{aligned} \tag{1.15}$$

Let us choose $k > N$. Then $\big(\langle v_q, \phi\rangle\big)_{q\in\mathbb{N}}$ is a convergent series, the sum of which is less than $C\mathcal{N}_M(\phi)$ for some integer M. Thus the formula

$$\langle v, \phi\rangle \stackrel{def}{=} \lim_{q\to\infty} \sum_{q'\leq q} \langle \Delta_{q'} v, \phi\rangle$$

defines a tempered distribution. □

2. Inhomogeneous Besov Spaces

2.1. Definitions.

DEFINITION 5.5. *Let s be a real number, and p and r two reals numbers greater than 1. The Besov spaces $\mathscr{B}^s_{p,r}$ is the space of all tempered distributions so that*

$$\|v\|_{\mathscr{B}^s_{p,r}} \stackrel{def}{=} \Big\| \big(2^{qs}\|\Delta_q v\|_{L^p}\big)_{q\in\mathbb{Z}} \Big\|_{\ell^r(\mathbb{Z})} < +\infty.$$

The following proposition (the proof of which is straightforward and omitted) describes the relations between homogeneous and inhomogeneous spaces.

PROPOSITION 5.6. *Let s be a negative number. Then $\dot{\mathscr{B}}^s_{p,r}$ is a subset of $\mathscr{B}^s_{p,r}$ and a constant C (independent of s) exists so that, for any v belonging to $\dot{\mathscr{B}}^s_{p,r}$, we have*

$$\|v\|_{\mathscr{B}^s_{p,r}} \leq \frac{C}{-s}\|v\|_{\dot{\mathscr{B}}^s_{p,r}}.$$

Let s be a positive number. Then $\mathscr{B}^s_{p,r}$ is a subset of $\dot{\mathscr{B}}^s_{p,r}$ when p is finite. $\mathscr{B}^s_{\infty,r} \cap \mathscr{S}'$ is a subset

LEMMA 5.7. *If r is finite, then for any v in $\mathscr{B}^s_{p,r}$, we have*

$$\lim_{q\to\infty} \|S_q v - v\|_{\mathscr{B}^s_{p,r}} = 0.$$

The proof of this proposition is an easy consequence of the definition of the norm of $\mathscr{B}^s_{p,r}$ and dyadic block S_q.
Let us give the first example for Besov space, the Sobolev spaces H^s. We have the following result.

THEOREM 5.8. *The two spaces H^s and $\mathscr{B}^s_{2,2}$ are equal and the two norms satisfies:*

$$\frac{1}{C^{|s|+1}}\|v\|_{\mathscr{B}^s_{2,2}} \le \|v\|_{H^s} \le C^{|s|+1}\|v\|_{\mathscr{B}^s_{2,2}}.$$

PROOF. As the support of the Fourier transform of $\Delta_q v$ is included in the ring $2^q\mathcal{C}$, it is clear, as $q \ge 0$, that a constant C exists such that, for any real s and any v such that $\hat{v}$ belongs to L^2_{loc},

$$\frac{1}{C^{|s|+1}}2^{qs}\|\Delta_q v\|_{L^2} \le \|\Delta_q v\|_{H^s} \le C^{|s|+1}2^{qs}\|\Delta_q v\|_{L^2}. \tag{2.1}$$

Using identity (1.8), we get

$$\frac{1}{3}\|v\|^2_{H^s} \le \int \chi^2(\xi)(1+|\xi|^2)^s|\hat{v}(\xi)|^2 d\xi + \sum_{q\ge 0}\int \varphi^2(2^{-q}\xi)(1+|\xi|^2)^s|\hat{v}(\xi)|^2 d\xi \le \|v\|^2_{H^s}.$$

which proves the theorem. □

PROPOSITION 5.9. *The space $\mathscr{B}^0_{p,1}$ is continuously embedded in L^p and the space L^p is continuously embedded in $\mathscr{B}^0_{p,\infty}$.*

PROOF. The proof is trivial. The first inclusion comes from the fact that the series $(\Delta_q v)_{q\in\mathbb{Z}}$ is convergent in L^p. The second one comes from the fact that for any p, we have $\|\Delta_q v\|_{L^p} \le C\|v\|_{L^p}$. □

