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#### Abstract

The purpose of this thesis is to study a collection of problems related to the Lebesgue spaces with variable exponent. In particular, we study the variable weighted Hardy space on domains, where, we explore its atomic decomposition and we also study its dual space. Next, we study the weighted variable Hardy space associated with operator $L$, more accurately, we establish the molecular characterization of the space $H_{w, L}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, then we explore its dual space. Finally, we study the weighted Hardy-Lorentz spaces with variable exponents, more precisely, we establish the atomic characterization of the variable weighted Hardy-Lorentz spaces.

\section*{Résumé}

L'objectif de cette thèse est d'étudier un ensemble de problèmes liés aux espaces de Lebesgue à exposant variable. Dans un premier temps, nous étudions les espaces de Hardy pondéré à exposant variable sur les domaines, où nous établissons sa décomposition atomique, et nous découvrons également son espace dual. Ensuite, nous étudions l'espace de Hardy pondéré à exposant variable associé à l'opérateur $L$, où, nous établissons la caractérisation moléculaire de l'éspace $H_{w, L}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, puis nous étudions son espace dual. Enfin, nous étudions les espaces de Hardy-Lorentz pondéré à exposant variable, où nous établissons la caractérisation atomique des espaces de Hardy-Lorentz pondéré à exposant variable. $$
\begin{aligned} & \text { ملخص } \\ & \text { في هذه الأطروحة نقوم بدراسة جموعة من المسائل المتعلقة بضضاءات لوبيغ بأس متغير. في البداية، نقوم } \end{aligned}
$$ $$
\begin{aligned} & \text { ذلك نقوم بدراسة فضاء هاردي الموزون و بأس هتغير المرفقة بمؤثر L ، أين، نقوم بالوصف الجزئي لوني } \end{aligned}
$$ $$
\begin{aligned} & \text { خصوصا، نؤسس التوصيف الذري لمساحات هاردي - لورينتز الموزونة بأس متغير. } \end{aligned}
$$


Keywords: weighted Hardy spaces, Hardy-Lorentz spaces, variable exponent, Davies-Gaffney estimates, atomic decomtosition, maximal function, Dual spaces, Hardy-Littlewood maximal operator.

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## INTRODUCTION

The theory of function spaces with variable exponents has evolved into an interesting field of research due to its connection with various domains such as, the modeling of electrorheological fluids, thermorheological fluids, in the study of image processing, in differential equations with nonstandard growth, see e.g. [12, 36, 79].

The starting point of this theory was in 1931 by Orlicz [77], who introduced a natural generalization for the well-known $L^{p}(\Omega)$-spaces via replacing the real exponent $p$ by a measurable function $p(\cdot)$. Kovćik and Rákosník [51] showed that $L^{p(\cdot)}(\Omega)$-spaces have many similar properties to the classical $L^{p}(\Omega)$-spaces, but different in subtle ways. After 10 years later, Fan and Zhao [26] proved the same properties and results in [51], where they have used a different method. The variable exponent function spaces have been widely investigated and used in harmonic analysis and partial differential equations, see for instance $[18,21,64]$ and the references therein.

In the last decades, the weighted variable exponent Lebesgue space, which is a generalization of both the weighted Lebesgue space and the variable Lebesgue space, has been intensively studied, see for example [22,30]. Finding the optimal condition for which the maximal function is bounded on this space is one of the most considerable problems in this space. Diening and Hästö [19] introduced the class $A_{p(\cdot)}$ as a generalization of the ordinary Mukenhoupt class $A_{p}$ and proved that the maximal operator is bounded on weighted variable exponent Lebesgue space with weights in this new class.

The real variable Lorentz spaces $L^{p, q}\left(\mathbb{R}^{n}\right)$ is firstly initiated by Lorentz in [63] and this
kind of space is considered as an extension for the classical Lebesgue spaces $L^{p}\left(\mathbb{R}^{n}\right)$. The study of the Lorentz spaces became an extensive field of research due to its applications in many fields of mathematics. We refer the reader to $[2,6,7,59,60]$ and the references therein. Concerning Lorentz spaces with variable exponent, Kempka and Vybíral [49], introduced and investigated the variable Lorentz spaces $L^{p(\cdot), q(\cdot)}\left(\mathbb{R}^{n}\right)$. One of the important result proved in [49] lies in the fact that the variable Lorentz space coincides with the variable Lebesgue space when $p(\cdot)=q(\cdot)$ i.e. $L^{p(\cdot), p(\cdot)}\left(\mathbb{R}^{n}\right) \equiv L^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, exactly as in the real variable case. Worth mentioning that Israfilov et al. [44] studied the mapping properties of classical operators arising in harmonic analysis in the weighted variable Lorentz spaces. Recently, Ö. Kulak [53] introduced a new weighted variable exponent Lorentz space, then he investigated the boundedness of the bilinear Littlewood-Paley square function and HardyLittlewood maximum function on these weighted variable exponent Lorentz spaces. Further historical details and recent developments related to the Lorentz spaces with variable exponents are reported in $[15,25,31,71]$.

Another proper substitute for the classical Lebesgue spaces is the classical Hardy spaces $H^{p}\left(\mathbb{R}^{n}\right)$, introduced and developed by E. Stein and G. Weiss [86]. Worth pointing out that the real variable Hardy space $H^{p}\left(\mathbb{R}^{n}\right)$ plays an important role in harmonic analysis and partial differential equations, see for instance [70, 82]. In [8], M. Bownik introduced and studied the anisotropic Hardy spaces associated with very general discrete groups of dilations. F. Weisz [87] investigated the atomic characterization of the Martingale Hardy Spaces in the case $0<p \leq 1$. It deserves to mention that Miyachi [68,69] introduced the Hardy space on open subset $\Omega$ of $\mathbb{R}^{n}$ via the maximal function, where he has studied the atomic decomposition and the duality theory of this kind of spaces. On the other hand, S. Wu [88] established a wavelet characterization for weighted Hardy spaces. Whilst, Lee and Lin [55] defined molecules belonging to weighted Hardy spaces $H_{w}^{p}\left(\mathbb{R}^{n}\right)$ and they have showed that each weighted atom is a weighted molecule, and each weighted molecule belongs to a weighted Hardy space. J. Huang and Y. Liu [43] explored the boundedness of intrinsic square functions on the weighted Hardy spaces, then they characterize the weighted Hardy spaces by the intrinsic square functions.

In the spirit of the ideas of Orlicz [77] and E. Stein and G. Weiss [86], Nakai and Sawano [74] introduced Hardy spaces with variable exponents on $\mathbb{R}^{n}$ via the grand maximal function, which can be seen as a generalization of the Hardy spaces and the variable Lebesgue spaces. Always in the same paper, Nakai and Sawano have investigated several properties
for the variable Hardy spaces, in particular, they have obtained the atomic characterization of the Hardy space $H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ with variable exponent, with this decomposition, they have proved the Littlewood-Paley characterization of $H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$. As an application of the atomic decomposition of $H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, the authors of [74] showed that the Campanato space with variable growth conditions is the dual space of the variable Hardy spaces. Independently, Cruz-Uribe and Wang [16] studied the variable Hardy space $H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ with $p(\cdot)$ satisfying some conditions slightly weaker than those used in [74]. Zhuo et al. [101] have proved the intrinsic square function characterizations of this space. Recently, Zhuo et al. [100], introduced the Hardy space with variable exponent on the $R D$-space with infinite measures, then they have obtained several characterizations for this kind of space. More recently, Liu [57] extended the result of Miyachi $[68,69]$ to the variable setting, where he introduced the Hardy spaces with variable exponents on domains and studied the atomic decomposition and the duality theory of the variable Hardy space. In the context of the weighted Hardy space with variable exponent, K-P. Ho investigated in [38] the weighted HardyMorrey spaces, then in [39], he studied the weighted variable Hardy spaces $H_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ and he introduced a new class of weights with variable exponent $W_{p(\cdot)}$. K-P. Ho in his work [39], by applying the extrapolation theory was able to obtain the Fefferman-Stein vector-valued maximal inequalities on the weighted variable Lebesgue spaces $L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, where $w$ belongs to the new class $W_{p(\cdot)}\left(\mathbb{R}^{n}\right)$, then he established the atomic decomposition of the weighted Hardy spaces with variable exponents (see the next chapter for the definition of $W_{p(\cdot)}\left(\mathbb{R}^{n}\right)$ ).

In recent years, there has been a lot of attention paid to the study of Hardy spaces associated with different operators, which is a very active research topic in harmonic analysis. X.T Duong and L. Yan [24] investigated the duality between the Hardy and the BMO space associated with operators fulfill the heat kernel bounds. Thereafter, S. Hofmann and S. Mayborod [41] explored the Hardy and BMO spaces associated to divergence elliptic operators. Subsequently, X. T. Duong and J. Li [23] studied the Hardy spaces associated to operators satisfying Davies-Gaffney estimates and bounded holomorphic functional calculus. The notion of the Davies-Gaffney estimates (or the so-called $L^{2}$ off-diagonal estimates) of the semigroup $\left\{e^{-t L}\right\}_{t \geq 0}$ was first introduced by Gaffney [46] and Davies [20], which serves as good generalization of the Gaussian upper bound of the associated heat kernel. Regarding the variable Hardy spaces, Yang and Zhuo [96] introduced the space $H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ associated with operators $L$ on $\mathbb{R}^{n}$, where $p(\cdot): \mathbb{R}^{n} \rightarrow(0,1]$ is a measurable function satisfying the globally log-Hölder continuous condition and $L$ is a linear operator on $L^{2}\left(\mathbb{R}^{n}\right)$, which generates an analytic semigroup $\left\{e^{-t L}\right\}_{t \geq 0}$ whose kernels have pointwise upper bounds. Furthermore,
as an applications Yang and Zhuo [96] studied the boundedness of the fractional integral on these Hardy spaces and the coincidence between the spaces $H_{L}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ and the variable exponent Hardy spaces $H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$. As a generalization of the results obtained in [96], Yang et al. [92] considered the variable Hardy spaces $H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ associated with operator $L$, which obeys the Davies-Gaffney estimates. More generally, Zuo et al. [99] investigated the variable Hardy-Lorentz spaces associated with operators satisfying Davies-Gaffney estimates, it deserves to point out that these results obtained in [99] are new even when the variable exponent $p(\cdot)$ is a constant. Many research works were devoted to the study of the Hardy spaces associated with different operators, we refer the reader to $[40,52,93]$ and the references therein.

