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Meriem ARAOUR

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THÈSE

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THESIS

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Option:
Operator Theory

Presented by

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On solving fuzzy integral equations

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THESIS

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Dedication

To my dear parents

To my brother

Contents

Introduction

Fuzzy theory plays an essential role in science and engineering. Several problems arise in a variety of scientific domains, including engineering, biological, and physical problems. Fuzzy integral and integro-differential equations can be used to model these situations. It is necessary to solve many fuzzy integro-differential equations numerically.

To introduce novel computational approach on fuzzy triangular numbers for the purpose of implementing fuzzy arithmetic calculus, the authors of [28] used the extension principle approximation based on product and Lukasiewicz t-norms.

We are accustomed to working with differential equations, but as we already know, they are not always difficult to train. Integral equations are a special variant of equation that is distinguished by its ease of solution and greater relevance to design phenomena. Integral equations are exciting in science. They are among mathematics' most important branches. They are known to impact various fields of applied mathematics and physics. Indeed, most models developed from industrial engineering and anatomy and physiology problems are best treated when presented as integral equations. Integral equation methods are particularly well suited to solving infinite news problems or where the boundaries are mobile or unknown. These methods are also exact.

Integral equations are equations in which the unknown function is placed under the integral sign. These are their typical forms:

$$\int_{\Omega} \psi(s, \varphi(\tau)) d\tau = g(s)$$
$$\alpha\varphi(s) + \beta \int_{\Omega} \psi(s, \varphi(\tau)) d\tau = g(s),$$

where φ is the unknown function, g is the known function called the right hand side and $\psi(.,.)$ is called the kernel.

Bernoulli applied integral equations for the first time around 1730 to examine the oscillations of a stretched cord. Nevertheless, Paul du Bois-Reymond was the first to use the term integral equation in 1888.

Numerous technical and theoretical studies can be formulated using differential equations or integral equations, particularly the values specific to the thermoplastic or the dynamics of structures. (see, [6, 7, 8, 16, 18, 19, 20, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58]). When integral equations are utilized, the boundary conditions are taken into account. The domain of integral equations is meant to be broader than that of differential equations. In fact, a differential equation containing an integral operation would be an integro-differential equation only; the differentiation operation disappears in front of the new operation, exactly in the same way that the solution of ordinary, algebraic, or transcendental equations comes after the differentiation operation; there is no reason to fear ambiguity.

In recent decades, the fields of fuzzy integral equations and fuzzy differential equations have grown rapidly. The fuzzy differential and integral equations are critical. In control theory, they have significant theoretical and practical value.

The aim of [30] is to introduce a new computational strategy as well as a kernel that reproduces the Hilbert space method is used to solve a system of fuzzy Volterra integro-differential equations via the Gram-Schmidt orthogonalization process. The authors of [78] exploited GH-differentiability of the first and second derivatives to convert a second-order implicit form of nonlinear fuzzy Volterra integro-differential equation into four different types of nonlinear integral equations.

The paper [41] aims to demonstrate a differential arithmetic in a quasilinear metric space. The authors applied derivative concepts in a more general way. In their paper, the researchers presented a new method of computing derivatives for quasilinear metric functions via the Hukuhara difference entirely.

The homotopy analysis approximation was proposed in [59] to solve a class of fuzzy linear Fredholm integral problem. In [63], the authors show that there is only one and unique solution to the fuzzy integral equation of Volterra type. The purpose of [25] is to approach the solution of the fuzzy differential and integral equations with arbitrary kernels. The authors used enough conditions to assure that the proposed methods converged.

The goal of [68] is to present a practical iterative procedure of successive approaches for numerically solving fuzzy two-dimensional integral equations of Hammerstein type using an ideal quadrature formula for Lipschitz-type fuzzy functions of two-dimensional type. A nonlinear fuzzy Hammerstein integral equations of Volterra type with constant delay has been examined in [69] by using Bernoulli wavelet approximation. A fuzzy nonlinear Hammerstein integral equation of Fredholm type has been considered in [79] by using an iterative numerical algorithm via the three-point quadrature formula.

The purpose of this thesis is to develop new methods for solving fuzzy integral and integro-differential equations.

The following is how the thesis is structured: To begin, in the first chapter, we present some fuzzy analysis concepts.

In Chapter 2, we demonstrate the existence of a solution for two classes of fuzzy Fredholm integro-differential systems. First, we use fixed point theory, the successive iteration method, and Gronwall's inequality to investigate a system of Volterra type integro-differential equations. Second, we investigate a system of integro-differential equations of the Fredholm type.

This third chapter presents and defends a practical method for solving fuzzy singular integro-differential equations. First, we show that solutions to two types of fuzzy singular integro-differential equations exist and are unique using different techniques: Picard's theorem for logarithmic kernels and Arzelà–Ascoli theorem for Cauchy ones. Then, using airfoil polynomials, we present a collocation method for numerically solving the current problems. We also look at the solutions to the approximate equations and introduce the concept of error analysis. We obtain two systems of linear equations using new procedures. These are the issues to be investigated. Finally, we use numerical examples to demonstrate the precision of the proposed approach.

Various arithmetic operations on intuitionistic fuzzy numbers are discussed in Chapter 4. We present some arithmetic operations as well as some differentiability properties for intuitionistic fuzzy functions. The average of (τ_1, τ_2)-cut method is also used to define the de-i-fuzzification of the corresponding intuitionistic fuzzy solution. We investigate intuitionistic fuzzy integral equations.

Chapter 1

Fuzzy Analysis

This chapter provides basic mathematical background on fuzzy analysis concepts.

Given a reference set X, one can indicate which elements of X belong to a certain class of X and which do not. This class is then a subset of X (in the usual sense of set theory), it is qualified as classical or ordinary in the sequel.
If the membership of certain elements of X to a class is not absolute, we can indicate to what degree each element belongs to this class. This is then a fuzzy subset of X.

Definition 1.1 *([74, 75, 76]) A classical subset A of X is defined by a characteristic function χ_A which takes the value 0 for the elements of X not belonging to A and the value 1 for those which belong to A :*

$$\chi_A : X \to [0, 1].$$

A fuzzy subset A is a classical subset of X in the particular case where f_A only takes values equal to 0 or 1. A classical subset is therefore a particular case of a fuzzy subset.

The extreme cases of a fuzzy subset of X are respectively X itself, associated with a membership function f_X taking the value 1 for all elements of X, and the empty set, associated with a membership function null on all X.

We often adopt tthe notation to represent the fuzzy subset A, which indicates for any element x of X its degree $f_{A(x)}$ of membership in A :

$$A = \sum_{x \in X} f_A(x)/x, \ if\ X\ is\ finite,$$

$$A = \int_x f_A(x)/x, \; if \; X \; is \; infinite.$$

To be able to easily describe a fuzzy subset A of X, we use some of its characteristics, essentially those which show to what extent it differs from a classical subset of X.

The first of these characteristics is the support of A, that is to say the set of elements of X which belong, at least a little, to A. it is denoted $supp(A)$ and it is the part of X on which the membership function of A is not zero:

$$supp(A) = \{x \in X; f_A(x) \neq 0\}$$

The second characteristic of A is its height, denoted $h(A)$, that is to say the strongest degree with which an element of X belongs to A. It is the greatest value taken by its membership function:

$$h(A) = \sup_{x \in X} f_A(x).$$

An important family of fuzzy subsets, which is used in possibility theory, corresponds to those which are normalized, i.e. for which there exists at least one element of X belonging absolutely (with a degree 1) to A. More precisely, A is normalized if its height $h(A)$ is equal to 1.
The set of all elements belonging absolutely (with degree 1) to A is called the kernel of A and denoted $ker(A)$:

$$ker(A) = \{x \in X; f_A(x) = 1\}.$$

If A is an ordinary subset of X, it is normalized and it is identical to its support and its kernel.

A last characteristic of the fuzzy subset A of X (when X is finite) is its cardinality, evaluating the global degree with which the elements of X belong to A. It is defined by:

If A is an ordinary subset of X, its ccardinality is the number of elements that compose it, according to the classical definition.

Let be consider the interval $\mathcal{I} := [-1, 1]$.

Definition 1.2 *([76]) A fuzzy number ρ is a function from $\mathbb{R}$ to $[0, 1]$ that meets the fol-*

lowing requirements:

(i) The function ρ *is normal, in other words,* $\exists\, t_0 \in \mathbb{R}:\ \rho(t_0) = 1$*;*

(ii) The function ρ *is a convex fuzzy set, specifically,*

$$\forall s, t \in \mathbb{R}, \forall \lambda \in [0,1]: \quad \rho(\lambda s + (1-\lambda) t) \geq \min\{\rho(s), \rho(t)\};$$

(iii) The function ρ *is upper semi-continuous on* $\mathbb{R}$*;*

(iv) The closure $\overline{\{s \in \mathbb{R} : \rho(s) > 0\}}$ *is a compact set.*

Denoting by $\mathcal{F}$ the set of all fuzzy numbers.

Definition 1.3 *([76]) Given* $\rho \in \mathcal{F}$*, the* r*-cut of* ρ *is defined by*

$$[\rho]_r := \{s \in \mathbb{R} : \rho(s) \geq r\},$$

with

$$[\rho]_0 := \overline{\{s \in \mathbb{R} : \rho(s) > 0\}}.$$

We note that for all $\rho_1, \rho_2 \in \mathcal{F}$ we have ρ_1 equal ρ_2 if and only if $[\rho_1]_r = [\rho_2]_r$.

A fuzzy number can be represented as parametric form as follows:

$$[\rho]_\alpha = [\underline{\rho}, \overline{\rho}]$$

for some two functions $\underline{\rho}, \overline{\rho} : \mathcal{I} \longrightarrow \mathbb{R}$ such that

1. The function $\underline{\rho}$ is a left continuous function bounded with a non-decreasing value;
2. The function $\overline{\rho}$ is a right continuous bounded function with a non-increasing value;
3. $\forall \tau \in \mathcal{I}:\ \underline{\rho}(\tau) \leq \overline{\rho}(\tau)$.

For two arbitrary fuzzy numbers $\rho_1 := [\underline{\rho_1}, \overline{\rho_1}]$ and $\rho_2 := [\underline{\rho_2}, \overline{\rho_2}]$, we define the following arithmetic operations: addition, scalar product, respectively in the following manner

$$\begin{aligned} \underline{(\rho_1 + \rho_2)} &= \underline{\rho_1} + \underline{\rho_2}, & \overline{(\rho_1 + \rho_2)} &= \overline{\rho_1} + \overline{\rho_2}, \\ \overline{k\rho_1} &= k\overline{\rho_1}, & \underline{k\rho_1} &= k\underline{\rho_1} \ \text{ for } \ k \geq 0, \\ \overline{k\rho_1} &= k\underline{\rho_1}, & \underline{k\rho_1} &= k\overline{\rho_1} \ \text{ for } \ k \leq 0. \end{aligned}$$

Definition 1.4 *([74, 75, 76]) Let* $\rho_1 := [\underline{\rho_1}, \overline{\rho_1}]$ *and* $\rho_2 := [\underline{\rho_2}, \overline{\rho_2}]$ *two fuzzy numbers. The Hausdorff distance between* ρ_1 *and* ρ_2 *is determined by*

$$D(\rho_1, \rho_2) = \sup_{0 \le \tau \le 1} \max \left\{ \left| \underline{\rho_2}(\tau) - \underline{\rho_1}(\tau) \right|, \ \left| \overline{\rho_2}(\tau) - \overline{\rho_1}(\tau) \right| \right\}.$$

Theorem 1.1 *([74, 75, 76]) The Hausdorff distance fulfills the following characteristics:*

1. *The metric space* $(\mathcal{F}, D)$ *is complete;*
2. $\forall \rho_1, \rho_2, \sigma \in \mathcal{F}, \quad D(\rho_1 + \sigma, \sigma + \rho_2) = D(\rho_1, \rho_2)$*;*
3. $\forall \rho_1, \rho_2, \sigma_1, \sigma_2 \in \mathcal{F}, \quad D(\rho_1 + \sigma_1, \rho_2 + \sigma_2) \le D(\rho_1, \rho_2) + D(\sigma_1, \sigma_2)$*;*
4. $\forall \rho_1, \rho_2 \in \mathcal{F}, \quad D(\rho_1 + \rho_2, \tilde{0}) \le D(\rho_1, \tilde{0}) + D(\rho_2, \tilde{0})$*;*
5. $\forall \rho_1, \rho_2 \in \mathcal{F}, k \in \mathbb{R}, \quad D(k\rho_1, k\rho_2) = |k| D(\rho_1, \rho_2)$*;*
6. $\forall \rho \in \mathcal{F}, k_1, k_2 \in \mathbb{R},$ *with* $k_1.k_2 \ge 0,$ *we have* $D(k_1\rho, k_2\rho) = |k_1 - k_2| D(\rho, \tilde{0}),$ *where* $\tilde{0} := \chi_{\{0\}}$.

Denoting by $\mathbb{F}$ the set of all fuzzy-number function over $\mathcal{I}$:

$$\mathbb{F} := \{\varphi : \mathcal{I} \to \mathcal{F}\}.$$

Definition 1.5 *([75, 79]) A function* $\varphi \in \mathbb{F}$ *is called continuous in* $s_0 \in \mathcal{I}$ *if,*

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 : D(\varphi(s), \varphi(s_0)) < \varepsilon \ \textit{ whenever } \ s \in \mathcal{I} \ \textit{ and } \ |s - s_0| < \delta.$$

If φ *is continuous at each* $s_0 \in \mathcal{I}$*, we call it fuzzy continuous on* $\mathcal{I}$*, we denote by* $\mathcal{C}_{\mathbb{F}}$ *the space of all such functions.*

Remark 1.1 *([75, 79])*

1. *A function* $\varphi \in \mathbb{F}$ *is referred to as bounded fuzzy function if and only if there is* $M \ge 0$ *in order for all* $t \in \mathcal{I}$*, we have* $D(\varphi(t), \tilde{0}) \le M$*. We denote by* $\mathcal{B}_{\mathbb{F}}$ *the space of all such functions.*
2. *We note that* $\mathcal{C}_{\mathbb{F}} \subset \mathcal{B}_{\mathbb{F}}$*.*

3. *For any* $\varphi \in \mathbb{F}$*, the functions* $\underline{\varphi}_\alpha(.),\ \bar{\varphi}_\alpha(.) : \mathcal{I} \to \mathbb{R}$ *are defined for all* $\alpha \in [0,1]$*. These functions are said the left and right* $\alpha-$*level functions of* φ*.*

Definition 1.6 *([13, 79]) Let* $\varphi \in \mathcal{B}_{\mathbb{F}}$*. Define the modulus of oscillation* $\omega_{\mathcal{I}}(\varphi, .) : \mathbb{R}^+ \to \mathbb{R}^+$ *of* φ *on* $\mathcal{I}$ *as follows*

$$\omega_{\mathcal{I}}(\varphi, \delta) := \sup\{D(\varphi(s), \varphi(t)) : s, t \in \mathcal{I} : |s-t| \leq \delta\}.$$

If $\varphi \in \mathcal{C}_{\mathbb{F}}$*, then* $\omega_{\mathcal{I}}(\varphi, \delta)$ *is known as* φ*'s uniform modulus of continuity.*

Theorem 1.2 *([13, 79]) The modulus of continuity has the following properties:*

1. $\forall s, t \in \mathcal{I} : \quad D(\varphi(s), \varphi(t)) \leq \omega_{\mathcal{I}}(\varphi, |s-t|)$*;*
2. *The function* $\omega_{\mathcal{I}}(\varphi, \delta)$ *is increasing of* δ*;*
3. $\omega_{\mathcal{I}}(\varphi, 0) = 0$*;*
4. $\forall \delta_1, \delta_2 \geq 0 : \quad \omega_{\mathcal{I}}(\varphi, \delta_1 + \delta_2) \leq \omega_{\mathcal{I}}(\varphi, \delta_1) + \omega_{\mathcal{I}}(\varphi, \delta_2)$;
5. $\forall \delta > 0\ \forall n \in \mathbb{N} : \quad \omega_{\mathcal{I}}(\varphi, n\delta) < n\omega_{\mathcal{I}}(\varphi, \delta)$*;*
6. $\forall \delta, \lambda \geq 0 : \quad \omega_{\mathcal{I}}(\varphi, \lambda\delta) \leq (\lambda+1)\omega_{\mathcal{I}}(\varphi, \delta)$*;*
7. *If* $\mathcal{J} \subseteq \mathcal{I}$*, then* $\omega_{\mathcal{J}}(\varphi, \delta) \leq \omega_{\mathcal{I}}(\varphi, \delta)$*.*

Definition 1.7 *([29, 23, 61]) Let* $f : [a, b] \to \mathcal{F}$*, for each partition* $P := \{t_0, \cdots, t_n\}$ *of* $[a, b]$ *and for arbitrary* $\xi_i \in [t_{i-1}, t_i]$, $1 \leq i \leq n$ *assume*

$$R_P = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

$$\int_a^b f(x)dx = \lim_{\Delta \to 0} R_P,$$

where

$$\Delta := \max\{|t_i - t_{i-1}|, i = 1, \cdots, n\}$$

provided that this limit exists in the metric D*.*

If the fuzzy function $f(.)$*is continuous in the metric* D *, its definite integral exists and also,*

$$\underline{(\int_a^b f(t;\alpha)dt)} = \int_a^b \underline{f}(t;\alpha)dt \ \text{ and } \ \overline{(\int_a^b f(t;\alpha)dt)} = \int_a^b \overline{f}(t;\alpha)dt.$$

Lemma 1.1 *([61, 74]) If f, g are Henstock integrable functions and if the function given by $D(f(.), g(.))$ is Lebesgue integrable, then*

$$D((FH)\int_a^b f(t)dt, (FH)\int_a^b g(t)dt) \leq (L)\int_a^b D(f(t), g(t))dt.$$

Definition 1.8 *([13, 79]) For $L \geq 0$, a function $f : [a, b] \to \mathcal{F}$ is $L-$Lipschitz if*

$$D(f(s), f(t)) \leq L|s - t|$$

for any $s, t \in [a, b]$.

Now, we recall the Hukuhara difference (H-difference) definition given in [15]. To this end, let $\rho_1, \rho_2 \in \mathcal{F}$. The H-difference has been introduced as a set σ for which $\rho_1 \ominus_H \rho_2 = \sigma \iff \rho_1 = \rho_2 + \sigma$. The H-difference is unique, but it does not always exist (a necessary condition for $\rho_1 \ominus_H \rho_2$ to exist is that ρ_1 contains a translate $c + \rho_2$ of ρ_2). A generalization of the Hukuhara definition is intended to remedy this situation.

Definition 1.9 *([15, 75]) The generalized Hukuhara difference between two fuzzy numbers $\rho_1, \rho_2 \in \mathcal{F}$ is defined as follows: $\rho_1 \ominus_{gH} \rho_2 = \sigma \iff \begin{cases} (i)\rho_1 = \rho_2 + \sigma; \\ or(ii)\rho_2 = \rho_1 + (-\sigma). \end{cases}$*

In terms of the $\alpha-$levels, we have $[\rho_1 \ominus_{gH} \rho_2]_\alpha = [min\{\underline{\rho_1}(\alpha) - \underline{\rho_2}(\alpha), \overline{\rho_1}(\alpha) - \overline{\rho_2}(\alpha)\}, max\{\underline{\rho_1}(\alpha) - \underline{\rho_2}(\alpha), \overline{\rho_1}(\alpha) - \overline{\rho_2}(\alpha)\}]$ and if the H-difference exists, then $\rho_1 \ominus_H \rho_2 = \rho_1 \ominus_{gH} \rho_2$; the conditions for the existence of $\sigma = \rho_1 \ominus_{gH} \rho_2 \in \mathcal{F}$ are

$$case(i) = \begin{cases} \underline{\sigma}(\alpha) = \underline{\rho_1}(\alpha) - \underline{\rho_2}(\alpha) \; and \; \overline{\sigma}(\alpha) = \overline{\rho_1}(\alpha) - \overline{\rho_2}(\alpha), \forall \in [0, 1]; \\ \\ with \; \underline{\sigma}(\alpha) \; increasing, \; \overline{\sigma}(\alpha) \; decreasing, \; \underline{\sigma}(\alpha) \leq \overline{\sigma}(\alpha). \end{cases}$$

$$case(ii) = \begin{cases} \underline{\sigma}(\alpha) = \overline{\rho_1}(\alpha) - \overline{\rho_2}(\alpha) \; and \; \overline{\sigma}(\alpha) = \underline{\rho_1}(\alpha) - \underline{\rho_2}(\alpha), \forall \in [0, 1]; \\ \\ with \; \underline{\sigma}(\alpha) \; increasing, \; \overline{\sigma}(\alpha) \; decreasing, \; \underline{\sigma}(\alpha) \leq \overline{\sigma}(\alpha). \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if σ is a crisp number. In the fuzzy case, it is possible that the gH-difference of two fuzzy numbers does not exist. To address this shortcoming, a new difference between fuzzy numbers was proposed in [15].

Definition 1.10 *([42, 61]) Let $f : [a, b] \to \mathcal{F}$. Fix $s_0 \in [a, b]$. We say f is differentiable at s_0, if there exists an element $f'(s_0) \in \mathcal{F}$ such that, the Hukuhara difference (H- difference) $f(s_0 + h) \ominus f(s_0)$, $f(s_0) \ominus f(s_0 - h)$ exist and the limits (in the metric D) presents as follows:*

$$\lim_{h \to 0^+} \frac{f(s_0 + h) \ominus f(s_0)}{h} = \lim_{h \to 0^+} \frac{f(s_0) \ominus f(s_0 - h)}{h} = f'(s_0)$$

Definition 1.11 *([31]) Let $f : (a,b) \to \mathcal{F}$ and $s \in (a,b)$. We say that f is strongly generalized differentiable at s_0, if there exists an element $f'(s_0) \in \mathcal{F}$, such that*

(i) For all $h > 0$ sufficiently small, $\exists f(s_0+h) \ominus f(s_0),\ \ \exists f(s_0) \ominus f(s_0-h)$and the following limits hold:

$$\lim_{h\to 0} \frac{f(s_0+h)\ominus f(s_0)}{h} = \lim_{h\to 0} \frac{f(s_0)\ominus f(s_0-h)}{h} = f^{'}(s_0)$$

or

(ii) for all $h > 0$ sufficiently small, $\exists f(s_0) \ominus f(s_0+h),\ \ \exists f(s_0-h) \ominus f(s_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(s_0)\ominus f(s_0+h)}{-h} = \lim_{h\to 0} \frac{f(s_0-h)\ominus f(s_0)}{-h} = f^{'}(s_0)$$

or

(iii) For all $h > 0$ sufficiently small, $\exists f(s_0+h) \ominus f(s_0),\ \ \exists f(s_0-h) \ominus f(s_0)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(s_0+h)\ominus f(s_0)}{h} = \lim_{h\to 0} \frac{f(s_0-h)\ominus f(s_0)}{-h} = f^{'}(s_0)$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f(s_0) \ominus f(s_0+h),\ \ \exists f(s_0) \ominus f(s_0-h)$ and the following limits hold:

$$\lim_{h\to 0} \frac{f(s_0)\ominus f(s_0+h)}{-h} = \lim_{h\to 0} \frac{f(s_0)\ominus f(s_0-h)}{h} = f^{'}(s_0).$$

Definition 1.12 *Let $f : (a,b) \to \mathcal{F}$. We say f is (i)-differentiable on (a,b) if f is differentiable in the sense (i) of Definition 1.11 and similarly for $(ii), (iii)$ and (iv) differentiability.*

Theorem 1.3 *([13, 61]) Let $f : [a,b] \to \mathcal{F}$ be a bounded and Henstock integrable function. Then for any partition $a = s_0 < s_1 < \cdots < s_n = b$ and $\zeta_i \in [s_{i-1}, s_i]$, we have*

$$D((FH)\int_a^b f(t)dt, \sum_{i=1}^{n}(s_i - s_{i-1})f(\zeta_i)) \leq \sum_{i=1}^{n}(s_i - s_{i-1})\omega_{[s_{i-1},s_i]}(f, s_i - s_{i-1}).$$

Particular election of the point ζ_i leads to the following result.

