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Thème

Opérateur linéaire borné à image fermée et transformation d'Aluthge

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Dedication

To my Parents

To my Husband

To my sisters and my brothers

To all my family and friends .

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Contents

Notations

Here we give the different notations used in this thesis.

$\mathcal{H}$	:	The complex Hilbert space.
$\mathcal{B}(\mathcal{H})$	:	The Banach algebra of all bounded linear operators on $\mathcal{H}$.
T^*	:	The adjoint of T.
T^+	:	The Moore-Penrose inverse of T.
$\mathcal{N}(T)$	:	The null subspace of T.
$\mathcal{R}(T)$	:	The range of T.
$\|T\|$	:	The norm of T.
$T^{\frac{1}{2}}$	:	The square root of T.
$\lvert A\rvert$	:	The modulus of T.
$\mathcal{N}(T)^{\perp}$	:	The orthogonal complement of $\mathcal{N}(T)$.
$\mathcal{R}(T)^{\perp}$	:	The orthogonal complement of $\mathcal{R}(T)$.
$\mathbb{C}$	:	The field of complex numbers.
$\langle x, y\rangle$	:	The inner product of $x,$ and y.
$\otimes$	:	The rank one operator.
$T = U\lvert T\rvert$	:	The polar decomposition of T.
$\Delta_1(T)$	:	The Duggal transform of T.
$\Delta(T)$	:	The Aluthge transform of T.
$\Delta_\lambda(T)$	:	λ-Aluthge transform of T.
$S(T)$	:	The transform $S(T)$.
$S_r(T)$	:	The transform $S_r(T)$.
$\widehat{T}$	:	The mean transform .
$\widehat{T_\lambda}$	:	The generalised mean transform .
$[S, T] = 0$	:	$ST - TS = 0$.
$\delta(\mathcal{H})$	:	$\{T \in \mathcal{B}(\mathcal{H})\, / U^2\lvert T\rvert = \lvert T\rvert U^2\}$.

Definitions

Let $\mathcal{H}$ be a complex Hilbert space, and let $T \in \mathcal{B}(\mathcal{H})$, then

- The spectrum of T, denoted by $\sigma(T)$, is defined as follows:

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible } \}.$$

- The spectral radius $r(T)$ of the operator T is defined by

$$r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

- The numerical range of an operator T, denoted by $W(T)$, is defined by

$$W(T) = \{< Tx, x >: \|x\| = 1\}.$$

- The numerical radius of an operator T, denoted by $w(T)$, is defined by

$$w(T) = sup\{|\lambda| : \lambda \in W(T)\}.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be;

1. Invertible, if there exists another operator $T^{-1} \in \mathcal{B}(\mathcal{H})$, such that, $TT^{-1} = T^{-1}T = I$ and T^{-1}, is called the inverse of T.
2. Positive, if $< Tx, x >\geq 0$ for all $x \in \mathcal{H}$.
3. Idempotent or projection, if $T^2 = T$.
4. An orthogonal projection, if $T^2 = T$ and $T^* = T$.
5. Self-adjoint, if $T^* = T$.
6. Unitary, if $T^*T = TT^* = I$.
7. Isometry, if $T^*T = I$.

8. Partial isometry, if $\|Tx\| = \|x\|$, for all $x \in (\mathcal{N}(T))^{\perp}$.

9. Normal, if $T^*T = TT^*$.

10. Quasinormal, if T and T^*T commute.

11. Binormal, if T^*T and TT^* commute.

12. Hyponormal, if $T^*T \geq TT^*$. Where $A \geq B$ means $A - B \geq 0$.

13. p-Hyponormal, if $(T^*T)^p \geq (TT^*)^p$ for $p \geq 0$.

14. semi-hyponormal, if $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$.

15. Paranormal, if $||T^2x|| \geq ||Tx||^2$ for every unit vector $x \in \mathcal{H}$.

16. normaloid, if $\|T\| = r(T)$.

17. spectraloid, if $w(T) = r(T)$.

18. Nilpotent, if there exists $n \in \mathbb{N}^*$ such that $T^n = 0$.

Introduction

This thesis is included in the conception of functional analysis and more precisely in the field of operator theory on a complex Hilbert space $\mathcal{H}$.

Most of the current research concerns the study of some operator classes which include the class of normal operators on a complex Hilbert space $\mathcal{H}$, Especially, the classes of normal, quasinormal, subnormal, hyponormal and paranormal operators are very famous. The structure of normal operators is well-known. Also the structure of quasi-normal and subnormal operators was given in ([8], [9]). On the other hand hyponormal operators are complicated and hard to study.

In order to study p-hyponormal and log-hyponormal operators, A. Aluthge [2] introduced in 1990 a transformation which is called Aluthge transform that is defined as

$$\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}, \quad T \in \mathcal{B}(\mathcal{H}),$$

where $T = U|T|$ is the polar decomposition of T. More generally, in 2003, K.Okubo [46] introduced the λ-Aluthge transform which has later been studied also in detail. This is defined for any $\lambda \in [0,1]$ by

$$\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}, \quad T \in \mathcal{B}(\mathcal{H}).$$

Clearly, for $\lambda = \frac{1}{2}$ we obtain the usual Aluthge transform. Also, $\Delta_1(T) = |T|U$ is known as Duggal's transform [21].

These transforms have been studied in many different contexts and considered by a number of authors (see for instance, [2, 29, 32, 41, 46, 47]). One of the interests of the Aluthge transform lies in the fact that it respects many properties of the original operator. For example T has a nontrivial invariant subspace if an only if $\Delta(T)$ does (see [32]). Another important property is that T and $\Delta_\lambda(T)$ have the same spectrum for a large family of spectra(see [32]). In particular T is invertible if and only if $\Delta_\lambda(T)$ is invertible, and in this case they are similar. Also, A.Aluthge [2] proved that if T is semi-hyponormal, then $\Delta(T)$ is hyponormal. It is well known that hyponormality of an operator implies semi-hyponormality, but the converse implication does not hold in general. So the Aluthge transform may have better properties than T. The idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one many spectral properties, but which is closer to being a normal operator. Recently, certain research in operator theory has been related to the relationship between operators and their Aluthge transforms. In this thesis, we present some new results that are in the same sense.

By interchanging U with $|T|^{\frac{1}{2}}$ in the Aluthge transform, a new operator transform was introduced in 2004 by S.M. Patel et al [48], and is given by

$$S(T) = U|T|^{\frac{1}{2}}U, \;\; T \in \mathcal{B}(\mathcal{H}).$$

In 2008, this concept was also generalized, for any scalar $r > 0$, by T.Furuta [22] as follows:

$$S_r(T) = U|T|^rU. \;\; T \in B(H)$$

Clearly, for $r = \frac{1}{2}$ we obtain the transform $S(T)$. In this work a number of questions about the transform $S_r(T)$ and its relationship to several classes of operators were addressed. Thus, some results obtained for $S(T)$ in [48], are extended to $S_r(T)$, $(r > 0)$.

Other kind of operator transform is the mean transform $\widehat{T}$ of T, introduced in 2013 by S.H.Lee et al [37]. The definition is

$$\widehat{T} = \frac{1}{2}(U|T| + |T|U),$$

it is the arithmetic mean of $T = U|T|$ and it's Duggal transform $\Delta_1(T) = |T|U$. As explained in [37], the mean transform may be useful than the Aluthge transform in the practical use. It has nice properties (see [13, 16, 33, 37]). The fixed point of mean transform are the quasinormal operators as the Aluthge transform.

Recently, in 2020, The generalized mean transform of an operator T was defined by C.Benhida et al [5] as follows

$$\widehat{T}_\lambda = \frac{1}{2}\left(\Delta_\lambda(T) + \Delta_{1-\lambda}(T)\right), \qquad \text{For } \lambda \in [0, \frac{1}{2}[.$$

where $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$ denotes the $\lambda-$Aluthge transform of T for some $\lambda \in [0, \frac{1}{2}]$. In particular, $\widehat{T}_0 = \widehat{T}$ is the mean transform of T and $\widehat{T}_{\frac{1}{2}} = \Delta(T)$ is the Aluthge transform of T. Also in this work we investigate the generalized mean transform of closed range operators on Hilbert space operators.

This thesis is devided into four chapters. We shall briefly outline the contents of each chapter of this thesis as follows.

Chapter one is about preliminaries that will be used throughout this thesis, it consists of five sections. In section 1, we introduce the most important properties of a special class of bounded linear operators acting on a complex Hilbert space, called positive operators. Section 2 is about polar decomposition, it discusses in details, several important and intersting well-know properties of such decomposition which are necessary to prove the next results of this thesis. section 3, is devoted to the study of the λ- Aluthge transform, we recall fundamontal results of this transformation on Hilbert space which are useful in the following. In section 4, we define the reduced minimum modulus of bounded linear operators, and state some of their properties. In Section 5, we introduce the notion of Moore-Penrose inverse and EP operators, and we collect some of well-know results about them.

Chapter 2 is divided into 3 sections. In section 1 we develop some basic properties of the class $\delta(\mathcal{H})$. In section 2, firstly, we provide a condition under which an operator in $\delta(\mathcal{H})$ becomes quasinormal. Secondly, we show that an invertible operator T belongs to the class $\delta(\mathcal{H})$ if and only if $\Delta_1(T^{-1}) = (\Delta_1(T))^{-1}$. Afterwards, we give examples and discuss how this class of operators is distinct from the class of binormal operators. We prove that, if T is invertible in $\delta(\mathcal{H})$, then T is binormal if and only if $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$, for $\lambda \in]0, 1[$. In [29], Ito et

al, prove that The binormality of an operator in $\mathcal{B}(\mathcal{H})$ does not imply the binormality of its Aluthge transform. However, the binormality of an invertible operator implies the binormality of its Duggal transform [41] . In the last part of this section, we show that if T is binormal in $\delta(\mathcal{H})$ such that the partial isometrie factor U of its polar decomposition is unitary, then $\Delta_\lambda(T)$ is binormal, for any $\lambda \in]0,1[$. Most results on the λ-Aluthge transform show that it generally has better properties than its original operator. However, an operator $T \in \mathcal{B}(\mathcal{H})$ may have a closed range without $\Delta_\lambda(T)$ having a closed range as shown in example 3.4. In section 2, firstly, we shall show a necessary and sufficient condition for the range of $\Delta_\lambda(T)$ to be closed. Secondly, we investigate when an operator and its λ-Aluthge transform both are EP. Finally, we give a formula for the Moore-Penrose inverse of $\Delta_\lambda(T)$ when T is a binormal operator with closed range and then show under some conditions that T^+ is nilpotent of order $d+1$ if and only if $(\triangle_\lambda(T)^+)^d = 0$. Section 2 and 3 of this chapter is the subject of the following publication [54]
(S.Zid and S.Menkad, (2022). The λ-Aluthge transform and its applications to some classes of operators. Filomat, 36(1), 289-301.)

In Chapter 3, we establish some useful results on the transform $S_r(T)$. These results are needed for proving our main theorems in Sections 3 and 4. It was proved in [48, Theorem 2.2] that if T is p-hyponormal, then $S(T)$ is 2p-hyponormal for $0 < p \leq \frac{1}{2}$ and hyponormal for $\frac{1}{2} < p \leq 1$. In section 3, we generalize this result to the transform $S_r(T)$, for $0 < r \leq \frac{1}{2}$. We find the form of the transform $S_r(T)$ ($r > 0$) of rank one operator. Moreover, we discuss the normality of $S(T)$ and provide a new characterization of invertible normal operators via the transform $S_r(T)$. In section 4, firstly, we show that if T is an operator such that the null subspace of its adjoint is contained in its own null subspace, then T has a closed range if and only if its transform $S_r(T)$ has a closed range too. Secondly, we investigate the transform $S_r(T)$ of EP operators. In particular, we prove that T is an EP operator if and only if $S_r(T)$ is too EP and the ranges of T and $S_r(T)$ are equal. As a consequence, the reverse order law for the Moore-Penrose inverse of an operator T and its transform $S_r(T)$ holds, Finally, we presente some relationships between an EP operator T, its transform $S_r(T)$ and the Moore-penrose T^+.

In Chapter 4, first we provide some properties of the mean transform obtained in [16], recently the autaurs in [13] proved that if range of $T \in \mathcal{B}(\mathcal{H})$ is closed then the range of it's mean transform is also closed. We generalize this results for $\widehat{T}_\lambda$ using the assumption $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. Also we prove that the reverse implication hold too under the same assumption. Moreover we prove that if $T \in \mathcal{B}(\mathcal{H})$ is binormal with closed range, then the range of it's generalized mean transform is closed too. Then We discuss the generalized mean transform of EP and binormal operators via Moore-Penrose inverse. In particular, we show that T is EP if and only if $\widehat{T}_\lambda$ is EP too and $\mathcal{R}(T) = \mathcal{R}(\widehat{T}_\lambda)$. Afterwards, we investigate the interconnections between the Moore-Penrose inverse of an EP operateur and both its generalized mean transform as well as the Moore-Penrose inverse of its generalized mean transform. finally, examining the reverse ordere low for the Moore-Penrose inverse, it is proved that it always holds for T and $\widehat{T}_\lambda$.

In the end of this thesis, we shall give some conclusion in suggested research questions that arose during our study.

Chapter 1

Prelimenaries

In this chapter, we recall some well-known definitions and notions in operator theory, We also take a look at several known facts to use later. Positive operators, polar decomposition, the reduced minimum modulus, Moore-Penrose inverse, EP operators, the λ-Aluthge transform, etc. is contained. .

Throughout this thesis, let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$.

1.1 Positive operators

In this section, we give some precise results on the class of positive operators and functional calculus for self-adjoint with some consequences. We start with the following inequalities of bounded linear operator for proofs see [24, 26, 28, 40, 42].

Lemma 1.1 *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that T is a positive operator, Then the following inequalities hold.*

(I) *Hölder-McCarthys inequality :*

$$\langle T^{\alpha}x, x\rangle \geq \langle Tx, x\rangle^{\alpha} \|x\|^{2(1-\alpha)}, \text{ for any } \alpha > 1 \text{ and any vector } x \in \mathcal{H}.$$

(II) *Lowner-Heinz inequality:*

$$\textit{If } T \geq S \geq 0, \ \textit{then } T^\alpha \geq S^\alpha, \ \textit{for } \alpha \in [0,1].$$

(III) *Hansen's Inequality :*

$$\textit{If } \|S\| \leq 1, \ \textit{then } (S^*TS)^\alpha \geq S^*T^\alpha S, \ \textit{for } \alpha \in [0,1].$$

'

Proposition 1.1 *[50] If $T \in \mathcal{B}(\mathcal{H})$ is positive, then T^α is positive, for all $\alpha \geq 0$.*

Proof. We shall prove the proposition by two cases, the first one for $\alpha \in [0,1]$, the second for $\alpha > 1$.
Let $T \geq 0$, and let $T' = \frac{T^{\frac{1}{2}}}{\|T^{\frac{1}{2}}\|}$, then $\|T'\| = 1$ and $T' \geq 0$, hence $(T')^* = T'$. Since $I \geq 0$, then by Lemma 1.1 (III) with $S = T'$ and $\alpha \in [0,1]$ we have
$\frac{T^\alpha}{\|T^{\frac{1}{2}}\|^{2\alpha}} = (T'IT')^\alpha \geq T'I^\alpha T' = (T')^2 = \frac{T}{\|T^{\frac{1}{2}}\|^2}$. Therefore, $T^\alpha \geq \|T^{\frac{1}{2}}\|^{2\alpha-2}T \geq 0$. Hence, for $\alpha \in [0,1]$ we have $T^\alpha \geq 0$.
Let $\alpha > 1$, then by Lemma 1.1(I) we have, $\langle T^\alpha x, x\rangle \geq \langle Tx, x\rangle^\alpha \|x\|^{2(1-\alpha)} \geq 0$, since $T \geq 0$, thus $T^\alpha \geq 0$ for $\alpha > 1$, hence $T^\alpha \geq 0$ for any $\alpha \geq 0$.

■

Proposition 1.2 *[38, 43] Let $T, S \in \mathcal{B}(\mathcal{H})$ and T be self-adjoint such that $[T, S] = 0$. Then $[f(T); S] = 0$ whenever f is a continuous complex-valued function on the interval $[-\|T\|; \|T\|]$.*

Proof. Choose any sequence $\{p_n\}_{n=1}^\infty$ of polynomials such that $p_n(t) \longrightarrow f(t)$ uniformly on the interval $[-\|T\|; \|T\|]$ Then $\|p_n(T) - f(T)\| \longrightarrow 0$ as $n \longrightarrow \infty$, hence

$$f(T)S = \lim_{n\to\infty} p_n(T)S = \lim_{n\to\infty} Sp_n(T) = Sf(T).$$

■

for each $\alpha > 0, f(t) = t^\alpha$ is continuous on $[0; +\infty)$, So frome Proposition 1.2 we have the following proposition:

Proposition 1.3 *[38] Let $S, T \in \mathcal{B}(\mathcal{H})$ and T be positive such that $[S, T] = 0$. Then $[S, T^\alpha] = 0$ for any $\alpha > 0$.*

The following, gives implication relations among several classes of operators.

Remark 1.1 *It is easy to see that*

$$\textit{positive} \implies \textit{self-adjoint} \implies \textit{normal} \implies \textit{quasinormal} \implies \textit{binormal}$$

and the inverse implications do not hold. However, every invertible quasinormal operator is normal. Also, in finite dimensional spaces every quasinormal operator is normal. For more details see [14, 24]. We note that if we intersect the class of binormal and paranormal operators, we obtain the class of hyponormal operators (see [11]).

Remark 1.2 *By Proposition 1.3, T is binormal if and only if $\forall \alpha, \beta > 0$, $|T|^\alpha$ and $|T^*|^\beta$ commute, if and only if there exist $\alpha > 0$ and $\beta > 0$ such that $|T|^\alpha$ and $|T^*|^\beta$ commute.*

The following lemma is well-known in operator theory, for proof see [45].

Lemma 1.2 *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then $\mathcal{N}(T^\lambda) = \mathcal{N}(T)$, for all $\lambda > 0$.*

Proof. If $\lambda \le \mu$, then $T^\mu = T^{\mu-\lambda} T^\lambda$, so $\mathcal{N}(T^\lambda) \subseteq \mathcal{N}(T^\mu)$. It follows from

$$||Tx||^2 = \langle Tx, Tx \rangle = \left\langle T^2 x, x \right\rangle$$

that $\mathcal{N}(T) = \mathcal{N}(T^2)$. For each $\lambda \ge 1$, there exists a positive integer n such that $\lambda \le 2^n$. Hence,

$$\mathcal{N}(T) \subseteq \mathcal{N}(T^\lambda) \subseteq \mathcal{N}(T^{2^n}) = \mathcal{N}(T).$$

Therefore, $\mathcal{N}(T) = \mathcal{N}(T^\lambda)$ for all $\lambda \geq 1$.

For each $0 < \lambda \leq 1$, we have $1/\lambda \geq 1$, and so

$$\mathcal{N}(T^\lambda) = \mathcal{N}(T^\lambda)^{\frac{1}{\lambda}} = \mathcal{N}(T).$$

■

Lemma 1.3 *([36, Proposition 3.7] and [38, Proposition 2.9]) Let $T \in \mathcal{B}(\mathcal{H})$. Then*

1. $\overline{R(T^*T)} = \overline{R(T^*)}$ *and* $\overline{R(TT^*)} = \overline{R(T)}$.
2. *If T is positive, then* $\overline{\mathcal{R}(T^\lambda)} = \overline{\mathcal{R}(T)}$, *for any $\lambda > 0$.*

Lemma 1.4 *([36, Theorem 3.2] and [52, Remark 1.1]) Let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathcal{R}(T) \text{ is closed} \iff \mathcal{R}(T^*) \text{ is closed} \iff \mathcal{R}(TT^*) \text{ is closed} \iff \mathcal{R}(T^*T) \text{ is closed.}$$

Remark 1.3 *From Lemma 1.3 and 1.4, we notice that*

$$\text{If } \mathcal{R}(T) \text{ is closed, then } \mathcal{R}(T) = \mathcal{R}(TT^*) \text{ and } \mathcal{R}(T^*) = \mathcal{R}(T^*T).$$

1.2 Polar decomposition of bounded linear operators on a Hilbert space

In this section, we state definition and properties of the polar decomposition. First, we recall the concept of partial isometry and some of it's properties.

Partial isometry operator

Definition 1.1 *An operator $U \in \mathcal{B}(\mathcal{H})$ is said to be a partial isometry operator if*

$$\|Ux\| = \|x\|, \text{ for all } x \in (\mathcal{N}(U))^\perp.$$

Proposition 1.4 *Let $U \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent:*

(i) *U is a partial isometry operator,*

(ii) *U^* is a partial isometry operator,*

(iii) *UU^* is an orthogonal projection,*

(iv) *U^*U is an orthogonal projection,*

(v) *$U^*UU^* = U^*$,*

(vi) *$UU^*U = U$.*

Proof. $(i) \Longrightarrow (v)$ Suppose that U is a partial isometry, then for $x \in \mathcal{B}(\mathcal{H})$ and $z \in \mathcal{N}(U)$, we have

$$< U^*UU^*z, x >=< UU^*z, Ux >= 0 =< z, Ux >=< U^*z, x > .$$

Since U is an isometry onto $N(U)^\perp = \overline{R(U^*)}$, then we obtain

$$< U^*UU^*z, x >=< UU^*z, Ux >=< U^*z, x > .$$

As $H = \mathcal{N}(U) \oplus \mathcal{N}(U)^\perp$, then we deduce that $< U^*UU^*z, x >=< U^*z, x >$. Hence $U^*UU^* = U^*$.

$(v) \Longrightarrow (vi)$ Trivial.

$(v) \Longrightarrow (iii)$ Suppose that $U^*UU^* = U^*$, then $(UU^*)^2 = UU^*$. Hence UU^* is an orthogonal projection, as it's selfadjoint.

$(v) \Longrightarrow (iv)$ The proof is similar to $(v) \Longrightarrow (iii)$.

$(iii) \Longrightarrow (i)$ Suppose that $UU^* = I$ and $x \in \mathcal{N}(U)^\perp = \overline{R(U^*)}$. then there existe a sequence $(x_n)_{n \geq 1}$, such that $\lim\limits_{n \to +\infty} U^*(x_n) = x$. As

$$\|Ux\|^2 = \lim_{n \to +\infty} \|UU^*x_n\|^2 = \lim_{n \to +\infty} \|U^*(x_n)\|^2 = \|x\|^2,$$

hence we deduce that U is a partial isometry.

The equivalence between (ii), (iv) and (vi) follows by replacing U with U^* in (iv). ∎

Polar Decomposition of an Operator

Recall that the polar decomposition of a nonzero complex number is $z = |z|e^{i\theta}$, $\theta \in \mathbb{R}$, where $|z| = (z^*z)^{\frac{1}{2}}$. Using this analogy, the polar decomposition of bounded linear operator consists of writing T as a product

$$T = U(T^*T)^{\frac{1}{2}}$$

where U is an isometry and $(T^*T)^{\frac{1}{2}}$ is well-defined, because (T^*T) is a positive operator which has a unique positive square root. So For any $T \in \mathcal{B}(\mathcal{H})$, let $|T|$ denote the square root of T^*T i.e. $|T| = (T^*T)^{\frac{1}{2}}$.

Theorem 1.1 *[24] For $T \in \mathcal{B}(\mathcal{H})$, there is a unique partial isometry $U \in \mathcal{B}(\mathcal{H})$, such that*

$$T = U|T| \text{ and } \mathcal{N}(U) = \mathcal{N}(T).$$

Definition 1.2 *[24] Let $T \in \mathcal{B}(\mathcal{H})$. When $T = U|T|$, with $\mathcal{N}(U) = \mathcal{N}(T)$, $T = U|T|$ is said to be the polar decomposition of T, and if the kernel condition $\mathcal{N}(U) = \mathcal{N}(T)$ is not necessarily satisfied, $T = U|T|$ is said to be merely a decomposition of T.*

Proposition 1.5 *[24, 38] Let $T = U|T|$ be the polar decomposition of T. Then the following statements are valid:*

(1) $\mathcal{N}(T) = \mathcal{N}(|T|)$.

(2) $UU^* = P_{N(T^*)^\perp} = P_{\overline{R(|T^*|)}} = P_{\overline{R(T)}}$.

(3) $U^*U = P_{N(T)^\perp} = P_{\overline{R(T^*)}} = P_{\overline{R(|T|)}}$.

(4) $T^* = U^*|T^*|$, *is the polar decomposition of* T^*.

Proof.

(1). Proof of (1) follows by

$$\begin{aligned} |T|x = 0 &\Longleftrightarrow |T|^2x = 0 \text{ since } <|T|^2x, x>= \||T|x\|^2 = 0 \\ &\Longleftrightarrow T^*Tx = 0 \Longleftrightarrow Tx = 0 \text{ since } <T^*Tx, x>= \|Tx\|^2 = 0. \end{aligned}$$

Hence $\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|)$ as $T = U|T|$ is the polar decomposition of T.

(2). By Lemma 1.3 and Lemma 1.4,, we have

$$\overline{R(|T^*|)} = \overline{R(TT^*)} = \overline{R(T)} = \mathcal{N}(T^*)^{\perp} = \mathcal{N}(U^*)^{\perp} = \mathcal{R}(U) = \mathcal{R}(UU^*).$$

(3). By Lemma 1.3 and Lemma 1.4, we have

$$\overline{R(|T|)} = \overline{R(T^*T)} = \overline{R(T^*)} = \mathcal{N}(T)^{\perp} = \mathcal{N}(U)^{\perp} = \mathcal{R}(U^*) = \mathcal{R}(U^*U).$$

(4). Using (3), we obtain $U^*U|T| = |T|$. So

$$TT^* = U|T||T|U^* = U|T|U^*U|T|U^* = (U|T|U^*)^2\,, \tag{1.1}$$

notice that TT^* and $U|T|U^*$ are positive operators, we get

$$|T^*| = U|T|U^*. \tag{1.2}$$

Hence

$$T^* = (U|T|)^* = |T|U^* = U^*U|T|U^* = U^*|T^*|.$$

So that we have only to show $\mathcal{N}(U^*) = \mathcal{N}(|T^*|)$. To this end, we have

$$\begin{aligned} U^*x = 0 &\Longleftrightarrow UU^*x = 0 \ since\ \langle UU^*x, x\rangle = \|U^*x\|^2 = 0 \\ &\Longleftrightarrow |T|U^*x = 0\ since\, \mathcal{N}(U) = \mathcal{N}(|T|) \\ &\Longleftrightarrow T^*x = 0\ since\ T^* = |T|U^* \\ &\Longleftrightarrow |T^*|x = 0 \text{ by (1).} \end{aligned}$$

■

Theorem 1.2 *[38] Let $T = U|T|$ be the polar decomposition of T. Then for any $\alpha > 0$, the* *following statements are valid:*

(i) $U^*U|T|^\alpha = |T|^\alpha$;

(ii) $|T^*|^\alpha = U|T|^\alpha U^*$;

(iii) $U^*|T^*|^\alpha U = |T|^\alpha$;

(v) $U|T|^\alpha = |T^*|^\alpha U$.

