REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université Batna 2
Faculté de Mathématiques et d'Informatique
Département de Mathématiques

Thèse

Presentée pour l'obtention du diplôme de **Doctorat**
Option: **ANALYSE**
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STABILITÉ DES SYSTÈMES COUPLÉS DE DEUX ÉQUATIONS HYPERBOLIQUES

Soutenue le 01/12/2022 devant le jury composé de:

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ACKNOWLEDGMENTS

First of all, thank you Allah for everything. I feel so blessed to have been given the chance to make a difference.

This thesis would not have been possible without the support of many people. I would like to start with my lovely supervisor **Pr. Hamchi Ilhem** for her guidance, patience and support throughout preparing this thesis, she has been an ideal teacher, mentor and thesis supervisor, offering advice and encouragement, she was the first one who made me a junior researcher by giving me the life hacks. With her I could participate in international mathematical conferences with confidence. I'm proud of , and grateful for my time working with her.

My Sincere thanks go to my Jury members. the president of the Jury **Pr. Youkana Amar** from University Batna 2, and examiners: **Pr. Lombarkia Farida** from University Batna 2, **Dr. Lakhdari Imed Eddine** from University of Biskra and **Dr. Houas Amrane** from University of Biskra, for agreeing to review and report on my thesis. I appreciate their time and efforts they spent in reading my thesis. I would like to thank again, **Pr. Youkana Amar** who I have benefited greatly from his wealth of knowledge. Many thanks to **Dr. Chaabane Abdou** from the University Batna 2 for his support and encouragement. Thanks go to **Dr. Ghoggali Salim** from the Higher national school of renewable energies, envirement and sustainable developement, for his time and detailed explanation on the damping terms. Also, thanks go to my friends in the Mathematics

Department for our Mathematical discussions.

I would like to express my gratitude towards my parents for their endless patience and support with out them I would never be here, they have always stood behind me. Mom and dad thank you for being there for me and for always reminding me of the end goal. My thanks and appreciations go to my only sister Khalissa for believing in me even when I didn't believe in my self. Thank you to my brothers: Nesser Eddin, Yassine, Amine and Toufik for their support. Thank you to my nieces and nephews: Douàa, Wassim, Oussama, Karam, Tassnim for being around me while I was preparing my thesis. Thank you to all my familly and friends.

In the end, I would like to thank my self for not stoping even when it was hard to continue, until I reached my goal.

الملخص

الهدف الرئيسي من هذه الأطروحة هو دراسة خصائص حلول ثلاثة أنواع من جمل المعادلات التفاضلية الزائدية الغير خطية. في الدراسة الأولى نعتبر معادلة أمواج غير خطية مرفقة بمعامل منبع داخلي و معامل كبح خارجي. في البداية, نستخدم طريقة المجموعة المستقرة لإثبات وجود الحل الضعيف الشامل. ثم, نستخدم متراجحات تكامل كومورنيك لإثبات استقرار هذا الحل. أما بالنسبة إلى الدراسة الثانية فنقدم نتيجة الوجود الشامل للحل الضعيف لمعادلة الأمواج الغير خطية المرفقة بمعامل كبح ذو الأس المتغير. ثم, نبرهن ان هذا الحل الضعيف الشامل مستقر. هذه الدراسة تعتمد على نظرية نصف زمرة وعلى متراجحات التكامل. في الدراسة الثالثة، نعتبر جملة معادلتي الأمواج بمعاملات المنبع و الكبح داخلية/خارجية. الهدف الرئيسي من هذه الدراسة, هو برهنة نتيجتين للانفجار في وقت منتهي: النتيجة الأولى متعلقة بالحل الضعيف ذو طاقة ابتدائية سالبة، أما النتيجة الثانية فمتعلقة بالحل الضعيف ذو طاقة ابتدائية موجبة.

الكلمات المفتاحية: معادلة الأمواج؛ جملة معادلتين؛ معامل منبع؛ معامل كبح؛ الأس الثابت؛ الأس المتغير؛ الوجود المحلي؛ الوجود الشامل؛ الاستقرار؛ الانفجار.

Résumé

L'objectif principal de cette thèse est d'étudier les propriétés de la solution de trois types de systèmes d'équations hyperboliques non linéaires. Dans la première étude, on considère l'équation des Ondes avec un terme source interne et un terme dissipatif frontière. Au début, on utilise la méthode de l'ensemble stable pour prouver l'existence de la solution faible globale. Ensuite, on utilise les inégalités intégrales de Komornik pour montrer la stabilité de cette solution. Quant au deuxième étude, on présente le résultat de l'existence globale de la solution faible pour l'équation des Ondes avec terme dissipatif variable frontière. Ensuite, on prouve que cette solution faible globale est stable. Cette étude est basée sur la théorie des semi groupes et certaines inégalités intégrales. Pour la troisième étude, on considère un système de deux équations des Ondes avec des termes dissipatifs interne/frontière et des termes sources. L'objectif majeur de cette étude est de montrer deux résultats d'explosion en un temps fini : le premier concerne la solution faible avec une énergie initiale négative. Le deuxième concerne la solution faible à énergie initiale positive.

Mots clés: Équation des Ondes; Système couplé; Terme de source; Terme dissipatif; Exposant constant; Exposant variable; Solution locale; Solution globale; Stabilité; Explosion.

Abstract

The main purpose of this thesis is to study the properties of the solution for three types of systems of nonlinear hyperbolic equations. In the first study, we consider the wave equation with internal source and boundary damping terms. In the beginning, we use the stable set method to prove the existence of the global weak solution. Then, we use some integral inequalities due to Komornik to prove the stability of this solution. As for the second study, we present the result of the global existence of a weak solution for the wave equation with boundary variable damping term. Then, we prove that this global weak solution is stable. This study is based on the semi groups theory and some integral inequalities. For the third study, we consider a system of two wave equations with internal/boundary damping and source terms. The major aim of this study, is to prove two blow up results in finite time: the first one is concerned with weak solution with negative initial energy. The second one is concerned with weak solution with positive initial energy.

keywords: Wave equation; Coupled system; Source term; Damping term; Constant exponent; Variable exponent; Local solution; Global solution; Stability; Blow up.

CONTENTS

GENERAL INTRODUCTION

Literature review

During the last few decades, many researchers have been interested to the existence and behavior of the solution for nonlinear wave systems. In the case of one wave equation, the system with $internal$ damping and source terms have been dealt by a lot of authors. In 1977, Ball [4] showed that, in the absence of the damping term, the source term causes a finite time blow up of solutions with negative initial energy. Haraux and Zuazua [14], in 1988, proved that, in the absence of the source term, the damping term assures the global existence for arbitrary initial data. In the linear damping case, Levine [23], in 1974, established a finite time blow up result for negative initial energy. In 1994, Georgiev and Todorova [17] extended Levine's result to the nonlinear damping case. They gave two results :

• if the damping term dominated the source term then the global solution exists for arbitrary initial data,

• if the source term dominated the damping term then the solution with sufficiently negative initial energy blows up in finite time.

In 2001, Messaoudi [24] improved the result of Georgiev and Todorova and proved a finite time blow up result for solutions with negative initial energy only. Ikehata [16], in 1995, used the stable set method, introduced by Sattinger [38] in 1968, to show that the global solution exists for small enough initial energy. In addition, authors in [12], [43], [44] and [45] have addressed this issue.

Park and Ha [36], in 2008, used the multiplier method to get the existence and the uniform decay rates concerning the semilinear wave equation with $boundary$ damping and source terms. In 2015, Fiscella and Vittilaro [11] showed the blow up in finite time of the solution with positive initial energy.

Recently, many efforts have been devoted on the Mathematical models equations of elliptic, parabolic and hyperbolic equations with $internal\ variable\ exponent$ which are including in the models of various physical phenomena such as flows of electro rheological fluids or fluids with temperature dependent viscosity and filtration processes through a porous media and image processing. Among which, we mention some studies in this direction. In 2011, Antontsev [2] obtained, under suitable conditions on the parameters in the system of wave equation with $p(x,t)-$ Laplacian and damping term, the existence and the blow up of solution. Next, in 2015, Sun and al. [41] discussed the lower and upper boundes for the blow up time results of the nonlinear hyperbolic equation with $internal\ variable$ damping and source terms under appropriate assumptions on the initial data. After that, Messaoudi and Talahmeh [27], in 2017, extended the result of Korpusov in [20] with $internal\ constant\ exponent$. They proved that a certain solution with arbitrary positive energy blows up in finite time. Also, in the same year in [28], they discussed the blow up result with suitable conditions on the $variable\ exponent$ and on the initial data for a different nonlinear equation. In 2017, Messaoudi and al. [30] considered the wave equation with $internal$ damping term and source term with $variable\ exponent$. They proved the local existence using the Faedo Galerkin method. Then, under some conditions on the $variable\ exponent$ and the initial data, they obtained the blow up result of the solution.

Ghegal and al. [18], in 2018, considered the same equation and proved, by using the stable set method, that the global solution with suitable assumptions on the initial data exists. Beside that, they showed that this solution is stable by applying the integral inequality due to Komornik [19]. In 2021, Mustafa and al. [32] considered the wave equation with $internal$ variable exponent and time dependent nonlinear damping. They used the multiplier method to obtain an energy decay results. For more studies in this direction, we refer the readers to [6] and [29].

In the case of two coupled wave equations, Agre and Rammaha [1], in 2006, proved several results concerning the local and the global existence, uniqueness of the weak solution to systems of nonlinear wave equations with $internal$ damping and source terms. Then, by using some technics as in [5], [17] and [37], they showed that any weak solution with negative initial energy blows up in finite time.

Houari [15], in 2010, extended the blow up result which proved in [1] for solution with positive initial data by employing the same method as in [43] with some needed modifications. In the same year and with the presence of the viscoelastic term, Messaoudi and Houari, in [26], also, proved the blow up result for some solutions with positive initial energy, using the same technics as in [15] and some estimates obtained in [25].

Later, Yanqiu and Rammaha [47], in 2013, studied the systems of nonlinear wave equations with nonlinearities supercritical interior and boundary sources and the boundary and interior damping functions. They proved that under some restrictions on the parameters in the system, every weak solution with negative initial energy blows up in finite time. In addition, in 2014, they obtained in [48] for the same system, the local and the global existence, the uniqueness results of the weak solutions using the nonlinear semi groups and the theory of the monotone operators. They, also, showed that such solution depend continuously on the initial data.

By applying the Galerkin and the energy methods, Hao and Cai [13], in 2016, proved several results on the local and the global existence, the blow up of solutions with positive initial energy for nonlinear coupled wave equations with viscoelastic terms.

Main Contribution

Our results, in this thesis, are conducted under the aim of studing the existence and the behavior of solution for different types of nonlinear hyperbolic systems. First, we consider a nonlinear wave equation with internal source and boundary damping terms, both terms are with a constant exponent. We apply the stable set method to prove the existence of the global weak solution. Then, we use some integral inequalities to obtain the stability of this solution. To the best of our knowledge, the application of those technics is new for this kind of problems. Second, we shed some light on the systems with variable exponent in the boundaries. We state and prove the global existence result of the weak solution. Then, we prove the stability of this solution. We note, here, that no study was given, in the literature, for the case of boundary variable exponent damping terms and no result of existence of systems with variable exponent was proved by using the semi groups theory method. In this part, we find some difficulties on the variable exponent nonlinearity comparing with the constant exponent in the first part, especially on the boundary term. Third, we focuse on two coupled wave equations with constant exponent. We prove two main results. We start with showing that the weak solution with negative initial energy blows up in finite time. Then, we prove the same result for weak solution with positive initial energy.

Organization of the thesis

This dissertation is divided into four chapters, in addition to the general introduction, conclusion and perspective.

▷ **Chapter 1:** This chapter consists on three sections: in $Section$ 1.1, we recall some useful preliminaries on the Lebesgue and on the Sobolev spaces with constant exponent, their definitions and some results needed in our proofs later. $Section$ 1.2 is concerned with spaces with variable exponent, which include the history of the Lebesgue and Sobolev spaces, also, we mention some definitions and properties of those spaces. In $Section$ 1.3, we give the most important results that we will use them later in our studies.

▷ **Chapter 2:** Here, we deal with a system of wave equation with nonlinear internal source and boundary damping terms. The study consists on the following: In $Section$ 2.2, by assuming some hypothesis on the parameters in the system, we state the existence result of the maximal weak solution. Then, we show that the energy of the solution is a decreasing function. $Section$ 2.3 is concerned with the global property of the maximal weak solution by using the stable set method. In $Section$ 2.4, we apply the multiplier method and the integral inequalities due to Komornik to prove that this solution is stable.

▷ **Chapter 3:** In this chapter, we consider a system of wave equation with variable exponent in the boundary damping term. In $Section$ 3.1, we present and prove the global existence result of the weak solution by the semi groups theory. In $Section$ 3.2, we prove that the energy associated to the weak solution is a decreasing function. After that, we give and prove the stability of the obtained solution.

▷ **Chapter 4:** At last, we study a system of two coupled wave equations with internal/boundary damping and source terms in the case of constant exponent. In $Section$ 4.2, we present the existence result of the weak maximal solution, we also, give the energy identity associated to the solution. In $Section$ 4.3, we state and prove our first blow up result in the case of negative initial energy. In $Section$ 4.4, we give and prove the second blow up result in the case of positive energy.

CHAPTER 1

PRELIMINARIES

In this chapter, we recall the definitions of the Lebesgue and Sobolev spaces with constant/variable exponent. Then, we present some useful inequalities and formulas that are related to this spaces in which we will need them later in our proofs. After that, we present some required results.

1.1 The constant exponent spaces

Let Ω be a domain of $\mathbb{R}^n (n \in \mathbb{N}^*)$ with sufficiently smooth boundary $\partial\Omega$.

1.1.1 Lebesgue space with constant exponent

Definition 1.1.
Let $p \in \mathbb{R}^*$.
• *For* $1 \leq p < \infty$, *the Lebesgue space is defined as:*

$$L^p(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \ \ measurable \ \ and \int\limits_\Omega |u|^p dx < +\infty\}.$$

$L^p(\Omega)$ *is equipped with the norm*

$$\|u\|_{L^p(\Omega)} = \|u\|_p = (\int\limits_\Omega |u|^p dx)^{\frac{1}{p}}.$$

• *For* $p = \infty$, $L^\infty(\Omega)$ *is given by*

$$L^\infty(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \;\; measurable \;\; and \;\; \exists C > 0 : |u| \leq C \;\; a.e \;\; on \;\; \Omega\},$$

$L^\infty(\Omega)$ *is equipped with the following norm*

$$\|u\|_{L^\infty(\Omega)} = \|u\|_\infty = inf\{C > 0 : |u| \leq C \;\; a.e \;\; on \;\; \Omega\}.$$

1.1.2 Sobolev space with constant exponent

Definition 1.2.
Let $m \in \mathbb{N}^*$.
• *For* $p \in [1, +\infty[$, *the Sobolev space* $W^{m,p}(\Omega)$ *is defined as follows:*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega), \partial^\alpha u \in L^p(\Omega); \alpha \in \mathbb{N} : |\alpha| \leq m\}.$$

Where ∂^α *is the generalised derivative in the distribution sense.*
$W^{m,p}(\Omega)$ *is endowed with the norm bellow*

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_p^p \right)^{\frac{1}{p}}.$$

• *For* $p = +\infty$, $W^{m,p}(\Omega)$ *is defined as follows*

$$W^{m,\infty}(\Omega) = \{u \in L^\infty(\Omega), \partial^\alpha u \in L^\infty(\Omega); \alpha \in \mathbb{N} : |\alpha| \leq m\},$$

$W^{m,\infty}(\Omega)$ *is endowed with the norm*

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{\|\alpha\| \leq m} \|\partial^\alpha u\|_\infty.$$

Remark 1.1.
For $p = 2$ *and* $m = 1$, *we note* $W^{1,2}(\Omega) = H^1(\Omega)$. *So*

$$H^1(\Omega) = \{u \in L^2(\Omega) / \frac{\partial u}{\partial x_i} \in L^2(\Omega) \; for \; all \; i = \overline{1, n}\}.$$

Remark 1.2.
We note by $H_0^1(\Omega)$ *and* $H^2(\Omega)$ *the spaces given by*

$$H_0^1(\Omega) = \{u \in H^1(\Omega) / u_{/\partial\Omega} = 0\}$$

and

$$H^2(\Omega) = \{u \in L^2(\Omega) / \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega) \; for\ all\ i, j = \overline{1, n}\}.$$

Lemma 1.1.
The Sobolev space $H_0^1(\Omega)$ is a Hilbert space with the scalar product defined by

$$\langle u, v\rangle_{H_0^1(\Omega)} = \int_\Omega \nabla u.\nabla v dx \quad for\ all\ \ u, v \in H_0^1(\Omega)$$

and with the norm

$$\|u\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}} \quad for\ all\ \ u \in H_0^1(\Omega).$$

1.1.3 Important inequalities with constant exponent

Lemma 1.2. ***(Poincare's inequality)***
There exists a positive constant, depending on Ω, such that

$$\|u\|_{L^2(\Omega)} \leq C\|u\|_{H_0^1(\Omega)} \quad for\ all\ \ u \in H_0^1(\Omega). \tag{1.1}$$

Lemma 1.3. ***(Holder's inequality)***
Let $0 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^r(\Omega)$ and we have

$$\|fg\|_r \leq \|f\|_p \|g\|_q. \tag{1.2}$$

Lemma 1.4. ***(Green formula)***
For all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have

$$\int_\Omega \Delta u v dx = -\int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\nu. \tag{1.3}$$

Proposition 1.1. ***(Young's inequality)***
Let $a, b \geq 0$ and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
• *We have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.4}$$

• *For all $\epsilon > 0$, we have*

$$ab \leq \epsilon a^p + \frac{b^q}{\epsilon^{\frac{q}{p}}}. \tag{1.5}$$

• *For all $\delta > 0$, we have*

$$ab \leq \frac{\delta^p}{p}a^p + \frac{\delta^{-q}}{q}b^q. \tag{1.6}$$

Lemma 1.5. ***(Algebric inequality)***
Let $p \geq 1$. For all $a, b > 0$, we have

$$(a+b)^p \leq 2^{p-1}(a^p + b^p). \tag{1.7}$$

1.2 The variable exponent spaces

In 1931, Orlicz was the first one who present the variable exponent Lebesgue spaces in his paper [35], where, he asked about the necessary and the sufficient conditions on a real sequence (y_i) for which $\sum_i x_i y_i$ converges, for sequences of real numbers (p_i) with $p_i > 1$ and (x_i) such that $\sum_i x_i^{p_i}$ converges. Also, he considered the variable exponent function space $L^{p(\cdot)}$ on the real line, the function spaces bear his name after he concentrated to the theory of this spaces. The space $L^{\varphi}(\Omega)$ is constituted by measurable function $u : \Omega \longrightarrow \mathbb{R}$ for which

$$\varrho(\lambda u) = \int_{\Omega} \varphi(\lambda |u(x)|)dx < +\infty,$$

for some $\lambda > 0$ and φ is a real valued function that may depend and must satisfies some conditions, putting certain properties of ϱ in an abstract setting. Nakano in [33] and [34] was the first who studied a more general class of functions spaces called modular spaces, where, the work of Nakano and the modular spaces were investigated by several people. An explicit version of these spaces was investigated by Polish Mathematicians, like Hidzik. The interested reader can see the monograph [31] of Musielak and Orlicz for more details.