The Hardy-Lorentz spaces $H^{p, q}\left(\mathbb{R}^{n}\right)$ can be seen as interpolators between the Lorentz spaces and the Hardy spaces, with the convenient parameters of the Lorentz spaces $L^{p, q}\left(\mathbb{R}^{n}\right)$ and the Hardy spaces $H^{p}\left(\mathbb{R}^{n}\right)$. In [27], Fefferman et al. investigated the real interpolation of the Hardy-Lorentz spaces $H^{p, q}\left(\mathbb{R}^{n}\right)$. Fefferman and Soria [28] proved the atomic decomposition of Hardy-Lorentz space with $p=1$ and $q=\infty$. This result was extended later by Abu-Shammala and Torchinsky [1] to the space $H^{p, q}\left(\mathbb{R}^{n}\right)$, where they have established its atomic characterizations, and proved the boundedness of singular integrals for $p \in(0,1]$ and $q \in(0, \infty]$. Recently, Grafakos and He [47] established various results for the WeakHardy space, corresponding to the case $H^{p, \infty}$, with $p \in(0, \infty)$. Lately, L. Jun et al.[59] investigated the anisotropic Hardy-Lorentz spaces, thereafter, L. Jun et al. [60] studied the Littlewood-Paley characterizations of anisotropic Hardy-Lorentz spaces.

As a generalization for the classical weak Hardy-Lorentz spaces, Yan et al. [90] introduced the variable weak Hardy space on $\mathbb{R}^{n}$, via the radial grand maximal function, and they have established its radial or non-tangential maximal function characterizations (they have used the notation $W H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ instead of $H^{p(\cdot), \infty}\left(\mathbb{R}^{n}\right)$ ). Moreover, they have also obtained various equivalent characterizations of the $W H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, by means of atoms, molecules, the Lusin area function, and the Littlewood-Paley $g$-function or $g_{\lambda}^{*}$-function, respectively. As an application of these results, Yan et al. [90] established the boundedness of convolutional $\delta$-type and non-convolutional $\gamma$-order Calderón-Zygmund operators from $H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ to $W H^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ including the critical case when $p_{-}=n /(n+\delta)$ or when $p_{-}=n /(n+\gamma)$, where $p_{-}:=\operatorname{essinf}_{x \in \mathbb{R}^{n}} p(x)$. J. Liu et al. [58] investigated the anisotropic variable Hardy-Lorentz spaces. Thereafter, Jiao et al. in [32] were able to extend the results stated in [90] to the variable Hardy-Lorentz spaces $H^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, with $q \in(0, \infty]$. In addition,
they have obtained a new John-Nirenberg theorem, and the boundedness of singular operators on $H^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$. Recently, J. Liu [61] studied the Littlewood-Paley and finite atomic characterizations of anisotropic variable Hardy-Lorentz spaces. More recently, K. Saibi [80] established the intrinsic square function characterizations of the variable Hardy-Lorentz spaces and X. Liu et al. [62] proved the molecular characterization of anisotropic HardyLorentz spaces with variable exponent.

In line with the above works, this thesis aims to study a class of function spaces with variable exponent. More precisely, in Chapter 2, we introduce and study the weighted Hardy spaces $H_{w}^{p(\cdot)}(\Omega)$ on a proper open subsets, more specifically, we assume that $\Omega$ is a proper open subset of $\mathbb{R}^{n}$ and $p(\cdot)$ is a variable exponent with $p: \Omega \rightarrow(0, \infty)$. We first introduce the weighted hardy spaces, then we characterize these spaces via the grand maximal function and we establish the atomic decomposition of these spaces. Moreover, we introduce the weighted variable Hölder spaces $\Lambda_{w}^{p(\cdot), q, d}(\Omega)$ over the set $\Omega$, then we show that $\Lambda_{w}^{p(\cdot), q, d}(\Omega)$ represents the dual space of the variable weighted Hardy space $H_{w}^{p(\cdot)}(\Omega)$.

Chapter 3 is devoted to the study of the weighted variable Hardy $H_{w, L}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ associated with operators, we assume that this kind of operator obeys the Davies-Gaffney estimates. We start by investigating the molecular characterization of these spaces by mean of the atomic decomposition of the weighted tent spaces, then as an application of this molecular characterization, we prove that $B M O_{L^{*}, w}^{p(\cdot), M}$ is the dual space of the variable weighted Hardy space $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, where $L^{*}$ denotes the adjoint operator of $L$ on $L^{2}\left(\mathbb{R}^{n}\right)$.

The last Chapter is concerned with the study of the weighted Hardy-Lorentz spaces with variable exponents. In particular, we introduce the weighted Hardy-Lorentz spaces $H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, where $q \in(0, \infty]$ via the radial or non-tangential maximal functions. Thereafter, we define the variable weighted atomic Hardy-Lorentz spaces $H_{w, \text { atom, }, s}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ by means of $(p(\cdot), r, s)$-atoms (see definition 4.1.2). As first step, we prove the boundedness of the Hardy-Littelwood maximal operator on the variable weighted Lorentz spaces by using an interpolation theorem of sublinear operators. Next, under the assumption that $p(\cdot)$ satisfies the log-Hölder condition, $q \in(0, \infty]$ and $r \in\left(\left(k_{w}^{1 / b}\right)^{\prime}, \infty\right]$ such that $\frac{1}{b} \in \mathrm{~S}_{w}$ (see definition 1.2.2) we prove the following identity,

$$
H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)=H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right),
$$

with equivalent quasi-norms.

## List of notations

Throughout this thesis the following notations will be used.

- $C$ stands for a positive constant which may be different from line to line.
- $A \lesssim B$ means that there exists a positive constant $C$ such that $A \leq C B$.
- The symbol $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.
- For an open set $\Omega \subset \mathbb{R}^{n}$, we denote by $\mathcal{D}(\Omega)$ the set of infinitely differentiable functions with compact supports in $\Omega$.
- $\mathcal{D}^{\prime}(\Omega)$ is the topological dual space of $\mathcal{D}(\Omega)$ equipped with the weak $-*$ topology.
- For a measurable subset $\Omega \subset \mathbb{R}^{n}$ we denote by $|\Omega|$ and $\chi_{\Omega}$ the Lebesgue measure of $\Omega$ and the characteristic function of $\Omega$, respectively.
- We denote by $\mathbb{N}$ the set $\{1,2, \cdots\}$ and by $\mathbb{Z}_{+}$the set $\mathbb{N} \cup\{0\}$.
- The space $\mathcal{S}\left(\mathbb{R}^{n}\right)$ denotes the space of all Schwartz functions.
- The space $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ denotes the topological dual space of $\mathcal{S}\left(\mathbb{R}^{n}\right)$ called also by (the space of tempered distributions).


## CHAPTER 1

## PRELIMINARIES AND BASIC PROPERTIES

In this chapter, we give some definitions, notations and some mathematical backgrounds on the variable Lebesgue spaces, Lorentz spaces with variable exponents, the weighted variable Lebesgue spaces, weighted variable Lorentz spaces and compile some useful lemmas.

### 1.1 Variable exponent Lebesgue space

A measurable function $p(\cdot): \Omega \rightarrow(0, \infty)$ is called a variable exponent. For a variable exponent $p(\cdot)$, we define

$$
p_{-}=\operatorname{essinf}_{x \in \Omega} p(x) \quad \text { and } \quad p_{+}=\operatorname{esssup}_{x \in \Omega} p(x)
$$

and by $\mathcal{P}(\Omega)$ we denote the collection of all variable exponents such that $0<p_{-} \leq p_{+}<\infty$.
Example 1.1.1. Here two examples for the variable exponents:
(i) $\Omega=\mathbb{R}$ and $p(x)=2+\sin (x)$,
(ii) $\Omega=(1, \infty)$ and $p(x)=x$.

We define some notations to describe the range of exponent functions. Let $p(\cdot) \in \mathcal{P}(\Omega)$
and a set $E \subset \Omega$, let

$$
p_{-}(E)=\operatorname{ess}_{\inf _{x \in E}} p(x), \quad p_{+}(E)=\operatorname{ess} \sup _{x \in E} p(x)
$$

For the sake of simplicity, we write $p_{-}=p_{-}(\Omega)$, and $p_{+}(\Omega)=p_{+}$. We define three canonical subsets of :

$$
\begin{gathered}
\Omega_{\infty}^{p(\cdot)}=\{x \in \Omega: p(x)=+\infty\} \\
\Omega_{1}^{p(\cdot)}=\{x \in \Omega: p(x)=1\} \\
\Omega_{*}^{p(\cdot)}=\{x \in \Omega: p(x) \in(1, \infty)\} .
\end{gathered}
$$

We denote by $p^{\prime}(\cdot)$ the conjugate of variable exponent $p(\cdot)$, where

$$
\frac{1}{p(\cdot)}+\frac{1}{p^{\prime}(\cdot)}=1
$$

The notation $p^{\prime}$ will also be used to denote the conjugate of a constant exponent. We say that the variable exponent $p(\cdot)$ fulfills the globally log-Hölder continuity condition and we write $p(\cdot) \in C^{\log }(\Omega)$, if there exists a constants $C_{p(\cdot)}, C_{\infty}$ and $p_{\infty}$ such that, for any $x, y \in \mathbb{R}^{n}$,

$$
\begin{gathered}
|p(x)-p(y)| \leq \frac{C_{p(\cdot)}}{\log \left(e+\frac{1}{|x-y|}\right)}, \quad x \neq y \\
\left|p(x)-p_{\infty}\right| \leq \frac{C_{\infty}}{\log (e+|x|)}
\end{gathered}
$$

To define the Lebesgue spaces with variable Lebesgue spaces, we need first to define the following modular function,

Definition 1.1.2. Given $\Omega, p(\cdot) \in \mathcal{P}(\Omega)$ and a Lebesgue measurable function $f$, define the modular associated with $p(\cdot)$ by

$$
\rho_{p(\cdot), \Omega}(f):=\int_{\Omega \backslash \Omega_{\infty}}|f(x)|^{p(x)} d x+\|f\|_{L^{\infty}\left(\Omega_{\infty}\right)} .
$$

If $f \notin L^{\infty}\left(\Omega_{\infty}\right)$ or $f^{p(\cdot)} \notin L^{1}\left(\Omega \backslash \Omega_{\infty}\right)$, we define $\rho_{p(\cdot), \Omega}(f)=\infty$. When $\left|\Omega_{\infty}\right|=0$, in particular, when $p_{+}<+\infty$, we let $\|f\|_{L^{\infty}\left(\Omega_{\infty}\right)}=0$, when $\left|\Omega \backslash \Omega_{\infty}\right|$, then $\rho_{p(\cdot)}(f)=\|f\|_{L^{\infty}(\Omega)}$. In the
situation, when there is no confusing we write $\rho_{p(\cdot)}(f)$ or $\rho(f)$ instead of $\rho_{p(\cdot), \Omega}(f)$.
Now, we state some properties of Lebesgue spaces with variable exponent. For more details, see for example [18] and [21].

