République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université de Mostafa Ben Boulaid Batna 2
Faculté de Mathématiques et de l'informatique
Département de Mathématiques
Laboratoire des Techniques Mathématiques (LTM)

THÈSE

présenté en vue de l'obtention du

DIPLÔME DE DOCTORAT EN MATHÉMATIQUES

Option: Théorie de Contrôle

Présenté par : **Amira KAMECHE**

Sur le rayon de stabilité des systèmes stochastiques de dimension infinie

Soutenu le 07/12/2022 devant le jury composé de

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Année universitaire: 2021-2022

People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University of Mostafa Ben Boulaid Batna 2
Faculty of Mathematics and C omputer Science
Departement of Mathematics
Laboratory of Mathematical Techniques (LTM)

THESIS

Presented for the degree of

DOCTORAT IN MATHEMATICS

Specialty: Control Theory

Presented by : **Amira KAMECHE**

On the stability radius of infinite dimensional stochastic systems

Defended 07/12/2022

The Jury:

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University year: 2021-2022

Acknowledgments

I want to start by giving thanks to Almighty Allah, my parents, and my siblings for providing me the encouragement and fortitude I needed to succeed at this point in my life.
I also want to sincerely thank my supervisor Dr. Kada Maissa *and my co-supervisor Prof.* Rebiai Salah-Eddine *for their encouragement and ongoing assistance. Without their guidance and support I wouldn't be able to accomplish my Phd thesis.*
My profound appreciation to the president of the jury: Prof. Lombarkia Farida *and to the examiners: Prof.* Mennouni Abdelaziz *and Prof.* Khalfallah Nabil *for the valuable criticisms and suggestions.*

Kameche Amira ✍

Publications

1. Kameche Amira & Kada Maissa. Robust stabilization of infinite dimensional systems subjected to stochastic and deterministic perturbations. *2021 International Conference on Recent Advances in Mathematics and Informatics (ICRAMI), IEEE* . ISBN:978-1-6654-4172-8 (2021)

2. Amira Kameche, Hocine Makheloufi, Maissa Kada & Salah-Eddine Rebiai. New general decay rates of solutions for an abstract semilinear stochastic evolution equation with an infinite memory. *Nonlinear Studies.* www.nonlinearstudies.com, ISSN :1359-8678, **29**, No.2, 1-18, (2022).

Conferences

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2. **Amira Kameche & Amira Kada, Robust stability of infinite dimensional stochastic systems subjected to stochastic perturbation,** FIRST ONLINE CONFERENCE ON MODERN FRACTIONAL CALCULUS AND ITS APPLICATIONS (OCMFCA-2020) Biruni University, Istanbul, Turkey, 4- 6 December 2020, ISTANBUL TIME.
3. **Amira Kameche & Maissa Kada, Robust stabilization of stochastic infinite dimensional systems subjected to deterministic and stochastic perturbations,** The Third International Conference on Research in Applied Mathematics and Computer Science ICRAMCS'2021 26-27 March, Marocco.

Contents

Notations

To simplify exposition of the material in this thesis and to keep compactness of presentation the following notations will be used. Let H, U, V be real separable Hilbert spaces.

$L(U,H)$	The space of linear operators from U to H.
$\mathscr{L}(U,H)$	The space of bounded linear operators form U to H.
$\mathscr{L}(H)$	The space of bounded linear operators form H to H.
$\mathscr{L}_1(H)$	The space of all nuclear operators from H to H.
$\mathscr{L}_2^0(K_0,H)$	denote the sapce of all Hilbert-Schmidt operators from K_0 to H.
$\|\cdot\|$	The norm in H.
$\langle .,. \rangle$	The inner product in H.
$P \succ 0$	$P \in \mathscr{L}(H)$ is positive ($\langle Pz, z \rangle > 0$, for all $z \in H, z \neq 0$).
$P \succeq 0$	$P \in \mathscr{L}(H)$ is nonnegative ($\langle Pz, z \rangle \geq 0$, for all $z \in H$).
$P \gg 0$	$P \in \mathscr{L}(H)$, $\langle Pz, z \rangle > \gamma \|z\|^2$, $\gamma > 0$ for all $z \in H, z \neq 0$).
$\mathscr{L}^+(H)$	The set of self-adjoint linear bounded operators $P \in \mathscr{L}(H)$ such that $P \succeq 0$
$\mathrm{Lip}(Y,U)$	The set of Lipschitzian functions $\Delta : Y \longrightarrow U$.
$L^p((0,T),H), p \geq 1$	The space of functions $f(t)$ with $\int_0^T \|f(t)\|^p dt < +\infty$.
$C^1((0,T),H)$	The space of strongly continuously differentiable functions on $(0,T)$ with values in H.

$L^2(\Omega,H)$	The space of square integrable H-valued functions on the probability space $(\Omega,\mathscr{F},\mathbb{P})$
$L^2_w(\mathbb{R}^+,L^2(\Omega,H))$	The space of predictable stochastic processes $z(t)=(z(t))_{t\in\mathbb{R}^+}$ with respect to the $\sigma-algebras$ $\{\mathscr{F}_t\}_{t\in\mathbb{R}_+}\subset\mathscr{F}$ satisfying $$\|z\|^2_{L^2_w}=\mathbb{E}\int_0^{+\infty}\|z(t)\|^2_H dt=\int_0^{+\infty}\mathbb{E}(\|z(t)\|^2)dt<+\infty$$
$\mathbb{E}(x)$	The expectation of x.

Introduction

Stochastic equations in infinite dimension are natural generalizations of stochastic differential equations and their theory has motivations coming both from mathematics and the natural sciences: physics, chemistry and biology (see e.g. [13]). Stochastic stability for stochastic equations in Hilbert spaces has been studied extensively in the literature (see e.g [5, 6, 7, 42]).

In engineering, physics and economics, many dynamical systems involving stochastic process and random noise are often modeled by stochastic models. The stochastic effects of these models are then used to describe the uncertainty about the dynamic system parameters.

Modeling of any practical system for control design invariably involves uncertainty. Since the exact model of a process may be difficult or even impossible to determine, the logical approach is to design a control strategy based on a suitable nominal (i.e., most likely) model. Once a system has been identified and a nominal model established along with the availability of an associated uncertainty description, there are two main tasks to achieve. The first task is the analysis of the uncertain system, then the following issue arise naturally and concerns the question of robustness analysis. A system is said to be robustly stable if it can sustain stability in spite of suffering from uncertainty. The main question combining the overall problem is: What is the measure of the maximum uncertainty that a system can tolerate and still sustain stability and performance? Alternatively, given the size of the uncertainty, one might wish to study the effect of the uncertainty on the stability and performance of a system.

Robust control has been studied in extensive works due to its applications in many industrial control problems, e.g. power electronics systems, flight control systems , motion control systems and networked control systems. Indeed, a control law is typically designed from an idealized and simplified model of the applied control system. The potential problem of controller without considering the uncertainties is that closed-loop systems performance and stability are easier to be affected, which indicates that the controller is not robust enough to suppress the introduced disturbances. The goal of a robust control is therefore, to generate a suitable control law to overcome the imperfection of a model and assuring a certain performance level against the presence of uncertainties or external disturbances.

The aspect of developing measures of stability robustness for linear uncertain systems with state space description has received significant attention in system and control theory. These measures can be characterized by the stability radius. The problem of evaluating and calculating this stability radius is of great importance, from both theoretical and practical points of view and has attracted a lot of attention from researchers see [1, 14, 30, 34, 37, 38, 39, 51].

Stability radii for linear state space systems subjected to structured perturbations have been introduced in [27]. The authors considered continuous finite dimensional systems subjected to deterministic perturbations. They established characterizations of the stability radius in terms of a Riccati equation. El Bouhtouri and Pritchard used the framework of stability radii to analyse robust stability and robust stabilization of linear systems subjected to stochastic perturbations [15] [16].

Hinrichsen and Pritchard [29] obtained lower bounds and the perturbations which guarantee the stability of a linear system which is subjected to deterministic and stochastic perturbations.

El Bouhtouri, Hinrichsen and Pritchard [17] considered stochastic linear systems subjected to stochastic perturbation. For which they established lower bounds for stochastic perturbation. Authors in [35], considered continuous infinite dimensional systems subjected to stochastic multi-perturbations. They derived a lower bounds for the(supremal) stability radii.

If both the deterministic and the stochastic parameters are perturbed the stability radius problem is more complicated and at present it is far from been resolved. The results established in the finite dimensional case yield only conservative estimate.

Our first purpose in this thesis is to study the stability radii framework robust stability and robust stabilization of a class of infinite dimensional linear deterministic or stochastic systems subjected to deterministic and/or stochastic purpose.

The second objective of our research concerns the study of the stability behavior of an abstract semilinear stochastic evolution equation with an infinite memory that includes several equations coming from elasticity such as the wave and Petrovsky equations in the presence of additive noise. More specifically, we are concerned with the following stochastic evolution equation

$$\begin{cases} u_{tt}+Au(t)-\int_0^{+\infty}h(s)A^{\alpha}u(t-s)ds+f(u(t))=\sigma(t)W_t(t) \qquad t,s \ \text{ in}[0,+\infty[, \\ u(-t)=u_0(t), \quad u_t(0)=u_1. \end{cases} \tag{0.1}$$

When $\sigma\equiv 0$ and $\alpha=1$, problem (0.1) has been considered in a series of papers, most of them adressed the issues of the global existence of the solutions and their stability. Regarding the stability property, we can state [3, 4, 8, 20, 21, 25, 49] and the references therein. In [12], the author considered a second-order evolution with infinite memory of the form

$$u_{tt}(t)+Au(t)-\int_0^{+\infty}h(s)Au(t-s)ds=0 \quad \text{in } [0,+\infty[, \tag{0.2}$$

and proved that the solutions decay to 0 as t tends ∞ but no explicit rate of the decay was given. Later, it was shown that the solutions decay exponentially if the kernel function h decays exponentially and polynomially if h does, see for instance [24]. Messaoudi [44] generalized this result by introducing new conditions on the function h that leads to a general decay of solutions where the exponential, or polynomial decay rates are considered as special cases. In fact, he studied a wave equation with finite memory of the form

$$u_{tt}(x,t)-\Delta u(x,t)-\int_0^{t}h(t-s)\Delta u(x,s)ds=0, \tag{0.3}$$

together with Dirichlet boundary condition in $\Omega \times [0,+\infty[$ where Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 1$, and proved, under the condition

$$h'(t)\leq -\xi(t)h(t), \quad \forall t\in\mathbb{R}^+, \tag{0.4}$$

where $\xi:\mathbb{R}^+\to\mathbb{R}^+$ is a non-increasing differentiable function, that the solution has the following decay property

$$\|u(t)\|\leq k_1\exp\left(-k_2\int_0^t\xi(s)ds\right), \quad \forall t\in\mathbb{R}^+, \tag{0.5}$$

for some positive constants k_1 and k_2. As was mentioned, the above stability estimate recover the usual exponential and polynomial decay rates.

A few works studied stability problems for stochastic evolution equations with memory, see [31, 41, 52]. In [52], the authors showed the existence and uniqueness of solution for a class of stochastic wave equations with memory and they obtained a decay estimate of the energy function. Liang and Gao [41] considered a nonlinear stochastic viscoelastic wave equation with linear damping. By an appropriate energy inequality and estimations, they showed that the local solution of the stochastic equations will blow up with positive probability or explosive in L_2 sense under some sufficient conditions. Yang et al [31] proved global existence and asymptotic stability for the solution of a second order quasilinear stochastic viscoelastic evolution equation with memory.

The main aim is to address the global well-posedness and to study the stability of the global solution of (0.1). Regarding the stability problem, we use the general assumption (0.4) but we do not require that the function ξ to be decreasing which improves many results such as in [22, 43, 45, 46, 47]. It is worth to note that this general condition has never been employed for the stability of stochastic evolution systems.

The monograph is organized in five chapters that are structured as follows.

Chapter 1

In this chapter we survey the necessary notations and the main tools needed throughout this thesis. We firstly introduce some basic definitions and preliminaries in stochastic differential equations in Hilbert spaces. Then, we give concepts of solutions of deterministic and stochastic systems.

Chapter 2

This chapter studies the stability radius of deterministic systems subjected to both deterministic and stochastic perturbations.

First we establish characterizations of the stability radius, then we consider the maximization problem. We investigate controlled systems with bounded an unbounded input operator. The results are giving in terms of Lyapunov and Riccati equations.

Chapter 3

The goal of this chapter is to study the stability radius and its maximization for stochastic systems subjected to stochastic perturbations. The results are giving in terms of Lyapunov and Riccati stochastic equations.

Chapter 4

This chapter considers the general case. It deals with stochastic systems subjected to both deterministic and stochastic perturbations. We establish robustness results via a linear operator inequality.

Chapter 5

This chapter presents an important contribution on the stability of solutions for a class of stochastic equations with infinite memory. We generalize the works [31, 41, 52] by establishing a general stability result that allows a larger class of relaxation functions and improves many previous works.

We illustrate the methods developed in the thesis on several examples of stochastic partial differential equations.

Chapter 1

Stochastic differential equations in infinite dimension

In this chapter we introduce some basic definitions and state known results needed in our exposition. Definitions and results on stochastic processus are recalled in section 1.1. For a substantial treatment of these results see the monographs [13], [42]. Then, we recall some concepts from control theory of linear deterministic systems in Hilbert spaces. In particular, exponential stability, stabilization, Lyapunov and Riccati equations. Infinite dimensional stochastic equations are considered. Concepts of strong and mild solutions, immportant stochastic stability theorems are presented. Finally, we present some useful inequalities.

1.1 Nuclear and Hilbert-Schmidt operators

Let $\mathscr{E},\mathbb{G}$ be Banach spaces and let $\mathscr{L}(\mathscr{E},\mathbb{G})$ be the Banach spaces of all linear bounded operators from $\mathscr{E}$ into $\mathbb{G}$ endowed with the usual supremum norm. We denote by $\mathscr{E}^*$ and $\mathbb{G}^*$ the dual spaces of $\mathscr{E}$ and $\mathbb{G}$ respectively. An element $T\in\mathscr{L}(\mathscr{E},\mathbb{G})$is said to be a nuclear operator if there exist two sequences $\{a_j\}\subset\mathbb{G}$, $\{b_j\}\subset\mathscr{E}^*$ such that

$$\sum_{j=1}^{\infty}\|a_j\|.\|b_j\|<+\infty$$

and T has the representation

$$Tx=\sum_{j=1}^{\infty}a_jb_j(x),\quad x\in\mathscr{E}$$

The space of all nuclear operators from $\mathscr{E}$ into $\mathbb{G}$, endowed with the norm

$$\|T\|_1=inf\Big\{\sum_{j=1}^{\infty}\|a_j\|.\|b_j\|:\quad Tx=\sum_{j=1}^{\infty}a_jb_j(x),\;x\in\mathscr{E}\Big\}$$

is a Banach space , and will be denoted $\mathscr{L}_1(\mathscr{E},\mathbb{G})$. Let $\mathbb{K}$ be another Banach space; it is clear that if $T\in\mathscr{L}_1(\mathscr{E},\mathbb{G})$ and $S\in\mathscr{L}(\mathbb{G},\mathbb{K})$ then $TS\in\mathscr{L}_1(\mathscr{E},\mathbb{K})$ and $\|TS\|_1\le\|T\|\|S\|_1$.

Let H be a separable Hilbert space and let $\{e_k\}$ be a complete orthonormal system in H. If $T\in\mathscr{L}(H,H)$ then we define trace of T:

$$TrT=\sum_{j=1}^{\infty}\langle Te_j,e_j\rangle$$

Proposition 1.1.1 *[13]*

If $T \in \mathscr{L}_1(H)$ then TrT is a well-defined number independent of the choice of the orthonormal basis $\{e_k\}$.

Proposition 1.1.2 *[13]*

A nonnegative operator $T \in \mathscr{L}(H)$ is nuclear if and only if for an orthonormal basis $\{e_k\}$ on H

$$\sum_{j=1}^{\infty} \langle Te_j, e_j \rangle \leq +\infty$$

Moreover in this case $TrT = \|T\|_1$.

Let $\mathscr{E}$ and $\mathbb{F}$ be two separable Hilbert spaces with complete orthonormal bases $\{e_k\} \in H, \{f_j\} \subset \mathbb{F}$. A linear bounded operator $T : H \to \mathscr{E}$ is said to be Hilbert-Schmidt if

$$\sum_{k=1}^{\infty} |Te_k|^2 < \infty$$

The definition of Hilbert-Schmidt operator, and the number

$$\|T\|_2 = \left(\sum_{k=1}^{\infty} |Te_k|^2 \right)^{\frac{1}{2}}$$

are independent of the choice of the basis $\{e_k\}$

Proposition 1.1.3 *[13]*

Let$\mathscr{E}, \mathbb{F}, \mathbb{G}$ be separable Hilbert spaces. If $T \in \mathscr{L}_2(\mathscr{E}, \mathbb{F})$ and $S \in \mathscr{L}_2(\mathbb{F}, \mathbb{G})$, then $ST \in \mathscr{L}_1(\mathscr{E}, \mathbb{G})$ and

$$\|ST\|_1 \leq \|T\|_2 \|S\|_2$$

1.2 Hilbert space valued Wiener processes

A measurable space is a pair $(\Omega, \mathscr{F})$ where Ω is a set and $\mathscr{F}$ is a σ-field, also called a σ-algebra, of subsets of Ω. This means that the family $\{\mathscr{F}\}$ contains the set Ω and is closed under the operation of taking complements and countable unions of its elements. If $(\Omega, \mathscr{F})$ and $(S, \mathscr{S})$ are two measurable spaces, then a mapping ξ from Ω into S such that the set $\{\omega \in \Omega : \xi(\omega) \in A\} = \{\xi \in A\}$ belongs to $\mathscr{F}$ for arbitrary $A \in \mathscr{S}$ is called a random variable from $(\Omega, \mathscr{F})$ into $(S, \mathscr{S})$.

A probability measure on a measurable space $(\Omega, \mathscr{F})$ is a σ-additive function $\mathbb{P}$ from $\mathscr{F}$ into $[0,1]$ such that $\mathbb{P}(\Omega) = 1$. The triplet $(\Omega, \mathscr{F}, \mathbb{P})$ is called a probability space. If $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, we set

$$\overline{\mathscr{F}} = \{A \subset \Omega : \exists B, C \in \mathscr{F}, B \subset A \subset C, \mathbb{P}(B) = \mathbb{P}(C)\}$$

Then $\overline{\mathscr{F}}$ is a σ-field, called the completion of $\mathscr{F}$. If $\mathscr{F} = \overline{\mathscr{F}}$, the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be complete.

Let $(\Omega,\mathscr{F},\mathbb{P})$ denote a complete probability space. A family $\{\mathscr{F}_t\}, t \geq 0$, for which all the $\mathscr{F}_t$ are sub- σ-fields of $\mathscr{F}$ and form an increasing family of σ-fields, is called a filtration if $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}$ for $s \leq t$.

We assume that the probability space $(\Omega,\mathscr{F},\mathbb{P})$ is equipped with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that $\mathscr{F}_0$ contains all sets of P-measure zero. We consider two Hilbert spaces K and H, and a symmetric nonnegative operator $Q \in \mathscr{L}_1(K)$. If $\operatorname{Tr} Q < +\infty$, then there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in K, and a bounded sequence of positive real numbers $\{\lambda_k\}_{k\geq 1}$ such that

$$Qe_k = \lambda_k e_k, \quad k = 1,2,\cdots$$

Definition 1.2.1 *(H-valued Q-Wiener process) [53]*

A H-valued stochastic process $\{w(t)\}_{t\geq 0}$, is called a Q-Wiener process if

(a) . $w(0) = 0$,

(b) . $w(t)$ has continuous trajectories,

(c) . $w(t)$ has independent increments,

(d) . $\mathbb{E}(w(t)) = 0$ *and* $\operatorname{Cov}(w(t)-w(s)) = (t-s)Q$, *for all $t \geq s \geq 0$, where* $\operatorname{Cov}(x)$ *denotes the covariance operator of $x \in H$.*

If the covariance Q is the identity operator I, then the Wiener process $\{w(t)\}_t \geq 0$ is called a cylindrical Wiener process in H.

Proposition 1.2.1 *[53]*

Assume that $\{w(t)\}_{t\geq 0}$ is a Q-Wiener process with $\operatorname{Tr} Q < +\infty$. *Then the following statements hold:*

- *$\{w(t)\}_{t\geq 0}$ is a Gaussian process on H and*

$$\mathbb{E}(w(t)) = 0, \quad \operatorname{Cov}(w(t)) = tQ, \quad t \geq 0 \tag{1.1}$$

- *For arbitrary $t \geq 0, \{w(t)\}$ has the expansion*

$$w(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j \tag{1.2}$$

where

$$\beta_j(t) = \frac{1}{\sqrt{\lambda_j}}\left\langle w(t), e_j\right\rangle, \quad j = 1,2,\cdots \tag{1.3}$$

are real valued Brownian motions mutually independent on $(\Omega,\mathscr{F},\mathbb{P})$ *and the series in* (1.2) *is convergent in* $L^2(\Omega,\mathscr{F},\mathbb{P})$.

Theorem 1.2.1 *(**Stochastic Fubini Theorem**)[35]*

Let (H,Ω,μ) be a measurable space and $\left(\Phi(t,x)_{(t,x)\in[0,T]\times H}\right)$ be a $\mathscr{L}_2^0$-valued stochastic process. Assume that

$$\int_H \int_0^T \|\Phi(s,x)\|^2_{\mathscr{L}_2^0} ds\mu(dx) < +\infty, \tag{1.4}$$

then with probability one

$$\int_H \left(\int_0^T \Phi(s,x) dw(s) \right) \mu(dx) = \int_0^T \left(\int_H \Phi(s,x) \mu(dx) \right) dw(s). \tag{1.5}$$

Lemma 1.2.1 ***(Burkholder-Davis-Gundy)****[13]*

For arbitrary $p \geq 0$, then there exists a constant $C_p > 0$, dependent only on p such that for any $T \geq 0$,

$$\mathbb{E}\left\{ \sup_{0\leq t\leq T} \left\| \int_0^t \Phi(s,\omega) dw(s) \right\|_H^p \right\} \leq C_p \mathbb{E}\left\{ \int_0^T \|\Phi(s,\omega)\|^2_{\mathscr{L}_2^0} ds \right\}^{\frac{p}{2}}. \tag{1.6}$$

Theorem 1.2.2 ***(Itô Formula)****[19]*

Let Q be a symmetric nonnegative trace-class operator on a separable Hilbert space K, and let $\{w_t\}_{0\leq t\leq T}$ be a Q-Wiener process on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\leq t\leq T}, \mathbb{P})$. Assume that a stochastic process $X(t), 0 \leq t \leq T$, is given by

$$x(t) = x(0) + \int_0^t \Psi(s) ds + \int_0^t \Phi(s) dw_s \tag{1.7}$$

where $x(0)$ is an $\mathscr{F}_0$-measurable H-valued random variable, $\Psi(s)$ is an H-valued $\mathscr{F}_s$-measurable P-a.s. Bochner-integrable process on $[0,T]$,

$$\int_0^T \|\Psi(s)\|_H ds < \infty \quad P-a.s,$$

and $\Phi \in \mathscr{L}_2^0-$ valued process stochastically integrable in $[0,T]$.

Assume that a function $F : [0,T] \times H \to \mathbb{R}$ is such that F is continuous and its Fréchet partial derivatives F_t, F_x, F_{xx} are continuous and bounded on bounded subsets of $[0,T] \times H$. Then the following Itô's formula holds:

$$\begin{aligned} F(t,x(t)) =& F(0,x(0)) + \int_0^t \langle F_x(s,x(s)), \Phi(s) dw(s)\rangle_H \\ &+ \int_0^t \Big\{ F_t(s,x(s)) + \langle F_x(s,x(s)), \Psi(s)\rangle_H \\ &+ \frac{1}{2} \mathrm{Tr}\left[F_{xx}(s,x(s)) \left(\Phi(s)Q^{1/2}\right) \left(\Phi(s)Q^{1/2}\right)^* \right] \Big\} ds \end{aligned}$$

P-a.s. for all $t \in [0,T]$.

1.3 Semigroup approach of evolution equations

Let H be a real separable Hilbert space. We recall at first the definition of a semigroup.

Definition 1.3.1 *[33]*

A strongly continuous semigroup is an operator-valued function $S(t)$ from $\mathbb{R}^+$ to $\mathscr{L}(H)$ that satisfies the following properties:

- *$S(t+s) = S(t)S(s)$ for any s, $t \geq 0$,*
- *$S(0) = I_H$,*

- $\|S(t)z - z\| \to 0$ *as* $t \to 0^+$, *for any* $z \in H$.

We shall use the standard abbreviation C_0-semigroup for a strongly continuous semigroup.

Example 1.3.1 *[33]*
Let $A \in \mathscr{L}(H)$, *then*

$$S(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{fact(n)}$$

is a C_0*-semigroup.*

Theorem 1.3.1
Let $S(t)$ *be a* C_0*-semigroup. Then*

- *It exist constants* $\omega \geq 0$ *and* $M \geq 1$ *such that*

$$\|S(t)\| \leq Me^{\omega t}, \quad for\ 0 \leq t \leq \infty \tag{1.8}$$

- *For each* $x \in H, t \longrightarrow S(t)x$ *is a continuous function from* $[0,\infty)$ *into* H.
- *For* $x \in D(A), S(t)x \in D(A)$ *and*

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax \tag{1.9}$$

- *The domain of* A *is dense in* X *and* A *is a closed linear operator.*
- *If* B *is a bounded linear operator on* H, *then* $A + B$ *is the infinitesimal generator of a* C_0- *semigroup* $T(t)$ *on* H *satisfying*

$$\|T(t)\| \leq Me^{(\omega + M\|B\|)t}, t \geq 0.$$

.

Theorem 1.3.2
If A *is the infinitesimal generator of a* C_0*-semigroup* $S(t)$ *on* H, *then*

- *for every* $x \in D(A)$ *the abstract Cauchy problem:*

$$\begin{cases} u_t(t) = Au(t), 0 < t < T, \\ u(0) = x, \end{cases} \tag{1.10}$$

  has a unique strong solution given by $u(t) = S(t)x$.

- *for all* $x \in H$ *the abstract Cauchy problem* (1.10) *has a unique weak solution given by* $u(t) = S(t)x$.

Remark 1.3.1
If $\omega = 0$ *in* (1.8) *then the corresponding semigroup is uniformly bounded. If moreover* $M = 1$ *then* $(S(t))_{t\geq 0}$ *is called a* C_0*-semigroup of contractions.*

Definition 1.3.2 *[33]*

An analytic semigroup on a Hilbert space H is a family of continuous linear operators on $H, (S(t))_{t\geq 0}$, satisfying

- . $S(0) = I_H$ *and* $S(t+s) = S(t)S(s)$ *for any* $s,t > 0$.
- *The map* $t \to S(t)z$ *is real analytic on* $0 < t < \infty$ *for all* $z \in H$.
- $\lim_{t\to 0^+} S(t)z = z$, *for any* $z \in H$.

Asume that A generates an exponentinlly stable analytic semigroup and the spectrum of A lies entirely in the (open) left half-plane. For any $\beta \in (0,1)$, we define

$$(-A)^{-\beta} = -\frac{1}{2\pi i}\int_{\Gamma} \lambda^{-\beta}(\lambda + A)^{-1} d\lambda$$

where Γ is a curve from $\infty e^{-i\theta}$ to $\infty e^{i\theta}, \theta \in (\pi/2, \pi/2+b)$ for some $b > 0$, such that the spectrum of $-A$ lies to the right and the origin lies to the left of Γ. It can be shown that $(-A)^{-\beta}$ is bounded and one-to-ane. The arverse $(-A)^{\beta}$ of $(-A)^{-\beta}$ is called fractional power of $-A$ with domnin $D\left((-A)^{\beta}\right)$.

We conclude this section with some results relating $(-A)^{\alpha}$ and the analytic semigroup $S(t)$

Theorem 1.3.3 *[53]*

Let A be the inftenitesimal generator of an exponentially stable analytic semigromp $S(t)$. For any $0 < \beta < 1$, the following equality holds:

$S(t) : H \to D\left((-A)^{\beta}\right)$ *for every* $t > 0$ *and* $\alpha > 0$

- *For every* $x \in D\left((-A)^{\beta}\right)$ *we have*

$$S(t)(-A)^{\beta}x = (-A)^{\beta}S(t)x, \quad t > 0$$

- *For every* $t > 0$ *the operator* $(-A)^{\beta}S(t)$ *is bounded. There exist numbers* $M_{\beta} > 0, \gamma > 0$ *such that*

$$\|(-A)\beta S(t)\| \leq M_{\beta}t^{-\beta}e^{-\gamma t}$$

- Let $0 < \beta \leq 1$, *and* $x \in D\left((-A)^{\beta}\right)$ *then*

$$\|S(t)x - x\| \leq C_{\beta}t^{\mathrm{N}} \left\|(-A)^{\beta}\right\|_1 \quad t > 0$$

where $C_{\beta} > 0$ *is a constant dependent on* β.

1.4 Stability of infinite dimensional differential equations

Consider in the Hilbert H space, the differential equation

$$\frac{dz(t)}{dt} = Az(t), \tag{1.11}$$

where A is an unbounded operator with domain $D(A) \subset H$. Suppose that the above diferential equation subject to the condition $z(0) = z_0$ is uniquely solvable and that $z \equiv 0$ is an equilibrium point for (1.11).

Definition 1.4.1

Assume that A is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on H. We say that the C_0-semigroup $(S(t))_{t\geq 0}$ is exponentially stable if there exist two positive constants M and ω such that

$$\|S(t)z\|_H \leq Me^{-\omega t}\|z\|_H, \quad t>0,\ z\in H$$

Theorem 1.4.1

Suppose that A is the infinitesimal generator of the C_0- semigroup $S(t)$ on the Hilbert space H. The following statements are equivalent.

- *$S(t)$ is exponentially stable,*
- *There exists a self-adjoint nonnegative operator $P\in\mathscr{L}(H)$ which satisfies the Lyapunov equation*
$$\langle Az,Pz\rangle+\langle Pz,Az\rangle=-\langle z,z\rangle \text{ for all } z\in D(A)$$
- *For every $z\in H$ there exists a positive constant $\gamma_z>0$ such that*
$$\int_0^{+\infty}\|S(t)z\|^2dt\leq\gamma_z$$

Explicit formula for the solution of the Lyapunov equation is given in the following lemma in .

Lemma 1.4.1 *[33]*

Let $S(t)$ be an exponentially stable semigroup on H with infinitesimal generator A and let $Q\in L(H)$ be a nonnegative operator. Then the operator P defined by

$$Pz=\int_0^{+\infty}S^*(t)QS(t)zdt$$

is well-defined, nonnegative and satisfies the equation

$$\langle Az,Pz\rangle+\langle Pz,Az\rangle+\langle Qz,z\rangle=0 \text{ for all } z\in D(A) \tag{1.12}$$

Conversely, if P is self-adjoint and satisfies the equation (1.12), P is represented by the above integral.