The Russian researchers have independently developed the variable exponent Lebesgue spaces on the real line. In 1961, Tsenov [42] originated their results. Sharapudinov in [39] and [40] introduced the Luxembourg norm for the Lebesgue space and showed that if the exponent satisfies $1 < ess \inf p \leq p \leq ess \sup p < +\infty$ then this space is a Banach space. By considering variational integrals with non standard growth conditions, Zhikov [46] in the mid-80's, started a new line of investigation of variable exponent spaces. Next, Kovacik and Rakosnik [21] in the early 90's, established many of the basic properties of Lebesgue and Sobolev spaces in $\mathbb{R}^n$. A big developmenet has been made in the beginning of the millennium, for the rigorous study of variable exponent spaces. In particular, a relation was made between the variable exponent spaces and the variational integrals with non standard growth and coercivity conditions. Also, modelling of some physical phenomena such as flows of electro-rheological, nonlinear viscoelasticity and many other examples.

1.2.1 Lebesgue spaces with variable exponent

Let Ω be a domain of $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$.

Definition 1.3.
Let $\mathbb{P}(\Omega, \Sigma, \mu)$ be a $\alpha - finite$, complete measurable space. Let $\mathbb{P}(\Omega, \mu)$ be the set of all $\mu - measurable$ functions $p : \Omega \longrightarrow [1, \infty)$. The function $p \in \mathbb{P}(\Omega, \mu)$ is called a variable exponent on Ω. We define

$$p^- := ess \inf_{x\in\Omega} p(x) \;\; and \;\; p^+ := ess \sup_{x\in\Omega} p(x).$$

If $p^+ < +\infty$, then p is said to be a bounded variable exponent. If $p \in \mathbb{P}(\Omega, \mu)$, then, we define $p' \in \mathbb{P}(\Omega, \mu)$ by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \;\; where \;\; \frac{1}{\infty} := 0.$$

The function p' is called the dual variable exponent of p.

Definition 1.4.
Let $p : \Omega \longrightarrow [1, \infty)$ be a measurable function. We define the Lebesgue space with variable exponent $p(.)$ by

$$L^{p(.)}(\Omega) := \{u : \Omega \longrightarrow \mathbb{R} \; measurable : \quad \varrho_{p(.)}(\lambda u) = \int_{\Omega} |\lambda u(x)|^{p(x)} dx < \infty, \; for \; some \; \lambda > 0\}.$$

or equivalently

$$L^{p(.)}(\Omega) := \{u : \Omega \longrightarrow \mathbb{R}; \; measurable \; in \; \Omega \; and \; \lim_{\lambda \longrightarrow 0} \varrho_{p(.)}(\lambda u) = 0\}.$$

$L^{p(.)}(\Omega)$ is equipped with the following Luxembourg type norm

$$\|u\|_{L^{p(x)}(\Omega)} := inf\{\lambda > 0 : \int_{\Omega} |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}.$$

Lemma 1.6. *[3]*
If $p(.) \equiv p$, then

$$\|u\|_{p(.)} = \|u\|_p.$$

Definition 1.5.
We say that a function $q : \Omega \longrightarrow \mathbb{R}$ $(n \in \mathbb{N}^*)$ is log Holder continuous on Ω if there exists a constant $\theta > 0$ such that for all $0 < \delta < 1$, we have

$$|q(x) - q(y)| \leq -\frac{\theta}{log|x-y|} \;\; for\ a.e. x, y \in \Omega\ with\ |x-y| < \delta.$$

Lemma 1.7. *[22]*
If $p : \Omega \longrightarrow \mathbb{R}$ is a Lipschitz function on Ω, then, p is log Holder continuous on Ω.

Remark 1.3.
The log Holder continuity condition on p can be replaced by $p \in C(\overline{\Omega})$ if Ω is bounded.

Theorem 1.1. *[22]*
If $p \in \mathbb{P}(\Omega, \mu)$, then, $L^{p(.)}(\Omega, \mu)$ is a Banach space.

Lemma 1.8.
If $p : \Omega \longrightarrow [1, \infty)$ is a measurable function with $p' < +\infty$, then, $C_0^{\infty}(\Omega)$ is dense in $L^{p(.)}(\Omega)$.

Lemma 1.9.
If $1 < p^- \leq p(x) \leq p^+ < +\infty$, then

$$\min\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\} \leq \varrho_{p(x)}(u) \leq \max\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\},$$

for any $u \in L^{p(.)}(\Omega)$.

Remark 1.4.
If the exponent p is constant then $p^- = p^+$ and hence $\varrho_{p(x)}(u) = \|u\|_p^p$.

1.2.2 Sobolev spaces with variable exponent

Let Ω be a domain of $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$.

Definition 1.6.
Let $\alpha := (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ be a multi-index. Assume that $u \in L^1_{loc}(\Omega)$. If there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u \frac{\partial^{\alpha_1+...+\alpha_n}\psi}{\partial^{\alpha_1}x_1...\partial^{\alpha_n}x_n}dx = (-1)^{\alpha_1+...+\alpha_n}\int_\Omega \psi g dx \ \ for\ all\ \ \psi \in C_0^\infty(\Omega).$$

Then g is called a weak partial derivative of u of order α. The function g denoted by $\partial_\alpha u$ or $\frac{\partial^{\alpha_1+...+\alpha_n}u}{\partial^{\alpha_1}x_1...\partial^{\alpha_n}x_n}$.

Definition 1.7.
Let $m \in \mathbb{N}$ and $p(.)$ be a variable exponent. $W^{m,p(.)}(\Omega)$ is defined as follows

$$W^{m,p(.)}(\Omega) = \{u \in L^{p(.)}(\Omega)\ such\ that\ \partial^{|\alpha|}u \in L^{p(.)}(\Omega)\ with\ |\alpha| \leq m\},$$

Where $|\alpha| = \alpha_1 + ... + \alpha_n$.
$W^{m,p(.)}(\Omega)$ is equipped with the following norm

$$\|u\|_{W^{m,p(.)}(\Omega)} := \inf\{\lambda > 0 : \varrho_{W^{m,p(.)}(\Omega)}(\frac{u}{\lambda}) \leq 1\} = \sum_{0\leq|\alpha|\leq m} \|\partial_\alpha u\|_{p(.)},$$

with

$$\varrho_{W^{m,p(.)}(\Omega)}(u) = \sum_{0\leq|\alpha|\leq m} \varrho_{L^{p(.)}(\Omega)}(\partial_\alpha u).$$

Clearly

$$W^{0,p(.)}(\Omega) = L^{p(.)}(\Omega).$$

Remark 1.5.
We have

$$W^{1,p(.)}(\Omega) := \{u \in L^{p(.)}(\Omega)\ such\ that\ \nabla u\ exists\ and\ |\nabla u| \in L^{p(.)}(\Omega)\}.$$

$W^{1,p(.)}(\Omega)$ is equipped with the following norm

$$\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

Theorem 1.2. *[3]*
Let $p \in \mathbb{P}(\Omega, \mu)$. The space $W^{m,p(.)}(\Omega)$ is a Banach space, which is separable if p is bounded and reflexive if $1 < p^- \leq p^+ < +\infty$.

Definition 1.8.
The closure of the set of $W^{m,p(.)}(\Omega)$- functions with compact support in $W^{m,p(.)}(\Omega)$ is the Sobolev space $W_0^{m,p(.)}(\Omega)$ " with zero boundary trace."
Furtheremore, we denote by $H_0^{m,p(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(.)}(\Omega)$ and by $W^{-1,p'(.)}(\Omega)$ the dual space of $W^{1,p(.)}(\Omega)$, in the same way as the usual Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$.

Lemma 1.10.
We have
• $H_0^{m,p(.)}(\Omega) \subset W_0^{m,p(.)}(\Omega)$.
• *If p is log Holder continuous on Ω then $H_0^{m,p(.)}(\Omega) = W_0^{m,p(.)}(\Omega)$.*
• *If $p(.) = 2$ and $m = 1$ then $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.*

Theorem 1.3. *[3]*
Let $p \in \mathbb{P}(\Omega, \mu)$. The space $W_0^{m,p(.)}(\Omega)$ is a Banach space, which is separable if p is bounded and reflexive if $1 < p^- \leq p^+ < +\infty$.

Lemma 1.11. *[3, 22]**(Embedding Proberty)***
Assume that Ω is a bounded with sufficiently smooth boundary $\partial\Omega$ and $p, q \in C(\overline{\Omega})$ such that

$$1 < p^- \leq p^+ < +\infty \ \ and \ \ 1 < q^- \leq q^+ < +\infty \ \ for \ all \ \ x \in \Omega$$

and $p(x) < q^*(x)$ in $\overline{\Omega}$ with $q^*(x) = \begin{cases} \frac{nq(x)}{n-q(x)} & if \ q^+ < n, \\ +\infty & if \ q^+ \geq n. \end{cases}$
Then, the embedding $W_0^{1,q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

Corollary 1.1.
Assume that Ω is a bounded with sufficiently smooth boundary $\partial\Omega$ and $p : \overline{\Omega} \longrightarrow (1, \infty)$ is a continuous function such that

$$1 < p^- \leq p^+ < \frac{2n}{n-2} \ \ if \ \ n \geq 3.$$

Then, the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

1.2.3 Important inequalities with variable exponent

Theorem 1.4. *[22] **(Poincaré's Inequality)***
Assume that Ω is a bounded. If p satisfies the log Holder inequality on Ω, then

$$\|u\|_{p(.)} \leq C\|\nabla u\|_{p(.)} \ \ for \ all \ \ u \in W_0^{1,p(.)}(\Omega),$$

where C is a positive constant depending on Ω and $p(.)$. In particular, the space $W_0^{1,p(.)}(\Omega)$ has an equivalent norm given by

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.)}.$$

Lemma 1.12. ***(Holder's Inequality)***
Let $p, q, r \geq 1$ be a measurable functions defined on Ω satisfying

$$\frac{1}{r(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \ \ for \ a.e \ y \in \Omega.$$

If $f \in L^{p(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$ then $fg \in L^{r(.)}(\Omega)$ and

$$\|fg\|_{r(.)} \leq \|f\|_{p(.)}\|g\|_{q(.)}.$$

*Case $p = q = 2$ yields the **Cauchy Schwarz inequality**.*

Lemma 1.13. ***(Young's Inequality)***
Let $a, b \geq 0$. Let $p, q, r \geq 1$ be a measurable functions defined on Ω, such that

$$\frac{1}{r(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \ \ for \ a.e \ y \in \Omega.$$

Then

$$\frac{(ab)^{r(.)}}{r(.)} \leq \frac{a^{p(.)}}{p(.)} + \frac{b^{q(.)}}{q(.)}.$$

By taking $r = 1$ and $1 < p, q < +\infty$, it follows that for any $\epsilon > 0$, we have

$$ab \leq \epsilon a^{p(.)} + C_\epsilon b^{q(.)},$$

where $C_\epsilon = \frac{1}{q(\epsilon p)^{\frac{q}{p}}}$.
For $p = q = 2$, it comes that for all $\epsilon > 0$, we have

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

1.3 Useful Theorems

In this section, we give some important results that we will apply later.

Theorem 1.5. *[19]*
Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a nonincreasing function and assume that there exists a constants* $C > 0$ *and* $\alpha \geq 0$ *such that*

$$\int_S^{\infty} E(t)^{1+\alpha} dt \leq CE(S), \ \ 0 \leq S < \infty.$$

Then, there exists a positive constants c*,* w *and* $t_0 \geq 0$ *such that, for all* $t \geq t_0$*, we have*

$$E(t) \leq \begin{cases} E(0)e^{-wt} & if \quad \alpha = 0, \\ ct^{\frac{-1}{\alpha}} & if \quad \alpha > 0. \end{cases}$$

Theorem 1.6. *[10]*
Let $\mathcal{A}$ *be a maximal monotone operator in a Hilbert space* $\mathcal{H}$ *with domine* $D(\mathcal{A})$*. Then,*
• *If* $U^0 \in \overline{D(\mathcal{A})}$*, then, the problem*

$$U' + \mathcal{A}U = 0 \ \ in \, \mathbb{R}_+, \; U(0) = 0,$$

has a unique solution

$$U \in C(\mathbb{R}_+, \mathcal{H}).$$

• *If* $U^0 \in D(\mathcal{A})$*, then, the solution is more regular:*

$$U \in W^{1,\infty}(\mathbb{R}_+, \mathcal{H}).$$

CHAPTER 2

WAVE EQUATION WITH INTERNAL SOURCE AND BOUNDARY DAMPING TERMS: GLOBAL EXISTENCE AND STABILITY

This chapter is the subject of the following accepted publication:
Wave equation with internal source and boundary damping terms: Global existence and stability. By Boulmerka Imane and Hamchi Ilham.

In this chapter, we consider the following system

$$\begin{cases} u_{tt} - \Delta u = f(x,u) & in \;\; (0,T)\times\Omega, \\ u = 0 & on \;\; (0,T)\times\Gamma_0, \\ \partial_\nu u = -(h.\nu)g(x,u_t) & on \;\; (0,T)\times\Gamma_1, \\ u(0) = u_0 \; and \; u_t(0) = u_1 & in \;\; \Omega. \end{cases} \tag{2.1}$$

Where $T > 0$, Ω is a bounded domain of $\mathbb{R}^n(n \geq 1)$ with sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. f is a nonlinear internal source function, g represents a nonlinear boundary damping function, and $h \in \mathbb{R}^n$. Δ is the Laplacien with respect to the *spatial variables.* $\partial_\nu v = \nabla v.\nu$ *where* ν *is the unit outward normal vector to* Γ *and* ∇v *is the gradient with respect to the spatial variables.*

The objective of this chapter is to apply the stable set method to prove the existence of the global weak solution of (2.1) then to use some integral inequalities to obtain the stability of this solution.

This chapter is organized as follows: section 2.1 *contains assumptions on the parameters of (2.1) needed to obtain our results. In section* 2.2*, we present the results of the existence of the maximal weak solution of our problem and the decreasing of the energy of this solution. In section* 2.3*, we prove that this weak maximal solution is global. In section* 2.4*, we prove that the obtained global weak solution is stable.*

The following assumptions are made:

2.1 Assumptions

$(A1)$ ***Assumptions on the partition*** $\{\Gamma_0, \Gamma_1\}$ ***of*** Γ***:***

• *Let* $x_0 \in \mathbb{R}^n$ *and* $h_0 > 0$*. Put*

$$h = h(x) = x - x_0 \qquad for\ all\ x \in \overline{\Omega},$$

$$\Gamma_0 = \{x \in \Gamma / h.\nu \leq 0\} \neq \emptyset$$

and

$$\Gamma_1 = \{x \in \Gamma / h.\nu \geq h_0\}.$$

$(A2)$ ***Assumptions on the source term*** f***:***

• *We assume that the function* f *is countinous in* $\Omega \times \mathbb{R}$ *where,* $f(x, 0) = 0$ *and there exists* $C_1, C_2, p > 0$*, with*

$$\begin{cases} 2 \leq p & if\ \ n = 1, 2, \\ 2 \leq p \leq 2\frac{n-1}{n-2} & if\ \ n \geq 3, \end{cases}$$

such that

$$|f(x, u) - f(x, v)| \leq C_1 |u - v|(1 + |u|^{p-2} + |v|^{p-2}) \ \ for\ all\ x \in \Omega\ and\ u, v \in \mathbb{R}$$

and

$$F(x,u) \leq \frac{C_2}{p}|u|^p \quad for\ all\ x \in \Omega\ and\ u \in \mathbb{R}, \tag{2.2}$$

where F is the primitive of f defined by

$$F(x,u) = \int_0^u f(x,\tau)d\tau \quad for\ all\ x \in \Omega\ and\ u \in \mathbb{R}.$$

$(A3)$ ***Assumptions on the damping term*** g:

• *We shall assume that the function g is countinous on $\Gamma_1 \times \mathbb{R}$ and there exists $C_3, C_4, C_5, C_6, m > 0$, where*

$$\begin{cases} 2 \leq m \quad if\ n = 1,2, \\ 2 \leq m \leq \frac{2n}{n-2} \quad if\ n \geq 3, \end{cases}$$

such that, for all $x \in \Gamma_1$, we have

$$C_3|u|^{m-1} \leq |g(x,u)| \leq C_4|u|^{\frac{1}{m-1}} \quad if\ |u| \leq 1,$$

$$C_5|u| \leq |g(x,u)| \leq C_6|u| \quad if\ |u| > 1 \tag{2.3}$$

and

$$g(x,u)u \geq 0 \quad for\ all\ x \in \Gamma_1\ and\ u \in \mathbb{R}.$$

2.2 Existence of the maximal weak solution

This section is concerned with the existence of the maximal weak solution of (2.1) and the decreasing of the usual energy associated to this solution.

According to [36], we obtain the following result.

Theorem 2.1.
• *If $u^0 \in H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u = 0\ on\ \Gamma_0\}$ and $u^1 \in L^2(\Omega)$, then, there exists $T > 0$ and a maximal weak solution u in $(0,T)$ of the problem (2.1), such that*

$$u \in C^0((0,T), H^1_{\Gamma_0}(\Omega)) \cap C^1((0,T), L^2(\Omega)).$$

• *If $u^0 \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ and $u^1 \in H^1_{\Gamma_0}(\Omega)$, such that*

$$\frac{\partial u^0}{\partial \nu} + (h.\nu)g(x,u^1) = 0 \quad on\ \ \Gamma_1.$$

Then, there exists $T > 0$ and a unique maximal solution of the problem (2.1), such that

$$u \in L^\infty((0,T), H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)),$$

$$u_t \in L^\infty((0,T), H^1_{\Gamma_0}(\Omega))$$

and

$$u_{tt} \in L^\infty((0,T), L^2(\Omega)).$$

Next, we consider the energy functional E associated with our system defined by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(x,u)dx \quad for\ all\ t \in (0,T).$$

We have the following derivative energy identity, which shows that the above energy is a decreasing function.

Lemma 2.1. *[11]*
Let $u_0 \in H^1_{\Gamma_0}(\Omega)$ and $u_1 \in L^2(\Omega)$, we have

$$E(t) - E(s) = -\int_s^t \int_{\Gamma_1} (h.\nu)g(x,u_t)u_t d\Gamma d\tau \qquad for\ all\ 0 \leq s \leq t \leq T.$$

2.3 Global property of the maximal weak solution

In this section, we prove the global property of the weak solution of our system. For this end, we introduce the following functionals, associated to the maximal weak solution given in $Theorem$ 2.1, defined by

$$J(t) = J(u(t)) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(x,u)dx\ for\ all\ t \in (0,T)$$

and

$$K(t) = K(u(t)) = \|\nabla u\|_2^2 - p\int_\Omega F(x,u)dx\ for\ all\ t \in (0,T).$$

We consider the set

$$\mathbb{H} = \{w \in H^1_{\Gamma_0}(\Omega)/K(w) > 0\}. \tag{2.4}$$

Let C_* be the best constant such that

$$\|u\|_p \leq C_*\|\nabla u\|_2 \quad for\ all\ u \in H^1_{\Gamma_0}(\Omega). \tag{2.5}$$

We have next, the property of the set $\mathbb{H}$.

Theorem 2.2.
If $u_0 \in \mathbb{H}$ and $u_1 \in L^2(\Omega)$ with

$$\beta = C_2 C_*^p (\frac{2p}{p-2} E(0))^{\frac{p-2}{2}} < 1, \tag{2.6}$$

then, the maximal weak solution u of (2.1) is global.