Proof.

(i). By Lemma 1.3, we have

$$\overline{R(|T|^\alpha)} = \overline{R(T^*T)} = \overline{R(T^*)};$$

and thus $U^*U|T|^\alpha = |T|^\alpha$.

(ii). Since $U^*U|T| = |T|$, we have

$$(U|T|U^*)^n = U|T|^n U^*, \text{ for any } n \in \mathbb{N}. \tag{1.3}$$

Let $f(t) = t^\alpha$ and choose any sequence $(P_n)_{n>0}$ of polynomials such that $P_n(0) = 0 (\forall m \in \mathbb{N})$, and $(P_n)(t) \longrightarrow f(t)$ uniformly on the interval $[0, \||T|\|]$. Then from (1.2) and (1.3), we have

$$U|T|^\alpha U^* = Uf(|T|)U^* = \lim_{n\to\infty} UP_n(|T|)U^* = \lim_{n\to\infty} P_n(U|T|U^*)$$

$$= f(U|T|U^*) = (U|T|U^*)^\alpha = |T^*|^\alpha.$$

(iii). Since $T^* = U^*|T^*|$ is the polar decomposition of T^* , the conclusion follows immediately from (1.2) by replacing the pair $(U;T)$ with $(U^*;T^*)$.

(v). From (i), we have $U^*U|T|^\alpha = |T|^\alpha$. Taking *-operation, we get $|T|^\alpha = |T|^\alpha U^*U$. It follows from (ii) that

$$U|T|^\alpha = U(|T|^\alpha U^*U) = (U|T|^\alpha U^*)U = |T^*|^\alpha U.$$

■

Theorem 1.3 *[24] Let $T = U|T|$ be the polar decomposition of T. Then T is quasinormal if and only if $U|T| = |T|U$.*

Proof. ($\Longleftarrow$). Assume $U|T| = |T|U$. Then $T(T^*T) - (T^*T)T = U|T||T|^2 - |T|^2U|T| = 0$ so that T is quasi-normal.
($\Longrightarrow$). If T is quasinormal , then

$$0 = T(T^*T) - (T^*T)T = U|T||T|^2 - |T|^2U|T| = (U|T|^2 - |T|^2U)|T|.$$

This equality means that $U|T|^2 = |T|^2U$ on $\overline{\mathcal{R}(|T|)}$ by continuity of an an operator, and also $U|T|^2 = |T|^2U$ on $\mathcal{N}(|T|)$ since $\mathcal{N}(U) = \mathcal{N}(|T|)$, so that $U|T|^2 = |T|^2U$ on $mathcal H = \mathcal{N}(|T|) \oplus \overline{\mathcal{R}(|T|)}$, that is, $U|T|^2 = |T|^2U$. Consequently $U|T| = |T|U$.

■

Lemma 1.5 *[23, Theorem 2] Let $T, S \in \mathcal{B}(\mathcal{H})$ are positive operators and $[T, S] = 0$. Then*

$$[P_{\mathcal{N}(T)^\perp}, P_{\mathcal{N}(S)^\perp}] = [P_{\mathcal{N}(T)^\perp}, S] = [T, P_{\mathcal{N}(S)^\perp}] = 0,$$

As an application of Lemma 1.5 and Remark 1.2, we obtain the following result.

Corollary 1.1 *Let $T \in \mathcal{B}(\mathcal{H})$ be binormal and $T = U|T|$ be it's polar decomposition. Then*

$$[UU^*, U^*U] = [|T|^\alpha, UU^*] = [|T^*|^\alpha, U^*U] = 0, \ \textit{for each } \alpha > 0.$$

Next, we state some results about polar decomposition of an invertible operators $T \in \mathcal{B}(\mathcal{H})$.

Proposition 1.6 *[13] Let $T = U|T|$ be the polar decomposition of T. Then, we have*

$$T \text{ is injective } \Longleftrightarrow U \text{ is isometry } (i.e. U^*U = I).$$

$$\mathcal{N}(T) = \mathcal{N}(T^*) \Longleftrightarrow U \text{ is normal } (i.e. UU^* = U^*U).$$

$$T \text{ and } T^* \text{ are injective } \Longleftrightarrow U \text{ is unitary.}$$

Proof.

- Since two orthogonal projection are equal if and only if they have the same null subspace, then we have

$$T \text{ is injective } \Longleftrightarrow \mathcal{N}(U^*U) = \mathcal{N}(U) = \mathcal{N}(T) = \{0\} = \mathcal{N}(I) \Longleftrightarrow U^*U = I. \tag{1.4}$$

- Notice that

$$\mathcal{N}(T) = \mathcal{N}(T^*) \Longleftrightarrow \mathcal{N}(U) = \mathcal{N}(U^*) \Longleftrightarrow \mathcal{N}(U^*U) = \mathcal{N}(UU^*) \Longleftrightarrow U^*U = UU^*. \tag{1.5}$$

- The last statement follows from (1.4) and (1.5).

■

Remark 1.4 *If T is invertible, then $|T|$ is invertible and U is unitary.*

Proposition 1.7 *Let T be an invertible operator and let $T = U|T|$ be it's polar decomposition. Then the following statements hold*

(a) $|T^{-1}| = |T^*|^{-1}$;

(b) $|T|^{-1} = |(T^*)^{-1}|$;

(c) $T^{-1} = U^*|T^{-1}|$ *is the polar decomposition of* T^{-1};

(d) $|T|^{-\alpha} = U^*|T^{-1}|^{\alpha}U$, *for* $\alpha > 0$,

(e) $|T^{-1}|^{\alpha} = U|T|^{-\alpha}U^*$, *for* $\alpha > 0$.

Proof.

(a). Notice that

$$|T^{-1}|^2 = (T^{-1})^*T^{-1} = (T^*)^{-1}T^{-1} = (TT^*)^{-1} = |T^*|^{-2}, \tag{1.6}$$

so $|T^{-1}| = |T^*|^{-1}$.

(b). It follows from (a), by replacing T with T^*.

(c). From $T = U|T|$ we get $T^{-1} = |T|^{-1}U^*$; and then

$$\begin{aligned} T^{-1} &= |T|^{-1}U^* \\ &= (U^*|T^*|U)^{-1}U^* \quad \textit{by} \quad \text{Theorem 1.2 (iii)} \\ &= U^*|T^{-1}|UU^* \qquad \textit{by } \text{(a)} \\ &= U^*|T^{-1}| \qquad\qquad \text{since U is a unitary .} \end{aligned}$$

(d). Using Theorem 1.2(v) and (a), we get

$$U^*|T^{-1}|^{\alpha}U = |(T^*)^{-1}|^{\alpha}U^*U = |T|^{-\alpha}U^*U = |T|^{-\alpha};$$

(e). Follow from (v) by replacing T by T^*.

■

1.3 The reduced minimum modulus of operators

The reduced minimum modulus of operators on a Hilbert space measure the clodness of the range of operators.

Definition 1.3 *(see [25]) Let $T \in \mathcal{B}(\mathcal{H})$. Then :*

$$\gamma(T) := \begin{cases} \inf\{\|Tx\|; \|x\| = 1,\, x \in \mathcal{N}(T)^{\perp}\} & if\ T \neq 0 \\ +\infty & if\ T = 0. \end{cases}$$

The notion of the reduced minimum module is motivated by the following characterization:

Proposition 1.8 *[25] Let $T \in \mathcal{B}(\mathcal{H})$, then we have*

$$\gamma(T) > 0 \Longleftrightarrow \mathcal{R}(T) \text{ is closed.}$$

The reduced minimum modulus of $T \in \mathcal{B}(\mathcal{H})$ has the following properties.

Proposition 1.9 *(see [1, 20]) Let $T \in \mathcal{B}(\mathcal{H})$. Then*

(i) $\gamma(T)^2 = \gamma(T^*T)$.

(ii) $\gamma(T) = \gamma(T^*)$.

1.4 The Moore-Penrose inverse of operators

Moore-Penrose inverse is a generalization of the inverse of operator. It was independently described by Moore in 1920, Bjerhammar in 1951 and Penrose in 1955. This inverse is a powerful tool for solving matrix and operators equations.

Definition 1.4 *Let $T \in \mathcal{B}(\mathcal{H})$, the Moore-Penrose inverse of T denoted by $T^+ \in \mathcal{B}(\mathcal{H})$ is the unique solution of the following set of equations*

$$TT^+T = T,\ T^+TT^+ = T^+,\ TT^+ = (TT^+)^*,\ T^+T = (T^+T)^*.$$

Notice that T^+ exists if and only if $R(T)$ is closed [27]. In this case TT^+ and T^+T are the orthogonal projections onto $R(T)$ and $R(T^*)$ respectively.

Properties of Moore-Penrose inverse

Now, we state some properties of the Moore-Penrose inverse needed in this work.

Proposition 1.10 *[51, Section 1] Let $T \in \mathcal{B}(H)$. If $R(T)$ is closed, then we have:*

(a) $\mathcal{R}(T^+) = \mathcal{R}(T^*) = \mathcal{N}(T)^\perp$

(b) $\mathcal{N}(T^+) = \mathcal{N}(TT^+) = \mathcal{N}(T^*) = \mathcal{R}(T)^\perp$

(c) $\mathcal{R}(T) = \mathcal{R}(TT^+) = \mathcal{R}((T^+)^*)$,

(d) $\mathcal{N}(T) = \mathcal{N}(T^+T) = \mathcal{N}((T^+)^*)$,

(e) $(TT^*)^+ = (T^+)^*T^+$,

(f) $(T^*T)^+ = T^+(T^+)^*$,

(g) $(T^*)^+ = (T^+)^*$,

(h) $(T^+)^+ = T$.

Next, we recall some results on the polar decomposition of closed range operators.

Lemma 1.6 *[30, 31] Let $T \in \mathcal{B}(\mathcal{H})$ with closed range and $T = U|T|$ be its polar decomposition. Then*

(i) $T^+ = U^*|T^+|$ *is the polar decomposition of* T^+;

(ii) $|T^+| = |T^*|^+$;

(iii) $|T|^+ = |(T^*)^+|$;

(iv) $\left(|T|^+\right)^\alpha = U^*\left(|T^+|\right)^\alpha U$, *for each* $\alpha > 0$;

(v) $\left(|T^+|\right)^\alpha = U\left(|T|^+\right)^\alpha U^*$, *for each* $\alpha > 0$.

Proof.

(i) Put $S = |T^*|^+ U$, since $\mathcal{R}(U) = \mathcal{R}(T) = \mathcal{R}(|T^*|)$ and $\mathcal{R}(|T^*|) = \mathcal{N}(|T^*|)^\perp$, so $\mathcal{R}(S) = \mathcal{R}(U) = \mathcal{R}(|T^*|)$. Moreover, we have

$$\begin{aligned} T^*ST^* &= U^*|T^*|(|T^*|^+U)U^*|T^*| \\ &= U^*|T^*||T^*|^+|T^*| \\ &= U^*|T^*|, \\ ST^*S &= |T^*|^+(UU^*|T^*|)|T^*|^+U \\ &= |T^*|^+|T^*||T^*|^+U \\ &= |T^*|^+U = S, \\ T^*S &= U^*|T^*||T^*|^+U \\ &= U^*P_{R(T)}U \\ &= U^*U \\ &= P_{R(T^*)}, \end{aligned}$$

and

$$\begin{aligned} ST^* &= |T^*|^+UU^*|T^*| \\ &= |T^*|^+P_{R(T)}|T^*| \\ &= |T^*|^+|T^*| \\ &= P_{R(|T^*|^+)} = P_{R(|T^*|)} = P_{R(S)}. \end{aligned}$$

This equalities show that $(T^+)^* = (T^*)^+ = S$, and hence $T^+ = S^* = U^*|T^*|^+$ is the polar decomposition for T^+

(ii) It follows from Proposition 1.10(e).

(iii) It follows from (ii) by replacing T with T^*.

(iv) By (iii), we have

$$\begin{aligned}(|T|^+)^2 &= (|(T^*)^+|)^2 = (|(T^+)^*|)^2 = T^+(T^+)^* \\ &= U^*|T^*|^+|T^*|^+U \\ &= U^*|T^*|^+UU^*|T^*|^+U \\ &= (U^*|T^*|^+U)^2\end{aligned}$$

So $|T|^+ = U^*|T^*|^+U$, thus $(|T|^+)^n = U^*(|T^*|^+)^nU$. Hence $(|T|^+)^\alpha = U^*\left(|T^+|\right)^\alpha U$.

(v) It follows from (iv) by replacing T with T^* .

■

Lemma 1.7 *[30] Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator with closed range. Then T^+ is positive and $\mathcal{N}((T^+)^\alpha) = \mathcal{N}(T^+) = \mathcal{N}(T^*)$.*

Proof. Let $x \in \mathcal{H}$. Then $< T^+x, x >=< T^+TT^+x, x >=< T^+TT^+x, T^+x >\geq 0$ and so $< T^+x, x >\geq 0$. So from Lemma 1.2 and Proposition 1.10(b), we have $\mathcal{N}((T^+)^\alpha) = \mathcal{N}(T^+) = \mathcal{N}(T^*)$. ■ Using the functional calculus, we have the following lemma

Lemma 1.8 *[43] Let $T \in \mathcal{B}(\mathcal{H})$ with closed range. Then $(|T|^+)^\alpha = (|T|^\alpha)^+$, for all $\alpha > 0$.*

Remark 1.5 *By Proposition (1.4), we have U is a partial isometry if and only if $U^+ = U^*$.*

Next lemma gives a nessesery and suffissant condition which guarantee the reverse order law for the Moore-Penrose inverse.

Lemma 1.9 *[19] Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ be such that T, S, TS have closed ranges. Then the following statements are equivalent:*

(i) $(TS)^+ = S^+T^+$.

(ii) $\mathcal{R}(T^*TS) \subset \mathcal{R}(S)$ *and* $\mathcal{R}(SS^*T^*) \subset \mathcal{R}(T^*)$.

Lemma 1.10 *[44] Let $T \in \mathcal{B}(\mathcal{H})$ with closed range. If $S \in \mathcal{B}(\mathcal{H})$ suth that $ST = TS$ and $ST^* = T^*S$, then $ST^+ = T^+S$.*

Corollary 1.2 *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that T have closed range. Then $S|T|^+ = |T|^+S$.*

EP operators

Definition 1.5 *Let $T \in \mathcal{B}(\mathcal{H})$ have closed range. T is EP operator if, $TT^+ = T^+T$.*

Theorem 1.4 *let $T \in \mathcal{B}(\mathcal{H})$ have closed range. Then the following assertions are equivalent:*

1. *T is EP operator.*
2. *$\mathcal{R}(T) = \mathcal{R}(T^*)$.*
3. *$\mathcal{N}(T) = \mathcal{N}(T^*)$.*
4. *$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{R}(T) \oplus \mathcal{N}(T^*) \longrightarrow \mathcal{R}(T) \oplus \mathcal{N}(T^*)$, or $T_1 \in \mathcal{B}(\mathcal{H})$ invertible.*

Remark 1.6 *From the last theorem, notice that normal operators with closed range are EP but the converse is not true even in a finite dimensional space as shown by the following example.*

Example 1.1 *Let $T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then T is not normal. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is invertible and $A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Then $T^+ = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It is easy to see that $TT^+ = T^+T$.*

For more detail on EP operators see [10, 17, 18].

1.5 Some properties of the λ-Aluthge transform

In this section, we explore the λ-Aluthge transform by giving its basic definitions and its related properties.

Definition 1.6 *[46] Let $T = U|T|$ and $\lambda \in [0,1]$. Then the λ-Aluthge transform of T is defined by*

$$\Delta_\lambda(T) = |T|^\lambda U |T|^{1-\lambda},$$

In the case $\lambda = \frac{1}{2}$, *the* λ*-Aluthge transform becomes the Aluthge transform* $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. *Also,* $\Delta_1(T) = |T|U$ *is known as Duggal's transform [21].*

Example 1.2 *Consider the weighted shift* $T : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ *given by*

$$T(x_1, x_2, x_3, ...)=(0, \tfrac{1}{2}x_1, 2x_2, \tfrac{1}{2}x_3, ...).$$

Then

$$T^*(x_1, x_2, x_3, ...)=(\tfrac{1}{2}x_2, 2x_3, \tfrac{1}{2}x_4, ...), \text{ and } \quad T^*T(x_1, x_2, x_3, ..) = (\tfrac{1}{4}x_1, 4x_2, \tfrac{1}{4}x_3, ...).$$

So

$$|T|(x_1, x_2, x_3, ...)=(T^*T)^{\frac{1}{2}}(x_1, x_2, x_3, ...) = (\tfrac{1}{2}x_1, 2x_2, \tfrac{1}{2}x_3, ...).$$

Hence,

$$|T|^{\frac{1}{2}}(x_1, x_2, x_3, ..) = (\frac{1}{\sqrt{2}}x_1, \sqrt{2}x_2, ...).$$

Since $|T|$ *is invertible and* $|T|^{-1}(x_1, x_2, x_3, ...) = (2x_1, \frac{1}{2}x_2...)$, *we have* $U(x_1, x_2, x_3, ...) = T|T|^{-1}(x_1, x_2, x_3, ...) = (0, x_1, x_2, ...)$, *therefore*

$$\Delta(T)(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...).$$

In general if

$$T(x_1, x_2, x_3, ...) = (0, \alpha_1 x_1, \alpha_2 x_2, ...),$$

then

$$\Delta_\lambda(T)(x_1, x_2, x_3, ...) = (0, \alpha_1^{1-\lambda}\alpha_2^{\lambda}x_1, \alpha_2^{1-\lambda}\alpha_3^{\lambda}x_2, ...).$$

Remark 1.7 *We note that the Aluthge transform of any operators does not depend on the choice of the partial isometry, indeed, suppose that* U_1 *and* U_2 *are two partial isometries such that* $T = U_1|T| = U_2|T|$ *be an arbitrary polar decomposition. Then*

$$U_1|T|^{\frac{1}{2}} = U_2|T|^{\frac{1}{2}} \text{ on } \mathcal{R}(|T|^{\frac{1}{2}}) \text{ and thus}$$

$$U_1|T|^{\frac{1}{2}} = U_2|T|^{\frac{1}{2}} = 0 \text{ on } N(|T|^{\frac{1}{2}}),$$

Hence $U_1|T|^{\frac{1}{2}} = U_2|T|^{\frac{1}{2}}$ *on* $\mathcal{H}$. *Therefore by multiplying this relation on the left by* $|T|^{\frac{1}{2}}$, *we have :*

$$|T|^{\frac{1}{2}}U_1\ |T|^{\frac{1}{2}}\ = |T|^{\frac{1}{2}}U_2|T|^{\frac{1}{2}}$$

If $\lambda = 1$, *then the Duggal's transform depend on the choice of the partial isometry of such a factorisation, as shown by the following example*

Example 1.3 *Let* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in B(\mathbb{C}^2)$. *Then* $|T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
On the other hand, let

$$U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence U_1 *and* U_2 *are partial isometries such that* $T = U_1|T| = U_2|T|$. *By a simple computation, we get*

$$\Delta_1 = |T|U_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |T|U_1.$$

In 1990, A. Aluthge [2] proved the next Theorem, which shows that Aluthge transform is a good tool for the study of hyponormal operators.

Theorem 1.5 *[2] Let* $T \in \mathcal{B}(\mathcal{H})$ *be a p-hyponormal operator and* $p \in]0,1]$. *Then the following assertions hold*

(i) $\Delta_\lambda(T)$ *is* $(p+\frac{1}{2})$ *hyponormal if* $0 < p \leq \frac{1}{2}$.

(ii) $\Delta_\lambda(T)$ *is hyponormal if* $\frac{1}{2} < p \leq 1$.

The next proposition contains some easy properties of $\lambda-$Aluthge transform.

Proposition 1.11 *[3, 4, 32, 14,* **?***] Let* $T = U|T|$*be the polar decomposition of* T *and* $\lambda \in [0,1]$. *Then*

1. $\Delta_\lambda(\alpha T) = \alpha\Delta_\lambda(T)$, *for all* $\alpha \in \mathbb{C}$.

2. *$\Delta_\lambda(VTV^*) = V\Delta_\lambda(T)V^*$ such that V unitary.*

3. *$\|\Delta_\lambda(T)\| \leq \|T\|$.*

4. *T and $\Delta_\lambda(T)$ have the same spectrum.*

5. *If $T = T_1 \oplus T_2$, then $\Delta_\lambda(T) = \Delta_\lambda(T_1) \oplus \Delta_\lambda(T_2)$ (orthogonal decompositions).*

Proof.

1. We have $T = U|T|$ then $\alpha T = \alpha U|T|$. Since $\alpha = \frac{|\alpha|^2}{\overline{\alpha}}$. Hence

$$\begin{aligned} \alpha T &= \frac{|\alpha|^2}{\overline{\alpha}}T \\ &= \frac{|\alpha|}{\overline{\alpha}}U|\alpha||T| \end{aligned}$$

As $|\alpha T| = |\alpha||T|$. Then $\alpha T = \frac{|\alpha|}{\overline{\alpha}}U|\alpha T|$ and for $x \in \mathcal{H}$

$$\left\|\frac{|\alpha|}{\overline{\alpha}}U(x)\right\| = \|U(x)\| = \|x\|.$$

Then the operator $V = \frac{|\alpha|}{\overline{\alpha}}U$ is a partial isometry.

$$\begin{aligned} \Delta_\lambda(\alpha T) &= |\alpha T|^\lambda V|\alpha T|^{1-\lambda} \\ &= |\alpha||T|^\lambda \frac{|\alpha|}{\overline{\alpha}}U|T|^{1-\lambda} \\ &= \frac{|\alpha|^2}{\overline{\alpha}}\Delta_\lambda(T) \\ &= \alpha\Delta_\lambda(T) \end{aligned}$$

2. Let $T \in \mathcal{B}(\mathcal{H})$. It is easy to check that

$$|VTV^*| = V|T|V^* \text{ and } |VTV^*|^\lambda = V|T|^\lambda V^*, \lambda \in [0,1]$$

Now, let $T = U|T|$ be a polar decomposition. Then

$$VTV^* = VU|T|V^* = (VUV^*)(V|T|V^*) = \tilde{U}|VTV^*|,$$

where $\tilde{U} = VUV^*$ is a partial isometry and we have $\mathcal{N}(VTV^*) = \mathcal{N}(\tilde{U})$. Hence $\tilde{U}|VTV^*|$ is the polar decompositon of VTV^*. This implies that

$$\begin{aligned}\Delta_\lambda(VTV^*) &= |VTV^*|^\lambda \tilde{U}|VTV^*|^{1-\lambda}\\ &= V|T|^\lambda V^*\tilde{U}V|T|^{1-\lambda}V^*\\ &= V|T|^\lambda U|T|^{1-\lambda}V^*\\ &= V\Delta_\lambda(T)V^*\end{aligned}$$

3. As $\|T\| = \||T|\|$ and $\||T|^\alpha\| = \||T|\|^\alpha$, for $\alpha > 0$, so we obtain

$$\begin{aligned}\|\Delta_\lambda(T)\| &= \||T|^\lambda U|T|^{1-\lambda}\|\\ &\leq \||T|^\lambda\|\|U\|\||T|^{1-\lambda}\|\\ &\leq \||T|^\lambda\|\||T|^{1-\lambda}\| \qquad \text{Since } \|U\| = 1\\ &\leq \||T|\|^\lambda\||T|\|^{1-\lambda}\\ &= \||T|\| = \|T\|.\end{aligned}$$

For $\lambda = 1$, we obtain

$$\|\Delta_1(T)\| = \||T|U\| \leq \||T|\|\|U\| = \||T|\| = \|T\|.$$

4. Since T is positive, then T is normal. Then

$$\sigma(|T|^\lambda U|T|^{1-\lambda}) = \sigma(U|T|^{1-\lambda}|T|^\lambda) = \sigma(U|T|) = \sigma(T).$$

5. Let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then the canonical polar decomposition of T can be decomposed as follows

$$T = U|T| = \begin{pmatrix} U_A & 0 \\ 0 & U_B \end{pmatrix} \begin{pmatrix} |A| & 0 \\ 0 & |B| \end{pmatrix}.$$

As a consequence,

$$\Delta_\lambda(T) = \begin{pmatrix} |A|^\lambda U_A |A|^{1-\lambda} & 0 \\ 0 & |B|^\lambda U_B |B|^{1-\lambda} \end{pmatrix} = \begin{pmatrix} \Delta_\lambda(A) & 0 \\ 0 & \Delta_\lambda(B) \end{pmatrix} = \Delta_\lambda(A) \oplus \Delta_\lambda(B).$$

■

Remark 1.8 *If $T = U|T|$ is invertibe, then $|T|^\lambda$ is invertible for every $\lambda \in]0,1[$, and*

$$\Delta(T) = |T|^\lambda U |T|^{1-\lambda} = |T|^\lambda T |T|^{-\lambda}.$$

So, $\Delta_\lambda(T)$ and T are similar.

The following shows that every quasinormal operators is fixed point for $\lambda-$Aluthge transform.