2.2. Basic properties. The first point to look at is the invariance with respect to the choice of the dyadic partition of unity chosen to define the space. Most of the properties of the Besov spaces are based on the following lemma:

LEMMA 5.10. *Let $\mathcal{C}'$ be a ring in $\mathbb{R}^d$; let s be a real number and p and r two real numbers greater than 1. Let $(v_q)_{q\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\text{supp } \hat{u}_q \subset 2^q\mathcal{C}' \quad \text{and} \quad \left\|\left(2^{qs}\|v_q\|_{L^p}\right)_{q\in\mathbb{N}}\right\|_{\ell^r} < +\infty.$$

Then we have

$$v = \sum_{q\in\mathbb{N}} v_q \in \mathscr{B}^s_{p,r} \quad \text{and} \quad \|v\|_{\mathscr{B}^s_{p,r}} \le C_s\left\|\left(2^{qs}\|v_q\|_{L^p}\right)_{q\in\mathbb{N}}\right\|_{\ell^r}.$$

PROOF. In order to prove the lemma, let us first observe that $(v_q)_{q\in\mathbb{N}}$ is a convergent series in $\mathscr{S}'$. Indeed using Lemma 5.1, we get that $\|v_q\|_{L^\infty} \leq C2^{q\left(\frac{d}{p}-s\right)}$. Proposition 5.4 implies that $(v_q)_{q\in\mathbb{N}}$ is a convergent series in $\mathscr{S}'$. Then, let us study $\Delta_{q'}v$. As $\mathcal{C}$ and $\mathcal{C}'$ are two rings, an integer N_0 exists so that

$$|q' - q| \geq N_0 \Rightarrow 2^q\mathcal{C} \cap 2^{q'}\mathcal{C}' = \emptyset.$$

Here $\mathcal{C}$ is the ring defined in the Proposition 5.2. Now, it is clear that

$$\begin{aligned} |q' - q| \geq N_0 \ &\Rightarrow \ \mathscr{F}(\Delta_{q'}v_q) = 0 \\ &\Rightarrow \ \Delta_{q'}v_q = 0. \end{aligned}$$

Now, we can write that

$$\begin{aligned} \|\Delta_{q'}v\|_{L^p} &= \Big\| \sum_{|q-q'|<N_0} \Delta_{q'}v_q \Big\|_{L^p} \\ &\leq C \sum_{|q-q'|<N_0} \|v_q\|_{L^p}. \end{aligned}$$

So, we obtain that

$$\begin{aligned} 2^{q's}\|\Delta_{q'}v\|_{L^p} &\leq C \sum_{\substack{q'\geq -1 \\ |q-q'|\leq N_0}} 2^{q's}\|\Delta_{q'}v_q\|_{L^p} \\ &\leq C \sum_{\substack{q'\geq -1 \\ |q-q'|\leq N_0}} 2^{q's}\|v_q\|_{L^p}. \end{aligned}$$

We deduce from this that

$$2^{q's}\|\Delta_{q'}v\|_{L^p} \leq (c_k)_{k\in\mathbb{Z}} \star (d_l)_{l\in\mathbb{Z}}$$

with $c_k = \mathbf{1}_{[-N_0,N_0]}(k)$ and $d_l = \mathbf{1}_{\mathbb{N}}(l)2^{ls}\|v_l\|_{L^p}$. The classical property of convolution between $\ell^1(\mathbb{Z})$ and $\ell^r(\mathbb{Z})$ gives that

$$\|v\|_{\mathscr{B}^s_{p,r}} \leq C\Big(\sum_{q\in\mathbb{N}} 2^{rqs}\|v_q\|^r_{L^p}\Big)^{\frac{1}{r}}$$

which proves the lemma. □

The following theorem is the equivalent of Sobolev embedding:

THEOREM 5.12. *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then for any real number s, the space $\mathscr{B}^s_{p_1,r_1}$ is continuously embedded in $\mathscr{B}^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$.*

In order to prove this result, we again apply Lemma 5.1 which tells us that

$$\|S_0v\|_{L^{p_2}} \leq C\|v\|_{L^{p_1}} \quad \text{and} \quad \|\Delta_qv\|_{L^{p_2}} \leq C2^{dq\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|\Delta_qv\|_{L^{p_1}}.$$