Let U and V be Hilbert spaces and B,C,R linear bounded operators belonging respectively to the spaces $\mathscr{L}(U,H),\mathscr{L}(H,V)$ and $\mathscr{L}(U,U)$, where R is assumed to be an invertible positive operator. Consider the system

$$\frac{dz(t)}{dt}=Az(t)+Bu(t),z(0)=z_0\in H$$

where A is the infinitesimal generator of a C_0 - semigroup $S(t),t>0$, on the Hilbert space H and $u\in\mathscr{L}^2(0,\infty;U)$. We recall now the definitions of the stabilizability and detectability.

Definition 1.4.2

If there exists an $F\in\mathscr{L}(H,U)$ such that $A+BF$ generates an exponentially stable C_0-semigroup $S_{BF}(t)$, then we say that (A,B) is exponentially stabilizable.

Definition 1.4.3

If there exists $L \in \mathscr{L}(V,H)$ such that $A+LC$ generates an exponentially stable C_0-semigroup $S_{LC}(t)$, then we say that (A,C) is exponentially detectable.

Theorem 1.4.2

If the pair (A,C) is detectable then the Riccati equation

$$\langle Az,Pz\rangle + \langle Pz,Az\rangle - \left\langle PBR^{-1}B^{*}Pz,z\right\rangle + \langle Cz,Cz\rangle = 0, z \in D(A), \tag{1.13}$$

has at most one nonnegative solution and if P is the solution then the operator $A-BR^{-1}B^{*}P$ is stable. If, in addition, the pair $(A;B)$ is stabilizable then the equation (1.13) has exactly one solution.

1.5 Semigroup approach and mild solutions of stochastic differential equations

Let $\left(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P}\right)$ a complet probability space. In this section, we consider the following semilinear stochastic differential equation on $I=[0,T], T\geq 0$,

$$\begin{cases} dx(t) = (Ax(t)+F(t,x(t)))dt + G(t,x(t))dw(t), \\ x(0) = x_0 \in H \end{cases} \tag{1.14}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t), t\geq 0$, of bounded linear operators on the Hilbert space H. The coefficients F and G are two nonlinear measurable mappings from $[0,T]\times H \to H$ and $[0,T]\times H \to \mathscr{L}(K,H)$, respectively, satisfying the following Lipschitz continuity conditions:

$$\begin{array}{ll} \|F(t,y)-F(t,z)\|_H \leq \alpha(T)\|y-z\|_H, & \alpha(T)>0, y,z\in H, t\in[0,T], \\ \|G(t,y)-G(t,z)\|_{\mathscr{L}_2^0} \leq \beta(T)\|y-z\|_H, & \beta(T)>0, y,z\in H, t\in[0,T]. \end{array} \tag{1.15}$$

Definition 1.5.1

A stochastic process $\{x(t)\}_{t\in I}$, is called a strong solution of equation (1.14) if

- *$x(t)\in D(A), 0\leq t\leq T$, almost surely and is adapted to $\mathscr{F}_t, t\in I$;*
- *$x(t)$ is continuous in $t\in I$ almost surely. For arbitrary $0\leq t\leq T$,*

$$\mathbb{P}\left\{\omega : \int_0^t \|x(s,\omega)\|_H^2 ds < \infty\right\} = 1$$

  and

$$x(t) = x_0 + \int_0^t (Ax(s)+F(s,x(s)))\,ds + \int_0^t G(s,x(s))dw(s).$$

  for any $x_0\in D(A)$ almost surely.

In most situations, one finds that the concept of strong solution is too limited to include important examples. There is a weaker concept, mild solution, which is found to be more appropriate for practical purposes.

Definition 1.5.2

A stochastic process $\{x(t)\}_{t\in I}$, define on $\left(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P}\right)$ is called a mild solution of equation (1.14) if

- *$x(t)$ is adapted to $\mathscr{F}_t, t\geq 0$;*
- *For arbitrary $0\leq t\leq T$,*

$$\mathbb{P}\left\{\omega:\int_0^t \|x(s,\omega)\|_H^2 ds<\infty\right\}=1,$$

and

$$x(t)=S(t)x_0+\int_0^t S(t-s)F(s,x(s))ds+\int_0^t S(t-s)G(s,x(s))dw(s),$$

for any $x_0\in H$ almost surely.

1.6 Stability of stochastic equations

Assume a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$, equipped with a normal filtration $\{\mathscr{F}_t\}_{t\geq 0}$ with respect to which $\{w_t\}_{t\geq 0}$, is some given Q Wiener process with $\mathrm{Tr}\,Q<\infty$ in the Hilbert space K. Consider the following linear stochastic integral equation on the Hilbert space H

$$\begin{cases} x(t)=T(t)x_0+\int_0^t T(t-s)B\left(x(s)\right)dw(s), \\ x(0)=x_0\in H \end{cases} \tag{1.16}$$

where $T(t), t\geq 0$, is a strongly continuous semigroup with its infinitesimal generator A on the Hilbert space H and $B\in\mathscr{L}(K,H)$). From Theorem 1.3.4 [42] that the equation (1.16) has a unique (mild) solution $x_t\in C\left(0,\infty;L^2(\Omega;H)\right), t\geq 0$.

Theorem 1.6.1 *[42]*

Suppose $x(t), t\geq 0$, is the unique solution of (1.16) with initial datum $x_0\in H$. Then the following statements are equivalent:

- *The solution $x(t), t\geq 0$, satisfies*

$$\int_0^\infty \mathbb{E}\,\|x(t)\|_H^2\,dt<\infty \quad \textit{for} \quad x_0\in H$$

- *There exists a nonnegative, self-adjoint operator $P\in\mathscr{L}(H)$ such that*

$$2\langle Ax,Px\rangle_H+\langle\Delta(P)x,x\rangle_H=-\langle x,x\rangle_H \quad \textit{for any} \quad x\in D(A),$$

where $\langle\Delta(P)x,x\rangle_H=\mathrm{Tr}\,\{B^*(x)PB(x)Q\}$.

- *There exist positive numbers $M\geq 1,\mu>0$ such that for all $t\geq 0$,*

$$\mathbb{E}\,\|x(t)\|_H^2\leq M\cdot e^{-\mu t}\,\|x_0\|_H^2$$

Remark 1.6.1

If

$$\int_0^\infty \mathbb{E}\,\|x(t)\|_H^2\,dt<\infty \quad \textit{for} \quad x_0\in H$$

we said that the system (1.16) is L^2-stable.

1.7 Some useful Inequalities

Young's inequality

Let a,b and p be fixed positive constants and $m,n \geq 1$, $\frac{1}{m}+\frac{1}{n}=1$. Then we have the inequality

$$ab \leq \frac{p^m a^m}{m} + \frac{b^n}{np^n}.$$

Jensen's inequality

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, such that $\mu(\Omega)=1$. If g is a real-valued function that is μ-integrable, and if it is a convex function on the real line, then

$$\varphi\left(\int_\Omega g d\mu\right) \leq \int_\Omega \varphi \circ g d\mu.$$

In real analysis, we may require an estimate on $\varphi\left(\int_a^b g(x)dx\right)$ where a,b are real numbers, and g is a non-negative real-valued function that is Lebesgue-integrable. In this case, the Lebesgue measure of $[a,b]$ don't need to be unity. However, by integration by substitution, the interval can be rescaled so that it has measure unity. Then Jensen's inequality can be applied to get

$$\varphi\left(\int_a^b g(x)dx\right) \leq \frac{1}{b-a}\int_a^b \varphi((b-a)g(x))dx.$$

Hölder's inequality

Let $1<p<\infty$ and $1<q<\infty$ be real values, such that $\frac{1}{p}+\frac{1}{q}=1$. If $f(\cdot)\in L^p(X)$ and $g(\cdot)\in L^q(X)$ then $f(\cdot)g(\cdot)\in L^1(X)$ and

$$\begin{aligned}\int_X \|f(x)g(x)\| m(dx) &\leq \left(\int_X \|f(x)\|^p m(dx)\right)^{1/p} \cdot \left(\int_X \|g(x)\|^q m(dx)\right)^{1/q}\\ &= \|f(x)\|_p \|g(x)\|_q.\end{aligned}$$

In particular, if $p=q=2$, Hölder's inequality is the so-called Schwarz's inequality.

Chapter 2

Robust stability and robust stabilization of systems subjected to stochastic and deterministic perturbations

2.1 Introduction

Our objective in this chapter is to establish characterizations of the stability radius for an infinite dimensional system subjected to both deterministic and stochastic perturbations.
Firstly, we give the system description, then we define the stability radius. We establish some results which enables us to derive bounds for the stability radius. We end with an example to illustrate the theory.
Secondly, we investigate the robust stabilization problem. First, we give conditions providing the stability of the parameterized system. Then, we investigate the maximization of the stability radius by state feedback. We establish conditions for the existence of suboptimal controllers. Using these conditions we characterize the supreme achievable stability radius via an infinite dimensional Riccati equation.

2.2 Robust stability

2.2.1 System description

Let A be the infinitesimal generator of an exponentially stable semigroup $S(t)$ on a real separable Hilbert space H. Moreover, let $B \in \mathscr{L}(U_1, H)$, $D \in \mathscr{L}(U_2, H)$ and $E \in \mathscr{L}(H, Y)$. Consider the nominal system

$$\begin{cases} dx(t) = Ax(t)dt,\ t > 0, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

and assume that (2.1) is subjected to structured perturbations as follows

$$\begin{cases} dx(t) = Ax(t)dt + B\Delta_1(Ex(t))dt + D\Delta_2(Ex(t))dw(t),\ t > 0, \\ x(0) = x_0, \\ \|\Delta\| < \sigma, \end{cases} \tag{2.2}$$

where x_0 varies in H, Δ_1, Δ_2 are unknown Lipschitzian nonlinearities, $\{w(t)\}_{t\in\mathbb{R}_+}$ is a real Wiener process on a probability space (Ω , $\mathscr{F}$, P) equipped with a filtration $\{\mathscr{F}_t\}_{t\geq 0} \subset \mathscr{F}$, $\theta > 0$ denotes the variance of $\{w(t)\}_{t\in\mathbb{R}_+}$.
The disturbance Δ_1 varies in $Lip(Y,U_1)$, (B,E) determines the structure of the deterministic perturbation.
The disturbance Δ_2 varies in $Lip(Y,U_2)$, (D,E) determines the structure of the stochastic perturbation.
The size of each $\Delta_i \in Lip(Y,U_i), i = 1,2$, is measured by the Lipschitz norm

$$\|\Delta\|_{Lip} = inf\{\gamma > 0; \forall y,\ \ \widehat{y} \in Y : \|\Delta(y) - \Delta(\widehat{y})\|_U \leq \gamma\|y - \widehat{y}\|_Y\}.$$

Set

$$\Delta = (\Delta_1, \Delta_2), \|\Delta\| = max\{\|\Delta_1\|, \|\Delta_2\|\}.$$

Definition 2.2.1

The stability radius of A with respect to the perturbations structures $(D,E),(B,E)$ and the Wiener process $\{w(t)\}_{t\in\mathbb{R}_+}$ is

$$r^w(A,D,B,E) = inf\Big\{\|\Delta\|; \Delta_i \in Lip(Y,U_i), i = 1,2 \text{ such that } (2.2) \text{ is not } L^2 - stable\Big\}.$$

2.2.2 Characterizations of the stability radius

The approach used in this work to characterize the stochastic and deterministic stability radius $r^w(A,D,B,E)$ is based on the following lemma.

Lemma 2.2.1

Suppose that $E \in \mathscr{L}(H,Y)$ and

$$y(t) = ES(t)x_0 + \int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau + \int_0^t ES(t-\tau)D\upsilon_2(\tau)dw(\tau)$$

where $\upsilon_i \in L^2_w(\mathbb{R}^+, L^2(\Omega, U_i)), i \in \{1,2\}$. Then $y(.) \in L^2_w(\mathbb{R}^+, L^2(\Omega, Y))$.

Proof.

We have

$$\begin{aligned}
\|y(t)\|^2 &= \langle y(t), y(t)\rangle \\
&= \Big\langle ES(t)x_0 + \int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau + \int_0^t ES(t-\tau)D\upsilon_2(\tau)]dw(\tau), ES(t)x_0 \\
&+ \int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau + \int_0^t ES(t-\tau)D\upsilon_2(\tau)dw(\tau)\Big\rangle.
\end{aligned}$$

Set

$$G_1 = ES(t-\tau)B\upsilon_1(\tau).$$
$$G_2 = ES(t-\tau)D\upsilon_2(\tau)).$$

Using Theorem 6.12 in [19], we get :

$$\begin{aligned}
\mathbb{E}\Big(\|y(t)\|^2\Big) &= \|ES(t)x_0\|^2 + \mathbb{E}\Big(\|\int_0^t G_1(\tau)d\tau\|^2\Big) + \theta\int_0^t \mathbb{E}\Big(\|G_2(\tau)\|^2\Big)d\tau \\
&+ 2\Big\langle ES(t)x_0, \mathbb{E}\Big(\int_0^t G_1(\tau)d\tau\Big)\Big\rangle,
\end{aligned}$$

then

$$\begin{aligned}\int_0^{+\infty}\mathbb{E}\Big(\|y(t)\|^2\Big)dt &= \int_0^{+\infty}\|ES(t)x_0\|^2dt+\int_0^{+\infty}\mathbb{E}\Big(\|\int_0^t G_1(\tau)d\tau\|^2\Big)dt\\ &+ \theta\int_0^{+\infty}\int_0^t\mathbb{E}\Big(\|G_2(\tau)\|^2\Big)d\tau dt+2\int_0^{+\infty}\Big\langle ES(t)x_0,\mathbb{E}\Big(\int_0^t G_1(\tau)d\tau\Big)\Big\rangle dt.\end{aligned}$$

Consider

$$\begin{aligned}T_1 &= \int_0^{+\infty}\|ES(t)x_0\|^2dt\\ T_2 &= \int_0^{+\infty}\mathbb{E}\Big(\|\int_0^t G_1(\tau)d\tau\|^2\Big)dt\\ T_3 &= \theta\int_0^{+\infty}\int_0^t\mathbb{E}\Big(\|G_2(\tau)\|^2\Big)d\tau dt\\ T_4 &= 2\int_0^{+\infty}\Big\langle ES(t)x_0,\mathbb{E}\Big(\int_0^t G_1(\tau)d\tau\Big)\Big\rangle dt.\end{aligned}$$

- Since $S(t)$ is an exponentially stable semigroup, there exist postive canstant ω and M such that :

  $$\|S(t)\| \leqslant Me^{(-2\omega t)},\ t\geq 0.$$

  Thus

  $$\begin{aligned}T_1 &\leq \int_0^{+\infty}Me^{(-2\omega t)}\|Ex_0\|^2dt\\ &\leq M\|Ex_0\|^2\int_0^{+\infty}e^{(-2\omega t)}dt\\ &\leq \frac{M\|Ex_0\|^2}{2\omega}.\end{aligned}$$

  Since E is a bounded operator, it follows that

  $$T_1\leq M_1\|x_0\|^2,\ M_1>0. \tag{2.3}$$

- For T_2 we have

  $$\begin{aligned}T_2 &= \int_0^{+\infty}\mathbb{E}\|\int_0^t ES(t-\tau)B\upsilon_1(\tau)\|^2d\tau dt\\ &\leq \int_0^{+\infty}\mathbb{E}\int_0^t\|E\|^2\|S(t-\tau)\|^2\|B\upsilon_1(\tau)\|^2d\tau dt\\ &\leq \int_0^{+\infty}\int_0^t\mathbb{E}\|E\|^2\|S(t-\tau)\|^2\|B\upsilon_1(\tau)\|^2d\tau dt\\ &\leq \int_0^{+\infty}\int_0^t\mathbb{E}[\|B\|^2\|E\|^2Me^{-2\omega(t-\tau)}\|\upsilon_1(\tau)\|^2]d\tau dt\\ &\leq \|B\|^2\|E\|^2M\int_0^{+\infty}\int_0^t\mathbb{E}[e^{-2\omega t}e^{2\omega\tau}\|\upsilon_1(\tau)\|^2]d\tau dt\\ &\leq \|B\|^2\|E\|^2M\int_0^{+\infty}\int_0^t e^{-2\omega t}e^{2\omega\tau}\mathbb{E}(\|\upsilon_1(\tau)\|^2)d\tau dt.\end{aligned}$$

  Since B and E are bounded operators, we get

  $$T_2\leq M_2\int_0^{+\infty}\int_0^t e^{-2\omega t}e^{2\omega\tau}\mathbb{E}(\|\upsilon_1(\tau)\|^2)d\tau dt.$$

Using Fubini Theorem, we obtain

$$\begin{aligned}
T_2 &\leq M_2\int_0^{+\infty}\int_\tau^{+\infty} e^{2\omega\tau}\mathbb{E}\Big(\|\upsilon_1(\tau)\|^2\Big)e^{-2\omega t}dtd\tau \\
&\leq M_2\int_0^{+\infty} e^{2\omega\tau}\mathbb{E}\Big(\|\upsilon_1(\tau)\|^2\Big)\Big(\int_\tau^{+\infty} e^{-2\omega t}dt\Big)d\tau \\
&\leq M_2\int_0^{+\infty} e^{2\omega\tau}\mathbb{E}(\|\upsilon_1(\tau)\|^2)[\frac{-1}{2\omega}e^{-2\omega t}]_\tau^{+\infty}d\tau \\
&\leq \frac{M_2}{2\omega}\int_0^{+\infty} e^{2\omega\tau}\mathbb{E}(\|\upsilon_1(\tau)\|^2)e^{-2\omega\tau}d\tau \\
&\leq M_3\|\upsilon_1(.)\|^2_{L^2_w},
\end{aligned}$$

hence

$$T_2 \leq M_3\|\upsilon_1(.)\|^2_{L^2_w},\ M_3 > 0. \tag{2.4}$$

- For T_3 we have

$$\begin{aligned}
T_3 &= \theta\int_0^{+\infty}\int_0^t \mathbb{E}\|G_2(\tau)d\tau\|^2dt \\
&\leq \theta\int_0^{+\infty}\int_0^t \mathbb{E}\|E\|^2\|S(t-\tau)\|^2\|D\upsilon_2(\tau)\|^2d\tau dt \\
&\leq \theta\int_0^{+\infty}\int_0^t \mathbb{E}\Big[\|D\|^2\|E\|^2Me^{-2\omega(t-\tau)}\|\upsilon_2(\tau)\|^2\Big]d\tau dt \\
&\leq \theta\|D\|^2\|E\|^2M\int_0^{+\infty}\int_0^t \mathbb{E}\Big[e^{-2\omega t}e^{2\omega\tau}\|\upsilon_2(\tau)\|^2\Big]d\tau dt \\
&\leq \theta\|D\|^2\|E\|^2M\int_0^{+\infty}\int_0^t e^{-2\omega t}e^{2\omega\tau}\mathbb{E}\Big(\|\upsilon_2(\tau)\|^2\Big)d\tau dt.
\end{aligned}$$

Using Fubini Theorem we get

$$\begin{aligned}
T_3 &\leq M_4\int_0^{+\infty}\int_\tau^{+\infty} e^{2\omega\tau}\mathbb{E}\Big(\|\upsilon_2(\tau)\|^2\Big)e^{-2\omega t}dtd\tau \\
&\leq M_4\int_0^{+\infty} e^{2\omega\tau}\mathbb{E}(\|\upsilon_1(\tau)\|^2)\Big(\int_\tau^{+\infty} e^{-2\omega t}dt\Big)d\tau \\
&\leq M_4\int_0^{+\infty} e^{2\omega\tau}\mathbb{E}(\|\upsilon_1(\tau)\|^2)[\frac{-1}{2\omega}e^{-2\omega t}]_\tau^{+\infty}d\tau \\
&\leq \frac{M_4}{2\omega}\int_0^{+\infty} e^{2\omega\tau}\mathbb{E}(\|\upsilon_1(\tau)\|^2)e^{-2\omega\tau}d\tau \\
&\leq \frac{M_4}{2\omega}\int_0^{+\infty} \mathbb{E}\Big(\|\upsilon_2(\tau)\|^2\Big)d\tau,
\end{aligned}$$

hence

$$T_3 \leq \frac{M_4}{2\omega}\|\upsilon_2(.)\|^2_{L^2_w},\ M_4 > 0. \tag{2.5}$$

- For T_4 we have

$$\begin{aligned} T_4 &= 2\int_0^{+\infty}\Big\langle ES(t)x_0, \mathbb{E}\Big(\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau\Big)\Big\rangle dt \\ &\leq \int_0^{+\infty}(\|ES(t)x_0\|^2 + \|\mathbb{E}\Big(\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau\Big)\|^2\Big)dt \\ &\leq \int_0^{+\infty}\|ES(t)x_0\|^2 dt + \int_0^{+\infty}\|\mathbb{E}\Big(\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau\Big)\|^2 dt \\ &\leq \int_0^{+\infty}\|ES(t)x_0\|^2 dt + \int_0^{+\infty}\mathbb{E}\|\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau\|^2 dt \\ &\leq M_1\|x_0\|^2 + M_3\|\upsilon_1(.)\|^2_{L^2_w}, \end{aligned}$$

hence

$$T_4 \leq M_1\|x_0\|^2 + M_3\|\upsilon_1(.)\|^2_{L^2_w}. \tag{2.6}$$

Using (2.3) , (2.4) , (2.5) and (2.6) we get

$$\|y(.)\|^2_{L^2_w} \leq 2M_1\|x_0\|^2 + 2M_3\|\upsilon_1(.)\|^2_{L^2_w} + \frac{M_4}{2\omega}\|\upsilon_2(.)\|^2_{L^2_w},$$

from which we deduce that

$$\int_0^{+\infty}\mathbb{E}\|y(t)\|^2 dt < +\infty.$$

■

The second lemma will be given in terms of the input-output operator

$$L : L^2_w(\mathbb{R}^+, L^2(\Omega, U)) \to L^2_w(\mathbb{R}^+, L^2(\Omega, Y))$$

defined by

$$L\upsilon(t) = \int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau + \int_0^t ES(t-\tau)D\upsilon_2(\tau)dw(\tau), \tag{2.7}$$

where $U = U_1 \times U_2$, $\upsilon = (\upsilon_1, \upsilon_2)$, $\upsilon_1 \in U_1$ and $\upsilon_2 \in U_2$.

Lemma 2.2.2

The input-output operator defined by (2.7) has the operator norm

$$\|L\| = [\theta\|D^*PD\| + \|B^*PB\|]^{\frac{1}{2}}, \tag{2.8}$$

where P satisfies the Lyapunov equation

$$2\langle Px, Ax\rangle + \langle Ex, Ex\rangle = 0, \quad x \in D(A). \tag{2.9}$$

Proof.

Let $\upsilon \in L^2_w(\mathbb{R}^+, L^2(\Omega, U))$. For $t > 0$, we have

$$\begin{aligned} \|L\upsilon(t)\|^2 &= \|\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau + \int_0^t ES(t-\tau)D\upsilon_2(\tau)dw(\tau)\|^2 \\ &= \|\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau\|^2 + \|\int_0^t ES(t-\tau)D\upsilon_2(\tau)dw(\tau)\|^2 \\ &+ 2\Big\langle\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau, \int_0^t ES(t-\tau)D\upsilon_2(\tau)dw(\tau)\Big\rangle, \end{aligned}$$

hence

$$\|L\upsilon(.)\|^2_{L^2_w} = \int_0^{+\infty}\mathbb{E}\Big(\|\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau\|^2+\|\int_0^t ES(t-\tau)D\upsilon_2(\tau)dw(\tau)\|^2\Big)dt.$$

Set

$$\begin{aligned} J_1 &= \int_0^{+\infty}\mathbb{E}\|\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau\|^2dt. \\ J_2 &= \int_0^{+\infty}\mathbb{E}\|\int_0^t ES(t-\tau)D\upsilon_2(\tau)dw(\tau)\|^2dt. \end{aligned}$$

We have

$$\begin{aligned} J_1 &= \int_0^{+\infty}\mathbb{E}\|\int_0^t ES(t-\tau)B\upsilon_1(\tau)d\tau\|^2dt \\ &\leq \int_0^{+\infty}\mathbb{E}\int_0^t \|ES(t-\tau)B\upsilon_1(\tau)\|^2d\tau dt \\ &\leq 2\int_0^{+\infty}\mathbb{E}\int_0^t \langle ES(t-\tau)B\upsilon_1(\tau),ES(t-\tau)B\upsilon_1(\tau)\rangle d\tau dt \\ &\leq \int_0^{+\infty}\mathbb{E}\int_0^t \Big\langle B\upsilon_1(\tau),S^*(t-\tau)E^*ES(t-\tau)B\upsilon_1(\tau)\Big\rangle d\tau dt. \end{aligned}$$

Using Fubini Theorem, we get

$$\begin{aligned} J_1 &\leq \int_0^{+\infty}\mathbb{E}\int_\tau^{+\infty}\Big\langle B\upsilon_1(\tau),S^*(t-\tau)E^*ES(t-\tau)B\upsilon_1(\tau)\Big\rangle dtd\tau \\ &\leq \int_0^{+\infty}\mathbb{E}\Big\langle B\upsilon_1(\tau),\int_\tau^{+\infty}S^*(t-\tau)E^*ES(t-\tau)\ dt\ \ B\upsilon_1(\tau)\Big\rangle d\tau \\ &\leq \int_0^{+\infty}\mathbb{E}\Big\langle B\upsilon_1(\tau),PB\upsilon_1(\tau)\Big\rangle d\tau. \end{aligned}$$

For J_2, we have

$$\begin{aligned} J_2 &= \int_0^{+\infty}\mathbb{E}\|\int_0^t ES(t-\tau)D\upsilon_2(\tau)dw(\tau)\|^2dt \\ &= \int_0^{+\infty}\theta\int_0^t \mathbb{E}\|ES(t-\tau)D\upsilon_2(\tau)\|^2d\tau dt \\ &= \theta\int_0^{+\infty}\int_0^t \mathbb{E}\langle ES(t-\tau)D\upsilon_2(\tau),ES(t-\tau)D\upsilon_2(\tau)\rangle d\tau dt \\ &= \theta\int_0^{+\infty}\int_0^t \mathbb{E}\Big\langle D\upsilon_2(\tau),S^*(t-\tau)E^*ES(t-\tau)D\upsilon_2(\tau)\Big\rangle d\tau dt. \end{aligned}$$

Using Fubini Theorem, we get

$$\begin{aligned} J_2 &= \theta\int_0^{+\infty}\mathbb{E}\Big\langle D\upsilon_2(\tau),\int_\tau^{+\infty}S^*(t-\tau)E^*ES(t-\tau)dtD\upsilon_2(\tau)\Big\rangle d\tau \\ &= \theta\int_0^{+\infty}\mathbb{E}\Big\langle D\upsilon_2(\tau),PD\upsilon_2(\tau)\Big\rangle d\tau. \end{aligned}$$

Therefore

$$\begin{aligned}
\|L\upsilon(.)\|^2_{L^2_w} &\leq \int_0^{+\infty}\mathbb{E}\langle B\upsilon_1(\tau), PB\upsilon_1(\tau)\rangle d\tau + \theta\int_0^{+\infty}\mathbb{E}\Big\langle D\upsilon_2(\tau), PD\upsilon_2(\tau)\Big\rangle d\tau\\
&\leq \int_0^{+\infty}\mathbb{E}\|B^*PB\|\ \|\upsilon_1(\tau)\|^2 d\tau + \theta\int_0^{+\infty}\mathbb{E}\|D^*PD\|\ \|\upsilon_2(\tau)\|^2 d\tau\\
&\leq \int_0^{+\infty}\mathbb{E}((\|B^*PB\| + \theta\|D^*PD\|)\|\upsilon(\tau)\|^2)d\tau\\
&\leq (\|B^*PB\| + \theta\|D^*PD\|)\int_0^{+\infty}\mathbb{E}(\|\upsilon(\tau)\|^2)d\tau\\
&\leq (\|B^*PB\| + \theta\|D^*PD\|)\|\upsilon(.)\|^2_{L^2_w}
\end{aligned}$$

where $\|\upsilon\| = max(\|\upsilon_1\|, \|\upsilon_2\|)$. We deduce that

$$\|L\|_{L^2_w} \leq (\|B^*PB\| + \theta\|D^*PD\|)^{\frac{1}{2}}.$$

Now we will show that there exists $\upsilon' \in L^2_w(R^+, L^2(\Omega, U))$ such that

$$\|L\upsilon'(.)\|_{L^2_w} = (\|B^*PB\| + \theta\|D^*PD\|)^{\frac{1}{2}}.$$

Assume that

$$\|D^*PD\|_U = \langle\upsilon_2^0, D^*PD\upsilon_2^0\rangle \quad /\|\upsilon_2^0\|_U = 1.$$

$$\|B^*PB\|_U = \langle\upsilon_1^0, B^*PB\upsilon_1^0\rangle \quad /\|\upsilon_1^0\|_U = 1.$$

Define ψ_1, ψ_2 as follows

$$\psi_1(.) = \beta(.)\upsilon_1^0,\ \psi_2(.) = \beta(.)\upsilon_2^0$$

where $\beta(.) \in L^2(\mathbb{R}^+, \mathbb{R})$, and $|\beta(.)|_{L^2(\mathbb{R}^+,\mathbb{R})} = 1$

Then

$$\begin{aligned}
\|\psi_1(.)\|^2_{L^2_w} &= \int_0^{+\infty}\mathbb{E}\|\psi_1(\tau)\|^2 d\tau\\
&= \|\upsilon_1^0\|^2\int_0^{+\infty}|\beta(\tau)|^2 d\tau\\
&= \|\upsilon_1^0\|^2 = 1,
\end{aligned}$$

and

$$\begin{aligned}
\|\psi_2(.)\|^2_{L^2_w} &= \int_0^{+\infty}\mathbb{E}\|\psi_2(.)\|^2 d\tau\\
&= \|\upsilon_2^0\|^2\int_0^{+\infty}|\beta(\tau)|^2 d\tau\\
&= \|\upsilon_2^0\|^2 = 1.
\end{aligned}$$

For $\psi = (\psi_1, \psi_2)$, we have

$$\begin{aligned}
\|L\psi(.)\|^2_{L^2_w} &= \int_0^{+\infty} \mathbb{E}\langle B\psi_1(\tau), PB\psi_1(\tau)\rangle d\tau + \theta \int_0^{+\infty} \mathbb{E}\Big\langle D\psi_2(\tau), PD\psi_2(\tau)\Big\rangle d\tau \\
&= \int_0^{+\infty} \mathbb{E}\Big\langle \beta(\tau)\upsilon_1^0, B^*PB\beta(\tau)\upsilon_1^0\Big\rangle d\tau + \theta \int_0^{+\infty} \mathbb{E}\Big\langle \beta(\tau)\upsilon_2^0, D^*PD\beta(\tau)\upsilon_2^0\Big\rangle d\tau \\
&= \|B^*PB\|^2 \int_0^{+\infty} \|\beta(\tau)\|^2 d\tau + \|D^*PD\|^2\theta \int_0^{+\infty} |\beta(\tau)|^2 d\tau \\
&= \|B^*PB\|^2 + \theta\|D^*PD\|^2.
\end{aligned}$$

Therefore

$$\|L\psi(.)\|_{L^2_w} = [\|B^*PB\|^2 + \theta\|D^*PD\|^2]^{\frac{1}{2}},$$

which concludes the proof.