Proof. **Firstly**, we have

$$u(t) \in \mathbb{H} \quad for\ all\ t \in (0, T).$$

Indeed, since

$$u_0 \in \mathbb{H},$$

then

$$K(u_0) > 0.$$

This implies that there exists $T' \leq T$ such that

$$K(t) \geq 0 \quad for\ all\ t \in [0, T']. \tag{2.7}$$

We have

$$J(t) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(x,u)dx$$

$$= \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}(\|\nabla u\|_2^2 - p\int_\Omega F(x,u)dx)$$

$$= \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}K(t).$$

By (2.7), we find

$$J(t) \geq \frac{p-2}{2p}\|\nabla u\|_2^2 \qquad for\ all\ t \in [0, T'].$$

Hence

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}J(t).$$

Moreover

$$J(t) = E(t) - \frac{1}{2}\|u_t\|_2^2 \leq E(t),$$

then

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}E(t). \tag{2.8}$$

Since E is a decreasing function, then we have

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}E(0). \tag{2.9}$$

By (2.2), we obtain

$$\int_\Omega F(x,u)dx \leq \frac{C_2}{p}\int_\Omega |u|^p dx = \frac{C_2}{p}\|u\|_p^p.$$

(2.5) leads to

$$\int_\Omega F(x,u)dx \leq \frac{C_2}{p}C_*^p\|\nabla u\|_2^p = \frac{C_2}{p}C_*^p\|\nabla u\|_2^{p-2}\|\nabla u\|_2^2.$$

Also, (2.9) gives

$$\int_\Omega F(x,u)dx \leq \frac{C_2}{p}C_*^p(\frac{2p}{p-2}E(0))^{\frac{p-2}{2}}\|\nabla u\|_2^2.$$

So

$$p\int_\Omega F(x,u)dx \leq C_2C_*^p(\frac{2p}{p-2}E(0))^{\frac{p-2}{2}}\|\nabla u\|_2^2 = \beta\|\nabla u\|_2^2.$$

We then use (2.6) to find

$$p\int_\Omega F(x,u)dx < \|\nabla u\|_2^2 \quad for\ all\ t \in [0,T'].$$

Hence

$$K(u(t)) = K(t) = \|\nabla u\|_2^2 - p\int_\Omega F(x,u)dx > 0 \quad for\ all\ t \in [0,T'].$$

(2.4) leads to

$$u(t) \in \mathbb{H} \quad for\ all\ t \in [0,T'].$$

By noting that

$$C_2C_*^p(\frac{2p}{p-2}E(T'))^{\frac{p-2}{2}} < 1,$$

we can repeat the proceedings above to extend T' to T.

Secondly, from the definition of E and K, we get for all $t \in (0,T)$

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(x,u)dx$$

$$= \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}K(t).$$

Since

$$K(t) > 0 \quad for\ all\ t \in (0,T),$$

then

$$E(t) \geq \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 \quad for\ all\ t \in (0,T).$$

This implies that there exists $C > 0$ such that

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \leq CE(t) \quad for\ all\ t \in (0,T). \tag{2.10}$$

Furthermore, E is a decreasing function, then

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \leq CE(0) \quad for\ all\ t \in (0,T).$$

By the alternative statement, we find the desired result. □

2.4 Stability of the global weak solution

We have the following stability result.

Theorem 2.3.
If $u_0 \in \mathbb{H}$ and $u_1 \in L^2(\Omega)$ with $\beta < 1$, then, there exists two positive constants C and w, such that the global weak solution of (2.1) satisfies for all $t \geq 0$

$$E(t) \leq Ce^{-wt} \qquad if\ \ m = 2,$$
$$E(t) \leq \frac{C}{t^{\frac{2}{m-2}}} \qquad if\ \ m > 2.$$

Proof. By the integral inequalities due to Komornik [19], it is sufficient to prove that, for all $0 \leq S \leq T \leq \infty$, there exist $C > 0$ such that

$$\int_S^T E^{\frac{m}{2}}(t)dx \leq CE(S). \tag{2.11}$$

For this end, we proceed in several steps.
Step 1: Energy identity
We put

$$Mu := 2h.\nabla u + (n-1)u.$$

We multiply the first equation of (2.1) by $E^{\frac{m-2}{2}}(t)Mu$. Then, we integrate the obtained result over $[S,T] \times \Omega$, we find

$$0 = \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega Mu(u_{tt} - \Delta u - f(x,u))dxdt$$

$$= I_1 + I_2 + I_3, \tag{2.12}$$

where

$$I_1 = \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega u_{tt} Mu dx dt,$$

$$I_2 = -\int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega \Delta u Mu dx dt$$

and

$$I_3 = -\int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega f(x,u) Mu dx dt.$$

We have

$$I_1 = \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega u_{tt} Mu dx dt = [E^{\frac{m-2}{2}}(t) \int_\Omega u_t Mu dx]_S^T$$

$$-\frac{m-2}{2} \int_S^T E^{\frac{m-4}{2}}(t) E_t(t) \int_\Omega u_t Mu dx dt - \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega u_t (Mu)_t dx dt.$$

But

$$-\int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega u_t (Mu)_t dx dt = -\int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega u_t (2h.\nabla u + (n-1)u)_t dx dt$$

$$= -2 \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega u_t (h.\nabla u)_t dx dt - \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega u_t ((n-1)u)_t dx dt,$$

which implies that

$$-\int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega u_t (Mu)_t dx dt = -2 \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega u_t (h.\nabla u_t) dx dt$$

$$-(n-1) \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega |u_t|^2 dx dt. \tag{2.13}$$

If we apply the following identity

$$\int_\Omega j_1 (k.\nabla j_2) dx = \int_\Gamma k.\nu (j_1 j_2) d\Gamma - \int_\Omega j_2 div(j_1 k) dx, \tag{2.14}$$

for all

$$j_1, j_2 \in C^1(\overline{\Omega}) \quad and \quad k \in (C^1(\overline{\Omega}))^n,$$

with

$$j_1 = j_2 = u_t \quad and \quad k = h,$$

we find

$$-\int_{\Omega} u_t(h.\nabla u_t)dx = -\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma + \int_{\Omega} u_t div(u_t h)dx.$$

Also, if we apply the following identity

$$div(jk) = jdivk + k.\nabla j \quad for\ all\ j \in C^1(\overline{\Omega})\ \ and\ \ k \in (C^1(\overline{\Omega}))^n, \tag{2.15}$$

with

$$j = u_t \quad and \quad k = h,$$

we obtain

$$-\int_{\Omega} u_t(h.\nabla u_t)dx = -\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma + \int_{\Omega} u_t(u_t divh + h.\nabla u_t)dx,$$

$$= -\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma + \int_{\Omega} divh|u_t|^2 dx + \int_{\Omega} u_t(h.\nabla u_t)dx,$$

this leads to

$$-2\int_{\Omega} u_t(h.\nabla u_t)dx = -\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma + n\int_{\Omega} |u_t|^2 dx.$$

Now, if we replace the above result in (2.13), we find

$$-\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Omega} u_t(Mu)_t dxdt = \int_S^T E^{\frac{m-2}{2}}(t)\int_{\Omega} |u_t|^2 dxdt$$

$$-\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma}(h.\nu)|u_t|^2 d\Gamma dt.$$

So, I_1 takes the forme

$$I_1 = [E^{\frac{m-2}{2}}(t)\int_{\Omega} u_t Mudx]_S^T - \frac{m-2}{2}\int_S^T E^{\frac{m-4}{2}}(t)E_t(t)\int_{\Omega} u_t Mudxdt$$

$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |u_t|^2 dxdt - \int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma (h.\nu)|u_t|^2 d\Gamma dt.$$

For I_2, we have

$$I_2 = -\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega \Delta u M u dxdt = -\int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma \frac{\partial u}{\partial \nu} M u d\Gamma dt$$

$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega \nabla u \nabla (Mu) dxdt.$$

For the second term in the above identity, we have

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega \nabla u \nabla (Mu) dxdt = \int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega \nabla u \nabla (2h.\nabla u + (n-1)u) dxdt,$$

it follows that

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega \nabla u \nabla (Mu) dxdt = 2\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega \nabla u.\nabla (h.\nabla u) dxdt$$

$$+(n-1)\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |\nabla u|^2 dxdt. \tag{2.16}$$

But

$$\int_\Omega \nabla u \nabla (h.\nabla u) dx = \int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega h.\nabla(|\nabla u|^2) dx.$$

Then, by the identity (2.14), we get

$$\int_\Omega \nabla u \nabla (h.\nabla u) dx = \int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma - \frac{1}{2}\int_\Omega divh|\nabla u|^2 dx$$

$$= \frac{2-n}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma.$$

So, by replacing it in (2.16), we find

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega \nabla u \nabla (Mu) dxdt = (2-n)\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |\nabla u|^2 dxdt$$

$$+(n-1)\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |\nabla u|^2 dxdt + \int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma dt$$
$$= \int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |\nabla u|^2 dxdt + \int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma dt.$$

Hence

$$I_2 = \int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |\nabla u|^2 dxdt + \int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma dt$$
$$-\int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma \frac{\partial u}{\partial \nu} Mud\Gamma dt.$$

Inserting I_1, I_2 and I_3 in (2.12) to find

$$0 = [E^{\frac{m-2}{2}}(t)\int_\Omega u_t Mudx]_S^T - \frac{m-2}{2}\int_S^T E^{\frac{m-4}{2}}(t)E_t(t)\int_\Omega u_t Mudxdt$$
$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |u_t|^2 dxdt + \int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |\nabla u|^2 dxdt - \int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega f(x,u)Mudxdt$$
$$-\int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma \frac{\partial u}{\partial \nu} Mud\Gamma dt + \int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma (h.\nu)(|\nabla u|^2 - |u_t|^2)d\Gamma dt.$$

Thus, we can write it as following

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega (|u_t|^2 + |\nabla u|^2)dxdt = I_\Omega + I_{[S,T]\times\Omega} + I_{[S,T]\times\Gamma}, \tag{2.17}$$

where

$$I_\Omega = -[E^{\frac{m-2}{2}}(t)\int_\Omega u_t Mudx]_S^T,$$

$$I_{[S,T]\times\Omega} = \frac{m-2}{2}\int_S^T E^{\frac{m-4}{2}}(t)E_t(t)\int_\Omega u_t Mudxdt + \int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega f(x,u)Mudxdt,$$

and

$$I_{[S,T]\times\Gamma} = \int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma \frac{\partial u}{\partial \nu} Mud\Gamma dt + \int_S^T E^{\frac{m-2}{2}}(t)\int_\Gamma (h.\nu)(|u_t|^2 - |\nabla u|^2)d\Gamma dt.$$

Step 2: Energy inequality

For the first term I_Ω, we can see that

$$\int_\Omega |Mu|^2 dx = \int_\Omega |2h.\nabla u + (n-1)u|^2 dx \leq \int_\Omega (|2h.\nabla u| + |(n-1)u|)^2 dx.$$

By the algebric inequality (1.7), we have

$$\int_\Omega |Mu|^2 dx \leq 2\int_\Omega |2h.\nabla u|^2 dx + 2\int_\Omega |(n-1)u|^2 dx.$$

In the rest of the proof, C represents a positive generic constant.

By the Poincare's inequality (1.1), we get

$$\int_\Omega |Mu|^2 dx \leq C\|\nabla u\|_2^2. \tag{2.18}$$

Hence, we have

$$|\int_\Omega u_t Mu dx| \leq \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\int_\Omega |Mu|^2 dx \leq C(\|u_t\|_2^2 + \|\nabla u\|_2^2),$$

but, by (2.10), we obtain

$$|\int_\Omega u_t Mu dx| \leq CE(t). \tag{2.19}$$

Then, the first term I_Ω became

$$\begin{aligned} I_\Omega &= E^{\frac{m-2}{2}}(S)\int_\Omega u_t(S)Mu(S)dx - E^{\frac{m-2}{2}}(T)\int_\Omega u_t(T)Mu(T)dx \\ &\leq CE^{\frac{m-2}{2}}(S)E(S) + CE^{\frac{m-2}{2}}(T)E(T). \end{aligned}$$

Since the energy E is a positive decreasing function, then

$$I_\Omega \leq CE^{\frac{m-2}{2}}(S)E(S) \leq CE(S). \tag{2.20}$$

For the second term $I_{[S,T]\times\Omega}$, we have

$$I_{[S,T]\times\Omega} = \frac{m-2}{2}\int_S^T E^{\frac{m-4}{2}}(t)E_t(t)\int_\Omega u_t Mu dx dt + \int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega f(x,u)Mu dx dt.$$

By (2.19) and the Young inequality (1.5), we get for all $\epsilon_1 > 0$

$$I_{[S,T]\times\Omega} \leq C\int_S^T E^{\frac{m-4}{2}}(t)(-E_t(t))E(t)dt + \frac{\epsilon_1}{2}\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |Mu|^2 dx dt$$

$$+\frac{1}{2\epsilon_1}\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega |f(x,u)|^2 dxdt.$$

Using the assumptions on f, (2.8) and (2.18), we find

$$I_{[S,T]\times\Omega} \leq C\int_S^T E^{\frac{m-2}{2}}(t)(-E_t(t))dt + \epsilon_1 C\int_S^T E^{\frac{m}{2}}(t)dt$$

$$+C(\epsilon_1)\int_S^T E^{\frac{m-2}{2}}(t)(\int_\Omega |u|^2 dx + \int_\Omega |u|^{2(p-1)}dx)dt$$

$$\leq C[E^{\frac{m}{2}}(S) - E^{\frac{m}{2}}(T)] + \epsilon_1 C\int_S^T E^{\frac{m}{2}}(t)dt$$

$$+C(\epsilon_1)\int_S^T E^{\frac{m-2}{2}}(t)(\int_\Omega |u|^2 dx + \int_\Omega |u|^{2(p-1)}dx)dt,$$

Since the energy is a positive decreasing function, then we obtain

$$I_{[S,T]\times\Omega} \leq CE(S) + \epsilon_1 C\int_S^T E^{\frac{m}{2}}(t)dt$$

$$+C(\epsilon_1)\int_S^T E^{\frac{m-2}{2}}(t)(\int_\Omega |u|^2 dx + \int_\Omega |u|^{2(p-1)}dx)dt, \tag{2.21}$$

We apply the interpolation inequality

$$\|u\|_r \leq \|u\|_2^\alpha \|u\|_\beta^{1-\alpha} \quad with \ \frac{1}{r} = \frac{\alpha}{2} + \frac{1-\alpha}{\beta} \ and \ \alpha \in [0,1].$$

For

$$r = 2(p-1), \ \alpha = \frac{1}{2(p-1)} \ and \ \beta = 2(2p-3),$$

we obtain

$$\|u\|_{2(p-1)} \leq \|u\|_2^{\frac{1}{2(p-1)}} \|u\|_{2(2p-3)}^{\frac{2p-3}{2(p-1)}},$$

then

$$\|u\|_{2(p-1)}^{2(p-1)} \leq \|u\|_2 \|u\|_{2(2p-3)}^{2p-3}.$$

We use the Young inequality (1.5) to find for all $\epsilon_2 > 0$

$$\|u\|_{2(p-1)}^{2(p-1)} \leq \frac{\epsilon_2}{2}\|u\|_{2(2p-3)}^{2(2p-3)} + \frac{1}{2\epsilon_2}\|u\|_2^2.$$

Using the embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{2(2p-3)}(\Omega)$ to have

$$\|u\|_{2(p-1)}^{2(p-1)} \leq \epsilon_2 C \|\nabla u\|_2^{2(2p-3)} + C(\epsilon_2)\|u\|_2^2.$$

By (2.8), we get

$$\|u\|_{2(p-1)}^{2(p-1)} \leq \epsilon_2 C E(t) + C(\epsilon_2)\|u\|_2^2.$$

Then, we replace it in (2.21), to find

$$I_{[S,T]\times\Omega} \leq CE(S) + \epsilon_1 C \int_S^T E^{\frac{m}{2}}(t)dt + \epsilon_2 C(\epsilon_1) \int_S^T E^{\frac{m}{2}}(t)dt$$

$$+C(\epsilon_1,\epsilon_2) \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega |u|^2 dxdt, \tag{2.22}$$

For the third term $I_{[S,T]\times\Gamma}$, we have

$$\begin{aligned}
I_{[S,T]\times\Gamma} &= \int_S^T E^{\frac{m-2}{2}}(t) \int_\Gamma \frac{\partial u}{\partial \nu} Mud\Gamma dt + \int_S^T E^{\frac{m-2}{2}}(t) \int_\Gamma (h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt \\
&= \int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_0} \frac{\partial u}{\partial \nu} Mud\Gamma dt + \int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_1} \frac{\partial u}{\partial \nu} Mud\Gamma dt \\
&\quad + \int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_0} (h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt \\
&\quad + \int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_1} (h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt \\
&= I_{[S,T]\times\Gamma_0} + I_{[S,T]\times\Gamma_1},
\end{aligned} \tag{2.23}$$

where

$$I_{[S,T]\times\Gamma_0} = \int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_0} \frac{\partial u}{\partial \nu} Mud\Gamma dt + \int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_0} (h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt$$

and

$$I_{[S,T]\times\Gamma_1} = \int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_1} \frac{\partial u}{\partial \nu} Mud\Gamma dt + \int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_1} (h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt.$$

For $I_{[S,T]\times\Gamma_0}$, we use the definition of Mu to get

$$I_{[S,T]\times\Gamma_0} = 2\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_0}(h.\nabla u)\frac{\partial u}{\partial \nu}d\Gamma dt + (n-1)\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_0} u\frac{\partial u}{\partial \nu}d\Gamma dt$$

$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_0}(h.\nu)(|u_t|^2 - |\nabla u|^2)d\Gamma dt.$$

On Γ_0, we have

$$\begin{cases} u = 0, \\ \\ \nabla u = \nu.\frac{\partial u}{\partial \nu}, \end{cases}$$

then

$$\begin{cases} u_t = 0, \\ |\nabla u|^2 = |\frac{\partial u}{\partial \nu}|^2. \end{cases}$$

So, we can write the term on $I_{[S,T]\times\Gamma_0}$ as following

$$I_{[S,T]\times\Gamma_0} = 2\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_0}(h.\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt - \int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_0}(h.\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt.$$

Then

$$I_{[S,T]\times\Gamma_0} = \int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_0}(h.\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt.$$

Since

$$h.\nu \leq 0 \;\; on \;\; \Gamma_0,$$

so, we arrive at

$$I_{[S,T]\times\Gamma_0} \leq 0. \tag{2.24}$$

For $I_{[S,T]\times\Gamma_1}$, we use the definition of Mu to find

$$I_{[S,T]\times\Gamma_1} = \int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}(2h.\nabla u)\frac{\partial u}{\partial \nu}d\Gamma dt$$

$$+(n-1)\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1} u\frac{\partial u}{\partial \nu}d\Gamma dt + \int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}(h.\nu)(|u_t|^2 - |\nabla u|^2)d\Gamma dt.$$

Then

$$I_{[S,T]\times\Gamma_1} = -\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}(2h.\nabla u)(h.\nu)g(x,u_t)d\Gamma dt$$

$$-(n-1)\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1} u(h.\nu)g(x,u_t)d\Gamma dt$$

$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt.$$

By the Young inequalities (1.5) and (1.4), we find for all $\epsilon_1 > 0$

$$I_{[S,T]\times\Gamma_1} \le \int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}[2(\frac{h^2}{2}|g(x,u_t)|^2+\frac{1}{2}|\nabla u|^2)$$

$$+(n-1)(\frac{\epsilon_1}{2}|u|^2+\frac{1}{2\epsilon_1}|g(x,u_t)|^2)](h.\nu)d\Gamma dt - \int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}(h.\nu)|\nabla u|^2 d\Gamma dt$$