Proposition 1.1.3. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. Then the following assertions hold true :
(1) For all $f, \rho(f) \geq 0$ and $\rho(|f|)=\rho(f)$,
(2) $\rho(f)=0$ if and only if $f(x)=0$ for a.e. $x \in \Omega$,
(3) if $\rho(f)<+\infty$, then $f(x)<\infty$ for a.e. $x \in \Omega$,
(4) the modular $\rho$ is convex, i.e. for $t>0$,

$$
\rho(t f+(1-t) g) \leq t \rho(f)+(1-t) \rho(g)
$$

(5) the modular $\rho$ is order preserving, i.e. if $|f(x) \leq|g(x)|$ a.e. then $\rho(f) \leq \rho(g)$,
(6) the modular $\rho$ has the continuity property. i.e. If $\rho(f / \sigma)<+\infty$, for some $\sigma>0$. Then the function $\lambda \rightarrow \rho(f / \lambda)$ is continuous and decreasing on $[\sigma, \infty)$.

After we have given definition and some properties of the modular $\rho$, we are now ready to provide the definition of the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\Omega)$.

Definition 1.1.4. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$.
Define the the space $L^{p(\cdot)}(\Omega)$ to be the set of Lebesgue measurable functions $f$ such that $\rho(f / \lambda)<+\infty$ for some $\lambda>0$.

Define the space $L_{\text {loc }}^{p(\cdot)}(\Omega)$ to be the set of measurable functions $f$ such that $f \in L_{\text {loc }}^{p(\cdot)}(K)$ for every compact set $K \subset \Omega$.

Remark 1.1.5. According to (3)-proposition 1.1.3, if $f \in L^{p(\cdot)}(\Omega)$ then $f$ is finite almost everywhere.

Example 1.1.6. Let $\Omega=(1, \infty), p(x)=x$ and $f(x)=1$. We can easily see that $\rho(f)=+\infty$, however

$$
\rho(f / \lambda)=\int_{1}^{\infty} e^{-x \log \lambda} d x=\frac{1}{\lambda \log \lambda}<+\infty .
$$

On the other hand, if $\Omega=(0,1), p(x)=\frac{1}{x}$ and $f(x)=1$. Then $\rho(f)<\infty$, but $\rho(f / \lambda)=$ $+\infty$ for all $\lambda<1$.

This condition is needed for the case, when $p(\cdot)$ is unbounded function. When $p_{+}(\Omega \backslash$ $\left.\Omega_{\infty}\right)<\infty$ then the space $L^{p(\cdot)}(\Omega)$ coincides with the set of functions such that $\rho(f)$ is finite.

The next proposition gives a relation between the modular function and the variable Lebesgue spaces which can be founded in [18], and for the convenience of the reader we give the proof here.

Proposition 1.1.7. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. Then $f \in L^{p(\cdot)}(\Omega)$ if and only if $\rho(f)<$ $\infty$ is equivalent to assuming that $p_{-}=\infty$ or $p_{+}\left(\Omega \backslash \Omega_{\infty}\right)<+\infty$.

Proof. In this proof we deal with the following implication :

$$
\rho(f) \Rightarrow p_{-}=\infty \text { or } p_{+}\left(\Omega \backslash \Omega_{\infty}\right)<+\infty
$$

Let $f \in L^{p(\cdot)}(\Omega)$, since the modular $\rho$ is order preserving then, $\rho(f / \lambda)<\rho(f)$ for some $\lambda>1$. Thus

$$
\rho(f)=\int_{\Omega \backslash \Omega_{\infty}}\left(\frac{|f(x)| \lambda}{\lambda}\right)^{p(x)} d x+\lambda\|f / \lambda\|_{L^{\infty}\left(\Omega_{\infty}\right)} \leq \lambda^{p_{+}\left(\Omega \backslash \Omega_{\infty}\right)} \rho(f / \lambda)<\infty .
$$

Now we assume that, $p_{-}<\infty$ and $p_{+}\left(\Omega \backslash \Omega_{\infty}\right)=\infty$. We construct a function $f$ such that $\rho(f)=\infty$ but $f \in L^{p(\cdot)}(\Omega)$. By the definition of essential supremum, there exists a decreasing sequence of sets $\left\{E_{k}\right\}_{k \in \mathbb{N}}$ with finite measure such that:
(i) $E_{k} \subset \Omega \backslash \Omega_{\infty}$,
(ii) $E_{k+1} \subset E_{k}$ and $\left|E_{k} \backslash E_{k+1}\right|>0$,
(iii) $\left|E_{k}\right| \rightarrow 0$,
(iv) if $x \in E_{k}, p(x) \geq p_{k}>k$.

Define the function $f$ by

$$
f(x):=\left(\sum_{k=1}^{\infty} \frac{1}{\left|E_{k} \backslash E_{k+1}\right|} \int_{E_{k} \backslash E_{k+1}} \chi_{E_{k} \backslash E_{k+1}}(x)\right)^{\frac{1}{p(x)}}
$$

Then, from the assumption (iv) we get, for any $\lambda>1$

$$
\begin{aligned}
\rho(f / \lambda) & =\sum_{k=1}^{\infty} \frac{1}{\left|E_{k} \backslash E_{k+1}\right|} \int_{E_{k} \backslash E_{k+1}} \lambda^{-p(x)} d x \\
& \leq \sum_{k=1}^{\infty} \lambda^{-k}<\infty
\end{aligned}
$$

By the same computation, we can find that $\rho(f)=\infty$. Thus the proof is achieved.
Proposition 1.1.8. [18] Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. If $p_{+}\left(\Omega \backslash \Omega_{\infty}\right)<\infty$, then for all $\lambda \geq 1$,

$$
\rho(\lambda f) \leq \lambda^{p_{+}\left(\Omega \backslash \Omega_{\infty}\right)} \rho(f)
$$

Furthermore, if $p_{+}<\infty$ and $\lambda \geq 1$, then

$$
\lambda^{p_{-}} \rho(f) \leq \rho(\lambda f) \leq \lambda^{p_{+}} \rho(f)
$$

Remark 1.1.9. If $\lambda \in(0,1)$, then the reverse inequalities are true.
The newt theorem shows that the variable Lebesgue spaces is a vector space, see e.g [18]. The proof is given here for the convenience of the reader.

Theorem 1.1.10. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega), L^{p(\cdot)}(\Omega)$ is a vector space.
Proof. We know that the set of Lebesgue measurable functions is itself a vector space, and since 0 belongs to $L^{p(\cdot)}(\Omega)$, it suffices to show that

$$
f, g \in L^{p(\cdot)}(\Omega) \Rightarrow \alpha f+\beta g \in L^{p(\cdot)}(\Omega), \quad \forall \alpha, \beta \in \mathbb{R}^{*}
$$

Let $\mu=(|\alpha|+|\beta|) \lambda$, for some $\lambda>0$.
From (1) and (5) in Proposition 1.1.3, we get

$$
\begin{aligned}
\rho\left(\frac{\alpha f+\beta g}{\mu}\right) & =\rho\left(\frac{|\alpha f+\beta g|}{\mu}\right) \\
& \leq \rho\left(\frac{|\alpha|}{|\alpha|+|\beta|}(|f| / \lambda)+\frac{|\beta|}{|\alpha|+|\beta|}(|g| / \lambda)\right)
\end{aligned}
$$

We deduce from the convexity of the modular $\rho$,

$$
\rho\left(\frac{\alpha f+\beta g}{\mu}\right) \leq \frac{|\alpha|}{|\alpha|+|\beta|} \rho(|f| / \lambda)+\frac{|\beta|}{|\alpha|+|\beta|} \rho(|g| / \lambda)<\infty .
$$

## Luxemburg-Nakano Type Norm.

In the case of real variable Lebesgue spaces $L^{p}(\Omega)$, for $p \in[1, \infty]$ the norm derived directly from the modular

$$
\begin{equation*}
\|f\|_{L^{p}(\Omega)}=\left(\int_{\Omega}|f(x)|^{p} d x\right)^{1 / p} \tag{N}
\end{equation*}
$$

Due to the presence of the power $1 / p$ in (N) we cannot adopt this definition to the case of variable Lebesgue spaces, via replacing $1 / p$ by a variable function $1 / p(x)$. Then we need to use the so-called Luxemburg quasi-norm, also known as Luxemburg-Nakano quasi-norm.

Definition 1.1.11. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. For a measurable function $f$, we define

$$
\begin{equation*}
\|f\|_{L^{p(\cdot)}(\Omega)}:=\inf \left\{\lambda>0: \rho_{p(\cdot), \Omega}(f / \lambda) \leq 1\right\} \tag{LN}
\end{equation*}
$$

The next Lemma gives some characterizations for the Luxemburg quasi-norm, we refer the reader to the books $[18,21]$, here we give the proof for the convenience of the reader.

Lemma 1.1.12. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. The function $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ defines a norm on $L^{p(\cdot)}(\Omega)$. i.e.
(i) $\|f\|_{L^{p(\cdot)}(\Omega)}=0$ if and only if $f \equiv 0$,
(ii) for all $\gamma \in \mathbb{R},\|\gamma f\|_{L^{p(\cdot)}(\Omega)}=|\gamma|\|f\|_{L^{p(\cdot)}(\Omega)}$,
(iii) $\|f+g\|_{L^{p(\cdot)}(\Omega)} \leq\|f\|_{L^{p(\cdot)}(\Omega)}+\|g\|_{L^{p(\cdot)}(\Omega)}$.

Proof. We start by proving (i). When $f \equiv 0$, then $\rho(f / \lambda)=0 \leq 0$, consequently $\|f\|_{L^{p(\cdot)}(\Omega)}=$ 0 . Conversely, we suppose that $\|f\|_{L^{p(\cdot)}(\Omega)}=0$, then for all $\lambda>0$,

$$
\rho(f / \lambda)=\underbrace{\int_{\Omega \backslash \Omega_{\infty}}|f(x) / \lambda|^{p(x)} d x}_{I}+\underbrace{\|f / \lambda\|_{L^{\infty}(\Omega)}}_{I I} \leq 1
$$

We observe that $I I \leq 1$ thus $\|f\|_{L^{\infty}\left(\Omega_{\infty}\right)} \leq \lambda$, thus $f(x)=0$ for a.e. $x \in \Omega_{\infty}$. If $\lambda<1$, by virtue of Proposition 1.1.8, we have

$$
\lambda^{-p-} \int_{\Omega \backslash \Omega_{\infty}}|f(x)|^{p(x)} d x \leq I \leq 1
$$

Hence,

$$
\int_{\Omega \backslash \Omega_{\infty}}|f(x)|^{p(x)} d x \leq \lambda^{p_{-}}
$$

Consequently, $\left\|f^{p(\cdot)}\right\|_{L^{1}\left(\Omega \backslash \Omega_{\infty}\right)}=0$, which means that $f(x)=0$ for a.e. $x \in \Omega \backslash \Omega_{\infty}$. Now we turn to prove (ii). Let $\gamma \in \mathbb{R}^{*}$, then we have

$$
\|\gamma f\|_{L^{p(\cdot)}(\Omega)}=\inf \{\lambda>0: \rho(|\gamma| f / \lambda) \leq 1\}
$$

by a change of variable, we infer that

$$
\begin{aligned}
\|\gamma f\|_{L^{p(\cdot)}(\Omega)} & =|\gamma| \inf \left\{\frac{\lambda}{|\gamma|}>0: \rho(|\gamma| f /(\lambda /|\gamma|)) \leq 1\right\} \\
& =|\gamma| \inf \{\delta>0: \rho(|\gamma| f / \delta)) \leq 1\}
\end{aligned}
$$

Finally, we move to prove (iii). Fix $\lambda_{f}>\|f\|_{L^{p(\cdot)}(\Omega)}$ and $\lambda_{g}>\|g\|_{L^{p(\cdot)}(\Omega)}$, then $\rho\left(f / \lambda_{f}\right) \leq 1$ and $\rho\left(g / \lambda_{g}\right) \leq 1$. Let $\lambda=\lambda_{f}+\lambda_{g}$.
(3) -of Proposition 1.1.3 and the convexity of the modular $\rho$ leads to

$$
\rho\left(\frac{f+g}{\lambda}\right)=\rho\left(\frac{\lambda_{f} f}{\lambda_{f} \lambda}+\frac{\lambda_{g} g}{\lambda_{g} \lambda}\right) \leq \frac{\lambda_{f}}{\lambda} \rho\left(f / \lambda_{f}\right)+\frac{\lambda_{g}}{\lambda} \rho\left(g / \lambda_{g}\right) \leq 1
$$

Thus $\|f+g\|_{L^{p(\cdot)}(\Omega)} \leq \lambda_{f}+\lambda_{g}$. By taking the infimum over $\lambda_{f}$ and $\lambda_{g}$ we get (iii).
The next proposition plays an important role in the study of function spaces with variable exponent, see for example [21]. For the convenience of the reader we present the proof of the proposition.