■

The main result of this section is giving in the following theorem.

Theorem 2.2.1

Let $\sigma > 0$. Suppose that there exists $P \in L^+(H)$ satisfying

$$2\langle Px, Ax\rangle + \langle Ex, Ex\rangle = 0, \quad x \in D(A), \tag{2.10}$$

$$1 - \sigma^2[\theta\|D^*PD\| + \|B^*PB\|] \geq 0, \tag{2.11}$$

then $r^w(A,B,D,E) \geq \sigma$.

Proof. Let $\Delta_1 \in Lip(Y,U_1)$ and $\Delta_2 \in Lip(Y,U_2)$ such that $\|\Delta\|_{Lip} < \sigma$ where $\Delta = (\Delta_1, \Delta_2)$. Let $x(t)$ the solution of the system (2.2).

Set $y(t) = Ex(t)$. We have

$$y(t) = ES(t)x_0 + \int_0^t ES(t-\tau)B\Delta_1(Ex(\tau))d\tau + \int_0^t ES(t-\tau)D\Delta_2(Ex(\tau))dw(\tau). \tag{2.12}$$

For $T > 0$, define the truncation

$$u_T^1 \in L^2_w(\mathbb{R}^+, L^2(\Omega, U_1)),$$

and

$$u_T^2 \in L^2_w(\mathbb{R}^+, L^2(\Omega, U_2)),$$

by

$$u_T^1(t) = \begin{cases} u_1(t) = \Delta_1(y(t)) & if\ t \in [0,T], \\ 0 & if\ \ t > T, \end{cases} \tag{2.13}$$

and

$$u_T^2(t) = \begin{cases} u_1(t) = \Delta_2(y(t)) & if\ t \in [0,T], \\ 0 & if\ \ t > T. \end{cases} \tag{2.14}$$

Then

$$\begin{aligned}
\|u_T^1(.)\| &= \int_0^{+\infty} \mathbb{E}(\|u_T^1(t)\|)^2 dt \\
&= \int_0^T \mathbb{E}(\|\Delta_1 y(t)\|)^2 dt \\
&\leq \|\Delta_1\|^2 \int_0^T (\mathbb{E}(\|y(t)\|)^2) dt,
\end{aligned}$$

and

$$\begin{aligned}\|u_T^2(.)\|_{L_w^2}^2 &= \int_0^{+\infty} \mathbb{E}(\|u_T^2(t)\|)^2 dt \\ &= \int_0^T \mathbb{E}(\|\Delta_2 y(t)\|)^2 dt \\ &\leq \|\Delta_2\|_{Lip}^2 \int_0^T (\mathbb{E}(\|y(t)\|)^2) dt.\end{aligned}$$

For $u_T = (u_T^1, u_T^2)$, we get

$$\|u_T(.)\|_{L_w^2}^2 \leq \|\Delta\|_{Lip}^2 \int_0^T \mathbb{E}(\|y(t)\|)^2 dt. \tag{2.15}$$

Set

$$y_T(t) = ES(t)x_0 + Lu_T(t), \quad t > 0. \tag{2.16}$$

From (2.12) - (2.16), we get

$$\begin{aligned}\left(\int_0^T \mathbb{E}\|y(t)\|^2 dt\right)^{\frac{1}{2}} &\leq \|y_T(.)\|_{L_w^2} \\ &\leq M\int_0^{+\infty} e^{(-2\omega t)} dt \|Ex_0\| + \|L\|\|u_T(.)\|_{L_w^2}.\end{aligned}$$

Thus

$$\left(\int_0^T \mathbb{E}\|y(t)\|^2 dt\right)^{\frac{1}{2}} \leq M\int_0^{+\infty} e^{(-2\omega t)} dt \|Ex_0\| + \|L\|\|\Delta\|_{Lip}\left(\int_0^T \mathbb{E}\|y(t)\|^2 dt\right)^{\frac{1}{2}}. \tag{2.17}$$

Condition (2.11) implies that

$$1 - \sigma^2[\theta\|D^*PD\| + \|B^*PB\|] \geq 0.$$

Thus

$$[\theta\|D^*PD\| + \|B^*PB\|] \leq \sigma^{-2}.$$

By the previous lemma, it follows that

$$\|L\|^2 \leq \sigma^{-2}.$$

Now since $\|\Delta\|_{Lip} < \sigma$, the operator $L\Delta$ is a contraction on $L_w^2(\mathbb{R}^+, L^2(\Omega, Y))$ with $\beta = \|L\|\|\Delta\| < 1$. From (2.17) we get

$$\left(\int_0^T \mathbb{E}\|y(t)\|^2 dt\right)^{\frac{1}{2}} \leq (1-\beta)^{-1} M e^{(-2\omega t)} \|Ex_0\|,$$

for all $T > 0$. Therefore $y \in L_w^2(\mathbb{R}^+, L^2(\Omega, Y))$ and $u_1 = \Delta_1(y) \in L_w^2(\mathbb{R}^+, L^2(\Omega, U_1)), u_2 = \Delta_2(y) \in L_w^2(\mathbb{R}^+, L^2(\Omega, U_2))$.

By Lemma 2.2.1, the solution $x(.)$ belongs to $L_w^2(\mathbb{R}^+, L^2(\Omega, H))$. We conclude then that $r^w(A, B, D, E) \geq \sigma$.

■

As a consequence of this Theorem we have the following corollary which enables us to obtain a lower bound for the stability radius.

Corollary 2.2.1
Suppose that there exists $P \in \mathscr{L}^+(H)$ a solution of the Lyapunov equation (2.10). Then

$$r^w(A,B,D,E)) \geq \Big[(\theta\|D^*PD\| + \|B^*PB\|)\Big]^{-\frac{1}{2}}. \tag{2.18}$$

Proof.

1. if $\|D^*PD\| = 0$ and $\|B^*PB\| = 0$, then

$$1-\sigma^2[\theta\|D^*PD\| + \|B^*PB\|] > 0,\ for\,all\,\sigma > 0.$$

From the above Theorem, it follows that

$$r^w(A,B,D,E) \geq \sigma,\ for\,all\,\sigma > 0.$$

From which we deduce that $r^w(A,B,D,E) = +\infty$.

2. Assume that $\|D^*PD\| = 0$ and $\|B^*PB\| \neq 0$. We have

$$\|u\|^2 - ((\|B^*PB\|)^{\frac{-1}{2}})^2\langle B^*PBu,u\rangle \geq 0, \text{ for all } u \in U_1.$$

By the previous Theorem we deduce that

$$r^w(A,B,D,E) \geq (\|B^*PB\|)^{\frac{-1}{2}}.$$

Therefore

$$r^w(A,B,D,E) \geq \Big(\|B^*PB\| + \theta\|D^*PD\|\Big)^{\frac{-1}{2}}.$$

3. Assume that $\|B^*PB\| = 0$ and $\|D^*PD\| \neq 0$. We have

$$\|u\|^2 - \|D^*PD\|^{-1}\langle D^*PDu,u\rangle \geq 0$$

for all $u \in U_2$. Hence

$$\|u\|^2 - \left(\Big(\theta\|D^*PD\|\Big)^{\frac{-1}{2}}\right)^2\theta\langle D^*PDu,u\rangle \geq 0\,for\,all\,u \in U_2.$$

By the previous Theorem we deduce that

$$r^w(A,B,D,E) \geq \Big(\theta\|D^*PD\|\Big)^{\frac{-1}{2}}.$$

Therefore

$$r^w(A,B,D,E) \geq (\|B^*PB\| + \theta\|D^*PD\|)^{\frac{-1}{2}}.$$

4. Assume that $\|B^*PB\| \neq 0$ and $\|D^*PD\| \neq 0$.

By the previous Theorem we deduce that

$$r^w(A,B,D,E) \geq (\|B^*PB\| + \theta\|D^*PD\|)^{\frac{-1}{2}}.$$

■

In the following result we give a characterization of the stability radius in terms of the Lyapunov inequality.

Corollary 2.2.2

Suppose that there exists $P \in L(H^+)$ satisfying

$$2\langle Px, Ax\rangle + \langle Ex, Ex\rangle \leq 0, \quad x \in D(A). \tag{2.19}$$

$$\begin{aligned} &1 - \sigma^2[\theta\|D^*PD\| + \|B^*PB\|] \geq 0, \\ &\big(resp. 1 - \sigma^2[\theta\|D^*PD\| + \|B^*PB\|] > 0\big). \end{aligned} \tag{2.20}$$

Then $r^w(A,B,D,E) \geq \sigma$, $\big(resp. r^w(A,B,D,E) > \sigma\big)$.. In this case the Lyapunov equation (2.10) has a solution $P_0 \in L(H^+)$ such that $P \succeq P_0$.

Proof. Because $S(t)$ is exponentially stable there exists a solution P_0 of the Lyapunov equation (2.10). Set $X = P - P_0$, then

$$2\langle Xx, Ax\rangle \leq 0, \quad x \in D(A).$$

Applying Lemma 2.1 in [10] we obtain that $X \succeq 0$, thus $P \succeq P_0$.

Using condition (2.20), it follows that

$$0 < 1 - \sigma^2[\theta\|D^*PD\| + \|B^*PB\|] \leq 1 - \sigma^2[\theta\|D^*P_0D\| + \|B^*P_0B\|].$$

Hence conditions (2.10) and (2.11) are satisfied. By applying Theorem 2.2.1 we deduce that $r^w(A,B,D,E) \geq \sigma$.

■

Remark 2.2.1

In the particular case where $B = 0$, we obtain the same results established in [35],

$$r^w(A,(D,E)) \geq \Big(\theta\|D^*PD\|\Big)^{-\frac{1}{2}}.$$

2.2.3 Example

Consider the system

$$\begin{cases} dy(t) = \dfrac{\partial^2 y(x,t)}{\partial x^2}dt + c_1 y(x,t)dt + c_2 y(x,t)dw(t)\ , 0 < x < 1, t > 0, \\ y(x,0) = y_0(x), \quad 0 < x < 1, \\ y(0,t) = y(1,t), \quad t > 0. \end{cases} \tag{2.21}$$

To put the problem (2.21) into the abstract setting we introduce the self-adjoint operator $Ah = \frac{d^2h}{dx^2}$ in the real Hilbert space $H = L^2(0,1)$ with $D(A) = H_0^1(0,1) \cap H^2(0,1)$ The operator A generates an exponentially stable semigroup $S(t)$. The eigenvalues and the eigenvectors of A are given by [9]

$$\lambda_n = -n^2\pi^2, \ \psi_n(x) = \sqrt{2}sin(n\pi x), n \geq 1.$$

Setting $B = I_{L^2(0,1)}$, $D = I_{L^2(0,1)}$, $\Delta_1 = c_1 \in \mathbb{R}$, $\Delta_2 = c_2 \in \mathbb{R}$, $E = I_{L^2(0,1)}$ such that $Ez = z$. In the abstract from system (2.21) can be presented as follows

$$\begin{cases} dz(t) = Az(t)dt + B\Delta_1(Ez(t))dt + D\Delta_2(Ez(t))dw(t), \\ z(0) = z_0. \end{cases} \tag{2.22}$$

The Lyapunov equation corresponding to this system is

$$2\langle Pz, Az\rangle + \langle Ez, Ez\rangle = 0, z \in D(A). \tag{2.23}$$

Suppose we can express the solution P of (2.23) by

$$Pz = \sum_{n,i=1}^{+\infty} P_{in}\langle z, \psi_n\rangle\psi_i, \quad z \in H. \tag{2.24}$$

Then since

$$Az = \sum_{n=1}^{+\infty} \lambda_n\langle z, \psi_n\rangle\psi_n, \quad z \in D(A). \tag{2.25}$$

It follows that

$$\langle Pz, Az\rangle \quad = \quad \textstyle\sum_{i,n=1}^{+\infty} P_{in}\lambda_n\langle z, \psi_n\rangle^2. \tag{2.26}$$

For the second term of the Lyapunov equation (2.23) we have

$$\langle Ez, Ez\rangle = \|z\|^2.$$

Equation (2.23) is then equivalent to

$$2\textstyle\sum_{i,,n=1}^{+\infty} P_{in}\lambda_n\langle z, \psi_n\rangle^2 + \|z\|^2 = 0$$

Assume that $P_{in} = 0$ for $i \neq n$. For $z = \psi_k, k \geq 1$, we get

$$2P_{kk}\lambda_k + 1 = 0.$$

From which we obtain

$$P_{kk} = -\tfrac{1}{2\lambda_k}.$$

We deduce that the solution of (2.23) is given by

$$Pz = \sum_{k=1}^{+\infty} P_k\langle z, \psi_k\rangle\psi_k, z \in H$$

where

$$P_k = \frac{1}{2k^2\pi^2}, \quad k \geq 1.$$

We have

$$\|B^*PB\|_{U_1} = \|P\| = \sum_{k=1}^{+\infty} P_k = \sum_{k=1}^{+\infty} \frac{1}{2k^2\pi^2},$$

$$\|D^*PD\|_{U_2} = \|P\| = \sum_{k=1}^{+\infty} P_k = \sum_{k=1}^{+\infty} \frac{1}{2k^2\pi^2}.$$

But

$$\sum_{k=1}^{+\infty} \frac{1}{k^2\pi^2} = \frac{1}{\pi^2}\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{1}{6}.$$

Therefore

$$\|B^*PB\| = \frac{1}{12} \quad and \quad \|D^*PD\| = \frac{1}{12},$$

thus

$$[\|B^*PB\| + \|D^*PD\|]^{-1} = 6.$$

Set $c = \max\{c_1, c_2\}$. We deduce that the system (2.21) is stable for all $c < \sqrt{6}$.

2.3 Robust stabilization

2.3.1 Robust stabilization with bounded input operator

In this section we consider the controlled system

$$\begin{cases} dx(t) = Ax(t)dt + B\Delta_1(Ex(t))dt + D\Delta_2(Ex(t))dw(t) + B_0u(t)dt, \; t > 0, \\ x(0) = x_0, \end{cases} \tag{2.27}$$

where u takes its values in the real separable Hilbert space Z, $B_0 \in \mathcal{L}(Z,H)$. In addition we assume that (A,B_0) is stabilizable.
Our aim is to characterize the supremum of the stability radii which can be achieved by linear state feedback $u = Fx$, where $F \in \mathcal{L}(H,Z)$.
Let

$$\overline{\mathscr{Z}} = \left\{ \begin{array}{c} F \in \mathcal{L}(H,Z); A + B_0F \; is \; the \; infinitesimal \\ generator \; of \; an \; exponentially \; stable \; semigroup \; S_F(t) \end{array} \right\}.$$

and define

$$\overline{r^w}(A,D,B,E) = sup\Big\{ r^w(A+B_0F,D,B,E); \; F \in \overline{\mathscr{F}} \Big\}.$$

For $F \in \overline{\mathscr{Z}}$, $\varepsilon > 0$, consider the Lyapunov inequality

$$2\langle P(A+B_0F)x,x\rangle + \langle Ex,Ex\rangle + \varepsilon^2\langle Fx,Fx\rangle \leq 0, \qquad x \in D(A). \tag{2.28}$$

In order to establish conditions for the existence of suboptimal controllers $u(t) = Fx(t)$ such that $F \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A+B_0F,D,B,E)$, for $\sigma > 0$, we need the following Lemmas.

Lemma 2.3.1

Let $\varepsilon > 0$. If there exists $P \in L^+(H)$ such that

$$2\langle Px,(A-\varepsilon^{-2}B_0B_0^*P)x\rangle + \varepsilon^{-2}\langle PB_0B_0^*Px,x\rangle + \langle Ex,Ex\rangle \leq 0, \qquad x \in D(A), \tag{2.29}$$

$$1 - \sigma^2[\theta\|D^*PD\| + \|B^*PB\|] \geq 0, \tag{2.30}$$

then $A_\varepsilon = A - \varepsilon^{-2}B_0B_0^*P$ generates an exponentially stable semigroup and $\sigma \leq r^w(A_\varepsilon,D,B,E)$.

Proof.

Consider the initial value problem

$$\begin{cases} \frac{d}{dt}x(t) = A_\varepsilon x(t), \qquad t \in \mathbb{R}^+ \\ x(0) = x_0, \; x_0 \in H. \end{cases} \tag{2.31}$$

For $x_0 \in D(A_\varepsilon), V(x) = \langle x,Px\rangle$ is differentiable and

$$\frac{d}{dt}V(x(t)) = 2\langle PA_\varepsilon x,x\rangle$$

From the inequality (2.29) we obtain

$$\tfrac{d}{dt}V(x(t)) \;\; \leq \;\; -\varepsilon^{-2}\langle PB_0B_0^*Px,x\rangle - \langle Ex,Ex\rangle \leq -\varepsilon^{-2}\langle PB_0B_0^*Px,x\rangle.$$

Thus

$$\int_0^T \frac{d}{dt}V(x(t))dt \;\; \leq \;\; -\varepsilon^{-2}\int_0^T \langle PB_0B_0^*Px,x\rangle dt,$$

Hence

$$V(x(T))-V(x(0)) \quad \leq \quad -\varepsilon^{-2}\int_0^T \|B_0^*Px(t)\|^2 dt.$$

Using the fact that $P \succ 0$ we get

$$\varepsilon^{-2}\int_0^T \|B_0^*Px(t)\|^2 dt \leq V(x_0), \quad for \quad all \quad T>0.$$

Therefore

$$\varepsilon^{-2}\int_0^T \|B_0^*Px(t)\|^2 dt \leq k\|x_0\|^2.$$

which implies that $B_0^*Px(t) \in L^2(\mathbb{R}^+,Z)$. The solution $x(t)$ of the system (2.31) is given by

$$x(t) = S(t)x_0 - \varepsilon^{-2}\int_0^t S(t-s)B_0B_0^*Px(s)ds$$

We have

$$\begin{aligned}\|x(t)\| \quad &\leq \quad \|S(t)x_0\| + \varepsilon^{-2}\left\|\textstyle\int_0^t S(t-s)B_0B_0^*Px(s)ds\right\| \\ &\leq \quad Me^{-\omega t}\|x_0\| + \varepsilon^{-2}M\|B_0\|\int_0^t e^{-\omega(t-s)}\|B_0^*Px(s)\|ds.\end{aligned}$$

from which we get

$$\begin{aligned}\|x(t)\|^2 \quad &\leq \quad 2M^2e^{-2\omega t}\|x_0\|^2 + 2\varepsilon^{-4}M^2\|B_0\|^2\left[\int_0^t e^{-\omega(t-s)}\|B_0^*Px(s)\|ds\right]^2 \\ &\leq \quad K_1e^{-2\omega t} + K_2\int_0^t e^{-2\omega(t-s)}\|B_0^*Px(s)\|^2 ds,\end{aligned}$$

where $K_1 = 2M^2\|x_0\|^2$, $K_2 = 2\varepsilon^{-4}M^2\|B_0\|^2$. It follows then that

$$\int_0^{+\infty}\|x(t)\|^2 dt \quad \leq \quad \int_0^{+\infty} K_1e^{-2\omega t}dt + \int_0^{+\infty} K_2\int_0^t e^{-2\omega(t-s)}\|B_0^*Px(s)\|^2 dsdt.$$

Thus

$$\int_0^{+\infty}\|x(t)\|^2 dt \quad \leq \quad \tfrac{K_1}{2\omega} + \int_0^{+\infty} K_2e^{2\omega s}\|B_0^*Px(s)\|^2\left(\int_s^{+\infty} e^{-2\omega t}dt\right)ds.$$

which implies that

$$\int_0^{+\infty}\|x(t)\|^2 dt \quad \leq \quad \tfrac{K_1}{2\omega} + \tfrac{K_2}{2\omega}\int_0^{+\infty}\|B_0^*Px(s)\|^2 ds.$$

Since $B_0^*Px(t) \in L^2(\mathbb{R}^+,Z)$, we deduce that $x(t)$ belongs to the space $L^2(\mathbb{R}^+,H)$. Applying Corollary 2.2.2 with

$$F_\varepsilon = -\varepsilon^{-2}B_0^*P \in \overline{\mathscr{Z}}$$

we get that $\sigma \leq r^w(A_\varepsilon,D,B,E)$.

■

Lemma 2.3.2

Let $\varepsilon>0$ and $F \in \overline{\mathscr{Z}}$. If the inequality (2.28) has a solution $P_1 \in L^+(H)$ satisfying condition (2.30) then $F_1 = -\varepsilon^{-2}B_0^*P_1 \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A+B_0F_1,D,B,E)$. Moreover, there exists $P_2 \in L^+(H)$ such that

$$2\langle P_2(A+B_0F_1)x,x\rangle + \langle Ex,Ex\rangle + \varepsilon^{-2}\langle P_1B_0B_0^*P_1x,x\rangle = 0,$$

$$x \in D(A),$$

$$1-\sigma^2[\theta\|D^*P_2D\| + \|B^*P_2B\|] \geq 0.$$

$$P_2 \preceq P_1$$

Proof.

Set $F^{'} = \varepsilon F + \varepsilon^{-1} B_0^* P_1$, then

$$\langle F^{'}x, F^{'}x\rangle - \varepsilon^{-2}\langle B_0^* P_1 x, B_0^* P_1 x\rangle = \varepsilon^2 \langle Fx, Fx\rangle + 2\langle B_0^* P_1 x, Fx\rangle.$$

Since P_1 is a solution of the inequality (2.28) it follows that

$$2\langle P_1 Ax, x\rangle + \langle Ex, Ex\rangle - \varepsilon^{-2}\langle B_0^* P_1 x, B_0^* P_1 x\rangle + \langle F^{'}x, F^{'}x\rangle \leq 0. \tag{2.32}$$

Set $A_0 = A + B_0 F_1$ where $F_1 = -\varepsilon^{-2} B_0^* P_1$, then

$$2\left\langle P_1 A_0 x, x\right\rangle + \langle Ex, Ex\rangle + \varepsilon^{-2}\left\langle P_1 B_0 B_0^* P_1 x, x\right\rangle \leq 0. \tag{2.33}$$

Applying Lemma 2.3.1 we conclude that $F_1 \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A_0, D, B, E)$.

Now since P_1 is a solution of the inequality (2.33), then it satisfies the following inequality

$$2\left\langle P_1 A_0 x, x\right\rangle + \left\langle \widehat{E}x, \widehat{E}x\right\rangle \leq 0,$$

where

$$\widehat{E} = \begin{pmatrix} E \\ \varepsilon^{-1} B_0^* P_1. \end{pmatrix}$$

By Corollary 2.2.2 there exists $P_2 \in L^+(H)$ such that

$$2\left\langle P_2 A_0 x, x\right\rangle + \left\langle \widehat{E}x, \widehat{E}x\right\rangle = 0,$$

with $P_2 \preceq P_1$. Therefore

$$2\left\langle P_2 A_0 x, x\right\rangle + \varepsilon^{-2}\left\langle x, P_1 B_0 B_0^* P_1 x\right\rangle + \langle Ex, Ex\rangle \leq 0$$

and

$$1 - \sigma^2[\theta\|D^* P_2 D\| + \|B^* P_2 B\|] \geq 0.$$

■

Applying this Lemma iteratively we show in the following Theorem that there exists $P \in L^+(H)$ such that

$$2\left\langle Ax, Px\right\rangle + \langle Ex, Ex\rangle - \varepsilon^{-2}\left\langle x, PB_0 B_0^* Px\right\rangle = 0, \quad x \in D(A) \tag{2.34}$$

Theorem 2.3.1

Let $F \in \overline{\mathscr{Z}}$. Suppose that there exist $\varepsilon > 0$ such that the Lyapunov inequality (2.28) has a solution $P_0 \in L^+(H)$ which satisfies condition (2.30) then the Riccati equation (2.34) has a solution $P \in L^+(H)$ satisfying

$$1 - \sigma^2[\theta\|D^* PD\| + \|B^* PB\|] \geq 0,$$
$$F_\varepsilon = -\varepsilon^{-2} B_0^* P \in \overline{\mathscr{Z}},$$
$$\sigma \leq r^w(A - \varepsilon^{-2} B_0 B_0^* P, D, B, E).$$

Proof.

Applying the above Lemma iteratively we construct a sequence of linear operators $(P_k)_{k\in\mathbb{N}} \in L^+(H)$ which satisfies

$$2\langle P_{k+1}A_kx,x\rangle + \langle Ex,Ex\rangle + \varepsilon^{-2}\langle x,P_kB_0B_0^*P_kx\rangle = 0,$$

$$x \in D(A),$$
$$1-\sigma^2[\theta\|D^*P_{k+1}D\| + \|B^*P_{k+1}B\|] \geq 0,$$
$$P_{k+1} \preceq P_k,$$

where P_1 is a solution of the inequality (2.28) and $A_k = A-\varepsilon^{-2}B_0B_0^*P_k$.

Let $P = \lim_{k\to+\infty} P_k$ then

$$2\langle PA_\varepsilon x,x\rangle + \langle Ex,Ex\rangle + \varepsilon^{-2}\langle x,PB_0B_0^*Px\rangle = 0, \quad x\in D(A),$$

$$1-\sigma^2[\theta\|D^*PD\| + \|B^*PB\|] \geq 0,$$

where $A_\varepsilon = A-\varepsilon^{-2}B_0B_0^*P$.

Using Lemma 2.3.2 we deduce that $F_\varepsilon = -\varepsilon^{-2}B_0^*P \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A-\varepsilon^{-2}B_0B_0^*P,D,B,E)$.

Finally since

$$2\langle PA_\varepsilon x,x\rangle + \langle Ex,Ex\rangle + \varepsilon^{-2}\langle x,PB_0B_0^*Px\rangle = 2\langle PAx,x\rangle + \langle Ex,Ex\rangle - \varepsilon^{-2}\langle x,PB_0B_0^*Px\rangle,$$

then P satisfies the Riccati equation (2.34).

■

Proposition 2.3.1

Let $\sigma,\varepsilon > 0$. Suppose that the Riccati equation (2.34) has a solution $P \in L^+(H)$ such that

$$1-\sigma^2[\theta\|D^*PD\| + \|B_0^*PB\|] \geq 0,$$

then $F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A+B_0F_\varepsilon,D,B,E)$.

Proof. Since P is a solution of the Riccati equation (2.34) then

$$2\langle P(A-\varepsilon^{-2}B_0B_0^*P)x,x\rangle) + \langle Ex,Ex\rangle + \varepsilon^{-2}\langle x,PB_0B_0^*Px\rangle = 0, \quad x\in D(A),$$

$$1-\sigma^2[\theta\|D^*PD\| + \|B^*PB\|] \geq 0.$$

From Lemma 2.3.2 we obtain $F_\varepsilon = -\varepsilon^{-2}B_0^*P \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A+B_0F_\varepsilon,D,B,E)$.

■

As a consequence of the above proposition we characterize the supremal achievable stability radius via the Riccati equation (2.34) as follows.

Corollary 2.3.1 *We have*

$$\overline{r^w}(A,D,B,E) \geq \sup\left\{ \begin{array}{l} \sigma>0;\ \textit{there exist } \varepsilon>0 \textit{ such that } (2.34))\ \textit{has a solution} \\ P\in L^+(H) \textit{with } 1-\sigma^2(\theta\|D^*PD\| + \|B^*PB\|) \geq 0. \end{array} \right\}.$$

2.3.2 Robust stabilization with unbounded input operator

In this section we consider the controlled system

$$\begin{cases} dx(t) = Ax(t)dt + B\Delta_1(Ex(t))dt + D\Delta_2(Ex(t))dw(t) + B_0u(t)dt,\ t > 0, \\ x(0) = x_0 \in H, \\ \|\Delta\| < \sigma, \end{cases} \tag{2.35}$$

under the asumption that A generates an exponentially stable **analytic** semigroup $S(t)$ and B_0 is a linear operator from Z to H, (B_0 is generally unbounded as an operator from Z to H), such that $(-A)^{-\eta}B_0 \in \mathscr{L}(Z,H)$ for some fixed η , $0 \leq \eta < \frac{1}{2}$.

We assume that (A, B_0) is stabilizable.

2.3.2.1 Existence and uniqueness

In this Theorem we establish the existence and uniqueness of the solution to the problem (2.35).

Theorem 2.3.2 *For any $T > 0$, there exists a unique mild solution of the equation (2.35) in $\mathbb{C}\left([0,T], L^2(\Omega,H)\right)$, satisfying the initial condition $x(0) = x_0$*

Proof.
The approach adopted to prove this Theorem is based on the classical fixed point Theorem for contractions and on the analytic estimates.
Set

$$\chi = \mathbb{C}\left([0,T], L^2(\Omega,H)\right),$$

and define the corresponding norm by

$$\|x\|_\chi = \left(\sup_{t\in[0,T]} \left(\mathbb{E}\|x(t)\|^2\right)\right)^{\frac{1}{2}} < +\infty.$$

The solution of the system (2.35) is given by

$$x(t) = S(t)x_0 + \int_0^t S(t-s)B\Delta_1(Ex(s))\,ds + \int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s) + \int_0^t S(t-s)B_0u(s)ds.$$

We have

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-s)B\Delta_1(Ex(s))\,ds + \int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s) \\ &+ \int_0^t S(t-s)(-A)^{\eta}(-A)^{-\eta}B_0u(s)ds, \end{aligned}$$

$$x(t) = S(t)x_0 + \int_0^t S(t-s)B\Delta_1(Ex(s))\,ds + \int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s) + \int_0^t S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds.$$

where $\widehat{B_0} = (-A)^{-\eta}B_0 \in L(Z,H)$.
Let

$$\Upsilon : \chi \to \chi$$

defined by

$$\begin{aligned} \Upsilon(x(t)) &= S(t)x_0 + \int_0^t S(t-s)B\Delta_1(Ex(s))\,ds \\ &+ \int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s) + \int_0^t S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds. \end{aligned}$$

In order to establish existence and uniqueness for (2.35), we proceed in three steps.
Step 1: Υ is well defined as a mapping from χ to χ.