Here, we present the quadrature rules obtained in [13], which contain as particular cases with the three point, middle point and trapezoidal rules.

Corollary 1.1 *([13, 61]) Let $f : [a, b] \to \mathcal{F}$ be a bounded and Henstock integrable function. Then:*

1. $D((FH)\int_a^b f(t)dt, (b-a)f(\frac{(a+b)}{2})) \leq \frac{(b-a)}{2}\omega_{[a,b]}(f, \frac{(a-b)}{2})$;
2. $D((FH)\int_a^b f(t)dt, \frac{(a-b)}{2}[f(a) + f(b)] \leq \frac{(b-a)}{2}\omega_{[a,b]}(f, \frac{(b-a)}{2})$,
3. $D((FH)\int_a^b f(t)dt, \frac{(b-a)}{6}[f(a) + 4f(\frac{(a+b)}{2}) + f(b)]) \leq 3(b-a)\omega_{[a,b]}(f, \frac{(b-a)}{6})$.

Let $(\mathcal{X}; D)$ be a metric space. Consider the operator $T : \mathcal{X} \to \mathcal{X}$ and the following fixed points set $\mathrm{F_T}$ of T

$$\mathrm{F_T} := \{x \in \mathcal{X}, \ \ T(x) = x\}.$$

Define the iterate operators of T as follows

$$T^0 := I_{\mathcal{X}}, \ \ T^1 := T, \ \ \text{and} \ \ T^{n+1} := TT^n, \ \ \text{for all} \ \ n \in \mathbb{N}.$$

Following are the definitions for the Picard, c-Picard, and weakly Picard operators.

Definition 1.13 *([23]) We say that T is Picard operator if there exists $x^* \in \mathcal{X}$ such that:*

(a) $\mathrm{F_T} = \{x^*\}$;

(b) The sequence $(T^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in \mathcal{X}$.

Definition 1.14 *([23]) We say that T is c-Picard operator if T is Picard operator and*

$$d(x, x^*) \leq cd(x, T(x)), \ \ \textit{for all} \ \ x \in \mathcal{X}, \ \ \textit{with} \ \ c > 0.$$

Definition 1.15 *([23]) We say that T is weakly Picard operator if the sequence $(T^n(x))_{n\in\mathbb{N}}$ converges to x for all $x \in \mathcal{X}$, moreover, the limit x is a fixed point of T.*

Theorem 1.4 ***(Contraction Principle).****([23]) We assume that $T : \mathcal{X} \to \mathcal{X}$ is an $\alpha-$contraction* $(\alpha < 1)$. *Subject to these conditions, we have:*

(i) $F_T = \{x^*\}$;

(ii) $x^* = \lim_{n\to\infty} T^n(x_0)$, *for all $x_0 \in \mathcal{X}$;*

(iii) $D(x^*, T^n(x_0)) \leq \frac{\alpha^n}{1-\alpha} D(x_0, T(x_0))$.

Definition 1.16 *([4]) We assume that $\mathcal{X}$ is a Banach space. We say that T is compact, if it maps bounded sets of $\mathcal{X}$ into relatively compact sets. Moreover, T is said to be completely continuous, if it is continuous and compact.*

In the special case, where $\mathcal{X} = \mathcal{C}_{\mathbb{F}}$; we use the Arzela-Ascoli's Theorem to demonstrate the compactness of T.

Theorem 1.5 *([4]) A family of continuous functions on $\mathcal{I}$ is compact in $\mathcal{C}_{\mathbb{F}}$ if and only if it's equicontinuous and uniformly bounded.*

The Schauder's fixed point Theorem is presented at the end of this section.

Theorem 1.6 *([4]) Let $\mathcal{X}$ be a Banach space with a closed convex subset $\mathcal{K}$. If $T : \mathcal{X} \to \mathcal{X}$ is continuous and $\mathcal{K} = T(\mathcal{K})$ is compact, then T has a fixed point in $\mathcal{K}$.*

Chapter 2

On the existence and uniqueness of solutions to two fuzzy integro-differential systems

In recent decades, the fields of fuzzy integral equations and fuzzy differential equations have grown rapidly. The fuzzy differential and integral equations are critical. In control theory, they have significant theoretical and practical value.

In this chapter, we prove some results concerning the existence of a solution of two classes of fuzzy Fredholm integro-differential systems. First we examine a system of Volterra type integro-differential equations using fixed point theory, the successive iteration method, and Gronwall's inequality. Second, we investigate a system of Fredholm type integro-differential equations.

Let us begin by recalling the concept of vector-valued metric spaces, (see [2]).
Let X be a nonempty set. A mapping $D_v : X \times X \to \mathbb{R}^n_+$ is called a vector-valued metric on X if the following conditions are satisfied:

1. $D_v(x, y) = 0_n \in \mathbb{R}^n_+ \Leftrightarrow x = y$, for all $x, y \in X$;
2. $D_v(x, y) = D_v(y, x)$, for all $x, y \in X$;
3. $D_v(x, y) \leq D_v(x, z) + D_v(z, y)$, for all $x, y, z \in X$;

The following are examples of vector-valued metrics:

Example 2.1 *Let* $X := (C[a, b])^2$ *and* $D' : ((C[a, b])^2 \times (C[a, b])^2) \to \mathbb{R}^2_+$, *defined* *for all* $x = (x_1, x_2), y = (y_1, y_2) \in (C[a,b])^2$ *by*

$$D'(x,y) := \left(\max_{t\in[a,b]} |x_1(t) - y_1(t)|, \max_{t\in[a,b]} |x_2(t) - y_2(t)| \right).$$

Example 2.2 *Let* $X := (C[a,b])^2$ *and* $\Lambda' : ((C[a,b])^2 \times (C[a,b])^2) \to \mathbb{R}^2_+$, *defined by* $\Lambda'(x,y) := \left(\left(\int_a^b |x_1(t) - y_1(t)|^2 dt\right)^{\frac{1}{2}}, \left(\int_a^b |x_2(t) - y_2(t)|^2 dt\right)^{\frac{1}{2}} \right)$, *for all* $x = (x_1, x_2), y = (y_1, y_2) \in (C[a,b])^2$. *A nonempty set* X *endowed with a vector-valued metric* D' *is also called a* $\mathbb{R}^n_+-$*metric space and it is denoted by the pair* (X, D').

The notions of convergent sequence, Cauchy sequence, completeness, open and closed subset, etc. are similar to those described for conventional metric spaces.

2.1 Fuzzy Volterra integro-differential system

This section addresses the fuzzy Volterra integro-differential equation system of the form

$$\begin{aligned} &U'(s) = F(s, U(s)) + \int_a^s K(s,t,U(t))dt, \quad s \in \mathcal{J} = [a,b], \\ &U(a) = U_0, \end{aligned} \tag{2.1}$$

where the fuzzy functions are given by:

$$\begin{aligned} U'(s) &:= [u_1'(s), \cdots, u_n'(s)]^T, \\ U(s) &:= [u_1(s), \cdots, u_n(s)]^T, \\ K(s,t,U(t)) &:= [k_1(s,t,u_1(t), \cdots, u_n(t)), \cdots, k_n(s,t,u_1(t), \cdots, u_n(t))]^T, \\ F(s,U(s)) &:= [f_1(s,,u_1(s), \cdots, u_n(s)), \cdots, f_n(s,,u_1(s), \cdots, u_n(s))]^T, \\ U(a) &:= [u_{0,1}, \cdots, u_{0,n}]^T, \end{aligned}$$

Furthermore, F, K are a known functions in $C(\mathcal{J} \times \mathcal{F}^n, \mathcal{F}^n)$ and $C(\mathcal{J} \times \mathcal{J} \times \mathcal{F}^n, \mathcal{F}^n)$, respectively, while U is the unknown.

The purpose of this work is to prove that the problem (2.1) has a solution $U \in C^1(\mathcal{J}, \mathcal{F}^n)$. In order to accomplish this goal, it is important to present some definitions for the function $U \in C^1(\mathcal{J}, \mathcal{F}^n)$.

Definition 2.1 *1. The function* U *is called a proper solution of* (2.1) *if it is either (i) or (ii)-differentiable on* $\mathcal{J}$. *Moreover,* U' *is also a solution of* (2.1).

2. *The function U is called a mixed solution of (2.1) if $\mathcal{J}$ is partitioned into a finite number of nonempty sub-intervals such that on some of them, U is (i)-differentiable and on remainders (ii)-differentiable and also it satisfies (2.1).*

As in [5], we have the following Lemma:

Lemma 2.1 *The problem (2.1) is equivalent to one of the following fuzzy integral equations system*

(E_1)

$$U(s) = U_0 + \int_a^s F(t, U(t))dt + \int_a^s \int_a^t K(t, \tau, U(\tau))d\tau dt, \quad s \in \mathcal{J},$$

if U is (i)-differentiable;

(E_2)

$$U(s) = U_0 \ominus (-1) \int_a^s F(t, U(t))dt \ominus (-1) \int_a^s \int_a^t K(t, \tau, U(\tau))d\tau dt \quad s \in \mathcal{J},$$

if U is (ii)-differentiable;

(E_3)

$$U(s) = \begin{cases} U_0 + \int_a^s F(t, U(t))dt + \int_a^s \int_a^t K(t, \tau, U(\tau))d\tau dt, & s \in [a, c], \\ U(c) \ominus (-1) \int_c^s F(t, U(t))dt \ominus (-1) \int_c^s \int_a^t K(t, \tau, U(\tau))d\tau dt, & s \in [c, b], \end{cases}$$

if there exists a point $c \in (a, b)$ such that U is (i)-differentiable on $[a, c]$ and (ii)-differentiable on $[c, b]$.

Let us consider the nonlinear mappings $A : C(\mathcal{J}, \mathcal{F}^n) \to C(\mathcal{J}, \mathcal{F}^n)$, corresponding with (E_1) in Lemma 2.1. Define

$$(A\Phi)(s) := U_0 + \int_a^s F(t, \Phi(t))dt + \int_a^s \int_a^t K(t, \tau, \Phi(\tau))d\tau dt, \quad s \in \mathcal{J}.$$

where, $(A\Phi_i)(s) = u_{0_i} + \int_a^s f_i(t, \varphi_1(t), \cdots, \varphi_n(t))dt + \int_a^s \int_a^t k_i(t, \tau, \varphi_1(\tau), \cdots, \varphi_n(\tau))d\tau dt, \quad i = 1, \cdots, n,$ with $\Phi = [\varphi_1, \cdots, \varphi_n]^t$.

Lemma 2.2 *If the valued functions $K : \mathcal{J} \times \mathcal{J} \times \mathcal{F}^n \to \mathcal{F}^n$ and $F : \mathcal{J} \times \mathcal{F}^n \to \mathcal{F}^n$ are continuous and bounded. Then, A is compact.*

Proof : Let Φ be an arbitrary bounded set in $C(\mathcal{J}, \mathcal{F}^n)$. We will prove that $A\Phi$ is relatively compact.

Since f_i and k_i are bounded, there exist $M_i, N_i \geq 0$ such that

$$D(f_i(t, \varphi_1(t), \cdots, \varphi_n(t)), \tilde{0}) \leq M_i, \quad \text{for all } \ t \in J, \quad i = 1, \cdots, n,$$

and

$$D(k_i(t, \tau, \varphi_1(\tau), \cdots, \varphi_n(\tau)), \tilde{0}) \leq N_i, \quad \text{for all } \ t, \tau \in J, \quad i = 1, \cdots, n.$$

Assume that $s_1 \geq s_2$. So,

$$D_v\left(\left(A\Phi\right)(s_1), \left(A\Phi\right)(s_2)\right) = \begin{pmatrix} D\left(\left(A\Phi_1\right)(s_1), \left(A\Phi_1\right)(s_2)\right) \\ \vdots \\ D\left(\left(A\Phi_n\right)(s_1), \left(A\Phi_n\right)(s_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} D(u_{0_1} + \int\limits_a^{s_1} f_1(t, \varphi_1(t), \cdots, \varphi_n(t))dt + \int\limits_a^{s_1}\int\limits_a^{t} k_1(t, \tau, \varphi_1(\tau), \cdots, \varphi_n(\tau))d\tau dt, \\ \\ u_{0_1} + \int\limits_a^{s_2} f_1(s, \varphi_1(t), \cdots, \varphi_n(t))dt + \int\limits_a^{s_2}\int\limits_a^{t} k_1(t, \tau, \varphi_1(\tau), \cdots, \varphi_n(\tau))d\tau dt) \\ \vdots \\ \\ D(u_{0_n} + \int\limits_a^{s_1} f_n(t, \varphi_1(t), \cdots, \varphi_n(t))dt + \int\limits_a^{s_1}\int\limits_a^{t} k_n(t, \tau, \varphi_1(\tau), \cdots, \varphi_n(\tau))d\tau dt, \\ \\ u_{0_n} + \int\limits_a^{s_2} f_n(t, \varphi_1(t), \cdots, \varphi_n(t))dt + \int\limits_a^{s_2}\int\limits_a^{t} k_n(t, \tau, \varphi_1(\tau), \cdots, \varphi_n(\tau))d\tau dt) \end{pmatrix}$$

$$\leq \begin{pmatrix} D(\int_a^{s_1} f_1(t,\varphi_1(t),\cdots,\varphi_n(t))dt, \int_a^{s_2} f_1(t,\varphi_1(t),\cdots,\varphi_n(t))dt) \\ +D(\int_a^{s_1}\int_a^{t} D(k_1(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)),)d\tau dt, \int_a^{s_2}\int_a^{t} D(k_1(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)))d\tau dt) \\ \vdots \\ D(\int_a^{s_1} f_n(t,\varphi_1(t),\cdots,\varphi_n(t))dt, \int_a^{s_2} f_n(t,\varphi_1(t),\cdots,\varphi_n(t))dt) \\ +D(\int_a^{s_1}\int_a^{t} D(k_n(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)),)d\tau dt, \int_a^{s_2}\int_a^{t} D(k_n(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)))d\tau dt) \end{pmatrix}$$

$$\leq \begin{pmatrix} D(\int_{s_2}^{s_1} f_1(t,\varphi_1(t),\cdots,\varphi_n(t))dt,\tilde{0}) + D(\int_{s_2}^{s_1}\int_a^{t} D(k_1(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)),)d\tau dt,\tilde{0}) \\ \vdots \\ D(\int_{s_2}^{s_1} f_n(t,\varphi_1(t),\cdots,\varphi_n(t))dt,\tilde{0}) + D(\int_{s_2}^{s_1}\int_a^{t} D(k_n(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)),)d\tau dt,\tilde{0}) \end{pmatrix}$$

$$\leq \begin{pmatrix} \int_{s_2}^{s_1} D(f_1(t,\varphi_1(t),\cdots,\varphi_n(t)),\tilde{0})dt + \int_{s_2}^{s_1}\int_a^{t} D(D(k_1(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)),\tilde{0})d\tau dt \\ \vdots \\ \int_{s_2}^{s_1} D(f_n(t,\varphi_1(t),\cdots,\varphi_n(t)),\tilde{0})dt + \int_{s_2}^{s_1}\int_a^{t} D(D(k_n(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)),\tilde{0})d\tau dt \end{pmatrix}$$

$$\begin{aligned} &\leq \begin{pmatrix} (s_1-s_2)M_1 \\ \vdots \\ (s_1-s_2)M_n \end{pmatrix} + \begin{pmatrix} (s_1-s_2)(b-a)N_1 \\ \vdots \\ (s_1-s_2)(b-a)N_n \end{pmatrix} \\ &= (s_1-s_2)\begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix} + (s_1-s_2)(b-a)\begin{pmatrix} N_1 \\ \vdots \\ N_n \end{pmatrix} \\ &\leq (s_1-s_2)[M+(b-a)N], \end{aligned}$$

where

$$M := \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}, \text{ and } N = \begin{pmatrix} N_1 \\ \vdots \\ N_n \end{pmatrix}.$$

We have to prove the uniformly boundedness of

$$D_v\left((A\Phi)(s),\tilde{0}_n\right) = \begin{pmatrix} D\left((A\varphi_1)(s),\tilde{0}\right) \\ \vdots \\ D\left((A\varphi_n)(s),\tilde{0}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} D(u_{0_1} + \int\limits_a^s f_1(t,\varphi_1(t),\cdots,\varphi_n(t))dt + \int\limits_a^s\int\limits_a^t k_1(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau))d\tau dt,\tilde{0}) \\ \vdots \\ D(u_{0_n} + \int\limits_a^s f_n(t,\varphi_1(t),\cdots,\varphi_n(t))dt + \int\limits_a^s\int\limits_a^t k_n(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau))d\tau dt,\tilde{0}) \end{pmatrix}$$

$$\leq \begin{pmatrix} D(u_{0_1},\tilde{0}) + \int\limits_a^s D(f_1(t,\varphi_1(t),\cdots,\varphi_n(t)),\tilde{0})dt + \int\limits_a^s\int\limits_a^t D(k_1(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)),\tilde{0})d\tau dt \\ \vdots \\ D(u_{0_n},\tilde{0}) + \int\limits_a^s D(f_n(t,\varphi_1(t),\cdots,\varphi_n(t)),\tilde{0})dt + \int\limits_a^s\int\limits_a^t D(k_n(t,\tau,\varphi_1(\tau),\cdots,\varphi_n(\tau)),\tilde{0})d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} D(u_{0_1},\tilde{0}) \\ \vdots \\ D(u_{0_n},\tilde{0}) \end{pmatrix} + (b-a)\left[\begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix} + (b-a)\begin{pmatrix} N_1 \\ \vdots \\ N_n \end{pmatrix}\right]$$

$$\leq W + (b-a)(M+(b-a)N),$$

where

$$W := \begin{pmatrix} D(u_{0_1},\tilde{0}) \\ \vdots \\ D(u_{0_n},\tilde{0}) \end{pmatrix}, \text{ and } \tilde{0}_n := \begin{bmatrix} \tilde{0} \\ \vdots \\ \tilde{0} \end{bmatrix}.$$

□

We will prove the following Theorem:

Theorem 2.1 *Let $f_i : \mathcal{J}\times\mathcal{F}^n \to \mathcal{F}$ and $k_i : \mathcal{J}\times\mathcal{J}\times\mathcal{F}^n \to \mathcal{F}$, for all $i = 1,..,n$ be bounded continuous functions. Then the problem (2.1) has at least a proper solution which is (i)-differentiable on $\mathcal{J}$. Moreover, if f_i and k_i are Lipschitz continuous relative to their last argument, for all $i = 1,..,n$, i.e. there exist the real numbers $L_{ij},\ C_{ij} >$*

0 $i, j = 1, .., n$, *such that for all* $s, t \in I$, $u_i, v_i \in \mathcal{F}$, $i = 1, .., n$, *we have*

$$D(f_i(s, u_1, \cdots, u_n), f_i(s, v_1, \cdots, v_n) \leq L_{i1}D(u_1, v_1) + \cdots + L_{in}D(u_n, v_n),$$

$$D(k_i(s, t, u_1, \cdots, u_n), k_i(s, t, v_1, \cdots, v_n) \leq C_{i1}D(u_1, v_1) + \cdots + C_{in}D(u_n, v_n).$$

Then, the proper solution of the problem (2.1) *is unique on* $\mathcal{J}$.

Proof : Let us define the closed and convex ball of $C(\mathcal{J}, \mathcal{F}^n)$.

$$\mathcal{B} := \{U \in C(\mathcal{J}, \mathcal{F}^n) : D_v^* \left(U, \tilde{0}_n\right) \leq W + (b-a)\left(M + (b-a)N\right)\}, \quad \text{where}$$

$$D_v^* \left(U, \tilde{0}_n\right) = \sup_{s \in \mathcal{J}} D_v \left(U(s), \tilde{0}_n\right)$$

From Lemma 2.2, we deduce the continuity and compactness of $K : \mathcal{J} \times \mathcal{J} \times \mathcal{F}^n \to \mathcal{F}^n$.

In order to use Schauder's fixed point Theorem, we have to show that $A\mathcal{B} \subseteq \mathcal{B}$.

Or equivalently,

$$D_v(AU(s), \tilde{0}_n) \leq W + (b-a)\left(M + (b-a)N\right), \quad \text{for all } U \in \mathcal{B}.$$

So,

$$D_v^*(AU, \tilde{0}_n) \leq W + (b-a)\left(M + (b-a)N\right).$$

Thus, we conclude that $AU \in \mathcal{B}$.

From fixed point Theorem, A has at least one fixed point U, corrresponding to proper solution of (2.1) .

We have to prove uniqueness of solution, let $U, V \in C(\mathcal{J}, \mathcal{F}^n)$ are two solutions of (2.1). Then,

$$D_v\left(U(s), V(s)\right) = \begin{pmatrix} D(u_{0_1} + \int_a^s f_1(t, u_1(t), \cdots, u_n(t))dt + \int_a^s \int_a^t k_1(t, \tau, u_1(\tau), \cdots, u_n(\tau))d\tau dt, \\ u_{0_1} + \int_a^s f_1(t, v_1(t), \cdots, v_n(t))dt + \int_a^s \int_a^t k_1(t, \tau, v_1(\tau), \cdots, v_n(\tau))d\tau ds) \\ \vdots \\ D(u_{0_n} + \int_a^s f_n(t, u_1(t), \cdots, u_n(t))dt + \int_a^s \int_a^t k_n(t, \tau, u_1(\tau), \cdots, u_n(\tau))d\tau dt, \\ u_{0_n} + \int_a^s f_n(t, v_1(t), \cdots, v_n(t))dt + \int_a^s \int_a^t k_n(t, \tau, v_1(\tau), \cdots, v_n(\tau))d\tau dt)) \end{pmatrix}$$

$$\leq \begin{pmatrix} \int_a^s D(f_1(t,u_1(t),\cdots,u_n(t)), f_1(t,v_1(t),\cdots,v_n(t)))dt \\ \vdots \\ \int_a^s D(f_n(t,u_1(t),\cdots,u_n(t)), f_n(t,v_1(t),\cdots,v_n(t)))dt \end{pmatrix}$$

$$+ \begin{pmatrix} \int_a^s\int_a^t D(k_1(t,\tau,u_1(\tau),\cdots,u_n(\tau)), k_1(t,\tau,v_1(\tau),\cdots,v_n(\tau)))d\tau dt \\ \vdots \\ \int_a^s\int_a^t D(k_n(t,\tau,u_1(\tau),\cdots,u_n(\tau)), k_n(t,\tau,v_1(\tau),\cdots,v_n(\tau)))d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} \int_a^s \sum_{j=1}^n L_{1j} D(u_j(t),v_j(t))dt \\ \vdots \\ \int_a^s \sum_{j=1}^n L_{nj} D(u_j(t),v_j(t))dt \end{pmatrix} + \begin{pmatrix} \int_a^s\int_a^t \sum_{j=1}^n C_{1j} D(u_j(\tau),v_j(\tau))d\tau dt \\ \vdots \\ \int_a^s\int_a^t \sum_{j=1}^n C_{nj} D(u_j(\tau),v_j(\tau))d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} \sum_{j=1}^n L_{1j} \int_a^s D(u_j(t),v_j(t))dt \\ \vdots \\ \sum_{j=1}^n L_{nj} \int_a^s D(u_j(t),v_j(t))dt \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^n C_{1i} \int_a^t\int_a^s D(u_j(t),v_j(t))dt d\tau \\ \vdots \\ \sum_{j=1}^n C_{nj} \int_a^t\int_a^s D(u_j(t),v_j(t))dt d\tau \end{pmatrix}$$

$$\leq \begin{pmatrix} \sum_{j=1}^n \Big(L_{1j} + C_{1j}(b-a)\Big) \int_a^s D(u_j(t),v_j(t))dt \\ \vdots \\ \sum_{j=1}^n \Big(L_{nj} + C_{nj}(b-a)\Big) \int_a^s D(u_j(t),v_j(t))dt \quad . \end{pmatrix}$$

The Gronwall's inequality for the relation

$$D_v(U(s),V(s)) \leq \begin{pmatrix} \sum_{j=1}^n \Big(L_{1j} + C_{1j}(b-a)\Big) \int_a^s D(u_j(t),v_j(t))dt \\ \vdots \\ \sum_{j=1}^n \Big(L_{nj} + C_{nj}(b-a)\Big) \int_a^s D(u_j(t),v_j(t))dt \end{pmatrix}$$

implies that $D_v(U(s),V(s)) \leq 0$ on the interval $\mathcal{J}$. Thus, $U(s) = V(s), \quad \text{for all} \;\; s \in \mathcal{J}$. $\square$

Theorem 2.2 *Let $f_i : \mathcal{J} \times \mathcal{F}^n \to \mathcal{F}$ and $k_i : \mathcal{J} \times \mathcal{J} \times \mathcal{F}^n \to \mathcal{F}$, $i = 1, .., n$ be bounded continuous and Lipschitz continuous functions as mentioned in Theorem 2.1. Let the sequence $U_n : \mathcal{J} \to \mathcal{F}^n$ defined by $U_0(s) = U_0$ and*

$$U_{n+1}(s) = U_0 \ominus (-1). \int_a^s F(t, U_n(t))dt \ominus (-1). \int_a^s \int_a^t K(t, \tau, U_n(\tau))d\tau dt, \quad n \in \mathbb{N}.$$

Then, the problem (2.1) has a unique proper solution which is (ii)-differentiable on $\mathcal{J}$. Furthermore, the successive iteration

$$\begin{aligned} &U_0(s) = U_0, \\ &U_{n+1}(s) = U_0 \ominus (-1). \int_a^s F(t, U_n(t))dt \ominus (-1). \int_a^s \int_a^t K(t, \tau, U_n(\tau))d\tau dt, \end{aligned} \tag{2.2}$$

converges to this solution, where $U_0(s) = U_0 = \begin{bmatrix} u_{0_1} \\ \vdots \\ u_{0_n} \end{bmatrix}$, *and* $U_n = \begin{bmatrix} u_{n_1} \\ \vdots \\ u_{n_n} \end{bmatrix}$.