PROOF. By definition $\mathscr{B}^s_{p,r}$ is a subspace of $\mathscr{S}'$. Thus we have only to prove of a constant C and an integer M exists such that for any test function ϕ in $\mathscr{S}$ we have:

$$\langle v, \phi \rangle \leq C\|v\|_{\mathscr{B}^s_{p,r}} \mathscr{N}_M(\phi). \tag{2.2}$$

Using the above Theorem 5.12 and the relation (1.15), we can write, if N is a large enough integer,

$$\begin{aligned} \langle \Delta_q v, \phi \rangle &= 2^{-q(N+1)} \sum_{|\alpha|=N+1} \langle \Delta_q v, 2^{dq} g_\alpha(2^q.) \star \partial^\alpha \phi \rangle \\ &\leq 2^{-q}\|v\|_{\mathscr{B}^{-N}_{\infty,\infty}} \sup_{|\alpha|=N+1} \|\partial^\alpha \phi\|_{L^1} \\ &\leq C2^{-q}\|v\|_{\mathscr{B}^s_{p,r}} \mathscr{N}_M(\phi). \end{aligned} \tag{2.3}$$

Now Proposition 5.3 implies the inequality (2.2). □

THEOREM 5.14. *The space $\mathscr{B}^s_{p,r}$ equipped with the norm $\|\cdot\|_{\mathscr{B}^s_{p,r}}$ is a Banach*[2] *space and satisfies the Fatou properties, i.e. if $(v_n)_{n\in\mathbb{N}}$ is a bounded sequence of $\mathscr{B}^s_{p,r}$, then an element v of $\mathscr{B}^s_{p,r}$ and a subsequence $v_{\rho(n)}$ exist such that:*

$$\lim_{n\to\infty} v_n = v \text{ in } \mathscr{S}' \quad \text{and} \quad \|v_n\|_{\mathscr{B}^s_{p,r}} \leq \liminf_{n\to\infty} \|v_n\|_{\dot{\mathscr{B}}^s_{p,r}}.$$

PROOF. Let us first prove the Fatou's property. Using Lemma 5.1, we claim that, for any q, the sequence $(\Delta_q v_n)_{n\in\mathbb{N}}$ is bounded in $L^p \cap L^\infty$. Then, using Cantor's diagonal process, we infer the existence of a subsequence $(v_{\rho(n)})_{n\in\mathbb{N}}$ and a sequence $(\tilde{v}_q)_{q\in\mathbb{Z}}$ such that, for any $q \in \mathbb{Z}$ and any $\phi \in \mathscr{S}$,

$$\lim_{n\to\infty} \int \Delta_q v_{\rho(n)}(x)\phi(x)dx = \int \tilde{v}_q(x)\phi(x)dx \quad \text{and} \quad \|\tilde{v}_q\|_{L^p} \leq \lim_{n\to\infty} \|\Delta_q v_n\|_{L^p}.$$

As the Fourier transform of $(\Delta_q v_n)_{n\in\mathbb{N}}$ is supported in $2^q\widetilde{\mathcal{C}}$, the same holds for $\tilde{v}_q$. Then, let us observe that the sequence $\big((2^{qs}\|\Delta_q v_n\|_{L^p})_q\big)_{n\in\mathbb{N}}$ is bounded in ℓ^r, an element $(\tilde{c}_q)_q$ of ℓ^r such that (up to an omitted extraction), we have, for any sequence $(d_q)_q$ of non negative real numbers different from 0 only for a finite number of index q,

$$\lim_{n\to\infty} \sum_q 2^{qs}\|\Delta_q v_{\rho(n)}\|_{L^p} d_q = \sum_q \tilde{c}_q d_q \quad \text{and} \quad \|(\tilde{c}_q)_q\|_{\ell^r} \leq \lim_{n\to\infty} \|v_{\rho(n)}\|_{\dot{\mathscr{B}}^s_{p,r}}.$$

Going to the limit in the sum gives that $\big(2^{qs}\|\tilde{v}_q\|_{L^p}\big)_q$ belongs to $\ell^r(\mathbb{Z})$. Using Lemma 5.1 and Proposition 5.9 implies that the series $(\tilde{v}_q)_{q\in\mathbb{Z}}$ converges to some v in $\mathscr{B}^s_{p,r}$ such that:

$$\|v\|_{\mathscr{B}^s_{p,r}} \leq C_s \Big\| \big(2^{qs}\|\tilde{u}_q\|_{L^p}\big)_q \Big\|_{\ell^r}.$$

This is proves the first part of the theorem.