Proposition 1.1.13. Given $\Omega$ and a function $p(\cdot)$ such that $\left|\Omega_{\infty}\right|=0$, then for all $s \in\left[\frac{1}{p_{-}}, \infty\right)$,

$$
\begin{equation*}
\left\||f|^{s}\right\|_{L^{p(\cdot)}(\Omega)}=\|f\|_{L^{s p(\cdot)}(\Omega)}^{s} \tag{1.1.1}
\end{equation*}
$$

Proof. By setting $\mu^{s}=\lambda$, then from the definition of the norm and the fact that $\left|\Omega_{\infty}\right|=0$, we infer that

$$
\begin{aligned}
\left\||f|^{s}\right\|_{L^{p(\cdot)}(\Omega)} & =\inf \left\{\lambda>0: \int_{\Omega}\left(\frac{|f(x)|^{s}}{\lambda}\right)^{p(x)} d x \leq 1\right\} \\
& =\inf \left\{\mu^{s}>0: \int_{\Omega}\left(\frac{|f(x)|}{\mu}\right)^{s p(x)} d x \leq 1\right\}=\|f\|_{L^{s p(\cdot)}(\Omega)}^{s}
\end{aligned}
$$

Remark 1.1.14. If $|\Omega|>0$ and the exponent $p(\cdot)$ is not identically infinite, then (1.1.1) do not always hold. Indeed, let $\Omega=[-1,1]$, and define

$$
p(x)=\left\{\begin{array}{cc}
1 & \text { if } \quad x \in[-1,0] \\
\infty & \text { if }
\end{array} \quad x \in(0,1]\right.
$$

and

$$
f(x)=\left\{\begin{array}{lll}
1 & \text { if } & x \in[-1,0] \\
2 & \text { if } & x \in(0,1]
\end{array}\right.
$$

and $s=2$, then we observe that

$$
\rho_{p(\cdot)}\left(f^{2} / \lambda\right)=\int_{-1}^{0} \frac{1}{\lambda} d x+2^{2} \frac{1}{\lambda}=\frac{5}{\lambda}
$$

thus $\left\|f^{2}\right\|_{L^{p(\cdot)}(\Omega)}=5$. But, on the other hand,

$$
\rho_{2 p(\cdot)}(f / \lambda)=\left(\frac{1}{\lambda}\right)^{2}+2\left(\frac{1}{\lambda}\right),
$$

if we solve the equation $\left(\frac{1}{\lambda}\right)^{2}+2\left(\frac{1}{\lambda}\right)-1=0$, we find out that $\|f\|_{L^{2 p(\cdot)}(\Omega)}^{2}=\left(\frac{1}{\sqrt{2}-1}\right)^{2}$ which is different to $\left\|f^{2}\right\|_{L^{p(\cdot)}(\Omega)}$.

The next results show some properties of the Luxemburg quasi-norm and the modular function.

Proposition 1.1.15. [18] Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$, if $f \in L^{p(\cdot)}(\Omega)$ and $\|f\|_{L^{p(\cdot)}(\Omega)}>0$ then $\rho\left(f /\|f\|_{L^{p(\cdot)}(\Omega)}\right) \leq 1$. Moreover, $\rho\left(f /\|f\|_{L^{p(\cdot)}(\Omega)}\right)=1$ for all non-trivial $f \in L^{p(\cdot)}(\Omega)$ if and only if $p_{+}\left(\Omega \backslash \Omega_{\infty}\right)$ is finite.

Corollary 1.1.16. [18] Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$.
(i) If $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$, then $\rho(f) \leq\|f\|_{L^{p(\cdot)}(\Omega)^{\prime}}$,
(ii) if $\|f\|_{L^{p(\cdot)}(\Omega)}>1$, then $\rho(f) \geq\|f\|_{L^{p(\cdot)}(\Omega)}$.

Furthermore, if we suppose that $\left|\Omega_{\infty}\right|=0$, then
(iii) if $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$, then $\rho(f)^{1 / p_{-}} \leq\|f\|_{L^{p(\cdot)}(\Omega)} \leq \rho(f)^{1 / p_{+}}$,
(iv) if $\|f\|_{L^{p(\cdot)}(\Omega)}>1$, then $\rho(f)^{1 / p_{+}} \leq\|f\|_{L^{p(\cdot)}(\Omega)} \leq \rho(f)^{1 / p_{-}}$.

## Hölder Inequality.

The Hölder inequality is one of important inequalities in the analysis of partial differential equations and harmonic analysis, then it would be reasonable to ask whether Hölder's inequality remains true in the case of variable exponent, more precisely, we have the following theorem.

Theorem 1.1.17. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. For all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p^{\prime}(\cdot)}(\Omega)$. Then $f g \in L^{1}(\Omega)$ and we have

$$
\begin{equation*}
\int_{\Omega}|f(x) g(x)| d x \leq C_{p(\cdot)}\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p^{\prime}(\cdot)}(\Omega)^{\prime}} \tag{1.1.2}
\end{equation*}
$$

where $C_{p(\cdot)}=\left(1+\frac{1}{p_{-}}-\frac{1}{p_{+}}\right)\left\|\chi_{\Omega_{*}}\right\|_{L^{\infty}(\Omega)}+\left\|\chi_{\Omega_{\infty}}\right\|_{L^{\infty}(\Omega)}+\left\|\chi_{\Omega_{1}}\right\|_{L^{\infty}(\Omega)}$.
The next result is the generalized Hölder inequality
Corollary 1.1.18. Given $\Omega$ and a functions $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$. Define $r(\cdot)$ by

$$
\frac{1}{r(x)}=\frac{1}{p(x)}+\frac{1}{q(x)} .
$$

Then, there exists a constant $C>0$ such that, for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then we have $f g \in L^{r(\cdot)}(\Omega)$ and

$$
\begin{equation*}
\|f g\|_{L^{(\cdot)}(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{g \cdot(\cdot)}(\Omega)} . \tag{1.1.3}
\end{equation*}
$$

For the proof of theorem 1.1.2 and corollary 1.1.3, we refer the reader to [17, 21].

Definition 1.1.19. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$.

$$
\pi f \|_{L^{p(\cdot)}(\Omega)}:=\sup _{g \in L^{p^{\prime}(\cdot)}(\Omega),\|g\|_{L^{p^{\prime}} \cdot()(\Omega)} \leq 1} \int_{\Omega} f(x) g(x) d x .
$$

Theorem 1.1.20. [18] Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. Then, $f \in L^{p(\cdot)}(\Omega)$ if and only if $\pi f \|_{L^{p(\cdot)}(\Omega)}$ is finite, moreover, we have

$$
c_{p(\cdot)}\|f\|_{L^{p(\cdot)}(\Omega)} \leq \tilde{\|} f\left\|_{L^{p(\cdot)}(\Omega)} \leq C_{p(\cdot)}\right\| f \|_{L^{p(\cdot)}(\Omega)^{\prime}}
$$

where $1 / c_{p(\cdot)}=\left\|\chi_{\Omega_{*}}\right\|_{L^{\infty}(\Omega)}+\left\|\chi_{\Omega_{\infty}}\right\|_{L^{\infty}(\Omega)}+\left\|\chi_{\Omega_{1}}\right\|_{L^{\infty}(\Omega)}$.

## Convergence and Completeness of $L^{p(\cdot)}(\Omega)$.

In this subsection, we consider the convergence in the Lebesgue spaces with variable exponent.

Definition 1.1.21. Given $\Omega, p(\cdot) \in \mathcal{P}(\Omega)$ and given a sequence of functions, $\left\{f_{k}\right\}_{k \in \mathbb{N}} \subset$ $L^{p(\cdot)}(\Omega)$.
(i) (convergence in modular) We say that $f_{k}$ converges to $f$ in modular if for some $\alpha>0$, $\rho\left(\alpha\left(f-f_{k}\right)\right) \rightarrow 0$ as $k \rightarrow \infty$,
(ii) (convergence in norm) We say that $f_{k}$ converges to $f$ in norm if $\left\|f-f_{k}\right\|_{L^{p(\cdot)}(\Omega)} \rightarrow 0$ as $k \rightarrow \infty$.

The presence of the constant $\alpha$ in the above definition is to preserve the homogeneity of convergence i.e. If $f_{k}$ converges to $f$ then $2 f_{k}$ converges also to $2 f$.

Proposition 1.1.22. Given $\Omega$, and a function $p(\cdot) \in \mathcal{P}(\Omega)$, the sequence of functions, $\left\{f_{k}\right\}_{k \in \mathbb{N}} \subset$ $L^{p(\cdot)}(\Omega)$ converges to $f$ in norm if and only if for every $\alpha>0, \rho\left(\alpha\left(f-f_{k}\right)\right) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. Assume that $f_{k}$ converges to $f$ in norm. Let $\alpha>0$. The homogeneity of the norm gives,

$$
\left\|\alpha\left(f_{k}-f\right)\right\|_{L^{p(\cdot)}(\Omega)}=\alpha\left\|f_{k}-f\right\|_{L^{p(\cdot)}(\Omega)} \underset{k \rightarrow \infty}{\rightarrow} 0
$$

By using corollary 1.1.16, for all $k$ sufficiently large, we obtain

$$
\rho\left(\alpha\left(f_{k}-f\right)\right) \leq \alpha\left\|f_{k}-f\right\|_{L^{p(\cdot)}(\Omega)} .
$$

Hence,

$$
\rho\left(\alpha\left(f_{k}-f\right)\right) \underset{k \rightarrow \infty}{\rightarrow} 0
$$

Conversely, let $\lambda>0$ and let $\alpha=\frac{1}{\lambda}$. Then for all $k$ sufficiently large, $\rho\left(\left(f_{k}-f\right) / \lambda\right) \leq 1$, thus $\left\|f_{k}-f\right\|_{L^{p(\cdot)}(\Omega)} \leq \lambda$. Since this inequality is true for any $\lambda>0$ we find out

$$
\left\|f_{k}-f\right\|_{L^{p(\cdot)}(\Omega)} \rightarrow \underset{k \rightarrow \infty}{ } 0 .
$$

The next result shows the relation between the convergence in norm and the convergence in modular. For more details we refer the reader to [18].