$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}(h.\nu)|u_t|^2 d\Gamma dt$$

$$= \epsilon_1 C\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}|u|^2(h.\nu)d\Gamma dt + C(\epsilon_1)\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}|g(x,u_t)|^2(h.\nu)d\Gamma dt$$

$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}(h.\nu)|u_t|^2 d\Gamma dt.$$

We put $d\Gamma_h = (h.\nu)d\Gamma$ to obtain

$$I_{[S,T]\times\Gamma_1} \le \epsilon_1 C\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}|u|^2 d\Gamma_h dt$$

$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}|u_t|^2 d\Gamma_h dt + C(\epsilon_1)\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}|g(x,u_t)|^2 d\Gamma_h dt.$$

We have

$$\int_{\Gamma_1}|u|^2 d\Gamma_h \le \frac{1}{2}\int_\Omega |\nabla u|^2 dx \le CE(t).$$

So, the term on Γ_1 became

$$I_{[S,T]\times\Gamma_1} \leq \epsilon_1 C \int_S^T E^{\frac{m}{2}}(t)dt + C(\epsilon_1)\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}(|g(x,u_t)|^2 + |u_t|^2)d\Gamma_h dt. \qquad (2.25)$$

We have

$$\Gamma_1 = \Gamma_2 \cup \Gamma_3,$$

with

$$\Gamma_2 = \{x \in \Gamma_1; |u_t| \leq 1\}$$

and

$$\Gamma_3 = \{x \in \Gamma_1; |u_t| > 1\}.$$

Then, we obtain

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_1}|(g(x,u_t)|^2 + |u_t|^2)d\Gamma_h dt = \int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_2}(|g(x,u_t)|^2 + |u_t|^2)d\Gamma_h dt$$

$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_3}(|g(x,u_t)|^2 + |u_t|^2)d\Gamma_h dt. \qquad (2.26)$$

We have

$$\begin{aligned}|g(x,u_t)|^2 &= |g(x,u_t)|^{\frac{2}{m}}(|g(x,u_t)|^{\frac{-2}{m}}|g(x,u_t)|^2)\\ &= |g(x,u_t)|^{\frac{2}{m}}|g(x,u_t)|^{\frac{2(m-1)}{m}}\end{aligned}$$

and

$$|u_t|^2 = |u_t|^{\frac{2}{m}}(|u_t|^{\frac{-2}{m}}|u_t|^2) = |u_t|^{\frac{2}{m}}|u_t|^{\frac{2(m-1)}{m}}.$$

For $|u_t| \leq 1$, we use the assumptions on g (2.3) to find

$$|g(x,u_t)|^2 \leq C_4^{\frac{2(m-1)}{m}}|g(x,u_t)u_t|^{\frac{2}{m}}$$

and

$$|u_t|^2 \leq \frac{1}{C_3^{\frac{2}{m}}}|g(x,u_t)u_t|^{\frac{2}{m}},$$

then

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_2}(|g(x,u_t)|^2 + |u_t|^2)d\Gamma_h dt \leq C\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_2\subset\Gamma_1}|g(x,u_t)u_t|^{\frac{2}{m}}d\Gamma_h dt.$$

By the embedding $L^1(\Gamma_1)$ in $L^{\frac{2}{m}}(\Gamma_1)$, we get

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_2}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \le C\int_S^T E^{\frac{m-2}{2}}(t)[\int_{\Gamma_1} g(x,u_t)u_t d\Gamma_h]^{\frac{2}{m}}dt$$

$$\le C\int_S^T E^{\frac{m-2}{2}}(t)(-E_t(t))^{\frac{2}{m}}dt.$$

If $m>2$, we apply the Young inequality (1.5) for

$$a = E^{\frac{m-2}{2}}(t), \quad b=(-E_t(t))^{\frac{2}{m}}, \quad p=\frac{m}{m-2} \quad and \quad q=\frac{m}{2},$$

to find for all $\epsilon_2>0$

$$\int_S^T E^{\frac{m-2}{2}}(t)(-E_t(t))^{\frac{2}{m}}dt \le \epsilon_2\int_S^T E^{\frac{m}{2}}(t)dt + C(\epsilon_2)\int_S^T(-E_t(t))dt.$$

This implies that

$$\int_S^T E^{\frac{m-2}{2}}(t)(-E_t(t))^{\frac{2}{m}}dt \le \epsilon_2\int_S^T E^{\frac{m}{2}}(t)dt + C(\epsilon_2)E(S).$$

If $m=2$, we obtain

$$\int_S^T E^{\frac{m-2}{2}}(t)(-E_t(t))^{\frac{2}{m}}dt \le CE(S).$$

Hence

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_2}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \le \epsilon_2 C\int_S^T E^{\frac{m}{2}}(t)dt + C(\epsilon_2)E(S). \tag{2.27}$$

Now, for $|u_t|>1$, we use the assumption on g (2.3) to obtain

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_3}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \le (\frac{1}{C_5}+C_6)\int_S^T E^{\frac{m-2}{2}}(t)\int_{\Gamma_3\subset\Gamma_1} g(x,u_t)u_t d\Gamma_h dt$$

$$\le C\int_S^T E^{\frac{m-2}{2}}(t)(-E_t(t))dt,$$

then

$$\int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_3} (|g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt \leq CE(S). \tag{2.28}$$

We insert (2.27), (2.28) in (2.26) to find

$$\int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_1} |(g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt \leq \epsilon_2 C \int_S^T E^{\frac{m}{2}}(t) dt + C(\epsilon_2) E(S). \tag{2.29}$$

After that, we put the result (2.29) in (2.25) to obtain

$$I_{[S,T]\times\Gamma_1} \leq \epsilon_1 C \int_S^T E^{\frac{m}{2}}(t) dt + \epsilon_2 C(\epsilon_1) \int_S^T E^{\frac{m}{2}}(t) dt + C(\epsilon_1, \epsilon_2) E(S). \tag{2.30}$$

Putting (2.24) and (2.30) in (2.23) we find

$$I_{[S,T]\times\Gamma} \leq \epsilon_1 C \int_S^T E^{\frac{m}{2}}(t) dt + \epsilon_2 C(\epsilon_1) \int_S^T E^{\frac{m}{2}}(t) dt + C(\epsilon_1, \epsilon_2) E(S). \tag{2.31}$$

Combining (2.20), (2.22) and (2.31) in (2.17), we get

$$\int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega (|u_t|^2 + |\nabla u|^2) dx dt \leq C(\epsilon_1, \epsilon_2) \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega |u|^2 dx dt$$

$$+(\epsilon_1 C + \epsilon_2 C(\epsilon_1)) \int_S^T E^{\frac{m}{2}}(t) dt + C(\epsilon_1, \epsilon_2) E(S).$$

Taking ϵ_1 sufficiently small, then, ϵ_2 sufficiently small and using the definition of the energy, to obtain

$$\int_S^T E^{\frac{m}{2}}(t) dt \leq C \int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega |u|^2 dx dt + CE(S). \tag{2.32}$$

Step 3: End of the proof

By the uniqueness compacteness argument, we can prove that

$$\int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega |u|^2 dx dt \leq C \int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_1} (|g(x,u_t)|^2 + |u_t|^2) d\Gamma dt.$$

Then, we have for all $\epsilon_3 > 0$

$$\int_S^T E^{\frac{m-2}{2}}(t) \int_\Omega |u|^2 dx dt \leq \epsilon_3 C \int_S^T E^{\frac{m}{2}}(t) dt + C(\epsilon_3) E(S).$$

Replacing it in (2.32) and taking ϵ_3 sufficiently small, then, the result (2.11) is finally obtained.

□

CHAPTER 3

WAVE EQUATION WITH BOUNDARY VARIABLE DAMPING TERM: GLOBAL EXISTENCE AND STABILITY

This result was submitted by Boulmerka and Hamchi.

In this chapter, we study the following wave equation with variable exponent in the boundary damping term:

$$\begin{cases} u_{tt} - \Delta u = 0 & in \quad \Omega \times \mathbb{R}_+, \\ u = 0 & on \ \ \Gamma_0 \times \mathbb{R}_+, \\ \partial_\nu u + (h.\nu)g(.,u_t) = 0 & on \ \ \Gamma_1 \times \mathbb{R}_+, \\ u(0) = u^0 \ and \ u_t(0) = u^1 & in \ \ \Omega. \end{cases} \tag{3.1}$$

Here, Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. ν is the unit outward normal to Γ. $h \in \mathbb{R}^n$, g represents the boundary damping function with variable exponent.

The main purpose of this chapter is to present and prove the global existence result by the *semi groups theory, then, to obtain the stability of the weak global solution of the problem (3.1).*

This study is organized as follows: in section 3.1 *we present and prove the global existence result of the weak solution of the problem (3.1). For the section* 3.2*, we prove that the energy associated to the weak solution of the problem (3.1) is a decreasing function. After that, we state and prove the stability of the obtained global weak solution.*

In order to state the corresponding results we have the following assumptions:

3.1 Assumptions

$(A1)$***Assumptions on the partition*** $\{\Gamma_0, \Gamma_1\}$ ***of*** Γ***:***

• *Let* $x_0 \in \mathbb{R}^n$ *and* $h_0 > 0$, *we have*

$$\Gamma_0 = \{x \in \Gamma / h.\nu \leq 0\} \neq \emptyset \ \ and \ \ \Gamma_1 = \{x \in \Gamma / h.\nu \geq h_0\},$$

where

$$h(x) = x - x_0 \ \ for \ all \ x \in \overline{\Omega}.$$

$(A2)$ ***Assumptions on the damping term*** g***:***

• *Assuming that the function* g *is continuous on* $\Gamma_1 \times \mathbb{R}$ *and for all* $x \in \Gamma_1$, $g(x, .)$ *is an increasing and globally Lipschitzian function on* $\mathbb{R}$ *with*

$$g(x, 0) = 0, \ \ g(x, v)v \geq 0 \ \ \ for \ all \ x \in \Gamma_1 \ and \ v \in \mathbb{R} \tag{3.2}$$

and there exists $C_1, C_2, C_3, C_4 > 0$ *such that, for all* $x \in \Gamma_1$ *and* $v \in \mathbb{R}$, *we have*

$$\begin{cases} C_1|v|^{m(x)-1} \leq |g(x, v)| \leq C_2|v|^{\frac{1}{m(x)-1}} & if \ |v| \leq 1, \\ C_3|v| \leq |g(x, v)| \leq C_4|v| & if \ |v| > 1. \end{cases} \tag{3.3}$$

Here, the exponent $m(.)$ *is a measurable function on* Γ_1, *such that*

$$2 \leq m_1 \leq m(.) \leq m_2 \ \ \ if \ n = 1, 2$$

and

$$2 \leq m_1 \leq m(.) \leq m_2 \leq \frac{2n}{n-2} \ \ \ if \ n \geq 3,$$

where

$$m_1 := ess \inf_{x\in\Gamma_1} m(x) \quad and \quad m_2 := ess \sup_{x\in\Gamma_1} m(x).$$

As an exemple of the boundary term g, we take for all $x \in \Gamma_1$

$$g(x,v) = \begin{cases} \alpha(x)|v|^{m(x)-2}v & if \quad |v| \leq 1, \\ \alpha(x)v & if \quad |v| > 1, \end{cases}$$

in which α *is a positive bounded and continuous function on* Γ_1.

3.2 Existence of the global weak solution

In this section, we prove the global existence of the weak solution to the problem (3.1).

Theorem 3.1.
• *If* $u^0 \in V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$ *and* $u^1 \in L^2(\Omega)$*, then, there exists a unique weak solution* u *of the problem (3.1), such that*

$$u \in C(\mathbb{R}_+, V) \cap C^1(\mathbb{R}_+, L^2(\Omega)).$$

• *If* $u^0 \in H^2(\Omega) \cap V$ *and* $u^1 \in V$*, such that*

$$\frac{\partial u^0}{\partial \nu} + (h.\nu)g(x,u^1) = 0 \quad on \ \Gamma_1.$$

Then, there exists a unique strong solution of the problem (3.1), such that

$$u \in L^\infty(\mathbb{R}_+, H^2(\Omega)),$$

$$u_t \in L^\infty(\mathbb{R}_+, V)$$

and

$$u_{tt} \in L^\infty(\mathbb{R}_+, L^2(\Omega)).$$

Proof. Consider the following operators

$$A, B : V \longrightarrow V',$$

defined for all $v, w \in V$ by

$$\langle Av, w\rangle_{V',V} = \int_\Omega \nabla v.\nabla w dx$$

and

$$\langle Bv, w\rangle_{V',V} = \int_{\Gamma_1} (h.\nu)g(x, v)wd\Gamma.$$

The problem (3.1) can be rewritten as following

$$\begin{cases} U_t + \mathcal{A}U = 0 \;\; in \;\; \mathbb{R}_+, \\ U(0) = (u^0, u^1), \end{cases}$$

where

$$\mathcal{A} : D(\mathcal{A}) \subset V \times L^2(\Omega) \longrightarrow V \times L^2(\Omega),$$

is the operator defined as following

$$\mathcal{A}(u_1, u_2) = (-u_2, Au_1 + Bu_2) \;\; for\ all\ (u_1, u_2) \in D(\mathcal{A}),$$

with

$$D(\mathcal{A}) = \{(u_1, u_2) \in V \times V : Au_1 + Bu_2 \in L^2(\Omega)\}.$$

Since

$$D(\mathcal{A}) = \{(u_1, u_2) \in V \times V : u_1 \in H^2(\Omega)\ and\ \frac{\partial u_1}{\partial \nu} + (h.\nu)g(x, u_2) = 0\ on\ \Gamma_1\},$$

$D(\mathcal{A})$ is dense in $V \times L^2(\Omega)$ and $\mathcal{A}$ is an maximal monotone operator in $V \times L^2(\Omega)$, then, $Theorem$ 7.1 in Komornik [19] gives us the desired result. □

3.3 Stability of the solution

Bellow, we state and prove that our global weak solution is stable. This will be accomplished by using the multiplier method and some integral inequalities.

Before proving the main result, we need to prove the following.

Lemma 3.1.
The energy E associated with the problem (3.1) defined as follows

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \quad for\ all\ t \in \mathbb{R}_+$$

is a decreasing function.
Moreover, the derivative is given by

$$E_t(t) = -\int_{\Gamma_1} (h.\nu)g(x, u_t)u_t d\Gamma \leq 0 \quad for\ all\ t \in \mathbb{R}_+. \tag{3.4}$$

Proof. Firstly, we prove (3.4) for the strong solution. For this end, we multiply the differential equation in (3.1) by u_t and integrate over Ω, we obtain

$$\int_\Omega u_{tt}u_t dx - \int_\Omega \Delta u u_t dx = 0,$$

then, for the first term in the left hand side of the equation, we have

$$\int_\Omega u_{tt}u_t dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx.$$

For the second term in the right hand side, we use the Green formula (1.3) and the boundary conditions to obtain

$$-\int_\Omega \Delta u u_t dx = \int_\Omega \nabla u.\nabla u_t dx - \int_\Gamma \frac{\partial u}{\partial \nu} u_t d\Gamma = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \int_{\Gamma_1} (h.\nu) g(x,u_t) u_t d\Gamma.$$

Adding the two above results, we find

$$\frac{1}{2}\frac{d}{dt}(\|u_t\|_2^2 + \|\nabla u\|_2^2) = -\int_{\Gamma_1} (h.\nu) g(x,u_t) u_t d\Gamma,$$

which means

$$\frac{d}{dt}E(t) = -\int_{\Gamma_1} (h.\nu) g(x,u_t) u_t d\Gamma.$$

Using the assumption (3.2) on the damping term to get

$$E_t(t) \leq 0 \ \ for \ all \ t \in \mathbb{R}_+.$$

Secondly, by density, we obtain (3.4) for the weak solution. □

We are now ready to give the stability result.

Theorem 3.2.
There exists two positive constants c and w such that, for all $t \in \mathbb{R}_+$, we have

$$\begin{cases} E(t) \leq ce^{-wt} & if \ \ m_2 = 2 \\ E(t) \leq \frac{c}{t^{\frac{2}{m_2-2}}} & if \ \ m_2 > 2. \end{cases} \tag{3.5}$$

Proof. We prove (3.5) for strong solutions of (3.1) and by density we can extend our results to weak solutions. So, to reach this end, we proceed in several steps.

Step 1: Energy identity

Let

$$0 \leq S < T < +\infty$$

and

$$Mu := 2h.\nabla u + (n-1)u.$$

We multiply the first equation of (3.1) by $E^{\frac{m_2-2}{2}}(t)Mu$. Then, we integrate the obtained result over $[S,T] \times \Omega$ to find

$$0 = \int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega Mu(u_{tt} - \Delta u)dxdt$$

$$= I_1 + I_2, \tag{3.6}$$

where

$$I_1 = \int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega u_{tt} Mu dxdt$$

and

$$I_2 = -\int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega \Delta u Mu dxdt.$$

We have

$$I_1 = \int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega u_{tt} Mu dxdt = [E^{\frac{m_2-2}{2}}(t) \int_\Omega u_t Mu dx]_S^T$$

$$-\frac{m_2-2}{2} \int_S^T E^{\frac{m_2-4}{2}}(t) E_t(t) \int_\Omega u_t Mu dxdt - \int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega u_t (Mu)_t dxdt.$$

But

$$-\int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega u_t (Mu)_t dxdt = -\int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega u_t (2h.\nabla u + (n-1)u)_t dxdt$$

$$= -2\int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega u_t (h.\nabla u)_t dxdt - \int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega u_t ((n-1)u)_t dxdt,$$

which implies that

$$-\int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega u_t (Mu)_t dxdt = -2\int_S^T E^{\frac{m_2-2}{2}}(t) \int_\Omega u_t (h.\nabla u_t) dxdt$$

$$-(n-1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u_t|^2 dxdt. \tag{3.7}$$

But

$$-\int_\Omega u_t(h.\nabla u_t)dx = -\int_\Gamma (h.\nu)|u_t|^2 d\Gamma + \int_\Omega u_t div(u_t h)dx.$$

If we apply the identity (2.15) for

$$j = u_t \quad and \quad k = h,$$

we obtain

$$-\int_\Omega u_t(h.\nabla u_t)dx = -\int_\Gamma (h.\nu)|u_t|^2 d\Gamma + \int_\Omega u_t(u_t divh + h.\nabla u_t)dx$$

$$= -\int_\Gamma (h.\nu)|u_t|^2 d\Gamma + \int_\Omega divh|u_t|^2 dx + \int_\Omega u_t(h.\nabla u_t)dx,$$

this leads to

$$-2\int_\Omega u_t(h.\nabla u_t)dx = -\int_\Gamma (h.\nu)|u_t|^2 d\Gamma + n\int_\Omega |u_t|^2 dx.$$

Now, if we replace the above result in (3.7) we find

$$-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega u_t(Mu)_t dxdt = \int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u_t|^2 dxdt$$

$$-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma (h.\nu)|u_t|^2 d\Gamma dt.$$

So, I_1 takes the form

$$I_1 = [E^{\frac{m_2-2}{2}}(t)\int_\Omega u_t Mudx]_S^T - \frac{m_2-2}{2}\int_S^T E^{\frac{m_2-4}{2}}(t)E_t(t)\int_\Omega u_t Mudxdt$$

$$+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u_t|^2 dxdt - \int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma (h.\nu)|u_t|^2 d\Gamma dt.$$