Proposition 1.12 *[7] Let $T \in \mathcal{B}(\mathcal{H})$ and $\lambda \in]0,1]$. Then*

$$T \textit{ is quasinormal } \Longleftrightarrow \Delta_\lambda(T) = T.$$

Proof. Consider the polar decomposition $T = U|T|$. We have $\Delta_\lambda(T) = T$ if and only if

$$(|T|^\lambda U - U|T|^\lambda)|T|^{1-\lambda} = 0. \tag{1.7}$$

If $\lambda = 1$, then this is equivalent to $|T|U - U|T| = 0$. If $\lambda < 1$, then for every $x \in \mathcal{R}(|T|^{1-\lambda})^\perp = \mathcal{N}(|T|^{1-\lambda}) = \mathcal{N}(|T|) = \mathcal{N}(|T|^\lambda) = \mathcal{R}(|T|)^\perp$, we have $Ux = 0 = |T|^\lambda x$ implying $(|T|^\lambda U - U|T|^\lambda)x = 0$. Therefore, (1.7) holds if and only if $|T|^\lambda U - U|T|^\lambda = 0$ which is equivalent to the commutativity of U and $|T|$. But as already mentioned earlier, this holds exactly when T is quasinormal.

■ The proof of Proposition 1.12 in the case $\lambda = \frac{1}{2}$ can be found in [32]. Next, we give the λ-Aluthge transform of partial isometry.

Proposition 1.13 *[41] If U is a partial isometry, then $|U| = U^*U$ and $U = U|U|$ is the polar decomposition of U. Also, $\Delta(T) = U^*UU$.*

Proof. Since U^*U is an orthogonal projection, $(U^*U)^2 = U^*U$ and $U^*U \geq 0$. Therefore, $|U| = (U^*U)^{1/2} = U^*U$. So $U|U| = U$. The kernel condition for the polar decomposition is satisfied automatically. Hence $U = U|U|$ is the polar decomposition of U.
Since $|U| = U^*U = (U^*U)^2$ and $U^*U \geq 0$, we have $|U|^{\frac{1}{2}} = U^*U$. Therefore, $\Delta(T) = |U|^{\frac{1}{2}}U|U|^{\frac{1}{2}} = U^*UUU^*U = U^*UU$. ∎

Remark 1.9 *By the functional calculus, we obtain $|U|^{\alpha} = U^*U$, for $\alpha > 0$ and so $\Delta_\lambda(U) = U^*UU$, for $\lambda \in]0,1]$.*

Now, we give the λ-Aluthge transform of nilpotent operators.

Theorem 1.6 *[12] Let $T \in \mathcal{B}(\mathcal{H})$,$\lambda \in]0,1]$ and $d \geq 1$. Then*

$$T^{d+1} = 0 \Longleftrightarrow \Delta_\lambda(T)^d = 0.$$

Proof. Let $T = U|T|$ be the polar decomposition of T. Observe that

$$\Delta_\lambda(T)^d = (|T|^\lambda U|T|^{1-\lambda})^d = |T|^\lambda T^{d-1}U|T|^{1-\lambda}.$$

Thus,

$$\begin{aligned} \Delta_\lambda(T)^d = 0 \quad &\Rightarrow \quad |T|^\lambda T^{d-1}U|T|^{1-\lambda} = 0 \\ &\Rightarrow \quad U|T|^{1-\lambda}(|T|^\lambda T^{d-1}U|T|^{1-\lambda})|T|^\lambda = 0 \\ &\Rightarrow \quad T^{d+1} = 0 \end{aligned}$$

Conversely, we have

$$T^{d+1} = (U|T|)^{d+1} = U|T|^{1-\lambda}\Delta_\lambda(T)^d|T|^\lambda.$$

Thus,

$$\begin{aligned} T^{d+1}=0 &\Rightarrow U|T|^{1-\lambda}\Delta_\lambda(T)^d|T|^\lambda=0 \\ &\Rightarrow U^*U|T|^{1-\lambda}\Delta_\lambda(T)^d|T|^\lambda=0 \\ &\Rightarrow |T|^{1-\lambda}\Delta_\lambda(T)^d|T|^\lambda=0 \\ &\Rightarrow |T|\Delta_\lambda(T)^d|T|=0 \\ &\Rightarrow \langle\Delta_\lambda(T)^d|T|x,|T|x\rangle=0\ for\ all\ x\in mathcalH \end{aligned}$$

Since $N(|T|)\subseteq\mathcal{N}(\Delta_\lambda(T))$, $\langle\Delta_\lambda(T)^d x,x\rangle=0$ for all $x\in mathcalH$. Hence $\Delta_\lambda(T)^d=0$. ■

For $x,y\in\mathcal{H}$ $(x\neq 0, y\neq 0)$, we define the rank one operator $x\otimes y$ in $B(\mathcal{H})$ by

$$(x\otimes y)z=<z,y>x, \text{ for } z\in\mathcal{H}.$$

Then $(x\otimes y)^*=(y\otimes x)$. The next result gives the transform $\Delta_\lambda(T)$ of rank one operators.

Proposition 1.14 *[15] let $\lambda\in]0,1]$ and $T=x\otimes y$ be the rank one operator. Then, we have*

$$\Delta_\lambda(x\otimes y)=\tfrac{\langle x,y\rangle}{\|y\|^2}(y\otimes y).$$

Proof. Denote $T=x\otimes y$. First note that

$$T^*T=|T|^2=\|x\|^2(y\otimes y)=(\frac{\|x\|}{\|y\|}(y\otimes y))^2\ and\ |T|=\sqrt{T^*T}=\frac{\|x\|}{\|y\|}(y\otimes y).$$

It follows that $|T|^2=\|x\|\|y\||T|$. Hence $|T|^\gamma=(\|x\|\|y\|)^{\gamma-1}|T|$ for any $\gamma>0$.

Now, let $T=U|T|$ be the polar decomposition of T, then we have

$$\begin{aligned} \Delta_\lambda(T) &= |T|^\lambda U|T|^{1-\lambda} \\ &= (\|x\|\|y\|)^{\lambda-1}(\|x\|\|y\|)^{-\lambda}|T|U|T| \\ &= \frac{1}{\|x\|\|y\|}|T|T \\ &= \frac{1}{\|y\|^2}(y\otimes y)\circ(x\otimes y)=\frac{\langle x,y\rangle}{\|y\|^2}(y\otimes y). \end{aligned}$$

■

Chapter 2

The λ-Aluthge transform and its applications to some classes of operators

In this chapter, we characterize the invertible, binormal, and EP operators and its intersection with a special class ($\delta(H)$) of introduced operators via the λ-Aluthge transformation. We start with the definition of δ-class which is given by S. Lee, W. Lee and J. Yoon in 2013.(see[37]).

Definition 2.1 *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. If $U^2|T| = |T|U^2$, then T will said to be in the δ-class, denoted by $T \in \delta(H)$.*

In the following, we establish some new and basic properties of δ-class.

2.1 Properties of δ-class

Theorem 2.1 *Let $T \in \delta(\mathcal{H})$. Then*

1. *$\alpha T \in \delta(\mathcal{H})$, for $\alpha \in \mathbb{C}$,*

2. *If $N(T^*) \subset \mathcal{N}(T)$, then $T^* \in \delta(\mathcal{H})$,*

3. *If T^{-1} exists, then $T^{-1} \in \delta(\mathcal{H})$.*

4. *If $S \in \mathcal{B}(\mathcal{H})$ is unitary equivalent to T, then $S \in \delta(\mathcal{H})$.*

Proof.

1. Let $\alpha = |\alpha|e^{(i\theta)} \in \mathbb{C}$ and $T = U|T|$ be the polar decomposition of α and T, respectively. Then $\alpha T = e^{(i\theta)}U|\alpha T| = W|\alpha T|$ is the polar decomposition of αT. Since $T \in \delta(\mathcal{H})$, then

$$\begin{aligned} W^2|\alpha T| &= e^{(2i\theta)}U^2|\alpha T| \\ &= e^{(2i\theta)}|\alpha T|U^2 \\ &= |\alpha T|e^{(2i\theta)}U^2 \\ &= |\alpha T|W^2. \end{aligned}$$

So $\alpha T \in \delta(\mathcal{H})$, for $\alpha \in \mathbb{C}$.

2. Let $T = U|T|$ be the polar decomposition of T. Then $T^* = U^*|T^*|$ is the polar decomposition of T^*. Since $T \in \delta(\mathcal{H})$, then $U^2|T| = |T|U^2$. From the fact $U|T| = |T^*|U$, we obtain $U|T^*|U = |T|UU$. Hence

$$U|T^*| = |T|U,$$

on $\mathcal{R}(U)$. Since $\mathcal{N}(T^*) \subset \mathcal{N}(T)$, we get

$$U|T^*| = |T|U,$$

on $\mathcal{N}(U^*)$and so $U|T^*| = |T|U$, on H. Then, we have

$$|T^*|UU = U|T|U = UU|T^*|.$$

Therefore

$$|T^*|U^2 = U^2|T^*|,$$

by taking the *-operation, we get that $T^* \in \delta(\mathcal{H})$.

3. Let $T = U|T|$ be the polar decomposition of T. Then $T^{-1} = U^*|T^{-1}|$ is the polar decomposition of T^{-1}. Since T is invertible, U is unitary and $|T|$ is ivertible. Also, notice that $|T^{-1}| = |T^*|^{-1}$. Then by Theorem 2.1 (2) , we get that $U^2|T^*|^{-1} = |T^*|^{-1}U^2$. Hence $T^{-1} \in \delta(\mathcal{H})$.

4. Let $T \in \delta(\mathcal{H})$ and S be unitary equivalent of T. Then there exists unitary operator V such that $S = VTV^*$. Since V is unitary, then

$$|VTV^*| = V|T|V^*,$$

and $VTV^* = VUV^*V|T|V^*$ is the canonical polar decomposition of VTV^*. Therefore $S = W|S| = VUV^*|VTV^*|$. From the uniqueness of the polar decomposition, we have $W = VUV^*$ and $|S| = |VTV^*| = V|T|V^*$. Hence

$$\begin{aligned}
W^2|S| &= VUV^*VUV^*V|T|V^* \\
&= VU^2|T|V^* \\
&= V|T|U^2V^* \\
&= V|T|V^*VUV^*VUV^* \\
&= |VTV^*|(VUV^*)^2 \\
&= |S|W^2.
\end{aligned}$$

So $S \in \delta(\mathcal{H})$.

■ Now, we present an example to show that in Theorem 2.1(2) the condition "$\mathcal{N}(T^*) \subset \mathcal{N}(T)$" is essential.

Example 2.1 *Consider the right shift operator S, defined on the Hilbert space $\ell^2(\mathbb{N})$ by $S(x_1, x_2, ...) = (0, x_1, x_2, ...)$. Then $S^*(x_1, x_2, ...) = (x_2, x_3, ...)$ and so $S^*S = I$. Hence S is an isometry and $|S| = I$, it follows that $S = S|S|$ is the polar decomposition of S. Therefore $S \in \delta(\mathcal{H})$. On the other hand, as S^* is a partial isometry, $S^* = S^*|S^*| = S^*SS^*$. A simple calculation shows that*

$$(S^*)^2SS^* = (S^*)^2 \neq SS^*(S^*)^2.$$

Hence $S^* \notin \delta(\mathcal{H})$

Proposition 2.1 *Let $T \in \delta(\mathcal{H})$ with closed range. If $\mathcal{N}(T^*) \subset \mathcal{N}(T)$, then $T^+ \in \delta(\mathcal{H})$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Then $T^+ = U^*|T^+|$ is the polar decomposition of T^+. From Theorem 2.1 (2), we have

$$(U^*)^2|T^*| = |T^*|(U^*)^2.$$

Then

$$(U^*)^2|T^*|^+ = |T^*|^+(U^*)^2.$$

As $|T^+| = |T^*|^+$, it follows

$$(U^*)^2|T^+| = |T^+|(U^*)^2.$$

Therefore $T^+ \in \delta(\mathcal{H})$. ■

Theorem 2.2 *Let $T \in \mathcal{B}(\mathcal{H})$ with closed range. Then the following statements are equivalent:*

(a) $T^* \in \delta(H)$.

(b) $T^+ \in \delta(H)$.

Proof. We have

$$\begin{aligned} T^* \in \delta(H) &\Longleftrightarrow (U^*)^2|T^*| = |T^*|(U^*)^2 \\ &\Longleftrightarrow (U^*)^2|T^*|^+ = |T^*|^+(U^*)^2 \quad \text{by corollary 1.2} \\ &\Longleftrightarrow (U^*)^2|T^+| = |T^+|(U^*)^2 \quad \text{by Lemma 1.6} \quad \text{(ii)} \\ &\Longleftrightarrow T^+ \in \delta(H). \end{aligned}$$

■ The next example shows that sum of two operators in $\delta(\mathcal{H})$ need not be in $\delta(\mathcal{H})$, even when the operators commute.

Example 2.2 *Let* $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Then*

$$T = U|T| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$S = V|S| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$T + S = W|T + S| = \begin{pmatrix} \frac{2\sqrt{5}}{5} & \frac{-\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{pmatrix} \begin{pmatrix} \frac{3\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{pmatrix},$$

are the polar decomposition of T, S and $T+S$, respectively. By an easy calculation,

$$U^2|T| = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = |T|U^2, \ \ so \ \ T \in \delta(\mathcal{H}).$$

And

$$V^2|S| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |S|V^2, \ \ so \ \ S \in \delta(\mathcal{H}).$$

But

$$W^2|T+S| = \begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{-\sqrt{5}}{5} \\ \frac{3\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{pmatrix} \neq \begin{pmatrix} \frac{13\sqrt{5}}{25} & \frac{-9\sqrt{5}}{25} \\ \frac{11\sqrt{5}}{25} & \frac{2\sqrt{5}}{25} \end{pmatrix} = |T+S|W^2, \;\; so \;\; T+S \notin \delta(\mathcal{H}),$$

Also, the above example shows that the class $\delta(\mathcal{H})$ may not have the translation-invariant property. (i.e. if $T \in \delta(\mathcal{H})$, then $T-\lambda$ need not be in $\delta(\mathcal{H})$ for every $\lambda \in \mathbb{C}$);

Definition 2.2 *Let $T, S \in \mathcal{B}(\mathcal{H})$, we say that T Doubly commutes with S if $TS = ST$ and $TS^* = S^*T$.*

Theorem 2.3 *If $T \in \delta(\mathcal{H})$ and $S \in \delta(\mathcal{H})$ such that T is Doubly commutes with S, then $TS \in \delta(\mathcal{H})$. In order to prove this theorem, we need the .*

Theorem 2.4 *[23] Let $T = U|T|$ and $S = V|S|$ be he polar decomposition of T and S respectively. Then the following conditions are equivalent:*

(i) *T doubly commutes with S.*

(ii) *The following five equation are satisfied*

$$(1)[|T|, |S|] = 0 \;\; (2)[U, |S|] = 0 \;\; (3)[V, |T|] = 0, \;\; (4)[U, V] = 0, \;\; (5)[U^*, V] = 0.$$

Proof. of theorem 2.3. Let $T = U|T|$ and $S = V|S|$ be the polar decomposition of T and S, respectively. As T is Doubly commutes with S, then $TS = UV|TS| = UV|T||S|$ is the polar decomposition of TS (see [24]) . Using Theorem 2.4, we obtain

$$\begin{aligned}
(UV)^2|TS| &= UVUV|T||S| \\
&= UVU|T|V|S| \\
&= UUV|T|V|S| \\
&= UU|T|VV|S| \\
&= |T|UU|S|VV \qquad\qquad \textit{since } T,S \in \delta(\mathcal{H}) \\
&= |T||S|UUVV \\
&= |T||S|UVUV \\
&= |TS|(UV)^2
\end{aligned}$$

Hence $TS \in \delta(\mathcal{H})$. ■

Proposition 2.2 *Let $T = U|T|$ be the polar decomposition of T. If $T \in \delta(\mathcal{H})$, then $U \in \delta(\mathcal{H})$.*

In order to prove Proposition 2.2, we need the following theorem.

Theorem 2.5 *[24] Let $S, T \in \mathcal{B}(\mathcal{H})$ such that T Doubly commutes with S. Let $T = U|T|$ be the polar decomposition of T, then U and $|T|$ Doubly commutes with S.*

Proof. (of Propositon 2.2) Since U is a partial isometry. Then $U = U|U| = UU^*U$ is the polar decomposition of U. It is sufficient to show that $U^2 = U^*UU^2$. As $T \in \delta(\mathcal{H})$, it follows $U^2|T| = |T|U^2$. Hence $|T|(U^*)^2 = (U^*)^2|T|$. Since $|T| = U^*U|T|$ is the polar decomposition of $|T|$. Then by using Theorem 2.5, we obtain

$$U^*UU^2 = U^2U^*U = U^2,$$

Therefore $U \in \delta(\mathcal{H})$. ■

2.2 On the class $\delta(\mathcal{H})$, binormal operators and λ-Aluthge transform

In this section, first we show that there is no relationship between $\delta(\mathcal{H})$ and sevral major classes of operators.

Example 2.3 *Let* $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ *, Then* $\|T\| = 1, w(T) = \frac{1}{2}$ *and* $r(T) = 0$ *. Therefore* T *is not normaloid because* $\|T\| \neq r(T)$ *also it's not spectraloid because* $w(T) \neq r(T)$*. But* T *satisfy the equation* $U^2|T| = |T|U^2$ *(i.e.* $T \in \delta(\mathcal{H})$*).*

Next, we give a condition under which an operator in $\delta(\mathcal{H})$ becomes quasinormal.

Proposition 2.3 *Let* n *be a positive integer and* $T \in \delta(\mathcal{H})$*, with polar decomposition* $T = U|T|$*. If* $U^{2n+1} = I$*, then* T *is quasinormal.*

Proof. From $U^2|T| = |T|U^2$, we get $U^{2n}|T| = |T|U^{2n}$. This implies $U^{2n+1}|T|U = U|T|U^{2n+1}$. If $U^{2n+1} = I$, then $U|T| = |T|U$. Hence, T is quasinormal. ■

The following is a characterization of invertible operators in $\delta(\mathcal{H})$ via Duggal transform.

Proposition 2.4 *Let* $T \in \mathcal{B}(\mathcal{H})$ *be invertible. Then*

$$T \in \delta(\mathcal{H}) \iff \Delta_1(T^{-1}) = (\Delta_1(T))^{-1}.$$

Proof. Suppose that $T = U|T|$ is the polar decomposition of T. Since T is invertible, it follows that

$$T \in \delta(\mathcal{H}) \iff U^2|T| = |T|U^2 \iff U^2|T|^{-1} = |T|^{-1}U^2.$$

By Proposition 1.7 (d), $U^2|T|^{-1} = U^2U^*|T^{-1}|U$. Since U is unitary, then

$$\begin{aligned} T \in \delta(\mathcal{H}) &\iff U^2U^*|T^{-1}|U = |T|^{-1}U^2 \\ &\iff U|T^{-1}|U = |T|^{-1}U^2 \\ &\iff U|T^{-1}| = |T|^{-1}U \\ &\iff |T^{-1}|U^* = U^*|T|^{-1} \\ &\iff \Delta_1(T^{-1}) = (\Delta_1(T))^{-1}. \end{aligned}$$

■

Example 2.4 *Proposition 2.4 is not valid when the Duggal transform is replaced by the Aluthge transform. To see this let* $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$*, where* A *and* B *are invertible positive operators such that* $AB \neq BA$*. Then* T *is invertible and*

$$T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = U|T|$$

is the polar decomposition of T*. Since* $U^2 = I$*, it follows that* $U^2|T| = |T|U^2$ *and so* $T \in \delta(\mathcal{H} \oplus \mathcal{H})$*. On the other hand, since*

$$\Delta(T) = \begin{pmatrix} 0 & B^{\frac{1}{2}}A^{\frac{1}{2}} \\ A^{\frac{1}{2}}B^{\frac{1}{2}} & 0 \end{pmatrix}, \text{ we obtain } (\Delta(T))^{-1} = \begin{pmatrix} 0 & B^{-\frac{1}{2}}A^{-\frac{1}{2}} \\ A^{-\frac{1}{2}}B^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Using Proposition 1.7 (a) and (d), we have

$$\Delta(T^{-1}) = |T^{-1}|^{\frac{1}{2}}U^*|T^{-1}|^{\frac{1}{2}} = |T^*|^{-\frac{1}{2}}U^*|T^*|^{-\frac{1}{2}} = \begin{pmatrix} 0 & A^{-\frac{1}{2}}B^{-\frac{1}{2}} \\ B^{-\frac{1}{2}}A^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Hence $\Delta(T^{-1}) \neq (\Delta(T))^{-1}$.

It is well known that every quasi-normal operator is binormal. Hence one might expect that there is a relationship between $\delta(\mathcal{H})$ and binormal operators. But in the example 2.4, $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \delta(\mathcal{H} \oplus \mathcal{H})$ and T is not binormal because $AB \neq BA$.
Next, we shall show the following result on the binormality of an invertible operator T in $\delta(\mathcal{H})$.

Theorem 2.6 *Let $T \in \delta(\mathcal{H})$ be an invertible operator. Then the following statements are equivalent.*

(1) *T is binormal.*

(2) *$\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$ for all $\lambda \in]0,1[$.*

(3) *$\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$ for some $\lambda \in]0,1[$.*

Proof.
First, if $T \in \delta(\mathcal{H})$, then by the functional calculus, we obtain $U^2|T|^\lambda = |T|^\lambda U^2$ for all $\lambda > 0$. This implies $U|T^*|^\lambda U = |T|^\lambda U^2$, by Theorem 1.2(v). Multiplying this equality by U^* on the right side and since U is unitary, we get

$$U|T^*|^\lambda = |T|^\lambda U \quad \textit{for all} \;\; \lambda \in]0,1[. \tag{2.1}$$

(1) $\Longrightarrow$ (2). Suppose that T is binormal and invertible. From $Theorem 1.2(v)$, we get

$$\begin{aligned}\left(\Delta_\lambda(T^{-1})\right)^{-1} &= \left(|T^*|^{-\lambda}U^*|T^*|^{-(1-\lambda)}\right)^{-1} \\ &= |T^*|^{1-\lambda}U|T^*|^\lambda \\ &= U|T|^{1-\lambda}|T^*|^\lambda.\end{aligned}$$

Since T is binormal, then $|T||T^*| = |T^*||T|$. Also by functional calculus, we get $|T|^{1-\lambda}|T^*|^{\lambda} = |T^*|^{\lambda}|T|^{1-\lambda}$ for $\lambda \in]0,1[$. Then, by using this equality and (2.1), we deduce that

$$\begin{aligned}\left(\Delta_\lambda(T^{-1})\right)^{-1} &= U|T^*|^{\lambda}|T|^{1-\lambda}\\ &= |T|^{\lambda}U|T|^{1-\lambda}\\ &= \Delta_\lambda(T).\end{aligned}$$

Hence, $\left(\Delta_\lambda(T)\right)^{-1} = \Delta_\lambda(T^{-1})$, for all $\lambda \in]0,1[$.

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (1). Assume that $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$ for some $\lambda \in]0,1[$. From (2.1), we obtain

$$\Delta_\lambda(T) = |T|^{\lambda}U|T|^{1-\lambda} = U|T^*|^{\lambda}|T|^{1-\lambda}.$$

On the other hand, by Theorem 1.2(v) we have

$$(\Delta_\lambda(T^{-1}))^{-1} = U|T|^{1-\lambda}|T^*|^{\lambda}.$$

Using our assumption, we get that

$$U|T^*|^{\lambda}|T|^{1-\lambda} = U|T|^{1-\lambda}|T^*|^{\lambda}, \quad for\ \ some\ \ \lambda \in]0,1[.$$

Since U is unitary, we conclude that

$$|T^*|^{\lambda}|T|^{1-\lambda} = |T|^{1-\lambda}|T^*|^{\lambda}, \quad for\ \ some\ \ \lambda \in]0,1[.$$

By the continuous functional calculus, we obtain $|T^*||T| = |T||T^*|$. So T is binormal. ■ The following corollary generalizes one implication of [47, Theorem 3.7] to infinite-dimensional Hilbert space.

Corollary 2.1 *Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. Then*

$$\Delta(T) = T \implies \Delta_\lambda(T^{-1}) = \left(\Delta_\lambda(T)\right)^{-1}, \ \textit{for all } \lambda \in]0,1[.$$

Proof. Let $T = U|T|$ be the polar decomposition of T. Since $\Delta(T) = T$, then T is normal. It follows that $U|T| = |T|U$ and so $U^2|T| = |T|U^2$. Hence, $T \in \delta(\mathcal{H})$. Moreover, since T is normal, T is binormal and by Theorem 2.6, we deduce that $\Delta_\lambda(T^{-1}) = \left(\Delta_\lambda(T)\right)^{-1}$, for all $\lambda \in]0,1[$.

■

Remark 2.1 *In Corollary 2.1, the reverse implication is false even in finite dimentionel space as shown by the following example.*

Example 2.5 *Let* $T = \begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, *where* $P \geq 0$ *and* $P \neq I$ *is invertible. The polar decomposition of* T *is* $T = U|T|$, *where*

$$|T| = (T^*T)^{\frac{1}{2}} = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} \quad \textit{and} \quad U = T|T|^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

For any $\lambda \in]0,1[$, *we have*

$$\begin{aligned} \Delta_\lambda(T) &= |T|^\lambda U |T|^{1-\lambda} \\ &= \begin{pmatrix} P^\lambda & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} P^{1-\lambda} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 0 & P^\lambda \\ P^{1-\lambda} & 0 \end{pmatrix}. \end{aligned}$$

It follows that

$$\left(\Delta_\lambda(T)\right)^{-1} = \begin{pmatrix} 0 & P^{-(1-\lambda)} \\ P^{-\lambda} & 0 \end{pmatrix}.$$

Also we have

$$\begin{aligned}\Delta_\lambda(T^{-1}) &= |T^*|^{-\lambda}U^*|T^*|^{-(1-\lambda)} \\ &= \begin{pmatrix} I & 0 \\ 0 & P^{-\lambda} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P^{-(1-\lambda)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & P^{-(1-\lambda)} \\ P^{-\lambda} & 0 \end{pmatrix}.\end{aligned}$$

Hence, $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$, *while* $\Delta_\lambda(T) \neq T$.

Now we add some conditions under which the reverse implication of Corollary 2.1 hold.