Theorem 1.1.23. Given $\Omega$, and a function $p(\cdot) \in \mathcal{P}(\Omega)$. Convergence in norm is equivalent to convergence in modular if and only if either $p_{-}=\infty$ or $p\left(\Omega \backslash \Omega_{\infty}\right)$ is finite.

The following two theorems represent respectively the monotone convergence theorem and the Fatou-like Lemma in the variable Lebesgue spaces.

Theorem 1.1.24. [18] Given $\Omega$, and a function $p(\cdot) \in \mathcal{P}(\Omega)$. Let $\left\{f_{k}\right\}_{k \in \mathbb{N}} \subset L^{p(\cdot)}(\Omega)$ be a sequence of non-negative functions such that $f_{k}$ increases to a function $f$ pointwise a.e. Then either $f \in L^{p(\cdot)}(\Omega)$ and $\left\|f_{k}\right\|_{L^{p(\cdot)}(\Omega)} \rightarrow\|f\|_{L^{p(\cdot)}(\Omega)}$ or $f \notin L^{p(\cdot)}(\Omega)$ and $\left\|f_{k}\right\|_{L^{p(\cdot)}(\Omega)} \rightarrow \infty$.
Theorem 1.1.25. [18] Given $\Omega$, and a function $p(\cdot) \in \mathcal{P}(\Omega)$. Let $\left\{f_{k}\right\}_{k \in \mathbb{N}} \subset L^{p(\cdot)}(\Omega)$ such that $f_{k}$ converges pointwise almost everywhere to $f$. If $\liminf _{k \rightarrow \infty}\left\|f_{k}\right\|_{L^{p(\cdot)}(\Omega)}<\infty$, then $f \in L^{p(\cdot)}(\Omega)$ and

$$
\|f\|_{L^{p(\cdot)}(\Omega)} \leq \liminf _{k \rightarrow \infty}\left\|f_{k}\right\|_{L^{p(\cdot)}(\Omega)}
$$

Now, we state an extension of the called Lebesgue's dominated convergence theorem from the classical Lebesgue spaces to the variable Lebesgue spaces. For the proof we refer the reader to [18].

Theorem 1.1.26. Given $\Omega, p(\cdot) \in \mathcal{P}(\Omega)$, and assume that $p_{+}<\infty$. If the sequence $f_{k}$ converges pointwise almost everywhere to $f$ and there exists a function $g \in L^{p(\cdot)}(\Omega)$ such that $\left|f_{k}(x)\right| \leq g(x)$ a.e. Then $f \in L^{p(\cdot)}(\Omega)$ and

$$
\left\|f-f_{k}\right\|_{L^{p(\cdot)}(\Omega)} \rightarrow \underset{k \rightarrow \infty}{ } 0 .
$$

If $p_{+}=\infty$, then this result is always false.

The following result gives the relation between the convergence in norm and the convergence in measure.

Theorem 1.1.27. [18] Given $\Omega$, and a function $p(\cdot) \in \mathcal{P}(\Omega)$, If the sequence $\left\{f_{k}\right\}_{k \in \mathbb{N}} \subset L^{p(\cdot)}(\Omega)$ converges to $f$ in norm, then it converges in measure.

As in the case of the real variable Lebesgue spaces $L^{p}(\Omega)$, we have the following completeness theorem, for further details we refer the reader to [18].
Theorem 1.1.28. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. The variable Lebesgue space $L^{p(\cdot)}(\Omega)$ is complete.

Remark 1.1.29. From the fact that the modular $\rho$ is order preserving and the above theorem, then the space $L^{p(\cdot)}(\Omega)$ is ideal.

Theorem 1.1.30. Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. Suppose that $p_{+}<\infty$. Then the set of bounded functions of compact support with suppf $\Subset \Omega$ is dense in $L^{p(\cdot)}(\Omega)$.

The next theorem shows the non-density result of the set of bounded functions in the variable Lebesgue spaces.

Theorem 1.1.31. [18] Given $\Omega$ and a function $p(\cdot) \in \mathcal{P}(\Omega)$. If $p_{+}\left(\Omega \backslash \Omega_{\infty}\right)=\infty$, then the set of bounded functions is not dense in $L^{p(\cdot)}(\Omega)$.

Remark 1.1.32. It is well-known that the variable exponents Lebesgue space is a special case of Musielak-Orlicz spaces (see [73]).

## The Hardy-Littlewood Maximal Operator

The maximal function $M f$ represents the largest average value of $f$ at each point. In particular, we have the following definition

Definition 1.1.33. Given a function $f \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$, then $M f$, the Hardy-Littlewood maximal function of $f$, is defined for all $x \in \Omega$

$$
M(f)(x):=\sup _{B \ni x} \frac{1}{|B|} \int_{B}|f(y)| d y,
$$

where the supremum is taken over all balls $B$ of $\Omega$ containing $x$.
The following result gives some properties of the Hardy-Littlewood maximal operator, see for example $[21,19]$.

Proposition 1.1.34. The Hardy-Littlewood maximal operator has the following properties
(1) The operator $M$ is sublinear, i.e.

$$
M(f+g)(x) \leq M f(x)+M g(x)
$$

(2) the operator $M$ is homogeneous, i.e. for all $\alpha \in \mathbb{R}, M(\alpha f)(x)=|\alpha| M f(x)$,
(3) for all $f,|f(x)| \leq M f(x)$,
(4) if $f \in L^{\infty}\left(\mathbb{R}^{n}\right)$, then $M f \in L^{\infty}\left(\mathbb{R}^{n}\right)$ and $\|M f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}=\|f\|_{L^{\infty}\left(\mathbb{R}^{n}\right)}$,
(5) if $f(x) \neq 0$ on a set of positive measure, then on any bounded set $\Omega$ there exists $\varepsilon>0$ such that $M f(x) \geq \varepsilon, x \in \Omega$,
(5) if $f(x) \neq 0$ on a set of positive measure, then $M f \notin L^{1}\left(\mathbb{R}^{n}\right)$.

Theorem 1.1.35. The function $M f$ is lower semi-continuous and therefore measurable.
The following Theorem gives some classical norm inequalities for the Hardy-Littlewood maximal operator.

Theorem 1.1.36. Given $f \in L^{p}\left(\mathbb{R}^{n}\right), 1 \leq p<\infty$, for every $t>0$,

$$
\left|\left\{x \in \mathbb{R}^{n}: M f(x)>t\right\}\right| \leq \frac{C}{t^{p}} \int_{\mathbb{R}^{n}}|f(x)|^{p} d x
$$

Further, if $p \in(1, \infty]$, then

$$
\|M f\|_{L^{p}\left(\mathbb{R}^{n}\right)} \leq C_{1}\|f\|_{L^{p}\left(\mathbb{R}^{n}\right)}
$$

The next lemma shows the boundedness of the Hardy-Littlewood maximal operator on variable Lebesgue spaces, we refer the reader to [20].

Lemma 1.1.37. Let $p \in C^{\log }\left(\mathbb{R}^{n}\right)$ with $1<p_{-}<p_{+}<\infty$. Then there exists a positive constant $C$ such that for all $f \in L^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ and we have

$$
\|M f\|_{L^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{L^{p(\cdot)}\left(\mathbb{R}^{n}\right)}
$$

### 1.2 Variable weighted Lebesgue space

In this section, we recall the definitions and some properties of the weighted variable Lebesgue spaces. We start with the definition of the Muckenhoupt class of weight functions.

Definition 1.2.1. For $p \in(1, \infty)$. A locally integrable function $w: \Omega \rightarrow(0, \infty)$ is said to be an

- $A_{p}$-weight if

$$
[w]_{A_{p}}:=\sup _{B \in B(z, r)}\left(\frac{1}{|B|} \int_{B} w(x) d x\right)\left(\frac{1}{|B|} \int_{B} w(x)^{-\frac{p^{\prime}}{p}} d x\right)^{p / p^{\prime}}<\infty .
$$

- $A_{1}$-weight, if for all balls $B$

$$
\frac{1}{|B|} \int_{B} w(x) d x \leq C w(x), \quad \text { a.e. } x \in B
$$

for some a constant $C>0$.

- for the case of $p=\infty$ we have the following definition of the $A_{\infty}$-weight,

$$
A_{\infty}:=\cup_{p \geq 1} A_{p}
$$

Example 1.2.2. Here we give a classical examples for the Muckenhoupt class of weight. The function $w(x)=|x|^{\sigma}$ is an $A_{p}$-weight if and only if $\sigma \in(-n, n(p-1))$. Another classical example is $w(x)=d^{\gamma}(x, \partial \Omega)$, where for $x \in \Omega$ the function $d$ denotes the distance from the point $x$ and the boundary $\partial \Omega$, this function $d$ belongs to $A_{2}$ if and only if $\gamma \in(-n, n)$.

Now we recall the definition of the weighted variable exponent Lebesgue space $L_{w}^{p(\cdot)}(\Omega)$ is defined as

$$
L_{w}^{p(\cdot)}(\Omega):=\left\{f: \Omega \rightarrow \mathbb{R}^{m}: \rho_{p(\cdot), w}(f)=\int_{\Omega}|f(x) w(x)|^{p(x)} d x<\infty\right\}
$$

For a function $f \in L_{w}^{p(\cdot)}(\Omega)$, define

$$
\|f\|_{L_{w}^{p(\cdot)}(\Omega)}:=\inf \left\{\lambda>0: \rho_{p(\cdot), w}\left(\frac{f}{\lambda}\right) \leq 1\right\}
$$

In particular, if $w=1$, it is well known that $L_{w}^{p(\cdot)}(\Omega)=L^{p(\cdot)}(\Omega)$ and if $p(\cdot)$ is a constant, i.e., $p(\cdot) \equiv p$, then $L_{w}^{p(\cdot)}(\Omega)$ is the classical weighted Lebesgue space $L_{w}^{p}(\Omega)$.

Next, we give a definition of a class of weights which is more general compared with the Muckenhoupt class of weight.