Since $(S(t))_{t\geq 0}$ is an analytic exponentially stable semigroup there exist positive constants M, M_η, η and ω such that

$$\|S(t)\| \leq Me^{-\omega t} \;\;,\;\; t>0 \;\;,\;\; \omega>0,$$

and

$$\left\|(-A)^{\eta} S(t)\right\| \leq M_\eta t^{-\eta} e^{-\omega t}.$$

Now since $B\in L(U_1,H)\,,D\in L(U_2,H)\,,E\in L(H,Y)$, and $\widehat{B_0}\in L(Z,H)$ there exist constants m_1,m_2,m_3 and m_4 such that

$$\|B\|_{L(U_1,H)} \leq m_1,\;\|D\|_{L(U_2,H)} \leq m_2,\;\|E\|_{L(H,Y)} \leq m_3 \text{ and } \left\|\widehat{B_0}\right\|_{L(Z,H)} \leq m_4.$$

we have

$$\begin{aligned}
\|\Upsilon(x(t))\|^2 &= \langle \Upsilon(x(t)),\Upsilon(x(t))\rangle \\
&= \left\|S(t)x_0+\int_0^t S(t-s)B\Delta_1(Ex(s))\,ds+\int_0^t S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds\right\|^2 \\
&+ 2\left\langle S(t)x_0,\int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s)\right\rangle \\
&+ 2\left\langle \int_0^t S(t-s)B\Delta_1(Ex(s))\,ds,\int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s)\right\rangle \\
&+ \left\|\int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s)\right\|^2 \\
&+ 2\left\langle \int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s)\,,\int_0^t S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds\right\rangle.
\end{aligned}$$

We have

$$\mathbb{E}\left(\left\langle \int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s)\,,S(t)x_0\right\rangle\right)=0$$

$$\mathbb{E}\left(\left\langle \int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s),\int_0^t S(t-s)B\Delta_1(Ex(s))\,ds\right\rangle\right)=0$$

$$\mathbb{E}\left(\left\langle \int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s)\,,\int_0^t S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds\right\rangle\right)=0$$

Let $G_2(s)=S(t-s)D\Delta_2(Ex(s))$. By Burkholder-Davis-Gundy inequality, there exist positive constants c_2 such that

$$\mathbb{E}\left(\left\|\int_0^t G_2(s)\,dw(s)\right\|^2\right)\leq c\int_0^t \mathbb{E}\,\|G_2(s)\|^2\,ds.$$

Then

$$\begin{aligned}
\mathbb{E}\left(\|\Upsilon(x(t))\|^2\right) &= \mathbb{E}\left(\left\|S(t)x_0+\int_0^t S(t-s)B\Delta_1(Ex(s))\,ds+\int_0^t S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds\right\|^2\right) \\
&+ \mathbb{E}\left(\left\|\int_0^t S(t-s)D\Delta_2(Ex(s))\,dw(s)\right\|^2\right)
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left(\|\Upsilon(x(t))\|^2\right) &\leq 3\|S(t)x_0\|^2+3\mathbb{E}\left(\left\|\int_0^t S(t-s)B\Delta_1(Ex(s))\,ds\right\|^2\right)\\
&+ 3\left\|\int_0^t S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds\right\|^2+c\int_0^t \mathbb{E}\|S(t-s)D\Delta_2(Ex(s))\,dw(s)\|^2\\
&\leq 3M^2e^{-2\omega t}\|x_0\|^2+3\int_0^t M^2m_1^2K_1^2m_3^2e^{-2\omega(t-s)}\mathbb{E}\|x(s)\|^2\\
&+ 3\int_0^t M_\eta^2m_4^2(t-s)^{-2\eta}e^{-2\omega(t-s)}\|u(s)\|^2ds\\
&+ c\int_0^t M^2m_2^2K_2^2m_3^2e^{-2\omega(t-s)}\mathbb{E}\|x(s)\|^2ds.
\end{aligned}
$$

By Hölder inequality we obtain

$$
\begin{aligned}
\mathbb{E}\left(\|\Upsilon(x(t))\|^2\right) &\leq 3M^2e^{-2\omega t}\|x_0\|^2+(3M^2m_1^2K_1^2m_3^2+cM^2m_2^2K_2^2m_3^2)\int_0^t e^{-2\omega(t-s)}\mathbb{E}\|x(s)\|^2ds\\
&+ 3M_\eta^2m_4^2\int_0^t(t-s)^{-2\eta}e^{-2\omega(t-s)}ds\int_0^t\|u(s)\|^2ds.
\end{aligned}
$$

But

$$
\int_0^t(t-s)^{-2\eta}e^{-2\omega(t-s)}ds\leq\frac{1}{1-2\eta}T^{1-2\eta}
$$

Then

$$
\begin{aligned}
\mathbb{E}\left(\|\Upsilon(x(t))\|^2\right) &\leq 3M^2e^{-2\omega t}\|x_0\|^2+(3M^2m_1^2K_1^2m_3^2\\
&+ c_2M^2m_2^2K_2^2m_3^2)\frac{1-e^{-2\omega t}}{2\omega}\sup_{s\in[0,T]}\mathbb{E}\|x(s)\|^2\\
&+ 3M_\eta^2m_4^2\frac{1}{1-2\eta}T^{1-2\eta}(\|u(.)\|^2)_{L^2}.
\end{aligned}
$$

We conclude that Υ is well defined on χ.

Step 2: Now we show that Υ maps χ into χ.

For $h \in [0,T]$ and $t \in [0,T-h]$, we have

$$
\begin{aligned}
\Upsilon(x(t+h)) - \Upsilon(x(t)) &= S(t+h)x_0 + \int_0^{t+h} S(t+h-s)D\Delta_2(Ex(s))\,dw(s) \\
&+ \int_0^{t+h} S(t+h-s)B\Delta_1(Ex(s))\,ds + \int_0^{t+h} S(t+h-s)(-A)^{\eta}\widehat{B_0}u(s)ds \\
&- (S(t)x_0 + \int_0^{t} S(t-s)D\Delta_2(Ex(s))dw(s) + \int_0^{t} S(t-s)B\Delta_1(Ex(s))ds \\
&+ \int_0^{t} S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds) \\
&= (S(t+h)x_0 - S(t)x_0) + \int_0^{t} S(t+h-s)D\Delta_2(Ex(s))\,dw(s) \\
&+ \int_0^{t} S(t+h-s)B\Delta_1(Ex(s))\,ds - \int_0^{t} S(t-s)B\Delta_1(Ex(s))\,ds \\
&+ \int_0^{t} S(t+h-s)(-)^{\eta}\widehat{B_0}u(s)ds - \int_0^{t} S(t-s)D\Delta_2(Ex(s))\,dw(s) \\
&- \int_0^{t} S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds + \int_t^{t+h} S(t+h-s)B\Delta_1(Ex(s))\,ds \\
&+ \int_t^{t+h} S(t+h-s)D\Delta_2(E_ix(s))\,dw(s) + \int_t^{t+h} S(t+h-s)(-A)^{\eta}\widehat{B_0}u(s)ds \\
&= (S(t+h)x_0 - S(t)x_0) + \int_0^{t} [S(t+h-s) - S(t-s)]D\Delta_2(Ex(s))\,dw(s) \\
&+ \int_0^{t} [S(t+h-s) - S(t-s)]B\Delta_1(Ex(s))\,ds \\
&+ \int_0^{t} [S(t+h-s) - S(t-s)](-A)^{\eta}\widehat{B_0}u(s)ds + \int_t^{t+h} S(t+h-s)B\Delta_1(Ex(s))\,ds \\
&+ \int_t^{t+h} S(t+h-s)D\Delta_2(Ex(s))\,dw(s) \\
&+ \int_t^{t+h} S(t+h-s)(-A)^{\eta}\widehat{B_0}u(s)ds.
\end{aligned}
$$

Assume that $\Upsilon(x(t+h)) - \Upsilon(x(t)) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7$ such that

$$
\begin{aligned}
I_1 &= S(t+h)x_0 - S(t)x_0 \\
I_2 &= \int_0^{t} (S(t+h-s) - S(t-s))\,B\Delta_1(Ex(s))\,ds \\
I_3 &= \int_0^{t} (S(t+h-s) - S(t-s))\,D\Delta_2(Ex(s))\,dw(s) \\
I_4 &= \int_0^{t} (S(t+h-s) - S(t-s))\,(-A)^{\eta}\widehat{B_0}u(s)ds \\
I_5 &= \int_t^{t+h} S(t+h-s)B\Delta_1(Ex(s))\,ds \\
I_6 &= \int_t^{t+h} S(t+h-s)D\Delta_2(Ex(s))\,dw(s) \\
I_7 &= \int_t^{t+h} S(t+h-s)(-A)^{\eta}\widehat{B_0}u(s)ds.
\end{aligned}
$$

Therefore

$$
\|\Upsilon(x(t+h)) - \Upsilon(x(t))\|^2 = \|I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7\|^2
$$

Let $G_3(s) = (S(t+h-s) - S(t-s)) D\Delta_2(Ex(s))$. By Burkholder-Davis-Gundy inequality, there exist positive constants c' such that

$$\mathbb{E}\left(\|I_3\|^2\right) = \mathbb{E}\left(\left\|\int_0^t G_3(s)\, dw(s)\right\|^2\right) \leq c_1 \int_0^t \mathbb{E}\, \|G_3(s)\|^2\, ds.$$

Let $g(s) = S(t+h-s)D\Delta_2(Ex(s))$. By Burkholder-Davis-Gundy inequality, there exist positive constants c" such that

$$\mathbb{E}\left(\|I_6\|^2\right) = \mathbb{E}\left(\left\|\int_0^t g(s)\, dw(s)\right\|^2\right) \leq c_2 \int_0^t \mathbb{E}\, \|g(s)\|^2\, ds.$$

Since $(S(t))_{t\geq 0}$ is strongly continuous, we have

$$\lim_{h\to 0} \|I_1\| = \lim_{h\to 0} \|S(t+h)z_0 - S(t)z_0\| = 0, \tag{2.36}$$

and we have

$$S(t+h-s) = S(t-s)S(h)$$
$$S(t+h-s) - S(t-s) = S(t-s)S(h) - S(t-s) = S(t-s)\,[S(h) - I]$$

For I_2 we have

$$\begin{aligned}
\mathbb{E}\left(\|I_2\|^2\right) &= \mathbb{E}\left(\left\|\int_0^t (S(t+h-s) - S(t-s))\, B\Delta_1(Ex(s))\, ds\right\|^2\right) \\
&\leq c_3 \int_0^t \|\,(S(t+h-s) - S(t-s))\, B\Delta_1(Ex(s))\,\|^2 ds \\
&= c_3 \int_0^t \|(S(t-s)\,[S(h)-I])\, B\Delta_1(Ex(s))\|^2\, ds \\
&\leq c_3 \int_0^t \left(\|S(h)-I\|^2\, \|S(t-s)B\Delta_1(Ex(s))\|^2\right) ds \\
&\leq c_3 \int_0^t \left(\|S(h)-I\|^2\, \|S(t-s)\|^2\, \|B\|^2\, \|\Delta_1\|^2\, \|(Ex(s))\|^2\right) ds \\
&\leq c_3\, \|S(h)-I\|^2 \int_0^t M^2 e^{-2\omega(t-s)} m_1^2 K_1^2 m_3^2 \left(\|x(s)\|^2\right) ds \\
&\leq c_3\, \|S(h)-I\|^2\, M^2 m_1^2 K_1^2 m_3^2 \int_0^t e^{-2\omega(t-s)} \left(\|x(s)\|^2\right) ds
\end{aligned}$$

$$\mathbb{E}\, \|I_2\|^2 \leq c_3\, \|S(h)-I\|^2\, M^2 m_1^2 K_1^2 m_3^2 \int_0^t e^{-2\omega(t-s)} \mathbb{E}\left(\|x(s)\|^2\right) ds.$$

But

$$\int_0^t e^{-2\omega(t-s)} ds = \frac{1-e^{-2\omega t}}{2\omega}$$

hence

$$\mathbb{E}\, \|I_2\|^2 \leq \frac{1-e^{-2\omega t}}{2\omega}\, \|S(h)-I\|^2\, c_3 M^2 m_1^2 K_1^2 m_3^2 \sup_{s\in[0,T]} \mathbb{E}\left(\|x(t)\|^2\right).$$

Set $M_2 = c_3 M^2 m_1^2 K_1^2 m_3^2$ then

$$\mathbb{E}\, \|I_2\|^2 \leq \left(\frac{1-e^{-2\omega t}}{2\omega}\right) \|S(h)-I\|^2\, M_2 \sup_{s\in[0,T]} \mathbb{E}\left(\|x(s)\|^2\right). \tag{2.37}$$

For I_3 we have

$$\begin{aligned}
\int_0^t \mathbb{E}\left\|G_3(s)\right\|^2 ds &= \int_0^t \mathbb{E}\left\|(S(t+h-s)-S(t-s))\,D\Delta_2(Ex(s))\right\|^2 ds \\
&= \int_0^t \mathbb{E}\left\|(S(t-s)\,[S(h)-I])\,D\Delta_2(Ex(s))\right\|^2 ds \\
&\leq \int_0^t \mathbb{E}\left(\|S(h)-I\|^2\left\|S(t-s)D\Delta_2(Ex(s))\right\|^2\right) ds \\
&\leq \int_0^t \mathbb{E}\left(\|S(h)-I\|^2\|S(t-s)\|^2\|D\|^2\|\Delta_2\|^2\|(Ex(s))\|^2\right) ds \\
&\leq \|S(h)-I\|^2\int_0^t M^2e^{-2\omega(t-s)}m_2^2K_2^2m_3^2\mathbb{E}\left(\|x(s)\|^2\right) ds \\
&\leq \|S(h)-I\|^2 M^2m_2^2K_2^2m_3^2\int_0^t e^{-2^{\omega}(t-s)}\mathbb{E}\left(\|x(s)\|^2\right) ds
\end{aligned}$$

But

$$\int_0^t e^{-2\omega(t-s)}ds = \frac{1-e^{-2\omega t}}{2\omega}$$

hence

$$\int_0^t \mathbb{E}\left\|G_3(s)\right\|^2 ds \leq \frac{1-e^{-2\omega t}}{2\omega}\|S(h)-I\|^2 M^2m_2^2K_2^2m_3^2 \sup_{s\in[0,T]}\mathbb{E}\left(\|x(s)\|^2\right)$$

Therefore

$$c_1\int_0^t \mathbb{E}\left\|G_3(s)\right\|^2 ds \leq \left(\frac{1-e^{-2\omega t}}{2\omega}\right)\|S(h)-I\|^2 c'M^2m_2^2K_2^2m_3^2 \sup_{s\in[0,T]}\mathbb{E}\left(\|x(s)\|^2\right).$$

Set $M_3 = c_1M^2m_2^2K_2^2m_3^2$ then

$$c_1\int_0^t \mathbb{E}\left\|G_3(s)\right\|^2 ds \leq M_3\left(\frac{1-e^{-2\omega t}}{2\omega}\right)\|S(h)-I\|^2 \sup_{s\in[0,T]}\mathbb{E}\left(\|x(s)\|^2\right)$$

$$\mathbb{E}\left\|I_3\right\|^2 \leq \left(\frac{1-e^{-2\omega t}}{2\omega}\right)\|S(h)-I\|^2 M_3 \sup_{s\in[0,T]}\mathbb{E}\left(\|x(s)\|^2\right). \tag{2.38}$$

For I_4 and applying Hölder inequality, we get

$$\begin{aligned}
\|I_4\|^2 &= \left\|\int_0^t (S(t+h-s)-S(t-s))\,(-A)^{\eta}\widehat{B_0}u(s)ds\right\|^2 \\
&\leq \int_0^t \left\|(S(t+h-s)-S(t-s))\,(-A)^{\eta}\widehat{B_0}u(s)\right\|^2 ds \\
&\leq \int_0^t \left\|(S(t-s)\,[S(h)-I])\,(-A)^{\eta}\widehat{B_0}u(s)\right\|^2 ds \\
&\leq \int_0^t \|S(h)-I\|^2.\left\|S(t-s)\,(-A)^{\eta}\right\|^2.\left\|\widehat{B_0}\right\|^2 ds\int_0^t \|u(s)\|^2 ds \\
&\leq \int_0^t \|S(h)-I\|^2 M_{\eta}^2(t-s)^{-2\eta}e^{-2\omega(t-s)}m_4^2 ds\int_0^t \|u(s)\|^2 ds \\
&\leq \|S(h)-I\|^2 M_{\eta}^2m_4^2\int_0^t (t-s)^{-2\eta}e^{-2\omega(t-s)}ds\int_0^t \|u(s)\|^2 ds.
\end{aligned}$$

But
$$\int_0^t (t-s)^{-2\eta} e^{-2\omega(t-s)} ds \leq \frac{1}{1-2\eta} t^{1-2\eta} \leq \frac{1}{1-2\eta} T^{1-2\eta}.$$
Set $M_4 = M_\eta^2 m_4^2$
$$\|I_4\|^2 \leq \|S(h)-I\|^2 M_4 \left(\frac{1}{1-2\eta} T^{1-2\eta}\right) \|u(.)\|_{L^2}^2 \tag{2.39}$$
for I_5 we have
$$\begin{aligned}
\mathbb{E}\|I_5\|^2 &= \mathbb{E}\left\|\int_t^{t+h} S(t+h-s)B\Delta_1(Ex(s))\,ds\right\|^2 \\
&\leq \int_t^{t+h} \mathbb{E}\left(\|S(t+h-s)\|.\|B\|.\|\Delta_1\|.\|(Ex(t))\|^2\right) ds \\
&\leq \int_t^{t+h} M^2 e^{-2\omega(t+h-s)} m_1^2 K_1^2 m_3^2 \mathbb{E}\|x(s)\|^2 ds \\
&\leq M^2 m_1^2 K_1^2 m_3^2 \int_t^{t+h} e^{-2\omega(t+h-s)} \mathbb{E}\|x(s)\|^2 ds.
\end{aligned}$$
But
$$\int_t^{t+h} e^{-2\omega(t+h-s)} ds = \frac{1-e^{-2\omega t}}{2\omega}.$$
Or
$$\mathbb{E}\|I_5\|^2 ds \leq M^2 m_2^2 K_2^2 m_1^2 \left(\frac{1-e^{-2\omega t}}{2\omega}\right) \sup_{s\in[t,t+h]} \mathbb{E}\|x(s)\|^2.$$
Set $M_5 = M^2 m_1^2 K_1^2 m_3^2$ then
$$\mathbb{E}\|I_5\|^2 ds \leq M_5 \left(\frac{1-e^{-2\omega t}}{2\omega}\right) \sup_{s\in[t,t+h]} \mathbb{E}\|x(s)\|^2$$
$$\mathbb{E}\|I_5\|^2 \leq \left(\frac{1-e^{-2\omega t}}{2\omega}\right) M_5 \sup_{s\in[t,t+h]} \mathbb{E}\|x(s)\|^2 \tag{2.40}$$
for I_6 we have
$$\begin{aligned}
\int_t^{t+h} \mathbb{E}\|g(s)\|^2 ds &= \int_t^{t+h} \mathbb{E}\|S(t+h-s)D\Delta_2(Ex(s))\|^2 ds \\
&\leq \int_t^{t+h} \mathbb{E}\left(\|S(t+h-s)\|.\|D\|.\|\Delta_2\|.\|(Ex(s))\|^2\right) ds \\
&\leq \int_t^{t+h} M^2 e^{-2\omega(t+h-s)} m_2^2 K_2^2 m_3^2 \mathbb{E}\|x(s)\|^2 ds \\
&\leq M^2 m_2^2 K_2^2 m_3^2 \int_t^{t+h} e^{-2\omega(t+h-s)} \mathbb{E}\|x(s)\|^2 ds.
\end{aligned}$$
But
$$\int_t^{t+h} e^{-2\omega(t+h-s)} ds = \frac{1-e^{-2\omega t}}{2\omega}.$$
Or
$$\int_t^{t+h} \mathbb{E}\|g(s)\|^2 ds \leq M^2 m_2^2 K_2^2 m_1^2 \left(\frac{1-e^{-2\omega t}}{2\omega}\right) \sup_{s\in[t,t+h]} \mathbb{E}\|x(s)\|^2.$$

Therefore

$$c_2 \int_t^{t+h} \mathbb{E}\|g(s)\|^2 ds \leq c_2 M^2 m_2^2 K_2^2 m_3^2 \left(\frac{1-e^{-2\omega t}}{2\omega}\right) \sup_{s\in[t,t+h]} \mathbb{E}\|x(s)\|^2$$

Set $M_6 = c_2 M^2 m_2^2 K_2^2 m_3^2$ then

$$c_2 \int_t^{t+h} \mathbb{E}\|g(s)\|^2 ds \leq M_6 \left(\frac{1-e^{-2\omega t}}{2\omega}\right) \sup_{s\in[t,t+h]} \mathbb{E}\|x(s)\|^2$$

$$\mathbb{E}\|I_6\|^2 \leq \left(\frac{1-e^{-2\omega t}}{2\omega}\right) M_6 \sup_{s\in[t,t+h]} \mathbb{E}\|x(s)\|^2 \tag{2.41}$$

For I_7 and applying Hölder inequality, we get

$$\begin{aligned}
\|I_7\|^2 &= \left\|\int_t^{t+h} S(t+h-s)(-A)^\eta \widehat{B}_0 u(s) ds\right\|^2 \\
&\leq \int_t^{t+h} \left\|S(t+h-s)(-A)^\eta \widehat{B}_0 u(s)\right\|^2 ds \\
&\leq \int_t^{t+h} \|S(t+h-s)(-A)^\eta\| . \left\|\widehat{B}_0\right\| ds \int_t^{t+h} \|u(s)\|^2 ds \\
&\leq \int_t^{t+h} M_\eta^2 (t+h-s)^{-2\eta} e^{-2\omega(t+h-s)} m_4^2 ds \int_t^{t+h} \|u(s)\|^2 ds \\
&\leq M_\eta^2 m_4^2 \int_t^{t+h} (t+h-s)^{-2\eta} e^{-2\omega(t+h-s)} ds \int_t^{t+h} \|u(s)\|^2 ds \\
&\leq M_\eta^2 m_4^2 \left(\frac{1}{1-2\eta}\right) h^{1-2\eta} \|u(.)\|_{L^2}^2 .
\end{aligned}$$

Set $M_7 = M_\eta^2 m_4^2$ then

$$\|I_7\|^2 \leq M_\eta^2 m_4^2 \left(\frac{1}{1-2\eta}\right) h^{1-2\eta} \|u(.)\|_{L^2}^2 \tag{2.42}$$

Using (2.36), (2.37), (2.38), (2.39), (2.40), (2.41), (2.42), we deduce that

$$\lim_{h\to 0^+} \mathbb{E}\left(\|(\Upsilon x)(t+h) - (\Upsilon x)(t)\|^2\right) = 0.$$

In order to prove the left continuity of Υ we have, for every $t\in[0,T]$, $h\in[0,t]$,

$$\begin{aligned}
(\Upsilon x)(t-h)-(\Upsilon x)(t) &= S(t-h)x_0+\int_0^{t-h}S(t-h-s)D\Delta_2(Ex(s))\,dw(s)\\
&+ \int_0^{t-h}S(t-h-s)B\Delta_1(Ex(s))\,ds+\int_0^{t-h}S(t-h-s)(-A)^{\eta}\widehat{B}_0u(s)ds\\
&- \left(S(t)x_0+\int_0^{t}S(t-s)D\Delta_2(Ex(s))\,dw(s)+\int_0^{t}S(t-s)B\Delta_1(Ex(s))\,ds\right.\\
&+ \left.\int_0^{t}S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds\right)\\
&= (S(t-h)x_0-S(t)x_0)+\int_0^{t-h}[S(t-h-s)-S(t-s)]D\Delta_2(Ex(s))\,dw(s)\\
&+ \int_0^{t-h}[S(t-h-s)-S(t-s)]B\Delta_1(Ex(s))\,ds\\
&+ \int_0^{t-h}[S(t-h-s)-S(t-s)](-A)^{\eta}\widehat{B_0}u(s)ds-\int_{t-h}^{t}S(t-s)B\Delta_1(Ex(s))\,ds\\
&- \int_{t-h}^{t}S(t-s)D\Delta_2(Ex(s))\,dw(s)-\int_{t-h}^{t}S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds.
\end{aligned}$$

Using as for the right continuity, we get the left continuity

$$\lim_{h\to 0^+}\mathbb{E}\left(\|(\Upsilon x)(t-h)-(\Upsilon x)(t)\|^2\right)=0 \qquad (2.43)$$

Step 3: It remains to verify that Υ is a contraction.
Let $x_1,x_2\in\chi$, then

$$\begin{aligned}
(\Upsilon x_1)(t)-(\Upsilon x_2)(t) &= S(t)x_0+\int_0^{t}S(t-s)B\Delta_1(Ex_1(s))\,ds+\int_0^{t}S(t-s)D\Delta_2(Ex_1(s))\,dw(s)\\
&+ \int_0^{t}S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds-S(t)x_0-\int_0^{t}S(t-s)B\Delta_1(Ex_2(s))\,ds\\
&- \int_0^{t}S(t-s)D\Delta_2(Ex_2(s))\,dw(s)-\int_0^{t}S(t-s)(-A)^{\eta}\widehat{B_0}u(s)ds\\
&= \int_0^{t}S(t-s)D(\Delta_2Ex_1(s)-\Delta_2Ex_2(s))\,dw(s)\\
&+ \int_0^{t}S(t-s)B(\Delta_1Ex_1(s)-\Delta_1Ex_2(s))\,ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\|(\Upsilon x_1)(t)-(\Upsilon x_2)(t)\|^2\\
=\ &\left\|\int_0^{t}S(t-s)D(\Delta_2Ex_1(s)-\Delta_2Ex_2(s))\,dw(s)+\int_0^{t}S(t-s)B(\Delta_1Ex_1(s)-\Delta_1Ex_2(s))\,ds\right\|^2\\
\le\ & 2\left\|\int_0^{t}S(t-s)D(\Delta_2Ex_1(s)-\Delta_2Ex_2(s))\,dw(s)\right\|^2+2\left\|\int_0^{t}S(t-s)B(\Delta_1Ex_1(s)-\Delta_1Ex_2(s))\,ds\right\|^2.
\end{aligned}$$

By Burkholder-Davis-Gundy inequality and the Lipschizianity of Δ_i, there exist positive constants c_4 such that

$$\begin{aligned}
\mathbb{E}\left(\|\Upsilon(x_1(t))-\Upsilon(x_2(t))\|^2\right) &\le 2c_4\int_0^t \|S(t-s)\|^2\,\mathbb{E}\,\|D(\Delta_2(Ex_1(s))-\Delta_2(Ex_2(s)))\|^2\,ds \\
&+ 2\left\|\int_0^t S(t-s)B(\Delta_1 Ex_1(s)-\Delta_1 Ex_2(s))\,ds\right\|^2 \\
&\le 2c_4M^2\int_0^t e^{-2\omega(t-s)}m_2^2K_2^2m_3^2 \sup_{s\in[0,T]}\mathbb{E}\,\|x_1(s)-x_2(s)\|^2\,ds \\
&+ 2M^2\int_0^t e^{-2\omega(t-s)}m_1^2K_1^2m_3^2 \sup_{s\in[0,T]}\mathbb{E}\,\|x_1(s)-x_2(s)\|^2\,ds \\
&\le (2c_4M^2m_2^2K_2^2m_3^2+2M^2m_1^2K_1^2m_3^2)(\frac{1-e^{-2\omega T}}{2\omega})\,\|x_1(s)-x_2(s)\|_\chi^2 .
\end{aligned}$$

Therefore Υ is contractive for enough small $T>0$. For large T we can proceed in a usual way by considering the equation on intervals $[0,\widetilde{T}]$, $[\widetilde{T},2\widetilde{T}]$, ... with $\widetilde{T}$ enough small.

■

2.3.2.2 Maximization of the stability radius

Our aim is to characterize the supremum of the stability radii which can be achieved by linear state feedback $u=Fx$, where $F\in\mathscr{L}(H,Z)$.
Let

$$\overline{\mathscr{F}}=\left\{\begin{array}{c} F\in\mathscr{L}(H,Z); A+B_0F \text{ is the infinitesimal generator of an} \\ \text{exponentially stable semigroup } S_F(t)\end{array}\right\}.$$

and define

$$\overline{r^w}(A,D,B,E)=sup\left\{r^w(A+B_0F,D,B,E);\ F\in\overline{\mathscr{F}}\right\}.$$

Lemma 2.3.3

Let $\varepsilon>0$. If there exists $P\in L^+(H)$ such that

$$2\langle Px,(A-\varepsilon^{-2}B_0B_0^*P)x\rangle+\varepsilon^{-2}\langle PB_0B_0^*Px,x\rangle+\langle Ex,Ex\rangle\le 0,\qquad x\in D(A),\tag{2.44}$$

$$1-\sigma^2[\theta\|D^*PD\|+\|B^*PB\|]\ge 0,\tag{2.45}$$

then $F_\varepsilon=-\varepsilon^{-2}B_0^*P\in\overline{\mathscr{F}}$ and $\sigma\le r^w(A+B_0F_\varepsilon,D,B,E)$.

Proof.