Proof : We have

$$U_1 = \begin{bmatrix} u_{1_1} \\ \vdots \\ u_{1_n} \end{bmatrix}, \quad U_0 = \begin{bmatrix} u_{0_1} \\ \vdots \\ u_{0_n} \end{bmatrix}.$$

Hence

$$D_v\left(U_1, U_0\right) =$$

$$= \begin{pmatrix} D(u_{0_1} \ominus (-1) \int\limits_a^s f_1(t, u_{0_1}(t), \cdots, u_{0_n}(t))dt \ominus (-1) \int\limits_a^s \int\limits_a^t k_1(t, \tau, u_{0_1}(\tau), \cdots, u_{0_n}(\tau))d\tau dt, u_{0_1}) \\ \vdots \\ D(u_{0_n} \ominus (-1) \int\limits_a^s f_n(t, u_{0_1}(t), \cdots, u_{0_n}(t))dt \ominus (-1) \int\limits_a^s \int\limits_a^t k_n(t, \tau, u_{0_1}(\tau), \cdots, u_{0_n}(\tau))d\tau dt, u_{0_n}) \end{pmatrix}$$

$$\leq \begin{pmatrix} \int\limits_a^s D(f_1(t, u_{0_1}(t), \cdots, u_{0_n}(t)), \tilde{0})dt \\ \vdots \\ \int\limits_a^s D(f_n(t, u_{0_1}(t), \cdots, u_{0_n}(t)), \tilde{0})dt \end{pmatrix} + \begin{pmatrix} \int\limits_a^s \int\limits_a^t D(k_1(t, u_{0_1}(\tau), \cdots, u_{0_n}(\tau)), \tilde{0})d\tau dt \\ \vdots \\ \int\limits_a^s \int\limits_a^t D(k_n(t, u_{0_1}(\tau), \cdots, u_{0_n}(\tau)), \tilde{0})d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} M_1(s-a) \\ \vdots \\ M_n(s-a) \end{pmatrix} + \begin{pmatrix} \frac{(s-a)^2}{2!} N_1 \\ \vdots \\ \frac{(s-a)^2}{2!} N_n \end{pmatrix}.$$

Moreover,

$$\begin{aligned} D_v\left(U_{n+1}(s), U_n(s)\right) &= \begin{pmatrix} D(u_{n+1_1}(s), u_{n_1}(s)) \\ \vdots \\ D(u_{n+1_n}(s), u_{n_n}(s)) \end{pmatrix} \\ &\leq \begin{pmatrix} D(u_{0_1} \ominus (-1) \int_a^s f_1(t, u_{n_1}(t), \cdots, u_{n_n}(t))dt \ominus \\ (-1) \int_a^s \int_a^t k_1(t, \tau, u_{n_1}(\tau), \cdots, u_{n_n}(\tau))d\tau dt, \\ u_{0_1} \ominus (-1) \int_a^s f_1(t, u_{n-1_1}(t), \cdots, u_{n-1_n}(t))dt \ominus \\ (-1) \int_a^s \int_a^t k_1(t, \tau, u_{n-1_1}(\tau), \cdots, u_{n-1_n}(\tau))d\tau dt) \\ \vdots \\ D(u_{0_n} \ominus (-1) \int_a^s f_n(t, u_{n_1}(t), \cdots, u_{n_n}(t))dt \ominus \\ (-1) \int_a^s \int_a^t k_n(t, \tau, u_{n_1}(\tau), \cdots, u_{n_n}(\tau))d\tau dt, \\ u_{0_n} \ominus (-1) \int_a^s f_n(t, u_{n-1_1}(t), \cdots, u_{n-1_n}(t))dt \ominus \\ (-1) \int_a^s \int_a^t k_n(t, \tau, u_{n-1_1}(\tau), \cdots, u_{n-1_n}(\tau))d\tau dt) \end{pmatrix} \end{aligned}$$

$$\leq \begin{pmatrix} \int_a^s D(f_1(t,u_{n_1}(t),\cdots,u_{n_n}(t)),f_1(t,u_{n-1_1}(t),\cdots,u_{n-1_n}(t)))dt \\ \vdots \\ \int_a^s D(f_n(t,u_{n_1}(t),\cdots,u_{n_n}(t)),f_n(t,u_{n-1_1}(t),\cdots,u_{n-1_n}(t))dt \end{pmatrix}$$

$$+ \begin{pmatrix} \int_a^s\int_a^t D(k_1(t,\tau,u_{n_1}(\tau),\cdots,u_{n_n}(\tau)),k_1(t,\tau,u_{n-1_1}(\tau),\cdots,u_{n-1_n}(\tau))d\tau dt \\ \vdots \\ \int_a^s\int_a^t D(k_n(t,\tau,u_{n_1}(\tau),\cdots,u_{n_n}(\tau)),k_n(t,\tau,u_{n-1_1}(\tau),\cdots,u_{n-1_n}(\tau)))d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} \int_a^s \sum_{j=1}^n L_{1j}D(u_{n_j}(t),u_{n-1_j}(t))dt \\ \vdots \\ \int_a^s \sum_{j=1}^n L_{nj}D(u_{n_j}(t),u_{n-1_j}(t))dt \end{pmatrix} + \begin{pmatrix} \int_a^s\int_a^t \sum_{j=1}^n C_{1j}D(u_{n_j}(\tau),u_{n-1_j}(\tau))d\tau dt \\ \vdots \\ \int_a^s\int_a^t \sum_{j=1}^n C_{nj}D(u_{n_j}(\tau),u_{n-1_j}(\tau))d\tau dt \end{pmatrix}$$

$$\begin{aligned} D_v(U_{n+1}(s),U_n(s)) &\leq \begin{pmatrix} \sum_{j=1}^n L_{1j}\int_a^s D(u_{n_j}(t),u_{n-1_j}(t))dt \\ \vdots \\ \sum_{j=1}^n L_{nj}\int_a^s D(u_{n_j}(t),u_{n-1_j}(t))dt \end{pmatrix} \\ &+ \begin{pmatrix} \sum_{j=1}^n C_{1j}\int_a^t\int_a^s D(u_{n_j}(t),u_{n-1_j}(t))dtd\tau \\ \vdots \\ \sum_{j=1}^n C_{nj}\int_a^t\int_a^s D(u_{n_j}(t),u_{n-1_j}(t))dtd\tau \end{pmatrix}. \end{aligned}$$

Also,

$$D_v(U_{n+1}(s),U_n(s)) \leq \begin{pmatrix} \sum_{j=1}^n (L_{1j}+C_{1j}(b-a))\int_a^s D(u_{n_j}(t),u_{n-1_j}(t)))dt \\ \vdots \\ \sum_{j=1}^n (L_{nj}+C_{nj}(b-a))\int_a^s D(u_{n_j}(t),u_{n-1_j}(t)))dt \end{pmatrix}$$

$$\leq \begin{pmatrix} \sum_{j=1}^n (L_{1j}+C_{1j}(b-a))^n\left[\frac{(s-a)^{n+1}}{(n+1)!}M_1+\frac{(s-a)^{n+2}}{(n+2)!}N_1\right] \\ \vdots \\ \sum_{j=1}^n (L_{nj}+C_{nj}(b-a))^n\left[\frac{(s-a)^{n+1}}{(n+1)!}M_n+\frac{(s-a)^{n+2}}{(n+2)!}N_n\right] \end{pmatrix}.$$

This shows that $(U_n)_{n\geq 0}$ is a Cauchy sequence in $C(\mathcal{J},\mathcal{F}^n)$. Thus, there exists $U\in C(\mathcal{J},\mathcal{F}^n)$ such that $(U_n)_{n\geq 0}$ converges to U.

Now, we show that U is a solution of the problem (2.1). We have

$$\begin{pmatrix} D\Big(u_1(s)+(-1)\int_a^s f_1(t,u_1(t),\cdots,u_n(t))dt+(-1)\int_a^s\int_a^t k_1(t,\tau,u_1(\tau),\cdots,u_n(\tau))d\tau dt, u_{0_1}\Big) \\ \vdots \\ D\Big(u_n(s)+(-1)\int_a^s f_n(t,u_1(t),\cdots,u_n(t))dt+(-1)\int_a^s\int_a^t k_n(t,\tau,u_1(\tau),\cdots,u_n(\tau))d\tau dt, u_{0_n}\Big) \end{pmatrix}$$

$$= \begin{pmatrix} D\Big(u_1(s)+(-1)\int_a^s f_1(t,u_1(t),\cdots,u_n(t))dt+(-1)\int_a^s\int_a^t k_1(t,\tau,u_1(\tau),\cdots,u_n(\tau))\Big)d\tau dt, \\ u_{n+1_1}(s)+(-1)\int_a^s f_1(t,u_{n_1}(t),\cdots,u_{n_n}(t))dt)+(-1)\int_a^s\int_a^t k_1(t,\tau,u_{n_1}(\tau),\cdots,u_{n_n}(\tau))\Big)d\tau dt \\ \vdots \\ D\Big(u_n(s)+(-1)\int_a^s f_n(t,u_1(t),\cdots,u_n(t))dt+(-1)\int_a^s\int_a^t k_n(t,\tau,u_1(\tau),\cdots,u_n(\tau))d\tau dt, \\ u_{n+1_n}(s)+(-1)\int_a^s f_n(t,u_{n_1}(t),\cdots,u_{n_n}(t))dt+(-1)\int_a^s\int_a^t k_n(t,\tau,u_{n_1}(\tau),\cdots,u_{n_n}(\tau))\Big)d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} D\Big(u_1(s),u_{n+1_1}(s)\Big)+\int_a^s D\Big(f_1(t,u_1(t),\cdots,u_n(t)),f_1(s,u_{n_1}(t),\cdots,u_{n_n}(t))\Big)dt \quad + \\ \int_a^s\int_a^t D\Big(k_1(t,\tau,u_1(\tau),\cdots,u_n(\tau)),k_1(t,\tau,u_{n_1}(\tau),\cdots,u_{n_n}(\tau))\Big)d\tau dt \\ \vdots \\ D\Big(u_n(s),u_{n+1_n}(s)\Big)+\int_a^s D\Big(f_n(t,u_1(t),\cdots,u_n(t)),f_n(t,u_{n_1}(t),\cdots,u_{n_n}(t))\Big)dt \quad + \\ \int_a^s\int_a^t D\Big(k_n(t,\tau,u_1(\tau),\cdots,u_n(\tau)),k_n(t,\tau,u_{n_1}(\tau),\cdots,u_{n_n}(\tau))\Big)d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} D\Big(u_1(s),u_{n+1_1}(s)\Big)+\int_a^s\Big[L_{11}D\Big(u_1(t),u_{n_1}(t)\Big)+\cdots+L_{1n}D\Big(u_n(t),u_{n_n}(t)\Big)\Big]dt+ \\ \int_a^s\int_a^t\Big[C_{11}D\Big(u_1(\tau),u_{n_1}(\tau)\Big)+\cdots+C_{nn}D\Big(u_n(\tau),u_{n_n}(\tau)\Big)\Big]d\tau dt \\ \vdots \\ D\Big(u_n(s),u_{n+1_n}(s)\Big)+\int_a^s D\Big[L_{n1}D\Big(u_1(t),u_{n_1}(t)\Big)+\cdots+L_{nn}\Big(u_n(t),u_{n_n}(t)\Big)\Big]dt+ \\ \int_a^s\int_a^t\Big[C_{n1}D\Big(u_1(\tau),u_{n_1}(\tau)\Big)+\cdots+C_{nn}D\Big(u_n(\tau),u_{n_n}(\tau)\Big)\Big]d\tau dt \end{pmatrix}.$$

The right-hand side tends to $\tilde{0}$ as $n\to\infty$. Hence,

$$u_i(s)+(-1)\int_a^s f_i(t,u_1(t),\cdots,u_n(t))dt+(-1)\int_a^s\int_a^t k_i(t,\tau,u_1(\tau),\cdots,u_n(\tau))d\tau dt=u_{0_i}.$$

The uniqueness is proven by using Gronwall's inequality, which is similar to the proof of Theorem 2.1. □

Now, we prove the following Theorem:

Theorem 2.3 *Let $f_i : \mathcal{J} \times \mathcal{F}^n, \mathcal{F}^n$ and $k_i : \mathcal{J} \times \mathcal{J} \times \mathcal{F}^n \to \mathcal{F}^n, i = 1, \cdots, n$ be bounded continuous functions. Let $\underline{U}$ be a solution of the problem (2.1) on $[a, c]$ which is (i)-differentiable. Assume that the functions $f_i : [c, b] \times \mathcal{F}^n \to \mathcal{F}$ and $k_i : [c, b] \times [c, b] \times \mathcal{F}^n \to \mathcal{F}$, $i = 1, .., n$ are Lipschitz continuous relative to their last argument. In addition, the sequence*

$$\begin{aligned} U_0(s) &= \underline{U}(c), \\ U_{n+1}(s) &= \underline{U}(c) \ominus (-1). \int_c^s F(t, U_n(t))dt \ominus (-1). \int_c^s \int_a^c K(t, \tau, \underline{U}(\tau))d\tau dt \\ &\ominus \; (-1) \int_c^s \int_c^t K(t, \tau, U_n(\tau))d\tau dt, \quad n \in \mathbb{N} \end{aligned}$$

is well-defined. Then, the problem 2.1 has a mixed solution.

Proof : Because all of the conditions of Theorem 2.1 are satisfied on $[a, c]$, there exists a $\underline{U}$ solution to the problem (2.1) on $[a, c]$. We show that the introduced sequence in the Theorem is a Cauchy sequence in $C([c, b], \mathcal{F}^n)$.

$$D'(U_1(s), U_0(s)) = \begin{pmatrix} D\Big(\underline{u}_{1_1}(s), \underline{u}_{0_1}\Big) \\ \vdots \\ D\Big(\underline{u}_{1_n}(s), \underline{u}_{0_n}\Big) \end{pmatrix} =$$

$$\begin{pmatrix} D\Big(\underline{u}_{0_1} \ominus (-1) \int_c^s f_1(s, \underline{u}_1(t), \cdots, \underline{u}_n(t))dt \ominus (-1) \int_c^s \int_a^t k_1(t, \tau, \underline{u}_1(\tau), \cdots, \underline{u}_n(\tau))d\tau dt, \underline{u}_{0_1}\Big) \\ \vdots \\ D\Big(\underline{u}_{0_n} \ominus (-1) \int_c^s f_n(s, \underline{u}_1(t), \cdots, \underline{u}_n(t))dt \ominus (-1) \int_c^s \int_a^t k_n(t, \tau, \underline{u}_1(\tau), \cdots, \underline{u}_n(\tau))d\tau dt, \underline{u}_{0_n}\Big) \end{pmatrix} \leq$$

$$\begin{pmatrix} \int_c^s D\Big(f_1(t, \underline{u}_1(t), \cdots, \underline{u}_n(t)), \tilde{0}\Big)dt + \int_c^s \int_a^t D\Big(k_1(t, \tau, \underline{u}_1(\tau), \cdots, \underline{u}_n(\tau)), \tilde{0}\Big)d\tau dt \\ \vdots \\ \int_c^s D\Big(f_n(t, \underline{u}_1(t, \cdots, \underline{u}_n(t)), \tilde{0}\Big)dt + \int_c^s \int_a^t D\Big(k_n(t, \tau, \underline{u}_1(\tau), \cdots, \underline{u}_n(\tau)), \tilde{0}\Big)d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} \int_c^s M_1 dt \\ \vdots \\ \int_c^s M_n dt \end{pmatrix} + \begin{pmatrix} \int_c^s \int_a^t N_1 d\tau dt \\ \vdots \\ \int_c^s \int_a^t N_n d\tau dt \end{pmatrix}$$

$$= \begin{pmatrix} \int_c^s M_1 dt \\ \vdots \\ \int_c^s M_n dt \end{pmatrix} + \begin{pmatrix} \int_c^s \int_a^c N_1 d\tau dt + \int_c^s \int_c^t N_1 d\tau dt \\ \vdots \\ \int_c^s \int_a^c N_n d\tau dt + \int_c^s \int_c^t N_n d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} M_1(s-c) \\ \vdots \\ M_n(s-c) \end{pmatrix} + \begin{pmatrix} N_1(s-c)(c-a) + N_1(s-c)(t-c) \\ \vdots \\ N_n(s-c)(c-a) + N_n(s-c)(t-c) \end{pmatrix}$$

$$\leq \begin{pmatrix} M_1 + N_1(t-a) \\ \vdots \\ M_n + N_n(t-a) \end{pmatrix} (s-c).$$

We will continue this pattern for $n \in \mathbb{N}$, to obtain

$$D_v\left(U_{n+1}(s), U_n(s)\right) = \begin{pmatrix} D\Big(u_{n+1_1}(s), u_{n_1}(s)\Big) \\ \vdots \\ D\Big(u_{n+1_n}(s), u_{n_n}(s)\Big) \end{pmatrix}$$

$$\leq \begin{pmatrix} D\Big(u_{0_1}(s) \ominus (-1)\int_c^s f_1(t, u_{n_1}(t), \cdots, u_{n_n}(t))dt) \\ \ominus(-1)\int_c^s \int_c^t k_1(t, \tau, u_{n_1}(\tau), \cdots, u_{n_n}(\tau))d\tau dt, \\ u_{0_1}(s) \ominus (-1)\int_c^s f_1(t, u_{n-1_1}(t), \cdots, u_{n-1_n}(t))dt) \\ \ominus(-1)\int_c^s \int_c^t k_1(t, \tau, u_{n-1_1}(\tau), \cdots, u_{n-1_n}(\tau))d\tau dt\Big) \\ \vdots \\ D\Big(u_{0_n}(s) \ominus (-1)\int_c^s f_n(t, u_{n_1}(t), \cdots, u_{n_n}(t))dt) \\ \ominus(-1)\int_c^s \int_c^t k_n(t, \tau, u_{n_1}(\tau), \cdots, u_{n_n}(\tau))d\tau dt, \\ u_{0_n}(s) \ominus (-1)\int_c^s f_n(t, u_{n-1_1}(t), \cdots, u_{n-1_n}(t))dt) \\ \ominus(-1)\int_c^s \int_c^t k_n(t, \tau, u_{n-1_1}(\tau), \cdots, u_{n-1_n}(\tau))d\tau dt\Big) \end{pmatrix}$$

$$\leq \begin{pmatrix} \int_c^s D\Big(f_1(t, u_{n_1}(t), \cdots, u_{n_n}(t)), f_1(t, u_{n-1_1}(t), \cdots, u_{n-1_n}(t))\Big)dt+ \\ \int_c^s \int_c^t D\Big(k_1(t, \tau, u_{n_1}(\tau), \cdots, u_{n_n}(\tau)), k_1(t, \tau, u_{n-1_1}(\tau), \cdots, u_{n-1_n}(\tau))\Big)d\tau dt \\ \vdots \\ \int_c^s D\Big(f_n(t, u_{n_1}(t), \cdots, u_{n_n}(t)), f_n(t, u_{n-1_1}(t), \cdots, u_{n-1_n}(t))\Big)dt+ \\ \int_c^s \int_a^t D\Big(k_n(t, \tau, u_{n_1}(\tau), \cdots, u_{n_n}(\tau)), k_n(t, \tau, u_{n-1_1}(\tau), \cdots, u_{n-1_n}(\tau))\Big)d\tau dt \end{pmatrix}$$

$$\leq \begin{pmatrix} \int_c^s \Big[L_{11}D\Big(u_{n_1}(t), u_{n-1_1}(t)\Big) + \cdots + L_{1n}D\Big(u_{n_n}(t), u_{n-1_n}(t)\Big)\Big]dt+ \\ \int_c^s \int_c^t \Big[C_{11}D\Big(u_{n_1}(\tau), u_{n-1_1}(\tau)\Big) + \cdots + C_{1n}D\Big(u_{n_n}(\tau), u_{n-1_n}(\tau)\Big)\Big]d\tau dt \\ \vdots \\ \int_c^s D\Big[L_{n1}D\Big(u_{n_1}(t), u_{n-1_1}(t)\Big) + \cdots + L_{nn}D\Big(u_{n_n}(t), u_{n-1_n}(t)\Big)\Big]dt+ \\ \int_c^s \int_c^t \Big[C_{n1}D\Big(u_{n_1}(\tau), u_{n-1_1}(\tau)\Big) + \cdots + C_{nn}D\Big(u_{n_n}(\tau), u_{n-1_n}(\tau)\Big)\Big]d\tau dt \end{pmatrix}$$

$$D_v(U_{n+1}(s),U_n(s)) \leq \begin{pmatrix} \int_c^s \Big[L_{11}D\Big(u_{n_1}(t),u_{n-1_1}(t)\Big)+\cdots+L_{1n}D\Big(u_{n_n}(t),u_{n-1_n}(t)\Big)\Big]dt+ \\ \int_c^t\int_c^s \Big[C_{11}D\Big(u_{n_1}(t),u_{n-1_1}(t)\Big)+\cdots+C_{1n}D\Big(u_{n_n}(t),u_{n-1_n}(t)\Big)\Big]dtd\tau \\ \vdots \\ \int_c^s D\Big[L_{n1}D\Big(u_{n_1}(t),u_{n-1_1}(t)\Big)+\cdots+L_{nn}D\Big(u_{n_n}(t),u_{n-1_n}(t)\Big)\Big]dt+ \\ \int_c^t\int_c^s \Big[C_{n1}D\Big(u_{n_1}(t),u_{n-1_1}(t)\Big)+\cdots+C_{nn}D\Big(u_{n_n}(t),u_{n-1_n}(t)\Big)\Big]dtd\tau \end{pmatrix}$$

$$\leq \begin{bmatrix} \Big[L_{11}+C_{11}(b-a)\Big]\int_c^s D\Big(u_{1_n}(t),u_{1_{n-1}}(t)\Big)dt+\cdots+\Big[L_{1n}+C_{1n}(b-a)\Big]\int_c^s D\Big(u_{n_n}(t),u_{n_{n-1}}(t)\Big)\Big]dt \\ \vdots \\ \Big[L_{n1}+C_{n1}(b-a)\Big]\int_c^s D\Big(u_{1_n}(t),u_{1_{n-1}}(t)\Big)dt+\cdots+\Big[L_{nn}+C_{nn}(b-a)\Big]\int_c^s D\Big(u_{n_n}(t),u_{n_{n-1}}(t)\Big)\Big]dt \end{bmatrix}.$$

Also,

$$D_v(U_{n+1}(s),U_n(s)) \;\leq\; \frac{(s-c)^{n+1}}{(n+1)!}\begin{bmatrix} \Big(M_1+N_1(b-a)\Big)\sum_{j=1}^n\Big(L_{1j}+C_{1j}(b-a)\Big)^n \\ \vdots \\ \Big(M_n+N_n(b-a)\Big)\sum_{j=1}^n\Big(L_{nj}+C_{nj}(b-a)\Big)^n \end{bmatrix}.$$

This proves that $(U_n)_{n\geq 0}$ is a Cauchy sequence in $C([c,b],\mathcal{F}^n)$. Then, there exists $\bar{U}\in C([c,b],\mathcal{F}^n)$ such that $(U_n)_{n\geq 0}$ converges to $\bar{U}$.