Theorem 2.7 *Let* $\lambda \in]0,1[$ *and let* $T \in \mathcal{B}(\mathcal{H})$ *be invertible and paranormal. Suppose that* $U^2|T| = |T|U^2$. *Then*

$$\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1} \Rightarrow T \quad is \ \ quasinormal.$$

We start by the following Theorem , and then give the proof of Theorem 2.7.

Theorem 2.8 *[11] A binormal operator is hyponormal if and only if it is paranormal.*

Proof. (of theorem 2.7) Let $T = U|T|$ be the polar decomposition of T. We have

$$|T^*|^2 = U|T|^2U^*.$$

Since T is invertible, we obtain that U is unitary and then

$$U^*U|T^*|^2 = U|T|^2U^*.$$

Since $T \in \delta(H)$ and by using the functional calculus, we get

$$U^*|T|^2U = U|T|^2U^* \tag{2.2}$$

On the other hand, since $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$, thus it follows from Theorem 2.8 and Theorem 2.6 that T is hyponormal (*i.e.* $TT^* \leq T^*T$). So $U|T|^2U^* \leq |T|^2$. Multiplying this inequality by U^* on the left and by U on the right, we get that

$$U|T|^2U^* \leq |T|^2 \leq U^*|T|^2U. \tag{2.3}$$

Then

$$|T|^2 = U^*|T|^2U \qquad by\ (2.3)$$

By multiplying this relation on the left by U, we obtain that

$$U|T|^2 = |T|^2U.$$

Therefore

$$U|T| = |T|U.$$

Hence T is quasinormal. ■

Corollary 2.2 *Let $T = U|T|$ be the polar decomposition of a binormal operator T. Suppose that $U^{2n+1} = I$ for some positive integer n. Then*

$$\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1} \Longrightarrow\ T\ is\ quasinormal.$$

Remark 2.2 *example 2.5 show that in Corollary 2.2 and Proposition 2.3 the condition $U^{2n+1} = I$ is essential.*

The following is an example of a binormal operator which is not in $\delta(\mathcal{H})$.

Example 2.6 *Consider* $T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^4$. *Then T is invertible and binormal since*

$$TT^*T^*T = T^*TTT^* = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

By a direct calculation, we have

$$|T| = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ |T^*| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \ \text{ and } \ U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that $U^2|T| \neq |T|U^2$, then $T \notin \delta(\mathcal{H})$. Moreover, since

$$\Delta_\lambda(T) = \begin{pmatrix} 0 & 0 & 2^\lambda & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2^{1-\lambda} & 0 & 0 & 0 \end{pmatrix}, \ \text{ then } \ (\Delta_\lambda(T))^{-1} = \begin{pmatrix} 0 & 0 & 0 & 2^{-(1-\lambda)} \\ 0 & 0 & 1 & 0 \\ 2^{-\lambda} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Also, we have

$$\Delta_\lambda(T^{-1}) = |T^*|^{-\lambda}U^*|T^*|^{-(1-\lambda)} = \begin{pmatrix} 0 & 0 & 0 & 2^{-(1-\lambda)} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2^{-\lambda} & 0 & 0 \end{pmatrix}.$$

Hence, $\Delta_\lambda(T^{-1}) \neq (\Delta_\lambda(T))^{-1}$ *for any* $\lambda \in]0,1[$.

Now, we provide equivalent conditions under which an invertible binormal operator belongs to $\delta(\mathcal{H})$.

Theorem 2.9 *Let* $T \in \mathcal{B}(\mathcal{H})$ *be an invertible binormal operator and* $T = U|T|$ *be its polar decomposition. Then the following statements are equivalent.*

1. $T \in \delta(\mathcal{H})$.

2. $\Delta(T^{-1}) = (\Delta(T))^{-1}$.

3. $U\Delta(T) = \Delta(T)U$.

Proof. (1) $\Rightarrow$ (2). The proof follows from Theorem 2.6.

(2) $\Rightarrow$ (3). Since T is invertible, U is unitary. Using Proposition1.7(d), we get

$$\begin{aligned}
\Delta(T^{-1})U\Delta(T) &= |T^{-1}|^{\frac{1}{2}}U^*|T^{-1}|^{\frac{1}{2}}U|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
&= |T^{-1}|^{\frac{1}{2}}|T|^{-\frac{1}{2}}|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
&= |T^{-1}|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
&= UU^*|T^{-1}|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
&= U|T|^{-\frac{1}{2}}|T|^{\frac{1}{2}} \\
&= U.
\end{aligned}$$

Thus, the condition $\Delta(T^{-1}) = (\Delta(T))^{-1}$ implies that $U\Delta(T) = \Delta(T)U$.

(3) $\Rightarrow$ (1). Assume that $U\Delta(T) = \Delta(T)U$. Then we have

$$\begin{aligned} U|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}U &\Longrightarrow U|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U = |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U^2 \quad && \textit{by } 1.2(v) \\ &\Longrightarrow |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}U = |T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}U^2 && \textit{since } T \textit{ is binormal} \\ &\Longrightarrow U|T^*|^{\frac{1}{2}}U = |T|^{\frac{1}{2}}U^2 \\ &\Longrightarrow U^2|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U^2 && \textit{by } 1.2(v) \\ &\Longrightarrow U^2|T| = |T|U^2. \end{aligned}$$

Hence, $T \in \delta(\mathcal{H})$. ■

Now, we focus on the binormalilty of $\Delta_\lambda(T)$ when T is binormal. In [29], Ito et al, gave an example of a binormal invertible operator T such that its Aluthge transform $\Delta(T)$ is not binormal. However, it was proved in [41] that if T is a binormal invertible operator, then its Duggal transform is binormal.

Theorem 2.10 *Let $T \in \delta(\mathcal{H})$ and $T = U|T|$ be its polar decomposition. If U is unitary, then for $\lambda \in]0,1[$, we have*

$$T \textit{ is binormal} \Longrightarrow \Delta_\lambda(T) \textit{ is binormal.}$$

Proof. Since U is unitary, we obtain that

$$\begin{aligned} |\Delta_\lambda(T)^*|^2|\Delta_\lambda(T)|^2 &= |T|^\lambda U|T|^{2(1-\lambda)}U^*|T|U^*|T|^{2\lambda}U|T|^{1-\lambda} \\ &= |T|^\lambda|T^*|^{2(1-\lambda)}|T|U^*|T|^{2\lambda}U|T|^{1-\lambda} \quad \textit{by } 1.2(v) \\ &= |T|^\lambda|T^*|^{2(1-\lambda)}|T||T^*|^{2\lambda}U^*U|T|^{1-\lambda} \quad \textit{by } (2.1) \\ &= |T|^2|T^*|^2 \quad \textit{since } T \textit{ is binormal.} \end{aligned}$$

And

$$\begin{aligned}
|\Delta_\lambda(T)|^2|\Delta_\lambda(T)^*|^2 &= |T|^{1-\lambda}U^*|T|^{2\lambda}U|T|U|T|^{2(1-\lambda)}U^*|T|^\lambda \\
&= |T|^{1-\lambda}U^*|T|^{2\lambda}U|T||T^*|^{2(1-\lambda)}UU^*|T|^\lambda \quad by\ \ 1.2(v) \\
&= |T|^{1-\lambda}U^*|T|^{2\lambda}U|T||T^*|^{2(1-\lambda)}|T|^\lambda \\
&= |T|^{1-\lambda}U^*U|T^*|^{2\lambda}|T||T^*|^{2(1-\lambda)}|T|^\lambda \qquad by\,(2.1) \\
&= |T|^2|T^*|^2 \quad since\ \ T\ \ is\ \ binormal.
\end{aligned}$$

Hence, $\Delta_\lambda(T)$ is binormal. ■

Remark 2.3 *(i) The reverse implication of the previous Theorem is false. Indeed if we take the example 2.4, we obtain $T \in \delta(\mathcal{H})$ and $\Delta(T)$ is binormal but T is not binormal.*

(ii) If $T \in \mathcal{B}(\mathcal{H})$ is a binormal invertible operator such that $\Delta_\lambda(T)$ is binormal then T need not be in $\delta(\mathcal{H})$. To see this, consider the example 2.6. Then T is binormal and $\Delta_\lambda(T)$ is also binormal as

$$|\Delta_\lambda(T)^*|^2|\Delta_\lambda(T)|^2 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4^\lambda & 0 \\ 0 & 0 & 0 & 4^{1-\lambda} \end{pmatrix} = |\Delta_\lambda(T)|^2|\Delta_\lambda(T)^*|^2$$

but $T \notin \delta(\mathcal{H})$.

Proposition 2.5 *[29] Let $T = U|T|$ be the polar decomposition of a binormal operator T. Then $\Delta_\lambda(T) = U^*UU|\Delta_\lambda(T)|$ is the polar decomposition of $\Delta_\lambda(T)$.*

The following result state that the λ-Aluthge transform of binormal operators in $\delta(\mathcal{H})$ also belongs to $\delta(\mathcal{H})$.

Theorem 2.11 *Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$ where U is unitary.*

Suppose that T is binormal and $\lambda \in]0,1[$. Then

$$T \in \delta(\mathcal{H}) \Longrightarrow \Delta_\lambda(T) \in \delta(\mathcal{H}).$$

Proof. First we prove that

$$U^*|T|^\lambda U = U|T|^\lambda U^* \quad for \ any \ \lambda > 0. \tag{2.4}$$

We have $|T^*|^\lambda = U|T|^\lambda U^*$. Since $U^*U = I$, then $U^*U|T^*|^\lambda = U|T|^\lambda U^*$. Now as $U^2|T| = |T|U^2$, we get $U^2|T|^\lambda = |T|^\lambda U^2$ for any $\lambda > 0$ by the functional calculus. This implies that $U^*|T|^\lambda U = U^*U|T^*|^\lambda = U|T|^\lambda U^*$. Therefore

$$\begin{aligned}
U^2|\Delta_\lambda(T)|^2 &= U^2\Delta_\lambda(T)^*\Delta_\lambda(T) \\
&= U^2|T|^{1-\lambda}U^*|T|^{2\lambda}U|T|^{1-\lambda} \\
&= |T|^{1-\lambda}U^2U^*|T|^{2\lambda}U|T|^{1-\lambda} \quad since \ T \in \delta(\mathcal{H}) \\
&= |T|^{1-\lambda}U|T|^{2\lambda}U|T|^{1-\lambda} \\
&= |T|^{1-\lambda}|T^*|^{2\lambda}U^2|T|^{1-\lambda} \qquad by \ \ 1.2(v) \\
&= |T|^{1-\lambda}U|T|^{2\lambda}U^*|T|^{1-\lambda}U^2 \\
&= |T|^{1-\lambda}U^*|T|^{2\lambda}U|T|^{1-\lambda}U^2 \ \ by \ \ (2.4) \\
&= |\Delta_\lambda(T)|^2U^2.
\end{aligned}$$

Hence $\Delta_\lambda(T) \in \delta(\mathcal{H})$. ∎

2.3 The λ-Aluthge transform of closed range operators

We start this section by giving a new proof to the following lemma from [49].

Lemma 2.1 *Let $T \in \mathcal{B}(\mathcal{H})$ be positive and $\alpha > 0$. Then $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T^\alpha)$ is closed. In this case $\mathcal{R}(T) = \mathcal{R}(T^\alpha)$.*

Proof. $(\Rightarrow)$. Suppose that $\mathcal{R}(T)$ is closed and $\mathcal{R}(T^\alpha)$ is not closed, for some $\alpha > 0$. Then $\gamma(T^\alpha) = 0$ and so there existe a sequence of unit vectors $x_n \in \mathcal{N}(T^\alpha)^\perp$ such that $T^\alpha x_n \longrightarrow 0$. Since $\mathcal{N}(T^\alpha) = \mathcal{N}(T)$, $x_n \in \mathcal{N}(T)^\perp$, for all n. In case $\alpha \in]0, 1]$, we have

$$Tx_n = T^{1-\alpha}T^\alpha x_n \longrightarrow 0,$$

Now, in case $\alpha > 1$, by Hölder-McCarthy inequality, we have

$$\|T^{\frac{1}{2}}x_n\|^{2\alpha} = \langle Tx_n, x_n\rangle^\alpha \leq \langle T^\alpha x_n, x_n\rangle \leq \|T^\alpha x_n\|,$$

for all n. Hence $T^{\frac{1}{2}}x_n \longrightarrow 0$, so $Tx_n \longrightarrow 0$. Therefore, in both cases the sequence $(Tx_n)_n$ converges to 0, which is a contradiction with the fact that $\mathcal{R}(T)$ is closed.

$(\Leftarrow)$. Suppose that $\mathcal{R}(T^\alpha)$ is closed, for $\alpha > 0$, by the above implication we obtain $\mathcal{R}((T^\alpha)^{\frac{1}{\alpha}}) = R(T)$ is also closed and then $\mathcal{R}(T) = \mathcal{R}(T^\alpha)$. ■

The λ-Aluthge transform preserves many properties of the original operator. However, an operator $T \in \mathcal{B}(\mathcal{H})$ may have a closed range without $\Delta_\lambda(T)$ having a closed range as shown by the following example.

Example 2.7 *Let* $T = \begin{pmatrix} A & 0 \\ (I - A^*A)^{\frac{1}{2}} & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, *where* A *is a contraction and* $\mathcal{R}(A)$ *is not closed. Then*

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

is an orthogonal projection. Hence T *is a partial isometry. This implies that* $\mathcal{R}(T)$ *is closed and* $T = T|T| = TT^*T$ *is the polar decomposition of* T. *Therefore, for* $\lambda \in]0, 1]$, *we have*

$$\Delta_\lambda(T) = (T^*T)^\lambda T (T^*T)^{1-\lambda} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

So $\mathcal{R}(\Delta_\lambda(T))$ *is not closed.*

The next result provide a necessary and sufficient condition for the range of $\Delta_\lambda(T)$ to be closed .

Proposition 2.6 *Let $\lambda \in]0,1]$ and $T \in \mathcal{B}(\mathcal{H})$ with closed range. Let P be an idempotent with range $\mathcal{R}(T)$ and Q be an idempotent with kernel $\mathcal{N}(T)$. Then*

$$\mathcal{R}(\Delta_\lambda(T)) \textit{ is closed if and only if } \mathcal{R}(QP) \textit{ is closed.}$$

Proof. Assume that $R(T)$ is closed. Since $\mathcal{R}(P) = \mathcal{R}(T)$ and $\mathcal{N}(Q) = \mathcal{N}(T)$, then for $\lambda \in]0,1]$, we get

$$\begin{aligned}
\mathcal{R}(\Delta_\lambda(T)) \textit{ is closed} &\iff \mathcal{R}(|T|^\lambda U|T|^{1-\lambda}) \textit{ is closed} \\
&\iff \mathcal{R}(|T|^\lambda |T^*|^{1-\lambda} U) \textit{ is closed} \qquad \textit{by } \text{Theorem1.2}(v) \\
&\iff |T|^\lambda |T^*|^{1-\lambda} \mathcal{R}(|T^*|) \textit{ is closed} \\
&\iff |T|^\lambda |T^*|^{1-\lambda} \mathcal{R}(|T^*|^\lambda) \textit{ is closed} \qquad \textit{by lemma } 2.1 \\
&\iff |T|^\lambda \mathcal{R}(|T^*|) \textit{ is closed} \\
&\iff |T|^\lambda \mathcal{R}(T) \textit{ is closed} \\
&\iff |T|^\lambda \mathcal{R}(P) \textit{ is closed} \\
&\iff \mathcal{R}(P^*|T|^\lambda) \textit{ is closed} \\
&\iff P^*\mathcal{R}(|T|) \textit{ is closed} \\
&\iff P^*\mathcal{R}(T^*) \textit{ is closed} \\
&\iff \mathcal{R}(P^*Q^*) \textit{ is closed} \qquad \textit{since } \mathcal{R}(Q^*) = \mathcal{N}(Q)^\perp = \mathcal{R}(T^*) \\
&\iff \mathcal{R}(QP) \textit{ is closed.}
\end{aligned}$$

■

λ-Aluthge transform and EP operators

The following result, which is one of the main results of this section, generalizes Theorems 3.3 and 3.15 obtained for complex matrices in [47] to the closed range operators on an arbitrary Hilbert space.

Theorem 2.12 *For $T \in \mathcal{B}(\mathcal{H})$ with closed range and $\lambda \in]0,1]$, we have*

$$T \text{ is an EP operator} \iff \Delta_\lambda(T) \text{ is EP and } \mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T)).$$

Proof. ($\Longrightarrow$). We assume that T is EP. Then $\mathcal{R}(T)$ is closed and $\mathcal{R}(T) = \mathcal{R}(T^*)$. This implies that $P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)} = P_{\mathcal{R}(T)}$. Then $\mathcal{R}(P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)})$ is closed and by Proposition 2.6, we deduce that $\mathcal{R}(\Delta_\lambda(T))$ is closed. Now we show that $\mathcal{N}(\Delta_\lambda(T)) = \mathcal{N}(\Delta_\lambda(T)^*)$. Since $\mathcal{R}(T) = \mathcal{R}(T^*)$, it follows that

$$\mathcal{N}(|T|) = \mathcal{N}(T) = \mathcal{N}(T^*) = \mathcal{N}(|T^*|).$$

Since, for $\lambda \in]0,1]$, $\mathcal{N}(|T|) = \mathcal{N}(|T|^\lambda)$ and $\mathcal{N}(|T^*|) = \mathcal{N}(|T^*|^\lambda)$, then we get $\mathcal{N}(|T|^\lambda) = \mathcal{N}(|T^*|^\lambda)$. Let $x \in \mathcal{H}$. Hence, for $\lambda \in]0,1]$ we have

$$\begin{aligned}
|T|^\lambda U|T|^{1-\lambda}x = 0 &\iff |T^*|^\lambda U|T|^{1-\lambda}x = 0 \\
&\iff U|T|^\lambda|T|^{1-\lambda}x = 0 \qquad \textit{by} \quad \text{Theorem } 1.2(v) \\
&\iff Tx = 0 \\
&\iff |T|x = 0 \\
&\iff |T|^\lambda x = 0 \\
&\iff |T|^{1-\lambda}U^*|T|^\lambda x = 0 \\
&\iff \Delta_\lambda(T)^*x = 0.
\end{aligned}$$

Therefore, $\mathcal{N}(\Delta_\lambda(T)) = \mathcal{N}(\Delta_\lambda(T)^*) = \mathcal{N}(T)$. Consequently, $\Delta_\lambda(T)$ is also EP. By taking the orthogonal complements in the relation $\mathcal{N}(\Delta_\lambda(T)) = \mathcal{N}(T)$ and since T and $\Delta_\lambda(T)$ are EP, we conclude that $\mathcal{R}(\Delta_\lambda(T)) = \mathcal{R}(T)$.

($\Longleftarrow$). We suppose that $\Delta_\lambda(T)$ is EP. Since $\mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T))$, then $\mathcal{R}(\Delta_\lambda(T))$ is closed and $\mathcal{R}(\Delta_\lambda(T)) = \mathcal{R}(\Delta_\lambda(T)^*)$. Thus,

$$\mathcal{N}(T^*) = \mathcal{N}(\Delta_\lambda(T)) = \mathcal{N}(\Delta_\lambda(T)^*).$$

Since $\mathcal{N}(T) \subset \mathcal{N}(\Delta_\lambda(T))$, then $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. Hence, to prove T is EP, it is enough to prove that $\mathcal{N}(\Delta_\lambda(T)) \subset \mathcal{N}(T)$. Let $x \in \mathcal{N}(\Delta_\lambda(T))$. This implies that $U|T|^{1-\lambda}x \in \mathcal{N}(|T|^\lambda) = \mathcal{N}(T)$. Hence $|T^*|^\lambda U|T|^{1-\lambda}x = 0$, because $\mathcal{N}(T) \subset \mathcal{N}(T^*) = \mathcal{N}(|T^*|^\lambda)$. According to Theorem 1.2(v) , we get $T(x) = 0$. Therefore $\mathcal{N}(\Delta_\lambda(T)) \subset \mathcal{N}(T)$. Finally T is EP. ■

Remark 2.4 *Without the condition $\mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T))$, the reverse implication does not hold, as the following example shows.*

Example 2.8 *let $T = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then $\mathcal{R}(T)$ is closed and $T^+ = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$. Furthermore, $T^2 = 0$. Hence $\Delta_\lambda(T) = 0$ is EP, while T is not EP because*

$$TT^+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = T^+T.$$

As a consequence of the previous theorem, we have the following result.

Corollary 2.3 *When $T \in \mathcal{B}(\mathcal{H})$ is EP, then the following operators are also EP:*

$$\Delta(T^+) = \begin{pmatrix} \Delta(A^{-1}) & 0 \\ 0 & 0 \end{pmatrix} \quad (\Delta(T))^+ = \begin{pmatrix} (\Delta(A))^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad and \quad \Delta_\lambda(T^*).$$

It well known that the reverse order law for the Moore-Penrose inverse does not necessarily hold. In [18], Djordjević showed that if A and B are two EP operators such that $\mathcal{R}(A) = \mathcal{R}(B)$, then $(AB)^+ = B^+A^+$. Combining Theorem 2.12 and Djordjević result, we can directly obtain the following corollary.

Corollary 2.4 *If $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$ are EP operators such that $\mathcal{R}(A) = \mathcal{R}(B)$, then $\triangle_\lambda(AB)$ and $\triangle_\lambda(A) \triangle_\lambda (B)$ are also EP.*

Proof. Since $(AB)^+ = B^+A^+$. Thus it is easy to show that $(AB)^+AB = AB(AB)^+$, therefore AB is EP operator. So by theorem we deduce that $\triangle_\lambda(AB)$ is EP operator. Using Theorem 2.12 we obtain that $\triangle_\lambda(A) \triangle_\lambda (B)$ is also EP operator. ■

The statement of proposition 2.7 can be derived from Theorem 2.12.

Proposition 2.7 *If $T \in \mathcal{B}(\mathcal{H})$ with closed range is EP operator, then*

$$(T \triangle_\lambda (T))^+ = \triangle_\lambda(T)^+ T^+ \textit{ and } (\triangle_\lambda(T)T)^+ = T^+ \triangle_\lambda (T)^+.$$

Now we prove a version of Theorem 2.6 for closed range operators.

Corollary 2.5 *Let $T \in \delta(\mathcal{H})$ with closed range. If T is EP, then the following statements are equivalent.*

(1) *T is binormal.*

(2) $\Delta_\lambda(T^+) = (\Delta_\lambda(T))^+$ *for all* $\lambda \in]0, 1[$.

(3) $\Delta_\lambda(T^+) = (\Delta_\lambda(T))^+$ *for some* $\lambda \in]0, 1[$.

Proof.

Since T is an EP operator, then $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ and T has the following matrix form

$$T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where the operator $A : \mathcal{R}(T) \longrightarrow \mathcal{R}(T)$ is invertible. Now it is known that

$$U = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \textit{ and } |T| = \begin{pmatrix} |A| & 0 \\ 0 & 0 \end{pmatrix},$$

where $A = V|A|$ is the polar decomposition of A. Then for $\lambda \in]0, 1[$

$$\Delta_\lambda(T) = \begin{pmatrix} \Delta_\lambda(A) & 0 \\ 0 & 0 \end{pmatrix} \textit{ and } (\Delta(T))^+ = \begin{pmatrix} (\Delta_\lambda(A))^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

also we have

$$T^+ = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} \textit{ and } \Delta(T^+) = \begin{pmatrix} \Delta_\lambda(A^{-1}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$(\Delta_\lambda(T))^+ = \Delta_\lambda(T^+) \iff (\Delta_\lambda(A))^{-1} = \Delta_\lambda(A^{-1}).$$

Hence, the implications (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), and (3) $\Longrightarrow$ (1) holds by using Theorem 2.6.

■ The assumption T is an EP operator is necessary in the previous theorem as shown by the following example.

Example 2.9 *Consider the right shift operator S, defined on the Hilbert space $\ell^2(\mathbb{N})$ by $S(x_1, x_2, ...) = (0, x_1, x_2, ...)$. Then $S^*(x_1, x_2, ...) = (x_2, x_3, ...)$ and so $S^*S = I$. Hence S is an isometry, which implies that $S^* = S^+$ and S is not EP because $S^*S \neq SS^*$. Since $|S| = I$, it follows that $S = S|S|$ is the polar decomposition of S and $S \in \delta(\mathcal{H})$. On the other hand a simple calculation shows that*

$$(\Delta(S))^+ = S^+ = S^* \neq SS^*S^* = \Delta(S^+).$$

Next three propositions generalise some results from section 2 to closed range operators.

Theorem 2.13 *Let $T = U|T|$ be the polar decomposition of an EP operator T such that $U^{2n+1} = I$ for some positive integer n. Then*

$$\Delta_1(T^+) = (\Delta_1(T))^+ \iff T \text{ is quasinormal.}$$

Proof. The proof is analogous to the proof of Theorem 2.6 and it is left to the reader. ■

Proposition 2.8 *Let $\lambda \in]0, 1[$ and $T \in \mathcal{B}(\mathcal{H})$ be EP and paranormal. Suppose that $U^2|T| = |T|U^2$. Then*

$$\Delta_\lambda(T^+) = (\Delta_\lambda(T))^+ \Rightarrow T \text{ is quasinormal.}$$

Proof. The proof is analogous to the proof of Theorem 2.6 and it is left to the reader. ■

Proposition 2.9 *Let $T = U|T|$ be the polar decomposition of a binormal operator T. Suppose* *that $U^{2n+1} = I$ for some positive integer n. Then*

$$\Delta_\lambda(T^+) = (\Delta_\lambda(T))^+ \Longrightarrow \ T \ \textit{is quasinormal}.$$

The next proposition was etablished by Jabbarzadeh and Bakhshkandi in the case $\lambda = \frac{1}{2}$, (see [30, Theorem 2.5]).