Now, let us check that $\dot{\mathscr{B}}^s_{p,r}$ is complete. Let us consider a Cauchy sequence $(v_n)_{n\in\mathbb{N}}$. This sequence is of course bounded. Thus v exists in $\dot{\mathscr{B}}^s_{p,r}$ such that a subsequence $(v_{\rho(n)})_{n\in\mathbb{N}}$ converges to v in $\mathscr{S}'$. Using that, for any positive ε, an integer n_ε exists such that:

Applying the above method to the sequence $(v_m - v_{\rho(n)})_{n\in\mathbb{N}}$, we infer that

$$\forall m \geq n_\varepsilon \Rightarrow \|v_m - v\|_{\dot{\mathscr{B}}^s_{p,r}} < \varepsilon.$$

The theorem is proved. □

3. Paradifferential Calculus

In this section, we are going to study the way how the product acts on Besov spaces. Of course, we shall use the dyadic decomposition constructed in the Section 5.1.2.

3.1. Bony's decomposition. Let us consider two tempered distributions u and v, we write:

$$u = \sum_{q'} \Delta_{q'} u \quad \text{and} \quad v = \sum_q \Delta_q v$$

Formally, the product can be written as:

$$uv = \sum_{q,q'} \Delta_{q'} u \Delta_q v.$$

Now, let us introduce Bony's decomposition:

DEFINITION 5.15. *We shall designate paraproduct by u and shall denote by $T_u v$ the following bilinear operator:*

$$T_u v \stackrel{def}{=} \sum_q S_{q-1} u \Delta_q v.$$

We shall designate remainder of u and v and shall denote by $R(u,v)$ the following bilinear operator:

$$R(u,v) = \sum_{|q-q'|\leq 1} \Delta_q u \Delta_{q'} v.$$

Just by looking at the definition, it is clear that

$$uv = T_u v + T_v u + R(u,v). \tag{3.1}$$

The way how paraproduct and remainder act on Besov spaces is described by the following theorem:

LEMMA 5.16. *For any s, a constant C exists such that, for any $(p,r) \in [1,\infty]^2$, we have*

$$\forall (u,v) \in L^\infty \times \mathscr{B}^s_{p,r}, \ \|T_u v\|_{\mathscr{B}^s_{p,r}} \leq C\|v\|_{L^\infty}\|v\|_{\mathscr{B}^s_{p,r}}.$$

PROOF. From the assertion (1.7), the Fourier transform of $S_{q-1}u\Delta_q v$ is supported in $2^q\widetilde{\mathcal{C}}$. Then, let us write that:

$$\|S_{q-1}\Delta_q v\|_{L^p} \leq \|u\|_{L^\infty}\|\Delta_q v\|_{L^p}.$$

Theorem 5.12 implies the result. □

Now we shall study the behavior of operators R. Here we have to consider terms of the

LEMMA 5.17. *Let B be a ball of $\mathbb{R}^d$, s a positive real number and $(p,r) \in [1,\infty]^2$. Let $(v_q)_{q\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\text{supp } \hat{u}_q \subset 2^q B \quad \text{and} \quad \left\| \left(2^{qs} |v_q\|_{L^p} \right)_{q\in\mathbb{N}} \right\|_{\ell^r} < +\infty.$$

Then we have:

$$v = \sum_{q\in\mathbb{N}} v_q \in \mathscr{B}^s_{p,r} \quad \text{and} \quad \|v\|_{\mathscr{B}^s_{p,r}} \leq C_s \left\| \left(2^{qs} \|v_q\|_{L^p} \right)_{q\in\mathbb{N}} \right\|_{\ell^r}.$$