We claim that $\bar{U}$ satisfies the integral equation:

$$\begin{aligned}\bar{U} \;&+\; (-1)\Big(\int_c^s f_1(t,\bar{u}_1(t),\cdots,\bar{u}_n(t))dt+\int_c^s\int_c^t k_1(t,\tau,\bar{u}_1(\tau),\cdots,\bar{u}_n(\tau))d\tau dt \\ &+\; \int_c^s\int_a^c k_1(t,\tau,\underline{u}_1(\tau),\cdots,\underline{u}_n(\tau))d\tau dt\Big)=\underline{U}(c)\end{aligned}$$

We have to prove

$$\begin{bmatrix} \bar{u}_1(s) \\ \vdots \\ \bar{u}_n(s) \end{bmatrix} + \begin{bmatrix} (-1)\Big(\int_c^s f_1(t,\bar{u}_1(t),\cdots,\bar{u}_n(t))dt + \int_c^s \int_c^t k_1(t,\tau,\bar{u}_1(\tau),\cdots,\bar{u}_n(\tau))d\tau dt + \\ \int_c^s \int_a^c k_1(t,\tau,\underline{u}_1(\tau),\cdots,\underline{u}_n(\tau))d\tau dt\Big) \\ \vdots \\ (-1)\Big(\int_c^s f_n(t,\bar{u}_1(t),\cdots,\bar{u}_n(t))dt + \int_c^s \int_c^t k_n(t,\tau,\bar{u}_1(\tau),\cdots,\bar{u}_n(\tau))d\tau dt + \\ \int_c^s \int_a^c k_n(t,\tau,\underline{u}_1(\tau),\cdots,\underline{u}_n(\tau))d\tau dt\Big) \end{bmatrix}$$

$$= \begin{bmatrix} \underline{u}_{0_1}(c) \\ \vdots \\ \underline{u}_{0_n}(c) \end{bmatrix}. \tag{2.3}$$

To prove the assertion, we have

$$\begin{bmatrix} D\Big(\bar{u}_1(s) + (-1)\Big(\int_c^s f_1(t,\bar{u}_1(t),\cdots,\bar{u}_n(t))dt + \int_c^s \int_c^t k_1(t,\tau,\bar{u}_1(\tau),\cdots,\bar{u}_n(\tau))d\tau dt + \\ \int_c^s \int_a^c k_1(t,\tau,\underline{u}_1(\tau),\cdots,\underline{u}_n(\tau))d\tau dt\Big), \underline{u}_{0_1}(c)\Big) \\ \vdots \\ D\Big(\bar{u}_n(s) + (-1)\Big(\int_c^s f_n(t,\bar{u}_1(t),\cdots,\bar{u}_n(t))dt + \int_c^s \int_c^t k_n(t,\tau,\bar{u}_1(\tau),\cdots,\bar{u}_n(\tau))d\tau dt + \\ \int_c^s \int_a^c k_n(t,\tau,\underline{u}_1(\tau),\cdots,\underline{u}_n(\tau))d\tau dt\Big), \underline{u}_{0_n}(c)\Big) \end{bmatrix}$$

$$\leq \begin{bmatrix} D\Big(\bar{u}_1(s), u_{n+1_1}(s)\Big) + \Big(\int_c^s \Big[L_{11} D\Big(\bar{u}_1(t), u_{n_1}(t)\Big) + \cdots + L_{1n} D\Big(\bar{u}_n(\tau), u_{n_n}(t)\Big)\Big] dt + \\ \Big(\int_c^s \int_c^t \Big[C_{11} D\Big(\bar{u}_1(t), u_{n_1}(\tau)\Big) + \cdots + C_{1n} D\Big(\bar{u}_n(\tau), u_{n_n}(\tau)\Big)\Big] d\tau dt \\ \vdots \\ D\Big(\bar{u}_n(s), u_{n+1_n}(s)\Big) + \Big(\int_c^s \Big[L_{n1} D\Big(\bar{u}_1(t), u_{n_1}(t)\Big) + \cdots + L_{nn} D\Big(\bar{u}_n(t), u_{n_n}(t)\Big)\Big] dt + \\ \Big(\int_c^s \int_c^t \Big[C_{n1} D\Big(\bar{u}_1(\tau), u_{n_1}(\tau)\Big) + \cdots + C_{nn} D\Big(\bar{u}_n(\tau), u_{n_n}(\tau)\Big)\Big] d\tau dt \end{bmatrix}.$$

As $n \to \infty$, the last term tends to zero. Consequently, $\bar{U}$ satisfies the (2.3) for all $s \in [c, b]$. □

2.2 Fuzzy Fredholm integro-differential system

This section examines the fuzzy Fredholm integro-differential system of the form:

$$\Phi'(s) - \lambda \int_{-1}^{1} K(s,t)\Phi(t)dt = G(s), \quad -1 < s < 1, \tag{2.4}$$

where

$$\begin{aligned}
\Phi'(s) &= [\varphi_1'(s), \cdots, \varphi_n'(s)]^t, \\
\Phi(s) &= [\varphi_1(s), \cdots, \varphi_n(s)]^t, \\
K(s,t) &= [k_{ij}(s,t)]; \quad i,j = 1, \cdots, n, \\
G(s) &= [g_1(s), \cdots, g_n(s)]^t,
\end{aligned}$$

subject to

$$\varphi_i(-1) = \varphi_{0i}, \quad \text{for all} \quad i = 1, \cdots, n.$$

Theorem 2.4 *Assume that*

$$\forall \epsilon_i > 0,\ \exists \delta > 0,\ \forall s_1, s_2 \in [a,b] \ \textit{with}\ |s_1 - s_2| \leq \delta :\ D(k_{ij}(t,\tau)\varphi_j(\tau), \tilde{0}) < \frac{\epsilon_i}{2n\delta}, i = 1, \cdots, n,$$

$$\forall i = 1, \cdots, n,\ \exists \alpha_i > 0 : D(g_i(t), \tilde{0}) \leq \frac{\epsilon_i'}{\delta}$$

and

$$\forall i = 1, \cdots, n,\ \exists M_{ij} > 0 : D(k_{ij}(t,\tau)\varphi_j(\tau), k_{ij}(t,\tau)\psi_j(\tau)) \leq M_{ij} D^*(\varphi_j, \psi_j),\ j = 1, \cdots, n.$$

Then, the problem (2.4) has a unique continuous solution $\Phi^* \in (C(\mathcal{I}))^n$, where

$$\Phi^*(s) = [\varphi_1^*(s), \cdots, \varphi_n^*(s)]^t.$$

Proof : The system (2.4) reads as

$$\begin{cases}
\varphi_1'(s) - \lambda \sum_{j=1}^{n} \int_{-1}^{1} k_{1j}(s,t)\varphi_j(t)dt = g_1(s), \\
\vdots \\
\varphi_n'(s) - \lambda \sum_{j=1}^{n} \int_{-1}^{1} k_{nj}(s,t)\varphi_j(t)dt = g_n(s).
\end{cases}$$

Hence

$$\begin{cases} \int_{-1}^{s} \varphi_1'(t)dt - \lambda \sum\limits_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{1j}(t,\tau)\varphi_j(\tau)d\tau dt = \int_{-1}^{s} g_1(t)dt, \\ \vdots \\ \int_{-1}^{s} \varphi_n'(t)dt - \lambda \sum\limits_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{nj}(t,\tau)\varphi_j(\tau)d\tau dt = \int_{-1}^{s} g_n(t)dt, \end{cases}$$

and hence

$$\begin{cases} \varphi_1(s) = \varphi_{01} + \lambda \sum\limits_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{1j}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s} g_1(t)dt, \\ \vdots \\ \varphi_n(s) = \varphi_{0n} + \lambda \sum\limits_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{nj}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s} g_n(t)dt. \end{cases} \tag{2.5}$$

The system (2.4) is equivalent to

$$\Phi(s) = \Phi_0 + \lambda \int_{-1}^{s} \int_{-1}^{1} K(t,\tau)\Phi(\tau)d\tau dt + \int_{-1}^{s} G(t)dt, \quad -1 < s < 1.$$

Letting

$$(A\Phi)(s) := \Phi_0 + \lambda \int_{-1}^{s} \int_{-1}^{1} K(t,\tau)\Phi(\tau)d\tau dt + \int_{-1}^{s} G(t)dt, \quad -1 < s < 1,$$

where

$$\begin{cases} (A\varphi_1)(s) = \varphi_{01} + \lambda \sum\limits_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{1j}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s} g_1(t)dt, \\ \vdots \\ (A\varphi_n)(s) = \varphi_{0n} + \lambda \sum\limits_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{nj}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s} g_n(t)dt. \end{cases}$$

We consider the metric $D_v : \mathcal{F}^n \times \mathcal{F}^n \to \mathbb{R}_+^n$, where

$$D_v(\Phi(s),\Psi(s)) = \begin{pmatrix} D(\varphi_1(s),\psi_1(s)) \\ \vdots \\ D(\varphi_n(s),\psi_n(s)) \end{pmatrix}.$$

We have to prove that $A((C(\mathcal{I}))^n) \subset (C(\mathcal{I}))^n$. To this aim, for all $\Phi, \Psi \in (C(\mathcal{I}))^n$, and $s_1, s_2 \in \mathcal{I}$, with $s_1 > s_2$, we have

$$D_v((A\Phi)(s_1),(A\Phi)(s_2)) = \begin{pmatrix} D((A\varphi)_1(s_1),(A\varphi)_1(s_2)) \\ \vdots \\ D((A\varphi)_n(s_1),(A\varphi)_n(s_2)) \end{pmatrix}$$

$$= \begin{pmatrix} D(\varphi_{01} + \lambda \sum\limits_{j=1}^{n} \int_{-1}^{s_1}\int_{-1}^{1} k_{1j}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s_1} g_1(t)dt, \\ \varphi_{01} + \lambda \sum\limits_{j=0}^{n} \int_{-1}^{s_2}\int_{-1}^{1} k_{1j}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s_2} g_1(t)dt) \\ \vdots \\ D(\varphi_{0n} + \lambda \sum\limits_{j=1}^{n} \int_{-1}^{s_1}\int_{-1}^{1} k_{nj}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s_1} g_n(t)dt, \varphi_{0n}+ \\ +\lambda \sum\limits_{j=0}^{n} \int_{-1}^{s_2}\int_{-1}^{1} k_{nj}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s_2} g_n(t)dt) \end{pmatrix}$$

$$= \begin{pmatrix} D(\lambda \sum\limits_{j=1}^{n} \int_{s_2}^{s_1}\int_{-1}^{1} k_{1j}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{s_2}^{s_1} g_1(t)dt, \tilde{0}) \\ \vdots \\ D(\lambda \sum\limits_{j=1}^{n} \int_{s_2}^{s_1}\int_{-1}^{1} k_{nj}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{s_2}^{s_1} g_n(t)dt, \tilde{0})) \end{pmatrix}$$

$$\leq \begin{pmatrix} \lambda \sum\limits_{j=1}^{n} \int_{s_2}^{s_1}\int_{-1}^{1} D(k_{1j}(t,\tau)\varphi_j(\tau),\tilde{0})d\tau dt + \int_{s_2}^{s_1} D(g_1(t),\tilde{0})dt \\ \vdots \\ \lambda \sum\limits_{j=1}^{n} \int_{s_2}^{s_1}\int_{-1}^{1} D(k_{nj}(t,\tau)\varphi_j(\tau),\tilde{0})d\tau dt + \int_{s_2}^{s_1} D(g_n(t),\tilde{0})dt \end{pmatrix}$$

$$\leq \begin{pmatrix} 2\lambda(s_1 - s_2) \sum\limits_{j=1}^{n} \frac{\epsilon_1}{2n\delta} + (s_1 - s_2)\frac{\epsilon_1^{'}}{\delta} \\ \vdots \\ 2\lambda(s_1 - s_2) \sum\limits_{j=1}^{n} \frac{\epsilon_n}{2n\delta} + (s_1 - s_2)\frac{\epsilon_n^{'}}{\delta} \end{pmatrix}$$

$$\leq \lambda \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix} + \begin{pmatrix} \epsilon_1^{'} \\ \vdots \\ \epsilon_n^{'} \end{pmatrix}$$
$$\leq \lambda\epsilon + \epsilon',$$

where

$$\epsilon = [\epsilon_1, \cdots, \epsilon_n]^t,$$
$$\epsilon' = [\epsilon', \cdots, \epsilon']^t.$$

Thus, the operator A is uniformly continuous. It follows $A((C(\mathcal{I}))^n) \subset (C(\mathcal{I}))^n$.

Now, we will examine the continuous of A on $(C(\mathcal{I}))^n$. Let $\Phi, \Psi \in (C(\mathcal{I}))^n$, $s \in \mathcal{I}$. In fact,

$$D_v((A\Phi)(s), (A\Psi)(s)) = \begin{pmatrix} D((A\varphi_1)(s), (A\psi_1)(s)) \\ \vdots \\ D((A\varphi_n)(s), (A\psi_n)(s)) \end{pmatrix}$$

$$= \begin{pmatrix} D(\varphi_{01} + \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{1j}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s} g_1(t)dt, \\ \varphi_{01} + \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{1j}(t,\tau)\psi_j(\tau)d\tau dt + \int_{-1}^{s} g_1(t)dt) \\ \vdots \\ D(\varphi_{0n} + \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{nj}(t,\tau)\varphi_j(\tau)d\tau dt + \int_{-1}^{s} g_n(t)dt, \\ \varphi_{0n} + \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{nj}(t,\tau)\psi_j(\tau)d\tau dt + \int_{-1}^{s} g_n(t)dt) \end{pmatrix}$$

$$\leq \begin{pmatrix} \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} D(k_{1j}(t,\tau)\varphi_j(\tau), k_{1j}(t,\tau)\psi_j(\tau))d\tau dt \\ \vdots \\ \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} D(k_{nj}(t,\tau)\varphi_j(\tau), k_{nj}(t,\tau)\psi_j(\tau))d\tau dt \end{pmatrix}$$

$$D_v^*(A\Phi, A\Psi) \leq 2\lambda\delta \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix} \begin{pmatrix} D^*(\varphi_1, \psi_1) \\ \vdots \\ D^*(\varphi_n, \psi_n) \end{pmatrix}.$$

Thus, A is continuous.

Now, in order to prove the compactness of the operator A, we use the Arzelà-Ascoli theorem.

Let $F := \{\varphi_{im}, n \in \mathbb{N};\ \forall i = 1, \cdots, n\}$ be a bounded set of $(C(\mathcal{I}))^n$ with the constant c.

So, $\forall i = 1, \cdots, n,\ \exists c_i > 0; \forall m \in\ \mathbb{N}, ||\varphi_{im}(.)||_{\mathcal{F}} \leq c_i$, where

$$\Phi_m = [\varphi_{1m}, \cdots, \varphi_{nm}]^t, \quad \text{and}$$

$$||\Phi_m(.)||_v = \begin{bmatrix} ||\varphi_{i1}(.)||_{\mathcal{F}} \\ \vdots \\ ||\varphi_{in}(.)||_{\mathcal{F}} \end{bmatrix} = D_v(\Phi_m, \tilde{0}_m) = \begin{bmatrix} D(\varphi_{i1}, \tilde{0}) \\ \vdots \\ D(\varphi_{im}, \tilde{0}) \end{bmatrix},$$

and hence

$$D_v(\tilde{0}, (A\Phi_m)(s)) = \begin{pmatrix} D(\tilde{0}, (A\varphi_{1m})(s)) \\ \vdots \\ D(\tilde{0}, (A\varphi_{nm})(s)) \end{pmatrix}$$

$$= \begin{pmatrix} D(\tilde{0}, \varphi_{01} + \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{1j}(t,\tau)\varphi_{jm}(\tau)d\tau dt + \int_{-1}^{s} g_1(t)dt) \\ \vdots \\ D(\tilde{0}, \varphi_{0n} + \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} k_{nj}(t,\tau)\varphi_{jm}(\tau)d\tau dt + \int_{-1}^{s} g_n(t)dt) \end{pmatrix}$$

$$\leq \begin{pmatrix} D(\tilde{0}, \varphi_{01}) + \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} D(\tilde{0}, k_{1j}(t,\tau)\varphi_{jm}(\tau))d\tau dt + \int_{-1}^{s} D(\tilde{0}, g_1(t))dt) \\ \vdots \\ D(\tilde{0}, \varphi_{0n}) + \lambda \sum_{j=1}^{n} \int_{-1}^{s} \int_{-1}^{1} D(\tilde{0}, k_{nj}(t,\tau)\varphi_{jm}(\tau))d\tau dt + \int_{-1}^{s} D(\tilde{0}, g_n(t))dt) \end{pmatrix}$$

$$\leq \begin{pmatrix} D(\tilde{0}, \varphi_{01}) + 4\lambda \sum_{j=1}^{n} \frac{\epsilon_1}{2n\delta} + 2\frac{\epsilon_1^{'}}{\delta} \\ \vdots \\ D(\tilde{0}, \varphi_{0n}) + 4\lambda \sum_{j=1}^{n} \frac{\epsilon_n}{2n\delta} + 2\frac{\epsilon_n^{'}}{\delta} \end{pmatrix}$$

$$\leq \begin{pmatrix} D(\tilde{0}, \varphi_{01}) \\ \vdots \\ D(\tilde{0}, \varphi_{0n}) \end{pmatrix} + \frac{2\lambda}{\delta} \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix} + \frac{2}{\delta} \begin{pmatrix} \epsilon_1^{'} \\ \vdots \\ \epsilon_n^{'} \end{pmatrix}$$

Thus, $A(F)$ is bounded.

We prove that $A(F)$ is equicontinuous, that is

$$\forall \epsilon > 0 \, \exists \delta > 0; \; \forall s_1, s_2 \in \mathcal{I}, \; A\Phi_m \in A(F): \; |s_1 - s_2| < \delta \Rightarrow D^{'}(A\Phi_m(s_1), A\Psi_m(s_2)) < \epsilon.$$

In the same manner as previously, it follows that $A(F)$ is equicontinuous. Therefore, according to the Arzelà-Ascoli, A is compact, and so that A from $(C(\mathcal{I}))^n$ into itself is completely continuous.

According to Schauder fixed point Theorem the system (2.4) has a continuous solution. □

Chapter 3

Two classes of fuzzy singular integro-differential equations

3.1 Introduction

Many studies discuss numerical methods for solving differential and integro-differential problems (see, [26, 27, 34, 35, 36]). More recently, reference [48] investigated the approximate solution of Cauchy integro-differential equations using the Legendre projection approximation. The reference [49] describes a collocation approach for solving logarithmic singular integro-differential equations utilizing airfoil polynomials. Refer to [50] as well. The fundamental idea driving [52] is to use the Kulkarni approach in combination with Legendre polynomials rather than piecewise ones to extend and improve the results of [47, 51].

Motivated by the above reasons, this work aims to consider two classes of fuzzy integro-differential equations, the fuzzy logarithmic integro-differential equation, and the fuzzy Cauchy one respectively. Firstly, we clearly show that solutions to these equations exist and are unique. We use Picard's theorem for the logarithmic fuzzy integro-differential equation, while Arzelà–Ascoli theorem for the Cauchy one. Secondly, we introduce a collocation method to solve the considered equations via airfoil polynomials numerically. Also, we show that there are solutions to approximation concerns and provide error analysis.

3.2 Airfoil polynomials

We recall that in steady or unsteady subsonic flow, the so-referred as airfoil polynomials are utilized as expansion functions to calculate the pressure on an airfoil. These polynomials play a pivotal part in approximation theory, including in the solution of integral and integro-differential equations.
The intention of [49] is to establish a collocation method via airfoil polynomials for the approximate solution of integro-differential equations with a logarithmic kernel in the classical situation. We demonstrated the existence of a solution to the approximation equation and conducted an error analysis. This section extends the method and employs a new procedure to numerically solve two classes of fuzzy singular integro-differential equations: logarithmic fuzzy integro-differential equations and Cauchy ons.

The airfoil polynomials t_n and u_n of the first and second kind, respectively, are defined as follows

$$t_n(\tau) = \frac{\cos[(n+\frac{1}{2})\arccos\tau]}{\cos(\frac{1}{2}\arccos\tau)}, \quad u_n(\tau) = \frac{\sin[(n+\frac{1}{2})\arccos\tau]}{\sin(\frac{1}{2}\arccos\tau)}.$$

3.3 Logarithmic Fuzzy Fredholm integro-differential equation

Given a real constant λ and a fuzzy continuous function g, consider the problem of determining a fuzzy function φ that satisfies the equation bellow.

$$\varphi_l'(s) - \frac{\lambda}{\pi}\oint_{-1}^{1} \varphi_l(t)\ln|s-t|dt = g_l(s), \quad \varphi_l(-1) = 0, \quad -1 < s < 1. \tag{3.1}$$

The above equation is called the fuzzy Fredholm integro-differential equation with a logarithmic kernel. We assume λ is negative real throughout the paper. The other case will be treated similarly. The solution to (3.1) exists and is unique, as demonstrated in the following Theorem, which is based on Picard's Theorem. In [61], an overview of the key results of this Theorem can be found.

Theorem 3.1 *Assume that for equation (3.1) the following assumptions hold:*

H1. *There exists* $M > 0$*:* $D(\varphi(\tau)\ln|t-\tau|, \psi(\tau)\ln|t-\tau|) \leq MD^*(\varphi,\psi)$ *for all* $t,\tau \in \mathcal{I}$ *and* $\varphi,\psi \in \mathbb{F}$, *with* $|\lambda| < \frac{\pi}{4M}$;

H2. $\forall\varepsilon > 0,\ \exists\delta > 0,\ \forall s_1, s_2 \in \mathcal{I}$ *with* $|s_1 - s_2| \leq \delta :$ *for all* $t \in \mathcal{I},\ D(g_l(t),\tilde{0}) < \frac{\varepsilon}{\delta}$;

H3. $D(\varphi(\tau)\ln|t-\tau|,\tilde{0}) < \frac{\varepsilon}{2\delta}$, *for all* $\varphi \in \mathbb{F}$.