From the asumptions made on the operator B and since $S(t)$ is analytic and exponentially stable, it follows from [40] that $A+B_0F_\varepsilon$ generates an analytic semigroup. Set $V(x)=\langle x,Px\rangle$. Computing $\dot{V}(x(t))$ along solution of

$$\left\{\begin{array}{ll} \frac{d}{dt}x(t)=(A-\varepsilon^{-2}B_0B_0^*P)x(t), & t\in\mathbb{R}^+, \\ x(0)=x_0,\ x_0\in H. & \end{array}\right.\tag{2.46}$$

We get

$$\frac{d}{dt}V(x(t))=2\left\langle P(A-\varepsilon^{-2}B_0B_0^*P)x,x\right\rangle.$$

From the inequality (2.44) we obtain

$$\tfrac{d}{dt}V(x(t))\ \le\ -\varepsilon^{-2}\langle PB_0B_0^*Px,x\rangle.$$

Thus

$$\int_0^T \frac{d}{dt}V(x(t))dt \quad \leq \quad -\varepsilon^{-2}\int_0^T \langle PB_0B_0^*Px,x\rangle dt.$$

Hence

$$V(x(T))-V(x(0)) \quad \leq \quad -\varepsilon^{-2}\int_0^T \|B_0^*Px(t)\|^2dt.$$

Using the fact that $P \succ 0$ we get

$$\varepsilon^{-2}\int_0^T \|B_0^*Px(t)\|^2dt \leq V(x_0), \quad for \quad all \quad T>0.$$

Therefore

$$\varepsilon^{-2}\int_0^T \|B_0^*Px(t)\|^2dt \leq k\|x_0\|^2.$$

which implies that $B_0^*Px(t) \in L^2(\mathbb{R}^+,Z)$. The solution $x(t)$ of the system (2.46) is given by

$$x(t) = S(t)x_0 - \varepsilon^{-2}\int_0^t S(t-s)B_0B_0^*Px(s)ds$$

We have

$$\begin{aligned}
\|x(t)\| &\leq \|S(t)x_0\| + \varepsilon^{-2}\left\|\int_0^t S(t-s)B_0B_0^*Px(s)ds\right\| \\
&\leq \|S(t)x_0\| + \varepsilon^{-2}\int_0^t \|S(t-s)(-A)^{\eta}\| \left\|\widehat{B_0}\right\| \|B_0^*Px(s)\|\, ds \\
&\leq Me^{-wt}\|x_0\| + \varepsilon^{-2}\left\|\widehat{B_0}\right\| M_\eta \int_0^t \frac{e^{-w(t-s)}}{(t-s)^{-\eta}} \|B_0^*Px(s)\|.
\end{aligned}$$

Then

$$\|x(t)\|^2 \leq 2M^2e^{-2\omega t}\|x_0\|^2 + 2\varepsilon^{-4}\left\|\widehat{B_0}\right\|^2 M_\eta^2 \int_0^t \frac{e^{-2\omega(t-s)}}{(t-s)^{-2\eta}}\|B_0^*Px(s)\|^2ds.$$

Therefore

$$\int_0^{+\infty}\|x(t)\|^2dt \leq 2M^2\|x_0\|^2\int_0^{+\infty}e^{-2\omega t}dt + 2\varepsilon^{-4}\left\|\widehat{B_0}\right\|^2 M_\eta^2\int_0^{+\infty}\int_0^t \frac{e^{-2\omega(t-s)}}{(t-s)^{-2\eta}}\|B_0^*Px(s)\|^2dsdt. \tag{2.47}$$

By Fubbini theorem we obtain

$$\begin{aligned}
\textstyle\int_0^{+\infty}\|x(t)\|^2dt &\leq 2M^2\|x_0\|^2\int_0^{+\infty}e^{-2\omega t}dt + 2\varepsilon^{-4}\left\|\widehat{B_0}\right\|^2 M_\eta^2\int_0^{+\infty}\int_s^{+\infty}\|B^*Px(s)\|^2\frac{e^{-2\omega(t-s)}}{(t-s)^{-2\eta}}dtds \\
&\leq \frac{2M^2\|x_0\|^2}{2\omega} + 2\varepsilon^{-4}\left\|\widehat{B_0}\right\|^2 M_\eta^2\int_0^{+\infty}\|B^*Px(s)\|^2\left(\int_s^{+\infty}\frac{e^{-2\omega(t-s)}}{(t-s)^{-2\eta}}dt\right)ds.
\end{aligned} \tag{2.48}$$

Since $\eta < \frac{1}{2}$, there exists $K>0$ such that

$$\int_0^{+\infty}\|x(t)\|^2dt \leq \frac{M^2\|x_0\|^2}{w} + 2\varepsilon^{-4}\left\|\widehat{B}\right\|^2 M_\eta^2 K\int_0^{+\infty}\|B_0^*Px(s)\|^2ds.$$

And since $B_0^*Px(t) \in L^2(\mathbb{R}^+,Z)$, we deduce that $x(.)$ belongs to the space $L^2(\mathbb{R}^+,H)$. Applying Corollary 2.2.2 with

$$F_\varepsilon = -\varepsilon^{-2}B_0^*P \in \overline{\mathscr{L}}$$

we get that $\sigma \leq r^w(A+B_0F_\varepsilon,D,B,E)$.

■

As for the bounded case, we can obtain the following results.

Lemma 2.3.4

Let $\varepsilon > 0$ and $F \in \overline{\mathscr{Z}}$. If there existe $P_1 \in L^+(H)$ such that

$$2\langle P_1(A+B_0F)x,x\rangle + \langle Ex,Ex\rangle + \varepsilon^2\langle Fx,Fx\rangle \leq 0, \qquad x \in D(A). \tag{2.49}$$

$$1-\sigma^2[\theta\|D^*P_1D\|+\|B^*P_1B\|] \geq 0, \tag{2.50}$$

then $F_1 = -\varepsilon^{-2}B_0^*P_1 \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A+B_0F_1,D,B,E)$. Moreover, there exists $P_2 \in L^+(H)$ such that

$$\begin{array}{l} 2\left\langle P_2(A+B_0F_1)x,x\right\rangle + \langle Ex,Ex\rangle + \varepsilon^{-2}\left\langle P_1B_0B_0^*P_1x,x\right\rangle = 0,\ x \in D(A), \\ 1-\sigma^2[\theta\|D^*P_2D\|+\|B^*P_2B\|] \geq 0. \\ P_2 \preceq P_1. \end{array} \tag{2.51}$$

Theorem 2.3.3

Let $F \in \overline{\mathscr{Z}}$. Suppose that there exist $\varepsilon > 0$ such that the Lyapunov inequality (2.49) has a solution $P_0 \in L^+(H)$ which satisfies condition (2.45) then the Riccati equation

$$2\left\langle Ax,Px\right\rangle + \langle Ex,Ex\rangle - \varepsilon^{-2}\left\langle x,PB_0B_0^*Px\right\rangle = 0, \quad x \in D(A) \tag{2.52}$$

has a solution $P \in L^+(H)$ satisfying

$$\begin{array}{l} 1-\sigma^2[\theta\|D^*PD\|+\|B^*PB\|] \geq 0, \\ F_\varepsilon = -\varepsilon^{-2}B_0^*P \in \overline{\mathscr{Z}}, \\ \sigma \leq r^w(A+B_0F_\varepsilon,D,B,E). \end{array} \tag{2.53}$$

Conditions for the existence of suboptimal controllers are given in the following proposition.

Proposition 2.3.2

Let $\sigma,\varepsilon > 0$. Assume that the Riccati equation (2.52) has a solution $P \in L^+(H)$ such that

$$1-\sigma^2[\theta\|D^*PD\|+\|B_0^*PB\|] \geq 0,$$

then $F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A+B_0F_\varepsilon,D,B,E)$.

Let us now state a corollary of Proposition 2.3.2 for characterizing the supremal achievable stability radius via the Riccati equation (2.52) as follows.

Corollary 2.3.2 *We have*

$$\overline{r^w}(A,D,B,E) \geq \sup\left\{ \begin{array}{l} \sigma > 0;\ \textit{there exist}\ \varepsilon > 0\ \textit{such that}\ (2.52)\ \textit{has a solution} \\ P \in L^+(H)\ \textit{with}\ 1-\sigma^2(\theta\|D^*PD\|+\|B^*PB\|) \geq 0 \end{array} \right\}.$$

2.3.2.3 Example

Consider the following stochastic parabolic equation with Newman boundary conditions

$$\left\{ \begin{array}{l} dy(x,t) = \pi^{-2}(\frac{\partial^2 y(x,t)}{\partial x^2})dt - y(x,t)dt + c_1y(x,t)dt + c_2y(x,t)dw(t), \\ 0 \leq x \leq 1, \quad 0 \leq t \leq T,\ c_1 \in \mathbb{R},\ c_2 \in \mathbb{R}. \\ y(x,0) = y_0(x), \\ \frac{\partial y}{\partial x}(0,t) = u(t), \quad \frac{\partial y}{\partial x}(1,t) = 0. \end{array} \right. \tag{2.54}$$

Consider the operator $Ah = \pi^{-2}\frac{d^2h}{dx^2} - h$ defined on $H = L^2(0,1)$ with

$$D(A) = \{\psi \in H^2(0,1), \dot{\psi}(0) = \dot{\psi}(1) = 0\}.$$

The operator A generates an exponentially stable semigroup $S(t)$, the eigenvalues of A are

$$\lambda_n = -n^2 - 1, \quad n \geq 0.$$

In addition, the corresponding eigenfunctions are

$$\phi_0 = 1 \text{ and } \phi_n(x) = \sqrt{2}\cos n\pi x, \text{ for all } n \geq 1.$$

Define the following operators:

$$\begin{aligned} D &= E = B = I. \\ \Delta_1 &= c_1, \quad \Delta_2 = c_2. \\ B_0^*\psi(x) &= -\frac{1}{\pi^2}\psi(0), \text{ with } \psi \in D(B_0^*). \end{aligned}$$

The problem (2.54) takes the following abstract form

$$\begin{cases} dx(t) = Ax(t)dt + B\Delta_1(Ex(t))dt + D\Delta_2(Ex(t))dw(t) + B_0u(t)dt, t > 0, \\ x(0) = x_0. \end{cases} \tag{2.55}$$

For this system we have

$$\eta = \frac{1}{4} + \frac{\beta}{2}, \ 0 < \beta < \frac{1}{2}.$$

Note that the corresponding Riccati equation is

$$2\langle A\mathscr{Y}, P\mathscr{Y}\rangle + \langle E\mathscr{Y}, E\mathscr{Y}\rangle - \varepsilon^{-2}\langle B_0^*P\mathscr{Y}, B_0^*P\mathscr{Y}\rangle = 0, \quad \mathscr{Y} \in D(A). \tag{2.56}$$

Assume that we can express the solution P of (2.54) as

$$P\mathscr{Y} = \sum_{n,j=0}^{+\infty} P_{nj}\langle \mathscr{Y}, \phi_n\rangle \phi_j, \quad \mathscr{Y} \in H. \tag{2.57}$$

Hence

$$A\mathscr{Y} = \sum_{n=0}^{+\infty} \lambda_n \langle \mathscr{Y}, \phi_n\rangle \phi_n, \text{ for } \mathscr{Y} \in D(A),$$

and hence

$$\langle A\mathscr{Y}, P\mathscr{Y}\rangle = \sum_{n=0}^{+\infty} \lambda_n P_{nn}(\langle \mathscr{Y}, \phi_n\rangle)^2.$$

We have

$$\langle E\mathscr{Y}, E\mathscr{Y}\rangle = \langle \mathscr{Y}, E\mathscr{Y}\rangle,$$

Since

$$\begin{aligned} B_0^*P\mathscr{Y} &= B_0^*\left(\sum_{n,j=0}^{+\infty} P_{nj}\langle \mathscr{Y}, \phi_n\rangle \phi_j\right) = \left(\sum_{n,j=0}^{+\infty} P_{nj}\langle \mathscr{Y}, \phi_n\rangle B_0^*\phi_j\right) \\ &= -\frac{1}{\pi^2}\left(\sum_{n,j=0}^{+\infty} P_{nj}\langle \mathscr{Y}, \phi_n\rangle \phi_j(0)\right) = -\frac{\sqrt{2}}{\pi^2}\left(\sum_{n,j=0}^{+\infty} P_{nj}\langle \mathscr{Y}, \phi_n\rangle\right). \end{aligned}$$

We get

$$\langle B_0^* P\mathscr{Y}, B_0^* P\mathscr{Y}\rangle = \frac{2}{\pi^4}\sum_{n,j=0}^{+\infty}\sum_{m,k=0}^{+\infty} P_{nj}P_{mk}\langle \mathscr{Y},\phi_n\rangle\langle \mathscr{Y},\phi_m\rangle$$

Equation (2.55) takes the form

$$2\sum_{n=0}^{+\infty} P_{nn}\lambda_n\langle \mathscr{Y},\phi_n\rangle^2 + \langle \mathscr{Y},\mathscr{Y}\rangle - \frac{2\varepsilon^{-2}}{\pi^4}\sum_{n,j=0}^{+\infty}\sum_{m,k=0}^{+\infty} P_{nj}P_{mk}\langle \mathscr{Y},\phi_n\rangle\langle \mathscr{Y},\phi_m\rangle = 0.$$

For $\mathscr{Y} = \phi_k$, $k \in \mathbb{N}$, we obtain

$$2\lambda_k P_{kk} + 1 - \frac{2\varepsilon^{-2}}{\pi^4}P_{kk}^2 = 0.$$

Or

$$-\frac{2\varepsilon^{-2}}{\pi^4}P_{kk}^2 + 2\lambda_k P_{kk} + 1 = 0,$$

we have

$$\Delta' = \lambda_k^2 + \frac{2\varepsilon^{-2}}{\pi^4} > 0$$

Then

$$P_{kk} = \frac{\pi^4(\lambda_k + \sqrt{\Delta'})}{2\varepsilon^{-2}} > 0$$

hence

$$P\mathscr{Y} = \sum_{n=0}^{+\infty} P_n\langle \mathscr{Y},\phi_n\rangle\phi_n, \quad \mathscr{Y} \in H$$

where $P_n = \frac{\pi^4(\lambda_n + \sqrt{\Delta'})}{2\varepsilon^{-2}}$.
Now we show that

$$1 - \sigma^2[\theta\|D^*PD\| + \|B^*PB\|] \geq 0, \tag{2.58}$$

for some $\sigma > 0$. We have

$$1 - \sigma^2[\theta\frac{\langle D^*PDz,z\rangle}{\|z\|^2} + \frac{\langle B^*PBz,z\rangle}{\|z\|^2}] \geq 0 \quad ,z \neq 0$$

$$\Leftrightarrow \quad \sigma^{-2} \geq (\theta+1)\frac{\langle Pz,z\rangle}{\|z\|^2} \quad ,z \neq 0$$

But

$$\begin{aligned}\langle Pz,z\rangle &= \langle \textstyle\sum_{n=0}^{+\infty} P_n\langle z,\phi_n\rangle\phi_n, z\rangle \\ &= \textstyle\sum_{n=0}^{+\infty} P_n\langle z,\phi_n\rangle\langle\phi_n,z\rangle \\ &= \textstyle\sum_{n=0}^{+\infty} P_n\langle z,\phi_n\rangle^2\end{aligned}$$

Then

$$\sigma^{-2} \geq (\theta+1)\frac{\langle Pz, z\rangle}{\|z\|^2}$$

is equivalent to

$$\sigma^{-2} \geq (\theta+1)\frac{\sum_{n=0}^{+\infty} P_n \langle z, \phi_n\rangle^2}{\|z\|^2}$$

For $z = \phi_k$, $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
\sigma^{-2} &\geq (\theta+1)P_k \\
\sigma^{-2} &\geq (\theta+1)\frac{\pi^2(\lambda_k+\sqrt{\Delta'})}{2\varepsilon^{-2}} \\
\sigma^{-2} &\geq (\theta+1)\frac{\pi^2(\lambda_k+\sqrt{\lambda_k^2+\frac{2\varepsilon^{-2}}{\pi^4}})}{2\varepsilon^{-2}} \times \frac{(\lambda_k-\sqrt{\lambda_k^2+\frac{2\varepsilon^{-2}}{\pi^4}})}{(\lambda_k-\sqrt{\lambda_k^2+\frac{2\varepsilon^{-2}}{\pi^4}})} \\
\sigma^{-2} &\geq -\frac{(\theta+1)}{\pi^2(\lambda_k-\sqrt{\lambda_k^2+\frac{2\varepsilon^{-2}}{\pi^4}})} \\
\sigma^{-2} &\geq \frac{(\theta+1)}{\pi^2(\sqrt{\lambda_k^2+\frac{2\varepsilon^{-2}}{\pi^4}}-\lambda_k)}
\end{aligned}$$

Let σ such that

$$\sigma \leq \sqrt{\frac{2\pi^2}{\theta+1}}$$

then

$$\sigma^2 \leq \frac{2\pi^2}{\theta+1}$$

or

$$\sigma^{-2} \geq \frac{\theta+1}{2\pi^2}$$

hence

$$\sigma^{-2} \geq -\frac{\theta+1}{2\pi^2\lambda_k}$$

because we have $\sqrt{\lambda_k^2+\frac{2\varepsilon^{-2}}{\pi^4}}-\lambda_k \geq 2\lambda_k$ or

$$\frac{1}{\sqrt{\lambda_k^2+\frac{2\varepsilon^{-2}}{\pi^4}}-\lambda_k} \leq \frac{1}{2\lambda_k}$$

then

$$\frac{(\theta+1)}{\pi^2(\sqrt{\lambda_k^2+\frac{2\varepsilon^{-2}}{\pi^4}}-\lambda_k)} \leq -\frac{\theta+1}{2\pi^2\lambda_k}$$

so

$$\sigma^{-2} \geq \frac{(\theta+1)}{\pi^2(\sqrt{\lambda_k^2+\frac{2\varepsilon^{-2}}{\pi^4}}-\lambda_k)}$$

thus for

$$\sigma \leq \sqrt{\frac{2\pi^2}{\theta+1}}$$

the Riccati equation has a solution P with

$$1-\sigma^2[\theta\|D^*PD\|+\|B^*PB\|] \geq 0$$

By Corrollary 2.3.2 we get

$$\overline{r^w}(A,D,B,E) \geq \sqrt{\frac{2\pi^2}{\theta+1}}$$

Chapter 3

Stability radii of stochastic systems subjected to stochastic perturbation and their optimization

3.1 Introduction

This chapter will investigate the robust stability analysis and the maximization of the stability radii by state feedback for stochastic systems. For the development of the theory we will follow the plan of chapter 2. Robust stability conditions for the considered systems are established on the basis of stochastic Lyapunov equation and some operator inequalities. Section 3 presents the state-feedback stabilization scheme. Proceeding as in chapter 2, the maximization of the stability radii is investigated and the results are given via a stochastic Riccati equation. An illustrative example is presented to demonstrate the effectiveness and applicability of the proposed methodologies.

3.2 Robust stability

3.2.1 System description

Let A be the infinitesimal generator of an exponentially stable semigroup $S(t)$ on a real separable Hilbert space H. Moreover let $D \in \mathscr{L}(U,H)$, $A_0 \in \mathscr{L}(H)$ and $E \in \mathscr{L}(H,Y)$. Consider the stochastic system

$$\begin{cases} dx(t) = Ax(t)dt + A_0x(t)dw_1(t),\ t \geq 0, \\ x(0) = x_0 \end{cases} \tag{3.1}$$

Assume that (3.1) is $L^2 - stable$ subjected to structured pertubations as follows

$$\begin{cases} dx(t) = Ax(t)dt + A_0x(t)dw_1(t) + D\Delta(Ex(t))dw_2(t),\ t \geq 0, \\ x(0) = x_0, \end{cases} \tag{3.2}$$

where x_0 varies in H and Δ is unknown Lipschitzian nonlinear perturbation, $\{w_i(t)\}_{t\in\mathbb{R}_+}, i \in \{1;2\}$ are independent zero mean real Wiener processes on a probability space $(\ \Omega,\ \mathscr{F},\ P\ )$ relative to a family $\{\mathscr{F}_t\}_{t\in\mathbb{R}_+}$ of $\sigma-$ algebras $\{\mathscr{F}_t\}_{t\in\mathbb{R}_+} \subset \mathscr{F}$ such that

$$\mathbb{E}\left(w_i(t) - w_i(s)\right)\left(w_j(t) - w_j(s)\right) = \delta_{ij}\theta_i\left(t-s\right),\ i,\ j \in \{1,2\},$$

where $\theta_i > 0$ denotes the variance of $\{w_i(t)\}_{t\in\mathbb{R}_+}$, (D,E) determines the structure of the stochastic perturbation.

The disturbance Δ vary in $Lip(Y,U)$,

The size of $\Delta \in Lip(Y,U)$ is measured by the Lipschitz norm

$$\|\Delta\|_{Lip} = inf\{\gamma > 0; \forall y,\ \widehat{y} \in Y : \|\Delta(y) - \Delta(\widehat{y})\|_U \leq \gamma \|y - \widehat{y}\|_Y\}.$$

3.2.2 Characterizations of the stability radius

The stability radius of the system (3.1) is defined as follows.

Definition 3.2.1

The stability radius of (3.1) whith respect to the pertubation structure (D,E) and the Wiener processes $\{w_i(t)\}_{t\in\mathbb{R}_+}; i \in \{1,2\}$ is

$$r^w(A,(D,E)) = inf\{\|\Delta\|; \Delta \in Lip(Y,U) \text{ such that (3.2) is not } L^2 - stable\}.$$

The approach used in this section to characterize the stability radius $r^w(A;(D,E))$ is based on the following lemma.

Lemma 3.2.1

Let $x(t)$ the solution of the system

$$\begin{cases} dx(t) = Ax(t)dt + A_0x(t)dw_1(t) + D\upsilon(t)dw_2(t), t \geq 0 \\ x(0) = x_0. \end{cases} \tag{3.3}$$

Where $\upsilon \in L^2_w(\mathbb{R}^+, L^2(\Omega,U))$. Set $y(t) = Ex(t)$, then $y \in L^2_w(\mathbb{R}^+, L^2(\Omega,Y))$ and

$$\|y\|^2_{L^2_w} \leq \|P\|.\|x_0\|^2 + \|D^*PD\|.\|\upsilon\|^2_{L^2_w},$$

where $P \in \mathscr{L}^+(H)$ is a self-adjoint nonnegative operator satisfying the Lyapunov stochastic equation

$$2\langle Px, Ax\rangle + \theta_1\langle A_0x, PA_0x\rangle + \langle Ex, Ex\rangle = 0, x \in D(A) \tag{3.4}$$

Proof.
Let $x(t)$ be the solution of the system (3.3). Using Itô formula with

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

we get

$$\begin{aligned} \langle x(t), Px(t)\rangle &= \langle x(0), Px(0)\rangle + \int_0^t \langle Px(\tau), (A_0x(\tau) \quad D\upsilon(\tau))dw(\tau)\rangle \\ &+ \int_0^t 2\langle Px(\tau), Ax(\tau)\rangle d\tau + \theta_1 \int_0^t \langle A_0x(\tau), PA_0x(\tau)\rangle d\tau + \theta_2 \int_0^t \langle D\upsilon(\tau), PD\upsilon(\tau)\rangle d\tau. \end{aligned}$$

Using Theorem 6.12 in [19], we get

$$\begin{aligned}\mathbb{E}\langle x(t),Px(t)\rangle &= \mathbb{E}\langle x(0),Px(0)\rangle+\mathbb{E}\int_0^t 2\langle x(\tau),PAx(\tau)\rangle d\tau \\ &+ \theta_1\mathbb{E}\int_0^t \langle A_0x(\tau),PA_0x(\tau)\rangle d\tau+\theta_2\mathbb{E}\int_0^t \langle D\upsilon(\tau),PD\upsilon(\tau)\rangle d\tau.\end{aligned}$$

Since P is the solution of the Lyapunov equation (3.4), it follows that

$$2\langle Px(t),Ax(t)\rangle+\theta_1\langle A_0x(t),PA_0x(t)\rangle=-\langle Ex(t),Ex(t)\rangle,$$

hence

$$2\langle Px(t),Ax(t)\rangle+\theta_1\langle A_0x(t),PA_0x(t)\rangle=-\langle y(t),y(t)\rangle,$$

thus

$$\mathbb{E}\langle x(t),Px(t)\rangle = \mathbb{E}\langle x_0,Px_0\rangle-\mathbb{E}\textstyle\int_0^t\langle y(\tau),y(\tau)\rangle d\tau+\theta_2\mathbb{E}\displaystyle\int_0^t \langle D\upsilon(\tau),PD\upsilon(\tau)\rangle d\tau,$$

therefore

$$\mathbb{E}\int_0^t \|y(\tau)\|^2 d\tau = \langle x_0,Px_0\rangle-\mathbb{E}\langle x(t),Px(t)\rangle+\theta_2\mathbb{E}\int_0^t \langle D\upsilon(\tau),PD\upsilon(\tau)\rangle d\tau.$$

since $P\succeq 0$ it follows that

$$\begin{aligned}\mathbb{E}\int_0^t \|y(\tau)\|^2 d\tau &\leq \langle x_0,Px_0\rangle+\theta_2\mathbb{E}\int_0^t \langle D\upsilon(\tau),PD\upsilon(\tau)\rangle d\tau \\ &\leq \|P\|\|x(0)\|^2+\theta_2\|D^*PD\|\mathbb{E}\int_0^t \|\upsilon(\tau)\|_H^2 d\tau,\end{aligned}$$

from which, we get

$$\|y\|^2_{L^2_w} \leq \|P\|\|x(0)\|^2+\theta_2\|D^*PD\|\|\upsilon\|^2_{L^2_w}.$$

Hence $y\in L^2_w(\mathbb{R}^+,L^2(\Omega,H))$.

■

The second lemma will be given in terms of the input-output operator

$$L: L^2_w(\mathbb{R}^+,L^2(\Omega,U))\to L^2_w(\mathbb{R}^+,L^2(\Omega,Y))$$

defined by

$$L\upsilon(t) = \int_0^t ES(t-\tau)A_0x(\tau)dw_1(\tau)+\int_0^t ES(t-\tau)D\upsilon(\tau)dw_2(\tau). \tag{3.5}$$

Lemma 3.2.2

The input-output operator defined by (3.5) has the operator norm

$$\|L\|=[\theta_2\|D^*PD\|]^{\frac{1}{2}}, \tag{3.6}$$

where P satisfies the Lyapunov equation (3.4).

Proof.
Let $\upsilon \in L^2_w(\mathbb{R}^+, L^2(\Omega, U))$. By the previous Lemma it follows that

$$\|L\upsilon\|^2_{L_{\omega^2}} \leq \theta_2\|D^*PD\|\|\upsilon\|^2_{L^2_w}.$$

therefore

$$\|L\|^2_{L^2_w} \leq \theta_2\|D^*PD\|.$$

Now we will show that there exists $\upsilon' \in L^2_w(R^+, L^2(\Omega, U))$ such that

$$\|L\upsilon'\|_{L^2_w} = (\theta_2\|D^*PD\|)^{\frac{1}{2}}$$

Let $\upsilon^0 \in U$, $\|\upsilon^0\|_U = 1$ such that

$$\|D^*PD\|_U = \max_{\|\upsilon\|=1} \langle \upsilon, D^*PD\upsilon\rangle_U = \langle \upsilon^0, D^*PD\upsilon^0\rangle.$$

Define ψ as follows

$$\psi(t) = \beta(t)\upsilon^0$$

where $\beta(.) \in L^2(\mathbb{R}^+, \mathbb{R})$ and $|\beta(.)|_{L^2(\mathbb{R}^*, \mathbb{R})} = 1$.
Then

$$\begin{aligned} \|\psi(.)\|^2_{L^2_w} &= \int_0^{+\infty} \mathbb{E}|\beta(\tau)\upsilon^0|^2 d\tau \\ &= \|\upsilon^0\|^2 \int_0^{+\infty} |\beta(\tau)|^2 d\tau \\ &= 1. \end{aligned} \tag{3.7}$$

Therefore

$$\begin{aligned} \|L\psi\|^2_{L^2_w} &= \theta_2 \int_0^{+\infty} \mathbb{E}\langle D\psi(\tau), PD\psi(\tau)\rangle d\tau \\ &= \theta_2 \int_0^{+\infty} \mathbb{E}\langle \beta(\tau)\upsilon^0(\tau), D^*PD\beta(\tau)\upsilon^0(\tau)\rangle d\tau \\ &= \|D^*PD\|\theta_2 \int_0^{+\infty} |\beta(\tau)|^2 d\tau \\ &= \theta_2\|D^*PD\|. \end{aligned} \tag{3.8}$$

Thus

$$\|L\psi\|_{L^2_w} = [\theta_2\|D^*PD\|]^{\frac{1}{2}},$$

which concludes the proof.
■
The main result of this section is giving in the following Theorem.

Theorem 3.2.1
Let $\sigma > 0$. Suppose that there exists $P \in L^+(H)$ satisfying

$$2\langle Px, Ax\rangle + \theta_1\langle A_0x, PA_0x\rangle + \langle Ex, Ex\rangle = 0, \quad x \in D(A), \tag{3.9}$$

$$1 - \sigma^2\theta_2\|D^*PD\| \geq 0. \tag{3.10}$$

then $r^w(A, D, E) \geq \sigma$.

Proof.

Let $\Delta \in Lip(Y,U)$ such that $\|\Delta\|_{Lip} < \sigma$. Suppose that $P \in \mathscr{L}^+(H)$ is such that (3.9) and (3.10) hold.

Set $y(t) = Ex(t)$ and $u(t) = \Delta(y(t)), t > 0$, where $x(t)$ is the solution of the system (3.2). We have

$$y(t) = ES(t)x_0 \quad + \quad \int_0^t ES(t-\tau)A_0 x(t)dw_1(\tau) + \int_0^t ES(t-\tau)D\Delta(Ex(\tau))dw_2(\tau). \qquad (3.11)$$

For every $T > 0$, define the truncation $u_T \in L^2_w(\mathbb{R}^+, L^2(\Omega, U))$ by

$$u_T(t) = \begin{cases} u(t) = \Delta(y(t)) & if\ t \in [0,T], \\ 0 & if\ \ t > T. \end{cases} \qquad (3.12)$$

Then

$$\begin{aligned} \|u_T(.)\|^2_{L^2_w} &= \int_0^{+\infty} \mathbb{E}(\|u_T(t)\|)^2 dt \\ &= \int_0^T \mathbb{E}(\|u(t)\|)^2 dt \\ &= \int_0^T \mathbb{E}(\|\Delta y(t)\|)^2 dt \\ &\leq \|\Delta\|^2_{Lip} \int_0^T (\mathbb{E}(\|y(t)\|)^2) dt. \end{aligned}$$

Hence

$$\|u_T(t)\|^2_{L^2_w} \leq \|\Delta\|^2_{Lip} \int_0^T \mathbb{E}(\|y(t)\|)^2 dt.$$

Now define y_T as the output of the system $(A,(D,E))$ generated by the input u_T with initial condition $x(0) = x_0$. Then

$$y_T(t) = ES(t)x_0 + Lu_T(t). \quad t \geq 0. \qquad (3.13)$$

From (3.11) - (3.13), we get

$$\begin{aligned} \left(\int_0^T \mathbb{E}\|y(t)\|^2 dt\right)^{\frac{1}{2}} &\leq \|y_T\|_{L^2_w} \\ &\leq M\int_0^{+\infty} e^{(-2\omega t)}\|Ex_0\|dt + \|L\|\|u_T\|_{L^2_w}. \end{aligned} \qquad (3.14)$$

Thus

$$\left(\int_0^T \mathbb{E}\|y(t)\|^2 dt\right)^{\frac{1}{2}} \leq \int_0^{+\infty} Me^{(-2\omega t)}\|Ex_0\|dt + \|L\|\|\Delta\|_{Lip}\left(\int_0^T \mathbb{E}\|y(t)\|^2 dt\right)^{\frac{1}{2}} \qquad (3.15)$$

Condition (3.10) implies that

$$\|L\|^2 < \sigma^{-2}$$

Now since $\|\Delta\|_{Lip} < \sigma$, the operator $L\Delta$ is contraction on $L^2_w(\mathbb{R}^+, L^2(\Omega, Y))$ with $\beta = \|L\|\|\Delta\| < 1$. From (3.15) we get

$$\left(\int_0^T \mathbb{E}\|y(t)\|^2 dt\right)^{\frac{1}{2}} \leq (1-\beta)^{-1}\int_0^{+\infty} Me^{(-2\omega t)} dt\|Ex_0\|, \text{ for all } T > 0.$$

Therefore $y \in L^2_w(\mathbb{R}^+, L^2(\Omega, Y))$ and $u = \Delta(y) \in L^2_w(\mathbb{R}^+, L^2(\Omega, U))$.