Definition 1.2.3. Let $p(\cdot): \Omega \rightarrow(0, \infty)$ be a measurable function such that $0<p_{-} \leq p_{+}<\infty$ and $w: \Omega \rightarrow(0, \infty)$ be a Lebesgue measurable function. We denote by $W_{p(\cdot)}(\Omega)$ the set of all Lebesgue measurable functions $w$ such that
(i) $\left\|\chi_{B}\right\|_{L_{w \underline{p}}^{p(\cdot) / p}}{ }_{(\Omega)}<\infty$ and $\left\|\chi_{B}\right\|_{L_{w^{-} \underline{p}}^{\left(\frac{p(\cdot)}{p}\right)^{\prime}}(\Omega)}<\infty$, for any $B \in \mathbb{B}$, where $\underline{p}=\min \left\{p_{-}, 1\right\}$;
(ii) there exists $k>1$ and $s>1$ such that the Hardy-Littlewood maximal operator is bounded on $L_{w^{-k / s}}^{(s p(\cdot))^{\prime} / k}(\Omega)$.

We introduce the following indices which will be used later. For any $w \in W_{p(\cdot)}$, we set

$$
\begin{equation*}
s_{w}=\inf \left\{s \geq 1: M \text { is bounded on } L_{w^{-1 / s}}^{(s p(\cdot))^{\prime}}(\Omega)\right\} \tag{1.2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{S}_{w}=\left\{s \geq 1: M \text { is bounded on } L_{w^{-k / s}}^{(s p(\cdot))^{\prime} / k}(\Omega), \text { for some } k>1\right\} \tag{1.2.2}
\end{equation*}
$$

For any fixed $s \in \mathrm{~S}_{w}$, we define

$$
k_{w}^{s}:=\sup \left\{k>1: M \text { is bounded on } L_{w^{-k / s}}^{(s p(\cdot))^{\prime} / k}\left(\mathbb{R}^{n}\right)\right\} .
$$

The index $k_{w}^{s}$ is used to measure the left-openness of the boundedness of $M$ on the family $\left\{L_{w^{-k / s}}^{(s p(\cdot))^{\prime} / k}\left(\mathbb{R}^{n}\right)\right\}_{k>1}$.

Next, we give some basic examples of functions belonging to the class defined above.
Example 1.2.4. 1. Let $p(\cdot): \Omega \rightarrow(0, \infty)$ such that $p(x)=2$ for any $x \in \Omega$ and $w(x): \Omega \rightarrow$ $(0, \infty)$ such that $w(x)=|x|^{-\frac{1}{4}}$. Then, one can easily show that $w$ satisfies the (i) of Definition 1.2.3, and by the fact that $w^{2} \in A_{q}$ for $q>\frac{2 n+1}{2 n}$ and [39, Proposition 2.4], we know that $w$ satisfies (ii) of Definition 1.2.3. Thus $w \in W_{p(\cdot)}(\Omega)$.
2. Suppose that $\Omega$ is an open bounded subset of $\mathbb{R}^{n}$. Let $p(\cdot): \Omega \rightarrow(0, \infty)$ defined by

$$
p(x)=\left\{\begin{array}{l}
\frac{2 \ln |x|+1}{3 \ln |x|+2} \quad \text { if } \quad|x| \geq e \\
\frac{3}{5} \quad \text { if } \quad|x|<e
\end{array}\right.
$$

and $w(x)=1+\operatorname{dist}(x, \partial \Omega)$. Obviously, $w$ satisfies the (1) of Definition 1.2.3, and by taking $s=k=2$ and using [75, Theorem 2.2] with a simple computation, we can show that

$$
\|M f\|_{L_{w-1}}^{\frac{(2 p(\cdot))^{\prime}}{2}} \leq C\|f\|_{L_{w-1}}^{\frac{(2 p(\cdot))^{\prime}}{2}}
$$

Thus $w \in W_{p(\cdot)}(\Omega)$.
Remark 1.2.5. Here we give some remarks on definition 1.2.3-(i) when the function exponent $p(\cdot)$ is a constant

- When $p(\cdot)=p, 1<p<\infty$, is a constant function, Definition 1.2.3-(i) is equivalent to the assumption that $w^{p}$ and $w^{-p^{\prime}}$ are locally integrable functions.
- When $p(\cdot)=p, 0<p \leq 1$, is a constant function, Definition 1.2.3-(i) is equivalent to the assumption that $w$ is locally integrable and $\frac{1}{w}$ is locally bounded.

Concerning definition 1.2.3-(ii), we have the following Proposition, for the proof see [39].
Proposition 1.2.6. Let $p \in(0, \infty)$, if $p(\cdot) \equiv p$, then a Lebesgue measurable function $w: \mathbb{R}^{n} \rightarrow$ $(0, \infty)$ fulfills (2) - in definition 1.2.3 if and only if $w^{p} \in A_{\infty}$.

For a general Lebesgue measurable function $p(\cdot): \mathbb{R}^{n} \rightarrow(0, \infty)$, we have the following result which guarantees that the weight $w$ satisfies the condition Definition 1.2.3-(i).

Lemma 1.2.7. Let $p(\cdot): \mathbb{R}^{n} \rightarrow(0, \infty)$, be a Lebesgue measurable function with $0<p_{-} \leq p_{+}<\infty$. If $w^{p_{+}}$is locally integrable, then for any $B \in \mathbb{B}$, the quantity $\left\|\chi_{B}\right\|_{L_{w} \frac{p(\cdot)}{\frac{p}{p}}}\left(\mathbb{R}^{n}\right)$ finite.
Proof. Since $w^{p_{+}}$is locally integrable, we have, for $B \in \mathbb{B}$

$$
\begin{aligned}
\rho_{p(\cdot) / \underline{p}}\left(\chi_{B} w^{p}\right)=\int_{B}(w(x))^{p(x)} d x & \leq|\{x \in B: w(x) \leq 1\}|+\int_{B} w(x)^{p_{+}} d x \\
& \leq|B|+\int_{B} w(x)^{p_{+}} d x \\
& <\infty
\end{aligned}
$$

Since the function $p(\cdot) / \underline{p}: \mathbb{R}^{n} \rightarrow[1, \infty)$, then [18, Proposition 2.12] ensures that $\chi_{B} w^{\underline{p}} \in$ $L^{p(\cdot) / \underline{p}}\left(\mathbb{R}^{n}\right)$, which implies that $\left\|\chi_{B}\right\|_{L_{w w^{\frac{p}{p}}}^{\frac{p(\cdot)}{p}}\left(\mathbb{R}^{n}\right)}<\infty$.

Remark 1.2.8. It is easy to see that $L_{w^{1 / s}}^{s p(\cdot)}(\Omega)$ is the $s$-convexification of $L_{w}^{p(\cdot)}(\Omega)$ and for any $f, g \in L_{w}^{p(\cdot)}(\Omega)$,

$$
\|f+g\| \frac{p}{L_{w}^{p(\cdot)}(\Omega)} \leq\|f\|_{L_{w}^{p(\cdot)}(\Omega)}^{\frac{p}{p}}+\|g\| \frac{L_{L_{w}}^{p(\cdot)}(\Omega)}{p} .
$$

Remark 1.2.9. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$, then we have,

$$
\left\||f|^{s}\right\|_{L_{w}^{p(\cdot)}}=\|f\|_{L_{w 1^{1 / s}}^{s p(\cdot)}}^{s}
$$

The following theorem gives the Fefferman-Stein vector valued maximal inequalities on $L_{w}^{p(\cdot)}(\Omega)$. For the proof, we refer to [39].

Theorem 1.2.10. Let $p(\cdot): \Omega \rightarrow(0, \infty)$ be a measurable function with $0<p_{-} \leq p_{+}<\infty$ and $q \in(1, \infty)$. If $w \in W_{p(\cdot)}(\Omega)$, then, for any $r>s_{w}$, we have

$$
\begin{equation*}
\left\|\left(\sum_{i \in \mathbb{N}}\left(M f_{i}\right)^{q}\right)^{1 / q}\right\|_{\substack{L_{w^{1 / r}}^{r p(\cdot)}(\Omega)}} \leq C\left\|\left(\sum_{i \in \mathbb{N}}\left|f_{i}\right|^{q}\right)^{1 / q}\right\|_{L_{w^{1 / r}}^{L^{r p(\cdot)}(\Omega)}} \tag{1.2.3}
\end{equation*}
$$

Remark 1.2.11. Let $\mu \in[1, \infty)$. Then by the above theorem and the fact that, for all balls $B \subset \mathbb{R}^{n}$ and $r \in\left(0, \min \left(\underline{p}, s_{w}\right)\right), \chi_{\mu B} \leq \mu^{\frac{n}{r}}\left(M\left(\chi_{B}\right)\right)^{\frac{1}{r}}$, we infer that there exists a positive constant $C$ such that, for any $\left\{B_{j}\right\}_{j \in \mathbb{N}}$ of balls of $\mathbb{R}^{n}$,

$$
\left\|\sum_{j \in \mathbb{N}} \chi_{\mu B_{j}}\right\|_{L_{w}^{p(\cdot)}(\Omega)} \leq C \mu^{\frac{n}{r}}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{j}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}
$$

We recall in the following lemma the Hölder inequality, for the proof see [39, Lemma 2.1].

Lemma 1.2.12. Let $p(\cdot): \mathbb{R}^{n} \rightarrow[1, \infty)$ be a measurable function and $w: \mathbb{R}^{n} \rightarrow(0, \infty)$ be a Lebesgue measurable function. Then

$$
\int_{\mathbb{R}^{n}}|f(x) g(x)| d x \leq 2\|f\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}\|g\|_{L_{w}^{p^{\prime}(\cdot)}\left(\mathbb{R}^{n}\right)}
$$

The following lemma presents the conjugate formula for $L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$.
Lemma 1.2.13. Let $p(\cdot): \mathbb{R}^{n} \rightarrow[1, \infty)$ be a measurable function and $w: \mathbb{R}^{n} \rightarrow(0, \infty)$ be a Locally
measurable function. Then

$$
\|f\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \approx \sup \left\{\int_{\mathbb{R}^{n}}|f(x) g(x)| d x: g \in L_{w^{-1}}^{p^{\prime}(\cdot)}\left(\mathbb{R}^{n}\right),\|g\|_{L_{w-1}^{p^{\prime}(\cdot)}\left(\mathbb{R}^{n}\right)} \leq 1\right\}
$$

### 1.3 Weighted Lorentz space with variable exponents

Before delving into the definition of the weighted variable Lorentz spaces, we first recall the definition of classical Lorentz spaces.