The problem (3.1) then has a continuous solution $\varphi_l^* \in \mathcal{C}_{\mathbb{F}}(\mathcal{I})$ *that is unique. Moreover,*

$$D(\varphi_l^*(s), \varphi_{l,n}(s)) \leq \frac{\left(\frac{4|\lambda|}{\pi}M\right)^n}{1-\frac{4|\lambda|}{\pi}M} D^*(\varphi_{l,0}, \varphi_{l,1}),$$

where $\varphi_{l,n}$ *is the approximate solution obtained through successive approaches with* $\varphi_{l,0} = \varphi_l(-1) = 0$ *and*

$$D^*(\varphi_{l,0}, \varphi_{l,1}) := \sup_{s\in\mathcal{I}} D(\varphi_{l,0}(s), \varphi_{l,1}(s)).$$

Proof : Equation (3.1) reads as

$$\varphi_l'(s) - \frac{\lambda}{\pi}\oint_{-1}^{1} \varphi_l(t)\ln|s-t|dt = g_l(s).$$

This shows that

$$\varphi_l(s) = \int_{-1}^{s} g_l(t)dt + \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1}\varphi_l(\tau)\ln|t-\tau|d\tau dt.$$

Letting

$$(A\varphi_l)(s) = \int_{-1}^{s} g_l(t)dt + \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1}\varphi_l(\tau)\ln|t-\tau|d\tau dt.$$

We have to prove that $A(\mathcal{C}_{\mathbb{F}}(\mathcal{I})) \subset \mathcal{C}_{\mathbb{F}}(\mathcal{I})$. For all $\varphi_l \in \mathcal{C}_{\mathbb{F}}(\mathcal{I})$, and $s_1, s_2 \in \mathcal{I}$, we have

$$\begin{aligned}
D((A\varphi_l)(s_1),(A\varphi_l)(s_2)) &= D\Big(\int_{-1}^{s_1} g_l(t)dt + \frac{\lambda}{\pi}\int_{-1}^{s_1}\oint_{-1}^{1}\varphi_l(\tau)\ln|t-\tau|d\tau dt,\\
&\quad \int_{-1}^{s_2} g_l(t)dt + \frac{\lambda}{\pi}\int_{-1}^{s_2}\oint_{-1}^{1}\varphi_l(\tau)\ln|t-\tau|d\tau dt\Big)\\
&= D\Big(\int_{-1}^{s_2} g_l(t)dt + \frac{\lambda}{\pi}\int_{-1}^{s_2}\oint_{-1}^{1}\varphi_l(\tau)\ln|t-\tau|d\tau dt\\
&+ \int_{s_2}^{s_1} g_l(t)dt + \frac{\lambda}{\pi}\int_{s_2}^{s_1}\oint_{-1}^{1}\varphi_l(\tau)\ln|t-\tau|d\tau dt,\\
&\quad \int_{-1}^{s_2} g_l(t)dt + \frac{\lambda}{\pi}\int_{-1}^{s_2}\oint_{-1}^{1}\varphi_l(\tau)\ln|t-\tau|d\tau dt\Big)\\
&= D\Big(\int_{s_2}^{s_1} g_l(t)dt + \frac{\lambda}{\pi}\int_{s_2}^{s_1}\oint_{-1}^{1}\varphi_l(\tau)\ln|t-\tau|d\tau dt, \tilde{0}\Big)\\
&\leq \int_{s_2}^{s_1} D(g_l(t),\tilde{0})dt + \frac{|\lambda|}{\pi}\int_{s_2}^{s_1}\oint_{-1}^{1} D(\varphi_l(\tau)\ln|t-\tau|,\tilde{0})d\tau dt\\
&\leq \varepsilon + \frac{|\lambda|}{\pi}\varepsilon\\
&\leq \varepsilon^{'}.
\end{aligned}$$

Thus, the operator $A\varphi_l$ is uniformly continuous. It follows $A(\mathcal{C}_{\mathbb{F}}(\mathcal{I})) \subset \mathcal{C}_{\mathbb{F}}(\mathcal{I})$.

We now study the continuous of A on $\mathcal{C}_{\mathbb{F}}(\mathcal{I})$. For this purpose, let $\varphi, \psi \in \mathcal{C}_{\mathbb{F}}(\mathcal{I})$, $s \in \mathcal{I}$, we have

$$\begin{aligned}
D\big((A\varphi)(s),(A\psi)(s)\big) &= D\Big(\int_{-1}^{s} g_l(t)dt + \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1}\varphi(\tau)\ln|t-\tau|d\tau dt, \int_{-1}^{s} g_l(t)dt\\
&+ \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1}\psi(\tau)\ln|t-\tau|d\tau dt\Big)\\
&= \frac{|\lambda|}{\pi}D\Big(\int_{-1}^{s}\oint_{-1}^{1}\varphi(\tau)\ln|t-\tau|d\tau dt, \int_{-1}^{s}\oint_{-1}^{1}\psi(\tau)\ln|t-\tau|d\tau dt\Big)\\
&\leq \frac{|\lambda|}{\pi}\int_{-1}^{s}\oint_{-1}^{1} D\big(\varphi(\tau)\ln|t-\tau|, \psi(\tau)\ln|t-\tau|\big)d\tau dt.
\end{aligned}$$

Hence

$$\begin{aligned}
D^*(A\varphi, A\psi) &\leq 2\frac{|\lambda|}{\pi}(s+1)MD^*(\varphi,\psi)\\
&\leq \frac{4|\lambda|}{\pi}MD^*(\varphi,\psi)\\
&\leq c_A D^*(\varphi,\psi), \quad \text{where} \quad c_A := \frac{4|\lambda|}{\pi}M.
\end{aligned}$$

Thus, A is an c_A $contraction.$

By the contraction principle, the operator A has a unique fixed point φ^*, then (3.1) has a unique continuous solution. Because A is a $c_A - Picard$ operator, it has a unique fixed point φ^*, and equation (3.1) has a unique continuous solution. Consequently,

$$D(\varphi_l^*(s), \varphi_{l,n}(s)) \leq \frac{\left(\frac{4|\lambda|}{\pi} M\right)^n}{1 - \frac{4|\lambda|}{\pi} M} D^*(\varphi_{l,0}, \varphi_{l,1}).$$

□

3.4 The approximate solution

We assume that the fuzzy numbers φ and g can be represented as parametric forms as follows:

$$\varphi_{\tau,l}(s) = [\underline{\varphi_l}(s,\tau), \overline{\varphi_l}(s,\tau)],$$

$$g_{\tau,l}(s) = [\underline{g_l}(s,\tau), \overline{g_l}(s,\tau)].$$

We recall that

$$\varphi'_{\tau,l}(s) = [\underline{\varphi_l}'(s,\tau), \overline{\varphi_l}'(s,\tau)].$$

Equation (3.1) can be rewritten in the following form

$$\underline{\varphi_l}'(s,\tau) - \frac{\lambda}{\pi} \oint_{-1}^{1} \underline{\varphi_l(t,\tau) \ln|s-t|} dt = \underline{g_l}(s,\tau), \quad -1 < s < 1, \tag{3.2}$$

$$\overline{\varphi_l}'(s,\tau) - \frac{\lambda}{\pi} \oint_{-1}^{1} \overline{\varphi_l(t,\tau) \ln|s-t|} dt = \overline{g_l}(s,\tau), \quad -1 < s < 1. \tag{3.3}$$

In order to simplify the above integrals, it is tempting to study the sign of the kernel $k(s,t) := \ln|s-t|$ for two cases as follows:
For $s \geq 0$, we have

$$\begin{aligned} k(s,t) > 0 \quad &\text{for} \quad t \in]-1, s-1[; \\ k(s,t) < 0 \quad &\text{for} \quad t \in]s-1, s[\cup]s, 1[; \\ k(s,t) = 0 \quad &\text{for} \quad t = s-1. \end{aligned}$$

For $s \leq 0$, we have

$$\begin{aligned} k(s,t) > 0 &\quad \text{for} \quad t \in]s+1, 1[; \\ k(s,t) < 0 &\quad \text{for} \quad t \in]-1, s[\cup]s, s+1[; \\ k(s,t) = 0 &\quad \text{for} \quad t = s+1. \end{aligned}$$

As in [1], letting

$$\varphi_l^c(s,\tau) := \frac{\overline{\varphi_l}(s,\tau) + \underline{\varphi_l}(s,\tau)}{2}, \quad \varphi_l^d(s,\tau) := \frac{\overline{\varphi_l}(s,\tau) - \underline{\varphi_l}(s,\tau)}{2},$$

$$g_l^c(s,\tau) := \frac{\overline{g_l}(s,\tau) + \underline{g_l}(s,\tau)}{2}, \quad g_l^d(s,\tau) := \frac{\overline{g_l}(s,\tau) - \underline{g_l}(s,\tau)}{2}.$$

Lemma 3.1 *For* $s \in]0, 1[$, *equation (3.1) can be rewritten as follows:*

$$\frac{\partial \varphi_l^c(s,\tau)}{\partial s} - \frac{\lambda}{\pi} \oint_{-1}^{1} \varphi_l^c(t,\tau) \ln|s-t| dt = g_l^c(s,\tau), \tag{3.4}$$

$$\begin{aligned} \frac{\partial \varphi_l^d(s,\tau)}{\partial s} - \frac{\lambda}{\pi} \oint_{-1}^{1} \varphi_l^d(t,\tau) \ln|s-t| dt &+ \frac{2\lambda}{\pi} \int_{s-1}^{1} \varphi_l^d(t,\tau) \ln|s-t| dt \\ &= g_l^d(s,\tau). \end{aligned} \tag{3.5}$$

Proof : We have

$$\underline{\varphi_l}(s,\tau) = \varphi_l^c(s,\tau) - \varphi_l^d(s,\tau), \quad \overline{\varphi_l}(s,\tau) = \varphi_l^c(s,\tau) + \varphi_l^d(s,\tau);$$

$$\underline{g_l}(s,\tau) = g_l^c(s,\tau) - g_l^d(s,\tau), \quad \overline{g_l}(s,\tau) = g_l^c(s,\tau) + g_l^d(s,\tau).$$

Substituting this into (3.2) and (3.3) respectively, leads to the system

$$\begin{aligned} \frac{\partial \varphi_l^c(s,\tau)}{\partial s} - \frac{\partial \varphi_l^d(s,\tau)}{\partial s} &- \frac{\lambda}{\pi} \int_{-1}^{s-1} \left[\varphi_l^c(t,\tau) - \varphi_l^d(t,\tau)\right] \ln|s-t| dt \\ &- \frac{\lambda}{\pi} \oint_{s-1}^{1} \left[\varphi_l^c(t,\tau) + \varphi_l^d(t,\tau)\right] \ln|s-t| dt \\ &= g_l^c(s,\tau) - g_l^d(s,\tau), \end{aligned} \tag{3.6}$$

$$\begin{aligned}\frac{\partial \varphi_l^c(s,\tau)}{\partial s}+\frac{\partial \varphi_l^d(s,\tau)}{\partial s} &- \frac{\lambda}{\pi}\int_{-1}^{s-1}\left[\varphi_l^c(t,\tau)+\varphi_l^d(t,\tau)\right]\ln|s-t|dt \\ &- \frac{\lambda}{\pi}\oint_{s-1}^{1}\left[\varphi_l^c(t,\tau)-\varphi_l^d(t,\tau)\right]\ln|s-t|dt \\ &= g_l^c(s,\tau)+g_l^d(s,\tau). \end{aligned} \tag{3.7}$$

By adding the two equations (3.6) and (3.7) together, we get (3.4). Again, by subtracting the (3.6) from the (3.7) we obtain (3.5). □

Lemma 3.2 *For* $s \in]-1,0]$, *equation (3.1) can be rewritten as follows:*

$$\frac{\partial \varphi_l^c(s,\tau)}{\partial s}-\frac{\lambda}{\pi}\oint_{-1}^{1}\varphi_l^c(t,\tau)\ln|s-t|dt = g_l^c(s,\tau), \tag{3.8}$$

$$\begin{aligned}\frac{\partial \varphi_l^d(s,\tau)}{\partial s}+\frac{\lambda}{\pi}\oint_{-1}^{1}\varphi_l^d(t,\tau)\ln|s-t|dt &- \frac{2\lambda}{\pi}\int_{s+1}^{1}\varphi_l^d(t,\tau)\ln|s-t|dt \\ &= g_l^d(s,\tau). \end{aligned} \tag{3.9}$$

Proof : Proceeding as the first case, we obtain the system

$$\begin{aligned}\frac{\partial \varphi_l^c(s,\tau)}{\partial s}-\frac{\partial \varphi_l^d(s,\tau)}{\partial s} &- \frac{\lambda}{\pi}\oint_{-1}^{s+1}\left[\varphi_l^c(t,\tau)+\varphi_l^d(t,\tau)\right]\ln|s-t|dt \\ &- \frac{\lambda}{\pi}\int_{s+1}^{1}\left[\varphi_l^c(t,\tau)-\varphi_l^d(t,\tau)\right]\ln|s-t|dt \\ &= g_l^c(s,\tau)-g_l^d(s,\tau), \end{aligned} \tag{3.10}$$

$$\begin{aligned}\frac{\partial \varphi^c(s,\tau)}{\partial s}+\frac{\partial \varphi_l^d(s,\tau)}{\partial s} &- \frac{\lambda}{\pi}\oint_{-1}^{s+1}\left[\varphi_l^c(t,\tau)-\varphi_l^d(t,\tau)\right]\ln|s-t|dt \\ &- \frac{\lambda}{\pi}\int_{s+1}^{1}\left[\varphi_l^c(t,\tau)+\varphi_l^d(t,\tau)\right]\ln|s-t|dt \\ &= g_l^c(s,\tau)+g_l^d(s,\tau). \end{aligned} \tag{3.11}$$

Hence, equation (3.8) follows by adding the two equations (3.10) and (3.11) together. Equation (3.9) succeeds by subtracting the (3.10) from the (3.11). □

We will propose an approximate solution for equation (3.1) via the approximate solutions for equations (3.4), (3.5), (3.8) and (3.9) respectively. For this purpose, we will introduce an approximation using the airfoil polynomials of the first kind t_n as

$$\varphi^c_{l,n}(s,\tau) = \omega(s)\sum_{i=0}^{n} a_{i,\tau} t_i(s),$$

$$\varphi^d_{l,n}(s,\tau) = \omega(s)\sum_{i=0}^{n} b_{i,\tau} t_i(s),$$

where

$$\omega(s) = \sqrt{\frac{1+s}{1-s}}.$$

Following ([22]), the formula

$$(1+s)t_i'(s) = (i+\frac{1}{2})u_i(s) - \frac{1}{2}t_i(s)$$

gives

$$\frac{\partial \varphi^c_{l,n}(s,\tau)}{\partial s} = \sum_{i=0}^{n} a_{i,\tau}\Big\{\omega'(s)t_i(s) + \frac{\omega(s)}{1+s}\Big[(i+\frac{1}{2})u_i(s) - \frac{1}{2}t_i(s)\Big]\Big\},$$

$$\frac{\partial \varphi^d_{l,n}(s,\tau)}{\partial s} = \sum_{i=0}^{n} b_{i,\tau}\Big\{\omega'(s)t_i(s) + \frac{\omega(s)}{1+s}\Big[(i+\frac{1}{2})u_i(s) - \frac{1}{2}t_i(s)\Big]\Big\}.$$

We recall that (cf. [22]),

$$\frac{1}{\pi}\oint_{-1}^{1}\sqrt{\frac{1+t}{1-t}}t_i(t)\ln|s-t|dt = \begin{cases} \frac{u_{i-1}(s)-u_i(s)}{2i} + \frac{u_i(s)-u_{i+1}(s)}{2(i+1)} & \text{if } i \neq 0 \\ -\ln 2 - s & \text{otherwise.} \end{cases} \quad (3.12)$$

For $s \in]0,1[$, by using (3.12), we get

$$\begin{cases} a^+_{0,\tau}A^+_0(s) + \sum_{i=1}^{n} a^+_{i,\tau}A^+_i(s) = g^c_l(s,\tau), \\ b^+_{0,\tau}B^+_0(s) + \sum_{i=1}^{n} b^+_{i,\tau}(\tau)B^+_i(s) = g^d_l(s,\tau), \end{cases}$$

with

$$\begin{aligned}
A_0^+(s) &:= \frac{1}{\omega(s)(1-s)^2} t_0(s) + \frac{\omega(s)}{2(1+s)}[u_0(s) - t_0(s)] + \lambda\,(\ln 2 + s)\,;\\
B_0^+(s) &:= A_0^+(s) + \frac{2\lambda}{\pi}\int_{s-1}^{1} \omega(t) t_0(t) \ln|s-t|dt;\\
A_i^+(s) &:= \frac{1}{\omega(s)(1-s)^2} t_i(s) + \frac{\omega(s)}{1+s}\left[(i+\frac{1}{2})u_i(s) - \frac{1}{2}t_i(s)\right]\\
&\quad - \lambda\left[\frac{u_{i-1}(s)-u_i(s)}{2i} + \frac{u_i(s)-u_{i+1}(s)}{2(i+1)}\right];\\
B_i^+(s) &:= A_i^+(s) + \frac{2\lambda}{\pi}\int_{s-1}^{1} \omega(t) t_i(t) \ln|s-t|dt.
\end{aligned}$$

For $s \in]-1, 0[$, again, by using (3.12), we get

$$\begin{cases}
a_{0,\tau}^- A_0^+(s) + \sum\limits_{i=1}^{n} a_{i,\tau}^- A_i^+(s) = g_l^c(s,\tau),\\
b_{0,\tau}^- B_0^-(s) + \sum\limits_{i=1}^{n} b_{i,\tau}^- B_i^-(s) = g_l^d(s,\tau),
\end{cases}$$

with

$$\begin{aligned}
B_0^-(s) &:= A_0^-(s) - \frac{2\lambda}{\pi}\int_{s+1}^{1} \omega(t) t_0(t) \ln|s-t|dt;\\
A_0^-(s) &:= \frac{1}{\omega(s)(1-s)^2} t_0(s) + \frac{\omega(s)}{2(1+s)}[u_0(s) - t_0(s)] - \lambda\,(\ln 2 + s)\,;\\
B_i^-(s) &:= A_i^-(s) - \frac{2\lambda}{\pi}\int_{s+1}^{1} \omega(t) t_i(t) \ln|s-t|dt;\\
A_i^-(s) &:= \frac{1}{\omega(s)(1-s)^2} t_i(s) + \frac{\omega(s)}{1+s}\left[(i+\frac{1}{2})u_i(s) - \frac{1}{2}t_i(s)\right]\\
&\quad + \lambda\left[\frac{u_{i-1}(s)-u_i(s)}{2i} + \frac{u_i(s)-u_{i+1}(s)}{2(i+1)}\right].
\end{aligned}$$

The collocation method leads to the following linear systems:
For $s \in]0, 1[$, we obtain

$$\begin{cases}
a_{0,\tau}^+ A_0^+(s_j) + \sum\limits_{i=1}^{n} a_{i,\tau}^+ A_i^+(s_j) = g_l^c(s_j,\tau),\\
b_{0,\tau}^+ B_0^+(s_j) + \sum\limits_{i=1}^{n} b_{i,\tau}^+ B_i^+(s_j) = g_l^d(s_j,\tau),
\end{cases}$$

For $s \in]-1,0[$, we have

$$\begin{cases} a^-_{0,\tau}A^+_0(s_j) + \sum_{i=1}^{n} a^-_{i,\tau}A^+_i(s_j) = g^c_l(s_j,\tau), \\ b^-_{0,\tau}B^-_0(s_j) + \sum_{i=1}^{n} b^-_{i,\tau}B^-_i(s_j) = g^d_l(s_j,\tau). \end{cases}$$

3.5 Cauchy Fuzzy Fredholm integro-differential equation

Let us consider the problem of finding a fuzzy function φ_c such that

$$\varphi'_c(s) - \frac{\lambda}{\pi}\oint_{-1}^{1} \frac{\varphi_c(t)}{s-t}dt = g_c(s), \quad \varphi_c(-1) = \varphi_{c,0}, \quad -1 < s < 1, \tag{3.13}$$

where λ is a know negative constant and g_c is a given a fuzzy function.

This equation called Cauchy Fuzzy Fredholm integro-differential equation.

Theorem 3.2 *Suppose that:*

H4. There exists $M_c > 0$*:* $D(\frac{\varphi(\tau)}{t-\tau}, \frac{\psi(\tau)}{t-\tau}) \leq M_c D^*(\varphi,\psi)$ *for all* $t,\tau \in \mathcal{I}$ *and* $\varphi,\psi \in \mathbb{F}$, *with* $|\lambda| < \frac{\pi}{4M_c}$;

H5. $\forall \varepsilon > 0,\ \exists \delta > 0,\ \ \forall s_1, s_2 \in \mathcal{I}$ *with* $|s_1 - s_2| \leq \delta :$ *for all* $t \in \mathcal{I},\ D(g_c(t),\tilde{0}) < \frac{\varepsilon}{\delta}$;

H6. $D(\frac{\varphi(\tau)}{t-\tau},\tilde{0}) < \frac{\varepsilon}{2\delta}$.
Then, problem (3.13) has a unique continuous solution $\varphi^*_c \in \mathcal{C}_{\mathbb{F}}(\mathcal{I})$ *. Moreover,*

$$D(\varphi^*_c(s), \varphi_{c,n}(s)) \leq \frac{\left(\frac{4|\lambda|}{\pi}M_c\right)^n}{1 - \frac{4|\lambda|}{\pi}M_c} D^*(\varphi_{c,0}, \varphi_{c,1}),$$

where $\varphi_{c,n}$ *is the approximate solution obtained through successive approaches with* $\varphi_{c,0} = \varphi_c(-1) = 0$ *and*

$$D^*(\varphi_{c,0}, \varphi_{c,1}) := \sup_{s\in\mathcal{I}} D(\varphi_{c,0}(s), \varphi_{c,1}(s)).$$

Proof : Equation (3.13) reads as

$$\int_{-1}^{s} \varphi'_c(t)dt - \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1} \frac{\varphi_c(\tau)}{t-\tau}d\tau dt = \int_{-1}^{s} g_c(t)dt.$$

This shows that

$$\varphi_c(s) = \varphi_{c,0} + \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt + \int_{-1}^{s} g_c(t)dt.$$

Letting

$$A_c\varphi(s) = \varphi_{c,0} + \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt + \int_{-1}^{s} g_c(t)dt.$$

We have to prove that $A_c(\mathcal{C}_{\mathbb{F}}(\mathcal{I})) \subset \mathcal{C}_{\mathbb{F}}(\mathcal{I})$. To this goal, for all $\varphi \in \mathcal{C}_{\mathbb{F}}(\mathcal{I})$, and $s_1, s_2 \in \mathcal{I}$, we have

$$\begin{aligned}
D(A_c\varphi(s_1), A_c\varphi(s_2)) &= D\Big(\varphi_{c,0} + \frac{\lambda}{\pi}\int_{-1}^{s_1}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt + \int_{-1}^{s_1} g_c(t)dt, \varphi_{c,0} \\
&+ \frac{\lambda}{\pi}\int_{-1}^{s_2}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt + \int_{-1}^{s_2} g_c(t)dt\Big) \\
&= D\Big(\frac{\lambda}{\pi}\int_{-1}^{s_1}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt + \int_{-1}^{s_1} g_c(t)dt, \\
&\quad \frac{\lambda}{\pi}\int_{-1}^{s_2}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt + \oint_{-1}^{s_2} g_c(t)dt\Big) \\
&= D\Big(\frac{\lambda}{\pi}\int_{-1}^{s_2}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt + \int_{-1}^{s_2} g_c(t)dt \\
&+ \frac{\lambda}{\pi}\int_{s_2}^{s_1}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt + \int_{s_2}^{s_1} g_c(t)dt, \frac{\lambda}{\pi}\int_{-1}^{s_2}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt \\
&+ \int_{-1}^{s_2} g_c(t)dt\Big)
\end{aligned}$$

$$\begin{aligned}
&= D\left(\frac{\lambda}{\pi}\int_{s_2}^{s_1}\oint_{-1}^{1}\frac{\varphi_c(\tau)}{t-\tau}d\tau dt + \int_{s_2}^{s_1} g_c(t)dt, \tilde{0}\right) \\
&\leq \frac{|\lambda|}{\pi}\int_{s_2}^{s_1}\oint_{-1}^{1} D\left(\frac{\varphi_c(\tau)}{t-\tau}, \tilde{0}\right) d\tau dt + \int_{s_2}^{s_1} D\left(g_c(.), \tilde{0}\right) dt \\
&\leq 2\frac{|\lambda|}{\pi}(s_2 - s_1) D\left(\frac{\varphi_c(\tau)}{t-\tau}, \tilde{0}\right) + (s_2 - s_1)\frac{\varepsilon}{\delta} \\
&\leq \frac{|\lambda|}{\pi}\varepsilon + \varepsilon \\
&\leq \varepsilon'.
\end{aligned}$$

Thus, the operator A_c is uniformly continuous. It follows $A_c(\mathcal{C}_{\mathbb{F}}(\mathcal{I})) \subset \mathcal{C}_{\mathbb{F}}(\mathcal{I})$.

We now study the continuous of A_c on $\mathcal{C}_{\mathbb{F}}(\mathcal{I})$.