By lemma 3.2.1, the solution $x(.)$ belongs to $L^2_w(\mathbb{R}^+, L^2(\Omega, H))$. We conclude then that

$$r^w(A,(D,E)) \geq \sigma.$$

■

As a consequence of this Theorem we have the following Corollary which enables us to obtain a lower bound for the stability radius.

Corollary 3.2.1

Suppose that there exists $P \in \mathscr{L}^+(H)$ a solution of the Lyapunov equation (3.9). Then

$$r^w(A,(D,E)) \geq \theta_2 \|D^*PD\|^{-\frac{1}{2}}. \tag{3.16}$$

Proof.

1. If $\|D^*PD\| = 0$, then $1 - \sigma^2\theta_2\|D^*PD\| > 0$, for all $\sigma > 0$. From the above Theorem, it follows that $r^w(A,(D,A)) \geq \sigma$, for all $\sigma > 0$. From which we deduce that $r^w(A,(D,A)) = +\infty$.

2. If $\|D^*PD\| \neq 0$, since

$$\|u\|^2 - \|D^*PD\|^{-1}\langle D^*PDu, u\rangle \geq 0 \text{ for all } u \in U.$$

it follows that

$$\|u\|^2 - ((\theta_2\|D^*PD\|)^{\frac{-1}{2}})^2\theta_2\langle D^*PDu, u\rangle \geq 0, \text{ for all } u \in U.$$

By the previous Theorem, we deduce that $r^w(A,(D,E)) \geq (\theta_2\|D^*PD\|)^{\frac{-1}{2}}$.

■

Corollary 3.2.2

Suppose that there exists $P \in L(H^+)$ satisfying

$$2\langle Px, Ax\rangle + \theta_1\langle A_0x, PA_0x\rangle + \langle Ex, Ex\rangle \leq 0, \quad x \in D(A), \tag{3.17}$$

$$\begin{gathered} 1 - \sigma^2\theta_2\|D^*PD\| \geq 0 \\ \left(resp. 1 - \sigma^2\theta_2\|D^*PD\| > 0.\right) \end{gathered} \tag{3.18}$$

Then $r^w(A,D,E) \geq \sigma$. In this case the Lyapunov equation (3.9) has a solution $P_0 \in L(H^+)$ such that $P \succeq P_0$.

Proof.

Since (3.1) is $L^2 - stable$, there exists a solution P_0 of the Lyapunov equation (3.9). Set $X = P - P_0$, then

$$2\langle Xx, Ax\rangle \leq 0, \quad x \in D(A).$$

Applying Lemma 2.1 in [10] we obtain that $X \succeq 0$, thus $P \succeq P_0$.

Using condition (3.18), it follows that

$$1 - \sigma^2\theta_2\|D^*PD\| \leq 1 - \sigma^2\theta_2\|D^*P_0D\|.$$

Hence conditions (3.9) and (3.10) are satisfied. By applying Theorem 3.2.1 we deduce that $r^w(A,D,E) \geq \sigma$.

■

Remark 3.2.1

In the case where $A_0 = 0$, we obtain the same result obtained in [35].

3.2.3 Example

Consider the stochastic system

$$\begin{cases} dy(t) = [\frac{\partial}{\partial x^2} + r_0]y(t)dt + r_1 y(t)dw_1(t) + r_2 y(t)dw_2(t), t > 0, 0 < x < \pi, \\ y(t,0) = y(t,\pi) = 0, \\ y(0,x) = y_0(x). \end{cases} \tag{3.19}$$

Where $r_0 < 1$, w_1, w_2 are independent zero mean real Wiener processes.
To put the problem (3.19) into the abstract setting we introduce the self-adjoint operator $Ah = \frac{d^2h}{dx^2} + r_0 h$ in the Hilbert space $H = L^2(0,\pi)$ with $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$.
Since $r_0 < 1$, A is the infinitesimal generator of an exponentially stable semigroup $S(t)$. The eigenvalues and the eigenvectors of A are given by [9].

$$\lambda_n = -n^2 + r_0;\ \psi_n(x) = \sqrt{2} sin(nx), n \geq 1$$

$A_0 = r_1 I$, $D = I$, $\Delta = r_2 \in \mathbb{R}$, $E = I$. In the abstract from system (3.19) can be presented as follows

$$\begin{cases} dz(t) = Az(t)dt + A_0 z(t)dw_1(t)dt + D\Delta_2(Ez(t))dw_2(t), \\ z(0) = z_0. \end{cases} \tag{3.20}$$

The stochastic Lyapunov equation corresponding to this system is

$$2\langle Pz, Az\rangle + \theta_1 \langle A_0 z, PA_0 z\rangle + \langle Ez, Ez\rangle = 0.$$

Setting

$$Pz = \sum_{k=1}^{+\infty} P_k \langle z, \phi_k\rangle \phi_k, \qquad Az = \sum_{k=1}^{+\infty} \lambda_k \langle z, \phi_k\rangle \phi_k$$

we get

$$\langle Pz, Az\rangle = \sum_{k=1}^{+\infty} P_k \lambda_k \langle z, \phi_k\rangle^2,$$

$$\langle A_0 z, A_0 z\rangle = r_1^2 \sum_{k=1}^{+\infty} P_k \langle z, \phi_k\rangle^2,$$

hence

$$2\sum_{k=1}^{+\infty} P_k \lambda_k \langle z, \phi_k\rangle^2 + \theta_1 r_1^2 \sum_{k=1}^{+\infty} P_k \langle z, \phi_k\rangle^2 + \langle z, z\rangle = 0.$$

For $z = \phi_k$ we obtain

$$2P_j \lambda_j + \theta_1 r_1^2 P_j + 1 = 0, \qquad j \geq 1,$$

therefore

$$P_j = \frac{1}{2(j^2 - r_0) - \theta_1 r_1^2}, \qquad j \geq 1.$$

If $\theta_1 r_1^2 < 2(1 - r_0)$ it follows that $P_j > 0$, for all $j \geq 1$.
For the condition (3.10), we have

$$\begin{aligned} & I - \gamma^2 \theta_2 D^* P D \geq 0 \\ \Leftrightarrow & \textstyle\sum_{k=1}^{+\infty} (1 - \gamma^2 \theta_2 P_k) \langle z, \phi_k\rangle^2 \quad \geq \quad 0, \text{for all}\, z \in H. \end{aligned}$$

For $\gamma^2 \leq \frac{1}{\theta_2}(2(1-r_0)-\theta_1 r_1^2)$, the Lyapounov equation has solution P with

$$I-\gamma^2\theta_2 D^*PD \geq 0.$$

Applying Theorem 3.2.1, we deduce that the system (3.19) is stable if

$$r_2^2 \leq \frac{1}{\theta_2}(2(1-r_0)-\theta_1 r_1^2).$$

If $[\theta_1=\theta_2=1]$, we get $r_2^2 \leq (2(1-r_0)-r_1^2)$.

3.3 Robust stabilization

In this section we consider the controlled system

$$\begin{cases} dx(t) = Ax(t)dt + A_0x(t)dw_1(t) + D\Delta(Ex(t))dw_2(t) + B_0u(t)dt, \ t>0, \\ x(0)=x_0. \end{cases} \tag{3.21}$$

where u takes its values in the real separable Hilbert space Z, $B_0 \in \mathscr{L}(Z,H)$. In addition we assume that (A,B_0) is stabilizable.
Our aim is to characterize the supremum of the stability radii which can be achieved by linear state feedback $u=Fx$, where $F \in \mathscr{L}(H,Z)$.
Let

$$\overline{\mathscr{Z}} = \left\{ \begin{array}{c} F \in \mathscr{L}(H,Z); A+B_0F \ is \ the \ infinitesimal \ generator \ of \ an \\ exponentially \ stable \ semigroup \ S_F(t) \end{array} \right\},$$

and define

$$\overline{r^w}(A,D,E) = sup\left\{ r^w(A+B_0F,D,E); \ F \in \overline{\mathscr{F}} \right\}.$$

For $F \in \overline{\mathscr{Z}}$, $\varepsilon>0$, consider the stochastic Lyapunov inequality

$$2\langle P(A+B_0F)x,x\rangle + \theta_1\langle A_0x, PA_0x\rangle + \langle Ex,Ex\rangle, +\varepsilon^2\langle Fx,Fx\rangle \leq 0, \qquad x \in D(A). \tag{3.22}$$

In order to establish conditions for the existence of suboptimal controllers $u(t)=Fx(t)$ such that $F \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A+B_0F,D,E)$, for $\sigma>0$, we need the following Lemmas.
This Lemma is of technical interest.

Lemma 3.3.1
Let $\varepsilon>0$. If there exists $P \in L^+(H)$ such that

$$2\langle Px,(A-\varepsilon^{-2}B_0B_0^*P)x\rangle + \theta_1\langle A_0x,PA_0x\rangle + \varepsilon^{-2}\langle PB_0B_0^*Px,x\rangle + \langle Ex,Ex\rangle \leq 0, \quad x\in D(A). \tag{3.23}$$

$$1-\sigma^2\theta_2\|D^*PD\| \geq 0, \tag{3.24}$$

then $A_\varepsilon = A-\varepsilon^{-2}B_0B_0^*P$ generates an exponentially stable semigroup and $\sigma \leq r^w(A_\varepsilon,D,E)$.

Proof.
Consider the initial value problem

$$\begin{cases} \frac{d}{dt}x(t) = A_\varepsilon x(t), \qquad t\in\mathbb{R}^+ \\ x(0)=x_0, \ x_0\in H. \end{cases} \tag{3.25}$$

For $x_0 \in D(A_\varepsilon), V(x) = \langle x, Px \rangle$ is differentiable and

$$\frac{d}{dt}V(x(t)) = 2\langle PA_\varepsilon x, x \rangle.$$

From the inequality (3.23) we obtain

$$\begin{aligned}\frac{d}{dt}V(x(t)) &\leq -\varepsilon^{-2}\langle PB_0B_0^*Px, x\rangle - \theta_1\langle A_0x, PA_0x\rangle - \langle Ex, Ex\rangle \\ &\leq -\varepsilon^{-2}\langle PB_0B_0^*Px, x\rangle.\end{aligned}$$

Thus

$$\int_0^T \frac{d}{dt}V(x(t))dt \leq -\varepsilon^{-2}\int_0^T \langle PB_0B_0^*Px, x\rangle dt.$$

Hence

$$V(x(T)) - V(x(0)) \leq -\varepsilon^{-2}\int_0^T \|B_0^*Px(t)\|^2 dt.$$

Using the fact that $P \succ 0$ we get

$$\varepsilon^{-2}\int_0^T \|B_0^*Px(t)\|^2 dt \leq V(x_0), \quad for \quad all \quad T > 0.$$

Therefore

$$\varepsilon^{-2}\int_0^T \|B_0^*Px(t)\|^2 dt \leq k\|x_0\|^2, \; k > 0.$$

which implies that $B_0^*Px(t) \in L^2(\mathbb{R}^+, Z)$. The solution $x(t)$ of the system (3.25) is given by

$$x(t) = S(t)x_0 - \varepsilon^{-2}\int_0^t S(t-s)B_0B_0^*Px(s)ds.$$

We have

$$\begin{aligned}\|x(t)\| &\leq \|S(t)x_0\| + \varepsilon^{-2}\left\|\int_0^t S(t-s)B_0B_0^*Px(s)ds\right\| \\ &\leq Me^{-\omega t}\|x_0\| + \varepsilon^{-2}M\|B_0\|\int_0^t e^{-\omega(t-s)}\|B_0^*Px(s)\|ds,\end{aligned}$$

from which we get

$$\begin{aligned}\|x(t)\|^2 &\leq 2M^2e^{-2\omega t}\|x_0\|^2 + 2\varepsilon^{-4}M^2\|B_0\|^2\left[\int_0^t e^{-\omega(t-s)}\|B_0^*Px(s)\|ds\right]^2 \\ &\leq K_1e^{-2\omega t} + K_2\int_0^t e^{-2\omega(t-s)}\|B_0^*Px(s)\|^2, ds\end{aligned}$$

where $K_1 = 2M^2\|x_0\|^2$, $K_2 = 2\varepsilon^{-4}M^2\|B_0\|^2$. It follows then that

$$\int_0^{+\infty} \|x(t)\|^2 dt \leq \int_0^{+\infty} K_1e^{-2\omega t}dt + \int_0^{+\infty} K_2\int_0^t e^{-2\omega(t-s)}\|B_0^*Px(s)\|^2 dsdt.$$

Thus

$$\int_0^{+\infty} \|x(t)\|^2 dt \leq \frac{K_1}{2\omega} + \int_0^{+\infty} K_2e^{2\omega s}\|B_0^*Px(s)\|^2\left(\int_s^{+\infty} e^{-2\omega t}dt\right)ds.$$

Which implies that

$$\int_0^{+\infty} \|x(t)\|^2 dt \leq \frac{K_1}{2\omega} + \frac{K_2}{2\omega}\int_0^{+\infty} \|B_0^*Px(s)\|^2 ds.$$

Since $B_0^*Px(t) \in L^2(\mathbb{R}^+,Z)$, we deduce that $x(t)$ belongs to the space $L^2(\mathbb{R}^+,H)$. Applying Corollary 3.2.2 with

$$F_\varepsilon = -\varepsilon^{-2}B_0^*P \in \overline{\mathscr{Z}}$$

we get that $\sigma \leq r^w(A_\varepsilon, D, E)$.

■

Lemma 3.3.2

Let $\varepsilon > 0$ and $F \in \overline{\mathscr{Z}}$. If the inequality (3.22) has a solution $P_1 \in L^+(H)$ satisfying condition (3.24) then $F_1 = -\varepsilon^{-2}B_0^*P_1 \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A+B_0F_1, D, B, E)$. Moreover, there exists $P_2 \in L^+(H)$ such that

$$2\langle P_2(A+B_0F_1)x,x\rangle + \langle Ex,Ex\rangle + \theta_1\langle A_0x,P_2A_0x\rangle + \varepsilon^{-2}\langle P_2B_0B_0^*P_2x,x\rangle = 0,$$

$$x \in D(A),$$

$$1-\sigma^2\theta_2\|D^*P_2D\| \geq 0,$$

$$P_2 \preceq P_1.$$

Proof.

Set $F' = \varepsilon F + \varepsilon^{-1}B_0^*P_1$, then

$$\langle F'x,F'x\rangle - \varepsilon^{-2}\langle B_0^*P_1x,B_0^*P_1x\rangle = \varepsilon^2\langle Fx,Fx\rangle + 2\langle B_0^*P_1x,Fx\rangle.$$

Since P_1 is a solution of the inequality (3.22) it follows that

$$2\langle P_1Ax,x\rangle + \theta_1\langle A_0x,P_1A_0x\rangle + \langle Ex,Ex\rangle - \varepsilon^{-2}\langle B_0^*P_1x,B_0^*P_1x\rangle + \langle F'x,F'x\rangle \leq 0. \tag{3.26}$$

Set $A_1 = A+B_0F_1$ where $F_1 = -\varepsilon^{-2}B_0^*P_1$, then

$$2\langle P_1A_1x,x\rangle + \theta_1\langle A_0x,PA_0x\rangle + \langle Ex,Ex\rangle + \varepsilon^{-2}\langle P_1B_0B_0^*P_1x,x\rangle \leq 0. \tag{3.27}$$

Applying Lemma 3.3.1 we conclude that $F_1 \in \overline{\mathscr{Z}}$ and $\sigma \leq r^w(A_1,D,E)$.

Now since P_1 is a solution of the inequality (3.27), then it satisfies the following inequality

$$2\langle P_1A_1x,x\rangle + \theta_1\langle A_0x,PA_0x\rangle + \left\langle \widehat{E}x,\widehat{E}x\right\rangle \leq 0,$$

where

$$\widehat{E} = \begin{pmatrix} E \\ \varepsilon^{-1}B_0^*P_1 \end{pmatrix}$$

By Corollary 3.2.2 there exists $P_2 \in L^+(H)$ such that

$$2\langle P_2A_0x,x\rangle + \theta_1\langle A_0x,P_2A_0x\rangle + \left\langle \widehat{E}x,\widehat{E}x\right\rangle = 0$$

with $P_2 \preceq P_1$. Therefore

$$2\langle P_2A_0x,x\rangle + \varepsilon^{-2}\langle x,P_1B_0B_0^*P_1x\rangle + \theta_1\langle A_0x,P_2A_0x\rangle + \langle Ex,Ex\rangle \leq 0$$

and

$$1-\sigma^2\theta_2\|D^*P_2D\| \geq 0.$$

■

Applying this Lemma iteratively we show in the following Theorem that there exists $P \in L^+(H)$ such that

$$2\langle Ax,Px\rangle + \theta_1\langle A_0x,PA_0x\rangle + \langle Ex,Ex\rangle - \varepsilon^{-2}\langle x,PB_0B_0^*Px\rangle = 0, \quad x \in D(A). \tag{3.28}$$

Theorem 3.3.1

Let $F \in \overline{\mathscr{L}}$. Suppose that there exist $\varepsilon > 0$ such that the Lyapunov inequality (3.22) has a solution $P_0 \in L^+(H)$ which satisfies condition (3.24) then the Riccati equation (3.28) has a solution $P \in L^+(H)$ satisfying

$$1 - \sigma^2\theta_2\|D^*PD\| \geq 0,$$
$$F_\varepsilon = -\varepsilon^{-2}B_0^*P \in \overline{\mathscr{L}},$$
$$\sigma \leq r^w(A - \varepsilon^{-2}B_0B_0^*P, D, E)$$

Proof.

Applying the above Lemma iteratively we construct a sequence of linear operators $(P_k)_{k\in\mathbb{N}} \in L^+(H)$ which satisfies

$$2\langle P_{k+1}A_kx, x\rangle + \theta_1\langle A_0x, PA_0x\rangle + \langle Ex, Ex\rangle + \varepsilon^{-2}\langle x, P_kB_0B_0^*P_k + 1x\rangle = 0,$$
$$x \in D(A),$$
$$1 - \sigma^2\theta_2\|D^*P_kD\| \geq 0,$$
$$P_{k+1} \preceq P_k,$$

where P_1 is a solution of the inequality (3.22) and $A_k = A - \varepsilon^{-2}B_0B_0^*P_k$.

Let $P = \lim_{k\to+\infty} P_k$ then

$$2\langle PA_\varepsilon x, x\rangle + \theta_1\langle A_0x, PA_0x\rangle + \langle Ex, Ex\rangle + \varepsilon^{-2}\langle x, PB_0B_0^*Px\rangle = 0, \quad x \in D(A),$$
$$1 - \sigma^2\theta_2\|D^*PD\| \geq 0,$$

where $A_\varepsilon = A - \varepsilon^{-2}B_0B_0^*P$.

Using Lemma 3.3.2 we deduce that $F_\varepsilon = -\varepsilon^{-2}B_0^*P \in \overline{\mathscr{L}}$ and $\sigma \leq r^w(A - \varepsilon^{-2}B_0B_0^*P, D, E)$.

Finally since

$$2\langle PA_\varepsilon x, x\rangle + \langle Ex, Ex\rangle + \varepsilon^{-2}\langle x, PB_0B_0^*Px\rangle$$
$$= 2\langle PAx, x\rangle + \theta_1\langle A_0x, PA_0x\rangle + \theta_1\langle A_0x, PA_0x\rangle + \langle Ex, Ex\rangle - \varepsilon^{-2}\langle x, PB_0B_0^*Px\rangle,$$

then P satisfies the Riccati equation (3.28).

■

Proposition 3.3.1

Let $\sigma, \varepsilon > 0$. Suppose that the Riccati equation (3.28) has a solution $P \in L^+(H)$ such that

$$1 - \sigma^2\theta_2\|D^*PD\| \geq 0$$

then $F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathscr{L}}$ and $\sigma \leq r^w(A + B_0F_\varepsilon, D, E)$.

Proof.

Since P is a solution of the Riccati equation (3.28) then

$$2\langle P(A - \varepsilon^{-2}B_0B_0^*P)x, x\rangle) + \theta_1\langle A_0x, PA_0x\rangle + \langle Ex, Ex\rangle + \varepsilon^{-2}\langle x, PB_0B_0^*Px\rangle = 0, \quad x \in D(A),$$
$$1 - \sigma^2\theta_2\|D^*PD\| \geq 0.$$

From Lemma 3.3.2 we obtain $F_\varepsilon = -\varepsilon^{-2}B_0^*P \in \overline{\mathscr{L}}$ and $\sigma \leq r^w(A + B_0F_\varepsilon, D, E)$.

■

As a consequence of the above proposition we characterize the supremal achievable stability radius via the Riccati equation (3.28) as follows.

Corollary 3.3.1 *We have*

$$\overline{r_w}(A, D, E) \geq \sup\left\{\begin{array}{l} \sigma > 0; \quad \text{there exist} \quad \varepsilon > 0 \text{such that} \quad (3.28) \quad \text{has} \\ \text{a solution} \quad P \in L^+(H) \quad \text{with} \quad 1 - \sigma^2\theta_2\|D^*PD\| \geq 0. \end{array}\right\}.$$

Chapter 4

Stability radii of stochastic systems subjected to stochastic and deterministic perturbations

4.1 Introduction

In this chapter, we consider stochastic systems with stochastic and deterministic uncertainties. Novel procedure for studying robust stability is presented. A novel characterization of the stability radius is proposed. The results generalize those of the finite dimensional case (see [17]). We procced as follows. Section 4.2 contains the mathematical formulation of the problem and basic definitions. In section 4.3 we establish some results which enable us to derive bounds for the stability radius. This bound is giving in terms of linear operator inequalities.

4.2 System description

Let A be the infinitesimal generator of an exponentially stable semigroup $S(t)$ on a real separable Hilbert space $H$and $A_0 \in \mathscr{L}(H)$.

Consider the stochastic system

$$\begin{cases} dx(t) = Ax(t)dt + A_0x(t)dw(t),\ t \geq 0, \\ x(0) = x_0. \end{cases} \tag{4.1}$$

Assume that (4.1) is L^2 stable and is subjected to structured perturbations as follows

$$\begin{cases} dx(t) = Ax(t)dt + A_0x(t)dw(t) + B\Delta(Ex(t))dt + D\Delta(Ex(t))dw(t),\ t \geq 0, \\ x(0) = x_0 \in H, \end{cases} \tag{4.2}$$

where

(H_1) $D, B \in \mathscr{L}(U,H)$ and $E \in \mathscr{L}(H,Y)$.

(H_2) $\Delta \in Lip(Y,U)$ is an unknown Lipschitzian nonlinear perturbation. The size of $\Delta \in Lip(Y,U)$ is measured by the Lipschitz norm

$$\|\Delta\|_{Lip} = inf\{\gamma > 0; \forall y,\ \ \widehat{y} \in Y : \|\Delta(y) - \Delta(\widehat{y})\|_U \leq \gamma \|y - \widehat{y}\|_Y\}.$$

(H_3) $\{w(t)\}_{t\in\mathbb{R}_+}$ is a real Wiener process on a probability space $(\ \Omega\ ,\ \mathscr{F},P\ )$ equipped with a filtration $\{\mathscr{F}_t\}_{t\geq 0}\subset\mathscr{F}$, whith variance θ.

(H_4) (B,E) determines the structure of the deterministic perturbation and (D,E) determines the structure of the stochastic perturbation.

By the asymptions H_1 H_2, there existe a unique solution x to (4.1) (see Theorem 7 page 186 in [13]).

4.3 Characterizations of the stability radius

Definition 4.3.1

The stability radius of A *whith respect to the perturbation structure* (D,B,E) *and the Wiener process* $\{w(t)\}_{t\in\mathbb{R}_+}$ *is*

$$r^w(A,A_0,D,B,E)=inf\{\|\Delta\|;\Delta\in Lip(Y,U)\ such\ that\ (4.2)\ is\ not\ L^2-stable\}.$$

Let $T>0$ and $\gamma>0$. For $\upsilon\in L^2_w(\mathbb{R}^+,L^2(\Omega,U))$, consider the following cost functional

$$J(x_0,\upsilon)=\int_0^T\mathbb{E}[\|Ex(t)\|^2-\gamma^{-2}\|\upsilon(t)\|^2]dt.$$

where $x(t)$ is the solution of the system

$$\begin{cases} dx(t)=Ax(t)dt+B\upsilon(t)dt+A_0x(t)dw(t)+D\upsilon(t)dw(t),t\geq 0,\\ x(0)=x_0.\end{cases}\tag{4.3}$$

The approach used in this section to characterize the stability radius $r^w(A,D,B,E)$ is based on the following Theorem.

Theorem 4.3.1

Let $\gamma>0$. *Suppose that there exists* $P\in L^+(H)$ *such that* $M(P)\prec 0$, *where*

$$M(P)=\begin{pmatrix} PA+A^*P+\theta A_0^*PA_0+E^*E & PB+\theta A_0^*PD\\ B^*P+\theta D^*PA_0 & -\gamma^{-2}I+\theta D^*PD\end{pmatrix}$$

then $\|L\|<\gamma^{-1}$.

Proof.

Let $x\in D(A)$ be the solution of the perturbed system (4.3). Using the Itô Formula with $F(t,x(t))=\langle x(t),Px(t)\rangle$, we get

$$\begin{aligned}\langle x(T),Px(T)\rangle &= \langle x(0),Px(0)\rangle+2\int_0^T\langle Px(t),(A_0x(t)+D\upsilon(t))dw(t)\rangle\\ &+ 2\int_0^T\langle Px(t),Ax(t)+B\upsilon(t)\rangle dt+\theta\int_0^T\langle A_0x(t),PA_0x(t)\rangle dt\\ &+ \theta\int_0^T\langle D\upsilon(t),PD\upsilon(t)\rangle dt+\theta\int_0^T\langle D\upsilon(t),PA_0x(t)\rangle dt\\ &+ \theta\int_0^T\langle A_0x(t),PD\upsilon(t)\rangle dt.\end{aligned}$$

Applying the expectation and using Theorem 6.12 in [19], we get

$$\begin{aligned}\mathbb{E}\langle x(T),Px(T)\rangle &= \mathbb{E}\langle x(0),Px(0)\rangle+\mathbb{E}\int_0^T 2\langle Px(t),Ax(t)\rangle dt+\theta\mathbb{E}\int_0^T\langle D\upsilon(t),PA_0x(t)\rangle dt\\&+ \theta\mathbb{E}\int_0^T\langle D\upsilon(t),PA_0x(t)\rangle dt+\mathbb{E}\int_0^T 2\langle Px(t),B\upsilon(t)\rangle dt\\&+ \theta\mathbb{E}\int_0^T\langle A_0x(t),PA_0x(t)\rangle dt+\theta\mathbb{E}\int_0^T\langle D\upsilon(t),PD\upsilon(t)\rangle dt,\end{aligned}.$$

then

$$\begin{aligned}\mathbb{E}\langle x(T),Px(T)\rangle &= \langle x(0),Px(0)\rangle+\mathbb{E}\int_0^T\langle PAx(t),x(t)\rangle dt+\theta\mathbb{E}\int_0^T\langle A_0^*PD\upsilon(t),x(t)\rangle dt\\&+ \theta\mathbb{E}\int_0^T\langle D^*PA_0x(t),\upsilon(t)\rangle dt+\mathbb{E}\int_0^T\langle x(t),PB\upsilon(t)\rangle dt\\&+ \theta\mathbb{E}\int_0^T\langle A_0^*PA_0x(t),x(t)\rangle dt+\theta\mathbb{E}\int_0^T\langle D^*PD\upsilon(t),\upsilon(t)\rangle dt\\&+ \mathbb{E}\int_0^T\langle A^*Px(t),x(t)\rangle dt+\mathbb{E}\int_0^T\langle B^*Px(t),\upsilon(t)\rangle dt,\end{aligned}$$

hence

$$\begin{aligned}\mathbb{E}\langle x(T),Px(T)\rangle &= \langle x(0),Px(0)\rangle+\mathbb{E}\int_0^T\langle (PA+A^*P+\theta A_0^*PA_0)x(t),x(t)\rangle dt\\&+ \mathbb{E}\int_0^T\langle (PB+\theta A_0^*PD)\upsilon(t),x(t)\rangle dt+\theta\mathbb{E}\int_0^T\langle D^*PD\upsilon(t),\upsilon(t)\rangle dt\\&+ \mathbb{E}\int_0^T\langle (B^*P+\theta D^*PA_0)x(t),\upsilon(t)\rangle dt,\end{aligned}$$

therefore

$$\mathbb{E}\langle x(T),Px(T)\rangle = \langle x(0),Px(0)\rangle+\mathbb{E}\int_0^T\langle\begin{pmatrix}x(t)\\\upsilon(t)\end{pmatrix},M_1(P)\begin{pmatrix}x(t)\\\upsilon(t)\end{pmatrix}\rangle dt,$$

where

$$M_1(P)=\begin{pmatrix}PA+A^*P+\theta A_0^*PA_0 & PB+\theta A_0^*PD\\ B^*P+\theta D^*PA_0 & \theta D^*PD,\end{pmatrix}$$

thus

$$J(x_0,\upsilon)+\mathbb{E}\langle x(T),Px(T)\rangle-\langle x(0),Px(0)\rangle = \mathbb{E}\int_0^T\langle\begin{pmatrix}x(t)\\\upsilon(t)\end{pmatrix},M_1(P)\begin{pmatrix}x(t)\\\upsilon(t)\end{pmatrix}\rangle dt\ +J(x_0,\upsilon),$$

$$=\mathbb{E}\int_0^T\langle\begin{pmatrix}x(t)\\\upsilon(t)\end{pmatrix},\begin{pmatrix}PA+A^*P+\theta A_0^*PA_0+E^*E & PB+\theta A_0^*PD\\ B^*P+\theta D^*PA_0 & -\gamma^{-2}I+\theta D^*PD\end{pmatrix}\begin{pmatrix}x(t)\\\upsilon(t)\end{pmatrix}\rangle dt,$$

Therefore

$$J(x_0,\upsilon)+\mathbb{E}\langle x(T),Px(T)\rangle-\langle x(0),Px(0)\rangle = \mathbb{E}\int_0^T\langle\begin{pmatrix}x(t)\\\upsilon(t)\end{pmatrix},M(P)\begin{pmatrix}x(t)\\\upsilon(t)\end{pmatrix}\rangle dt, \quad (4.4)$$

where

$$M(P)=\begin{pmatrix}PA+A^*P+\theta A_0^*PA_0+E^*E & PB+\theta A_0^*PD\\ B^*P+\theta D^*PA_0 & -\gamma^{-2}I+\theta D^*PD\end{pmatrix}.$$

Since $M(P) \prec 0$, it follows that

$$J(x_0,\upsilon)+\mathbb{E}\langle x(T),Px(T)\rangle-\langle x(0),Px(0)\rangle<0$$

For $x(0)=0$ we have $L\upsilon(t)=Ex(t)$, then

$$J(x_0,\upsilon)=\int_0^T \mathbb{E}[\|L\upsilon(t)\|^2-\gamma^{-2}\|\upsilon(t)\|^2]dt,$$

hence

$$\int_0^T \mathbb{E}[\|L\upsilon(t)\|^2-\gamma^{-2}\|\upsilon(t)\|^2]dt<-\mathbb{E}\langle x(T),Px(T)\rangle,$$

from which we obtain

$$\int_0^T \mathbb{E}\|L\upsilon(t)\|^2dt<\int_0^T \mathbb{E}\gamma^{-2}\|\upsilon(t)\|^2dt-\mathbb{E}\langle x(T),Px(T)\rangle$$

thus

$$\int_0^T \mathbb{E}\|L\upsilon(t)\|^2dt<\int_0^T \mathbb{E}\gamma^{-2}\|\upsilon(t)\|^2dt,$$

or

$$\|L\upsilon(t)\|^2_{L^2_w}<\gamma^{-2}\|\upsilon(t)\|^2_{L^2_w},$$

therefore $\|L\|_{L^2_w}<\gamma^{-1}$.