PROOF. We have for any q,

$$\|v_q\|_{L^p} \leq C 2^{-qs}.$$

As s is positive, $(v_q)_{q\in\mathbb{N}}$ is a convergent series in L^p. We then study $\Delta_{q'} v_q$. As $\mathcal{C}$ is a ring (defined in the proposition 5.2) and B is a ball, an integer N_1 exists so that

$$q' \geq q + N_1 \Rightarrow 2^{q'} \mathcal{C} \cap 2^q B = \emptyset.$$

So we obviously check

$$\begin{aligned} q' \geq q + N_1 \quad &\Rightarrow \quad \mathscr{F}(\Delta_{q'} v_q) = 0 \\ &\Rightarrow \quad \Delta_{q'} v_q = 0. \end{aligned}$$

Now, we write that

$$\begin{aligned} \|\Delta_{q'} v_q\|_{L^p} &= \Big\| \sum_{q \geq q' - N_1} \Delta_{q'} v_q \Big\|_{L^p} \\ &\leq \sum_{q \geq q' - N_1} \|\Delta_{q'} v_q\|_{L^p} \\ &\leq \sum_{q \geq q' - N_1} \|v_q\|_{L^p}. \end{aligned}$$

Therefore we deduce

$$\begin{aligned} 2^{q' s} \|\Delta_{q'} v_q\|_{L^p} &\leq \sum_{q \geq q' - N_1} 2^{q' s} \|\Delta_{q'} v_q\|_{L^p} \\ &\leq \sum_{q \geq q' - N_1} 2^{(q' - q)s} 2^{qs} \|v_q\|_{L^p} \\ &= (c_k) \star (d_l). \end{aligned}$$

with

$$c_k = \mathbf{1}_{[-N_1, +\infty[}(k) 2^{-ks}, \quad d_l = 2^{ls} \|v_l\|_{L^p}$$

The proof of the lemma is achieved. □

LEMMA 5.18. *For any (s_1, s_2) such that $s_1 + s_2 > 0$ a constant C exists such that, any $(p_1, p_2, r_1, r_2) \in [1,\infty]^4$ such that*

$$\frac{1}{p} \stackrel{def}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \quad \frac{1}{r} \stackrel{def}{=} \frac{1}{r_1} + \frac{1}{r_2} \leq 1.$$

PROOF. By definition of the remainder operator, we have:

$$R(u,v)=\sum_q R_q \quad \text{with} \quad R_q=\sum_{l=-1}^{1}\Delta_{q-l}u\Delta_q v.$$

By definition of Δ_q, the support of the Fourier transform of R_q is included in $2^qB(0,24)$. Moreover, Hölder inequalities implies that

$$2^{q(s_1+s_2)}\|R_q\|_{L^p}\leq \sum_{l=-1}^{1}\|\Delta_{q-l}u\|_{L^{p_1}}\|\Delta_q v\|_{L^{p_2}}.$$

Thus $2^{q(s_1+s_2)}\|R_q\|_{L^p}$ appears to be a sum of three series which are the product of a ℓ^{r_1} series by a ℓ^{r_2} series. Thus the lemma is proved. □

Now, we are going to deduce the following corollary:

COROLLARY 5.19. *For any positive s, the space $L^\infty\cap\mathscr{B}^s_{p,r}$ is an algebra. More precilesely, there exists constant C such that*

$$\|uv\|_{\mathscr{B}^s_{p,r}}\leq C\big(\|u\|_{L^\infty}\|v\|_{\mathscr{B}^s_{p,r}}+\|u\|_{\mathscr{B}^s_{p,r}}\|v\|_{L^\infty}\big).$$

The proof is nothing but the use of Bony's decomposition and the application of Lemmas 5.16 and 5.18.

3.2. Action of smooth functions. In this paragraph we shall study the action of smooth functions on the space $\mathscr{B}^s_{p,r}$. More precisely, if f is a smooth function vanishing at 0, and v a function of $\mathscr{B}^s_{p,r}$, does $f\circ v$ belongs to $\mathscr{B}^s_{p,r}$? The answer is given by the following theorem:

THEOREM 5.20. *Let f be a smooth function and s a positive real number and $(p,r)\in[1,\infty]^2$. If v belongs to $\mathscr{B}^s_{p,r}\cap L^\infty$, then $f\circ v$ belongs to $\mathscr{B}^s_{p,r}$ and we have:*

$$\|f\circ v\|_{\mathscr{B}^s_{p,r}}\leq C\big(s,f,\|v\|_{L^\infty}\big)\|v\|_{\mathscr{B}^s_{p,r}}.$$

Before proving this theorem, let us notice that if $s>d/p$ or if $s=d/p$ and $r=1$, then the space $\mathscr{B}^s_{p,r}$ is included into L^∞. Thus in those cases, the space $\mathscr{B}^s_{p,r}$ is stable under the action of f by composition. This is in particular the case for the Sobolev space H^s with $s>d/2$.