Proposition 2.10 *Let $T \in \mathcal{B}(\mathcal{H})$ be binormal with closed range and $T = U|T|$ be its polar decomposition. Then $\mathcal{R}(\Delta_\lambda(T))$ is closed and $(\Delta_\lambda(T))^+ = (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda$, for all $\lambda \in]0,1]$.*

Proof. First we show that $\mathcal{R}(\Delta_\lambda(T))$ is closed for $\lambda \in]0,1]$.
Since $\mathcal{R}(T)$ is closed and T is binormal, then $P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)} = P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)}$, by Lemma 1.5. Therefore $P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)}$ is a projection, it follows that $\mathcal{R}(P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)})$ is closed. So by using Proposition 2.6, we have $\mathcal{R}(\Delta_\lambda(T))$ is closed.
For $\lambda \in]0,1[$, we Put $S = (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda$. By Lemma 1.8, $(|T|^+)^\alpha = (|T|^\alpha)^+$, for all $\alpha > 0$, then

$$(|T|^+)^\lambda|T|^\lambda = (|T|^\lambda)^+|T|^\lambda = P_{\mathcal{R}(|T|^\lambda)},$$

and

$$|T|^{1-\lambda}(|T|^+)^{1-\lambda} = |T|^{1-\lambda}(|T|^{1-\lambda})^+ = P_{\mathcal{R}(|T|^{1-\lambda})}.$$

According to Lemma 2.1, $\mathcal{R}(|T|^\lambda) = \mathcal{R}(|T|^{1-\lambda}) = \mathcal{R}(T^*)$. So we deduce that

$$(|T|^+)^\lambda|T|^\lambda = |T|^{1-\lambda}(|T|^+)^{1-\lambda} = P_{\mathcal{R}(T^*)}.$$

Consequently

$$\begin{aligned}
S\Delta_\lambda(T)S &= (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda|T|^\lambda U|T|^{1-\lambda}(|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda \\
&= (|T|^+)^{1-\lambda}U^*P_{\mathcal{R}(T^*)}UP_{\mathcal{R}(T^*)}U^*(|T|^+)^\lambda \\
&= (|T|^+)^{1-\lambda}U^*P_{\mathcal{R}(T^*)}UU^*(|T|^+)^\lambda \\
&= (|T|^+)^{1-\lambda}U^*UU^*P_{\mathcal{R}(T^*)}(|T|^+)^\lambda \quad \textit{since } T \textit{ is binormal} \\
&= (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda = S,
\end{aligned}$$

$$\begin{aligned}
\Delta_\lambda(T)S\Delta_\lambda(T) &= |T|^\lambda U|T|^{1-\lambda}(|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda|T|^\lambda U|T|^{1-\lambda} \\
&= |T|^\lambda UU^*P_{\mathcal{R}(T^*)}U|T|^{1-\lambda} \\
&= |T|^\lambda P_{\mathcal{R}(T^*)}UU^*U|T|^{1-\lambda} \qquad \textit{since } T \textit{ is binormal} \\
&= (P_{\mathcal{R}(T^*)}|T|^\lambda)^*UU^*U|T|^{1-\lambda} \\
&= |T|^\lambda U|T|^{1-\lambda} = \Delta_\lambda(T),
\end{aligned}$$

and

$$\begin{aligned}
S\Delta_\lambda(T) &= (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda|T|^\lambda U|T|^{1-\lambda} \\
&= (|T|^+)^{1-\lambda}U^*P_{\mathcal{R}(T^*)}|T^*|^{1-\lambda}U \\
&= (|T|^+)^{1-\lambda}U^*|T^*|^{1-\lambda}P_{\mathcal{R}(T^*)}U \qquad \textit{by Lemma } 1.5 \\
&= (|T|^+)^{1-\lambda}|T|^{1-\lambda}U^*P_{\mathcal{R}(T^*)}U \qquad \textit{by} \ \text{Proposition 1.5 (4) and Theorem 1.2 (v)} \\
&= P_{\mathcal{R}(T^*)}U^*P_{\mathcal{R}(T^*)}U \\
&= U^*P_{\mathcal{R}(T^*)}U.
\end{aligned}$$

By similar computation we have $\Delta_\lambda(T)S = P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)}$. Hence $\Delta_\lambda(T)S$ and $S\Delta_\lambda(T)$ are self-adjoint operators. From the uniqueness of Moore-Penrose inverse we conclude that $(\Delta_\lambda(T))^+ = S$.

Now, we suppose that $\lambda = 1$. Since T is binormal, we have

$$\mathcal{R}(|T||T|U) = \mathcal{R}(|T||T||T^*|) \subset \mathcal{R}(|T^*|) = \mathcal{R}(U)$$

and

$$\mathcal{R}(UU^*|T|) = \mathcal{R}(|T|UU^*) \subset \mathcal{R}(|T|).$$

So, by Lemma 1.9 we obtain $(\Delta_1(T))^+ = (|T|U)^+ = U^*|T|^+$. ∎

Let $T \in \mathcal{B}(\mathcal{H})$ with closed range and $d \in \mathbb{N}^*$. By using [12, Theorem 2.5], we have $(T^+)^{d+1} = 0$ if and only if $(\Delta_\lambda(T^+))^d = 0$. But, what happens if we replace $\Delta_\lambda(T^+)$ by $\Delta_\lambda(T)^+$? The following last theorem gives the answer to this question.

Theorem 2.14 *Let $\lambda \in]0,1]$. Let $T \in \mathcal{B}(\mathcal{H})$ be a binormal operator with closed range and let $d \in \mathbb{N}^*$. Then*

$$(T^+)^{d+1} = 0 \iff (\Delta_\lambda(T)^+)^d = 0.$$

Proof. Let $T = U|T|$ be the polar decomposition of T. Let $d \in \mathbb{N}^*$. Since T is binormal with closed range, by Proposition 2.10, $(\Delta_\lambda(T))^+ = (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda$, for $\lambda \in]0,1]$. Thus

$$(\Delta_\lambda(T)^+)^d = ((|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda)^d = (|T|^+)^{1-\lambda}(U^*|T|^+)^{d-1}U^*(|T|^+)^\lambda.$$

This implies

$$(|T|^+)^\lambda(\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda}U^* = |T|^+(U^*|T|^+)^{d-1}U^*|T|^+U^*.$$

Since $T^+ = |T|^+U^*$, it follows that

$$(|T|^+)^\lambda(\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda}U^* = (T^+)^{d+1}.$$

Clearly, $(T^+)^{d+1} = 0$ if $(\Delta_\lambda(T)^+)^d = 0$. Conversely, for $\lambda =]0,1[$ we have

$$\begin{aligned}
(T^+)^{d+1} = 0 &\Longrightarrow (|T|^+)^{\lambda}(\Delta_{\lambda}(T)^+)^d(|T|^+)^{1-\lambda}U^* = 0\\
&\Longrightarrow (|T|^+)^{\lambda}(\lambda(T)^+)^d(|T|^+)^{1-\lambda}U^*U = 0\\
&\Longrightarrow (|T|^+)^{\lambda}(\Delta_{\lambda}(T)^+)^d(U^*U(|T|^+)^{1-\lambda})^* = 0\\
&\Longrightarrow (|T|^+)^{\lambda}(\Delta_{\lambda}(T)^+)^d(|T|^+)^{1-\lambda} = 0 \quad \textit{since } \mathcal{R}(U^*U) = \mathcal{R}(T^*) = \mathcal{R}((|T|^+)^{1-\lambda}).
\end{aligned}$$

Then, $\mathcal{R}(\Delta_{\lambda}(T)^+)^d(|T|^+)^{1-\lambda}) \subset \mathcal{N}((|T|^+)^{\lambda})$ and since $\mathcal{N}((|T|^+)^{\lambda}) = \mathcal{N}((|T|^+)^{1-\lambda}) = \mathcal{N}(T)$, it follows that

$$(|T|^+)^{1-\lambda}(\Delta_{\lambda}(T)^+)^d(|T|^+)^{1-\lambda} = 0.$$

Therefore, for all $x \in \mathcal{H}$ we have

$$\left\langle (|T|^+)^{1-\lambda}(\Delta_{\lambda}(T)^+)^d(|T|^+)^{1-\lambda}x, x\right\rangle = \left\langle (\Delta_{\lambda}(T)^+)^d(|T|^+)^{1-\lambda}x, (|T|^+)^{1-\lambda}x\right\rangle = 0.$$

So that $(\Delta_{\lambda}(T)^+)^d = 0$ on $\mathcal{R}((|T|)^{1-\lambda}) = \mathcal{R}(|T|)$. On the other hand, we have $\mathcal{N}(|T|) = \mathcal{N}((|T|^+)^{\lambda}) \subset \mathcal{N}(\Delta_{\lambda}(T)^+)$. Finally, $(\Delta_{\lambda}(T)^+)^d = 0$ on $\mathcal{H}$.

Now, For $\lambda = 1$, we have

$$\begin{aligned}
(T^+)^{d+1} = 0 &\Longrightarrow |T|^+(\Delta_1(T)^+)^dU^* = 0\\
&\Longrightarrow |T|^+(\Delta_1(T)^+)^dU^*U = 0\\
&\Longrightarrow |T|^+(\Delta_1(T)^+)^dU^*U|T|^+ = 0\\
&\Longrightarrow |T|^+(\Delta_1(T)^+)^d|T|^+ = 0 \quad \textit{since } \mathcal{R}(U^*U) = \mathcal{R}(T^*) = \mathcal{R}(|T|^+).
\end{aligned}$$

Hence, $(\Delta_1(T)^+)^d = 0$, on $\mathcal{R}(|T|)$. Also, we have $\mathcal{N}(|T|) \subset \mathcal{N}(\Delta_1(T)^+)$. Therefore $(\Delta_1(T)^+)^d = 0$ on $\mathcal{H}$. ■

Remark 2.5 *The assumption "T is binormal" is necessary in the previous theorem. Indeed,* *consider* $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ *acting on* $\mathbb{C}^3$*. Then* T *is not binormal and* $T^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$*. An easy calculation shows that* $T^3 = 0$*. This implies* $(\Delta_\lambda(T))^2 = 0$*. Then* $(\Delta_\lambda(T)^+)^2 = 0$ *but* $(T^+)^3 \neq 0$.

Recall that the numerical range $W(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined by $W(T) = \{< Tx, x >, \|X\| = 1\}$.

Proposition 2.11 *Let* $T \in \delta(\mathcal{H})$ *is binormal. Then the following statements Hold.*

(1) If T *is invertible, then* $W(\Delta(T^{-1})) = W(\Delta(T))^{-1}) \subseteq W(U^*)W(|T|^{-1})$

(2) If T *is invertible, then* $W(\Delta(T^{-1})) = W(\Delta(T))^{-1}) \subseteq W(U^*)W(|T^{-1}|)$.

(3) If T *is an EP operator, then* $W(\Delta(T^+)) = W(\Delta(T))^+) \subseteq W(U^*)W(|T|^{-1})$

(4) If T *is an EP operator, then* $W(\Delta(T^+)) = W(\Delta(T))^+) \subseteq W(U^*)W(|T^{-1}|)$.

(5) If T *is an EP operator, then* $W(\Delta(T) = W(\Delta^{(*)}(T)) \subseteq W(U)W(|T|)$

(6) If T *is an EP operator, then* $W(\Delta(T) = W(\Delta^{(*)}(T)) \subseteq W(U)W(|T^*|)$

Proof. (1) Let $x \in \mathcal{H}$, such that $||X|| = 1$. Then

$$\begin{aligned}
\left\langle \Delta(T^{-1})x, x \right\rangle &= \left\langle (\Delta(T))^{-1}x, x \right\rangle \\
&= \left\langle (|T|^{\frac{-1}{2}} U^* |T|^{\frac{-1}{2}} x, x \right\rangle \\
&= \left\langle U^* \frac{|T|^{\frac{-1}{2}} x}{|||T|^{\frac{-1}{2}} x||}, \frac{|T|^{\frac{-1}{2}} x}{|||T|^{\frac{-1}{2}} x||} \right\rangle \left\langle |T|^{-1} x, x \right\rangle
\end{aligned}$$

Thus $W(\Delta(T^{-1})) = W(\Delta(T))^{-1}) \subseteq W(U^*)W(|T|^{-1})$.

(2)Direct replacement shows that

$$\begin{aligned}
\left\langle (\Delta(T))^{-1}x, x\right\rangle &= \left\langle \Delta(T^{-1})x, x\right\rangle \\
&= \left\langle (|T^{-1}|^{\frac{1}{2}}U^*|T^{-1}|^{\frac{1}{2}}x, x\right\rangle \\
&= \left\langle U^*|T^{-1}|^{\frac{1}{2}}x, |T^{-1}|^{\frac{1}{2}}x\right\rangle \\
&= \left\langle U^*\frac{|T^{-1}|^{\frac{1}{2}}x}{|||T^{-1}|^{\frac{1}{2}}x||}, \frac{|T^{-1}|^{\frac{1}{2}}x}{|||T^{-1}|^{\frac{1}{2}}x||}\right\rangle \left\langle |T^{-1}|x, x\right\rangle
\end{aligned}$$

Hence $W(\Delta(T^{-1})) = W(\Delta(T))^{-1}) \subseteq W(U^*)W(|T^{-1}|)$.

with similar arguments, one proves (3),(4) ,(5) and (6). ■

In [53], Yamazaki introduce the notion of the *-Aluthge transform $\Delta^{(*)}(T)$ of T by setting $\Delta^{(*)}(T) = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$. With the motivation of this definition the +-Aluthge transform $\Delta^{(+)}(T)$ of T is defined in [30] by $\Delta^{(+)}(T) = (\Delta(T^+))^+$. In the same work they proved that if $T \in \mathcal{B}(\mathcal{H})$ is binormal with closed range, then *-Aluthge and +-Aluthge transformations coincide. The next result characterizes when Aluthge transform, *-Aluthge and +-Aluthge transformations coincide.

Corollary 2.6 *Let $T \in \delta(\mathcal{H})$ be an EP operator. Then*

$$T \ \textit{is binormal} \quad \Longleftrightarrow \quad \Delta(T) = \Delta^{(*)}(T) = \Delta^{(+)}(T).$$

Proof. The proof follows from Corollary 2.5 and [30, theorem 2.6] ■

Next proposition is a generalisation of [53, Proposition 2.4], for $\lambda \in [0, 1[$.

Proposition 2.12 *Let $\lambda \in [0, 1[, T = U|T| \in \mathcal{B}(\mathcal{H})$ and $|T|^\lambda|T^*|^{1-\lambda} = V||T|^\lambda|T^*|^{1-\lambda}|$ be the polar decompositions. Suppose that $R(\triangle_\lambda(T))$ is closed, then*

$\triangle_\lambda(T)^+ = U^*V^*| \triangle_\lambda (T)^+|$ is the polar decomposition.

Proof. It is sufficient to show that $\triangle_\lambda(T) = VU| \triangle_\lambda (T)|$ is the polar decomposition. By direct computations we obtain that

$$\begin{aligned}\triangle_\lambda(T)^* &= (|T|^\lambda U|T|^{1-\lambda})^* \\ &= (|T|^\lambda |T^*|^{1-\lambda}U)^* \\ &= U^*(|T|^\lambda |T^*|^{1-\lambda})^* \\ &= U^*V^*|(|T|^\lambda |T^*|^{1-\lambda})^*|\end{aligned}$$

But

$$\begin{aligned}|(|T|^\lambda |T^*|^{1-\lambda})^*|^2 &= |T|^\lambda |T^*|^{1-\lambda}(|T|^\lambda |T^*|^{1-\lambda})^* \\ &= |T|^\lambda |T^*|^{1-\lambda}|T^*|^{1-\lambda}|T|^\lambda \\ &= |T|^\lambda U|T|^{1-\lambda}U^*|T^*|^{1-\lambda}|T|^\lambda \\ &= |T|^\lambda U|T|^{1-\lambda}|T|^{1-\lambda}U^*|T|^\lambda \\ &= | \triangle_\lambda (T)^*|^2\end{aligned}$$

Hence $\triangle_\lambda(T)^* = U^*V^*| \triangle_\lambda (T)^*|$.

Now since

$$\mathcal{R}(V^*) = \overline{\mathcal{R}(|T^*|^{1-\lambda}|T|^\lambda)} \subset \overline{\mathcal{R}(|T^*|^{1-\lambda})} = \overline{\mathcal{R}(|T^*|)} = \mathcal{R}(U).$$

Then U^*V^* is partial isometry.

Also we have

$$\mathcal{N}(\triangle_\lambda(T)^*) = \mathcal{N}(|\triangle_\lambda(T)^*|) = \mathcal{N}(|(|T|^\lambda |T^*|^{1-\lambda})^*|) = \mathcal{N}(V^*) \subset \mathcal{N}(U^*V^*) \subset \mathcal{N}(UU^*V^*) = \mathcal{N}(V^*)$$

Consecontly $\triangle_\lambda(T)^* = U^*V^*| \triangle_\lambda (T)^*|$ is the polar decomposition, then $\triangle_\lambda(T)^+ = U^*V^*| \triangle_\lambda (T)^+|$ is the polar decomposition. ■

M. R. Jabbarzadeh, H. Emamalipour and M. Sohrabi Chegeni [31] proved the following theorem in case $\lambda = \frac{1}{2}$.

Theorem 2.15 *Let $T \in \mathcal{B}(\mathcal{H})$ with closed range and $\lambda \in]0,1[$. Suppose that T^+ is quasi-normal. Then*

$$(\triangle_\lambda(T^*))^+ = T^+ \Longrightarrow T^+ = (T^+)^*.$$

Proof. Let us consider the polar decomposition of T. Since T^+ is quasi-normal, $U^*|T^+|^\lambda = |T^+|^\lambda U^*$ and $U|T^+|^\lambda = |T^+|^\lambda U$. For every $\lambda > 0$. Hence, we have

$$\begin{aligned}(\triangle_\lambda(T^*))^+ = T^+ &\Longrightarrow |T^+|^{1-\lambda}U|T^+|^\lambda = U^*|T^+| \\ &\Longrightarrow |T^+|^{1-\lambda}U|T^+|^\lambda = U^*|T^+|^{1-\lambda}|T^+|^\lambda\end{aligned}$$

Thus $|T^+|^{1-\lambda}U = U^*|T^+|^{1-\lambda}$ on $\overline{\mathcal{R}(|T^+|)}$. Since T^+ is quasinormal, it follows that $|T^+|^{1-\lambda}U = U|T^+|^{1-\lambda} = U^*|T^+|^{1-\lambda}$ on $\mathcal{N}(|T^+|)$. Thus

$$\left(T^+\right)^* = \left(U^*|T^+|\right)^* = |T^+|U = U|T^+| = U|T^+|^{1-\lambda}|T^+|^\lambda = U^*|T^+|^{1-\lambda}|T^+|^\lambda = T^+.$$

So $(T^+)^* = T^+$. ■

Chapter 3

Some relationships between an operator and its $S_r(T)$ transform

In this section, we study another operator transform of a bounded linear operator on a complex Hilbert space, the definition of which is parallel to that of the Aluthge transform. Also we study the relationship between this transform and several classes of operators.

3.1 Some properties of the transform $S_r(T)$

We start with the following example that shows that the transform $S_r(T)$ does depend on the partial isometry factor of the polar decomposition of T, contrary to the Aluthge transform.

Example 3.1 *Let* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. *Then* $|T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *The canonical polar decomposition of* T *is* $T = U|T|$, *where* $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *Also we have* $T = V|T|$, *such that*

$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *is unitary. For* $r > 0$, *by a simple calculation, we have*

$$U|T|^r U = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = V|T|^r V.$$

Hence, the transform $S_r(T)$ *does depend on the polar decomposition. In the sequel we will always use the canonical polar decomposition.*

The following result gives interesting properties of the transform $S_r(T)$. Some of these properties were established by S.M. Patel et al, in case $r = \frac{1}{2}$, (see [48, Theorem 2.1]). First we need the next Lemma.

Lemma 3.1 *[24, 6] Let* $T, S \in \mathcal{B}(\mathcal{H})$. *Then* $\sigma(TS)\backslash\{0\} = \sigma(ST)\backslash\{0\}$.

Proposition 3.1 *let* $T \in \mathcal{B}(\mathcal{H})$ *and* $r > 0$. *Then*

1. $\|S_r(T)\|^{\frac{1}{r}} \leq \|T\|$.
2. $\mathcal{N}(S_r(T)) = \mathcal{N}(S(T)) = \mathcal{N}(U^2)$, *where* $T = U|T|$ *is the canonical polar decomposition of* T.
3. T *is invertible if and only if* $S_r(T)$ *is invertible.*
4. $\sigma(S_r(T^*)) = \sigma(S_r(T)^*)$.
5. $S_r(VTV^*) = VS_r(T)V^*$, *for all unitary operator* V.
6. $S_r(T_1 \oplus T_2) = S_r(T_1) \oplus S_r(T_2)$, *where* $T_1, T_2 \in \mathcal{B}(\mathcal{H})$.

Proof. Let $T = U|T|$ be the canonical polar decomposition of T and $r > 0$.

(1) Since U is a partial isometry, then $\|U\| = 1$. It follows that

$$\|U|T|^r U\| \leq \||T|^r\| = \||T|\|^r = \|T\|^r.$$

Therefore, $\|U|T|^rU\|^{\frac{1}{r}} \leq \|T\|$.

(2) First, we show that $\mathcal{N}(S_r(T)) = \mathcal{N}(S(T))$. Let $x \in \mathcal{H}$, then we have

$$\begin{aligned} x \in \mathcal{N}(S_r(T)) &\Longleftrightarrow U|T|^rUx = 0 \\ &\Longleftrightarrow |T^*|^rU^2x = 0 \qquad \text{by Theorem 1.2 } (v) \\ &\Longleftrightarrow |T^*|^{\frac{1}{2}}U^2x = 0 \\ &\Longleftrightarrow U|T|^{\frac{1}{2}}Ux = 0 \qquad \text{by Theorem 1.2 } (v) \\ &\Longleftrightarrow x \in \mathcal{N}(S(T)). \end{aligned}$$

The second equality has been shown in [48, Theorem 2.1].

(3) ($\Longrightarrow$). Trivial

($\Longleftarrow$). Suppose that $U|T|^rU$ is invertible. Then there existe $S \in \mathcal{B}(\mathcal{H})$ such that

$$\begin{cases} U|T|^rUS = I \\ SU|T|^rU = I, \end{cases}$$

it follows that U is invertible and so U is unitary. Now, multiplying these two equalities by U and U^*, we get

$$\begin{cases} |T|^rUSU = I \\ USU|T|^r = I. \end{cases}$$

Then, we deduce that $|T|^r$ is invertible, and so is $|T|$ by Lemma 2.1. Consequently, T is invertible.

(4) Since $S_r(T^*) = U^*|T^*|^rU^*$ and $S_r(T)^* = U^*|T|^rU^* = U^*U^*|T^*|^r$, by Theorem 1.2 (v), we have $\sigma(S_r(T^*)) \setminus \{0\} = \sigma(S_r(T)^*) \setminus \{0\}$. Now by (3), we obtain that $S_r(T)^*$ is invertible if and only T is invertible if and only T^* is. Using (3) again we get $S_r(T^*)$ is invertibe. Therefore $\sigma(S_r(T^*)) = \sigma(S_r(T)^*)$.

(5) Since V is unitary, it is easy to see that

$$|VTV^*| = V|T|V^* \text{ and so } |VTV^*|^r = V|T|^rV^*, \text{ for } r > 0.$$

Then, we have

$$VTV^* = VU|T|V^* = VUV^*V|T|V^* = VUV^*|VTV^*|,$$

where, VUV^* is a partial isometry and $\mathcal{N}(VTV^*) = \mathcal{N}(VUV^*)$. Hence, $VTV^* = VUV^*V|T|V^*$ is the canonical polar decomposition of VTV^*. Which implies that

$$\begin{aligned} S_r(VTV^*) &= VUV^*|VTV^*|^r VUV^* \\ &= VUV^*V|T|^r V^*VUV^* \\ &= VU|T|^r UV^* \\ &= VS_r(T)V^*. \end{aligned}$$

(6) Let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then the canonical polar decomposition of T can be decomposed as follows

$$T = U|T| = \begin{pmatrix} U_A & 0 \\ 0 & U_B \end{pmatrix} \begin{pmatrix} |A| & 0 \\ 0 & |B| \end{pmatrix}.$$

As a consequence,

$$S_r(T) = \begin{pmatrix} U_A|A|^r U_A & 0 \\ 0 & U_B|B|^r U_B \end{pmatrix} = \begin{pmatrix} S_r(A) & 0 \\ 0 & S_r(B) \end{pmatrix} = S_r(A) \oplus S_r(B).$$

■

The following gives the reduced minimum modulus of $S_r(T)$.

Proposition 3.2 *Let $r > 0$ and $T = U|T|$ be the canonical polar decomposition of T, then*

$$\gamma(S_r(T)) = \gamma(|T|^r U) = \gamma(U^*|T|^r).$$

Proof. We have

$$\begin{aligned}\gamma(S_r(T))^2 &= \gamma(S_r(T)^* S_r(T)) \\ &= \gamma(U^*|T|^r U^* U|T|^r U) \\ &= \gamma(U^*|T|^r |T|^r U) \\ &= \gamma((|T|^r U)^* |T|^r U) \\ &= \gamma(|T|^r U)^2 = \gamma(U^*|T|^r)^2.\end{aligned}$$

■ The next result shows when an operator and its $S_r(T)$ transform have the same null subspace.

Proposition 3.3 *Let $T = U|T|$ be the canonical polar decomposition of T and $r > 0$. Then the following statements hold,*

1. *If $\mathcal{N}(T) \subset \mathcal{N}(T^*)$, then $\mathcal{N}(S_r(T)) = \mathcal{N}(T)$.*
2. *If $\mathcal{N}(T^*) \subset \mathcal{N}(T)$, then $\mathcal{N}(S_r(T)^*) = \mathcal{N}(T^*)$.*

Proof. (1) The inclusion $\mathcal{N}(T) \subset \mathcal{N}(S_r(T))$ is obvious, so we focus on $\mathcal{N}(S_r(T)) \subset \mathcal{N}(T)$. Let $x \in \mathcal{N}(S_r(T))$. By Proposition 3.1, $U^2 x = 0$. It follows that

$$Ux \in \mathcal{N}(U) \subset \mathcal{N}(U^*).$$

Therefore, $U^*Ux = 0$. Hence $x \in \mathcal{N}(U^*U) = \mathcal{N}(T)$.