■

For the next, we need the following Schur Lemma.

Lemma 4.3.1

Let $X=\begin{bmatrix} S & T \\ T^* & Q \end{bmatrix}$*, where* $S\in L(H)$*,* $Q\in\mathcal{L}(H)$ *are linear Hermitian operators such that* Q *is coercive, and* $T\in L(H)$ *is a linear operator. We have*

$$X\succ 0 \Longleftrightarrow Q\succ 0 \text{ and } S-TQ^{-1}T^*\succ 0.$$

We have the following result.

Corollary 4.3.1

Let $\gamma>0$*. Assume that there exists* $P\in\mathcal{L}^+(H)$ *satisfying*

$$Q=\gamma^{-2}I-\theta D^*PD\gg 0,$$

and

$$\begin{aligned}&2\langle Px,Ax\rangle+\theta\langle PA_0x,A_0x\rangle+\langle Ex,Ex\rangle\\&-\langle(-Q)^{-1}\left(B^*Px+\theta D^*PA_0x\right),\left(B^*Px+\theta D^*PA_0x\right)\rangle<0,\quad x\in D(A).\end{aligned}$$

Then $\|L\|<\gamma^{-1}$.

Proof.

We have $Q\succ 0$ and $S-TQ^{-1}T^*\succ 0$. where

$$\begin{aligned} S &= -(PA+A^*P+\theta A_0^*PA_0+E^*E),\\ T &= -(PB+\theta A_0^*PD). \end{aligned} \tag{4.5}$$

Using Shur Lemma, we get $M(P)\prec 0$ and from the previous Theorem we conclude that $\|L\|<\gamma^{-1}$.

■

Now we obtain a bound for the stability radius.

Theorem 4.3.2 *Let $\gamma > 0$. Suppose that there exists $P \succ 0$ such that $M(P) \prec 0$ then $r^{\omega} \geq \gamma$.*

Proof.

Let Δ be such that $\|\Delta\| < \gamma$. For $T \geq 0$, the unique solution of (4.2) with initial condition $x(0) = x_0$ satisfies

$$x(T) = x_0 + \int_0^T [Ax(t) + B\Delta(Ex(t))]dt + \int_0^T [A_0x(t) + D\Delta(Ex(t))]dw(t).$$

Set $\upsilon(t) = \Delta(Ex(t))$. Since $M(P) \prec 0$ there exists $\sigma > 0$ such that $M(P) \preceq -\sigma^2 I$. By equation (4.4), we have

$$J(x_0, \upsilon) = \langle x(0), Px(0)\rangle - \mathbb{E}\langle x(T), Px(T)\rangle + \mathbb{E}\int_0^T \langle \begin{pmatrix} x(t) \\ \upsilon(t) \end{pmatrix}, M(P) \begin{pmatrix} x(t) \\ \upsilon(t) \end{pmatrix}\rangle dt,$$

then

$$J(x_0, \upsilon) \leq \langle x(0), Px(0)\rangle - \mathbb{E}\langle x(T), Px(T)\rangle - \int_0^T \sigma^2 \mathbb{E}\left(\|x(t)\|^2 + \|v(t)\|^2\right) dt,$$

which implies that

$$J(x_0, \upsilon) \leq \langle x(0), Px(0)\rangle - \mathbb{E}\langle x(T), Px(T)\rangle - \int_0^T \sigma^2 \mathbb{E}\left(\|x(t)\|^2 + \|v(t)\|^2\right) dt,$$

hence

$$\int_0^T \mathbb{E}[\|Ex(t)\|^2 - \gamma^{-2}\|\Delta(Ex(t))\|^2]dt \leq \langle x(0), Px(0)\rangle - \mathbb{E}\langle x(T), Px(T)\rangle - \int_0^T \sigma^2 \mathbb{E}\|x(t)\|^2 dt,$$

therefore

$$-\langle x(0), Px(0)\rangle + \mathbb{E}\langle x(T), Px(T)\rangle \leq \int_0^T [\gamma^{-2}\mathbb{E}\|\Delta(Ex(t))\|^2 - \mathbb{E}\|Ex(t)\|^2 - \sigma^2\mathbb{E}\|x(t)\|^2]dt.$$

Define the truncation $\upsilon_T \in L^2_w(\mathbb{R}^+, L^2(\Omega, U))$ by

$$\upsilon_T(t) = \begin{cases} v(t) = \Delta(Ex(t)) & if\ t \in [0, T], \\ 0 & if\ \ t > T, \end{cases} \tag{4.6}$$

Then

$$\begin{aligned} \|\upsilon_T(t)\|^2_{L^2_w} &= \int_0^{+\infty} \mathbb{E}(\|\upsilon_T(t)\|)^2 dt \\ &= \int_0^T \mathbb{E}(\|\upsilon(t)\|)^2 dt \\ &= \int_0^T \mathbb{E}(\|\Delta Ex(t)\|)^2 dt \\ &\leq \|\Delta\|^2_{Lip} \int_0^T (\mathbb{E}(\|Ex(t)\|)^2) dt. \end{aligned}$$

But

$$\gamma^{-2}\mathbb{E}\|\Delta(Ex(t))\|^2 \leq \gamma^{-2}\|\Delta\|^2\mathbb{E}\|Ex(t)\|^2 \leq \mathbb{E}\|Ex(t)\|^2.$$

Hence

$$-\langle x(0), Px(0)\rangle + \mathbb{E}\langle x(T), Px(T)\rangle \leq \int_0^T -\sigma^2\mathbb{E}\|x(t)\|^2 dt.$$

Thus

$$\int_0^T \sigma^2 \mathbb{E}\|x(t)\|^2 dt \leq \langle x(0), Px(0)\rangle - \mathbb{E}\langle x(T), Px(T)\rangle.$$

Therefore

$$\int_0^T \sigma^2 \mathbb{E}\|x(t)\|^2 dt \leq \langle x(0), Px(0)\rangle,$$

or

$$\int_0^T \mathbb{E}\|x(t)\|^2 dt \leq \sigma^{-2}\|P\|\|x(0)\|^2.$$

We deduce that

$$\int_0^\infty \mathbb{E}\|x(t)\|^2 dt \leq \sigma^{-2}\|P\|\|x(0)\|^2,$$

which implies that the solution $x(t)$ belongs to L^2_w. We conclude that $r^\omega \geq \gamma$.

■

Remark 4.3.1

From Theorem 4.3.1 and Theorem 4.3.2, we deduce that

$$r^\omega \geq \|L\|^{-1}$$

.

Remark 4.3.2

In the case where $A_0 = 0, B = 0$, we obtain the same result obtained in [35].

Chapter 5

New general decay rates of solutions for an abstract semilinear stochastic evolution equation with an infinite memory

5.1 Introduction

Our interest in this chapter is to analyse the asymptotic stability of the second-order stochastic evolution equation:

$$\begin{cases} u_{tt} + Au(t) - \int_0^{+\infty} h(s)A^{\alpha}u(t-s)ds + f(u(t)) = \sigma(t)W_t(t) \qquad t,s \ \text{in}[0,+\infty[, \\ u(-t) = u_0(t), \quad u_t(0) = u_1, \end{cases} \tag{5.1}$$

$A : D(A) \to H$ be a self-adjoint linear positive operator with domain $D(A) \subset H$ where $(H, \langle .,. \rangle, \|.\|)$ is a real Hilbert space and the embedding $D(A^{\gamma_0}) \hookrightarrow D(A^{\gamma_1})$ is compact for any $\gamma_0 > \gamma_1 \geq 0$.

$u : \mathbb{R}_+ \longrightarrow H$ is the displacement vector, $\alpha \in [0,1]$, the initial data (u_0, u_1) are given in suitable function spaces, $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ the kernel of memory term and the function $f : D(A^{\frac{1}{2}}) \longrightarrow H$ are subject to some assumptions to be specified later and $\sigma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a locally Lipschitz continuous function such that

$$\int_0^{+\infty} \sigma^2(t)dt < \infty. \tag{5.2}$$

W is a Q-Wiener process in H on some probability space $(\Omega, \mathscr{F}, P)$ with the variance operator Q satisfying

$$TrQ < \infty,$$

and $\{\mathscr{F}_t, t \geq 0\}$ as its natural filtration satisfying the usual conditions. Moreover, we assume that Q has the following form

$$Qe_i = \kappa_i e_i, \qquad i = 1, 2, ...,$$

where $\{\kappa_i\}$ are the eigenvalues of Q satisfying $\sum_{i=1}^{\infty} \kappa_i < \infty$ and $\{e_i\}$ are the corresponding eigenfunctions which form an orthonormal base of H. In this case

$$W(t) = \sum_{i=1}^{\infty} \kappa_i B_i(t) e_i,$$

where $\{\beta_i(t)\}$ be a sequence of real-valued one dimensional standard Brownian motions mutually independent over $(\Omega, \mathscr{F}, P)$. We first show that the system is well-posed by using the semi-group theory. Secondly, by assuming the general condition:

$$h'(t) \leq -\xi(t)h(t), \quad \forall t \geq 0,$$

where ξ is a positive function which is not necessarily monotone, we establish two stability results with decay rates depending on α and on the regularity of the initial data. Finally, we give some applications in order to illustrate our abstract results. This study improves and generalizes many previous ones in the literature. The chapter is planned as follows. In the first section, we introduce the needed assumptions and notations. In the next section, we study the well-posedness of the system. In the section5.4, we investigate the stablity of solution by using Lyapunov functionals. To illustrate our chapter result, some applications will be given in the last section.

5.2 Preliminaries

We introduce as in [12] the new variable

$$\eta(t,s) = u(t) - u(t-s), \qquad \forall t,s > 0,$$

which fulfills

$$\eta_t(t,s) + \eta_s(t,s) = u_t(t), \qquad \forall t,s > 0,$$

and so, our problem is equivalent to

$$\begin{cases} u_{tt} + Au(t) - h_0 A^{\alpha} u(t) + \displaystyle\int_0^{+\infty} h(s) A^{\alpha} \eta(t,s) ds + f(u(t)) = \sigma(t) W_t(t), \\ \eta_t(t,s) + \eta_s(t,s) = u_t(t), \\ u(-t) = u_0(t), \qquad u_t(0) = u_1, \\ \eta_0 = \eta(0,s) = u_0(0) - u_0(s), \end{cases} \tag{5.3}$$

where $h_0 = \displaystyle\int_0^{+\infty} h(s) ds < \infty$.

To establish our main results, we need the following assumptions:
(A_1) It exist three fixed positive constants a_0, a_1 and a_2 satisfying

$$\|u\|^2 \leq a_0 \|A^{\frac{\alpha}{2}} u\|^2 \qquad \forall u \in D(A^{\frac{\alpha}{2}}), \tag{5.4}$$

$$\|A^{\frac{\alpha}{2}} u\|^2 \leq a_1 \|A^{\frac{1}{2}} u\|^2 \qquad \forall u \in D(A^{\frac{1}{2}}), \tag{5.5}$$

and

$$\|A^{\frac{1}{2}} u\|^2 \leq a_2 \|A^{1-\frac{\alpha}{2}} u\|^2 \qquad \forall u \in D(A^{1-\frac{\alpha}{2}}). \tag{5.6}$$

(A_2) The function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is integrable, non-increasing and differentiable such that

$$h(0) > 0 \quad \text{and} \quad 1 - h_0 \max(a_1, a_2) = l > 0. \tag{5.7}$$

(A_3) There exists a C^0 function $\xi : \mathbb{R}_+ \to]0, +\infty[$ which is not necessarily monotone such that

$$h'(t) \leq -\xi(t) h(t) \quad \forall t \geq 0, \tag{5.8}$$

and

$$c_0 \leq \xi(t) \leq c_1 \quad \forall t \geq 0, \tag{5.9}$$

where c_0 and c_1 are two fixed positive constants.
(A_4) The function $f : D(A^{\frac{1}{2}}) \longrightarrow H$ is a global Lipschitz mapping with $f(0) = 0$ such that its potential function $F : D(A^{\frac{1}{2}}) \longrightarrow \mathbb{R}_+$ (i.e $DF = f$) satisfies

$$F(u) \leq \langle f(u), u \rangle . \tag{5.10}$$

Remark 5.2.1

i. The existence of the constants a_0, a_1 and a_2 is guaranteed by the compactness of the embedding $D(A^{\gamma_0}) \hookrightarrow D(A^{\gamma_1})$ for any $\gamma_0 > \gamma_1 \geq 0$.
ii. The condition $1 - h_0 \max(a_1, a_2)$ guarantees the positivity of the energy functional E and the modified energy functional E^* defined below.
iii. It follows from (A_4) that it exists a fixed positive constant L satisfying

$$\|f(u)\| \leq L\|u\| \quad \forall u \in D(A^{\frac{1}{2}}). \tag{5.11}$$

5.3 Well-posedness

In this section, we prove the existence and the uniqueness of solution for the problem (5.3). To this aim, we define the space

$$L^2_h\Big(\mathbb{R}_+, D(A^{\frac{\alpha}{2}})\Big) = \left\{ \eta : \mathbb{R}_+ \times \mathbb{R}_+ \to D(A^{\frac{\alpha}{2}}), \int_0^{+\infty} h(s) \|A^{\frac{\alpha}{2}} \eta(t,s)\|^2 ds < +\infty \right\},$$

which is a Hilbert space with respect to the following inner product

$$\langle u, v \rangle_{L^2_h\left(\mathbb{R}_+, D(A^{\frac{\alpha}{2}})\right)} = \int_0^{+\infty} h(s) \langle A^{\frac{\alpha}{2}} u(s), A^{\frac{\alpha}{2}} v(s) \rangle ds.$$

Consequently, the space

$$\mathscr{H} = D(A^{\frac{1}{2}}) \times H \times L^2_h\Big(\mathbb{R}_+, D(A^{\frac{\alpha}{2}})\Big),$$

equipped with the inner product

$$\langle U_1, U_2 \rangle_{\mathscr{H}} = \langle u_1, u_2 \rangle_{D(A^{\frac{1}{2}})} - h_0 \langle u_1, u_2 \rangle_{D(A^{\frac{\alpha}{2}})} + \langle v_1, v_2 \rangle + \langle \eta_1, \eta_2 \rangle_{L^2_h\left(\mathbb{R}_+, D(A^{\frac{\alpha}{2}})\right)}. \tag{5.12}$$

is also Hilbert space
Now, let $U(t) = (u, v, \eta)^T$, where $v = u_t$. Therefore, our system (5.3) can be rewritten abstractly as

$$\begin{cases} dU(t) = \big[\mathscr{A} U(t) + \mathscr{G}(U(t))\big] dt + \sigma_0(t) dW(t), \\ U(0) = U_0 = (u_0, u_1, \eta_0)^T, \end{cases} \tag{5.13}$$

where

$$\mathscr{A} \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -(A - h_0 A^{\alpha}) u - \displaystyle\int_0^{+\infty} h(s) A^{\alpha} \eta(., s) ds \\ v - \dfrac{\partial \eta}{\partial s} \end{pmatrix},$$

$$D(\mathscr{A}) = \left\{ \begin{array}{c} (u,v,\eta)^T \in \mathscr{H}, \\ (A - h_0 A^{\alpha})u + \int_0^{+\infty} h(s) A^{\alpha} \eta(.,s) ds \in H, \quad v \in D(A^{\frac{1}{2}}), \\ \frac{\partial \eta}{\partial s} \in L_h^2\Big(\mathbb{R}_+, D(A^{\frac{\alpha}{2}})\Big), \quad \eta(.,0) = 0 \end{array} \right\},$$

$$\sigma_0(t) = \begin{pmatrix} 0 \\ \sigma(t) \\ 0 \end{pmatrix},$$

and

$$\mathscr{G}(U(t)) = \begin{pmatrix} 0 \\ -f(u(t)) \\ 0 \end{pmatrix}.$$

The well-posedness result of problem (5.13) is the following:

Theorem 5.3.1

Assume that (5.2), (A_1), (A_2) and (A_4) hold. Then, for any initial datum $U_0 \in \mathscr{H}$, the system (5.13) has a unique mild solution $U \in C\big(0,T; L_p(\Omega, \mathscr{F}, \mu); \mathscr{H}\big)$.

Proof.

To prove the result given in Theorem 5.3.1 we will apply Theorem 2.1 in [32]. So, it suffices to show that the linear operator $\mathscr{A}$ generates a linear $C_0 - semigroup$ $\{S(t)\}_{t \geq 0}$ on $\mathscr{H}$. For that, let $U = (u,v,\eta)^T \in D(\mathscr{A})$, then

$$\langle \mathscr{A}U, U \rangle_{\mathscr{H}} = \langle v, u \rangle_{D(A^{\frac{1}{2}})} - h_0 \langle v, u \rangle_{D(A^{\frac{\alpha}{2}})} + \int_0^{+\infty} h(s) \langle v - \frac{\partial \eta}{\partial s}, \eta \rangle_{D(A^{\frac{\alpha}{2}})} ds$$
$$- \Big\langle (A - h_0 A^{\alpha})u - \int_0^{+\infty} h(s) A^{\alpha} \eta(s) ds, v \Big\rangle,$$

thanks to the definition of $A^{\frac{1}{2}}$ and $A^{\frac{\alpha}{2}}$, one has

$$\langle \mathscr{A}U, U \rangle_{\mathscr{H}} = -\int_0^{+\infty} h(s) \langle \frac{\partial \eta}{\partial s}, \eta \rangle_{D(A^{\frac{\alpha}{2}})} ds.$$

An integration by parts, using the fact that $\eta(0) = 0$, gives us

$$-\int_0^{+\infty} h(s) \Big\langle \frac{\partial \eta}{\partial s}, \eta \Big\rangle_{D(A^{\frac{\alpha}{2}})} ds \leq \frac{1}{2} \int_0^{+\infty} h'(s) \|A^{\frac{\alpha}{2}} \eta(s)\|^2 ds,$$

from which follows

$$\langle \mathscr{A}U, U \rangle \leq 0,$$

since h is nonincreasing. Consequently, the operator $\mathscr{A}$ is dissipative.

Now, we prove that $\kappa I - \mathscr{A}$ is surjective. Given $(g_1, g_2, g_3)^T \in \mathscr{H}$, we show that there exists $U = (u,v,\eta)^T \in D(\mathscr{A})$ satisfying

$$(\kappa I - \mathscr{A})(u,v,\eta)^T = (g_1, g_2, g_3)^T,$$

that is,

$$\left\{ \begin{array}{ll} \kappa u - v & = g_1, \\ \kappa v + (A - h_0 A^{\alpha})u - \int_0^{+\infty} h(s) A^{\alpha} \eta(s) ds & = g_2, \\ \kappa \eta - v - \frac{\partial \eta}{\partial s} & = g_3. \end{array} \right. \tag{5.14}$$

Suppose that we have found u. Then, we have

$$v = \kappa u - g_1. \tag{5.15}$$

Furthermore, one can easily show that Eq.(5.14)$_3$ with $\eta(0) = 0$ has a unique solution given by

$$\eta(s) = e^{-\kappa s}\int_0^s e^{\kappa y}(g_3(y) - g_1 + \kappa u)dy. \tag{5.16}$$

Plugging (5.15) and (5.16) into (5.14)$_2$, we can get

$$(A - \Psi_1 A^\alpha + \Psi_2 I)u = g_4, \tag{5.17}$$

where

$$\Psi_1 = h_0 - \kappa\int_0^\infty h(s)e^{-\kappa s}\Big(\int_0^s e^{\kappa y}\Big)dyds = \int_0^\infty h(s)e^{-\kappa s}ds, \qquad \Psi_2 = \kappa^2,$$

and

$$g_4 = g_2 + \lambda g_1 - \int_0^\infty h(s)e^{-\lambda s}\int_0^s A^\alpha e^{-\lambda y}(g_3(y) - g_1)dyds.$$

We should prove that (5.17) has a solution $u \in D(A^{\frac{1}{2}})$ and then replace in (5.15), (5.16) in order to get that $U \in D(\mathscr{A})$ satisfying (5.14). We have $\Psi_1 < h_0$, by (5.7) and (5.5), we conclude that $A - \Psi_1 A^\alpha$ is a positive definite operator. Then, taking the duality brackets $\langle .,. \rangle_{D(A^{\frac{1}{2}})' \times D(A^{\frac{1}{2}})}$ with $z \in D(A^{\frac{1}{2}})$, we obtain the following problem which is equivalent to (5.17)

$$\Lambda(u,z) = \mathscr{I}(z),$$

where the bilinear form $\Lambda : D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \to \mathbb{R}$ and the linear form $\mathscr{I} : D(A^{\frac{1}{2}}) \to \mathbb{R}$ are defined by

$$\Lambda(u,z) = \langle (A - \Psi_1 A^\alpha + \Psi_2 I)u, z \rangle_{D(A^{\frac{1}{2}})' \times D(A^{\frac{1}{2}})},$$

and

$$\mathscr{I}(z) = \langle g_4, z \rangle_{D(A^{\frac{1}{2}})' \times D(A^{\frac{1}{2}})}.$$

The bilinear form Λ is continuous and coercive on $D(A^{\frac{1}{2}})$. Indeed, we have

$$\left|\left\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}z \right\rangle\right| - \Psi_1\left|\left\langle A^{\frac{\alpha}{2}}u, A^{\frac{\alpha}{2}}z \right\rangle\right| + \Psi_2\langle u,z \rangle \le C\|u\|_{D(A^{\frac{1}{2}})}\|z\|_{D(A^{\frac{1}{2}})'},$$

and for $z = u \in D(A^{\frac{1}{2}})$ one can easily verify

$$\left|\left\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}u \right\rangle\right| - \Psi_1\left|\left\langle A^{\frac{\alpha}{2}}u, A^{\frac{\alpha}{2}}u \right\rangle\right| + \Psi_2\|u\|^2 \ge (1 - a\Psi_1)\left\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}u \right\rangle \ge c\|u\|^2_{D(A^{\frac{1}{2}})}.$$

Thus, applying the Lax-Milgram theorem and classical regularity arguments, we conclude that (5.14) has a unique solution $u \in D(\mathscr{A}^{\frac{1}{2}}))$ satisfying (5.14). Using (5.16), we obtain that

$$\left((A - A^{\frac{\alpha}{2}})u + \int_0^{+\infty} h(s)A^{\frac{\alpha}{2}}u(s)ds\right) \in H.$$

In conclusion, we have found $U = (u, v, \eta)^T \in D(\mathscr{A})$, which verifies (5.14), and thus, $\kappa I - \mathscr{A}$ is surjective for all $\kappa > 0$. Then, the Lumer-Phillips Theorem implies that $\mathscr{A}$ is an infinitesimal generator of a strongly continuous semigroup of contraction $\{S(t)\}_{t\ge 0}$ in $\mathscr{H}$.

■

5.4 Stability

In this section, we study the asymptotic behavior of solutions of the system (5.3). We start our investigation by introducing the following approximating system to system (5.13)

$$\begin{cases} dU(t) = [\mathscr{A}U(t) + R(\lambda)\mathscr{G}(U(t))]dt + R(\lambda)\sigma_0(t)dW(t), \\ U(0) = R(\lambda)U_0, \end{cases} \tag{5.18}$$

where $\lambda \in \rho(\mathscr{A})$ the resolvent set of $\mathscr{A}$, $R(\lambda) = \lambda R(\lambda, \mathscr{A})$ and $R(\lambda, \mathscr{A})$ is the resolvent of $\mathscr{A}$ given by $R(\lambda, \mathscr{A}) = (\lambda I - \mathscr{A})^{-1}$.

We have the following results regarding the existence and uniqueness of strong solution of system (5.18) and its relation to the mild solution of system (5.13)

Lemma 5.4.1 *[32]*

The stochastic differential equation (5.18) *has a unique strong solution* $U(t,\lambda)$ *which lies in* $C\big(0,T;L_p(\Omega,\mathscr{F},\mu);\mathscr{H}\big)$ *for all* T *and* $p \geqslant 2$*. Moreover,* $U(t,\lambda)$ *converges to the mild solution of* (5.13) *in* $C\big(0,T;L_p(\Omega,\mathscr{F},\mu);\mathscr{H}\big)$ *as* $\lambda \to \infty$ *for all* T *and* $p \geqslant 2$.

Then, system (5.18) can be rewritten as follows:

$$\begin{cases} du(t) = vdt, \\ dv(t) = \Big[-Au(t) + h_0 A^{\alpha} u(t) - \displaystyle\int_0^{+\infty} h(s) A^{\alpha}\eta(t,s)ds - R(\lambda) f(u(t))\Big]dt + R(\lambda)\sigma(t)dW(t), \\ d\eta(t,s) = \Big[v(t) - \dfrac{\partial \eta(t,s)}{\partial s}\Big]dt, \\ \eta_t(t,s) + \eta_s(t,s) = v(t), \\ u(-t) = u_0, \quad u_t(0) = u_1, \\ \eta(0,s) = u_0(0) - u_0(s). \end{cases} \tag{5.19}$$

Now, we define the energy functional corresponding to the solution of (5.19) as:

$$E_R(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|A^{\frac{1}{2}}u(t)\|^2 - \frac{h_0}{2}\|A^{\frac{\alpha}{2}}u(t)\|^2 + \frac{1}{2}\int_0^{+\infty} h(s)\|A^{\frac{\alpha}{2}}\eta(t,s)\|^2 ds + R(\lambda)F(u(t)). \tag{5.20}$$

Next, we shall establish a bound on the derivative of the modified energy functional E_R. So, we have the following estimate.

Lemma 5.4.2

Let $U(t,\lambda)$ *be the unique strong solution to the equation* (5.18)*. Then, the corresponding energy function satisfies the following estimate*

$$\frac{d}{dt}\mathbb{E}(E_R(t)) \leq \frac{1}{2}\mathbb{E}\left(\int_0^{+\infty} h'(s)\|A^{\frac{\alpha}{2}}\eta(t,s)\|^2 ds\right) + \frac{1}{2}\sum_{i=1}^{+\infty}\mathbb{E}\big(\kappa_i\sigma^2(t)\langle R(\lambda)e_i, R(\lambda)e_i\rangle\big), \tag{5.21}$$

where $\mathbb{E}$ *denotes the expectation.*

Proof.

Applying Ito's formula in the Hilbert space to $\|v(t)\|^2$, we get

$$\begin{aligned}\|v(t)\|^2 = \|v(0)\|^2 - 2\int_0^t \langle v(s), Au(s)\rangle ds + 2\int_0^t \langle v(s), h_0 A^{\alpha} u(s)\rangle ds \\ -2\int_0^t \langle v(s), \int_0^{+\infty} h(s)A^{\alpha}\eta(t,s)ds\rangle ds + \int_0^t \langle v(s), R(\lambda)\sigma(s)dw(s)\rangle \\ +\sum_{i=1}^{+\infty}\kappa_i\int_0^t \sigma^2(s)\langle R(\lambda)e_i, R(\lambda)e_i\rangle ds - 2R(\lambda)F(u(t)),\end{aligned} \tag{5.22}$$

that is,

$$\begin{aligned}\|v(t)\|^2 =& \|v(0)\|^2 + \|A^{\frac{1}{2}}u(0)\|^2 - h_0\|A^{\frac{\alpha}{2}}u(0)\|^2 - \|A^{\frac{1}{2}}u(t)\|^2 + h_0\|A^{\frac{\alpha}{2}}u(t)\|^2 \\ &-2\int_0^t \langle A^{\frac{\alpha}{2}}v(s), \int_0^{+\infty} h(s)A^{\frac{\alpha}{2}}\eta(t,s)ds\rangle dt + \int_0^t \langle v(s), R(\lambda)\sigma(s)dw(s)\rangle \\ &+\sum_{i=1}^{+\infty}\kappa_i\int_0^t \sigma^2(s)\langle R(\lambda)e_i, R(\lambda)e_i\rangle ds - 2R(\lambda)F(u(t)).\end{aligned} \tag{5.23}$$

Using $\eta(t,s) = u(t) - u(t-s)$, integrating by parts the first integral in (5.23) and then taking the expectation of both sides of the resulting equation we obtain (5.21) after recalling (5.20).
■

Remark 5.4.1

Let

$$S_R(t) = \frac{1}{2}\sum_{i=1}^{+\infty}\mathbb{E}\left(\kappa_i\int_0^t \sigma^2(s)\langle R(\lambda)e_i, R(\lambda)e_i\rangle ds\right).$$

Then it follows from (5.2) *that*

$$S_R(\infty) = S_{R1} < \infty. \tag{5.24}$$

By integrating (5.21) *over* $(0,t)$, *we get*

$$\mathbb{E}(E_R(t)) \leq \mathbb{E}(E_{R_1}(0)) + S_{R1}. \tag{5.25}$$

Lemma 5.4.3

Let $U(t,\lambda)$ *be a strong solution of* (5.18). *Then the functional*

$$\phi(t) = \langle u_t(t), u(t)\rangle,$$

satisfies, for all $t \geq 0$,

$$\begin{aligned}\frac{d}{dt}\phi(t) \leq& \|u_t\|^2 - \frac{l}{2}\|A^{\frac{1}{2}}u(t)\|^2 + \frac{a_1^2 h_0}{2l}\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ &-R(\lambda)F(u(t)) + \langle R(\lambda)\sigma(t)W_t(t), u(t)\rangle,\end{aligned} \tag{5.26}$$

and moreover

$$\frac{d}{dt}\mathbb{E}\big(\phi(t)\big) \leq \mathbb{E}\big(\|u_t\|^2\big) - \frac{l}{2}\mathbb{E}\big(\|A^{\frac{1}{2}}u(t)\|^2\big) + \frac{a_1^2 h_0}{2l}\mathbb{E}\left(\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds\right) - \mathbb{E}\big(R(\lambda)F(u(t))\big). \tag{5.27}$$

Proof.