Let $\varphi, \psi \in \mathcal{C}_{\mathbb{F}}(\mathcal{I})$, $s \in \mathcal{I}$. We have

$$\begin{aligned}
D(A_c\varphi(s), A_c\psi(s)) &= D\Big(\varphi_{c,0} + \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1}\frac{\varphi(\tau)}{t-\tau}d\tau dt + \int_{-1}^{s} g_c(t)dt, \varphi_{c,0} \\
&+ \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1}\frac{\psi(\tau)}{t-\tau}d\tau dt + \int_{-1}^{s} g_c(t)dt\Big) \\
&\leq \frac{|\lambda|}{\pi}\int_{-1}^{s}\oint_{-1}^{1} D\left(\frac{\varphi(\tau)}{t-\tau}, \frac{\psi(\tau)}{t-\tau}\right) d\tau dt \\
&\leq \frac{|\lambda|}{\pi} M_c \int_{-1}^{s}\oint_{-1}^{1} D^*(\varphi,\psi)\, d\tau dt \\
&\leq 2\frac{|\lambda|}{\pi}(s+1) M_c D^*(\varphi,\psi) \\
&\leq \frac{4|\lambda|}{\pi} M_c D^*(\varphi,\psi).
\end{aligned}$$

Thus, A_c is continuous.

Now, in order to demonstrate the compactness of the operator A_c, we use the Arzelà-Ascoli theorem.

Le $\mathcal{G} := \{\varphi_n,\ \ n \in \mathbb{N}\}$ be a bounded set of $\mathcal{C}_{\mathbb{F}}(\mathcal{I})$.

Hence

$$\forall n \in \mathbb{N},\ \ ||\varphi_n(.)||_{\mathcal{G}} \leq K, \ \ \text{for some positive constant } \ K,$$

and hence

$$\begin{aligned}
||A_c\varphi_n(s)||_{\mathcal{G}} &= D(\tilde{0}, A_c\varphi_n(s)) \\
&\leq D\Big(\tilde{0}, \varphi_{c,0} + \frac{\lambda}{\pi}\int_{-1}^{s}\oint_{-1}^{1}\frac{\varphi_n(\tau)}{t-\tau}d\tau dt + \int_{-1}^{s} g_c(t)dt\Big) \\
&\leq D(\tilde{0}, \varphi_{c,0}) + \frac{|\lambda|}{\pi}\int_{-1}^{s}\int_{-1}^{1} D(\tilde{0}, \frac{\varphi_n(\tau)}{t-\tau})d\tau dt + \int_{-1}^{s} D(\tilde{0}, g_c(t))dt \\
&\leq D(\tilde{0}, \varphi_{c,0}) + \frac{|\lambda|}{\pi} 2(s+1) D(\tilde{0}, \frac{\varphi_n(\tau)}{t-\tau}) + (s+1) D(\tilde{0}, g_c(t)) \\
&\leq D(\tilde{0}, \varphi_{c,0}) + \frac{2|\lambda|}{\pi}\frac{\varepsilon}{\delta} + 2\frac{\varepsilon}{\delta} =: \alpha,
\end{aligned}$$

so that $A_c(\mathcal{G})$ is bounded.

We prove that $A_c(\mathcal{G})$ is equicontinuous, that is

$$\forall \varepsilon > 0\ \exists \delta > 0;\ \forall s_1, s_2 \in \mathcal{I},\ A_c\varphi_n \in A_c(\mathcal{G}):\ |s_1 - s_2| < \delta \Rightarrow D\big((A_c\varphi_n)(s_1), (A_c\varphi_n)(s_2)\big) < \varepsilon.$$

Similarly as above, it follows that $A_c(\mathcal{G})$ is equicontinuous. Consequently, following Arzelà-Ascoli theorem A_c is compact, so that A_c from $\mathcal{C}_{\mathbb{F}}(\mathcal{I})$ into itself is completely continuous.

According to Schauder fixed point theorem equation (3.13) has a unique continuous solution. □

3.6 The approximate solution

We assume that the fuzzy numbers φ and g can be represented as parametric forms as follows:

$$\varphi_{\tau,c}(s) = [\underline{\varphi_c}(s,\tau), \overline{\varphi_c}(s,\tau)],$$

$$g_{\tau,c}(s) = [\underline{g_c}(s,\tau), \overline{g_c}(s,\tau)].$$

We recall that

$$\varphi'_{\tau,c}(s) = [\underline{\varphi_c}'(s,\tau), \overline{\varphi_c}'(s,\tau)].$$

Problem (3.13) can be rewritten in the following form

$$\frac{\partial \underline{\varphi_c}(s,\tau)}{\partial s} - \frac{\lambda}{\pi}\oint_{-1}^{1} \underline{\frac{\varphi_c(t,\tau)}{s-t}}dt = \underline{g_c}(s,\tau), \quad -1 < s < 1,$$

$$\frac{\partial \overline{\varphi_c}(s,\tau)}{\partial s} - \frac{\lambda}{\pi}\oint_{-1}^{1} \frac{\overline{\varphi_c(t,\tau)}}{s-t}dt = \overline{g_c}(s,\tau), \quad -1 < s < 1.$$

It follows that

$$\frac{\partial \underline{\varphi_c}(s,\tau)}{\partial s} - \frac{\lambda}{\pi}\oint_{-1}^{s} \frac{\underline{\varphi_c}(t,\tau)}{s-t}dt - \frac{\lambda}{\pi}\oint_{s}^{1} \frac{\overline{\varphi_c}(t,\tau)}{s-t}dt = \underline{g_c}(s,\tau), \tag{3.14}$$

$$\frac{\partial \overline{\varphi_c}(s,\tau)}{\partial s} - \frac{\lambda}{\pi}\oint_{-1}^{s} \frac{\overline{\varphi_c}(t,\tau)}{s-t}dt - \frac{\lambda}{\pi}\oint_{s}^{1} \frac{\underline{\varphi_c}(t,\tau)}{s-t}dt = \overline{g_c}(s,\tau). \tag{3.15}$$

In order to obtain an explicit system of equations, let us putting

$$\varphi_c^c(s,\tau) := \frac{\overline{\varphi_c}(s,\tau) + \underline{\varphi_c}(s,\tau)}{2}, \quad \varphi_c^d(s,\tau) := \frac{\overline{\varphi_c}(s,\tau) - \underline{\varphi_c}(s,\tau)}{2}$$

$$g_c^c(s,\tau) := \frac{\overline{g_c}(s,\tau) + \underline{g_c}(s,\tau)}{2}, \quad g_c^d(s,\tau) := \frac{\overline{g_c}(s,\tau) - \underline{g_c}(s,\tau)}{2}.$$

This Theorem makes it legitimate to apply the collocation method.

Theorem 3.3 *The problem (3.13) can be rewritten in the following form:*

$$\frac{\partial \varphi_c^c(s,\tau)}{\partial s} - \frac{\lambda}{\pi}\oint_{-1}^{1}\frac{\varphi_c^c(t,\tau)}{s-t}dt = g_c^c(s,\tau), \tag{3.16}$$

$$\frac{\partial \varphi_c^d(s,\tau)}{\partial s} - \frac{\lambda}{\pi}\oint_{-1}^{1}\frac{\varphi_c^d(t,\tau)}{s-t}dt + \frac{2\lambda}{\pi}\oint_{s}^{1}\frac{\varphi_c^d(t,\tau)}{s-t}dt = g_c^d(s,\tau). \tag{3.17}$$

Proof : We have

$$\underline{\varphi_c}(s,\tau) = \varphi_c^c(s,\tau) - \varphi_c^d(s,\tau), \quad \overline{\varphi_c}(s,\tau) = \varphi_c^c(s,\tau) + \varphi_c^d(s,\tau);$$

$$\underline{g_c}(s,\tau) = g_c^c(s,\tau) - g_c^d(s,\tau), \quad \overline{g_c}(s,\tau) = g_c^c(s,\tau) + g_c^d(s,\tau).$$

Substituting this into (3.14) and (3.15) respectively, yields

$$\begin{aligned}\frac{\partial \varphi_c^c(s,\tau)}{\partial s} - \frac{\partial \varphi_c^d(s,\tau)}{\partial s} &- \frac{\lambda}{\pi}\int_{-1}^{s}\frac{\varphi_c^c(t,\tau)-\varphi_c^d(t,\tau)}{s-t}dt \\ &- \frac{\lambda}{\pi}\oint_{s}^{1}\frac{\varphi_c^c(t,\tau)+\varphi_c^d(t,\tau)}{s-t}dt \\ &= g_c^c(s,\tau) - g_c^d(s,\tau),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\frac{\partial \varphi_c^c(s,\tau)}{\partial s} + \frac{\partial \varphi_c^d(s,\tau)}{\partial s} &- \frac{\lambda}{\pi}\int_{-1}^{s}\frac{\varphi_c^c(t,\tau)+\varphi_c^d(t,\tau)}{s-t}dt \\ &- \frac{\lambda}{\pi}\oint_{s}^{1}\frac{\varphi_c^c(t,\tau)-\varphi_c^d(t,\tau)}{s-t}dt \\ &= g_c^c(s,\tau) + g_c^d(s,\tau),\end{aligned} \tag{3.19}$$

By adding the two equations (3.18) and (3.19) together, we get (3.16). Again, by subtracting the (3.18) from the (3.19) we obtain (3.17). □

We'll use the approximate solutions for equations (3.16) and (3.17) to give an approach solution to the equation (3.13). To this end, the airfoil polynomials of the first kind t_n will be used to build an approximation as follows.

$$\varphi_{c,n}^c(s,\tau) = \omega(s)\sum_{i=0}^{n} c_{i,\tau} t_i(s),$$

$$\varphi_{c,n}^{d}(s,\tau)=\omega(s)\sum_{i=0}^{n}d_{i,\tau}t_i(s).$$

By using the formula (cf. [22]),

$$\frac{1}{\pi}\oint_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{t_i(t)}{t-s}dt=u_i(s)$$

we get

$$\sum_{i=0}^{n}c_{i,\tau}\Big\{\omega'(s)t_i(s)+\frac{\omega(s)}{1+s}[(i+\frac{1}{2})u_i(s)-\frac{1}{2}t_i(s)]+\lambda u_i(s)\Big\}=g_c^{c}(s,\tau),$$

$$\sum_{i=0}^{n}d_{i,\tau}\Big\{\omega'(s)t_i(s)+\frac{\omega(s)}{1+s}[(i+\frac{1}{2})u_i(s)-\frac{1}{2}t_i(s)]+\lambda u_i(s)+\frac{2\lambda}{\pi}\oint_{s}^{1}\frac{\omega(t)t_i(t)}{s-t}dt\Big\}=g_c^{d}(s,\tau).$$

Thus

$$\sum_{i=0}^{n}c_{i,\tau}\Big\{\omega'(s_j)t_i(s_j)+\frac{\omega(s_j)}{1+s_j}[(i+\frac{1}{2})u_i(s_j)-\frac{1}{2}t_i(s_j)]+\lambda u_i(s_j)\Big\}=g_c^{c}(s_j,\tau),$$

$$\begin{aligned}\sum_{i=0}^{n}d_{i,\tau}\Big\{\omega'(s_j)t_i(s_j) &+ \frac{\omega(s_j)}{1+s_j}[(i+\frac{1}{2})u_i(s_j)-\frac{1}{2}t_i(s_j)]+\lambda u_i(s_j)\\ &+ \frac{2\lambda}{\pi}\oint_{s_j}^{1}\frac{\omega(t)t_i(t)}{s_j-t}dt\Big\}=g_c^{d}(s_j,\tau),\quad j=0,1,\ldots,n.\end{aligned}$$

3.7 Convergence Analysis

In this section, we prove the convergence analysis of the current approximations. To accomplish this, we rewrite the obtained equations in operator forms.

Let us define the following operators:

$$
\begin{aligned}
(V\varphi)(s) &:= \int_{-1}^{s} \varphi(t)dt, \quad \varphi \in (C_{\mathbb{F}}(\mathcal{I}), \mathcal{F});\\
(T_c\varphi)(s) &:= \frac{\lambda}{\pi} \oint_{-1}^{1} \frac{\varphi(t)}{s-t} dt;\\
(T_l\varphi)(s) &:= \frac{\lambda}{\pi} \oint_{-1}^{1} \varphi(t) \ln|s-t| dt;\\
(V_l^{+}\varphi)(s) &:= \frac{2\lambda}{\pi} \int_{s-1}^{1} \varphi(t) \ln|s-t| dt;\\
(V_l^{-}\varphi)(s) &:= \frac{2\lambda}{\pi} \int_{s+1}^{1} \varphi(t) \ln|s-t| dt;\\
(V_c\varphi)(s) &:= \frac{2\lambda}{\pi} \oint_{s}^{1} \frac{\varphi(t)}{s-t} dt.
\end{aligned}
$$

Lemma 3.3 *Assume that*

$$
\exists r > 0, \quad \forall s \in \mathcal{I}; \quad D\big((V\varphi)(s), \tilde{0}\big) \le r,
$$

then V is compact from $(C_{\mathbb{F}}(\mathcal{I}), \mathcal{F})$ into itself.

Proof : We have

$$
\begin{aligned}
D^*(V\varphi, \tilde{0}) &= \sup_{s \in \mathcal{I}} D\big((V\varphi)(s), \tilde{0}\big)\\
&\le r,
\end{aligned}
$$

so that V is bounded.

Also,

$$
\begin{aligned}
D\big((V\varphi)(s), (V\psi)(s)\big) &= D\Big(\int_{-1}^{s} \varphi(t)dt, \int_{-1}^{s} \psi(t)dt\Big)\\
&\le \int_{-1}^{s} D\big(\varphi(t), \psi(t)\big)dt.
\end{aligned}
$$

Hence

$$\begin{aligned} D^*\big((V\varphi),(V\psi)\big) &\leq (s+1)D^*\big(\varphi,\psi\big) \\ &\leq 2D^*(\varphi,\psi) \\ &< \infty. \end{aligned}$$

Thus, V is continuous.

Letting

$$\Omega := \{\varphi \in (C_{\mathbb{F}}(\mathcal{I}),\mathcal{F});\ \ \exists\Lambda>0\ \ D(\varphi(s),\tilde{0}) \leq \Lambda; \quad \text{for all} \quad s\in\mathcal{I}\}.$$

We show that $V(\Omega)$ is equicontinuous,

$$\begin{aligned} D(V\varphi(s_1),V\varphi(s_2)) &= D(\int_{-1}^{s_1}\varphi(t)dt,\int_{-1}^{s_2}\varphi(t)dt) \\ &\leq D(\int_{-1}^{s_2}\varphi(t)dt+\int_{s_2}^{s_1}\varphi(t)dt,\int_{-1}^{s_2}\varphi(t)dt) \\ &\leq \int_{s_2}^{s_1}D(\varphi(t),\tilde{0})dt \to 0 \quad \text{as} \quad s_2\to s_1, \end{aligned}$$

so that $V(\Omega)$ is equicontinuous.

We prove that $V(\Omega)$ is bounded.

$$\begin{aligned} D^*\big((V\varphi),\tilde{0}\big) &= \sup_{s\in\mathcal{I}} D(\int_{-1}^{s}\varphi(t)dt,\tilde{0}) \\ &\leq \sup_{s\in\mathcal{I}}\int_{-1}^{s}D(\varphi(t),\tilde{0})dt \\ &\leq 2\Lambda, \end{aligned}$$

so that $V(\Omega)$ is bounded.

We conclude that V is compact from $\mathcal{C}_{\mathbb{F}}(\mathcal{I})$ into itself by applying the Arzelà-Ascoli Theorem. □

Suggest hat functions $e_0, e_1, e_2, \ldots, e_n$ in $C^0(\mathcal{I})$ subject to

$$e_j(s_k) = \delta_{j,k}.$$

Let us consider the projection operators π_n from $C^0(\mathcal{I})$ into the space of continuous functions by

$$\pi_n g(x) := \sum_{j=0}^{n} g(x_j) e_j(x).$$

Define the operators

$$\begin{aligned}
B_{c,n} &:= V\pi_n T_c, \quad B_c := VT_c, \\
B_{l,n} &:= V\pi_n T_l, \quad B_l := VT_l, \\
S_{c,n} &:= V\pi_n V_c, \quad S_c := VV_c, \\
S^-_{l,n} &:= V\pi_n V_l^-, \quad S_l^- := VV_l^-, \\
S^+_{l,n} &:= V\pi_n V_l^+, \quad S_l^+ := VV_l^+.
\end{aligned}$$

Consider the following approximate equations:

$$\begin{cases} \varphi^c_{l,n} - B_{l,n}\varphi^c_{l,n} = Vg^c_l, \\ \varphi^d_{l,n} - B_{l,n}\varphi^d_{l,n} + S^+_{l,n}\varphi^d_{c,n} = Vg^d_l, \end{cases}$$

$$\begin{cases} \varphi^c_{l,n} - B_{l,n}\varphi^c_{l,n} = Vg^c_l, \\ \varphi^d_{l,n} + B_{l,n}\varphi^d_{l,n} - S^-_{l,n}\varphi^d_{c,n} = Vg^d_l, \end{cases}$$

and

$$\begin{cases} \varphi^c_{c,n} - B_{c,n}\varphi^c_{c,n} = Vg^c_c, \\ \varphi^d_{c,n} - B_{c,n}\varphi^d_{c,n} + S_{c,n}\varphi^d_{c,n} = Vg^d_c. \end{cases}$$

Theorem 3.4 *Assume that $g^c_l, g^d_l, g^c_c, g^d_c \in \mathcal{C}_{\mathbb{F}}(\mathcal{I})$. There exist a positive constants $\gamma^c_l, \gamma^{d,+}_l, \gamma^{d,-}_l, \gamma^c_c, \gamma^d_c$, such that*

$$\begin{aligned}
\|\varphi^c_{l,n}(.,\tau) - \varphi^c_l(.,\tau)\|_\infty &\leq \gamma^c_l \|\left(B_l - B_{l,n}\right)\varphi^c_l(.,\tau)\|_\infty, \\
\|\varphi^d_{l,n}(.,\tau) - \varphi^d_l(.,\tau)\|_\infty &\leq \gamma^{d,+}_l \Big\|\Big((B_l - B_{l,n}) - (S^+_l - S^+_{l,n})\Big)\varphi^d_l(.,\tau)\Big\|_\infty, \\
\|\varphi^d_{l,n}(.,\tau) - \varphi^d_l(.,\tau)\|_\infty &\leq \gamma^{d,-}_l \Big\|\Big((S^-_l - S^-_{l,n}) - (B_l - B_{l,n})\Big)\varphi^d_l(.,\tau)\Big\|_\infty, \\
\|\varphi^d_{c,n}(.,\tau) - \varphi^d_c(.,\tau)\|_\infty &\leq \gamma^c_c \|\left(B_c - B_{c,n}\right)\varphi^c_c(.,\tau)\|_\infty, \\
\|\varphi^c_{c,n}(.,\tau) - \varphi^c_c(.,\tau)\|_\infty &\leq \gamma^d_c \Big\|\Big((B_c - B_{c,n}) - (S_c - S_{c,n})\Big)\varphi^d_c(.,\tau)\Big\|_\infty,
\end{aligned}$$

for n large enough.

Proof : We recall that $\|\pi_n\varphi - \varphi\|_\infty \to 0$, for all $\varphi \in \mathcal{C}_{\mathbb{F}}(\mathcal{I})$. Since V is compact, it is clear that B_l is compact. It is well-known that the inverse operator $(I - B_{l,n})^{-1}$ exists and is uniformly bounded for n large enough.

On the other hand,

$$\begin{aligned} \varphi_l^c(.,\tau) - \varphi_{l,n}^c(.,\tau) &= [Vg_l^c(.,\tau) + B_l\varphi_l^c(.,\tau)] - \left[Vg_l^c(.,\tau) + B_{l,n}\varphi_{l,n}^c(.,\tau)\right] \\ &= \left[B_l\varphi_l^c(.,\tau) - B_{l,n}\varphi_{l,n}^c(.,\tau)\right] \\ &= \left[(B_l - B_{l,n})\varphi_l^c(.,\tau) - B_{l,n}(\varphi_{l,n}^c(.,\tau) - \varphi_l^c)(.,\tau)\right]. \end{aligned}$$

This leads to

$$(I - B_{l,n})(\varphi_l^c(.,\tau) - \varphi_{l,n}^c(.,\tau)) = (B_l - B_{l,n})\varphi_l^c(.,\tau),$$

so that

$$\varphi_l^c(.,\tau) - \varphi_{l,n}^c(.,\tau) = (I - B_{l,n})^{-1}\left[(B_l - B_{l,n})\varphi_l^c(.,\tau)\right].$$

Consequently

$$\|\varphi_{l,n}^c(.,\tau) - \varphi_l^c(.,\tau)\|_\infty \leq \gamma_l^c\|(B_l - B_{l,n})\varphi_l^c(.,\tau)\|_\infty,$$

where

$$\gamma_l^c := \sup_{n \geq N}\left\|(I - B_{l,n})^{-1}\right\|,$$

which is finite. The other outcomes can be demonstrated in a similar manner to the one described above. □

Letting

$$\begin{aligned} \mathcal{R}_{l,n}^{d,+} &:= \gamma_l^{d,+}\left\|\Big((B_l - B_{l,n}) - (S_l^+ - S_{l,n}^+)\Big)\varphi_l^d(.,\tau)\right\|_\infty, \\ \mathcal{R}_{l,n}^{d,-} &:= \gamma_l^{d,-}\left\|\Big((S_l^- - S_{l,n}^-) - (B_l - B_{l,n})\Big)\varphi_l^d(.,\tau)\right\|_\infty, \\ \mathcal{R}_{l,n}^{d} &:= \max\left\{\mathcal{R}_{l,n}^{d,+}, \mathcal{R}_{l,n}^{d,-}\right\}. \end{aligned}$$

Theorem 3.5 *The following estimate hold*

$$\|\underline{\varphi}_l(.,\tau) - \underline{\varphi}_{l,n}(.,\tau)\|_\infty \;\leq\; \gamma_l^c \|\,(B_l - B_{l,n})\,\varphi_l^c(.,\tau)\|_\infty + \mathcal{R}_{l,n}^d,$$

$$\|\overline{\varphi}_l(.,\tau) - \overline{\varphi}_{l,n}(.,\tau)\|_\infty \;\leq\; \gamma_l^c \|\,(B_l - B_{l,n})\,\varphi_l^c\|_\infty + \mathcal{R}_{l,n}^d,$$

$$\begin{aligned}\|\underline{\varphi}_c(.,\tau) - \underline{\varphi}_{c,n}(.,\tau)\|_\infty &\leq \gamma_c^c \|\,(B_c - B_{c,n})\,\varphi_c^c(.,\tau)\|_\infty \\ &+ \gamma_c^d \Big\| \Big((B_c - B_{c,n}) - (S_c - S_{c,n})\Big)\varphi_c^d(.,\tau)\Big\|_\infty,\end{aligned}$$

$$\begin{aligned}\|\overline{\varphi}_c(.,\tau) - \overline{\varphi}_{c,n}(.,\tau)\|_\infty &\leq \gamma_c^c \|\,(B_c - B_{c,n})\,\varphi_c^c(.,\tau)\|_\infty \\ &+ \gamma_c^d \Big\| \Big((B_c - B_{c,n}) - (S_c - S_{c,n})\Big)\varphi_c^d(.,\tau)\Big\|_\infty,\end{aligned}$$

for n large enough.