(2) It is obvious that $\mathcal{N}(T^*) \subset \mathcal{N}(S_r(T)^*)$. Suppose that $x \in \mathcal{N}(S_r(T)^*)$. Then $U^*|T|^r U^* x = 0$. Therefore $UU^*|T|^r U^* x = 0$. Since the condition $\mathcal{N}(T^*) \subset \mathcal{N}(T)$ is equivalent to $\overline{\mathcal{R}(T^*)} \subset \overline{\mathcal{R}(T)}$ and by Lemma 2.1, it follows that

$$\mathcal{R}(|T|^r) \subset \overline{\mathcal{R}(|T|^r)} = \overline{\mathcal{R}(|T|)} = \overline{\mathcal{R}(T^*)} \subset \overline{\mathcal{R}(T)} = \mathcal{R}(UU^*).$$

Hence, the equality $UU^*|T|^r U^* x = 0$ implies $|T|^r U^* x = 0$ and so $T^* = |T|U^* x = 0$, by $P(1)$ and $P(2)$. Thus, $x \in \mathcal{N}(T^*)$. This complete the proof. ■ It is well known that the Aluthge transform preserves the spectrum (see [32]). However, this is not the case for $S(T)$ transform as shown by the following example.

Example 3.2 *Let* $T = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \in \mathbb{C}^2$. *A direct calculation shows that* $T = U|T|$ *is the canonical polar decomposition of* T*, with*

$$|T| = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \text{ and } U = T|T|^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We thus have that $S(T) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. *Then,* $\sigma(T) = \{1, 4\} \neq \{1, 2\} = \sigma(S(T))$.

3.2 Relationship between the transform $S_r(T)$ and several classes of operators

We prove below that $S_r(T)$ transform preserves some hyponormality proporties. The proof of Theorem 3.1, in the case $r = \frac{1}{2}$ can be found in [48].

Theorem 3.1 *Let* T *be a p-hyponormal operator with* $0 < p \leq 1$ *and* $0 < r \leq \frac{1}{2}$. *Then*

1. *If* $0 < p \leq \frac{1}{2}$*, then* $S_r(T)$ *is 2p-hyponormal.*
2. *If* $\frac{1}{2} < p \leq 1$*, then* $S_r(T)$ *is hyponormal.*

Proof. (1) Let $T = U|T|$, be the polar decomposition of T. Since T is p-hyponormal, Then $|T|^{2p} \geq |T^*|^{2p}$, According to P(3), $|T|^{2p} \geq U|T|^{2p}U^*$. As $U^*U|T|^{2p} = |T|^{2p}$, we have

$$(U|T|^{2p}U^*)^{2r} = U(|T|^{2p})^{2r}U^*.$$

Using lower-Heinz's inequality [28, 40],

$$(|T|^{2p})^{2r} \geq U(|T|^{2p})^{2r}U^*.$$

Multiplying this inequality by U^* on the left and U on the right, we obtain that $U^*(|T|^{2p})^{2r}U \geq (|T|^{2p})^{2r}$. Hence

$$U^*(|T|^{2p})^{2r}U \geq (|T|^{2p})^{2r} \geq U(|T|^{2p})^{2r}U^*. \tag{3.1}$$

On the other hand, we have

$$\begin{aligned}
(S_r(T)^*S_r(T))^{2p} &= (U^*|T|^rU^*U|T|^rU)^{2p} \\
&= (U^*|T|^{2r}U)^{2p} \\
&\geq U^*((|T|^{2r})^{2p})U \quad \text{by Hensen's inequality [26]} \\
&\geq U((|T|^{2r})^{2p})U^* \quad \text{by (3.1)} \\
&= (U|T|^{2r}U^*)^{2p}
\end{aligned}$$

Since $I - UU^*$ is a positive operator,

$$(U|T|^{2r}U^*)^{2p} = (U|T|^rI|T|^rU^*)^{2p} \geq (U|T|^rUU^*|T|^rU^*)^{2p} = (S_r(T)S_r(T)^*)^{2p}.$$

Hence, $S_r(T)$ is 2p-hyponormal.

(2) If $\frac{1}{2} \leq p \leq 1$, then T is semi-hyponormal. By (1), $S_r(T)$ is hyponormal. ■ For $x, y \in \mathcal{H}$ $(x \neq 0, y \neq 0)$, we define the rank one operator $x \otimes y$ in $B(\mathcal{H})$ by

$$(x \otimes y)z =< z, y > x, \text{ for } z \in \mathcal{H}.$$

Then $(x \otimes y)^* = (y \otimes x)$ and by [34, Theorem 3.1], we have

$$(x \otimes y)^+ = (\|x\|\|y\|)^{-2} y \otimes x.$$

The next result gives the transform $S_r(T)$ of rank one operators.

Proposition 3.4 *let $r > 0$ and $T = x \otimes y$ be the rank one operator. Then*

$$S_r(T) = (\|x\|\,\|y\|)^{r-2} \langle x, y \rangle (x \otimes y).$$

Proof. Since $T^*T = \|x\|^2(y \otimes y) = \left(\frac{\|x\|}{\|y\|}(y \otimes y)\right)^2$, Then $|T| = \frac{\|x\|}{\|y\|}(y \otimes y)$. Therefore,

$$|T|^2 = \|x\| \, \|y\| \, |T|.$$

By recurrence, we get

$$|T|^n = (\|x\| \, \|y\|)^{n-1} \, |T|, \text{ for } n \in \mathbb{N}.$$

Then by the functional calculus, we obtain

$$|T|^r = (\|x\| \, \|y\|)^{r-1} \, |T| \text{ for } \ r > 0.$$

Also

$$U = T|T|^+ = (x \otimes y) \left(\frac{1}{\|x\| \, \|y\|^3}(y \otimes y)\right) = \frac{1}{\|x\| \, \|y\|}(x \otimes y).$$

Hence

$$\begin{aligned} S_r(T) &= U|T|^r U \\ &= U \left(\|x\| \, \|y\|\right)^{r-1} |T| U \\ &= \left(\|x\| \, \|y\|\right)^{r-1} TU \\ &= \left(\|x\| \, \|y\|\right)^{r-1} (x \otimes y) \left(\|x\| \, \|y\|\right)^{-1} (x \otimes y) \\ &= \left(\|x\| \, \|y\|\right)^{r-2} \langle x, y \rangle \left(x \otimes y\right). \end{aligned}$$

■

Proposition 3.5 *Let $r > 0$ and $T \in \mathcal{B}(\mathcal{H})$. Then*

$$T \textit{ is self-adjoint } \Longrightarrow S_r(T) \textit{ is positive.}$$

Proof. Let $T = U|T|$ be the canonical polar decomposition of T. Since T is self-adjoint, then U is also self-adjoint. Let $x \in \mathcal{H}$. Then

$$\langle U|T|^r Ux, x \rangle = \langle |T|^r Ux, Ux \rangle \geq 0,$$

and so $\langle S_r(T)x, x\rangle \geq 0$. ■

Remark 3.1 *In Proposition 3.5, The reverse implication is false as shown by the following example.*

Example 3.3 *Let* $T = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \in \mathbb{C}^2$*. The canonical polar decomposition of* T *is* $T = U|T|$*, where*

$$|T| = (T^*T)^{\frac{1}{2}} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } U = T|T|^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For any $r > 0$*, we have*

$$\begin{aligned} S_r(T) &= U|T|^r U \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4^r & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 4^r \end{pmatrix}. \end{aligned}$$

Hence, $S_r(T) > 0$*, while* T *is not self-adjoint.*

The following theorem characterizes when the transform $S(T)$ is normal.

Theorem 3.2 *Let* $T = U|T|$ *be the canonical polar decomposition of* $T \in \mathcal{B}(\mathcal{H})$*. Suppose that* $\mathcal{N}((S(T)) \subset \mathcal{N}(T)$*. Then the following statements are equivalent.*

1. $S(T)$ *is normal.*
2. $U|T|U^* = U^*|T|U$*.*
3. $U^2|T| = |T|U^2$ *and* U *is normal.*
4. $U^2T = TU^2$ *and* U *is normal.*
5. $UTU^* = U^*TU$ *and* U *is normal.*

Proof. (1) $\Rightarrow$ (2). First, we show that

$$\mathcal{N}(T^*) \subset \mathcal{N}(T), \tag{3.2}$$

Since $S(T)$ is normal, we have $S(T)^*S(T) = S(T)S(T)^*$ i.e.

$$U^*|T|U = U|T|^{\frac{1}{2}}UU^*|T|^{\frac{1}{2}}U^* \tag{3.3}$$

Let $x \in \mathcal{N}(T^*)$, so that $T^*x = 0$. Then $U^*x = 0$ and therefore (3.3) implies

$$U^*|T|Ux = U^*|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}Ux = (|T|^{\frac{1}{2}}U)^*|T|^{\frac{1}{2}}Ux = 0,$$

or $|T|^{\frac{1}{2}}Ux = 0$. This in turn gives $S(T)x = 0$ and so by the kernal condition $Tx = 0$. Which establishes (3.2).

Note that by (3.2), $UU^*|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}$. Then (3.3) reduce to

$$U^*|T|U = U|T|U^* \tag{3.4}$$

(2) $\Rightarrow$ (3). From (3.4), we get $\mathcal{N}(T^*) \subset \mathcal{N}(T)$. Now suppose that $x \in \mathcal{N}(T)$. Then $T^*x = 0$ by (3.4). Thus $\mathcal{N}(T) = \mathcal{N}(T^*)$. Hence U is normal. Now, multiply the equality (3.4) on both side by U, we get $U^2|T| = |T|U^2$.

(3) $\Rightarrow$ (4). We have

$$U^2|T| = |T|U^2 \Rightarrow UT = |T|U^2 \Rightarrow U^2T = TU^2.$$

(4) $\Rightarrow$ (5). Multiply the equality $U^2T = TU^2$ on both sides by U^* and since U is normal, we obtain $UTU^* = U^*TU$.

(5) $\Rightarrow$ (1). Since U is normal, then

$$\begin{aligned}
UTU^* = U^*TU &\Rightarrow UU|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}U^* = U^*U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}U \\
&\Rightarrow UU|T|^{\frac{1}{2}}UU^*|T|^{\frac{1}{2}}U^* = U^*U|T|^{\frac{1}{2}}U^*U|T|^{\frac{1}{2}}U \\
&\Rightarrow U^*UU|T|^{\frac{1}{2}}UU^*|T|^{\frac{1}{2}}U^* = U^*U^*U|T|^{\frac{1}{2}}U^*U|T|^{\frac{1}{2}}U \\
&\Rightarrow UU^*U|T|^{\frac{1}{2}}UU^*|T|^{\frac{1}{2}}U^* = U^*UU^*|T|^{\frac{1}{2}}U^*U|T|^{\frac{1}{2}}U \\
&\Rightarrow S(T)S(T)^* = S(T)^*S(T).
\end{aligned}$$

So $S(T)S(T)^* = S(T)^*S(T)$. The proof is complete. ■ If in Theorem 3.3, we replace $\mathcal{N}(S(T)) \subset \mathcal{N}(T)$ by T is injective, we get the following result.

Corollary 3.1 *Let $T = U|T|$ be the canonical polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Suppose that T is injective. Then the following statements are equivalent.*

1. *$S(T)$ is normal.*
2. *$U|T|U^* = U^*|T|U$.*
3. *$U^2|T| = |T|U^2$ and U is unitary.*
4. *$U^2T = TU^2$ and U is unitary.*
5. *$UTU^* = U^*TU$ and U is unitary.*

Proof. As T is injective, Then, by Proposition 3.3, we get $\mathcal{N}(S(T)) = \mathcal{N}(T)$. Also T is injective implies U is an isometry and so U is unitary if it is normal. Consequently, by Theorem 3.3, it is easy to see that $(1) - (5)$ are all equivalent. ■

Theorem 3.3 *Let $T = U|T|$ be the canonical polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Suppose that T^* is injective. then the following statements are equivalent.*

1. *S(r,T) is normal.*
2. *$U|T|U^* = U^*|T|U$.*

3. $U^2|T| = |T|U^2$ *and* U *is unitary.*

4. $U^2T = TU^2$ *and* U *is unitary.*

5. $VTV^* = V^*TV$ *and* U *is unitary.*

Proof. We organize the proof as follows: (1) $\iff$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2). Since T^* is injective, we have $UU^* = I$. Therefore

$$\begin{aligned}
S(T^*)S(T) = S(T)S(T^*) &\iff U^*|T|^r U^*U|T|^r U = U|T|^r UU^*|T|^r U^* \\
&\iff U^*|T|^{2r} U = U|T|^{2r} U^* \\
&\iff (U^*|T|U)^{2r} = (U|T|U^*)^{2r} \\
&\iff U^*|T|U = U|T|U^* \; by \, Lowner - Heinz \, inequality.
\end{aligned}$$

(2) $\Rightarrow$ (3). Assume that

$$U|T|U^* = U^*|T|U. \tag{3.5}$$

If $Tx = 0$, then $Ux = 0$ and so $U^*x = 0$ or $T^*x = 0$ by (3.5). Hence $\mathcal{N}(T) = \mathcal{N}(T^*) = \{0\}$. Therefore U is unitary. On the other hand, multiply the equality (3.5) on both side by U, we see that $U^2|T| = |T|U^2$, and (3) is proved.

(3) $\Rightarrow$ (4). We have

$$U^2|T| = |T|U^2 \Rightarrow UT = |T|U^2 \Rightarrow U^2T = TU^2$$

Therefore $U^2T = TU^2$.

(4) $\Rightarrow$ (5). Multiply the equality $U^2T = TU^2$. on both sides by U^*, we obtain $UTU^* = U^*TU$.

(5) $\Rightarrow$ (2). Since U is unitary, then

$$UTU^* = U^*TU \Rightarrow UU|T|U^* = U^*U|T|U \Rightarrow UU|T|U^* = |T|U \Rightarrow U|T|U^* = U^*|T|U.$$

The proof is complete. ■ The following is a characterization of normal invertible operators via the transform $S(r,T)$.

Corollary 3.2 *Let n be a positive integer. Suppose that $T = U|T|$ is the canonical polar decomposition of T such that U is unitary and $U^{2n+1} = I$. Then the following statements are equivalent.*

1. *T is normal.*

2. *$S(r,T)$ is normal for $r > 0$.*

3. *$\Delta(T)$ is normal.*

Proof. $(1 \Longrightarrow 2)$. Since T is normal, then U is normal and $U|T|^r = |T|^r U$. Therefore $U|T|^r$ is normal and $U|T|^r U = UU|T|^r$. Hence $S(r,T)$ is normal.
$(2 \Longrightarrow 3)$. Since U is unitary, then T and T^* are injective. Using Theorem 3.3, we obtain $U^2|T| = |T|U^2$. Now from [35, Corollary 4.2], we have $\Delta(T)$ is normal.
$(3 \Longrightarrow 1)$. The normality of $\Delta(T)$ implies $U^2|T| = |T|U^2$, by [35]. So $U^{2n+1}|T|U = U|T|U^{2n+1}$. If $U^{2n+1} = I$, then $U|T| = |T|U$. Hence, T is normal. ■

Remark 3.2 *The assumption $U^{2n+1} = I$ is essential, indeed if we take example 3.3. Then it is easy to see that $U^2 = I$ and $S(T)$ is normal while T is not.*

We close this section with the following result that gives a new characterization of normal invertible operators.

Theorem 3.4 *Let $r > 0$ and $T \in \mathcal{B}(\mathcal{H})$ be invertible. Then*

$$T \text{ is normal} \iff S_r(T^{-1}) = S_r(T)^{-1}.$$

Proof. First, Let $T = U|T|$ be the canonical polar decomposition of T. Since T is invertible, U is unitary and $|T|$ is invertible. Then, by Proposition 1.7 (c), we get $S_r(T^{-1}) = U^*|T^{-1}|^r U^*$.
$(\Longrightarrow)$. Now, suppose that T is normal. According to Theorem 1.3, $U|T| = |T|U$ and by the

continuous functional calculus, we obtain $U|T|^r = |T|^r U$, for $r > 0$. This implies that $U^*|T|^r = |T|^r U^*$, so $U^*|T|^{-r} = |T|^{-r}U^*$. Thus, we have

$$\begin{aligned} S_r(T)^{-1} &= U^*|T|^{-r}U^* \\ &= |T|^{-r}U^*U^* \\ &= U^*|T^{-1}|^r UU^*U^* \qquad \text{by Proposition 1.7 (d)} \\ &= U^*|T^{-1}|^r U^* \\ &= S_r(T^{-1}). \end{aligned}$$

($\Longleftarrow$). The equation $S_r(T^{-1}) = S_r(T)^{-1}$ means $U^*|T^{-1}|^r U^* = U^*|T|^{-r}U^*$. From Proposition 1.7 (d), we obtain $U^*|T^{-1}|^r = |T|^{-r}U^*$. Then $|T|^{-r}U^*U^* = U^*|T|^{-r}U^*$. Multiplying this equation on the right by U, we get $|T|^{-r}U^* = U^*|T|^{-r}$. This implies $U|T|^r = |T|^r U$. Therefore T is normal. The proof is therefore complete. ∎

3.3 The transform $S_r(T)$ of closed range operators

In this section, first we show that if $T \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{R}(T)$ is closed, then $\mathcal{R}(S_r(T))$ need not be closed.

Example 3.4 *Let* $T = \begin{pmatrix} A & (I - AA^*)^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, *where A is a contraction and $\mathcal{R}(A)$ is not closed. Then $TT^*T = T$ Hence T is a partial isometry. This implies that $\mathcal{R}(T)$ is closed and $T = T|T| = TT^*T$ is the canonical polar decomposition of T. Therefore*

$$S(T) = T|T|^{\frac{1}{2}}T = TT^*TT = T^2 = \begin{pmatrix} A^2 & A((I - AA^*)^{\frac{1}{2}}) \\ 0 & 0 \end{pmatrix},$$

so

$$S(T) = FT = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & (I - AA^*)^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix},$$

Then $\mathcal{R}(S(T)) = \mathcal{R}(FT) = F\mathcal{R}(TT^*) = \mathcal{R}(FTT^*) = \mathcal{R}(F)$, *as*

$$TT^* = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $\mathcal{R}(S(T)) = \mathcal{R}(F) = \mathcal{R}(A)$ *which is not closed.*

Next, we investigate when $S_r(T)$ has closed ranges.

Proposition 3.6 *Let* $r > 0$ *and* $T \in \mathcal{B}(\mathcal{H})$ *with closed range. Let* P *be an idempotent with range* $\mathcal{R}(T^*)$ *and* Q *be an idempotent with kernel* $\mathcal{N}(T^*)$*. Then*

$$\mathcal{R}(S_r(T)) \; is \; closed \; if \; and \; only \; if \; \mathcal{R}(QP) \; is \; closed.$$

Proof. Assume that $R(T)$ is closed. Since $\mathcal{R}(P) = \mathcal{R}(T^*)$ and $\mathcal{N}(Q) = \mathcal{N}(T^*)$, then we get

$$\begin{aligned}
\mathcal{R}(S_r(T)) \text{ is closed} &\Longleftrightarrow \mathcal{R}(U|T|^rU) \text{ is closed} \\
&\Longleftrightarrow U|T|^r\mathcal{R}(U) \text{ is closed} \\
&\Longleftrightarrow U|T|^r\mathcal{R}(T) \text{ is closed} \\
&\Longleftrightarrow \mathcal{R}(T^*|T|^rU^*) \text{ is closed} \\
&\Longleftrightarrow T^*|T|^r\mathcal{R}(|T|) \text{ is closed} \\
&\Longleftrightarrow T^*\mathcal{R}(|T|) \text{ is closed} \qquad \text{by lemma } 2.1 \\
&\Longleftrightarrow T^*\mathcal{R}(P) \text{ is closed} \\
&\Longleftrightarrow \mathcal{R}(P^*T) \text{ is closed} \\
&\Longleftrightarrow \mathcal{R}(P^*Q^*) \text{ is closed} \qquad \text{since } \mathcal{R}(Q^*) = \mathcal{N}(Q)^\perp = \mathcal{R}(T) \\
&\Longleftrightarrow \mathcal{R}(QP) \text{ is closed.}
\end{aligned}$$

■

We now state and prove one of our main results of this section.

Theorem 3.5 *Let $r > 0$ and $T \in \mathcal{B}(\mathcal{H})$. If $\mathcal{N}(T^*) \subset \mathcal{N}(T)$, then*

$$\mathcal{R}(T) \text{ is closed } \iff \mathcal{R}(S_r(T)) \text{ is closed.}$$

Proof. Let $T = U|T|$ be the canonical polar decomposition of T.

($\Longrightarrow$). Suppose that $\mathcal{R}(T)$ is closed and $\mathcal{R}(S_r(T))$ is not closed. Then $\mathcal{R}(S_r(T)^*)$ is not closed and thus $\gamma(S_r(T)^*) = 0$. So we can choose a sequence of unit vectors $x_n \in \mathcal{N}(S_r(T)^*)^\perp$ such that $U^*|T|^r U^* x_n \longrightarrow 0$. By using P (1) and $P(2)$, we get $U^*U^*|T^*|^r x_n \longrightarrow 0$, this implies that

$$UU^*U^*|T^*|^r x_n \longrightarrow 0.$$

Since $\mathcal{N}(T^*) \subseteq \mathcal{N}(T)$, then $\mathcal{R}(T^*) \subseteq \mathcal{R}(T)$, By taking the orthogonal complements. It follows that $U^*|T^*|^r x_n \longrightarrow 0$. Hence

$$|T^*|^r x_n = UU^*|T^*|^r x_n \longrightarrow 0,$$

with $x_n \in \mathcal{N}(S_r(T)^*)^\perp = \mathcal{N}(T^*)^\perp = \mathcal{N}(|T^*|^r)^\perp$,, by Proposition 3.3 and Lemma 2.1. This contradict the fact that $\mathcal{R}(|T^*|^r) = \mathcal{R}(T)$ is closed.

($\Longleftarrow$). Suppose that $\mathcal{R}(S_r(T))$ is closed and $\mathcal{R}(T)$ is not closed. Then $\mathcal{R}(|T^*|)$ is not closed, Using Lemma 2.1, we have $\mathcal{R}(|T^*|^r)$ is not closed. Hence $\gamma(|T^*|^r) = 0$, so we can choose a sequence of unit vectors $x_n \in \mathcal{N}(|T^*|^r)^\perp = \mathcal{N}(T^*)^\perp$, such that $|T^*|^r x_n \longrightarrow 0$. Now, by using the hypothesis $\mathcal{N}(T^*) \subseteq \mathcal{N}(T)$ and Proposition 3.3, it follows that $x_n \in \mathcal{N}(S_r(T)^*)^\perp$ such that $U^*|T|^r U^* x_n = U^*U^*|T^*|^r x_n \longrightarrow 0$. This implies that $\mathcal{R}(S_r(T)^*)$ is not closed which is contradicts the fact that $\mathcal{R}(S_r(T))$ is closed.

■

As a direct consequence we have the following corollary :

Corollary 3.3 *Let $r > 0$ and $T \in \mathcal{B}(\mathcal{H})$. If $\mathcal{N}(T^*) = \mathcal{N}(T)$, then the following statements are equivalent.*

1. *$\mathcal{R}(T)$ is closed,*

2. *$\mathcal{R}(S_r(T))$ is closed,*

3. *$\mathcal{R}(S_r(T)^*)$ is closed,*

4. *$\mathcal{R}(S_r(T^*))$ is closed.*

Remark 3.3 *The assumption $\mathcal{N}(T^*) \subset \mathcal{N}(T)$ is necessary in the previous theorem. Indeed, consider the unilateral weighted shift T, defined on the Hilbert space $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, ...) = (0, x_1, 0, \frac{x_3}{3}, 0, \frac{x_5}{5}, ...)$. Clearly $T^2 = 0$. Then by [22, Theorem 6.1], $S(T) = 0$ has closed range while $\mathcal{R}(T)$ is not closed.*

In the next theorem, we present the relationship between an EP operator T and its $S_r(T)$ transform.

Theorem 3.6 *Let $T \in \mathcal{B}(\mathcal{H})$ and $r > 0$. Then*

$$T \text{ is EP} \iff S_r(T) \text{ is EP and } \mathcal{R}(T) = \mathcal{R}(S_r(T)).$$

Proof. ($\Longrightarrow$). We assume that T is EP. Then $\mathcal{R}(T)$ is closed and $\mathcal{N}(T) = \mathcal{N}(T^*)$. This implies that $\mathcal{R}(S_r(T))$ is closed, by Proposition 3.5. On the other hand, since $\mathcal{N}(T) = \mathcal{N}(T^*)$, by Proposition 3.3, it follows that

$$\mathcal{N}(S_r(T)) = \mathcal{N}(S_r(T)^*).$$

Consequently, $S_r(T)$ is also EP. Finally, by taking the orthogonal complements in the relation $\mathcal{N}(S_r(T)) = \mathcal{N}(T)$ and since T and $S_r(T)$ are EP, we conclude that $\mathcal{R}(S_r(T)) = \mathcal{R}(T)$.
($\Longleftarrow$). We suppose that $S_r(T)$ is EP. Then $\mathcal{R}(S_r(T))$ is closed and $\mathcal{R}(S_r(T)) = \mathcal{R}(S_r(T)^*)$. Since $\mathcal{R}(T) = \mathcal{R}(S_r(T))$, It follows that $\mathcal{R}(T)$ is closed and

$$\mathcal{N}(T^*) = \mathcal{N}(S_r(T)) = \mathcal{N}(S_r(T)^*).$$

As $\mathcal{N}(T) \subset \mathcal{N}(S_r(T))$, then $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. Hence, to prove T is EP, it is enough to prove that $\mathcal{N}(S_r(T)) \subset \mathcal{N}(T)$. Let $x \in \mathcal{N}(S_r(T))$. that's means $U|T|^r Ux = 0$. which yields

$U^*U|T|^rUx = 0$. According to Lemma 2.1, we obtain $|T|^rUx = 0$. Since

$$\mathcal{N}(|T|^r) = \mathcal{N}(|T|) = \mathcal{N}(T) \subset \mathcal{N}(T^*) = \mathcal{N}(|T^*|),$$

then we get $|T^*|Ux = 0$ and so $T(x) = 0$, by Theorem 1.2 (v). Therefore $\mathcal{N}(S_r(T)) \subset \mathcal{N}(T)$. Finally T is EP. $\blacksquare$

In the previous Theorem, the assumption $\mathcal{R}(T) = \mathcal{R}(S(r,T))$ is necessary. Without it the reverse implication is false, as shown in the following example.