PROOF. We shall use the argument of the so called "telescopic series". As the sequence $(S_qv)_{q\in\mathbb{N}}$ converges to v in L^p and $f(0)=0$, then we have:

$$f(v)=\sum_q f_q \quad \text{with} \quad f_q \stackrel{def}{=} f(S_{q+1}v)-f(S_qv). \tag{3.2}$$

Taylor formula at order 1 yields

$$f_q=m_q\Delta_q v \quad \text{with} \quad m_q\stackrel{def}{=}\int_0^1 f'\big(S_qv+t\Delta_qv\big)dt. \tag{3.3}$$

At this point of the proof, let us point out that there is no hope for the Fourier transform

LEMMA 5.21. *Let s be a positive real number and $(p,r) \in [1,+\infty]^2$. A constant C_s exists such that if $(v_q)_{q\in\mathbb{N}}$ is a sequence of smooth functions which satisfies:*

$$\Big(\sup_{|\alpha|\le[s]+1} 2^{q(s-|\alpha|)}\|\partial^\alpha v_q\|_{L^p}\Big)_q \in \ell^r.$$

Then we have:

$$v = \sum_{q\in\mathbb{N}} v_q \in \mathscr{B}^s_{p,r} \quad \text{and} \quad \|v\|_{\mathscr{B}^s_{p,r}} \le C_s \Big\|\Big(\sup_{|\alpha|\le[s]+1} 2^{q(s-|\alpha|)}\|\partial^\alpha v_q\|_{L^p}\Big)_q\Big\|_{\ell^r}.$$

PROOF. As s is positive, the series $(u_q)_{q\in\mathbb{N}}$ is convergent in L^p. Let us denote its sum by v and let us write that

$$\Delta_q v = \sum_{q'\le q} \Delta_q v_{q'} + \sum_{q'>q} \Delta_q v_{q'}.$$

Using that $\|\Delta_q v_{q'}\|_{L^p} \le \|v_{q'}\|_{L^p}$, we get that

$$\begin{aligned} 2^{qs}\Big\|\sum_{q'>q}\Delta_q v_{q'}\Big\|_{L^p} &\le 2^{qs}\sum_{q'>q}\|v_{q'}\|_{L^p} \\ &\le \sum_{q'>q} 2^{-(q'-q)s}2^{q's}\|v_{q'}\|_{L^p} \end{aligned} \tag{3.4}$$

Then the lemma 5.1 implies

$$\|\Delta_q v_{q'}\|_{L^p} \le C2^{-q([s]+1)} \sup_{|\alpha|=[s]+1} \|\partial^\alpha v_{q'}\|_{L^p}.$$

Then we write

$$2^{qs}\Big\|\sum_{q'\le q}\Delta_q v_{q'}\Big\|_{L^p} \le \sum_{q'\le q} 2^{(q'-q)([s]+1-s)} \sup_{|\alpha|=[s]+1} 2^{q'(s-|\alpha|)}\|\partial^\alpha v_{q'}\|_{L^p}.$$

This inequality together with (3.4) implies that

$$2^{qs}\|\Delta_q v\| \le (a\star b)_q$$

with

$$\begin{aligned} a_q &\stackrel{def}{=} \mathbf{1}_{\mathbb{N}}(q)2^{-qs} + \mathbf{1}_{\mathbb{N}}(q)2^{-q([s]+1-s)}, \\ b_q &\stackrel{def}{=} 2^{qs}\|v_q\|_{L^p} + \sup_{|\alpha|=[s]+1} 2^{q(s-|\alpha|)}\|\partial^\alpha v_q\|_{L^p}. \end{aligned}$$