Proof : To provide the desired results, we take into account that

$$\begin{aligned}\|\underline{\varphi}_l(.,\tau) - \underline{\varphi}_{l,n}(.,\tau)\|_\infty &= \|\,\big(\varphi_l^c(.,\tau) - \varphi_l^d(.,\tau)\big) - \big(\varphi_{l,n}^c(.,\tau) - \varphi_{l,n}^d(.,\tau)\big)\,\|_\infty \\ &\leq \|\varphi_l^c(.,\tau) - \varphi_{l,n}^c(.,\tau)\|_\infty + \|\varphi_l^d(.,\tau) - \varphi_{l,n}^d(.,\tau)\|_\infty, \\ \|\overline{\varphi}_l(.,\tau) - \overline{\varphi}_{l,n}(.,\tau)\|_\infty &= \|\,\big(\varphi_l^c(.,\tau) + \varphi_l^d(.,\tau)\big) - \big(\varphi_{l,n}^c(.,\tau) + \varphi_{l,n}^d(.,\tau)\big)\,\|_\infty \\ &\leq \|\varphi_l^c(.,\tau) - \varphi_{l,n}^c(.,\tau)\|_\infty + \|\varphi_l^d(.,\tau) - \varphi_{l,n}^d(.,\tau)\|_\infty,\end{aligned}$$

and

$$\begin{aligned}\|\underline{\varphi}_c(.,\tau) - \underline{\varphi}_{c,n}(.,\tau)\|_\infty &= \|\,\big(\varphi_c^c(.,\tau) - \varphi_c^d(.,\tau)\big) - \big(\varphi_{c,n}^c(.,\tau) - \varphi_{c,n}^d(.,\tau)\big)\,\|_\infty \\ &\leq \|\varphi_c^c(.,\tau) - \varphi_{c,n}^c(.,\tau)\|_\infty + \|\varphi_c^d(.,\tau) - \varphi_{c,n}^d(.,\tau)\|_\infty, \\ \|\overline{\varphi}_c(.,\tau) - \overline{\varphi}_{c,n}(.,\tau)\|_\infty &= \|\,\big(\varphi_c^c(.,\tau) + \varphi_c^d(.,\tau)\big) - \big(\varphi_{c,n}^c(.,\tau) + \varphi_{c,n}^d(.,\tau)\big)\,\|_\infty \\ &\leq \|\varphi_c^c(.,\tau) - \varphi_{c,n}^c(.,\tau)\|_\infty + \|\varphi_c^d(.,\tau) - \varphi_{c,n}^d(.,\tau)\|_\infty.\end{aligned}$$

□

Letting

$$\begin{aligned} d_{l,n}(\varphi_{l,\tau}) &:= \gamma_l^c \| \left(B_l - B_{l,n}\right) \varphi_l^c(.,\tau)\|_\infty + \mathcal{R}_{l,n}^d, \\ d_{c,n}^d(\varphi_{c,\tau}) &:= \gamma_c^c \| \left(B_c - B_{c,n}\right) \varphi_c^c(.,\tau)\|_\infty + \gamma_c^d \left\| \Big((B_c - B_{c,n}) - (S_c - S_{c,n}) \Big) \varphi_c^d(.,\tau) \right\|_\infty . \end{aligned}$$

We can now state the key result of convergence analysis is the following corollary.

Corollary 3.1 *The following estimate hold*

$$D^*\left(\varphi_l, \varphi_{l,n}\right) \leq \sup_\tau \left\{ d_{l,n}(\varphi_{l,\tau}) \right\},$$

$$D^*\left(\varphi_c, \varphi_{c,n}\right) \leq \sup_\tau \left\{ d_{c,n}(\varphi_{c,\tau}) \right\},$$

for n large enough.

Proof :

Since

$$\max \left\{ \left| \underline{\varphi}_l(s,\tau) - \underline{\varphi}_{l,n}(s,\tau) \right|, \left| \overline{\varphi}_l(s,\tau) - \overline{\varphi}_{l,n}(s,\tau) \right| \right\} \leq d_{l,n}(\varphi_{l,\tau}),$$

and since

$$\max \left\{ \left| \underline{\varphi}_c(s,\tau) - \underline{\varphi}_{c,n}(s,\tau) \right|, \left| \overline{\varphi}_c(s,\tau) - \overline{\varphi}_{c,n}(s,\tau) \right| \right\} \leq d_{c,n}(\varphi_{c,\tau}),$$

we get

$$\begin{aligned} \sup_{-1<s<1} D\left(\varphi_l(s), \varphi_{l,n}(s)\right) &\leq d_{l,n}(\varphi_{l,\tau}), \\ \sup_{-1<s<1} D\left(\varphi_c(s), \varphi_{c,n}(s)\right) &\leq d_{c,n}(\varphi_{c,\tau}). \end{aligned}$$

Consequently, we obtain the required estimates. □

3.8 Numerical examples

We give numerical results of two cases, selected integro-differential equations, solved by the methods of this work in this section to highlight the performance of our methods. Each table in these numerical computations displays the numerical error of our approximation. Letting

$$\mathcal{E}^c_{l,n}(\tau) := \left|\varphi^c_l(s_j,\tau) - \varphi^c_{l,n}(s_j,\tau)\right| \text{ and } \mathcal{E}^d_{l,n}(\tau) := \left|\varphi^d_l(s_j,\tau) - \varphi^d_{l,n}(s_j,\tau)\right|,$$

and

$$\mathcal{E}^c_{c,n}(\tau) := \left|\varphi^c_c(s_j,\tau) - \varphi^c_{c,n}(s_j,,\tau)\right| \text{ and } \mathcal{E}^d_{c,n}(\tau) := \left|\varphi^c_c(s_j,\tau) - \varphi^d_{c,n}(s_j,\tau)\right|,$$

Example 1

To begin, let's look at the logarithmic fuzzy Fredholm integro-differential equation (**??**) with $\lambda = -1$ and $g_l(.,\tau)$ such that

$$\varphi_l(s,\tau) = [\tau(s^2-1),(2-\tau)(s^2-1)].$$

It follows that

$$\varphi^c_l(s,\tau) = (s^2-1) \text{ and } \varphi^d_l(s,\tau) = (1-\tau)(s^2-1).$$

The numerical results for Example 1 are listed in Table (3.1) for $\tau = 0.1$.

n	$\mathcal{E}^d_{l,n}(\tau)$	$\mathcal{E}^c_{l,n}(\tau)$	$\mathcal{E}^d_{l,n}(\tau)+\mathcal{E}^c_{l,n}(\tau)$
15	4.248e-3	4.727e-3	8.975e-3
25	1.622e-3	1.802e-3	3.425e-3
35	9.184e-4	1.019e-3	1.938e-3
45	5.056e-4	5.637e-4	1.069e-3
55	4.017e-4	4.448e-4	0.846e-4
65	2.576e-4	3.200e-4	5.776e-4
75	3.780e-4	4.100e-4	7.880e-4
85	2.400e-4	3.880e-4	6.280e-4
100	1.900e-4	2.400e-4	4.300e-4

Table 3.1: *Example 1*

Example 2

The following Cauchy Fuzzy Fredholm integro-differential equation is the subject of the second example. Here, $\lambda = -1$ and the function $g_c(.,\tau)$ was chosen in such a way that

$$\varphi_c(s,\tau) = [\tau(s^3 - s), (2-\tau)(s^3 - s)].$$

This implies that

$$\varphi_c^c(s,\tau) = (s^3 - s) \text{ and } \varphi_c^d(s,\tau) = (1-\tau)(s^3 - s).$$

The method's rate of convergence is shown in table (3.2) for $\tau = 0.1$. The results back up the above-mentioned convergence features.

n	$\mathcal{E}_{c,n}^d(\tau)$	$\mathcal{E}_{c,n}^c(\tau)$	$\mathcal{E}_{c,n}^d(\tau) + \mathcal{E}_{c,n}^c(\tau)$
20	4.436e-3	1.329 e-3	5.765e-3
30	2.214e-3	6.630 e-4	2.877e-3
40	1.361e-3	4.049e-4	1.765e-3
50	9.550e-4	2.806e-4	1.235e-3
60	6.495e-4	1.933e-4	8.428e-4
70	5.312e-4	1.425e-4	6.737e-4
80	4.850e-4	1.270e-4	6.120e-4
90	3.080e-4	1.090e-4	4.170e-4
100	1.235e-4	8.990e-5	2.134e-4

Table 3.2: *Example 2*

3.9 Concluding remarks

To approximate two critical classes of fuzzy singular integro-differential equations with a logarithmic kernel and a Cauchy one, an efficient collocation approach based on airfoil polynomials was presented. Other types of equations can be generated and used with the approach. By presenting actual computational approaches, this work will help clarify the difference between theoretical fuzzy singular integro-differential equations research and practical applications currently used in the design of different fuzzy quantum systems.

Chapter 4

Intuitionistic fuzzy integral equations

4.1 Introduction

In this chapter, the term intuitionistic fuzzy set, which is a generalization of the term fuzzy set introduced by Zadeh [77, 76, 24, 38], is defined.

Nowadays, fuzzy theory and calculus are very popular topics. The papers [12, 72, 3] discussed various results on intuitionistic fuzzy set theory. In [11], the authors discussed intuitionistic fuzzy integrals. There are several literature sources where fuzzy integral equations are solved, such as fuzzy Fredholm integral equation, (see, [25, 37, 25]) and fuzzy Volterra integral equation, (see, [66, 67, 11])

Fuzzy set theory has long been used to handle fuzzy decision-making problems, but many researchers have recently taken an interest in intuitionistic fuzzy set (IFS) theory and applied it to the field of decision making. In cases where existing information is insufficient for the definition of an inexact concept using a conventional fuzzy set, the concept of an intuitionistic fuzzy set can be viewed as an alternative approach to acknowledging a fuzzy set.

Several authors consider intuitionistic fuzzy numbers in various articles and apply them in various fields. However, the point is that they only considered the intuitionistic fuzzy number with linear membership and nonmembership functions. However, this is not always necessary.

In this chapter, we present the various arithmetic operations on intuitionistic fuzzy numbers. We present all of the arithmetic operations as well as some properties of differentiability for intuitionistic fuzzy functions. The de-i-fuzzification of the corresponding intuitionistic fuzzy solution is also defined by the average of (τ_1, τ_2)-cut method. We examine an intuitionistic fuzzy integral equations.

4.2 Intuitionistic fuzzy analysis

Let X be the universal set.

Definition 4.1 *([62]) An intuitionistic fuzzy set (IFS) A in X is defined by*

$$A := \{(s, \rho(s), \varrho(s)), \quad s \in X\},$$

where the functions $\rho(s), \varrho(s) : X \to [0,1]$ represent respectively, the degree of membership and degree of non-membership of the element $s \in X$ to the set A, which is a subset of X, and for every $s \in X$, $0 \leq \rho(s) + \varrho(s) \leq 1$.

For each IFS A in X, we will call

$$\Pi(s) = 1 - \rho(s) - \varrho(s)$$

the intuitionistic fuzzy index of s in A. It is evident that

$$0 \leq \Pi(s) \leq 1, \quad \textit{for all} \quad s \in X.$$

Definition 4.2 *([62],([43])) An intuitionistic fuzzy set $A = \{(s, \rho(s), \varrho(s)), \quad s \in X\}$, of the real line is called an intuitionistic fuzzy number (IFN) if:*

(i) *A is IF-normal, i.e.there exist at least two points $s, s_0 \in X$ such that $\rho(s) = 1$ and $\varrho(s_0) = 1$,*

(ii) *ρ is a A is IF- convex, i.e. its membership function ρ is fuzzy convex, i.e.*

$$\rho\Big(\lambda s + (1-\lambda)s_0\Big) \geq \min\Big(\rho(s), \rho(s_0)\Big) \quad s, s_0 \in \mathbb{R}, \quad \lambda \in [0,1]\,;$$

and its non-membership function ϱ is fuzzy concave, i.e.

$$\varrho\Big(\lambda s + (1+\lambda\Big)s_0 \leq \max\Big(\varrho(s), \varrho(s_0)\Big) \quad s, s_0 \in \mathbb{R}, \quad \lambda \in [0,1]$$

(iii) *ρ is upper semi-continuous and ϱ is lower semi-continuous;*

(iv) *$suppA = \{s \in X, \quad \varrho(s) < 1\}$ is bounded.*

4.2.1 Arithmetic operations on interval-valued intuitionistic fuzzy numbers

Let $A = \{[a_1, a_2]; [a'_1, a'_2]\}$ and $B = \{[b_1, b_2]; [b'_1, b'_2]\}$ be two interval valued intuitionistic fuzzy numbers. Then, the following are the various arithmetic operations:

1. Addition:
$$A + B = \{[a_1 + b_1, a_2 + b_2]; [a'_1 + b'_1, a'_2 + b'_2]\};$$

2. Substraction:
$$A - B = \{[a_1 - b_2, a_2 - b_1]; [a'_1 - b'_2, a'_2 - b'_1]\};$$

3. Multiplication:
$$A \times B = \{\alpha; \beta\},$$
where
$$\alpha := [\min(a_1b_1, a_1b_2, a_2b_1, a_2b_2), \max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)],$$
and
$$\beta := [\min(a'_1b'_1, a'_1b'_2, a'_2b'_1, a'_2b'_2), \max(a'_1b'_1, a'_1b'_2, a'_2b'_1, a'_2b'_2)].$$

4. Division:
$$\begin{aligned} AB &= \left\{\left[\min\left(\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right), \max\left(\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right)\right];\right. \\ &\quad \left.\left[\min\left(\frac{a'_1}{b'_1}, \frac{a'_1}{b'_2}, \frac{a'_2}{b'_1}, \frac{a'_2}{b'_2}\right), \max\left(\frac{a'_1}{b'_1}, \frac{a'_1}{b'_2}, \frac{a'_2}{b'_1}, \frac{a'_2}{b'_2}\right)\right]\right\}. \end{aligned}$$

5. Scalar multiplication: Let $k \in \mathbb{R}$. Then,
$$kA = \begin{cases} \{[ka_1, ka_2]; [ka'_1, ka'_2]\} & \text{if } k \geq 0, \\ \{[ka_2, ka_1]; [ka'_2, ka'_1]\} & \text{if } k < 0. \end{cases}$$

More information concerning the arithmetic operations on interval-valued intuitionistic fuzzy numbers can be found in [73].

4.2.2 Intuitionistic fuzzy numbers

Definition 4.3 *([43]) A set of (r_1, r_2)-cuts, generated by IFS A, where $r_1, r_2 \in [0, 1]$ is a set of fixed numbers such that $r_1 + r_2 \leq 1$ is defined as*

$$A_{r_1,r_2} := \{(s, \rho(s), \varrho(s)) | s \in X \ \rho(s) \geq r_1, \varrho(s) \leq r_2, \ r_1, r_2 \in [0, 1]\}$$

(r_1, r_2)-cuts denoted by A_{r_1,r_2} is defined as the crisp set of elements s wich belong to A, at least to the degree r_1 and which does belong A at most to the degree r_2.

Denoting by $\mathcal{F}^i$ the set of all intuitionistic fuzzy numbers.

Let $x, y \in \mathcal{F}^i$, if there exists $z \in \mathcal{F}^i$ such that $x = y + z$ then z is called Hukuhara difference (H-difference) of x and y and is denoted by $x \ominus y$.([17])

Definition 4.4 *([17]) Let $f : (a, b) \to \mathcal{F}^i$ and $s_0 \in [a, b]$. We say that f is differentiable at s_0, if there exist an element $f'(s_0) \in \mathcal{F}^i$, such that*

1. *For all $h > 0$ sufficiently near to 0, there exist $f(s_0+h)\ominus f(s_0)$, $f(s_0)\ominus f(s_0-h)$, and the limits*

$$\lim_{h \to 0^+} \frac{f(s_0 + h) \ominus f(s_0)}{h} = \lim_{h \to 0^+} \frac{f(s_0) \ominus f(s_0 - h)}{h} = f'(s_0)$$

or

2. *for all $h < 0$ sufficiently near to 0, there exist $f(s_0+h)\ominus f(s_0)$, $f(s_0)\ominus f(s_0-h)$, and the limits*

$$\lim_{h \to 0^-} \frac{f(s_0 + h) \ominus f(s_0)}{h} = \lim_{h \to 0^-} \frac{f(s_0) \ominus f(s_0 - h)}{h} = f'(s_0).$$

in the case when f is intuitionistic fuzzy valued function, we have the following theorem

Theorem 4.1 *([17]) Let $f : \mathbb{R} \to \mathcal{F}^i$ be a intuitionistic fuzzy valued function with $(r_1, r_2)-$cut representation*

$$f_{r_1,r_2}(s) = \{\underline{f}(s, r_1), \overline{f}(s, r_2)\} = \{[\underline{f}_l(s, r_1), \underline{f}_r(s, r_1)]; [\overline{f}_l(s, r_2), \overline{f}_r(s, r_2)]\},$$

for each $(r_1, r_2) \in (0, 1)$. Then we have the following

1. *If f is differentiable in the first form (1) in Definition 4.4. Then*

$$\underline{f}_l(s, r_1), \underline{f}_r(s, r_1) \text{ and } \overline{f}_l(s, r_2), \overline{f}_r(s, r_2)$$

are differentiable functions and

$$f'_{r_1,r_2}(s) = \{[\underline{f}'_l(s,r_1), \underline{f}'_r(s,r_1)]; [\overline{f}'_l(s,r_2), \overline{f}'_r(s,r_2)]\}.$$

2. *If f is differentiable in the second form (2)in Definition 4.4. Then*

$$\underline{f}_l(s,r_1), \underline{f}_r(s,r_1) \text{ and } \overline{f}_l(s,r_2), \overline{f}_r(s,r_2)$$

are differentiable functions and

$$f'_{r_1,r_2}(s) = \{[\underline{f}'_r(s,r_1), \underline{f}'_l(s,r_1)]; [\overline{f}'_r(s,r_2), \overline{f}'_l(s,r_2)]\}.$$

Theorem 4.2 *([17]) Let $f : \mathbb{R} \to \mathcal{F}^i$ be a intuitionistic fuzzy valued function with (r_1, r_2)-cut representation*

$$f_{r_1,r_2}(s) = \{\underline{f}(s,r_1), \overline{f}(s,r_2)\} = \{[\underline{f}_l(s,r_1), \underline{f}_r(s,r_1)]; [\overline{f}_l(s,r_2), \overline{f}_r(s,r_2)]\},$$

for each $(r_1, r_2) \in (0,1)$. Then we have the following:

1. *If f and f' are differentiable in the first form (1) or If f and f' are differentiable in the second form (2) in Definition 4.4. Then*

$$f"_{r_1,r_2}(s) = \{\{[\underline{f}''_l(s,r_1), \underline{f}''_r(s,r_1)]; [\overline{f}''_l(s,r_2), \overline{f}''_r(s,r_2)]\}\}$$

2. *If f is differentiable in the first form (1) and f' are differentiable in the second form (2) or if f is differentiable in the second form (2) and f' are differentiable in the first form (1) in Definition 4.4. Then*

$$f"_{r_1,r_2}(s) = \{[\underline{f}''_r(s,r_1), \underline{f}''_l(s,r_1)]; [\overline{f}''_r(s,r_2), \overline{f}''_l(s,r_2)]\}.$$

4.2.3 Generalized Hukuhara distance on intuitionistic fuzzy-valued function

Definition 4.5 *[60] Let*

$$x_{r_1,r_2} = \{\underline{x}(r_1), \overline{x}(r_2)\} = \{[\underline{x}_l(r_1), \underline{x}_r(r_1)]; [\overline{x}_l(r_2), \overline{x}_r(r_2)]\}$$

and

$$y_{r_1,r_2} = \{\underline{y}(r_1), \overline{y}(r_2)\} = \{[\underline{y}_l(r_1), \underline{y}_r(r_1)]; [\overline{y}_l(r_2), \overline{y}_r(r_2)]\}$$

two intuitionistic fuzzy numbers. The Hausdorff distance between intuitionistic fuzzy numbers is given by $D^i : \mathcal{F}^i \times \mathcal{F}^i \to \mathbb{R}^+ \cup \{0\}$ *as in*

$$\begin{aligned} D^i(\underline{x}, \underline{y}; \overline{x}, \overline{y}) &= \sup_{r_1,r_2} D(\underline{x}(r_1), \underline{y}(r_1); \overline{x}(r_2), \overline{y}(r_2)) \\ &= \sup_{r_1,r_2} \max\left\{|\underline{x}_l(r_1) - \underline{y}_l(r_1)|, |\underline{x}_r(r_1) - \underline{y}_r(r_1)|, |\overline{x}_l(r_2) - \overline{y}_l(r_2)|, |\overline{x}_r(r_2) - \overline{y}_r(r_2)|\right\}, \end{aligned}$$

where D *is Hausdorff metric and metric space* $(\mathcal{F}^i, D^i)$ *is complete, separable, and locally compact, the following substances for metric* D^i *are tenable:*

1. $D^i(\underline{x} + w, \underline{y} + w; \overline{x} + z, \overline{y} + z) = D^i\left(\underline{x}, \underline{y}; \overline{x}, \overline{y}\right)$, *for all* $\underline{x}, \underline{y}, \overline{x}, \overline{y}, w, z \in \mathcal{F}^i$;
2. $D^i(k\underline{x}, k\underline{y}; k\overline{x}, k\overline{y}) = |k| D^i\left(\underline{x}, \underline{y}; \overline{x}, \overline{y}\right)$, *for all* $\underline{x}, \underline{y}, \overline{x}, \overline{y} \in \mathcal{F}^i, k \in R$;
3. $D^i(\underline{x}_l + \underline{x}_r, \underline{y}_l + \underline{y}_r; \overline{x}_l + \overline{x}_r, \overline{y}_l + \overline{y}_r) \le D^i\left(\underline{x}_l, \underline{y}_l; \overline{x}_l, \overline{y}_l\right) + D^i\left(\underline{x}_r, \underline{y}_r; \overline{x}_r, \overline{y}_r\right)$, *for all* $\underline{x}_l, \underline{y}_l, \overline{x}_l, \overline{y}_l, \underline{x}_r, \underline{y}_r, \overline{x}_r, \overline{y}_r \in \mathcal{F}^i$;
4. $D^i(\underline{x}_l \ominus \underline{x}_r, \underline{y}_l \ominus \underline{y}_r; \overline{x}_l \ominus \overline{x}_r, \overline{y}_l \ominus \overline{y}_r) \le D^i\left(\underline{x}_l, \underline{y}_l; \overline{x}_l, \overline{y}_l\right) + D^i\left(\underline{x}_r, \underline{y}_r; \overline{x}_r, \overline{y}_r\right)$, *as long as* $\underline{x}_l \ominus \underline{x}_r, \underline{y}_l \ominus \underline{y}_r, \overline{x}_l \ominus \overline{x}_r, \overline{y}_l \ominus \overline{y}_r$ *exists and for all* $\underline{x}_l, \underline{y}_l, \overline{x}_l, \overline{y}_l, \underline{x}_r, \underline{y}_r, \overline{x}_r, \overline{y}_r \in \mathcal{F}^i$.

4.2.4 Chebyshev polynomials

Definition 4.6 *[71] Let* $x = \cos(\theta), \quad \theta \in [0, \pi]$. *Then, the n-th degree Chebyshev polynomial* $T_n(.), n \in \mathbb{N}$, *on* $[-1, 1]$ *is defined by the relation*

$$T_n(x) := \cos(n\theta), \quad \textit{or explicitly,} \quad T_n(x) = \cos(n \arccos(x)).$$

The Chebyshev polynomials are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ *and the corresponding inner product.*

$$< f, g >= \int_{-1}^{1} w(x) g(x) f(x) dx, \quad \textit{where}, \quad f, g \in L^2(-1, 1).$$

The well-known recursive formula

$$T_{n+1}(x) = 2X T_n(x) - T_{n-1}(x), n \in \mathbb{N} \;\; \textit{where} \;\; T_0(x) = 1, \; T_1(x) = x$$

is important for numerical computation of these polynomials. Since it is more convenient to use range $[0, T]$ *than* $[-1, 1]$, *we transform* $[0, T]$ *into* $[-1, 1]$, *using linear transfor-*

mation $s = \frac{2}{T}x - 1$, *where* $x \in [0,T], s \in [-1,1]$. *This leads to a shifted Chebyshev polynomial (of the first kind)* $T_n^*(x)$ *of degree* n *in* x *on* $[0,T]$ *given by*

$$T_n^*(x) = T_n(\frac{2}{T}x - 1),$$

with the corresponding weight function $w^*(x) = w(\frac{2}{T}x - 1)$.