Example 3.5 *let* $T = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. *Then* $\mathcal{R}(T)$ *is closed and* $T^+ = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$. *Furthermore,* $T^2 = 0$. *Hence* $S(T) = 0$ *is EP, while* T *is not EP because*

$$TT^+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = T^+T.$$

Corollary 3.4 *Let* $r > 0$ *and* $T \in \mathcal{B}(\mathcal{H})$. *Then*

1. T *is EP.*

2. $S_r(T^*)$ *is EP and* $\mathcal{R}(T^*) = \mathcal{R}(S_r(T^*)) = \mathcal{R}(S_r(T)^*)$.

3. $S_r(T^*)$ *is EP and* $\mathcal{N}(S_r(T)) = \mathcal{N}(T)$.

Proof. (1) $\Longrightarrow$ (2). Suppose that T is EP. Then T^* is EP too. Hence, from Theorem 3.7, we deduce that $S_r(T^*)$ is EP and

$$\mathcal{R}(S_r(T^*)) = \mathcal{R}(T^*) = \mathcal{R}(T) = \mathcal{R}(S_r(T)) = \mathcal{R}(S_r(T)^*).$$

(2) $\Longrightarrow$ (3). It follows by taking the orthogonal complements in the relation $\mathcal{R}(T^*) = \mathcal{R}(S_r(T)^*)$.

(3) $\Longrightarrow$ (1). First we show that $\mathcal{N}(T^*) = \mathcal{N}(T)$. Let $T^*x = 0$. Then it is easy to see that

$S_r(T^*)x = 0$. Since $S_r(T^*)$ is EP, we obtain

$$S_r(T^*)^*x = U\,|T^*\,|^r Ux = 0.$$

Thus, $U^2\,|T\,|^r x = 0$, by Theorem 1.2(v). From the hypothesis $\mathcal{N}(S_r(T)) = \mathcal{N}(T)$ and Proposition 3.1, we have

$$|T\,|^r x \in \mathcal{N}(U^2) = \mathcal{N}(S_r(T)) = \mathcal{N}(T) = \mathcal{N}(|T|).$$

So $x \in \mathcal{N}(\,|T\,|^{r+1}) = \mathcal{N}(|T|) = \mathcal{N}(T)$. Hence,

$$\mathcal{N}(T^*) \subset \mathcal{N}(T) \tag{3.6}$$

Conversely, if $x \in \mathcal{N}(T) = \mathcal{N}(U)$, then $(S_r(T^*))^*x = U\,|T^*\,|^r Ux = 0$. Since $S_r(T^*)$ is EP, we deduce that $S_r(T^*)x = 0$ and by using again Proposition 3.1, we get $(U^*)^2x = 0$. Hence, the inclusion (3.6), gives

$$U^*x \in \mathcal{N}(U^*) = \mathcal{N}(T^*) \subset \mathcal{N}(T) = \mathcal{N}(U).$$

Then $UU^*x = 0$. Which yields $T^*x = 0$, because $\mathcal{N}(UU^*) = \mathcal{N}(T^*)$. This proves $\mathcal{N}(T) = \mathcal{N}(T^*)$. Now, since $\mathcal{R}(S_r(T^*))$ is closed, then by the previous equality and Theorem 3.5, $\mathcal{R}(T^*)$ is closed and so $\mathcal{R}(T)$ is closed too. Consequently, T is EP. ■

Combining Theorem 3.7, 2.12 and Djordjević result [19], we can directly obtain the following corollary.

Corollary 3.5 *Let $r > 0$ and $T \in \mathcal{B}(\mathcal{H})$ with closed range. Then*

1. $(TS_r(T))^+ = S_r(T)^+T^+$ *and* $(S_r(T)T)^+ = T^+S_r(T)^+$.

2. $(\Delta_\lambda(T)S_r(T))^+ = S_r(T)^+\Delta_\lambda(T)^+$ *and* $(S_r(T)\Delta_\lambda(T))^+ = \Delta_\lambda(T)^+S_r(T)^+$.

Next, we provide a conditions under which $S_r(T^+) = S_r(T)^+$.

Theorem 3.7 *Let $T \in \mathcal{B}(\mathcal{H})$ be binormal with closed range and let $T = U|T|$ be it's polar decomposition. Then $\mathcal{R}(S_r(T))$ is closed and $S_r(T)^+ = U^*(|T|^r)^+U^*$, for all $r > 0$.*

Proof. First we show that $\mathcal{R}(S_r(T))$ is closed.

Since $\mathcal{R}(T)$ is closed and T is binormal, then $P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)} = P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)}$, by Lemma 1.5. Therefore $P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)}$ is a projection, it follows that $\mathcal{R}(P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)})$ is closed. So by using Proposition 3.6, we have $\mathcal{R}(S_r(T))$ is closed.

Now, we Put $A = S_r(T)^+$. We have

$$\begin{aligned} S_r(T)AS_r(T) &= U|T|^rUU^*(|T|^r)^+U^*U|T|^rU \\ &= U|T|^rUU^*(|T|^r)^+|T|^rU \\ &= U|T|^rUU^*P_{\mathcal{R}(T^*)}U \\ &= U|T|^rP_{\mathcal{R}(T^*)}UU^*U \\ &= U|T|^rP_{\mathcal{R}(T^*)}U \\ &= |T^*|^rUU \\ &= U|T|^rU, \end{aligned}$$

$$\begin{aligned} AS_r(T)A &= U^*(|T|^r)^+U^*U|T|^rUU^*(|T|^r)^+U^* \\ &= U^*(|T|^r)^+|T|^rUU^*(|T|^r)^+U^* \\ &= U^*P_{\mathcal{R}(T^*)}UU^*(|T|^r)^+U^* \\ &= U^*UU^*P_{\mathcal{R}(T^*)}(|T|^r)^+U^* \\ &= U^*UU^*(|T|^r)^+U^* \\ &= U^*(|T|^r)^+U^* = S_r(T)^+, \end{aligned}$$

and

$$\begin{aligned} S_r(T)A &= U|T|^r UU^*(|T|^r)^+U^* \\ &= UUU^*|T|^r(|T|^r)^+U^* \\ &= UUU^*U^*. \end{aligned}$$

By similar computation we have $AS_r(T) = U^*P_{\mathcal{R}(T^*)}U$. Hence $S_r(T)A$ and $AS_r(T)$ are self-adjoint operators. From the uniqueness of Moore-Penrose inverse we conclude that $S_r(T)^+ = A = U^*(|T|^r)^+U^*$. ■

Next, we provide a conditions under which $S_r(T^+) = S_r(T)^+$.

Theorem 3.8 *Let $r > 0$ and $T \in \mathcal{B}(\mathcal{H})$ have closed range. Then*

$$T \textit{ is quasinormal} \implies S_r(T^+) = S_r(T)^+.$$

In order to prove Theorem 3.8, we need the following Lemma.

Proof. Since T is quasinormal, then $U|T| = |T|U$. Applying the involution to this, we obtain $U^*|T| = |T|U^*$. Hence, by the continuous functional calculus, we have

$$U|T|^r = |T|^r U \text{ and } U^*|T|^r = |T|^r U^*. \tag{3.7}$$

From Lemma 1.10, it follows that

$$U(|T|^r)^+ = (|T|^r)^+U \text{ and } U^*(|T|^r)^+ = (|T|^r)^+U^* \tag{3.8}$$

and from Lemma 1.6 (i) and (iii), we have $T^+ = U^*|T^+|$ is the polar decomposition of T^+ and $|(T^+)^*| = |T|^+$. So, we obtain

$$(|T|^+)^r U^* = U^*(|T^+|)^r \tag{3.9}$$

Thus,

$$\begin{aligned} S_r(T)^+ = U^*(|T|^r)^+U^* &= (|T|^r)^+U^*U^* && \text{by (3.8)} \\ &= (|T|^+)^rU^*U^* && \text{by Lemma 1.8} \\ &= U^*(|T^+|)^rU^* && \text{by (3.9)} \\ &= S_r(T^+). \end{aligned}$$

■

Example 3.6 *If we relax the condition T is quasinormal by assuming T is binormal , then the result is invalid. To see this let* $T = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \in \mathbb{C}^2$. *Then T is invertible, binormal and not quasinormal. Also*

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = U|T|,$$

is the canonical polar decomposition of T. Hence, we have

$$S(T) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and so } S(T)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

On the other hand, we have

$$S(T^{-1}) = U^*(|T^{-1}|)U^* = U^*U|T|^{-1}U^*U^* = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $S(T)^{-1} \neq S(T^{-1})$ while T is binormal.

Remark 3.4 *The reverse implication does not hold in the previous result. For example, consider the left shift operator A, defined on the Hilbert space $\ell^2(\mathbb{N})$ by*

$$A(x_1, x_2, ...) = (x_2, x_3, ...).$$

Then

$$A^*(x_1, x_2, ...) = (0, x_1, x_2, ...).$$

It is easy to see that

$$AA^*A(x_1, x_2, ...) = (x_2, x_3, ...) \neq (0, x_3, x_4, ...) = A^*AA.$$

Hence, A is not quasinormal. On the other hand since $AA^*A = A$, then A is a partial isometry, which implies that $A = A|A| = AA^*A$ is the canonical polar decomposition of A and $A^* = A^+$. Consequently, we get

$$S(A^+) = S(A^*) = A^*|A^*|^{\frac{1}{2}}A^* = A^*IA^* = (A^*)^2,$$

and

$$S(A)^+ = A^*(|A|^{\frac{1}{2}})^+A^* = A^*A^*AA^* = (A^*)^2.$$

Therefore, $S(A^+) = S(A)^+$, but A is not quasinormal.

the next generalizes Theorem 3.4 obtained for invertibles operators to EP operators.

Proposition 3.7 *Let $T \in \mathcal{B}(\mathcal{H})$ withclosed range and $r > 0$. If T is an EP operator, then*

$$T \text{ is normal} \iff S_r(T^+) = S_r(T)^+.$$

Proof. Since T is an EP operator, Then $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$. According to this deomposition, T has the following matrix form

$$T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where the operator $A : \mathcal{R}(T) \longrightarrow \mathcal{R}(T)$ is invertible.

Now it is known that the canonical polar decomposition of T is $T = U|T|$ such that

$$U = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad |T| = \begin{pmatrix} |A| & 0 \\ 0 & 0 \end{pmatrix}.$$

and $A = V|A|$ is the canonical polar decomposition of A. Then

$$S_r(T) = \begin{pmatrix} S_r(A) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (S_r(T))^+ = \begin{pmatrix} (S_r(A))^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Also we have

$$T^+ = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad S_r(T^+) = \begin{pmatrix} S_r(A^{-1}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$(S(r,T))^+ = S_r(T^+) \iff (S(r,A))^{-1} = S_r(A^{-1}).$$

So by using Theorem 3.4, we conclude that

$$S(r,T)^+ = S_r(T^+) \iff A \text{ is normal} \iff T \text{ is normal}.$$

■

Finally, we focus on the canonical polar decomposition of binormal operator via Moore-Penrose inverse and the transform $S_r(T)$. We recall the following Lemma needed in the sequel.

Lemma 3.2 *[24] Let $A > 0$ and $T = U|T|$ be the canonical polar decomposition of an operator T. Then for each $\alpha > 0$ and $\beta > 0$, the following statements hold:*

1. $(U|T|^\beta A|T|^\beta U^*)^\alpha = U(|T|^\beta A|T|^\beta)^\alpha U^*$.

2. $U^*U(|T|^\beta A|T|^\beta)^\alpha = (|T|^\beta A|T|^\beta)^\alpha$.

Theorem 3.9 *Let $T = U|T|$ have closed range and $U^*(|T|^+)^r = W|U^*(|T|^+)^r|$ be the polar decompositions. If T is binormal, then $S_r(T)^+ = WU^*|S_r(T)^+|$ is the canonical polar decomposition of $S_r(T)^+$.*

Proof. First we show that $S_r(T)^+ = WU^*|S_r(T)^+|$.

$$\begin{aligned} WU^*|S_r(T)^+| &= WU^*\left(U(|T|^+)^r UU^*(|T|^+)^r U^*\right)^{\frac{1}{2}} \\ &= WU^*U\left((|T|^+)^r UU^*(|T|^+)^r\right)^{\frac{1}{2}} U^* \qquad \textit{by Lemma } 3.2 \\ &= W|U^*(|T|^+)^r|U^* \\ &= U^*(|T|^+)^r U^* = S_r(T)^+. \end{aligned}$$

Now we show that WU^* is a partial isometry. Since

$$\mathcal{R}(W^*) = \overline{\mathcal{R}((|T|^+)^r U)} \subset \overline{\mathcal{R}((|T|^+)^r)} = \mathcal{R}(U^*),$$

then

$$WU^*UW^*WU^* = WW^*WU^* = WU^*.$$

Lastly, we prove that $\mathcal{N}(S_r(T)^+) = \mathcal{N}(WU^*)$. Let $WU^*x = 0$. Then $U^*x \in \mathcal{N}(W)$. Since $\mathcal{N}(W) = \mathcal{N}(U^*(|T|^+)^r)$, we have $U^*(|T|^+)^r U^*x = 0$. Hence $S_r(T)^+x = 0$. On the other hand if $S_r(T)^+x = 0$, then $U^*(|T|^+)^r U^*x = 0$. Thus $U^*x \in \mathcal{N}(W)$. This implies that $WU^*x = 0$. This completes the proof. ■

Theorem 3.10 *Let $r > 0$ and $T = U|T|$ be the canonical polar decomposition of a binormal operator with closed range. If $S_r(T)^+ = W|S_r(T)^+|$ is the polar decomposition. Then the operator $U^*(|T|^+)^r$ has the canonical polar decomposition given by $WU|U^*(|T|^+)^r|$.*

Proof. Since T is binormal, then we have

$$
\begin{aligned}
U^*(|T|^+)^r &= U^*(|T|^+)^r U^*U \\
&= S_r(T)^+U \\
&= W|S_r(T)^+|U \\
&= W\left(U(|T|^+)^r UU^*(|T|^+)^r U^*\right)^{\frac{1}{2}} U \\
&= WU\left((|T|^+)^r UU^*(|T|^+)^r\right)^{\frac{1}{2}} U^*U \qquad \textit{by Lemma } 3.2 \\
&= WU\left((|T|^+)^r UU^*(|T|^+)^r\right)^{\frac{1}{2}} \qquad \textit{by Lemma } 3.2 \\
&= WU|U^*(|T|^+)^r|.
\end{aligned}
$$

Now, we prove $\mathcal{N}(WU) = \mathcal{N}(U^*(|T|^+)^r)$.

$$
\begin{aligned}
WUx = 0 &\iff Ux \in \mathcal{N}(W) \\
&\iff Ux \in \mathcal{N}(S_r(T)^+) \\
&\iff U^*(|T|^+)^r U^*Ux = 0 \\
&\iff U^*(|T|^+)^r x = 0, \qquad \textit{as } T \textit{ is binormal}.
\end{aligned}
$$

Next we show WU is a partial isometry. Since $\mathcal{R}(W^*) = \mathcal{R}(S_r(T)) \subset \mathcal{R}(U)$, then $UU^*W^*W = W^*W$. Therefore

$$WU(WU)^*WU = WUU^*W^*WU = WW^*WU = WU.$$

■

Chapter 4

The generalized mean transform of closed range operators

In this chapter, we first provide several preliminary results of the mean transform, recently obtained in [16]. Then we shall obtain results about the closedness of T and $\widehat{T}_\lambda$, we prove under the assumption $\mathcal{N}(T) \subset \mathcal{N}(T^*)$ that range of T is closed if and only if the range of $\widehat{T}_\lambda$ is closed. Also we show that if T is binormal with closed range then the range of it's generalized mean transform is closed too. Afterwards, we see the generalized mean transform of EP and binormal operators via Moore-Penrose inverse. In particular, we show that T is EP if and only if $\widehat{T}_\lambda$ is EP too and $\mathcal{R}(T) = \mathcal{R}(\widehat{T}_\lambda)$. Also, we prove that the reverse order law hold for T and $\widehat{T}_\lambda$, when T is EP.

4.1 preliminary results

The mean transform does depend on the partial isometry of the polar decomposition T as shown by the following example

Example 4.1 *let* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *acting on* $\mathbb{C}^2$. *The canonical polar decomposition of* T *is*

$$T = V|T|, \quad where \quad |T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad and \quad V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

On the other hand, we can also write $T = U_{max}|T|$, *where* $U_{max} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *is unitary. This is the so-called maximal polar decomposition of T, since the partial isometry is unitary. in this case,*

$$\frac{1}{2}(U_{max}|T| + |T|U_{max}) = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \neq \frac{1}{2}(V|T| + |T|V) = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}.$$

This shows that the mean transform depends on the polar decomposition. (see[16])
In[16], F. Chabbabi et al, proved that T and $\widehat{T}$ have the same null subspaces.

Proposition 4.1 *[16] For any operator* $T \in \mathcal{B}(\mathcal{H})$, *we have*

$$\mathcal{N}(\widehat{T}) = \mathcal{N}(T).$$

Later, in 2020, C. Benhida et al [5] give this generalisation.

Proposition 4.2 *[5]Let* $T \in B(H)$ *and* $\lambda \in]0, \frac{1}{2}]$. *If* $N(T) \subset N(T^*)$, *then*

$$\mathcal{N}(\widehat{T}_\lambda) = \mathcal{N}(T).$$

4.2 The generalized mean transform of closed range operators

Recently, F.Chabbabi and M. Mebekhta in [13] proved that if range of $T \in \mathcal{B}(\mathcal{H})$ is closed then the range of it's mean transform is also closed. In the following, we generalize this result for $\widehat{T}_\lambda$ using the assumption $N(T) \subset N(T^*)$. Also in the next theorem, we prove that the reverse implication hold too under the same assumption.

Theorem 4.1 *Let $\lambda \in]0, \frac{1}{2}]$ and let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $N(T) \subset N(T^*)$. Then*

$$\mathcal{R}(T) \textit{ is closed } \iff \mathcal{R}(\widehat{T}_\lambda) \textit{ is also closed.}$$

Proof. ($\Rightarrow$). For $\lambda = \frac{1}{2}$ (see [13]). Let $\lambda \in]0, \frac{1}{2}[$. Suppose that $R(T)$ is closed and $\mathcal{R}(\widehat{T}_\lambda)$ is not closed. Then there exists a sequence of unit vectors $x_n \in (\mathcal{N}(\widehat{T}_\lambda))^\perp = N(T)^\perp$ such that

$$\widehat{T}_\lambda x_n = (|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda)x_n \longrightarrow 0.$$

Thus

$$|T|^\lambda(U|T|^{1-2\lambda} + |T|^{1-2\lambda}U)|T|^\lambda x_n \longrightarrow 0.$$

Therefore

$$(|T|^\lambda)^+|T|^\lambda(U|T|^{1-2\lambda} + |T|^{1-2\lambda}U)|T|^\lambda x_n \longrightarrow 0.$$

As

$$R(U) \subset R(T^*) = R(|T|^\lambda) \text{ and } R(|T|^{1-2\lambda}) = R(|T|^\lambda),$$

we have

$$(U|T|^{1-2\lambda} + |T|^{1-2\lambda}U)|T|^\lambda x_n \longrightarrow 0.$$

Put $y_n = |T|^\lambda x_n$, it follows that

$$U^*(U|T|^{1-2\lambda} + |T|^{1-2\lambda}U)y_n \longrightarrow 0.$$

Thus

$$(|T|^{1-2\lambda} + U^*|T|^{1-2\lambda}U)y_n \longrightarrow 0.$$

Since $|T|^{1-2\lambda}$ and $U^*|T|^{1-2\lambda}U$ are positive operators. Then

$$|T|^{1-2\lambda}y_n \longrightarrow 0 \text{ and } U^*|T|^{1-2\lambda}Uy_n \longrightarrow 0$$

Hence $|T|^{1-2\lambda}|T|^{\lambda}x_n \longrightarrow 0$. Then $|T|x_n \longrightarrow 0$, implies $U|T|x_n \longrightarrow 0$ which is a contradiction with $R(T)$ is closed.

($\Leftarrow$). Let $\lambda \in]0, \frac{1}{2}]$. Suppose that $R(\widehat{T_\lambda})$ is closed and $R(T)$ is not closed. Then by Lemma 2.1, $R(|T|)$ is not closed. Thus there exists a sequence of unit vectors $x_n \in \mathcal{N}(|T|)^\perp = \mathcal{N}(\widehat{T_\lambda})^\perp$, such that $|T|x_n \longrightarrow 0$. Hence

$$|T|^{\lambda}x_n \longrightarrow 0, \text{ and } |T|^{1-\lambda}x_n \longrightarrow 0.$$

It follows that

$$|T|^{\lambda}U|T|^{1-\lambda}x_n \longrightarrow 0 \ \textit{and} \ |T|^{1-\lambda}U|T|^{\lambda}x_n \longrightarrow 0.$$

therefore, $\widehat{T_\lambda}x_n \longrightarrow 0$. This contradicts the fact that $R(\widehat{T_\lambda})$ is closed. ■

Corollary 4.1 *Let $T \in B(H)$ be hyponormal. Then*

$$\mathcal{R}(T) \textit{ is closed } \iff \mathcal{R}(\widehat{T_\lambda}) \textit{ is also closed.}$$

Proof. Since T is hyponormal, then $N(T) \subset N(T^*)$. So, we from the previous theorem we have

$$\mathcal{R}(T) \textit{ is closed } \iff \mathcal{R}(\widehat{T_\lambda}) \textit{ is also closed.}$$

■

The assumption $N(T) \subset N(T^*)$ is necessary in the previous theorem as shown by the following example.

Example 4.2 *Let* $T = \begin{pmatrix} A & 0 \\ (I - A^*A)^{\frac{1}{2}} & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, *where* A *is a contraction and* $\mathcal{R}(A)$ *is not closed. Then*

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

is an orthogonal projection. Hence T *is a partial isometry. This implies that* $\mathcal{R}(T)$ *is closed and* $T = T|T| = TT^*T$ *is the polar decomposition of* T. *Since*

$$TT^*T = \begin{pmatrix} A & 0 \\ (I - A^*A)^{\frac{1}{2}} & 0 \end{pmatrix} \neq \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = T^*TT,$$

Then T *is not quasinormal, using [39, Proposition 2.1.] we obtain that* $N(T) \nsubseteq N(T^*)$.
On the other hand, for $\lambda \in]0, \frac{1}{2}]$, *we have*

$$\widehat{T_\lambda} = (T^*T)^\lambda\, T(T^*T)^{1-\lambda} + (T^*T)^{1-\lambda} T\, (T^*T)^\lambda = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

so $\mathcal{R}(\widehat{T_\lambda})$ *is not closed.*

Remark 4.1 *In Theorem 4.1, if* $\lambda = 0$, *The reverse implication is not valid as shown by [13]*

Proposition 4.3 *Let* $T \in B(H)$ *be binormal with closed range. Then*

1. $N(|T|^\alpha |T^*|^\beta) = N(P_{R(T)} R_{R(T^*)})$, *for all* $\alpha, \beta > 0$.

2. $N(\Delta_\lambda(T)) = \mathcal{N}(\widehat{T_\lambda})$ *for all* $\lambda \in]0, 1[$.

Proof. (1)Since T is binormal, $U^*U|T^*|^\beta x = |T^*|^\beta U^*Ux = 0$(see[]). Then, we have

$$\begin{aligned} |T|^\alpha |T^*|^\beta x = 0 &\iff U|T^*|^\beta x = 0, \qquad \text{as } N(|T|^\alpha) = N(U) \\ &\iff U^*U|T^*|^\beta x = 0 \\ &\iff |T^*|^\beta U^*Ux = 0 \\ &\iff (|T^*|^\beta)^+|T^*|^\beta U^*Ux = 0 \\ &\iff P_{R(T)}R_{R(T^*)}x = 0 \qquad \text{as } R(|T^*|^\beta) = R(|T^*|) = R(T). \end{aligned}$$

Hence $N(|T|^\alpha|T^*|^\beta) = N(P_{R(T)}R_{R(T^*)})$.

(2)Suppose that $x \in N(\Delta_\lambda(T))$, thus

$$|T|^\lambda U|T|^{1-\lambda}x = 0.$$

As $U|T|^{1-\lambda} = |T^*|^{1-\lambda}U$,

$$|T|^\lambda|T^*|^{1-\lambda}Ux = 0.$$

Then by (1),

$$Ux \in N(|T|^\lambda|T^*|^{1-\lambda}) = N(P_{R(T)}R_{R(T^*)}) = N(|T|^{1-\lambda}|T^*|^\lambda).$$

So

$$|T|^{1-\lambda}|T^*|^\lambda Ux = 0.$$

Consequetly

$$\widehat{T_\lambda}x = |T|^\lambda U|T|^{1-\lambda}x + |T|^{1-\lambda}U|T|^\lambda x = 0.$$

Hence $N(\Delta_\lambda(T)) \subset \mathcal{N}(\widehat{T_\lambda})$. Conversly, Let $X \in \mathcal{N}(\widehat{T_\lambda})$, then

$$\widehat{T_\lambda}x = (|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda)x = 0.$$

Thus

$$(|T|^{\lambda}|T^*|^{1-\lambda}+|T|^{1-\lambda}|T^*|^{\lambda})Ux_n=0,$$

It follows that

$$\left\langle |T|^{\lambda}|T^*|^{1-\lambda}Ux_n,Ux_n\right\rangle+\left\langle |T|^{1-\lambda}|T^*|^{\lambda}Ux,Ux\right\rangle=\left\langle (|T|^{\lambda}|T^*|^{1-\lambda}+|T|^{1-\lambda}|T^*|^{\lambda})Ux,Ux\right\rangle=0$$

As T is binormal, then $|T|^{\lambda}|T^*|^{1-\lambda}$ and $|T|^{1-\lambda}|T^*|^{\lambda}$ are positive operators and so

$$\|\left(|T|^{\lambda}|T^*|^{1-\lambda}\right)^{\frac{1}{2}}Ux\|=\left\langle |T|^{\lambda}|T^*|^{1-\lambda}Ux,Ux\right\rangle=0.$$

As a consequence,

$$x\in\mathcal{N}(\Delta_{\lambda}(T))$$

Hence

$$\mathcal{N}(\widehat{T_{\lambda}})=\mathcal{N}(\Delta_{\lambda}(T)).$$

■ In chapter two, Proposition 2.10, we proved that if $T\in B(H)$ is binormal with closed range. Then the range of it's λ-Aluthge transform is closed. It is equally true for the generalized mean transfom as the next theorem shows.