This proves the lemma. □

PROOF OF THEOREM 5.20. Let us admit for a while that

$$\forall \alpha\in\mathbb{N}^d, \quad \|\partial^\alpha m_q\|_{L^\infty} \le C_\alpha(f,\|v\|_{L^\infty})2^{q|\alpha|}. \tag{3.5}$$

Thus using Leibnitz formula and lemma 5.1, we get that

$$\|\partial^\alpha f_q\|_{L^p} \le \sum_{\beta\le\alpha} C^\alpha_\beta 2^{q|\beta|}C_\beta(f,\|v\|_{L^\infty})2^{q(|\alpha|-|\beta|)}\|\Delta_q v\|_{L^p}.$$

with $\|(c_q)\|_{\ell^r} = 1$. We apply lemma 5.21 and the theorem is proved provided we check (3.5). In order to do it, let us recall Faa-di-Bruno's formula:

$$\partial^\alpha g(a) = \sum_{\substack{\alpha_1+\cdots+\alpha_p=|\alpha| \\ |\alpha_q|\geq 1}} \Big(\prod_{k=1}^{p} \partial^{\alpha_k} a\Big) g^{(p)}(a).$$

From this formula, we find

$$\partial^\alpha m_q = \sum_{\substack{\alpha_1+\cdots+\alpha_p=|\alpha| \\ |\alpha_q|\geq 1}} \int_0^1 \Big(\prod_{k=1}^{p} \partial^{\alpha_k}(S_q v + t\Delta_q v)\Big) f^{(p+1)}(S_q v + t\Delta_q v) dt.$$

Using lemma 5.1, we get that

$$\begin{aligned} \|\partial^\alpha m_q\|_{L^\infty} &\leq C_\alpha(f) \sum_{\substack{\alpha_1+\cdots+\alpha_p=|\alpha| \\ |\alpha_q|\geq 1}} \int_0^1 \Big(\prod_{k=1}^{p} 2^{q|\alpha_p|}\|v\|_{L^\infty}\Big) \\ &\leq C_\alpha\big(f, \|v\|_{L^\infty}\big) 2^{q|\alpha|}. \end{aligned}$$

This proves (3.5) and thus the theorem. $\square$

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RÉSUMÉ. La première partie de cette thèse est conscrée a étudier le comportement asymptotique d'une fonctionnelle intégrale stochastique non convexe dépendant du second gradient dont l'intégrante est ceorcive, bornée, Lipschitzienne et vérifiant une condition de périodicité en loi. Pour identifier la $\Gamma-$limite, nous combinons le théorème ergodique des processus discret sous-addtif avec les techniques de la $\Gamma-$convergence on démontre le problème en question. La deuxième partie est composée de deux chapitres. Premièrement, on s'intéresse a étudier l'existence globale du système de Navier-Stokes lorsque les données initiales sont axisymétriques et dans des espaces de Besov critiques. Ensuite, on étudie la limite non-visqueuse du système de Navier-Stokes vers le système d'Euler, dont on estime le taux de convergence. Dans le deuxième chapitre, on établit l'existence et l'unicté du système d'Euler-Boussinesq avec une dissipation fractionnaire dans les espaces de Besov. La démonstration de ce résultat s'appuie sur le terme commutateur venant de la commutation entre le laplacien fractionnaire et le flot régularisé, puis l'effet régularisant de l'équation transport-diffusion régissant l'évolution de la température.

ABSTRACT. The first part of this thesis is devoted to study the asymptotic behavior of nonconvex random integral functionals depending on second gradient whose integrand is coercive, bounded, Liphschitzian and periodic in law. In order to identify the $\Gamma-$limit, we combine the ergodic theorem for discrete subadditive processes with $\Gamma-$ convergence techniques we prove the problem in question. The second part is composed of two parts. Firstly, we are interested to study the global well-posedness of incompressible Navier-Stokes equations with initial data is an axisymmetric vector fields and lying to critical Besov spaces. Afterward, we establish the inviscid limit of the Navier-Stokes equations toward Euler equations and we evaluate the rate of convergence. In the forth chapter we treat the global well-posedeness of Euler-Boussinesq system with fractional dissipation and initial data lying in critical Besov spaces. The proof of this result is based on the commutator term coming from the commutation between the fractional laplacian and the regularized flows, afterward the smoothing effects of the transport-diffusion equation governing the evolution of the temperature.