Let $u(x,y)$ be a bivariate function defined on $[0,T_1] \times [0,T_2]$. In the similar way, it can be expanded using Chebychev polynomials as follows

$$u(x,y) \simeq p_{N,M}(u)(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{M} u_{n,m} T_n^*(x)\bar{T}_m^*(y) = \Pi(x)^t U \overline{\Pi}(y),$$

where $p_{N,M} : C([0,T] \times [0,T]) \mapsto \pi_N \times \pi_M, (N, M \in \mathbb{N})$, is an orthogonal projection and we use $-$ to distinguish the shifted Chebyshev polynomials corresponding to different intervals. Here, $U = (u_{i,j})$ is a matrix of size $(N+1) \times (M+1)$ with the elements

$$\begin{aligned} u_{i,j} &= \frac{1}{\gamma_i \bar{\gamma}_j} \int_0^{T_1} \int_0^{T_2} w^*(x)\bar{w}^*(y)u(x,y)T_i^*(x)\bar{T}^*(y)dxdy \\ &\simeq \frac{T_1 T_2 \pi^2}{4\gamma_i\bar{\gamma}_j(N+1)^2} \sum_{n=0}^{N+1}\sum_{m=0}^{M+1} u(\frac{T_1}{2}(x_r+1), \frac{T_2}{2}(x_s+1))T_i(x_r)T_j(x_s). \end{aligned}$$

$\Pi(t) = [T_0^*(t), \cdots, T_N^*(t)]^t$.

Theorem 4.3 *[71] Let* $\Pi(x)$ *be the vector of shifted Chebyshev polynomials defined above. Let the* $(N+1) \times (M+1)$ *matrix* P *defined by* $P := \int_0^T \Pi(s)\Pi(s)^T ds$.

Then, the elements of this matrix can be determined by

$p_{00} = T,\ p_{11} = \frac{T}{3},\ p_{10} = p_{01} = 0,$

$p_{ij} = \frac{T}{4}(\frac{-1-(-1)^{i+j}}{(i+j-1)(i+j+1)}),\ \ for\ \ j = i+1, i-1, i \in \{1, \cdots, N\},\ \ and$

$$p_{ij} = \frac{T}{4}(\frac{-1-(-1)^{i+j}}{(i+j-1)(i+j+1)}) + \frac{-1-(-1)^{|i+j|}}{(|i+j|-1)(|i+j|+1)}),\ \ for\ \ j = i+1, i-1, i \in \{1, \cdots, N\}.$$

4.3 Intuitionistic fuzzy integral equation

Let us consider the following intuitionistic fuzzy integral equation

$$\varphi(s) = g(s) + \int_0^T h(s,t)\varphi(t)dt, \quad 0 < s,t < T. \tag{4.1}$$

where $h(.,.)$ and g are two known intuitionistic fuzzy numbers, φ is unknown intuitionistic fuzzy number.

4.4 The Approximate Solution

As in [71], we suppose that the intuitionistic fuzzy numbers φ and g can be described as described in the following:

$$\begin{aligned}
\varphi_{\tau_1,\tau_2}(s) &= \{\underline{\varphi}(s,\tau_1), \overline{\varphi}(s,\tau_2)\}, \\
&= \{[\underline{\varphi}_l(s,\tau_1), \underline{\varphi}_r(s,\tau_1)]; [\overline{\varphi}_l(s,\tau_2), \overline{\varphi}_r(s,\tau_2)]\} \\
g_{\tau_1,\tau_2}(s) &= \{\underline{g}(s,\tau_1), \overline{g}(s,\tau_2)\}, \\
&= \{[\underline{g}_l(s,\tau_1), \underline{g}_r(s,\tau_1)]; [\overline{g}_l(s,\tau_2), \overline{g}_r(s,\tau_2)]\}.
\end{aligned}$$

The equation (4.1) can be represented as follows:

$$\underline{\varphi}_l(s,\tau_1) = \underline{g}_l(s,\tau_1) + \int_0^T h_1(s,t)\underline{\varphi}_l(t,\tau_1)dt + \int_0^T h_2(s,t)\underline{\varphi}_r(t,\tau_1)dt, \quad , \quad 0 < s,t < T,$$

$$\underline{\varphi}_r(s,\tau_1) = \underline{g}_r(s,\tau_1) + \int_0^T h_1(s,t)\underline{\varphi}_r(t,\tau_1)dt + \int_0^T h_2(s,t)\underline{\varphi}_l(t,\tau_1)dt, \quad , \quad 0 < s,t < T,$$

$$\overline{\varphi}_l(s,\tau_2) = \overline{g}_l(s,\tau_2) + \int_0^T h_1(s,t)\overline{\varphi}_l(t,\tau_2)dt + \int_0^T h_2(s,t)\overline{\varphi}_r(t,\tau_2)dt, \quad , \quad 0 < s,t < T,$$

$$\overline{\varphi}_r(s,\tau_2) = \overline{g}_r(s,\tau_2) + \int_0^T h_1(s,t)\overline{\varphi}_r(t,\tau_2)dt + \int_0^T h_2(s,t)\overline{\varphi}_l(t,\tau_2)dt, \quad , \quad 0 < s,t < T,$$

Letting

$$h_1(s,t) = \begin{cases} h(s,t) & \text{if } h(s,t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$h_2(s,t) = \begin{cases} h(s,t) & \text{if } h(s,t) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\begin{bmatrix} \underline{\varphi}_l(s,\tau_1) \\ \underline{\varphi}_r(s,\tau_1) \end{bmatrix} = \begin{bmatrix} \underline{g}_l(s,\tau_1) \\ \underline{g}_r(s,\tau_1) \end{bmatrix} + \int_0^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_1 \end{pmatrix} \begin{bmatrix} \underline{\varphi}_l(t,\tau_1) \\ \underline{\varphi}_r(t,\tau_1) \end{bmatrix} dt$$

and

$$\begin{bmatrix} \overline{\varphi}_l(s,\tau_1) \\ \overline{\varphi}_r(s,\tau_1) \end{bmatrix} = \begin{bmatrix} \overline{g}_l(s,\tau_1) \\ \overline{g}_r(s,\tau_1) \end{bmatrix} + \int_0^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_1 \end{pmatrix} \begin{bmatrix} \overline{\varphi}_l(t,\tau_1) \\ \overline{\varphi}_r(t,\tau_1) \end{bmatrix} dt,$$

$$\begin{aligned}
\varphi_{\tau_1,\tau_2}(s) &\approx \{[\Pi^t(s)\underline{\Phi}_l\overline{\Pi}(\tau_1), \Pi^t(s)\underline{\Phi}_r\overline{\Pi}(\tau_1)]; [\Pi^t(s)\overline{\Phi}_l\overline{\Pi}(\tau_2), \Pi^t(s)\overline{\Phi}_r\overline{\Pi}(\tau_2)]\} \\
g_{\tau_1,\tau_2}(s) &\approx \{[\Pi^t(s)\underline{G}_l\overline{\Pi}(\tau_1), \Pi^t(s)\underline{G}_r\overline{\Pi}(\tau_1)]; [\Pi^t(s)\overline{G}_l\overline{\Pi}(\tau_2), \Pi^t(s)\overline{G}_r\overline{\Pi}(\tau_2)]\} \\
h_1(s,t) &\approx \Pi^t(s)H_1\Pi(t) \\
h_2(s,t) &\approx \Pi^t(s)H_2\Pi(t).
\end{aligned}$$

Hence,

$$\begin{bmatrix} \Pi^t(s)\underline{\Phi}_l\overline{\Pi}(\tau_1) \\ \\ \Pi^t(s)\underline{\Phi}_r\overline{\Pi}(\tau_1) \end{bmatrix} = \begin{bmatrix} \Pi^t(s)\underline{G}_l\overline{\Pi}(\tau_1) \\ \\ \Pi^t(s)\underline{G}_r\overline{\Pi}(\tau_1) \end{bmatrix} +$$

$$\int_0^T \begin{pmatrix} \Pi^t(s)H_1\Pi(t) & \Pi^t(s)H_2\Pi(t) \\ \\ \Pi^t(s)H_2\Pi(t) & \Pi^t(s)H_1\Pi(t) \end{pmatrix} \begin{bmatrix} \Pi^t(t)\underline{\Phi}_l\overline{\Pi}(\tau_1) \\ \\ \Pi^t(t)\underline{\Phi}_r\overline{\Pi}(\tau_1) \end{bmatrix} dt \tag{4.2}$$

and

$$\begin{bmatrix} \Pi^t(s)\overline{\Phi}_l\overline{\Pi}(\tau_2) \\ \\ \Pi^t(s)\overline{\Phi}_r\overline{\Pi}(\tau_2) \end{bmatrix} = \begin{bmatrix} \Pi^t(s)\overline{G}_l\overline{\Pi}(\tau_2) \\ \\ \Pi^t(s)\overline{G}_r\overline{\Pi}(\tau_2) \end{bmatrix} +$$

$$\int_0^T \begin{pmatrix} \Pi^t(s)H_1\Pi(t) & \Pi^t(s)H_2\Pi(t) \\ \\ \Pi^t(s)H_2\Pi(t) & \Pi^t(s)H_1\Pi(t) \end{pmatrix} \begin{bmatrix} \Pi^t(t)\overline{\Phi}_l\overline{\Pi}(\tau_2) \\ \\ \Pi^t(t)\overline{\Phi}_r\overline{\Pi}(\tau_2) \end{bmatrix} dt. \tag{4.3}$$

A similar analysis as [71] gives

$$\begin{bmatrix} \underline{\Phi}_l \\ \\ \underline{\Phi}_r \end{bmatrix} = \begin{bmatrix} \underline{G}_l + \int_0^T H_1\Pi(t)\Pi^t(t)\underline{\Phi}_l dt + \int_0^T H_2\Pi(t)\Pi^t(t)\underline{\Phi}_r dt \\ \\ \underline{G}_r + \int_0^T H_2\Pi(t)\Pi^t(t)\underline{\Phi}_l dt + \int_0^T H_1\Pi(t)\underline{\Phi}_r\Pi^t(t)\underline{\Phi}_r dt \end{bmatrix}, \tag{4.4}$$

$$\begin{bmatrix} \overline{\Phi}_l \\ \\ \overline{\Phi}_r \end{bmatrix} = \begin{bmatrix} \overline{G}_l + \int_0^T H_1\Pi(t)\Pi^t(t)\overline{\Phi}_l dt + \int_0^T H_2\Pi(t)\Pi^t(t)\overline{\Phi}_r dt \\ \\ \overline{G}_r + \int_0^T H_2\Pi(t)\Pi^t(t)\overline{\Phi}_l dt + \int_0^T H_1\Pi(t)\overline{\Phi}_r\Pi^t(t)\overline{\Phi}_r dt \end{bmatrix}. \tag{4.5}$$

Hence,

$$\begin{bmatrix} \underline{G}_l \\ \\ \underline{G}_r \end{bmatrix} = \begin{bmatrix} (I - H_1P) & -H_2P \\ \\ -H_2P & (I - H_1P) \end{bmatrix} \begin{bmatrix} \underline{\Phi}_l \\ \\ \underline{\Phi}_r \end{bmatrix}, \tag{4.6}$$

and

$$\begin{bmatrix} \overline{G}_l \\ \\ \overline{G}_r \end{bmatrix} = \begin{bmatrix} (I - H_1P) & -H_2P \\ \\ -H_2P & (I - H_1P) \end{bmatrix} \begin{bmatrix} \overline{\Phi}_l \\ \\ \overline{\Phi}_r \end{bmatrix}, \tag{4.7}$$

4.5 Existence and uniqueness

Theorem 4.4 *Assume that for equation (4.1) the following assumptions hold:*
$\underline{g}_l$, $\underline{g}_r$, $\overline{g}_l$, $\overline{g}_r$, h_1 *and* h_2 *are uniformly continuous with respect to* s *and there exist*

$\underline{c}_l > 0,\ \underline{c}_r > 0,\ \overline{c}_l > 0$ *and* $\overline{c}_r > 0$ *such that*

$$|\underline{g}_l(s_1,\tau_1) - \underline{g}_l(s_2,\tau_1)| \le \underline{c}_l|s_1 - s_2|,$$

$$|\underline{g}_r(s_1,\tau_1) - \underline{g}_r(s_2,\tau_1)| \le \underline{c}_r|s_1 - s_2|,$$

$$|\overline{g}_l(s_1,\tau_1) - \overline{g}_l(s_2,\tau_1)| \le \overline{c}_l|s_1 - s_2|,$$

$$|\overline{g}_r(s_1,\tau_1) - \overline{g}_r(s_2,\tau_1)| \le \overline{c}_r|s_1 - s_2|,$$

$$|h_1(s_1,t) - h_1(s_2,t)| \le k_1|s_1 - s_2|,$$

$$|h_2(s_1,t) - h_2(s_2,t)| \le k_2|s_1 - s_2|,$$

$$|\underline{\varphi}_l(s,\tau_1)| \le \underline{M}_l,\quad |\underline{\varphi}_r(s,\tau_1)| \le \underline{M}_r,$$

$$|\overline{\varphi}_l(s,\tau_1)| \le \overline{M}_l,\quad |\overline{\varphi}_r(s,\tau_1)| \le \overline{M}_r;$$

$$|h_1(s,t)| \le M_1,\quad |h_2(s,t)| \le M_2,\quad M = \max\{M_1, M_2\},\quad \textit{and}\ \ 2MT < 1.$$

Then, the problem (4.1) has a unique continuous solution $\varphi^* \in C([0,T])$.

Proof : We have to prove that $A(C([0,T])) \subset A(C([0,T]))$. To this goal, for all $\varphi \in A(C([0,T]))$, and $s_1, s_2 \in [0,T]$, we have

$$D^I((A\varphi)_{\tau_1,\tau_2}(s_1), (A\varphi)_{\tau_1,\tau_2}(s_2)) =$$

$$\begin{aligned}
&= \sup_{\tau_1,\tau_2} D(\underline{(A\varphi)}(s_1,\tau_1), \underline{(A\varphi)}(s_2,\tau_1); \overline{(A\varphi)}(s_1,\tau_2), \overline{(A\varphi)}(s_2,\tau_2)) \\
&= \sup_{\tau_1,\tau_2} \max\{|\underline{(A\varphi}_l)(s_1,\tau_1) - \underline{(A\varphi}_l)(s_2,\tau_1)|, |\underline{(A\varphi}_r)(s_1,\tau_1) - \underline{(A\varphi}_r)(s_2,\tau_1)|; \\
&\quad |\overline{(A\varphi_l})(s_1,\tau_2) - \overline{(A\varphi_l})(s_2,\tau_2)|, |\overline{(A\varphi_r})(s_1,\tau_2) - \overline{(A\varphi_r})(s_2,\tau_2)|\} \\
&\le \sup_{\tau_1,\tau_2} \max\{|\underline{g}_l(s_1,\tau_1) - \underline{g}_l(s_2,\tau_1)| + \int_0^T |h_1(s_1,t) - h_1(s_2,t)||\underline{\varphi}_l(t,\tau_1)|dt \\
&+ \int_0^T |h_2(s_1,t) - h_2(s_2,t)||\underline{\varphi}_r(t,\tau_1)|dt, |\underline{g}_r(s_1,\tau_1) - \underline{g}_r(s_2,\tau_1)| \\
&+ \int_0^T |h_1(s_1,t) - h_1(s_2,t)||\underline{\varphi}_r(t,\tau_1)|dt + \int_0^T |h_2(s_1,t) - h_2(s_2,t)||\underline{\varphi}_l(t,\tau_1)|dt; \\
&\quad |\overline{g}_l(s_1,\tau_2) - \overline{g}_l(s_2,\tau_2)| + \int_0^T |h_1(s_1,t) - h_1(s_2,t)||\overline{\varphi}_l(t,\tau_2)|dt
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^T |h_2(s_1,t)-h_2(s_2,t)||\overline{\varphi}_r(t,\tau_2)|dt, |\overline{g}_r(s_1,\tau_2)-\overline{g}_r(s_2,\tau_2)| \\
&+ \int_0^T |h_1(s_1,t)-h_1(s_2,t)||\overline{\varphi}_r(t,\tau_2)|dt + \int_0^T |h_2(s_1,t)-h_2(s_2,t)||\overline{\varphi}_l(t,\tau_2)|dt\} \\
&\leq \sup_{\tau_1,\tau_2}\max\{\underline{c}_l|s_1-s_2| + \int_0^T k_1|s_1-s_2||\underline{\varphi}_l(t,\tau_1)|dt + \int_0^T k_2|s_1-s_2||\underline{\varphi}_r(t,\tau_1)|dt, \\
&\quad \underline{c}_r|s_1-s_2| + \int_0^T k_1|s_1-s_2||\underline{\varphi}_r(t,\tau_1)|dt + \int_0^T k_2|s_1-s_2||\underline{\varphi}_l(t,\tau_1)|dt; \\
&\quad \overline{c}_l|s_1-s_2| + \int_0^T k_1|s_1-s_2||\overline{\varphi}_l(t,\tau_2)|dt + \int_0^T k_2|s_1-s_2||\overline{\varphi}_r(t,\tau_2)|dt, \\
&\quad \overline{c}_r|s_1-s_2| + \int_0^T k_1|s_1-s_2||\overline{\varphi}_r(t,\tau_2)|dt + \int_0^T k_2|s_1-s_2||\overline{\varphi}_l(t,\tau_2)|dt\} \\
&\leq \sup_{\tau_1,\tau_2}\max\{\underline{c}_l|s_1-s_2| + Tk_1\underline{M}_l|s_1-s_2| + Tk_2\underline{M}_r|s_1-s_2|, \\
&\quad \underline{c}_r|s_1-s_2| + Tk_1\underline{M}_r|s_1-s_2| + Tk_2\underline{M}_l|s_1-s_2|; \\
&\quad \overline{c}_l|s_1-s_2| + Tk_1\overline{M}_l|s_1-s_2| + Tk_2\overline{M}_r|s_1-s_2|, \\
&\quad \overline{c}_r|s_1-s_2| + Tk_1\overline{M}_r|s_1-s_2| + Tk_2\overline{M}_l|s_1-s_2|\} \\
&\leq \sup_{\tau_1,\tau_2}\max\{\underline{c}_l + Tk_1\underline{M}_l + Tk_2\underline{M}_r, \underline{c}_r + Tk_1\underline{M}_r + Tk_2\underline{M}_l; \\
&\quad \overline{c}_l + Tk_1\overline{M}_l + Tk_2\overline{M}_r, \overline{c}_r + Tk_1\overline{M}_r + Tk_2\overline{M}_l\}|s_1-s_2|
\end{aligned}
$$

Thus, the operator A is uniformly continuous. It follows $A(C([0,T])) \subset C([0,T])$.

We now study the continuous of A on $C([0,T])$. Let $\varphi, \psi \in C([0,T])$, $s \in [0,T]$. We have

$$
D^I((A\varphi)_{\tau_1,\tau_2}(s),(A\psi)_{\tau_1,\tau_2}(s)) = \sup_{\tau_1,\tau_2} D(\underline{(A\varphi)}(s,\tau_1),\underline{(A\psi)}(s,\tau_1);\overline{(A\varphi)}(s,\tau_2),\overline{(A\psi)}(s,\tau_2))
$$

$$
\begin{aligned}
&\leq \sup_{\tau_1,\tau_2}\max\{|\underline{(A\varphi)}_l(s,\tau_1)-\underline{(A\psi)}_l(s,\tau_1)|, |\underline{(A\varphi)}_r(s,\tau_1)-\underline{(A\psi)}_r(s,\tau_1)|; \\
&\quad |\overline{(A\varphi)}_l(s,\tau_2)-\overline{(A\psi)}_l(s,\tau_2)|, |\overline{(A\varphi)}_r(s,\tau_2)-\overline{(A\psi)}_r(s,\tau_2)|\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup_{\tau_1,\tau_2} \max\{\int_0^T |h_1(s,t)||\underline{\varphi}_l(t,\tau_1) - \underline{\psi}_l(t,\tau_1)|dt + \\
&\quad \int_0^T |h_2(s,t)||\underline{\varphi}_r(t,\tau_1) - \underline{\psi}_r(t,\tau_1)|dt, \\
&\quad \int_0^T |h_1(s,t)||\underline{\varphi}_r(t,\tau_1) - \underline{\psi}_r(t,\tau_1)|dt + \\
&\quad \int_0^T |h_2(s,t)||\underline{\varphi}_l(t,\tau_1) - \underline{\psi}_l(t,\tau_1)|dt; \\
&\quad \int_0^T |h_1(s,t)||\overline{\varphi}_l(t,\tau_2) - \overline{\psi}_l(t,\tau_2)|dt + \\
&\quad \int_0^T |h_2(s,t)||\overline{\varphi}_r(t,\tau_2) - \overline{\psi}_r(t,\tau_2)|dt, \\
&\quad \int_0^T |h_1(s,t)||\overline{\varphi}_r(t,\tau_2) - \overline{\psi}_r(t,\tau_2)|dt + \\
&\quad \int_0^T |h_2(s,t)||\overline{\varphi}_l(t,\tau_2) - \overline{\psi}_l(t,\tau_2)|dt\} \\
&\leq \sup_{\tau_1,\tau_2} \max\{M_1 T \sup_t |\underline{\varphi}_l(t,\tau_1) - \underline{\psi}_l(t,\tau_1)| + \\
&\quad M_2 T \sup_t |\underline{\varphi}_r(t,\tau_1) - \underline{\psi}_r(t,\tau_1)|, \\
&\quad M_1 T \sup_t |\underline{\varphi}_r(t,\tau_1) - \underline{\psi}_r(t,\tau_1)| + \\
&\quad M_2 T \sup_t |\underline{\varphi}_l(t,\tau_1) - \underline{\psi}_l(t,\tau_1)|; \\
&\quad M_1 T \sup_t |\overline{\varphi}_l(t,\tau_2) - \overline{\psi}_l(t,\tau_2)| + \\
&\quad M_2 T \sup_t |\overline{\varphi}_r(t,\tau_2) - \overline{\psi}_r(t,\tau_2)|, \\
&\quad M_1 T \sup_t |\overline{\varphi}_r(t,\tau_2) - \overline{\psi}_r(t,\tau_2)| + \\
&\quad M_2 T \sup_t |\overline{\varphi}_l(t,\tau_2) - \overline{\psi}_l(t,\tau_2)|\} \\
&\leq 2MT \sup_t D^i(\varphi_{\tau_1,\tau_2}(t), \psi_{\tau_1,\tau_2}(t)) \\
D^{I^*}(A\varphi_{\tau_1,\tau_2}, A\psi_{\tau_1,\tau_2}) &\leq 2MTD^{I^*}(\varphi_{\tau_1,\tau_2}, \psi_{\tau_1,\tau_2})
\end{aligned}
$$

Thus, A is a contraction.

By the contraction principle, the operator A has a unique fixed point φ^*, then (4.1) has a unique continuous solution. □

Conclusions and perspectives

In this dissertation, we have presented some modified methods for solving certain classes of fuzzy Fredholm integral and integro-differential equations, and we highlight our results with numerical examples.

Our work aims to develop an approximation for fuzzy linear integral and integro-differential equations using collocation methods based on some orthogonal polynomials. This work can be extended to fuzzy nonlinear integral and integro-differential equations as well as other fuzzy singular integral equation classes.

To determine, as a future project, the conditions under which the previous methods could be applied to fuzzy Volterra integral equations of the third kind. These techniques can also be used with nonlinear integrals and integro-differential equations, but some modifications are required.

Precisely, we aim to approximate the solution of fuzzy integral equations of the type :

$$\alpha\varphi(s) - \beta\sum_{k=1}^{m}\int_a^s H_k(s,t,\psi(t))\varphi(t)dt = g(s), \qquad m \in \mathbb{N}^*, \quad a \leq s \leq b,$$

$$\alpha\varphi(s) - \beta\sum_{k=1}^{m}\int_a^s H_k(s,t,\psi(t))\ln|s-t|\, h(s,t)\varphi(t)dt = g(s), \qquad m \in \mathbb{N}^*, \quad a \leq s \leq b,$$

$$\alpha\varphi(s) - \frac{\beta}{\pi}\int_0^1 \frac{h(s,t)k(s,t,\psi(t))}{s-t}\varphi(t)dt = g(s), \quad , \quad 0 \leq s \leq 1.$$

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