For I_2, we use the Green formula (1.3) to obtain

$$I_2 = -\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \Delta u Mudxdt = -\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma \frac{\partial u}{\partial \nu} Mud\Gamma dt$$

$$+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \nabla u\nabla(Mu)dxdt.$$

For the second term in the above identity, we have

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \nabla u\nabla(Mu)dxdt = \int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \nabla u\nabla(2h.\nabla u+(n-1)u)dxdt,$$

it follows that

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \nabla u\nabla(Mu)dxdt = 2\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \nabla u.\nabla(h.\nabla u)dxdt$$

$$+(n-1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\nabla u|^2 dxdt. \tag{3.8}$$

But

$$\nabla u\nabla(h.\nabla u) = |\nabla u|^2+\frac{1}{2}h.\nabla(|\nabla u|^2).$$

Therefore, we get

$$\int_\Omega \nabla u\nabla(h.\nabla u)dx = \int_\Omega |\nabla u|^2dx+\frac{1}{2}\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma-\frac{1}{2}\int_\Omega divh|\nabla u|^2dx$$

$$=\frac{2-n}{2}\int_\Omega |\nabla u|^2dx+\frac{1}{2}\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma.$$

Then, by replacing it in (3.8), we find

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega \nabla u\nabla(Mu)dxdt = (2-n)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\nabla u|^2dxdt$$

$$+(n-1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\nabla u|^2dxdt+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma dt$$

$$=\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\nabla u|^2dxdt+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma dt.$$

Hence

$$I_2=\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\nabla u|^2dxdt+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma (h.\nu)|\nabla u|^2 d\Gamma dt$$

$$-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma \frac{\partial u}{\partial \nu} Mud\Gamma dt.$$

Inserting I_1 and I_2 in (3.6) to find

$$0=[E^{\frac{m_2-2}{2}}(t)\int_\Omega u_t Mudx]_S^T-\frac{m_2-2}{2}\int_S^T E^{\frac{m_2-4}{2}}(t)E_t(t)\int_\Omega u_t Mudxdt$$
$$+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |u_t|^2dxdt+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Omega |\nabla u|^2dxdt$$
$$-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma \frac{\partial u}{\partial \nu}Mud\Gamma dt+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma (h.\nu)(|\nabla u|^2-|u_t|^2)d\Gamma dt.$$

Thus, we can write it as following

$$\int_S^T E^{\frac{m_2-2}{2}}(t)(\int_\Omega |u_t|^2dx+\int_\Omega |\nabla u|^2dx)dt=I_\Omega+I_{[S,T]\times\Omega}+I_{[S,T]\times\Gamma}, \tag{3.9}$$

where

$$I_\Omega=-[E^{\frac{m_2-2}{2}}(t)\int_\Omega u_t Mudx]_S^T,$$

$$I_{[S,T]\times\Omega}=\frac{m_2-2}{2}\int_S^T E^{\frac{m_2-4}{2}}(t)E_t(t)\int_\Omega u_t Mudxdt$$

and

$$I_{[S,T]\times\Gamma}=\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma \frac{\partial u}{\partial \nu}Mud\Gamma dt+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma (h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt.$$

Step 2: Energy inequality

First, we have

$$\int_\Omega |Mu|^2dx=\int_\Omega |2h.\nabla u+(n-1)u|^2dx\le\int_\Omega (|2h.\nabla u|+|(n-1)u|)^2dx.$$

By the algebric inequality (1.7), we obtain

$$\int_\Omega |Mu|^2dx\le 2\int_\Omega |2h.\nabla u|^2dx+2\int_\Omega |(n-1)u|^2dx.$$

In the rest of the proof, C represents a positive generic constant.
By the Poincare's inequality (1.1), we get

$$\int_\Omega |Mu|^2 dx \leq C\|\nabla u\|_2^2.$$

Hence, we have

$$|\int_\Omega u_t M u dx| \leq \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\int_\Omega |Mu|^2 dx \leq C(\|u_t\|_2^2 + \|\nabla u\|_2^2),$$

but, by the definition of the energy, we obtain

$$|\int_\Omega u_t M u dx| \leq CE(t). \tag{3.10}$$

Then, the first term I_Ω became

$$\begin{aligned} I_\Omega &= E^{\frac{m_2-2}{2}}(S)\int_\Omega u_t(S)Mu(S)dx - E^{\frac{m_2-2}{2}}(T)\int_\Omega u_t(T)Mu(T)dx \\ &\leq CE^{\frac{m_2-2}{2}}(S)E(S) + CE^{\frac{m_2-2}{2}}(T)E(T). \end{aligned}$$

Since the energy E is a decreasing function, then

$$I_\Omega \leq CE(S). \tag{3.11}$$

For the second term $I_{[S,T]\times\Omega}$, Thanks to (3.10) we have

$$\begin{aligned} I_{[S,T]\times\Omega} &\leq C\int_S^T E^{\frac{m_2-4}{2}}(t)(-E_t(t))E(t)dt = C\int_S^T E^{\frac{m_2-2}{2}}(t)(-E_t(t))dt \\ &= C[E^{\frac{m_2}{2}}(S) - E^{\frac{m_2}{2}}(T)]. \end{aligned}$$

This implies that

$$I_{[S,T]\times\Omega} \leq CE(S). \tag{3.12}$$

For the third term $I_{[S,T]\times\Gamma}$, we have

$$\begin{aligned} I_{[S,T]\times\Gamma} &= \int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma \frac{\partial u}{\partial \nu} Mu d\Gamma dt + \int_S^T E^{\frac{m_2-2}{2}}(t)\int_\Gamma (h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt \\ &= \int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0} \frac{\partial u}{\partial \nu} Mu d\Gamma dt + \int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1} \frac{\partial u}{\partial \nu} Mu d\Gamma dt \end{aligned}$$

$$+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt$$

$$+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt$$

$$=I_{[S,T]\times\Gamma_0}+I_{[S,T]\times\Gamma_1}, \tag{3.13}$$

where

$$I_{[S,T]\times\Gamma_0}=\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}\frac{\partial u}{\partial\nu}Mud\Gamma dt+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt$$

and

$$I_{[S,T]\times\Gamma_1}=\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1}\frac{\partial u}{\partial\nu}Mud\Gamma dt+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt.$$

For $I_{[S,T]\times\Gamma_0}$, we use the definition of Mu to get

$$I_{[S,T]\times\Gamma_0}=2\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}(h.\nabla u)\frac{\partial u}{\partial\nu}d\Gamma dt+(n-1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}u\frac{\partial u}{\partial\nu}d\Gamma dt$$

$$+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}(h.\nu)(|u_t|^2-|\nabla u|^2)d\Gamma dt.$$

But, we have

$$u=0,$$

then

$$u_t=0.$$

Hence, we obtain

$$I_{[S,T]\times\Gamma_0}=2\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}(h.\nabla u)\frac{\partial u}{\partial\nu}d\Gamma dt-\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}(h.\nu)|\nabla u|^2d\Gamma dt.$$

On the other hand, we have

$$\nabla u=\frac{\partial u}{\partial\nu}.\nu,$$

then

$$|\nabla u|^2 = |\frac{\partial u}{\partial \nu}|^2,$$

which implies that

$$I_{[S,T]\times\Gamma_0} = 2\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}(h.\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt - \int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}(h.\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt.$$

Therefore, we can write the term $I_{[S,T]\times\Gamma_0}$ as following

$$I_{[S,T]\times\Gamma_0} = \int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_0}(h.\nu)|\frac{\partial u}{\partial \nu}|^2 d\Gamma dt.$$

Since

$$h.\nu \leq 0 \ \ on \ \ \Gamma_0,$$

so, we arrive at

$$I_{[S,T]\times\Gamma_0} \leq 0. \tag{3.14}$$

For $I_{[S,T]\times\Gamma_1}$, we use the definition of Mu to find

$$I_{[S,T]\times\Gamma_1} = \int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1}(2h.\nabla u)\frac{\partial u}{\partial \nu} d\Gamma dt$$

$$+(n-1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1} u\frac{\partial u}{\partial \nu} d\Gamma dt + \int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1}(h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt.$$

Then

$$I_{[S,T]\times\Gamma_1} = -\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1}(2h.\nabla u)(h.\nu)g(x,u_t) d\Gamma dt$$

$$-(n-1)\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1} u(h.\nu)g(x,u_t) d\Gamma dt$$

$$+\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1}(h.\nu)(|u_t|^2 - |\nabla u|^2) d\Gamma dt.$$

By applying the Young inequality (1.5) for $a = u$, $b = g(x, u_t)$, $\epsilon = \epsilon_1 > 0$ and $p = q = 2$, we find for all $\epsilon_1 > 0$

$$I_{[S,T]\times\Gamma_1} \leq \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} \left[2(\frac{h^2}{2}|g(x,u_t)|^2 + \frac{1}{2}|\nabla u|^2) \right] (h.\nu) d\Gamma dt$$

$$+ \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} \left[(n-1)(\frac{\epsilon_1}{2}|u|^2 + \frac{1}{2\epsilon_1}|g(x,u_t)|^2) \right] (h.\nu) d\Gamma dt$$

$$- \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} (h.\nu)|\nabla u|^2 d\Gamma dt + \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} (h.\nu)|u_t|^2 d\Gamma dt$$

$$= \epsilon_1 C \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} |u|^2 (h.\nu) d\Gamma dt + C(\epsilon_1) \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} |g(x,u_t)|^2 (h.\nu) d\Gamma dt$$

$$+ \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} (h.\nu)|u_t|^2 d\Gamma dt.$$

We put $d\Gamma_h = (h.\nu)d\Gamma$ to obtain

$$I_{[S,T]\times\Gamma_1} \leq \epsilon_1 C \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} |u|^2 d\Gamma_h dt + \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} |u_t|^2 d\Gamma_h dt$$

$$+ C(\epsilon_1) \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} |g(x,u_t)|^2 d\Gamma_h dt.$$

We have

$$\int_{\Gamma_1} |u|^2 d\Gamma_h \leq \frac{1}{2} \int_\Omega |\nabla u|^2 dx \leq CE(t).$$

So, the term on Γ_1 became

$$I_{[S,T]\times\Gamma_1} \leq \epsilon_1 C \int_S^T E^{\frac{m_2}{2}}(t) dt + C(\epsilon_1) \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} (|g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt. \tag{3.15}$$

We put

$$\Gamma_1 = \Gamma_- \cup \Gamma_+,$$

with

$$\Gamma_- = \{x \in \Gamma_1; |u_t| \leq 1\}$$

and

$$\Gamma_+ = \{x \in \Gamma_1; |u_t| > 1\}.$$

Then, we obtain

$$\int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} |(g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt = \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_-} (|g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt$$

$$+ \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_+} (|g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt. \tag{3.16}$$

We have

$$\begin{aligned} |g(x,u_t)|^2 &= |g(x,u_t)|^{\frac{2}{m(x)}} (|g(x,u_t)|^{\frac{-2}{m(x)}} |g(x,u_t)|^2) \\ &= |g(x,u_t)|^{\frac{2}{m(x)}} |g(x,u_t)|^{\frac{2(m(x)-1)}{m(x)}} \end{aligned}$$

and

$$|u_t|^2 = |u_t|^{\frac{2}{m(x)}} (|u_t|^{\frac{-2}{m(x)}} |u_t|^2) = |u_t|^{\frac{2}{m(x)}} |u_t|^{\frac{2(m(x)-1)}{m(x)}}.$$

For $|u_t| \leq 1$, we use the assumptions (3.3) on the boundary term g to find

$$|g(x,u_t)|^2 \leq C |g(x,u_t)u_t|^{\frac{2}{m(x)}}$$

and

$$|u_t|^2 \leq C |g(x,u_t)u_t|^{\frac{2}{m(x)}},$$

which gives

$$\int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_-} (|g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt \leq C \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_-} |g(x,u_t)u_t|^{\frac{2}{m(x)}} d\Gamma_h dt.$$

Thus, we obtain

$$\int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_-} (|g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt \leq C \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_-} C_2^{\frac{2}{m(x)}} \left| \frac{g(x,u_t)u_t}{C_2} \right|^{\frac{2}{m(x)}} d\Gamma_h dt.$$

Since,

$$\frac{2}{m_2} \leq \frac{2}{m(.)} \ \ and \ \ \left| \frac{g(.,u_t)u_t}{C_2} \right| \leq 1 \ \ on\ \Gamma_-,$$

then

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_-}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \leq C\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_-\subset\Gamma_1} C_2^{\frac{2}{m(x)}}\left|\frac{g(x,u_t)u_t}{C_2}\right|^{\frac{2}{m_2}} d\Gamma_h dt$$

$$\leq C\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_1}|g(x,u_t)u_t|^{\frac{2}{m_2}} d\Gamma_h dt$$

By the embedding $L^1(\Gamma_1)$ in $L^{\frac{2}{m_2}}(\Gamma_1)$, we get

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_-}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \leq C\int_S^T E^{\frac{m_2-2}{2}}(t)[\int_{\Gamma_1} g(x,u_t)u_t d\Gamma_h]^{\frac{2}{m_2}} dt$$

$$\leq C\int_S^T E^{\frac{m_2-2}{2}}(t)(-E_t(t))^{\frac{2}{m_2}} dt. \tag{3.17}$$

If $m_2 > 2$, we apply the above Young inequality (1.5), with

$$a = E^{\frac{m_2-2}{2}}(t), \quad b = (-E_t(t))^{\frac{2}{m_2}}, \quad \epsilon = \epsilon_2, \quad p = \frac{m_2}{m_2-2} \quad and \quad q = \frac{m_2}{2},$$

to find

$$\int_S^T E^{\frac{m_2-2}{2}}(t)(-E_t(t))^{\frac{2}{m_2}} dt \leq \epsilon_2\int_S^T E^{\frac{m_2}{2}}(t)dt + C(\epsilon_2)\int_S^T(-E_t(t))dt \ \ for\ all\ \epsilon_2 > 0.$$

This implies that

$$\int_S^T E^{\frac{m_2-2}{2}}(t)(-E_t(t))^{\frac{2}{m_2}} dt \leq \epsilon_2\int_S^T E^{\frac{m_2}{2}}(t)dt + C(\epsilon_2)E(S).$$

If $m_2 = 2$, we obtain

$$\int_S^T E^{\frac{m_2-2}{2}}(t)(-E_t(t))^{\frac{2}{m_2}} dt \leq CE(S).$$

Hence, (3.17) became

$$\int_S^T E^{\frac{m_2-2}{2}}(t)\int_{\Gamma_-}(|g(x,u_t)|^2+|u_t|^2)d\Gamma_h dt \leq \epsilon_2 C\int_S^T E^{\frac{m_2}{2}}(t)dt + C(\epsilon_2)E(S). \tag{3.18}$$

Now, for $|u_t| > 1$, we use the assumption on g to obtain

$$\int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_+} (|g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt \le (\frac{1}{C_5} + C_6) \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_+ \subset \Gamma_1} g(x,u_t) u_t d\Gamma_h dt$$

$$\le C \int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_1} g(x,u_t) u_t d\Gamma_h dt \le C \int_S^T E^{\frac{m_2-2}{2}}(t)(-E_t(t)) dt,$$

then

$$\int_S^T E^{\frac{m_2-2}{2}}(t) \int_{\Gamma_+} (|g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt \le CE(S). \tag{3.19}$$

We insert (3.18), (3.19) in (3.16) to find

$$\int_S^T E^{\frac{m-2}{2}}(t) \int_{\Gamma_1} |(g(x,u_t)|^2 + |u_t|^2) d\Gamma_h dt \le \epsilon_2 C \int_S^T E^{\frac{m_2}{2}}(t) dt + C(\epsilon_2) E(S). \tag{3.20}$$

After that, we put the result (3.20) in (3.15) to obtain

$$I_{[S,T]\times\Gamma_1} \le \epsilon_1 C \int_S^T E^{\frac{m_2}{2}}(t) dt + \epsilon_2 C(\epsilon_1) \int_S^T E^{\frac{m_2}{2}}(t) dt + C(\epsilon_1, \epsilon_2) E(S)$$

$$\le (\epsilon_1 C + \epsilon_2 C(\epsilon_1)) \int_S^T E^{\frac{m_2}{2}}(t) dt + C(\epsilon_1, \epsilon_2) E(S). \tag{3.21}$$

(3.14) and (3.21) in (3.13) gives

$$I_{[S,T]\times\Gamma} \le (\epsilon_1 C + \epsilon_2 C(\epsilon_1)) \int_S^T E^{\frac{m_2}{2}}(t) dt + C(\epsilon_1, \epsilon_2) E(S). \tag{3.22}$$

Step 3: End of the proof

Combining (3.11), (3.12) and (3.22) in (3.9) and using the definition of the energy to get

$$\int_S^T E^{\frac{m_2}{2}}(t) dt \le (\epsilon_1 C + \epsilon_2 C(\epsilon_1)) \int_S^T E^{\frac{m_2}{2}}(t) dt + C(\epsilon_1, \epsilon_2) E(S).$$

Taking ϵ_1 sufficiently small, then ϵ_2 sufficiently small to find

$$\int_S^T E^{\frac{m_2}{2}}(t) dt \le CE(S).$$

Then, the result (3.5) is finally obtained by the integral inequalities due to Komornik [19].

□

CHAPTER 4

COUPLED WAVE EQUATIONS WITH INTERNAL/BOUNDARY DAMPING AND SOURCE TERMS: BLOW UP

This result was submitted by Boulmerka and Hamchi.

This chapter is devoted to the study of the following system

$$\begin{cases} u_{tt} - \Delta u + g_1(x, u_t) = f_1(u, v) & in \quad (0,T) \times \Omega, \\ v_{tt} - \Delta v = f_2(u, v) & in \quad (0,T) \times \Omega, \\ u = 0 & on \quad (0,T) \times \Gamma, \\ v = 0 & on \quad (0,T) \times \Gamma_0, \\ \partial_\nu v + g_2(x, v_t) = \psi(x, v) & on \quad (0,T) \times \Gamma_1, \\ (u(0), v(0)) = (u_0, v_0) \; and \; (u_t(0), v_t(0)) = (u_1, v_1) & in \quad \Omega. \end{cases} \tag{4.1}$$

Here, Ω *is a bounded domain of* $\mathbb{R}^n (n \geq 1)$ *with sufficiently smooth boundary* $\Gamma = \Gamma_0 \cup \Gamma_1$ *with* $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Δ *is the Laplacien with respect to the spatial variables.* $\partial_\nu v = \nabla v . \nu$ *where* ν *is the unit outward normal vector to* Γ *and* ∇v *is the gradient with respect to the spatial variables. Moreover,* f_1 *and* f_2 *are the coupling source terms,* g_1 *and* g_2 *represent the internal/boundary damping terms and* ψ *is the boundary source term.*

The principal goal of this work is to prove two main results concerning the system (4.1). First, we show that the weak solution with negative initial energy blows up in finite time. Then, we prove the same result for the weak solution with positive initial energy.