Taking the derivative of ϕ, we have

$$\frac{d}{dt}\phi(t) = \|u_t\|^2 + \langle u_{tt}(t), u(t)\rangle.$$

Using (5.18) and the definition of $A^{\frac{1}{2}}$ and $A^{\frac{\alpha}{2}}$, we obtain

$$\begin{aligned}\frac{d}{dt}\phi(t) = \|u_t(t)\|^2 - \|A^{\frac{1}{2}}u(t)\|^2 + h_0\|A^{\frac{\alpha}{2}}u(t)\|^2 - \langle \int_0^{+\infty} h(s)A^{\frac{\alpha}{2}}\eta(t,s)ds, A^{\frac{\alpha}{2}}u(t)\rangle \\ - \langle R(\lambda)f(u(t)), u(t)\rangle + \langle R(\lambda)\sigma(t)W_t(t), u(t)\rangle.\end{aligned} \tag{5.28}$$

By Cauchy-Schwarz's inequality, Young's inequality and (5.5), it follows that

$$\begin{aligned}-\Big\langle \int_0^{+\infty} h(s)A^{\frac{\alpha}{2}}\eta(t,s)ds, A^{\frac{\alpha}{2}}u(t)\Big\rangle &\le \frac{l}{2a_1}\|A^{\frac{\alpha}{2}}u(t)\|^2 + \frac{a_1}{2l}\left(\int_0^{+\infty}\sqrt{h(s)}\sqrt{h(s)}\|A^{\frac{\alpha}{2}}\eta(t,s)\|ds\right)^2 \\ &\le \frac{l}{2}\|A^{\frac{1}{2}}u(t)\|^2 + \frac{a_1^2 h_0}{2l}\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds.\end{aligned}$$

Substituting this last estimate into (5.28) and using (5.10), we can get

$$\begin{aligned}\frac{d}{dt}\phi(t) \le \|u_t\|^2 - \frac{l}{2}\|A^{\frac{1}{2}}u(t)\|^2 + \frac{a_1^2 h_0}{2l}\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ - R(\lambda)F(u(t)) + \langle R(\lambda)\sigma(t)W_t(t), u(t)\rangle.\end{aligned} \tag{5.29}$$

Taking the expectation of (5.29), we find the desired result (5.27).

■

Lemma 5.4.4

Le $U(t,\lambda)$ be a strong solution of (5.18). Then, the functional

$$\psi(t) = -\langle u_t(t), \int_0^{+\infty} h(s)\eta(t,s)ds\rangle,$$

satisfies, for any $\delta_1, \delta_2 > 0$, an estimate of the form

$$\begin{aligned}\frac{d}{dt}\psi(t) \le \delta_1\|A^{\frac{1}{2}}u\|^2 - \big(h_0 - \delta_2\big)\|u_t\|^2 + c_2\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 \\ - c_3\int_0^{+\infty} h'(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 - \langle R(\lambda)\sigma(t)W_t(t), \int_0^{+\infty} h(s)\eta(t,s)ds\rangle,\end{aligned} \tag{5.30}$$

where

$$c_2 = c_2(\delta_1) = \left(\frac{2 + 2a_1^2 + ca_0a_1L^2}{2\delta_1} + a_1\right)h_0, \qquad c_3 = \frac{h(0)}{4a_0\delta_2}.$$

Moreover

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\big(\psi(t)\big) \le \delta_1\mathbb{E}\big(\|A^{\frac{1}{2}}u\|^2\big) - \big(h_0 - \delta_2\big)\mathbb{E}\big(\|u_t\|^2\big) + c_2\mathbb{E}\left(\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds\right) \\ - c_3\mathbb{E}\left(\int_0^{+\infty} h'(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds\right).\end{aligned} \tag{5.31}$$

Proof.

A direct computation gives

$$\frac{d}{dt}\psi(t) = -\langle u_{tt}(t), \int_0^{+\infty} h(s)\eta(t,s)ds\rangle + \langle u_t(t), \int_0^{+\infty} h(s)\eta_s(t,s)ds\rangle - h_0\|u_t\|^2.$$

Integrating by parts with respect to s the second term on the right hand side of the above equality and using the fact that

$$\lim_{s\to+\infty} h(s) = 0,$$

$$\eta(t,0) = 0,$$

we get

$$\frac{d}{dt}\psi(t) = -\langle u_{tt}(t), \int_0^{+\infty} h(s)\eta(t,s)ds\rangle - \langle u_t(t), \int_0^{+\infty} h^{'}(s)\eta(t,s)ds\rangle - h_0\|u_t\|^2.$$

Using (5.18) and the definition of $A^{\frac{1}{2}}$ and $A^{\frac{\alpha}{2}}$, we obtain that

$$\begin{aligned}\frac{d}{dt}\psi(t) =& \underbrace{\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)A^{\frac{1}{2}}\eta(t,s)ds\rangle}_{I_1} \underbrace{-h_0\langle A^{\frac{\alpha}{2}}u(t), \int_0^{+\infty} h(s)A^{\frac{\alpha}{2}}\eta(t,s)ds\rangle}_{I_2} \\ & \underbrace{-\langle R(\lambda)f(u(t)), \int_0^{+\infty} h(s)A^{\frac{\alpha}{2}}\eta(t,s)ds\rangle}_{I_3} \\ & - \langle R(\lambda)\sigma(t)W_t(t), \int_0^{+\infty} h(s)\eta(t,s)ds\rangle \underbrace{-\langle u_t(t), \int_0^{+\infty} h^{'}(s)\eta(t,s)ds\rangle}_{I_4} \\ & - h_0\|u_t\|^2 + \underbrace{\Big\langle \int_0^{+\infty} h(s)A^{\frac{\alpha}{2}}\eta(t,s)ds, \int_0^{+\infty} h(s)A^{\frac{\alpha}{2}}\eta(t,s)ds\Big\rangle}_{I_5}.\end{aligned} \tag{5.32}$$

Making use of Cauchy-Schwarz's inequality, Young's inequality, (5.4) and (5.5), we get

$$I_1 \le \frac{\delta_1}{4}\|A^{\frac{1}{2}}u(t)\|^2 + \frac{h_0}{\delta_1}\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds, \tag{5.33}$$

$$\begin{aligned}I_2 &\le \frac{\delta_1}{a_1}\|A^{\frac{\alpha}{2}}u(t)\|^2 + \frac{a_1h_0}{\delta_1}\int_0^{+\infty} h(s)\|A^{\frac{\alpha}{2}}\eta(t,s)\|^2 ds \\ &\le \frac{\delta_1}{4}\|A^{\frac{1}{2}}u(t)\|^2 + \frac{a_1^2h_0}{\delta_1}\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds.\end{aligned} \tag{5.34}$$

$$\begin{aligned}I_3 &\le \frac{\delta_1}{2a_0a_1L^2}\|f(u(t))\|^2 + \frac{ca_0a_1L^2h_0}{2\delta_1}\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ &\le \frac{\delta_1}{2}\|u(t)\|^2 + \frac{ca_0a_1L^2h_0}{2\delta_1}\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ &\le \frac{\delta_1}{2}\|A^{\frac{1}{2}}u(t)\|^2 + \frac{ca_0a_1L^2h_0}{2\delta_1}\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds,\end{aligned} \tag{5.35}$$

$$\begin{aligned} I_4 &\leq \delta_2\|u_t(t)\|^2+\frac{1}{4\delta_2}\Big(\int_0^{+\infty}\sqrt{-h'(s)}\sqrt{-h'(s)}\|\eta(t,s)\|^2ds\Big)^2 \\ &\leq \delta_2\|u_t(t)\|^2-\frac{h(0)}{4a_0\delta_2}\int_0^{+\infty}h'(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2ds, \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} I_5 &\leq \Big(\int_0^{+\infty}\sqrt{h(s)}\sqrt{h(s)}\|A^{\frac{\alpha}{2}}\eta(t,s)\|ds\Big)^2 \\ &\leq a_1h_0\int_0^{+\infty}h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2ds. \end{aligned} \tag{5.37}$$

Insterting (5.33)-(5.37) into (5.32), we obtain

$$\begin{aligned} \frac{d}{dt}\psi(t) \leq\ &\delta_1\|A^{\frac{1}{2}}u\|^2-\big(h_0-\delta_2\big)\|u_t\|^2+c_2\int_0^{+\infty}h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 \\ &-c_3\int_0^{+\infty}h'(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2-\langle R(\lambda)\sigma(t)W_t(t),\int_0^{+\infty}h(s)\eta(t,s)ds\rangle. \end{aligned} \tag{5.38}$$

The desired estimate follows after taking the expectation of both sides of (5.38).

■

In order to state our stability results, we distinguish tow cases.

5.4.1 The first case: $\alpha = 1$

In this subsection we study the stability problem for (0.1) when $\alpha = 1$. For that, we define a Lyapunov functional $\mathscr{LY}$ as follows

$$\mathscr{LY}(t)=\mathbb{E}\Big(NE_R(t)+N_1\phi(t)+N_2\psi(t)\Big),$$

where N, N_1 and N_2 are positive constants to be selected later.

Lemma 5.4.5

For a suitable choice of N and N_i, $i=1,2$, there exist positive constants c_4, c_5 and m_0 such that the functional $\mathscr{LY}$ satisfies

$$c_4\mathbb{E}\big(E_R(t)\big)\leq\mathscr{LY}(t)\leq c_5\mathbb{E}\big(E_R(t)\big) \tag{5.39}$$

and

$$\frac{d}{dt}\mathscr{LY}(t)\leq -m_0\mathbb{E}\big(E_R(t)\big)+\frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\big(\kappa_i\sigma^2(s)\langle R(\lambda)e_i,R(\lambda)e_i\rangle\big). \tag{5.40}$$

Proof.

It is not hard to establish (5.39). Then, combining (5.21), (5.27) and (5.31), one has, for all $t\geq 0$,

$$\begin{aligned} \frac{d}{dt}\mathscr{LY}(t)\leq &-\Big[N_2\big(h_0-\delta_2\big)-N_1\Big]\mathbb{E}(\|u_t\|^2)-\Big[\frac{N_1l}{2}-N_2\delta_1\Big]\mathbb{E}\|A^{\frac{1}{2}}u\|^2 \\ &-N_1\mathbb{E}\big(R(\lambda)F(u(t))\big)+\Big[\frac{N}{2}-c_3N_2\Big]\mathbb{E}\Big(\int_0^{+\infty}h'(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2ds\Big) \\ &+\Big[\frac{a_1^2h_0}{2l}N_1+c_2N_2\Big]\mathbb{E}\Big(\int_0^{+\infty}h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2ds\Big) \\ &+\frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\big(\kappa_i\sigma^2(s)\langle R(\lambda)e_i,R(\lambda)e_i\rangle\big). \end{aligned}$$

At the first, we take $\delta_2 < h_0$, and N_2 large enough so that

$$N_2\big(h_0-\delta_2\big)-N_1>0.$$

As long as N_2 is fixed, we pick δ_1 sufficiently small such that

$$\frac{N_1 l}{2}-N_2\delta_1>0.$$

Thus, we can find a positive constant m_0 such that

$$\begin{aligned}\frac{d}{dt}\mathscr{L}\mathscr{Y}(t)\le&-m_0\mathbb{E}\big(E_R(t)\big)+\left[\frac{N}{2}-c_3N_2\right]\mathbb{E}\left(\int_0^{+\infty}h'(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2ds\right)\\&+\left[m_0+\frac{a_1^2h_0}{2l}N_1+c_2N_2\right]\mathbb{E}\left(\int_0^{+\infty}h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2ds\right)\\&+\frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\big(\kappa_i\sigma^2(s)\langle R(\lambda)e_i,R(\lambda)e_i\rangle\big),\end{aligned}\tag{5.41}$$

which, together with the assumption (A_3) and (5.9), gives

$$\begin{aligned}\frac{d}{dt}\mathscr{L}\mathscr{Y}(t)\le&-m_0\mathbb{E}\big(E_R(t)\big)+m_1\,\mathbb{E}\left(\int_0^{+\infty}h'(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2ds\right)\\&+\frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\big(\kappa_i\sigma^2(s)\langle R(\lambda)e_i,R(\lambda)e_i\rangle\big),\end{aligned}$$

where

$$m_1=\frac{N}{2}-c_3N_2-\frac{1}{c_0}\left(m_0+\frac{a_1^2h_0}{2l}N_1+c_2N_2\right).$$

Then, we choose N large enough so that

$$m_1\ge 0.$$

This finishes the proof.

■

To state our stability results, we need the following additional assumption:
(A_5) $S_R(t)\le\frac{c}{N^{\varpi}}$, $\varpi>1$.

Theorem 5.4.1

Assume that (5.2) and (A_1)-(A_5) are fulfilled. Then, there exist three fixed positive constants b_0, b_1 and b_2 such that the solution of (0.1) satisfies

$$\mathbb{E}\big(E(t)\big)\le b_0exp\left(-b_1\int_0^t\xi(s)ds\right)+b_2S_1,\tag{5.42}$$

where

$$\begin{aligned}E(t)=&\frac{1}{2}\|u_t(t)\|^2+\frac{1}{2}\|A^{\frac{1}{2}}u(t)\|^2-\frac{h_0}{2}\|A^{\frac{1}{2}}u(t)\|^2+\frac{1}{2}\int_0^{+\infty}h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2ds\\&+F(u(t)),\end{aligned}$$

$$S(t)=\frac{1}{2}\sum_{i=1}^{+\infty}\mathbb{E}\left(\kappa_i\int_0^t\sigma^2(s)\langle e_i,e_i\rangle ds\right),$$

and by (5.2)

$$S(\infty)=S_1<\infty.$$

Proof.

Because of lemma 5.4.1 and the dominated convergence theorem, it is sufficient to prove the result for the strong solution. It follows from (5.40) that

$$\frac{d}{dt}\Big(\mathscr{L}\mathscr{Y}-\frac{N}{2}S_R\Big)(t)\leq -m_0\mathbb{E}\big(E_R(t)\big),$$

and so

$$\frac{d}{dt}\Big(\mathscr{L}\mathscr{Y}-\frac{N}{2}S_R\Big)(t)\leq -m_0\Big(\mathbb{E}\big(E_R\big)-\frac{N}{2c_5}S_R\Big)(t),$$

using (5.39) and (5.9), we can get

$$\frac{d}{dt}\Big(\mathscr{L}\mathscr{Y}-\frac{N}{2}S_R\Big)(t)\leq -b_1\xi(t)\Big(\mathscr{L}\mathscr{Y}-\frac{N}{2}S_R\Big)(t),$$

where $b_1=\frac{m_0}{c_0c_5}$. Thanks to (A_5), we have that $(\mathscr{L}\mathscr{Y}-\frac{N}{2}S_R)(t)>0$, for all $t\geq 0$. Then, an integration over $(0,t)$ gives

$$\Big(\mathscr{L}\mathscr{Y}-\frac{N}{2}S_R\Big)(t)\leq \mathscr{L}\mathscr{Y}(0)\exp\Big(-b_1\int_0^t\xi(s)ds\Big),$$

which, together with (5.24), implies

$$\mathscr{L}\mathscr{Y}(t)\leq \mathscr{L}\mathscr{Y}(0)\exp\Big(-b_1\int_0^t\xi(s)ds\Big)+\frac{N}{2}S_{R1}.$$

This gives us when combined with (5.39)

$$\mathbb{E}\big(E_R(t)\big)\leq b_0\exp\Big(-b_1\int_0^t\xi(s)ds\Big)+b_2S_{R1},$$

where $b_0=\frac{\mathscr{L}\mathscr{Y}(0)}{c_4}$ and $b_2=\frac{N}{2c_4}$.

■

5.4.2 The second case: $0\leq\alpha<1$

Here in this subsection we investigate the asymptotic stability of system (0.1) with $f\equiv 0$ in case when $0\leq\alpha<1$. So, we have the following linear system

$$\begin{cases} dU(t)=\mathscr{A}U(t)dt+\sigma_0(t)dW(t),\\ U(0)=U_0.\end{cases}\tag{5.43}$$

Assume that $U_0\in D(\mathscr{A})$, hence by Theorem 3.2 pp 81 in [19], the system (5.43) has a unique strong solution. Defining then a modified energy functional corresponding to the solution of (5.43) by

$$E^*(t)=\frac{1}{2}\|A^{\frac{1-\alpha}{2}}u_t(t)\|^2+\frac{1}{2}\|A^{1-\frac{\alpha}{2}}u(t)\|^2-\frac{h_0}{2}\|A^{\frac{1}{2}}u(t)\|^2+\frac{1}{2}\int_0^{+\infty}h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2ds.\tag{5.44}$$

Lemma 5.4.6

Let U be a strong solution of (5.43). Then, the modified energy functional satisfies the following estimate

$$\frac{d}{dt}\mathbb{E}(E^*(t)) \leq \frac{1}{2}\mathbb{E}\Big(\int_0^{+\infty} h'(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds\Big) + \frac{1}{2}\sum_{i=1}^{+\infty}\mathbb{E}\Big(\lambda_i\langle A^{\frac{1-\alpha}{2}}e_i, A^{\frac{1-\alpha}{2}}e_i\rangle\sigma^2(t)\Big). \tag{5.45}$$

Proof.

Applying Itô's formula to $\|A^{\frac{1-\alpha}{2}}v(t)\|^2$ and proceeding in the same way as in the proof of Lemma 5.4.2 we obtain (5.45).

■

Remark 5.4.2

Define

$$Z(t) = \frac{1}{2}\sum_{i=1}^{+\infty}\mathbb{E}\Big(\kappa_i\int_0^t\langle A^{\frac{1-\alpha}{2}}e_i, A^{\frac{1-\alpha}{2}}e_i\rangle\sigma^2(s)ds\Big).$$

Then, it results from (5.2) that

$$Z(\infty) = Z_1 < \infty, \tag{5.46}$$

and so

$$\mathbb{E}(E^*(t)) \leq \mathbb{E}(E^*(0)) + Z_1. \tag{5.47}$$

The stability result in the case of $0 \leq \alpha < 1$ is given by the following Theorem.

Theorem 5.4.2

Assume that (5.2), (A_1)-(A_3) and (A_5) are fulfilled. Then, there exist two fixed positive constants b_3 and b_4 such that the solution of (0.1) satisfies

$$\mathbb{E}\big(E(t)\big) \leq \Big(b_3 + b_4\big(S_1 + Z_1\big)\Big)\Big(\int_0^t \xi(s)ds\Big)^{-1} + S_1. \tag{5.48}$$

Proof.

As usual, our proof is based on the construction a Lyapunov functional $\mathscr{LY}_1$ given by

$$\mathscr{LY}_1(t) = \mathbb{E}\Big(N\big(E+E^*\big)(t) + N_1\phi(t) + N_2\psi(t)\Big).$$

Before going further, it should be noticed that $\mathscr{LY}_1$ and $\mathbb{E}(E_1)$ are not equivalent. Then, gathering the estimates (5.21), (5.27), (5.31) and (5.45), we obtain, for all $t \geq 0$, that

$$\begin{aligned}\frac{d}{dt}\mathscr{LY}_1(t) \leq & -\Big[N_2\big(h_0-\delta_3\big) - N_1\Big]\mathbb{E}(\|u_t\|^2) - \Big[\frac{N_1 l}{2} - N_2\delta_1\Big]\mathbb{E}\|A^{\frac{1}{2}}u\|^2 \\ & + \Big[\frac{N}{2} - c_3N_2\Big]\mathbb{E}\Big(\int_0^{+\infty} h'(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds\Big) \\ & + \frac{N}{2}\mathbb{E}\Big(\int_0^{+\infty} h'(s)\|A^{\frac{\alpha}{2}}\eta(t,s)\|^2 ds\Big) \\ & + \Big[\frac{a_1^2h_0}{2l}N_1 + c_2'N_2\Big]\mathbb{E}\Big(\int_0^{+\infty} h(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds\Big) \\ & + \frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\big(\kappa_i\langle e_i,e_i\rangle\sigma^2(t)\big) + \frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\Big(\kappa_i\langle A^{\frac{1-\alpha}{2}}e_i, A^{\frac{1-\alpha}{2}}e_i\rangle\sigma^2(t)\Big),\end{aligned}$$

where $\delta_3 = \frac{\delta_2}{2}$ and $c_2' = c_2'(\delta_3) = \left(\frac{1+a_1^2}{\delta_3} + a_1\right)h_0$. Choosing N, N_1 and N_2 as in the proof of (5.40), we get

$$\begin{aligned}\frac{d}{dt}\mathscr{L}\mathscr{Y}_1(t) \leq & -m_0\mathbb{E}\big(E(t)\big) + \frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\big(\kappa_i\langle e_i, e_i\rangle\sigma^2(t)\big) \\ & + \frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\Big(\kappa_i\langle A^{\frac{1-\alpha}{2}}e_i, A^{\frac{1-\alpha}{2}}e_i\rangle\sigma^2(t)\Big),\end{aligned} \tag{5.49}$$

that is,

$$\begin{aligned}\frac{d}{dt}\mathscr{L}\mathscr{Y}_1(t) \leq & -m_0\Big(\mathbb{E}\big(E(t)\big) - S(t)\Big) + \frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\big(\kappa_i\langle e_i, e_i\rangle\sigma^2(t)\big) \\ & + \frac{N}{2}\sum_{i=1}^{+\infty}\mathbb{E}\Big(\kappa_i\langle A^{\frac{1-\alpha}{2}}e_i, A^{\frac{1-\alpha}{2}}e_i\rangle\sigma^2(t)\Big),\end{aligned}$$

the integration over $(0,t)$ yields

$$m_0\int_0^t\Big(\mathbb{E}\big(E(r)\big) - S(r)\Big)dr \leq \mathscr{L}\mathscr{Y}_1(0) + \frac{N}{2}S(t) + \frac{N}{2}Z(t). \tag{5.50}$$

Thanks to (A_5) and (5.21), the functional $\mathbb{E}\big(E(t)\big) - S(t)) > 0$ is non-increasing. And so

$$m_0(\mathbb{E}\big(E(t)\big) - S(t))t \leq m_0\int_0^t(\mathbb{E}\big(E(r)\big) - S(r))dr, \tag{5.51}$$

By (5.9), one has

$$\begin{aligned}\frac{m_0}{c_1}(\mathbb{E}\big(E(t)\big) - S(t))\int_0^t\xi(r)dr &\leq \frac{m_0}{c_1}(\mathbb{E}\big(E(t)\big) - S(t))\int_0^t c_1 dr \\ &= m_0(\mathbb{E}\big(E(t)\big) - S(t))t.\end{aligned} \tag{5.52}$$

Collecting Eqs (5.50)-(5.52) leads to

$$\frac{m_0}{c_1}(\mathbb{E}\big(E(t)\big) - S(t))\int_0^t\xi(r)dr \leq \mathscr{L}\mathscr{Y}_1(0) + \frac{N}{2}S(t) + \frac{N}{2}Z(t),$$

that is,

$$\mathbb{E}\big(E(t)\big) \leq \frac{c_1}{m_0}\left(\mathscr{L}\mathscr{Y}_1(0) + \frac{N}{2}S(t) + \frac{N}{2}Z(t)\right)\left(\int_0^t\xi(r)dr\right)^{-1} + S(t),$$

using then (5.24) and (5.46) we obtain (5.48) with $b_3 = \frac{c_1\mathscr{L}\mathscr{Y}_1(0)}{m_0}$ and $b_4 = \frac{c_1 N}{2m_0}$. That concludes the proof.

■

5.5 Applications

5.5.1 Wave equation

Let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary Γ. Our result (5.42) is valid for the following wave equation with Dirichlet boundary condition:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^{+\infty} h(s)\Delta u(x,t-s)ds + f(u) = \sigma(x,t)W_t(x,t) & \text{in } \Omega\times\mathbb{R}_+^*, \\ u = 0 & \text{in } \partial\Omega\times\mathbb{R}_+, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1 & \text{in } \quad \Omega, \end{cases}$$

which is (0.1) with $H = L^2(\Omega)$, $A = -\Delta$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $\alpha = 1$.

The same result be can obtained for the following Petrovsky equation with Dirichlet and Neumann boundary conditions:

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^{+\infty} h(s)\Delta u(x,t-s)ds + f(u) = \sigma(x,t)W_t(x,t) & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega \times \mathbb{R}_+, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1 & \text{in } \quad \Omega, \end{cases}$$

which is (0.1) with $H = L^2(\Omega)$, $A = \Delta^2$, $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$ and $\alpha = \frac{1}{2}$.

5.5.2 Coupled wave–wave system

One can derive the stability estimate obtained in (5.48) for the following system

$$\begin{cases} y_{tt} - \Delta y + \int_0^{+\infty} h(s)\Delta y(x,t-s)ds + z = \sigma(x,t)W_t(x,t) & \text{in } \Omega \times \mathbb{R}_+, \\ z_{tt} - \Delta z + \int_0^{+\infty} h(s)\Delta z(x,t-s)ds + y = \sigma(x,t)\widetilde{W}_t(x,t) & \text{in } \Omega \times \mathbb{R}_+, \\ y = z = 0 & \text{in } \partial\Omega \times \mathbb{R}_+, \\ y(x,-t) = y_0(x,t), \quad y_t(x,0) = y_1 & \text{in } \quad \Omega, \\ z(x,-t) = z_0(x,t), \quad z_t(x,0) = z_1 & \text{in } \quad \Omega. \end{cases}$$

which is (0.1) with $H = L^2(\Omega) \times L^2(\Omega)$, $A = \begin{bmatrix} -\Delta & 0 \\ 0 & -\Delta \end{bmatrix}$, $D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right)$, $D(A^{\frac{1}{2}}) = H_0^1(\Omega) \times) H_0^1(\Omega)$, and $\alpha = 1$.

Conclusion

The objective of this thesis is to investigate the stability of some stochastic infinite dimensional control systems. The background of differential equations in Hilbert space has been reviewed in Chapter 1. Some results on the stability of deterministic and stochastic equations have been presented. Robust stability analysis and synthesis for uncertain systems have been presented in Chapters 2, 3 and 4. Using the stability radius approach some improved stability conditions are established by utilizing deterministic and stochastic Lyapunov equations. State-feedback stabilization conroller which can achieve better system performances, are presented. The results are given via the solution of Riccati equations satisfying some operator inequalities. Chapter 5 considered an abstract semilinear stochastic evolution equation of second-order with an infinite memory term described by (0.1). We established existence and uniqueness of mild solution by means of semi-group theory. Also, we studied the asymptotic behavior of the solution where the obtained decay rates depend on the regularity of the solution, the exponent α and the nonlinearity f. In fact we proved that if $\alpha = 1$ then the mild solution of (5.3) has a general decay rate whereas if $0 < \alpha < 1$ we were able to obtain a weaker rate of decay for the strong solution of (0.1) with $f \equiv 0$ only.

Comments and open problems

Many questions remain to be solved. The first group of questions is for the robust stability problem:

1. The first question is the generalization of the results to the multipertubations case.

2. The theory developed assume that the operators defining the perturbations structure are bounded. However, often this class is too restrictive. Perturbed stochastic partial differential equations with boundary noise can not be considered. Consequently, it is of great practical significance to apply the developed methodology to systems with unbounded structure perturabtions.

3. Important results on the stability radius for finite dimensional jump linear systems where derived in [14]. The future work will focus on the extension of their results to infinite dimensional jump linear systems.

4. It is important to develop the counterparts of the obtained results for infinite dimensional discrete time systems..

5. Another possible direction for a future work is a transfer of the results presented in this thesis to time-varying and time-delay systems.

Another group of problems for abstract stochastic equations with infinite memory:

1. As it can be noticed from Section 5.4, it is a very interesting open problem to study stabillity problems for (5.3) when $0 \leq \alpha < 1$ and $f \neq 0$.

2. Another interesting problem concerns the stability of wave equations with boundary memory and internal stochastic terms, or conversely.

3. In [23], Guesmia considered the problem of stabilization for two linear wave equations with infinite memory in which he showed that the corresponding solutions are stable under a very much larger class of relaxation functions. To be specific, he assumed that the kernel function h satisfies for any $t \geq 0$:

$$h'(t) \leq -\xi(t)G(h(t)), \tag{5.53}$$

where $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+^*$ is a C^0 non-increasing function, $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an increasing convex function with $G(0) = G'(0) = 0$ and, he gave general and explicit formulas for the decay rates of solutions in terms of ξ and G. Motivated by this study, it is an important open problem to consider (5.3)) with the general condition (5.53).

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Abstract

This thesis deals with stochastic differential equations in infinite dimensional spaces. It mainly focuses on two issues.

The first issue is the analysis of robust stability and robust stabilization for stochastic differential equations with uncertainties. Characterizations of the stability radius are derived in terms of some Lyapunov equations. The maximization of the stability radius by state feedback is investigated. The supremal achievable stability radius is characterized via the resolution of a Riccati equation and some linear operator inequalities.

The second issue is about the analysis of the stability behavior of a semi-linear abstract stochastic evolution equation with an infinite memory. Existence and uniqueness of mild solution are established by means of semi-group theory, and the asymptotic behavior is studied.

Key-words: Stochastic evolution equations,Stabilityy radius, Robust stability, Robust stabilization, Riccati equation, Decay rate, Infinite memory.

Résumé

Cette thèse traite des équations différentielles stochastiques dans des espaces de dimension infinie. Elle se concentre principalement sur deux problèmes.

Le premier problème est l'analyse de la stabilité robuste et de la stabilisation robuste pour les équations différentielles stochastiques avec incertitudes. Des caractérisations du rayon de stabilité sont dérivées en termes de certaines équations de Lyapunov. La maximisation du rayon de stabilité par retour d'état est étudiée. Le rayon de stabilité suprême réalisable est caractérisé par la résolution d'une équation de Riccati et de certaines inégalités d'opérateurs linéaires.

Le deuxième problème porte sur l'analyse du comportement de stabilité d'une équation d'évolution stochastique abstraite semi-linéaire à mémoire infinie. L'existence et l'unicité de la solution faible sont établies au moyen de la théorie des semi-groupes, et le comportement asymptotique est étudié.

Mots-clés: Équations d'évolution stochastique, Rayon de stabilité, Stabilité robuste, Stabilisation robuste, Équation de Riccati, Taux de décroissance, Mémoire infinie.

الملخص

تتناول هذه الأطروحة المعادلات التفاضلية العشوائية في فضاءات أبعاد لا نهائية. انها تتركز بشكل رئيسي على مسألتين.

المسألة الأولى هي تحليل الاستقرار القوي والتثبيت القوي للمعادلات التفاضلية العشوائية الخاضعة لاضطرابات.يتم تعيين خصائص نصف قطر الاستقرار بواسطة معادلات ليابونوف. تتم دراسة تعظيم نصف قطر الإستقرار بواسطة معادلة ريكاتي وبعض متراجحات المؤثرات الخطية.

المسألة الثانية تتعلق بتحليل سلوك الاستقرار لمعادلة التطور العشوائي المجردة شبه الخطية ذات الذاكرة اللانهائية،يتم تحديد وجود ووحدانية الحل و كذلك دراسة السلوك المقارب.

الكلمات المفتاحية: معادلات التطور العشوائي،نصف قطر الاستقرار، الاستقرار القوي، التثبيت القوي، معادلة ريكاتي، معدل التناقص، الذاكرة اللانهائية.