Theorem 4.2 *Let $T\in B(H)$ be binormal with closed range. Then $\mathcal{R}(\widehat{T_{\lambda}})$ is closed .*

Proof. Suppose that T is binormal with closed range and $\mathcal{R}(\widehat{T_{\lambda}})$ is not closed. Then there exists a sequence of unit vectors $x_n\in(\mathcal{N}(\widehat{T_{\lambda}}))^{\perp}$ such that

$$\widehat{T_{\lambda}}x_n=(|T|^{\lambda}U|T|^{1-\lambda}+|T|^{1-\lambda}U|T|^{\lambda})x_n\longrightarrow 0.$$

Using Proposition 4.3, $x_n\in\mathcal{N}(\Delta_{\lambda}(T))^{\perp}$. Now, as T is binormal, we obtain

$$|T|^{\lambda}|T^*|^{1-\lambda}Ux_n\longrightarrow 0.$$

Therefore

$$\Delta_\lambda(T)x_n = |T|^\lambda U|T^*|^{1-\lambda}x_n \longrightarrow 0.$$

So $x_n \in \mathcal{N}(\Delta_\lambda(T))^\perp$ such that $\Delta_\lambda(T)x_n \longrightarrow 0$. This contradicts the fact that $R(\Delta_\lambda(T))$ is closed .

■ The reverse implication does not hold, as the following example shows.

Example 4.3 *Let $T : \ell^2 \longrightarrow \ell^2$, defined by*

$$T(x_1, x_2, x_3, ...)=(0, \tfrac{1}{2}x_1, 2x_2, \tfrac{1}{2}x_3...).$$

Its range is not closed in ℓ^2. By a simple computation, we have

$$T^*(x_1, x_2, x_3, ...)=(\tfrac{1}{2}x_2, 2x_3, \tfrac{1}{2}x_4, ...).$$

Then

$$T^*T(x_1, x_2, x_3, ..) = (\frac{1}{4}x_1, 4x_2, \frac{1}{4}x_3, ...),$$

and so $TT^*T = T^*TT$. Hence T is quasinormal, thus T is binormal. Also,

$$|T| = (T^*T)^{\frac{1}{2}} = (\frac{1}{2}x_1, 2x_2, \frac{1}{2}x_3, ...)$$

is invertible and

$$|T|^{-1}(x_1, x_2, x_3, ...) = (2x_1, \tfrac{1}{2}x_2, ...)$$

So $U = T|T|^{-1} = (0, x_1, x_2, ...)$. Thus,

$$\widehat{T}(x_1, x_2, x_3, ...) = (T + |T|U)(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...),$$

have closed range.

4.3 The generalized mean transform of EP operators

We know that the Aluthge transform of an EP operator T in $\mathcal{B}(\mathcal{H})$ is EP. We may ask whether the mean transform is too EP. The following result states that the mean transform of an EP operator T in $\mathcal{B}(\mathcal{H})$ is also EP, and in that case the range of T and its mean transform coincide.

Theorem 4.3 *Let $\lambda \in [0, \frac{1}{2}]$. Let $T \in \mathcal{B}(\mathcal{H})$ and $T = U|T|$ be its polar decomposition. Then*

$$T \textit{ is an EP operator} \iff \widehat{T_\lambda} \textit{ is EP and } \mathcal{R}(T) = \mathcal{R}(\widehat{T_\lambda}).$$

Proof. ($\Rightarrow$). According to theorem 4.1, $\mathcal{R}(\widehat{T_\lambda})$ is closed. Now we have to prove that $\mathcal{N}(\widehat{T_\lambda}) = \mathcal{N}(\widehat{T_\lambda}^*)$. For $\lambda = 0$. Suppose that T is EP operator and let $x \in \mathcal{N}(\widehat{T}) = \mathcal{N}(T) = \mathcal{N}(T^*)$. Then $U^*x = |T|x = 0$ and thus $\widehat{T}^*x = \dfrac{1}{2}(|T|U^* + U^*|T|)x = 0$. This shows that $\mathcal{N}(\widehat{T}) \subset \mathcal{N}(\widehat{T}^*)$. Conversely, if $x \in \mathcal{N}(\widehat{T}^*)$, then

$$|T|U^*x + U^*|T|x = 0,$$

and hence

$$U|T|U^*x + UU^*|T|x = U(|T|U^*x + U^*|T|)x = 0$$

Since $\mathcal{N}(T) = \mathcal{N}(T^*)$, then U is normal. So

$$|T^*|x + |T|x = 0.$$

It follows that

$$\langle |T^*|x, x\rangle + \langle |T|x, x\rangle = \langle |T^*|x + |T|x, x\rangle.$$

Since $|T^*|$ and $|T|$ are both positive, we have

$$|||T|^{\frac{1}{2}}x|| = \langle |T|x, x\rangle = 0,$$

As a consequence,

$$\mathcal{N}(\widehat{T}^*) \subseteq \mathcal{N}(T) = \mathcal{N}(\widehat{T}).$$

as desired.

On the other hand, taking orthogonal complements in the relation $\mathcal{N}(T) = \mathcal{N}(\widehat{T})$, and since T and $\widehat{T}$ are both EP operators. We deduce that $\mathcal{R}(T) = \mathcal{R}(\widehat{T})$.

$(\Leftarrow)$. By the assumption, we have $\mathcal{R}(T) = \mathcal{R}(\widehat{T})$, and then

$$\mathcal{N}(T^*) = \mathcal{N}(\widehat{T}^*).$$

Since $\widehat{T}$ is EP operator, we deduce that

$$\mathcal{N}(T^*) = \mathcal{N}(\widehat{T}) = \mathcal{N}(T).$$

Hence T is EP opertator.

For $\lambda \in]0, \frac{1}{2}[$. Let $x \in \mathcal{N}(\widehat{T_\lambda}) = \mathcal{N}(T) = \mathcal{N}(|T|^\lambda) = \mathcal{N}(|T|^{1-\lambda})$. Then

$$\widehat{T_\lambda}^* x = \frac{1}{2}(|T|^{1-\lambda}U^*|T|^\lambda + |T|^\lambda U^*|T|^{1-\lambda})x = 0.$$

This shows that $\mathcal{N}(\widehat{T_\lambda}) \subset \mathcal{N}(\widehat{T_\lambda}^*)$. Conversly, if $x \in \mathcal{N}(\widehat{T_\lambda}^*)$, then

$$(|T|^{1-\lambda}U^*|T|^\lambda + |T|^\lambda U^*|T|^{1-\lambda})x = 0,$$

and hence

$$|T|^\lambda(|T|^{1-2\lambda}U^* + U^*|T|^{1-2\lambda})|T|^\lambda x = 0.$$

So

$$U(|T|^{1-2\lambda}U^* + U^*|T|^{1-2\lambda})|T|^\lambda x = 0.$$

Put $S = |T|^{1-2\lambda}U^* + U^*|T|^{1-2\lambda}$ and $y = |T|^{\lambda}x$. Since $\mathcal{N}(T) = \mathcal{N}(T^*)$, then U is normal. So

$$
\begin{aligned}
< USy, y > &=< U(|T|^{1-2\lambda}U^* + U^*|T|^{1-2\lambda})y, y > \\
&=< (|T^*|^{1-2\lambda} + |T|^{1-2\lambda})y, y > \\
&=< |T^*|^{1-2\lambda}y, y > + < |T|^{1-2\lambda}y, y > \\
&= |||T^*|^{\frac{1-2\lambda}{2}}y||^2 + |||T|^{\frac{1-2\lambda}{2}}y||^2 = 0.
\end{aligned}
$$

Thus $|T|^{1-2\lambda}y = 0$. Therefore $|T|^{1-\lambda}x = 0$. Hence $x \in \mathcal{N}(T) = \mathcal{N}(\widehat{T_\lambda})$.

($\Leftarrow$). By the assumption, we have $\mathcal{R}(T) = \mathcal{R}(\widehat{T_\lambda})$, and then

$$\mathcal{N}(T^*) = \mathcal{N}(\widehat{T_\lambda}^*).$$

Since $\widehat{T}_\lambda$ is EP operator, we deduce that

$$\mathcal{N}(T^*) = \mathcal{N}(\widehat{T_\lambda}) = \mathcal{N}(T),$$

as $\mathcal{N}(T) = \mathcal{N}(|T|^{\lambda}) = \mathcal{N}(|T|^{1-\lambda}) \subset \mathcal{N}(\widehat{T_\lambda}^*) = \mathcal{N}(T^*)$. Hence T is EP opertator. This completes the proof.

For $\lambda = \frac{1}{2}$, see Theorem 2.12. ■

The following example shows that if $T \in \mathcal{B}(\mathcal{H})$ such that $\widehat{T}$ is EP, then T need not be EP.

Example 4.4 *Let* $T = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & 0 \end{pmatrix}$. *Then the polar decomposition* $T = U|T|$ *is given by* $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$ *and* $|T| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$. *Then* $\widehat{T} = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}$ *is EP matrix because*

$$\widehat{T}\widehat{T}^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \widehat{T}^+\widehat{T}.$$

While T is not, indeed

$$TT^{+} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = T^{+}T.$$

As a consequence of the previous theorem, we have the following resultat.

Corollary 4.2 *Let $T \in \mathcal{B}(\mathcal{H})$ is EP. Then the following operators are also EP.*

$$T^{+}, \widehat{T}^{+} = \begin{pmatrix} \widehat{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \widehat{T^{+}} = \begin{pmatrix} \widehat{A^{-1}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Theorem 4.4 *[33] Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. It hold that*

$$T = \widehat{T} \implies \widehat{T}^{-1} = \widehat{T^{-1}}.$$

Proposition 4.4 *Let $T \in \mathcal{B}(\mathcal{H})$ be an EP operator. It hold that*

$$T \text{ is quasinormal } \Longrightarrow T^{+} \text{ is quasinormal}$$

Proof. Since T is quasinormal, then $\Delta_{\lambda}(T) = T$. So $(\Delta_{\lambda}(T))^{+} = T^{+}$. Thus as $(\Delta_{\lambda}(T))^{+} = \Delta_{\lambda}(T^{+})$, we obtain $\Delta_{\lambda}(T^{+}) = T^{+}$. Hence T^{+} is quasinormal. ■

Corollary 4.3 *Let $T \in \mathcal{B}(\mathcal{H})$ be an EP operator. It hold that*

$$T \text{ is quasinormal } \implies \widehat{T}^{+} = \widehat{T^{+}}.$$

Proof. Since T is quasinormal, then $\widehat{T} = \Delta_{\lambda}(T)$. Then, we have $\widehat{T}^{+} = \widehat{T^{+}}$. ■

Corollary 4.4 *Let $T \in \mathcal{B}(\mathcal{H})$ be an EP operator. It hold that*

$$T \text{ is quasinormal } \implies \widehat{T_{\lambda}}^{+} = \widehat{T_{\lambda}^{+}}.$$

Proof.

Since T is an EP operator, then $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ and T has the following matrix form

$$T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where the operator $A : \mathcal{R}(T) \longrightarrow \mathcal{R}(T)$ is invertible. Now it is known that

$$U = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \quad and \; |T| = \begin{pmatrix} |A| & 0 \\ 0 & 0 \end{pmatrix},$$

where $A = V|A|$ is the polar decomposition of A. Then

$$\widehat{T_\lambda} = \begin{pmatrix} \widehat{A} & 0 \\ 0 & 0 \end{pmatrix} \quad and \;\; (\widehat{T_\lambda})^+ = \begin{pmatrix} (\widehat{A_\lambda})^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

also we have

$$T^+ = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} \quad and \;\; \widehat{T_\lambda^+} = \begin{pmatrix} \widehat{A_\lambda^{-1}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$(\widehat{T_\lambda})^+ = \widehat{T_\lambda^+} \iff (\widehat{A_\lambda})^{-1} = \widehat{A_\lambda^{-1}}.$$

T is quasinormal implies $(\widehat{T_\lambda})^{-1} = T^{-1} = \widehat{T_\lambda^{-1}}$ by [5, Theorem 1]. Hence T is quasinormal implies that $(\widehat{T_\lambda})^+ = \widehat{T_\lambda^+}$. ∎ The assumption T is an EP operator is necessary in the previous theorem as shown by the following example.

Example 4.5 *Consider the right shift operator S, defined on the Hilbert space $\ell^2(\mathbb{N})$ by $S(x_1, x_2, ...) = (0, x_1, x_2, ...)$. Then $S^*(x_1, x_2, ...) = (x_2, x_3, ...)$ and so $S^*S = I$. Hence S is quasinormal and isometry, which implies that $S^* = S^+$ and S is not EP because $S^*S \neq SS^*$. On the other hand a simple calculation shows that $S^*SS^* \neq SS^*S$, gives S^* is not quasinormal. So*

$$(\widehat{S})^+ = S^+ = S^* \neq (\widehat{S^*}) = (\widehat{S^+}).$$

The following example shows that if we relax the condition " T is a quasinormal operator and EP operator" by assuming " $T \in \mathcal{B}(\mathcal{H})$ is invertible and binormal", then $\widehat{T}^{-1} \neq \widehat{T^{-1}}$.

Example 4.6 *Let* $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathbb{C}^2$. *Then*

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = U|T|$$

is the polar decomposition of T. *A direct calculation shows that*

$$TT^*T^*T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T^*TTT^*.$$

So, T *is binormal, While*

$$\widehat{T}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \widehat{T^{-1}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In the following, we show that the reverse order law holds for T, $\widehat{T_\lambda}$ and $\Delta_\lambda(T)$ whenever T is EP with closed range.

Corollary 4.5 *When* $T \in \mathcal{B}(\mathcal{H})$ *is an EP operator, then it hold*

(1) $(T\widehat{T_\lambda})^+ = \widehat{T_\lambda}^+ T^+$ *and* $(\widehat{T_\lambda}T)^+ = T^+\widehat{T_\lambda}^+$.

(2) $(\widehat{T_\lambda}\Delta_\lambda(T))^+ = \Delta_\lambda(T)^+(\widehat{T_\lambda})^+$ *and* $(\Delta_\lambda(T)\widehat{T_\lambda})^+ = (\widehat{T_\lambda})^+\Delta_\lambda(T)^+$.

(3) $(S_r(T)\widehat{T_\lambda})^+ = (\widehat{T_\lambda})^+S_r(T)^+$ *and* $(\widehat{T_\lambda}S_r(T))^+ = S_r(T)^+(\widehat{T_\lambda})^+$.

Next, we present some results related to the reverse order law for the Moore-Penrose inverse of binormal operators with closed range.

Proposition 4.5 *Let $T \in \mathcal{B}(\mathcal{H})$ be a binormal operator with closed range and let $\alpha, \beta > 0$. Then*

1. $\mathcal{R}(|T|^\alpha |T^*|^\beta)$ *is closed and* $(|T|^\alpha |T^*|^\beta)^+ = (|T^*|^\beta)^+ (|T|^\alpha)^+$.

2. $\mathcal{R}(|T|^\alpha U)$ *is closed and* $(|T|^\alpha U)^+ = U^*(|T|^\alpha)^+$,

3. $\mathcal{R}(U|T^*|^\alpha)$ *is closed and* $(U|T^*|^\alpha)^+ = (|T^*|^\alpha)^+ U^*$,

4. $\mathcal{R}(T^*|T|^\alpha)$ *is closed and* $(T^*|T|^\alpha)^+ = (|T|^\alpha)^+ (T^*)^+$,

5. $(T^*|T|^\alpha T)^+ = T^+ (|T|^\alpha)^+ (T^*)^+$.

Proof.

(1) First we show that $\mathcal{R}(|T|^\alpha |T^*|^\beta)$ is closed.

Suppose that T is binormal with closed range and $\mathcal{R}(|T|^\alpha |T^*|^\beta)$ is not closed. Then, $\gamma(|T|^\alpha |T^*|^\beta) = 0$. So we can choose a sequence of unit vectors $x_n \in \mathcal{N}(|T|^\alpha |T^*|^\beta)^\perp \subseteq \mathcal{N}(|T^*|^\beta)^\perp$ such that $|T|^\alpha |T^*|^\beta x_n \longrightarrow 0$. Since $\mathcal{R}(|T^*|^\beta)$ is closed and each $x_n \in \mathcal{N}(|T^*|^\beta)^\perp$there exists $\eta > 0$ such that $|||T^*|^\beta x_n|| \geq \eta$ for all n. Put $y_n = \dfrac{|T^*|^\beta x_n}{|||T^*|^\beta x_n||}$ and note that $x_n \in \mathcal{N}(|T|^\alpha |T^*|^\beta)^\perp = \mathcal{N}(|T^*|^\beta |T|^\alpha)^\perp \subseteq \mathcal{N}(|T|^\alpha)^\perp = \mathcal{R}(|T|^\alpha)$,since T is binormal. and so

$$y_n \in |T^*|^\beta \mathcal{R}(|T|^\alpha) = \mathcal{R}(|T^*|^\beta |T|^\alpha) = \mathcal{R}(|T|^\alpha |T^*|^\beta) \subset \mathcal{R}(|T|^\alpha) = \mathcal{N}(|T|^\alpha)^\perp$$

for all n. it follows that

$$|||T|^\alpha y_n|| \leq \frac{1}{\eta} |||T|^\alpha |T^*|^\beta x_n||$$

for all n, and thus $|T|^\alpha y_n \longrightarrow 0$. this contrudicts the fact that $\mathcal{R}(T)$ is closed.

Now we prove that $(|T|^\alpha |T^*|^\beta)^+ = (|T^*|^\beta)^+ (|T|^\alpha)^+$.

Since T is binormal then we obtain that

$$\mathcal{R}(|T|^{\alpha}|T|^{\alpha}|T^{*}|^{\beta}) \subset \mathcal{R}(|T^{*}|^{\beta})$$

and

$$\mathcal{R}(|T^{*}|^{\beta}|T^{*}|^{\beta}|T|^{\alpha}) \subset \mathcal{R}(|T|^{\alpha})$$

.

So

$$(|T|^{\alpha}|T^{*}|^{\beta})^{+} = (|T^{*}|^{\beta})^{+}(|T|^{\alpha})^{+}.$$

(2) Since $\mathcal{R}(|T|^{\alpha}U) = \mathcal{R}(|T|^{\alpha}|T^{*}|^{\beta})$. Then by (1), we have $\mathcal{R}(|T|^{\alpha}U)$ is closed. Moreover,as T is binormal, we have

$$\mathcal{R}(|T|^{2\alpha}U) = \mathcal{R}(|T|^{2\alpha}|T^{*}|) \subset \mathcal{R}(|T^{*}|) = \mathcal{R}(U)$$

and

$$\mathcal{R}(UU^{*}|T|^{\alpha}) = R(|T|^{\alpha}).$$

Thus

$$(|T|^{\alpha}U)^{+} = U^{*}(|T|^{\alpha})^{+}.$$

(3) Since $\mathcal{R}(U|T^{*}|^{\alpha})$ is closed if and only if $\mathcal{R}(|T^{*}|^{\alpha}U^{*})$ and $\mathcal{R}(|T^{*}|^{\alpha}U^{*}) = \mathcal{R}(|T^{*}|^{\alpha}|T|)$. Then by (1), we have $\mathcal{R}(U|T^{*}|^{\alpha})$ is closed. Also, as T is binormal,

$$\mathcal{R}(U^{*}U|T^{*}|^{\alpha}) = \mathcal{R}(|T^{*}|^{\alpha}U^{*}U) \subset \mathcal{R}(|T^{*}|^{\alpha}),$$

and

$$\mathcal{R}(|T^{*}|^{2\alpha}U^{*}) = \mathcal{R}(|T^{*}|^{2\alpha}|T|) = \mathcal{R}(|T||T^{*}|^{2\alpha}) \subset \mathcal{R}(U).$$

+ Since $\mathcal{R}(T^*|T|^\alpha)$ is closed if and only if $\mathcal{R}(|T|^\alpha T)$ and $\mathcal{R}(|T|^\alpha T) = \mathcal{R}(|T|^\alpha |T^*|)$. Then by (1), we have $\mathcal{R}(T^*|T|^\alpha)$ is closed. Also, as T is binormal,

$$\mathcal{R}(|T^*|^2|T|^\alpha) = \mathcal{R}(|T|^\alpha|T^*|^2) \subset \mathcal{R}(|T|^\alpha),$$

and

$$\mathcal{R}(|T|^{2\alpha}T) = \mathcal{R}(|T|^{2\alpha}|T^*|) = \mathcal{R}(|T^*||T|^{2\alpha}) \subset \mathcal{R}(T).$$

(5) Let$S = T^+(|T|^\alpha)^+(T^*)^+$. Then

$$\begin{aligned}
(T^*|T|^\alpha T)S(T^*|T|^\alpha T) &= T^*|T|^\alpha TT^+|T|^\alpha)^+(T^*)^+T^*|T|^\alpha T \\
&= T^*|T|^\alpha P_{R(T)}(|T|^\alpha)^+P_{R(T)}|T|^\alpha T \\
&= T^*|T|^\alpha P_{R(T)}P_{R(T)}(|T|^\alpha)^+|T|^\alpha T \\
&= T^*|T|^\alpha P_{R(T)}P_{R(T^*)}T \\
&= T^*|T|^\alpha P_{R(T^*)}P_{R(T)}T \\
&= T^*|T|^\alpha P_{R(T^*)}T \\
&= T^*|T|^\alpha T,
\end{aligned}$$

$$\begin{aligned}
ST^*|T|^\alpha TS &= T^+(|T|^\alpha)^+(T^*)^+T^*|T|^\alpha TT^+(|T|^\alpha)^+(T^*)^+ \\
&= T^+(|T|^\alpha)^+P_{R(T)}|T|^\alpha P_{R(T)}(|T|^\alpha)^+(T^*)^+ \\
&= T^+(|T|^\alpha)^+P_{R(T)}P_{R(T^*)}(T^*)^+ \\
&= T^+(|T|^\alpha)^+P_{R(T^*)}P_{R(T)}(T^*)^+ \\
&= T^+(|T|^\alpha)^+(T^*)^+ = S,
\end{aligned}$$

and

$$\begin{aligned}
T^*|T|^\alpha TT^+(|T|^\alpha)^+(T^*)^+ &= T^*|T|^\alpha P_{R(T)}(|T|^\alpha)^+(T^*)^+ \\
&= T^* P_{R(T)}P_{R(T^*)}(T^*)^+ \\
&= T^* P_{R(T^*)}(T^*)^+ \\
&= U^*|T^*|U^*U(T^*)^+ \\
&= U^*U^*U|T^*|U|T|^+ \\
&= U^*U^*UUU^*U = U^*U^*UU
\end{aligned}$$

is self-adjoint. Finally,

$$\begin{aligned}
T^+(|T|^\alpha)^+(T^*)^+T^*|T|^\alpha T &= T^+(|T|^\alpha)^+P_{R(T)}|T|^\alpha T \\
&= T^+P_{R(T)}P_{R(T^*)}T \\
&= T^+P_{R(T^*)}T \\
&= U^*|T^+|P_{R(T^*)}U|T| \\
&= U^*P_{R(T^*)}|T^*|^+|T^*|U \\
&= U^*P_{R(T^*)}P_{R(T)}U
\end{aligned}$$

is self-adjoint, so $S = (T^*|T|^\alpha T)^+$. ∎

Remark 4.2 *In the prouvious result we show that the reverse order law holds for $|T|^\alpha$ and $|T^*|^\beta$ whenever T is binormal with closed range, where $\alpha > 0$ and $\beta > 0$. This result was proved in the case of $\alpha = \beta = \frac{1}{2}$ in [30] and in the case of adjointable operators from X to Y, where X and Y are Hilbert A-modules.(see[43]), here we give a new proof.*

Conclusion and Perspectives

Conclusion

- In this thesis we provided some relationships between a bounded linear operator with closed range T and its transforms:λ-Aluthge transform, $S_r(T)$ transform and generalized mean transform.

  - we characterize the invertible, binormal , and EP operators and its intersection with a special class of introduced operators via the λ-Aluthge transform. we present some applications of λ-Aluthge transform to closed range operators and others classes (the class : $\delta(H)$, quasinormal, binormal and EP operators).
  - we give some conection between an operator and its $S_r(T)$ transform.
  - we generalise some results,
  - we have studied the generalized mean transform of closed range operators.

Prospects

We cite some questions for future research.

(i) What is the connection between $\delta(H)$ and sevral classes of operators?

(ii) If T is binormal, is $S_r(T)$ binormal?

(iii) When T is binormal with closed range, what is the Moore-Penrose inverse of it's generalized mean transform?

(v) If $T \in \delta(H)$, is the generalized mean transform or $S_r(T)$ in $\delta(H)$?

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Abstract

This thesis is included in the conception of functional analysis and more precisely in the field of operator theory in Hilbert spaces. It's aim is to study the behavior of certain classes of operators with respect to some operator transformations, in particular the generalized Aluthge transformation which really had an impact on operator theory in the past ten years. We also study in this thesis the Moore-Penrose inverse of these transformations.

Résumé

Cette thèse se situe dans le cadre de l'analyse fonctionnelle et plus précisément dans le domaine de la théorie des opérateurs dans des espaces de Hilbert. Son objectif est d'étudier le comportement de certaines classes d'opérateurs vis-à-vis de quelques transformations d'opérateurs, en particulier la transformation de Aluthge généralisée qui a eu un impact important ces dernières années en théorie des opérateurs. Nous étudions également dans cette thèse l'inverse de Moore-Penrose de ces transformations.

ملخص

تقع هذه الأطروحة في إطار التحليل الدالي وبشكل أكثر دقة في مجال نظرية المؤثرات في فضاءات هلبرت. هدفها هو دراسة سلوك فئات معينة من المؤثرات بالنسبة لبعض تحولات المؤثرات ، وخاصة تحويل أليثج المعمم الذي كان له تأثير مهم في السنوات الأخيرة في نظرية المؤثرات . ندرس أيضا في هذه الأطروحة معكوس مور ـ بنروز لهذه التحولات.