This chapter is organized as follows: section 4.1 *contains assumptions on the parameters in (4.1) needed to obtain our results. In section* 4.2*, we present the preliminaries which are divided into two subsections: some useful Lemmas and the local existence result. In section* 4.3*, we give, first, some properties of the functional* H *in the case of negative initial energy. After that, we state and prove our first blow up result. In section* 4.4*, we give an important Lemmas and properties of the functional* H *when the initial energy data is positive and, also, state and prove the second blow up result.*

Throughout this chapter, we use the following assumptions:

4.1 Assumptions

$(A1)$ ***Assumptions on*** f_1 ***and*** f_2***:***

• *For all* $(u, v) \in \mathbb{R}^2$*, we have*

$$f_1(u, v) = |u + v|^{2(\rho+1)}(u + v) + |u|^{\rho}u|v|^{\rho+2} \tag{4.2}$$

and

$$f_2(u, v) = |u + v|^{2(\rho+1)}(u + v) + |v|^{\rho}v|u|^{\rho+2}, \tag{4.3}$$

where

$$\begin{cases} -1 < \rho & if \;\; n = 1.2, \\ -1 < \rho \leq \frac{3-n}{n-2} & if \;\; n \geq 3. \end{cases}$$

$(A2)$ ***Assumptions on*** g_1 ***and*** g_2 ***:***

• *We assume that the function* g_1 *is monotone, continuous in* $\Omega \times \mathbb{R}$ *with* $g_1(.,0) = 0$ *and there exists three positive constants* m_1*,* C_1 *and* C_2*, such that*

$$2 \leq m_1 < 2(\rho + 2)$$

and

$$C_1|u|^{m_1} \leq g_1(x, u)u \leq C_2|u|^{m_1} \quad for\ all \quad x \in \Omega \quad and \quad u \in \mathbb{R}. \tag{4.4}$$

• *We shall assume that the function g_2 is monotone, continuous function on $\Gamma_1 \times \mathbb{R}$ with $g_2(.,0) = 0$ and there exists three positive constants m_2 , C_3 and C_4, such that*

$$2 \leq m_2$$

and

$$C_3|v|^{m_2} \leq g_2(x,v)v \leq C_4|v|^{m_2} \quad for\ all \quad x \in \Gamma_1 \quad and \quad v \in \mathbb{R}. \tag{4.5}$$

$(A3)$ ***Assumptions on*** ψ ***:***

• *We asume that the function ψ is defined on $\Gamma_1 \times \mathbb{R}$ and there exists three positive constants k, C_5 and C_6, such that*

$$k > max\{m_2, \frac{2C_6}{C_5}\},\ C_5 \geq 1$$

and

$$C_5|v|^k \leq \psi(x,v)v \leq C_6|v|^k \quad for\ all \quad x \in \Gamma_1 \ \ and \ \ v \in \mathbb{R}. \tag{4.6}$$

4.2 Preliminaries

In this part, we recall some technical results that we will need them later.

4.2.1 Useful Lemmas

By the definition of f_1 and f_2, we find the following relation between them.

Lemma 4.1.
For all $(u,v) \in \mathbb{R}^2$, we have

$$uf_1(u,v) + vf_2(u,v) = 2(\rho+2)F(u,v), \tag{4.7}$$

where

$$F(u,v) = \frac{1}{2(\rho+2)}(|u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}). \tag{4.8}$$

Proof. We use the definitions (4.2) of f_1 and (4.3) of f_2, we find

$$uf_1(u,v) + vf_2(u,v) = |u+v|^{2(\rho+1)}(u+v)(u+v) + 2|u|^{\rho+2}|v|^{\rho+2}$$

$$= |u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}.$$

We multiply this result by $\frac{2(\rho+2)}{2(\rho+2)}$ to obtain

$$uf_1(u,v) + vf_2(u,v) = \frac{2(\rho+2)}{2(\rho+2)}(|u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}),$$

hence

$$uf_1(u,v) + vf_2(u,v) = 2(\rho+2)F(u,v).$$

The proof is, now, completed. □

Next, we present the following property of the function F.

Lemma 4.2. *[26]*
There exists two positive constants C_7 and C_8 such that, for all $(u,v) \in \mathbb{R}^2$, we have

$$\frac{C_7}{2(\rho+2)}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}) \leq F(u,v) \leq \frac{C_8}{2(\rho+2)}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}). \tag{4.9}$$

Proof. We start with the left hand side of (4.9), we have:
i) If $u = v = 0$, the result is trivial.
ii) If $v \neq 0$ and $|u| \leq |v|$, we have

$$F(u,v) = \frac{1}{2(\rho+2)}|v|^{2(\rho+2)}(|1+\frac{u}{v}|^{2(\rho+2)} + 2|\frac{u}{v}|^{\rho+2}).$$

We consider the following positive continuous function in $[-1,1]$ defined by

$$j(s) = |1+s|^{2(\rho+2)} + 2|s|^{\rho+2}.$$

We have

$$\min_{-1\leq s\leq 1} j(s) \geq 0.$$

If $\min_{-1\leq s\leq 1} j(s) = 0$ then, for some $s_0 \in [-1,1]$, we find

$$\min_{-1\leq s\leq 1} j(s) = j(s_0) = |1+s_0|^{2(\rho+2)} + 2|s_0|^{\rho+2} = 0.$$

This implies that

$$|1+s_0| = |s_0| = 0,$$

which is impossible. Thus

$$2C_7 = \min_{-1\leq s\leq 1} j(s) > 0,$$

therefore

$$F(u,v) \geq \frac{C_7}{\rho+2}|v|^{2(\rho+2)} \geq \frac{C_7}{\rho+2}|u|^{2(\rho+2)}.$$

Consequently,

$$2F(u,v) \geq \frac{C_7}{\rho+2}(|v|^{2(\rho+2)} + |u|^{2(\rho+2)}),$$

or

$$\frac{C_7}{2(\rho+2)}(|v|^{2(\rho+2)} + |u|^{2(\rho+2)}) \leq F(u,v).$$

iii) If $v \neq 0$ and $|u| > |v|$, we follow the same steps as in $ii)$ to obtain

$$F(u,v) = \frac{1}{2(\rho+2)}|u|^{2(\rho+2)}(|1+\frac{v}{u}|^{2(\rho+2)} + 2|\frac{v}{u}|^{\rho+2}) \geq \frac{C_7}{\rho+2}|u|^{2(\rho+2)} \geq \frac{C_7}{\rho+2}|v|^{2(\rho+2)},$$

then, we have

$$\frac{C_7}{2(\rho+2)}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}) \leq F(u,v).$$

At the end, in all cases, we find our goal.

Now, for the right hand side of inequality (4.9). So, we use the algebric inequality (1.7) for $a = |u|$, $b = |v|$ and $p = 2(\rho+2)$ to find

$$|u+v|^{2(\rho+2)} \leq (|u|+|v|)^{2(\rho+2)} \leq 2^{2\rho+3}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}). \tag{4.10}$$

We apply, also, the Young inequality (1.6) for $a = |u|$, $b = |v|$, $\delta = 1$ and $p = q = 2$ to get

$$|u||v| \leq \frac{1}{2}(|u|^2 + |v|^2),$$

then, we find

$$|u|^{\rho+2}|v|^{\rho+2} \leq (\frac{1}{2})^{\rho+2}(|u|^2 + |v|^2)^{\rho+2}.$$

We use, again, (1.7) to obtain

$$|uv|^{\rho+2} = |u|^{\rho+2}|v|^{\rho+2} \leq (\frac{1}{2})^{\rho+2}2^{\rho+1}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)})$$

$$= \frac{1}{2}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}), \tag{4.11}$$

(4.10) and (4.11) give

$$F(u,v) \leq \frac{1}{2(\rho+2)}\left(2^{2\rho+3}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}) + (|u|^{2(\rho+2)} + |v|^{2(\rho+2)})\right)$$

$$\leq \frac{2^{2\rho+3}+1}{2(\rho+2)}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}).$$

So, we deduce that

$$F(u,v) \leq \frac{C_8}{2(\rho+2)}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}),$$

where $C_8 = 2^{2\rho+3} + 1$. □

After that, we set the last important result in this section, by exploting the embedding $H_0^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$ *and* $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$.

Lemma 4.3. *[26]*
There esists $\eta > 0$ *such that, for any* $(u, v) \in H_0^1(\Omega) \times H^1_{\Gamma_0}(\Omega)$, *we have*

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \leq \eta(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\rho+2}, \tag{4.12}$$

where

$$\|u\|_{2(\rho+2)} = (\int_\Omega |u|^{2(\rho+2)}dx)^{\frac{1}{2(\rho+2)}}.$$

Proof. We have

$$\|u+v\|_{2(\rho+2)} \leq \|u\|_{2(\rho+2)} + \|v\|_{2(\rho+2)}.$$

By using (1.7) for

$$a = \|u\|_{2(\rho+2)},\ b = \|v\|_{2(\rho+2)}\ \ and\ \ p = 2,$$

we obtain

$$\|u+v\|_{2(\rho+2)}^2 \leq (\|u\|_{2(\rho+2)} + \|v\|_{2(\rho+2)})^2 \leq 2(\|u\|_{2(\rho+2)}^2 + \|v\|_{2(\rho+2)}^2).$$

Thus

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)} \leq 2^{\rho+2}(\|u\|_{2(\rho+2)}^2 + \|v\|_{2(\rho+2)}^2)^{\rho+2}. \tag{4.13}$$

Also, using the Hölder inequality (1.2)

$$for\ all\ \ 0 < p, q, r < \infty\ \ such\ that\ \ \frac{1}{p} + \frac{1}{q} = \frac{1}{r}\ \ and\ \ f \in L^p(\Omega)\ \ g \in L^q(\Omega),$$

for $p = 2(\rho+2)$, $q = 2(\rho+2)$, $r = \rho+2$ and $f = u$, $g = v$, to obtain

$$\|uv\|_{\rho+2} \leq \|u\|_{2(\rho+2)}\|v\|_{2(\rho+2)}.$$

Applying the Young inequality (1.4) to find

$$\|uv\|_{\rho+2} \leq \frac{1}{2}(\|u\|_{2(\rho+2)}^2 + \|v\|_{2(\rho+2)}^2).$$

Therefore

$$\|uv\|_{\rho+2}^{\rho+2} \leq \frac{1}{2^{\rho+2}}(\|u\|_{2(\rho+2)}^2 + \|v\|_{2(\rho+2)}^2)^{\rho+2}. \tag{4.14}$$

After that, suming up (4.13) and (4.14) to find

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \leq (2^{\rho+2} + \frac{1}{2^{\rho+1}})(\|u\|_{2(\rho+2)}^2 + \|v\|_{2(\rho+2)}^2)^{\rho+2}.$$

The embedding $H_0^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$ and $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$ yield to

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \leq \eta(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\rho+2},\ \ \eta > 0.$$

This conclude the proof. □

4.2.2 Existence of the local weak solution

In this subsection, we define the maximal weak solution of the system (4.1). Then, we present some important properties of this solution.

Definition 4.1.

A pair of functions (u, v) is said to be a weak solution of (4.1) on $(0, T)$, where $T > 0$, if

$$u \in C((0,T), H_0^1(\Omega)), u_t \in C((0,T), L^2(\Omega)) \cap L^{m_1}(\Omega \times (0,T)),$$

$$v \in C((0,T), H_{\Gamma_0}^1(\Omega)), v_t \in C((0,T), L^2(\Omega)) \cap L^{m_2}(\Gamma_1 \times (0,T)),$$

$$(u(0), v(0)) = (u_0, v_0) \in H_0^1(\Omega) \times H_{\Gamma_0}^1(\Omega),$$

$$(u_t(0), v_t(0)) = (u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$$

and, for all test functions:

$$\phi \in C((0,T), H_0^1(\Omega)) \cap L^{m_1}(\Omega \times (0,T)) \ \ with \ \ \phi_t \in L^1((0,T), L^2(\Omega))$$

and

$$\varphi \in C((0,T), H_{\Gamma_0}^1(\Omega)) \cap L^{m_2}(\Gamma_1 \times (0,T)) \ \ with \ \ \varphi_t \in L^1((0,T), L^2(\Omega)),$$

we have for all t $\in (0, T)$

$$\frac{d}{dt}\int_\Omega u_t\phi dx - \int_\Omega u_t\phi_t dx + \int_\Omega \nabla u \nabla \phi dx + \int_\Omega g_1(x, u_t)\phi dx = \int_\Omega f_1(u,v)\phi dx$$

and

$$\frac{d}{dt}\int_\Omega v_t\varphi dx - \int_\Omega v_t\varphi_t dx + \int_\Omega \nabla v \nabla \varphi dx + \int_{\Gamma_1} g_2(x, v_t)\varphi d\Gamma = \int_\Omega f_2(u,v)\varphi dx + \int_\Gamma \psi(x,v)\varphi d\Gamma.$$

Now, we present the definition of the maximal solution of the system (4.1).

Definition 4.2.

We say that a weak solution u is maximal if it cannot be a restriction of a weak solution in $(0, T')$, where $T < T'$.

By using the idea in [1], we can obtain the following result.

Theorem 4.1.
There exists a unique maximal weak solution (u, v) *of (4.1) defined on* $(0, T)$ *for some* $T > 0$*. Also, the following alternative holds:*

$$T = +\infty,$$

or

$$T < \infty \quad and \quad \lim_{t \to T}(\|u_t\|_2 + \|v_t\|_2 + \|\nabla u\|_2 + \|\nabla v\|_2) = +\infty.$$

Using the $Definition$ 4.1 *of the weak solution and the assumptions (4.4) and (4.5) on* g_1 *and* g_2*, we can obtain the decreasing of the energy functional of the system (4.1).*

Lemma 4.4.
We have

$$\frac{dE(t)}{dt} \leq -\int_{\Omega} g_1(x, u_t)u_t dx - \int_{\Gamma_1} g_2(x, v_t)v_t d\Gamma \leq 0 \quad for\ all\ t \in (0, T), \tag{4.15}$$

where E *is the energy functional associated to our system, defined for all* $t \in (0, T)$*, by*

$$E(t) = \frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2) + \frac{1}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) - \int_{\Omega} F(u, v)dx - \int_{\Gamma_1} \Psi(x, v)d\Gamma, \tag{4.16}$$

here

$$\Psi(x, v) = \int_0^v \psi(x, \tau)d\tau \quad for\ all \quad x \in \Gamma_1 \quad and \quad v \in \mathbb{R}.$$

4.3 First main result

In this section, we show that the weak solution defined in $Theorem$ 4.1 *blows up in finite time when the initial energy data is negative. To reach this end, we consider the following functional* H *defined by*

$$H(t) = -E(t) \quad for\ all\ t \in (0, T). \tag{4.17}$$

4.3.1 Properties of the functional H

In the following two Lemmas, we state and proof some important tools that will play a major role in the proof of our first main result.

Lemma 4.5.

We assume that

$$E(0) < 0. \tag{4.18}$$

Then

$$0 < H(0) \leq H(t) \leq \frac{C_8}{2(\rho+2)}(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}) + \frac{C_6}{k}\|v\|_{k,\Gamma_1}^k. \tag{4.19}$$

Here

$$\|v\|_{k,\Gamma_1}^k = \int_{\Gamma_1} |v|^k d\Gamma.$$

Proof. We use the definition of H in (4.17), the decreasing of the energy and the fact that the initial energy is negative (4.18) to get

$$0 < H(0) \leq H(t).$$

Now, we have

$$H(t) = -\frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2) - \frac{1}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \int_\Omega F(u,v)dx + \int_{\Gamma_1} \Psi(x,v)d\Gamma.$$

Which implies that

$$H(t) \leq \int_\Omega F(u,v)dx + \int_{\Gamma_1} \Psi(x,v)d\Gamma.$$

As a result of (4.6), we have

$$\Psi(x,v) \leq \frac{C_6}{k}|v|^k \; for \; all \; x \in \Gamma_1 \; and \; v \in \mathbb{R}, \tag{4.20}$$

hence, (4.9) and (4.20) give

$$H(t) \leq \frac{C_8}{2(\rho+2)}(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}) + \frac{C_6}{k}\|v\|_{k,\Gamma_1}^k \; for \; all \; t \in (0,T).$$

This concludes the proof. □

In the rest of this chapter, C represents a generic positive constant.

Lemma 4.6.

Let

$$0 < \sigma \leq min\{\frac{\rho+1}{2(\rho+2)}, \frac{2(\rho+2)-m_1}{2(m_1-1)(\rho+2)}, \frac{k-m_2}{(m_2-1)k}\}.$$

We have

$$H^{\sigma(m_1-1)}(t)\|u\|_{2(\rho+2)}^{m_1} + H^{\sigma(m_2-1)}(t)\|v\|_{k,\Gamma_1}^{m_2}$$

$$\leq C(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^{k} + H(t)) \tag{4.21}$$

and

$$\int_{\Omega} g_1(x,u_t)udx + \int_{\Gamma_1} g_2(x,v_t)vd\Gamma \leq C(M_1^{1-m_1} + M_2^{1-m_2})(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}$$

$$+\|v\|_{k,\Gamma_1}^{k} + H(t)) + CH^{-\sigma}(t)H'(t) \ \ for\ all\ \ M_1, M_2 > 0. \tag{4.22}$$

Proof. Thanks to (4.19) and the fact that $m_1 < 2(\rho+2)$, we obtain

$$H^{\sigma(m_1-1)}(t)\|u\|_{2(\rho+2)}^{m_1} \leq \left[\frac{C_8}{2(\rho+2)}(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}) + \frac{C_6}{k}\|v\|_{k,\Gamma_1}^{k}\right]^{\sigma(m_1-1)}$$

$$(\|u\|_{2(\rho+2)}^{2(\rho+2)})^{m_1/(2(\rho+2))} \leq C(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^{k})^{\sigma(m_1-1)}$$

$$(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^{k})^{m_1/2(\rho+2)},$$

so

$$H^{\sigma(m_1-1)}(t)\|u\|_{2(\rho+2)}^{m_1} \leq C(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^{k})^{\sigma(m_1-1)+m_1/2(\rho+2)}. \tag{4.23}$$

Now, we use the following algebric inequality

$$z^{\nu} \leq (1+\frac{1}{a})(z+a) \quad for\ all\ z \geq 0,\ 0 < \nu \leq 1\ and\ a > 0, \tag{4.24}$$

for

$$z = \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^{k},\ \nu = \sigma(m_1-1) + m_1/2(\rho+2)\ and\ a = H(0),$$

to obtain

$$(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^{k})^{\sigma(m_1-1)+m_1/2(\rho+2)}$$

$$\leq C(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^{k} + H(0)).$$

Since $H(0) \leq H(t)$ for all $t \geq 0$, then, (4.23) became

$$H^{\sigma(m_1-1)}(t)\|u\|_{2(\rho+2)}^{m_1} \leq C(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^{k} + H(t)). \tag{4.25}$$

Similarly, we find

$$H^{\sigma(m_2-1)}(t)\|v\|_{k,\Gamma_1}^{m_2} \leq C(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^{k} + H(t)). \tag{4.26}$$

By summing up (4.25) and (4.26), we find (4.21).
Next, according to (4.4) and (4.5), we have

$$\int_\Omega g_1(x,u_t)u dx \leq C_2 \int_\Omega |u||u_t|^{m_1-1} dx$$

and

$$\int_{\Gamma_1} g_2(x,v_t)v d\Gamma \leq C_4 \int_{\Gamma_1} |v||v_t|^{m_2-1} d\Gamma.$$

We apply, the Young inequality (1.6)
• First, for $a = |u|$, $b = |u_t|^{m_1-1}$, $p = m_1$ and $q = \frac{m_1}{m_1-1}$ to obtain

$$\int_\Omega g_1(x,u_t)u dx \leq \frac{\delta_1^{m_1}}{m_1}\|u\|_{m_1}^{m_1} + \frac{m_1-1}{m_1}\delta_1^{\frac{-m_1}{m_1-1}}\|u_t\|_{m_1}^{m_1} \quad for\ all\ \delta_1 > 0.$$