For a measurable function $f$ we define its non-increasing rearrangement by

$$
f^{*}(t):=\inf \{s,|\{x,|f(x)|>s\}| \leq t\}
$$

Definition 1.3.1. For $p, q \in(0, \infty]$, the Lorentz space $L^{p, q}\left(\mathbb{R}^{n}\right)$ is the set of functions $f$ such that $\|f\|_{L^{p, q}\left(\mathbb{R}^{n}\right)}$ is finite, with

$$
\|f\|_{L^{p, q}\left(\mathbb{R}^{n}\right)}= \begin{cases}\left(\int_{0}^{\infty}\left(t^{\frac{1}{p}} f^{*}(t)\right)^{q} \frac{d t}{t}\right)^{1 / q}, & \text { if } q<\infty  \tag{1.3.1}\\ \sup _{t>0} t^{1 / p} f^{*}(t), & \text { if } q=\infty\end{cases}
$$

The use of non-increasing rearrangement makes it rather difficult to extend the above definition to the variable setting. Fortunately, there is an equivalent characterization of Lorentz spaces $L^{p, q}\left(\mathbb{R}^{n}\right)$ which does not make use of the notion of non-increasing rearrangement, more precisely, we have the following characterization see [49]. For $p, q \in(0, \infty]$,

$$
\|f\|_{L^{p, q}\left(\mathbb{R}^{n}\right)} \approx \begin{cases}\left(\int_{0}^{\infty} \lambda^{q-1}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>\lambda\right\}}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}^{q} d \lambda\right)^{\frac{1}{q}}, & \text { if } 0<q<\infty,  \tag{1.3.2}\\ \sup _{\lambda>0} \lambda\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>\lambda\right\}}\right\|_{L^{p}\left(\mathbb{R}^{n}\right)}, & \text { if } q=\infty\end{cases}
$$

According to (1.3.2) and the [49, definition 2], we define the variable Lorentz spaces $L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$ as follows

Definition 1.3.2. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $0<q \leq \infty$. Then $L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$ is defined to be the set
of all measurable functions $f$ such that $\|f\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}<\infty$, where

$$
\|f\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}= \begin{cases}\left(\int_{0}^{\infty} \lambda^{q-1}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>\lambda\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} d \lambda\right)^{\frac{1}{q}}, & \text { if } 0<q<\infty,  \tag{1.3.3}\\ \sup _{\lambda>0} \lambda\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>\lambda\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)^{\prime}} & \text { if } q=\infty .\end{cases}
$$

Remark 1.3.3. When the weight $w \equiv 1$, then the definition 1.3.2 coincides with the definition of the variable Lorentz spaces $L^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$.

Below, we collect some properties for the weighted Lorentz spaces $L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$ that will be used later in this chapter. We start by the following Lemmas, where their proofs are similar to the classical one, see [49].

Lemma 1.3.4. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $q \in(0, \infty]$. Then $\|\cdot\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}$ defines a quasi-norm on $L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$.
Lemma 1.3.5. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right), w \in W_{p(\cdot)}\left(\mathbb{R}^{n}\right)$ and $q \in(0, \infty]$. Then for all $f \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$ and $s \in(0, \infty)$, it holds true that

$$
\left\||f|^{s}\right\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}=\|f\|_{L_{w w^{1 / s}}^{s p(\cdot), s q}}^{s}\left(\mathbb{R}^{n}\right)
$$

The next lemma presents an equivalent discrete characterization of the quasi-norm $\|\cdot\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}$. The proof is similar to [49, Lemma 2.4].
Lemma 1.3.6. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $0<q \leq \infty$. If $f \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, then

$$
\|f\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \sim \begin{cases}\left(\sum_{k \in \mathbb{Z}} 2^{k q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>2^{k}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{\frac{1}{q}}, & \text { if } 0<q<\infty,  \tag{1.3.4}\\ \sup _{k \in \mathbb{Z}} 2^{k}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>2^{k}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)^{\prime}} & \text { if } q=\infty .\end{cases}
$$

Lemma 1.3.7. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right), w \in W_{p(\cdot)}\left(\mathbb{R}^{n}\right)$ and let $0<q_{1} \leq q_{2} \leq \infty$. Then

$$
L_{w}^{p(\cdot), q_{1}}\left(\mathbb{R}^{n}\right) \subset L_{w}^{p(\cdot), q_{2}}\left(\mathbb{R}^{n}\right)
$$

Moreover, we have,

$$
\|f\|_{L_{w}^{p(\cdot), q_{2}}\left(\mathbb{R}^{n}\right)} \lesssim\|f\|_{L_{w}^{p(\cdot), q_{1}}\left(\mathbb{R}^{n}\right)}
$$

## CHAPTER 2

## WEIGHTED VARIABLE HARDY SPACES ON DOMAINS

Let $\Omega$ be a proper open subset of $\mathbb{R}^{n}$ and $p(\cdot): \Omega \rightarrow(0, \infty)$ a variable exponent obeys the globally log-Hölder continuity condition. In this chapter, we introduce the weighted variable Hardy space on $\Omega$ via the radial maximal function and then we characterize this spaces by the grand maximal functions. Moreover, we establish the atomic decomposition of the weighted variable Hardy space $H_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, and as application, we figure out its dual space.

### 2.1 Preparation and helpful results

Let $\phi \in \mathcal{D}\left(B\left(\mathbf{0}_{n}, 1\right)\right)$ such that $\int_{\mathbb{R}^{n}} \phi(x) d x=1$. For any $t \in(0, \infty)$ and $x \in \Omega$, we set $\phi_{t}(x)=$ $t^{-n} \phi\left(t^{-1} x\right)$. For any $f \in \mathcal{D}^{\prime}(\Omega)$, the radial maximal function $M_{\phi, \Omega}^{+}(f)$ is defined for any $x \in \Omega$ by

$$
\begin{equation*}
M_{\phi, \Omega}^{+}(f)(x):=\sup _{t \in\left(0, \operatorname{dist}\left(x, \Omega^{c}\right)\right)}\left|\left\langle f, \phi_{t}(x-\cdot)\right\rangle\right|, \tag{2.1.1}
\end{equation*}
$$

where $\Omega^{c}$ denotes the complementary set of $\Omega$ in $\mathbb{R}^{n}, \operatorname{dist}\left(x, \Omega^{c}\right):=\inf \left\{|x-y|: y \in \Omega^{c}\right\}$ and $\langle\cdot, \cdot\rangle$ denotes the duality between $\mathcal{D}^{\prime}(\Omega)$ and $\mathcal{D}(\Omega)$.

Definition 2.1.1. Let $\Omega$ be an open set of $\mathbb{R}^{n}$ and $p(\cdot) \in \mathcal{P}(\Omega)$. Then, the weighted variable Hardy space $H_{w}^{p(\cdot)}(\Omega)$ is defined to be the set of all $f \in \mathcal{D}^{\prime}(\Omega)$ such that $M_{\phi, \Omega}^{+}(f) \in L_{w}^{p(\cdot)}(\Omega)$,
where $M_{\phi, \Omega}^{+}$is as in (2.1.1), equipped with the quasi-norm

$$
\|f\|_{H_{w}^{p(\cdot)}(\Omega)}=\left\|M_{\phi, \Omega}^{+}(f)\right\|_{L_{w}^{p(\cdot)}(\Omega)} .
$$

Let $\Omega$ be an open subset of $\mathbb{R}^{n}$ and denote the set of all locally integrable functions on $\Omega$ by $L_{\text {loc }}^{1}(\Omega)$. Let $m \in \mathbb{Z}_{+}$. We denote the set of polynomial functions on $\mathbb{R}^{n}$ of degree less than $m$ by $\mathbb{P}_{m}$. For any $s \in(0, \infty)$ and $f \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{n}\right)$, let

$$
\|f\|_{\Lambda(s)}:=\sup _{B \subset \mathbb{R}^{n}}\left\{\inf _{P \in \mathbb{P}_{[s]}}\left[r_{B}^{-(n+s)} \int_{B}|f(x)-P(x)| d x\right]\right\}
$$

where the supremum is taken over all balls $B \subset \mathbb{R}^{n}$ and $r_{B}$ denotes the radius of $B$.
Let $\Omega$ be a proper open subset of $\mathbb{R}^{n}$ and $f$ a measurable function on $\Omega$. For any $x \in \Omega$, define $\tilde{f}$ by

$$
\tilde{f}=\left\{\begin{array}{l}
f(x) \quad \text { if } \quad x \in \Omega \\
0 \quad \text { if } \quad x \in \Omega^{c}
\end{array}\right.
$$

If $\tilde{f} \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$, set

$$
\|f\|_{\Lambda(s ; \Omega)}:=\|\tilde{f}\|_{\Lambda(s)}+\sup _{x \in \Omega}\left\{|f(x)|\left[\operatorname{dist}\left(x, \Omega^{c}\right)\right]^{-s}\right\} .
$$

Then

$$
\Lambda(s ; \Omega):=\left\{f \text { measurable on } \Omega: \tilde{f} \in L_{l o c}^{1}(\Omega) \text { and }\|f\|_{\Lambda(s ; \Omega)}<\infty\right\}
$$

Let $\Omega \in \mathbb{R}^{n}$ be an open set, $f \in \mathcal{D}^{\prime}(\Omega)$ and $s \in(0, \infty)$. For any $x \in \Omega$, the grand maximal function $f_{s, \Omega}^{*}$ is defined by

$$
f_{s, \Omega}^{*}(x):=\sup _{\phi}|\langle f, \phi\rangle|,
$$

where the supremum is taken over all those functions $\phi$ for which there exists $t_{\phi} \in(0, \infty)$ such that $\phi \in \mathcal{D}(B(x, t) \cap \Omega)$ and $\|\phi\|_{\Lambda(s ; \Omega)} \leq t^{-(n+s)}$.

Definition 2.1.2. Let $\Omega$ be an open set of $\mathbb{R}^{n}$ and $p(\cdot) \in \mathcal{P}(\Omega)$. Then, the weighted variable Hardy space $H_{w, \max , s}^{p(\cdot)}(\Omega)$ is defined to be the set of all $f \in \mathcal{D}^{\prime}(\Omega)$ such that $f_{s, \Omega}^{*} \in L_{w}^{p(\cdot)}(\Omega)$, equipped with the quasi-norm

$$
\|f\|_{H_{w, \text { max }, s}^{p(\cdot)}(\Omega)}=\left\|f_{s, \Omega}^{*}\right\|_{L_{w}^{p(\cdot)}(\Omega)} .
$$

Next, we give the definition of $(p(\cdot), r, w)$-atoms.
Definition 2.1.3. Let $\Omega$ be an open set of $\mathbb{R}^{n}, p(\cdot) \in \mathcal{P}(\Omega), w: \Omega \rightarrow(0, \infty), q \in(1, \infty]$ and

$$
\begin{equation*}
d_{w}=n\left(s_{w}-1\right) . \tag{2.1.2}
\end{equation*}
$$

1. A cube $Q \subset \mathbb{R}^{n}$ is called of type (a) if $4 Q \subset \Omega$ and $\widetilde{Q} \subset \mathbb{R}^{n}$ is called of type (b) if $2 \widetilde{Q} \cap \Omega^{c}=\varnothing$ and $4 \widetilde{Q} \cap \Omega^{c} \neq \varnothing$.
2. A measurable function $a$ on $\Omega$ is called a type (a) $(p(\cdot), q, w)_{\Omega}$-atom if there exists a cube $Q$ of type (a) such that
(1) $\operatorname{supp} a \subset Q$;
(2) $\|a\|_{L^{q}(\Omega)} \leq \frac{|Q|^{1 / q}}{\left\|\chi_{Q}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}$;
(3) there exist $s \geq d_{w}$ such that, $\int_{\mathbb{R}^{n}} a(x) x^{\alpha} d x=0$ for all $\alpha \in \mathbb{Z}_{+}^{n}$ with $|\alpha| \leq s$.
3. A measurable function $b$ on $\Omega$ is called a type (b) $(p(\cdot), q, w)_{\Omega}$-atom if there exists a cube $\widetilde{Q}$ of type (b) such that
(1) $\operatorname{supp} b \subset \widetilde{Q}$;
(2) $\|b\|_{L^{q}(\Omega)} \leq \frac{\mid \widetilde{\Omega} \|^{1 / q}}{\left\|\chi_{\Omega}\right\|_{L_{w}^{p(\cdot)}(\Omega)}} ;$

In what follows, we denote the set of all pairs $(a, Q)$ of type (a) $(p(\cdot), r, w)_{\Omega}-$ atoms and their supports by $\mathcal{A}(p(\cdot), r, w)$ and by $\mathcal{B}(p(\cdot), r, w)$ the set of all pairs $(a, \widetilde{Q})$ of type (b) $(p(\cdot), r)_{\Omega}-$ atoms and their supports.