• Second, for $a = |v|$, $b = |v_t|^{m_2-1}$, $p = m_2$ and $q = \frac{m_2}{m_2-1}$ to get

$$\int_{\Gamma_1} g_2(x,v_t)v d\Gamma \leq \frac{\delta_2^{m_2}}{m_2}\|v\|_{m_2,\Gamma_1}^{m_2} + \frac{m_2-1}{m_2}\delta_2^{\frac{-m_2}{m_2-1}}\|v_t\|_{m_2,\Gamma_1}^{m_2} \quad for\ all\ \delta_2 > 0.$$

By taking δ_1 and δ_2 such that

$$\delta_1^{m_1} = M_1^{1-m_1} H^{\sigma(m_1-1)}(t) \ \ and\ \ \delta_2^{m_2} = M_2^{1-m_2} H^{\sigma(m_2-1)}(t),$$

we find

$$\int_\Omega g_1(x,u_t)u dx \leq C M_1^{1-m_1} H^{\sigma(m_1-1)}(t)\|u\|_{m_1}^{m_1} + C H^{-\sigma}(t)\|u_t\|_{m_1}^{m_1}$$

and

$$\int_{\Gamma_1} g_2(x,v_t)v d\Gamma \leq C M_2^{1-m_2} H^{\sigma(m_2-1)}(t)\|v\|_{m_2,\Gamma_1}^{m_2} + C H^{-\sigma}(t)\|v_t\|_{m_2,\Gamma_1}^{m_2}.$$

Since $m_1 < 2(\rho+2)$ and $m_2 < k$, then

$$\int_\Omega g_1(x,u_t)u dx \leq C M_1^{1-m_1} H^{\sigma(m_1-1)}(t)\|u\|_{2(\rho+1)}^{m_1} + C H^{-\sigma}(t)\|u_t\|_{m_1}^{m_1} \tag{4.27}$$

and

$$\int_{\Gamma_1} g_2(x,v_t)v d\Gamma \leq C M_2^{1-m_2} H^{\sigma(m_2-1)}(t)\|v\|_{k,\Gamma_1}^{m_2} + C H^{-\sigma}(t)\|v_t\|_{m_2,\Gamma_1}^{m_2}. \tag{4.28}$$

(4.4), (4.15) and (4.17) gives

$$C_1\|u_t\|_{m_1}^{m_1} \le \int_\Omega g_1(x,u_t)u_t dx \le -E'(t) = H'(t),$$

which implies that

$$H^{-\sigma}(t)\|u_t\|_{m_1}^{m_1} \le CH^{-\sigma}(t)H'(t). \tag{4.29}$$

Similarly, we find

$$H^{-\sigma}(t)\|v_t\|_{m_2,\Gamma_1}^{m_2} \le CH^{-\sigma}(t)H'(t). \tag{4.30}$$

Therefore, we conclude by inserting (4.25) and (4.29) in (4.27) the estimate

$$\int_\Omega g_1(x,u_t)udx \le CM_1^{1-m_1}(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k + H(t))$$

$$+CH^{-\sigma}(t)H'(t). \tag{4.31}$$

Also, we put (4.26) and (4.30) in (4.28) to obtain

$$\int_{\Gamma_1} g_2(x,v_t)vd\Gamma \le CM_2^{1-m_2}(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k + CH(t))$$

$$+CH^{-\sigma}(t)H'(t). \tag{4.32}$$

Summing up (4.31) with (4.32), we get the wanted results (4.22). □

4.3.2 Blow up result

Now, we are in position to state and prove the first main result of this chapter.

Theorem 4.2.
Assume that

$$E(0) < 0.$$

Then, the weak solution of (4.1) blows up in finite time.

Proof. Let us introduce the following functional

$$L(t) = H^{1-\sigma}(t) + \epsilon\int_\Omega (uu_t + vv_t)dx \ \ for\ all\ \ t \in (0,T),$$

for ϵ small to be chosen later.
It is sufficient to prove that, for some $a_0 > 0$, we have

$$L'(t) \geq a_0 L^{1/(1-\sigma)}(t) \;\; for\; all \;\; t \in (0,T).$$

To reach this end, we proceed in several steps:
Step 1: By taking the derivative of L and using the $Definition$ 4.1 of the weak solution to the system (4.1), we get

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon(\|u_t\|_2^2 + \|v_t\|_2^2) - \epsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$
$$+\epsilon \int_\Omega (uf_1(u,v) + vf_2(u,v))dx - \epsilon \int_\Omega g_1(x,u_t)udx - \epsilon \int_{\Gamma_1} g_2(x,v_t)vd\Gamma + \int_{\Gamma_1} \psi(x,v)vd\Gamma.$$

Using (4.7), we obtain

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon(\|u_t\|_2^2 + \|v_t\|_2^2) - \epsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$
$$+\epsilon 2(\rho+2) \int_\Omega F(x,u)dx - \epsilon \int_\Omega g_1(x,u_t)udx - \epsilon \int_{\Gamma_1} g_2(x,v_t)vd\Gamma + \epsilon \int_{\Gamma_1} \psi(x,v)vd\Gamma.$$

Adding and substracting ϵpH for $2 < p < \min\{\frac{2(\rho+2)}{C_6}, \frac{kC_5}{C_6}\}$ to find

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon(\|u_t\|_2^2 + \|v_t\|_2^2) - \epsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$
$$+\epsilon 2(\rho+2) \int_\Omega F(x,u)dx - \epsilon \int_\Omega g_1(x,u_t)udx - \epsilon \int_{\Gamma_1} g_2(x,v_t)vd\Gamma$$
$$+\epsilon \int_{\Gamma_1} \psi(x,v)vd\Gamma + \epsilon pH(t) + \epsilon pE(t).$$

By the definition of E, we find

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon(1+\frac{p}{2})(\|u_t\|_2^2 + \|v_t\|_2^2) + \epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$
$$+\epsilon(2(\rho+2)-p) \int_\Omega F(x,u)dx - \epsilon \int_\Omega g_1(x,u_t)udx - \epsilon \int_{\Gamma_1} g_2(x,v_t)vd\Gamma$$
$$+\epsilon \int_{\Gamma_1} \psi(x,v)vd\Gamma - \epsilon p \int_{\Gamma_1} \Psi(x,v)d\Gamma + \epsilon pH(t).$$

(4.6) and (4.20) imply that

$$L'(t) \geq (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon(1+\frac{p}{2})(\|u_t\|_2^2 + \|v_t\|_2^2) + \epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$

$$+\epsilon pH(t) + \epsilon\overline{C}_1 \int_\Omega F(x,u)dx + \epsilon\overline{C}_2\|v\|_{k,\Gamma_1}^k - \epsilon\int_\Omega g_1(x,u_t)udx$$

$$-\epsilon\int_{\Gamma_1} g_2(x,v_t)vd\Gamma, \tag{4.33}$$

where

$$\overline{C}_1 = 2(\rho+2) - p > 0$$

and

$$\overline{C}_2 = C_5 - \frac{C_6 p}{k} > 0.$$

Inserting (4.22) in (4.33) to obtain

$$L'(t) \geq (1-\sigma-\epsilon C)H^{-\sigma}(t)H'(t) + \epsilon(1+\frac{p}{2})(\|u_t\|_2^2 + \|v_t\|_2^2) + \epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$

$$+\epsilon pH(t) + \epsilon\overline{C}_1\int_\Omega F(x,u)dx + \epsilon\overline{C}_2\|v\|_{k,\Gamma_1}^k - \epsilon C(M_1^{1-m_1} + M_2^{1-m_2})(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}$$

$$+\|v\|_{k,\Gamma_1}^k + H(t)).$$

Thanks to (4.9), we get

$$L'(t) \geq (1-\sigma-\epsilon C)H^{-\sigma}(t)H'(t) + \epsilon(1+\frac{p}{2})(\|u_t\|_2^2 + \|v_t\|_2^2) + \epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$

$$+\epsilon pH(t) + \epsilon\overline{C}_3(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{\rho+2}^{\rho+2}) + \epsilon\overline{C}_2\|v\|_{k,\Gamma_1}^k - \epsilon C(M_1^{1-m_1} + M_2^{1-m_2})(\|u\|_{2(\rho+2)}^{2(\rho+2)}$$

$$+\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k + H(t)),$$

where $\overline{C}_3 = \frac{C_7\overline{C}_1}{2(\rho+1)}$. Therefore

$$L'(t) \geq (1-\sigma-\epsilon C)H^{-\sigma}(t)H'(t) + \epsilon(1+\frac{p}{2})(\|u_t\|_2^2 + \|v_t\|_2^2) + \epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$

$$+\epsilon[p - C(M_1^{1-m_1} + M_2^{1-m_2})]H(t) + \epsilon[\overline{C}_3 - C(M_1^{1-m_1} + M_2^{1-m_2})](\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)})$$

$$+\epsilon[\overline{C}_2 - C(M_1^{1-m_1} + M_2^{1-m_2})]\|v\|_{k,\Gamma_1}^k. \tag{4.34}$$

First, we pick ϵ small enough so that $1-\sigma-\epsilon C \geq 0$. Then, we choose M_1 and M_2 sufficiently large to have

$$p - C(M_1^{1-m_1} + M_2^{1-m_2}) > 0, \;\; \overline{C}_3 - C(M_1^{1-m_1} + M_2^{1-m_2}) > 0$$

$$and \;\; \overline{C}_2 - C(M_1^{1-m_1} + M_2^{1-m_2}) > 0.$$

Consequentely, there exists $\zeta > 0$ such that (4.34) became

$$L'(t) \geq \zeta(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k). \tag{4.35}$$

Step 2: Now, according to (4.35), we have

$$L(t) \geq L(0) \quad for\ all \quad t \in (0, T),$$

where

$$L(0) = H^{1-\sigma}(0) + \epsilon \int_\Omega (u_0 u_1 + v_0 v_1)dx.$$

So,

• If

$$\int_\Omega (u_0 u_1 + v_0 v_1)dx \geq 0,$$

then

$$L(0) \geq 0.$$

• If

$$\int_\Omega (u_0 u_1 + v_0 v_1)dx < 0,$$

then, we choose ϵ such that

$$H^{1-\sigma}(0) + \epsilon \int_\Omega (u_0 u_1 + v_0 v_1)dx \geq 0,$$

as a consequence, we have

$$L(0) \geq 0.$$

So, L is a positive function.

Step 3: We have

$$L^{1/(1-\sigma)}(t) = [H^{1-\sigma}(t) + \epsilon \int_\Omega (uu_t + vv_t)dx]^{1/(1-\sigma)}$$

$$\leq [H^{1-\sigma}(t) + \epsilon| \int_\Omega (uu_t + vv_t)dx|]^{1/(1-\sigma)}.$$

Using the algebric inequality (1.7) for $a = H^{1-\sigma}(t), b = \epsilon|\int\limits_{\Omega}(uu_t + vv_t)dx|$ and $p = 1/(1-\sigma)$ to get

$$L^{1/(1-\sigma)}(t) \leq C[H(t) + |\int\limits_{\Omega}(uu_t + vv_t)dx|^{1/(1-\sigma)}],$$

Applying (1.7), again, to find

$$L^{1/(1-\sigma)}(t) \leq C[H(t) + (\int\limits_{\Omega}|u||u_t|dx)^{1/(1-\sigma)} + (\int\limits_{\Omega}|v||v_t|dx)^{1/(1-\sigma)}]. \tag{4.36}$$

But, using the Hölder inequality (1.2) and the fact that $2 \leq 2(\rho+2)$ give

$$\int\limits_{\Omega}|u||u_t|dx \leq (\int\limits_{\Omega}|u|^2dx)^{\frac{1}{2}}(\int\limits_{\Omega}|u_t|^2dx)^{\frac{1}{2}} \leq C\left[(\int\limits_{\Omega}|u|^{2(\rho+1)}dx)^{\frac{1}{2(\rho+1)}}(\int\limits_{\Omega}|u_t|^2dx)^{\frac{1}{2}}\right].$$

Therefore

$$(\int\limits_{\Omega}|u||u_t|dx)^{\frac{1}{1-\sigma}} \leq C\left[(\int\limits_{\Omega}|u|^{2(\rho+2)}dx)^{\frac{1}{2(1-\sigma)(\rho+2)}}(\int\limits_{\Omega}|u_t|^2dx)^{\frac{1}{2(1-\sigma)}}\right].$$

Using the Young inequality (1.6) to get

$$(\int\limits_{\Omega}|u||u_t|dx)^{\frac{1}{1-\sigma}} \leq C\left[(\int\limits_{\Omega}|u|^{2(\rho+2)}dx)^{\frac{p}{2(1-\sigma)(\rho+2)}} + (\int\limits_{\Omega}|u_t|^2dx)^{\frac{q}{2(1-\sigma)}}\right],$$

where

$$p = \frac{2(1-\sigma)}{1-2\sigma} \ \ and \ \ q = 2(1-\sigma),$$

so

$$(\int\limits_{\Omega}|u||u_t|dx)^{\frac{1}{1-\sigma}} \leq C((\|u\|_{2(\rho+2)}^{2(\rho+2)})^{\frac{1}{(1-2\sigma)(\rho+2)}} + \|u_t\|_2^2).$$

Now, we use the algebric inequality (4.24) for

$$z = \|u\|_{2(\rho+2)}^{2(\rho+2)}, \ \ \nu = \frac{1}{(1-2\sigma)(\rho+2)} \ \ and \ \ a = H(0),$$

to obtain

$$(\int\limits_{\Omega}|u||u_t|dx)^{\frac{1}{1-\sigma}} \leq C(H(0) + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2),$$

so

$$(\int\limits_{\Omega}|u||u_t|dx)^{\frac{1}{1-\sigma}} \leq C(H(t) + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2).$$

Similarly, we can prove that

$$(\int_{\Omega} |v||v_t|dx)^{\frac{1}{1-\sigma}} \le C(H(t) + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v_t\|_2^2).$$

Finally, (4.36) became

$$L^{1/(1-\sigma)}(t) \le C(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}),$$

so

$$L^{1/(1-\sigma)}(t) \le C(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k). \tag{4.37}$$

Step 4: In the last step, we combine (4.35) and (4.37) to find a positive constant a_0, such that

$$L'(t) \ge a_0 L^{1/(1-\sigma)}(t) \ \ for\ all \ \ t \in (0,T).$$

□

4.4 Second main result

In this section, we assume that the initial energy is stranded between the zero and a given positive number. With similar method as in the proof of the first main result, we get our second main result on the blow up of the weak solution in finite time.

To start, we need the following notations:

$\bullet B_1 = \eta^{1/2(\rho+2)}, \quad \bullet B_2^{-1} = inf\{\|\nabla v\|_2 : v \in H_{\Gamma_0}^1(\Omega) : \|v\|_{k,\Gamma_1} = 1\}$,

$\bullet Q(\alpha) = \frac{1}{2}\alpha^2 - \frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha^{2(\rho+2)} - \frac{C_6 B_2^k}{k}\alpha^k \quad for\ \ all \quad \alpha > 0$,

$\bullet\alpha_1$ *is the first positive zero of the function* Q',

$$\bullet E_1 = Q(\alpha_1), \qquad \bullet E_2 = \begin{cases} \frac{(C_5 k - C_6 p)\alpha_1^2}{2C_6 p(\rho+2)} & if\ C_5 k \le 2(\rho+2), \\ \\ \frac{(2(\rho+2) - C_6 p)\alpha_1^2}{C_5 C_6 p k} & if\ C_5 k > 2(\rho+2). \end{cases}$$

We note that $E_2 \le E_1$*. Indeed,*

• *If* $C_5 k \le 2(\rho+2)$*, then*

$$E_2 - E_1 = \frac{(C_5 k - C_6 p)\alpha_1^2}{2C_6 p(\rho+2)} - \frac{1}{2}\alpha_1^2 + \frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_1^{2(\rho+2)} + \frac{C_6 B_2^k}{k}\alpha_1^k$$

$$= \alpha_1^2 \left(\frac{C_5 k - C_6 p}{2C_6 p(\rho+2)} - \frac{1}{2} + \frac{B_1^{2(\rho+2)}}{2(\rho+2)} \alpha_1^{2\rho+2} + \frac{C_6 B_2^k}{k} \alpha_1^{k-2} \right),$$

since $k \leq C_5 k \leq 2(\rho+2)$, then

$$E_2 - E_1 \leq \alpha_1^2 \left(\frac{C_5 k - C_6 p}{C_6 p k} - \frac{1}{2} + \frac{B_1^{2(\rho+2)}}{k} \alpha_1^{2\rho+2} + \frac{C_6 B_2^k}{k} \alpha_1^{k-2} \right),$$

hence

$$E_2 - E_1 \leq \alpha_1^2 \left(\frac{C_5}{C_6 p} - \frac{1}{k} - \frac{1}{2} + \frac{B_1^{2(\rho+2)} \alpha_1^{2\rho+2} + C_6 B_2^k \alpha_1^{k-2}}{k} \right),$$

since α_1 is the zero of the function Q', then

$$B_1^{2(\rho+2)} \alpha_1^{2\rho+2} + C_6 B_2^k \alpha_1^{k-2} = 1. \tag{4.38}$$

Since $\frac{C_5}{C_6} \leq 1$, hence

$$E_2 - E_1 \leq \alpha_1^2 \left(\frac{C_5}{C_6 p} - \frac{1}{2} \right) \leq \alpha_1^2 \left(\frac{1}{p} - \frac{1}{2} \right).$$

So

$$E_2 - E_1 \leq 0.$$

• *If $C_5 k > 2(\rho+2)$, with the same way, we obtain*

$$E_2 - E_1 = \frac{(2(\rho+2) - C_6 p)\alpha_1^2}{C_5 C_6 p k} - \frac{1}{2}\alpha_1^2 + \frac{B_1^{2(\rho+2)}}{2(\rho+2)} \alpha_1^{2(\rho+2)} + \frac{C_6 B_2^k}{k} \alpha_1^k$$

$$\leq \alpha_1^2 \left(\frac{2(\rho+2) - C_6 p}{2C_6 p(\rho+2)} - \frac{1}{2} + \frac{B_1^{2(\rho+2)}}{2(\rho+2)} \alpha_1^{2\rho+2} + \frac{C_6 B_2^k}{k} \alpha_1^{k-2} \right).$$

Since, we have $\frac{1}{k} < \frac{C_5}{2(\rho+2)}$, then

$$E_2 - E_1 \leq \alpha_1^2 \left(\frac{C_5(2(\rho+2) - C_6 p)}{2C_6 p(\rho+2)} - \frac{1}{2} + \frac{C_5[B_1^{2(\rho+2)} \alpha_1^{2\rho+2} + C_6 B_2^k \alpha_1^{k-2}]}{2(\rho+2)} \right).$$

By (4.38), we obtain

$$E_2 - E_1 \leq \alpha_1^2 \left(\frac{C_5}{C_6 p} - \frac{C_5}{2(\rho+2)} - \frac{1}{2} + \frac{C_5}{2(\rho+2)} \right) = \alpha_1^2 \left(\frac{C_5}{C_6 p} - \frac{1}{2} \right)$$

$$\leq \alpha_1^2 \left(\frac{1}{p} - \frac{1}{2} \right).$$

So

$$E_2 - E_1 \leq 0.$$

4.4.1 Important Lemmas

Here, we are going to introduce and prove some useful results required to obtain our second main result.

Lemma 4.7.
Assume that

$$0 \leq E(0) < E_2 \ \ and \ \ (\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^{\frac{1}{2}} > \alpha_1.$$

Then, there exists a constant $\alpha_2 > \alpha_1$ such that, for all $t \in (0,T)$, we have

$$(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\frac{1}{2}} > \alpha_2 \tag{4.39}$$

and

$$\frac{1}{2(\rho+2)}(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}) + \frac{C_6}{k}\|v\|_{k,\Gamma_1}^k \geq \frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6 B_2^k}{k}\alpha_2^k. \tag{4.40}$$

Proof. Let us define the function

$$\gamma(t) := \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \ \ for \ all \ \ t \in (0,T).$$

By (4.8) and (4.16), we have

$$E(t) \geq \frac{1}{2}\gamma(t) - \frac{1}{2(\rho+2)}(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}) - \int_{\Gamma_1} \Psi(x,v)d\Gamma,$$

(4.12) and (4.20) give

$$E(t) \geq \frac{1}{2}\gamma(t) - \frac{\eta}{2(\rho+2)}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\rho+2} - \frac{C_6}{k}\|v\|_{k,\Gamma_1}^k.$$

By the definition of B_1 and B_2, we obtain

$$E(t) \geq \frac{1}{2}\gamma(t) - \frac{B_1^{2(\rho+2)}}{2(\rho+2)}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\rho+2} - \frac{C_6 B_2^k}{k}\|\nabla v\|_2^k.$$

Further, we find

$$E(t) \geq \frac{1}{2}\gamma(t) - \frac{B_1^{2(\rho+2)}}{2(\rho+2)}(\gamma(t))^{\rho+2} - \frac{C_6 B_2^k}{k}(\gamma(t))^{k/2}$$

$$= \frac{1}{2}\alpha^2 - \frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha^{2(\rho+2)} - \frac{C_6 B_2^k}{k}\alpha^k = Q(\alpha), \tag{4.41}$$

where $\alpha = (\gamma(t))^{1/2}$.
Clearly, we have

$$Q'(\alpha) = \alpha(1 - B_1^{2(\rho+2)}\alpha^{2(\rho+2)-2} - C_6 B_2^k \alpha^{k-2}).$$

We can easely verify that Q is increasing for $0 < \alpha < \alpha_1$, decreasing for $\alpha > \alpha_1$ and

$$Q(\alpha) \to -\infty \quad as \quad \alpha \to +\infty.$$

Since

$$Q(0) = 0 \leq E(0) < E_2 \leq E_1 = Q(\alpha_1),$$

then, there exists $\alpha_2 > \alpha_1$ such that

$$Q(\alpha_2) = E(0). \tag{4.42}$$

We set

$$\alpha_0 = (\gamma(0))^{1/2}.$$

Thanks to (4.41), we have

$$Q(\alpha_0) = Q((\gamma(0))^{1/2}) \leq E(0) = Q(\alpha_2).$$

Then

$$\alpha_0 \geq \alpha_2.$$

Now, we suppose that

$$(\gamma(t_0))^{1/2} \leq \alpha_2 \ \ for \ some \ \ t_0 > 0.$$

Using the continuity of the function γ, we can choose t_0 such that

$$(\gamma(t_0))^{1/2} > \alpha_1.$$

Again, by (4.41), we obtain

$$E(t_0) \geq Q(\gamma(t_0)) > Q(\alpha_2) = E(0)$$

and it's impossible since

$$E(t) \leq E(0) \quad for \ \ all \ t \in (0, T).$$

Therefore

$$(\gamma(t))^{\frac{1}{2}} > \alpha_2 \ \ for \ all \ \ t > 0.$$

Now, by (4.15), (4.16) and (4.20), we get

$$\frac{1}{2}\gamma(t) - E(0) \leq \frac{1}{2(\rho+2)}(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}) + \frac{C_6}{k}\|v\|_{k,\Gamma_1}^k.$$

Hence, (4.39) gives

$$\frac{1}{2(\rho+2)}(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}) + \frac{C_6}{k}\|v\|_{k,\Gamma_1}^k \geq \frac{\gamma(t)}{2} - E(0) \geq \frac{1}{2}\alpha_2^2 - E(0).$$

Using (4.42) and the definition of the functional Q to obtain

$$\frac{1}{2(\rho+2)}(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}) + \frac{C_6}{k}\|v\|_{k,\Gamma_1}^k \geq \frac{1}{2}\alpha_2^2 - Q(\alpha_2)$$

$$= \frac{1}{2}\alpha_2^2 - \frac{1}{2}\alpha_2^2 + \frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6 B_2^k}{k}\alpha_2^k = \frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6 B_2^k}{k}\alpha_2^k,$$

which gives (4.40). □

Now, we need the following new expression of the functional H:

$$H(t) = E_2 - E(t) \;\; for\ all\ t \in (0,T). \tag{4.43}$$

4.4.2 Properties of the functional H

Here, we are going to give the proof of the following useful Lemmas.

Lemma 4.8.
Assume that

$$0 \leq E(0) < E_2 \quad and \quad (\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^{\frac{1}{2}} > \alpha_1.$$

Then, for all $t \in (0,T)$, we have

$$0 < H(0) \leq H(t) \leq \frac{C_8}{2(\rho+2)}(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}) + \frac{C_6}{k}\|v\|_{k,\Gamma_1}^k.$$

Proof. The proof of

$$0 < H(0) \leq H(t),$$

is trivial.
We put (4.16) in (4.43) to get

$$H(t) = E_2 - \frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2) - \frac{1}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$

$$+\int_{\Omega} F(u,v)dx + \int_{\Gamma_1} \Psi(x,v)d\Gamma. \tag{4.44}$$

Using (4.39) and since $\alpha_2 > \alpha_1$, we find

$$E_2 - \frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2) - \frac{1}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) < E_1 - \frac{1}{2}\alpha_1^2.$$

Therefore, (4.44) became

$$\begin{aligned}
H(t) &\leq E_1 - \frac{1}{2}\alpha_1^2 + \int_{\Omega} F(u,v)dx + \int_{\Gamma_1} \Psi(x,v)d\Gamma \\
&= Q(\alpha_1) - \frac{1}{2}\alpha_1^2 + \int_{\Omega} F(u,v)dx + \int_{\Gamma_1} \Psi(x,v)d\Gamma \\
&= \frac{1}{2}\alpha_1^2 - \frac{1}{2}\alpha_1^2 - \frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_1^{2(\rho+2)} - \frac{C_6 B_2^k}{k}\alpha_1^k + \int_{\Omega} F(u,v)dx + \int_{\Gamma_1} \Psi(x,v)d\Gamma \\
&= -\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_1^{2(\rho+2)} - \frac{C_6 B_2^k}{k}\alpha_1^k + \int_{\Omega} F(u,v)dx + \int_{\Gamma_1} \Psi(x,v)d\Gamma \\
&\leq \int_{\Omega} F(u,v)dx + \int_{\Gamma_1} \Psi(x,v)d\Gamma.
\end{aligned}$$

By (4.9) and (4.20), we find

$$H(t) \leq \frac{C_8}{2(\rho+2)}(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}) + \frac{C_6}{k}\|v\|_{k,\Gamma_1}^k \ for\ all\ t \in (0,T).$$

The proof is now completed. □

By the same method as in the proof of Lemma 4.6*, we can obtain the following.*

Lemma 4.9.
Let

$$0 < \sigma \leq min\{\frac{\rho+1}{2(\rho+2)}, \frac{2(\rho+2)-m_1}{2(m_1-1)(\rho+2)}, \frac{k-m_2}{(m_2-1)k}\}.$$

We have for all $t \in (0,T)$

$$\begin{aligned}
&H^{\sigma(m_1-1)}(t)\|u\|_{m_1}^{m_1} + H^{\sigma(m_2-1)}(t)\|v\|_{m_2,\Gamma_1}^{m_2} \\
&\leq C(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k + H(t))
\end{aligned}$$

and

$$\int_\Omega g_1(x,u_t)udx + \int_{\Gamma_1} g_2(x,v_t)vd\Gamma \le C(M_1^{1-m_1} + M_2^{1-m_2})(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}$$

$$+\|v\|_{k,\Gamma_1}^k + H(t)) + CH^{-\sigma}(t)H'(t) \;\; for\ all \;\; M_1, M_2 > 0. \tag{4.45}$$

4.4.3 Blow up result

The corresponding second main result is presented in the below Theorem.

Theorem 4.3.
The weak solution of the system (4.1) with initial data satisfying

$$0 \le E(0) < E_2 \quad and \quad (\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^{\frac{1}{2}} > \alpha_1,$$

blows up in finite time.

Proof. We define the following function

$$L(t) = H^{1-\sigma}(t) + \epsilon \int_\Omega (uu_t + vv_t)dx \quad for \;\; all \quad t \in (0,T),$$

for ϵ small to be chosen later.

We proceed in several steps.
Step 1: As in the first step of the proof of $Theorem$ 4.2, we obtain

$$L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon(1+\frac{p}{2})(\|u_t\|_2^2 + \|v_t\|_2^2) + \epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$

$$+\epsilon(1-\frac{p}{2(\rho+2)})(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}) - \epsilon\int_\Omega g_1(x,u_t)udx - \epsilon\int_{\Gamma_1} g_2(x,v_t)vd\Gamma$$

$$+\epsilon\int_{\Gamma_1}\psi(x,v)vd\Gamma - \epsilon p\int_{\Gamma_1}\Psi(x,v)d\Gamma + \epsilon pH(t) - \epsilon pE_2. \tag{4.46}$$

For the last term, we can see that

$$-\epsilon pE_2 \ge -\epsilon pE_2(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6B_2^k}{k}\alpha_2^k)(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6B_2^k}{k}\alpha_2^k)^{-1}.$$

According to (4.40), we obtain

$$-\epsilon pE_2 \ge -\epsilon pE_2[\frac{1}{2(\rho+2)}(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2})$$

$$+\frac{C_6}{k}\|v\|_{k,\Gamma_1}^k](\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)}+\frac{C_6B_2^k}{k}\alpha_2^k)^{-1}.$$

However, using this result, (4.6) and (4.20), (4.46) becames

$$L'(t)\geq(1-\sigma)H^{-\sigma}(t)H'(t)+\epsilon(1+\frac{p}{2})(\|u_t\|_2^2+\|v_t\|_2^2)+\epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2+\|\nabla v\|_2^2)$$

$$+\epsilon pH(t)+\epsilon\overline{C}_4(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2})+\epsilon\overline{C}_5\|v\|_{k,\Gamma_1}^k-\epsilon\int\limits_\Omega g_1(x,u_t)udx$$

$$-\epsilon\int\limits_{\Gamma_1}g_2(x,v_t)vd\Gamma, \tag{4.47}$$

where

$$\overline{C}_4=1-\frac{p}{2(\rho+2)}-\frac{pE_2}{2(\rho+2)}(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)}+\frac{C_6B_2^k}{k}\alpha_2^k)^{-1}$$

and

$$\overline{C}_5=C_5-\frac{C_6p}{k}\left[1+E_2(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)}+\frac{C_6B_2^k}{k}\alpha_2^k)^{-1}\right].$$

Our next goal is to show that $\overline{C}_4$ and $\overline{C}_5$ are positive.

For $\overline{C}_4$, we have

• If $C_5k\leq 2(\rho+2)$, then

$$E_2=\frac{(C_5k-C_6p)\alpha_1^2}{2C_6p(\rho+2)}.$$

(4.38) leads to

$$\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_1^{2(\rho+2)}+\frac{C_6B_2^k}{2(\rho+2)}\alpha_1^k\geq\frac{\alpha_1^2}{2(\rho+2)},$$

since $\alpha_2>\alpha_1$, then, we find

$$\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)}+\frac{C_6B_2^k}{2(\rho+2)}\alpha_2^k>\frac{\alpha_1^2}{2(\rho+2)},$$

Since $\frac{1}{2(\rho+2)}\leq\frac{1}{C_5k}\leq\frac{1}{k}$, then

$$\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)}+\frac{C_6B_2^k}{k}\alpha_2^k>\frac{\alpha_1^2}{2(\rho+2)}, \tag{4.48}$$

then, we obtain

$$(1-\frac{p}{2(\rho+2)})(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)}+\frac{C_6B_2^k}{k}\alpha_2^k)>(1-\frac{p}{2(\rho+2)})\frac{\alpha_1^2}{2(\rho+2)}$$

$$= \frac{(2(\rho+2)-p)\alpha_1^2}{4(\rho+2)^2} \geq \frac{(2(\rho+2)-C_6p)\alpha_1^2}{4(\rho+2)^2}.$$

Since $2(\rho+2) \geq C_5k$, then

$$(1-\frac{p}{2(\rho+2)})(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6B_2^k}{k}\alpha_2^k) > \frac{(C_5k-C_6p)\alpha_1^2}{4(\rho+2)^2} = \frac{C_6pE_2}{2(\rho+2)} \geq \frac{pE_2}{2(\rho+2)},$$

consequently

$$\overline{C}_4 > 0.$$

• If $C_5k > 2(\rho+2)$, then

$$E_2 = \frac{(2(\rho+2)-C_6p)\alpha_1^2}{C_5C_6pk}.$$

We have

$$\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6B_2^k}{k}\alpha_2^k > \alpha_1^2\left(\frac{B_1^{2(\rho+2)}\alpha_1^{2(\rho+2)-2} + C_6B_2^k{\alpha_1}^{k-2}}{C_5k}\right),$$

using (4.38) to obtain

$$\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6B_2^k}{k}\alpha_2^k > \frac{\alpha_1^2}{C_5k}, \tag{4.49}$$

then

$$(1-\frac{p}{2(\rho+2)})(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6B_2^k}{k}\alpha_2^k) > \frac{(2(\rho+2)-p)\alpha_1^2}{2C_5k(\rho+2)}$$

$$\geq \frac{(2(\rho+2)-C_6p)\alpha_1^2}{2C_5k(\rho+2)} = \frac{C_6pE_2}{2(\rho+2)},$$

so

$$(1-\frac{p}{2(\rho+2)})(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6B_2^k}{k}\alpha_2^k) > \frac{pE_2}{2(\rho+2)},$$

hence

$$\overline{C}_4 > 0.$$

For $\overline{C}_5$, we have

• If $C_5k \leq 2(\rho+2)$, we use (4.48) to find

$$(C_5-\frac{C_6p}{k})(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6B_2^k}{k}\alpha_2^k) > (C_5-\frac{C_6p}{k})\frac{\alpha_1^2}{2(\rho+2)}$$

$$= \frac{(C_5k-C_6p)\alpha_1^2}{2k(\rho+2)} = \frac{C_6pE_2}{k}.$$

So

$$\overline{C}_5 > 0.$$

• If $C_5 k > 2(\rho+2)$, we use (4.49) to obtain

$$(C_5 - \frac{C_6 p}{k})(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6 B_2^k}{k}\alpha_2^k) > \frac{(C_5 k - C_6 p)\alpha_1^2}{C_5 k^2}.$$

The fact that $C_5 k > 2(\rho+2)$ gives

$$(C_5 - \frac{C_6 p}{k})(\frac{B_1^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)} + \frac{C_6 B_2^k}{k}\alpha_2^k) > \frac{(2(\rho+2) - C_6 p)\alpha_1^2}{C_5 k^2} = \frac{C_6 p E_2}{k}.$$

Then

$$\overline{C}_5 > 0.$$

Now, we insert (4.45) in (4.47) to obtain

$$L'(t) \geq (1-\sigma-\epsilon C)H^{-\sigma}(t)H'(t) + \epsilon(1+\frac{p}{2})(\|u_t\|_2^2 + \|v_t\|_2^2) + \epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$
$$+\epsilon p H(t) + \epsilon\overline{C}_4(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}) + \epsilon\overline{C}_5\|v\|_{k,\Gamma_1}^k$$
$$-\epsilon C(M_1^{1-m_1} + M_2^{1-m_2})(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k + H(t)),$$

using (4.9) to get

$$L'(t) \geq (1-\sigma-\epsilon C)H^{-\sigma}(t)H'(t) + \epsilon(1+\frac{p}{2})(\|u_t\|_2^2 + \|v_t\|_2^2) + \epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$
$$+\epsilon p H(t) + \epsilon\overline{C}_6(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{\rho+2}^{\rho+2}) + \epsilon\overline{C}_5\|v\|_{k,\Gamma_1}^k$$
$$-\epsilon C(M_1^{1-m_1} + M_2^{1-m_2})(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k + H(t)),$$

where $\overline{C}_6 = C_7\overline{C}_4$. Thus

$$L'(t) \geq (1-\sigma-\epsilon C)H^{-\sigma}(t)H'(t) + \epsilon(1+\frac{p}{2})(\|u_t\|_2^2 + \|v_t\|_2^2) + \epsilon(\frac{p}{2}-1)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)$$
$$+\epsilon[p - C(M_1^{1-m_1} + M_2^{1-m_2})]H(t) + \epsilon[\overline{C}_6 - C(M_1^{1-m_1} + M_2^{1-m_2})](\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)})$$
$$+\epsilon[\overline{C}_5 - C(M_1^{1-m_1} + M_2^{1-m_2})]\|v\|_{k,\Gamma_1}^k. \tag{4.50}$$

We take ϵ small enough so that $(1-\sigma) - \epsilon C \geq 0$. Then, we choose M_1 and M_2 sufficiently large to have

$$p - C(M_1^{1-m_1} + M_2^{1-m_2}) > 0, \ \ \overline{C}_6 - C(M_1^{1-m_1} + M_2^{1-m_2}) > 0$$

$$and \ \overline{C}_5 - C(M_1^{1-m_1} + M_2^{1-m_2}) > 0.$$

Therefore, there exists $\zeta' > 0$ such that for all $t \in (0, T)$, (4.50) can rewriting as follows

$$L'(t) \geq \zeta'(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k). \tag{4.51}$$

Step 2: Similarly, to the second step in the proof of $Theorem$ 4.2, we can find

$$L(t) \geq 0 \ \ for \ \ all \ \ t \in (0, T).$$

Step 3 : As in the third step in the proof of $Theorem$ 4.2, there exists $C > 0$ such that for all $t \in (0, T)$, we have

$$L^{1/(1-\sigma)}(t) \leq C(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{k,\Gamma_1}^k). \tag{4.52}$$

Step 4: At the end, we combine (4.51) with (4.52) to find a positive constant a_1, such that

$$L'(t) \geq a_1 L^{1/(1-\sigma)}(t) \ \ for \ all \ t \in (0, T).$$

□

CONCLUSION AND PERSPECTIVES

Conclusion

In this thesis, our results, concerning the existence and the behavior of solution of nonlinear hyperbolic systems, have been proved under suitable assumptions on the initial data and on the exponent nonlinearity.

In the first study, we have proved the global existence and stability of the weak solution for a nonlinear wave equation with the presence of the internal source and the boundary damping terms in the case of constant exponent. In the second study, we have obtained the global existence and the stability of the weak solution for a wave equation with variable boundary damping term. In the last study, we have showed that the weak solution of nonlinear coupled wave equations blows up in finite time for weak solution with positive or negative initial energy.

Perspectives

The following open questions can be made regarding the material presented in this thesis:

• *Study of the global existence of solution for the first system in the case of* $\beta \geq 1$.

• *Study of the global existence of solution for the first system in the case when the damping term dominated the source term* $(m \geq p)$.

• *Study of the blow up of solution for the first system.*

• *Study of the Stability of solution for the third system.*

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