Let $p(\cdot) \in \mathcal{P}(\Omega),\left\{\lambda_{i}\right\}_{i \in \mathbb{N}}$ be a sequence of numbers in $\mathbb{C},\left\{Q_{i}\right\}_{i \in \mathbb{N}}$ be a cube sequence of the supports of type (a) $(p(\cdot), r, w)_{\Omega}$ - atoms, $\left\{\kappa_{i}\right\}_{i \in \mathbb{N}}$ be a sequence of numbers in $\mathbb{C}$, $\left\{\widetilde{Q}_{i}\right\}_{i \in \mathbb{N}}$ be a cube sequence of the supports of type (b) $(p(\cdot), r, w)_{\Omega}$ - atoms. Define

$$
\mathcal{A}\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{Q_{i}\right\}_{i \in \mathbb{N}}\right):=\left\|\left\{\sum_{i \in \mathbb{N}}\left[\frac{\left|\lambda_{i}\right| \chi_{Q_{i}}}{\left\|\chi_{Q_{i}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\right]^{\theta}\right\}^{\frac{1}{\theta}}\right\|_{L_{w}^{p(\cdot)}(\Omega)},
$$

and

$$
\mathcal{B}\left(\left\{\kappa_{i}\right\}_{i \in \mathbb{N}},\left\{\widetilde{Q}_{i}\right\}_{i \in \mathbb{N}}\right):=\left\|\left\{\sum_{i \in \mathbb{N}}\left[\frac{\left|\kappa_{i}\right| \chi_{\widetilde{Q}_{i}}}{\left\|\chi_{\widetilde{Q}_{i}}\right\|_{L_{w}^{p, \cdot}(\Omega)}}\right]^{\theta}\right\}^{\frac{1}{\theta}}\right\|_{L_{w}^{p(\cdot)}(\Omega)},
$$

here and hereafter $\theta \in\left(0, s_{w}^{-1}\right)$.
Next, we give the definition of the atomic weighted variable Hardy spaces on domains.

Definition 2.1.4. Let $\Omega$ be an open set of $\mathbb{R}^{n}, p(\cdot) \in \mathcal{P}(\Omega)$ and $w: \Omega \rightarrow(0, \infty)$. The weighted variable atomic Hardy space $H_{w, \text { atom }}^{p(\cdot)}(\Omega)$ is defined to be the space of all functions $f \in \mathcal{D}^{\prime}(\Omega)$ which can be decomposed as

$$
\begin{equation*}
f=\sum_{i \in \mathbb{N}} \lambda_{i} a_{i}+\sum_{i \in \mathbb{N}} \kappa_{i} b_{i} \quad \text { in } \quad \mathcal{D}^{\prime}(\Omega) \tag{2.1.3}
\end{equation*}
$$

where $\left\{a_{i}\right\}_{i \in \mathbb{N}}$ is a sequence of type (a) $(p(\cdot), q, w)_{\Omega}$-atoms, associated with cubes $\left\{Q_{i}\right\}_{i \in \mathbb{N}}$, and $\left\{b_{i}\right\}_{i \in \mathbb{N}}$ is a sequence of type $(b)(p(\cdot), q, w)_{\Omega}$-atoms, associated with cubes $\left\{\widetilde{Q}_{i}\right\}_{i \in \mathbb{N}}$, satisfying that,

$$
\mathcal{A}\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{Q_{i}\right\}_{i \in \mathbb{N}}\right)+\mathcal{B}\left(\left\{\kappa_{i}\right\}_{i \in \mathbb{N}},\left\{\widetilde{Q}_{i}\right\}_{i \in \mathbb{N}}\right)<\infty .
$$

Moreover, for any $f \in H_{w, \text { atom }}^{p(\cdot), q}(\Omega)$, we define

$$
\|f\|_{H_{w, a \operatorname{tom}(\Omega)}^{p(,), q}(\Omega)}:=\inf \left\{\mathcal{A}\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{Q_{i}\right\}_{i \in \mathbb{N}}\right)+\mathcal{B}\left(\left\{\kappa_{i}\right\}_{i \in \mathbb{N}},\left\{\widetilde{Q}_{i}\right\}_{i \in \mathbb{N}}\right)\right\},
$$

where the infimum is taken over all the decompositions of $f$ as (4.1.5).
Remark 2.1.5. Let $\Omega$ be an open set of $\mathbb{R}^{n}$ and $p(\cdot) \in \mathcal{P}(\Omega)$. For $\left\{\lambda_{i}\right\}_{i \in \mathbb{N}} \subset \mathbb{C}$, and cubes $\left\{Q_{i}\right\}_{i \in \mathbb{N}} \subset \Omega$, let

$$
\mathcal{A}^{*}\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{Q_{j}\right\}_{i \in \mathbb{N}}\right):=\inf _{\lambda \in(0, \infty)}\left\{\int_{Q_{i}} \sum_{j \in \mathbb{N}}\left(\frac{\left|\lambda_{j}\right|}{\lambda\left\|\chi_{Q_{j}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\right)^{p(x)} w(x)^{p(x)} d x \leq 1\right\}
$$

Then from the embedding $\ell^{\theta} \hookrightarrow \ell^{\infty}$, we deduce that for any sequences of $\left\{\lambda_{i}\right\}_{i \in \mathbb{N}} \subset \mathbb{C}$, and cubes $\left\{Q_{i}\right\}_{i \in \mathbb{N}} \subset \Omega$,

$$
\mathcal{A}^{*}\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{Q_{i}\right\}\right) \leq \mathcal{A}\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{Q_{i}\right\}\right)
$$

### 2.2 Atomic decomposition

We start first by proving that the variable weighted Hardy spaces on domains can be characterized via the grand maximal function. More precisely, we have the following result.

Theorem 2.2.1. Let $\Omega$ be a proper open subset of $\mathbb{R}^{n}$ and $\phi \in \mathcal{D}\left(B\left(\mathbf{0}_{n}, 1\right)\right)$ such that $\int_{\mathbb{R}^{n}} \phi(x) d x=$ $1, p(\cdot) \in \mathcal{P}(\Omega)$ such that $n /(n+s)<p_{-} \leq p_{+}<\infty, w \in W_{p(\cdot)}(\Omega)$ and $s \in(0, \infty)$. Then, $H_{w}^{p(\cdot)}(\Omega) \approx H_{w, \text { max }, s}^{p(\cdot)}(\Omega)$ with equivalent quasi-norms.

Proof. Let $f \in \mathcal{D}^{\prime}(\Omega)$. By the definition of $M_{\phi, \Omega}^{+}(f)$ and $f_{s, \Omega}^{*}(f)$, we have

$$
\left\|M_{\phi, \Omega}^{+}(f)\right\|_{L_{w}^{p(\cdot)}(\Omega)} \leq\left\|f_{s, \Omega}^{*}(f)\right\|_{L_{w}^{p(\cdot)}(\Omega)} .
$$

On the other hand, by [68, corollary 1], we know that for any $x \in \Omega$,

$$
f_{s, \Omega}^{*}(f)(x) \lesssim M_{\phi, \Omega}^{+}(f)(x)
$$

Then, by the definitions of the spaces $H_{w}^{p(\cdot)}(\Omega)$ and $H_{w, \text { max }, S}^{p(\cdot)}(\Omega)$ and the above inequalities, we deduce that $H_{w}^{p(\cdot)}(\Omega) \approx H_{w, \max , s}^{p(\cdot)}(\Omega)$ with equivalent quasi-norms.

In the next theorem, we establish the atomic characterization of the weighted Hardy spaces on domains.

Theorem 2.2.2. Let $\Omega$ be a proper open subset of $\mathbb{R}^{n}$ and $\phi \in \mathcal{D}\left(B\left(\mathbf{0}_{n}, 1\right)\right)$ such that $\int_{\mathbb{R}^{n}} \phi(x) d x=$ $1, p(\cdot) \in \mathcal{P}(\Omega), w \in W_{p(\cdot)}(\Omega)$ and $q \in\left(\max \left\{1, p_{+}\right\}, \infty\right]$. Then, $H_{w}^{p(\cdot)}(\Omega) \approx H_{w, \text { atom }}^{p(\cdot), q}(\Omega)$ with equivalent quasi-norms. In particular, if $n /(n+s)<p_{-}$then

$$
H_{w}^{p(\cdot)}(\Omega) \approx H_{w, \text { atom }}^{p(\cdot)}(\Omega) \approx H_{w, \max , s}^{p(\cdot)}(\Omega)
$$

with equivalent quasi-norms.
Before giving the proof of the above theorem, we give some useful lemmas. We begin by a result obtained in [69, pp. 211-212]. We denote the set of cubes, with sidelength 1 and center $\left(x_{1}, \cdots, x_{n}\right)$ with $x_{j} \in \mathbb{Z}$ for any $j \in\{1, \cdots, n\}$, by $\mathbf{Q}_{0}$. For $i \in \mathbb{Z}$, we denote the set of cubes of the form $2^{i} Q$, with $Q \in \mathbf{Q}_{0}$, by $\mathbf{Q}_{i}$. Let $\mathbf{Q}=\cup_{i \in \mathbb{Z}} \mathbf{Q}_{i}$. For a proper open set $\Omega$ of $\mathbb{R}^{n}$, we set

$$
W(\Omega):=\left\{Q\left(c_{Q}, l_{Q}\right) \in \mathbf{Q}: 20 l_{Q}<\operatorname{dist}\left(c_{Q}, \Omega^{c}\right) \leq 43 l_{Q}\right\}
$$

Lemma 2.2.3. Let $\Omega$ be a proper open subset of $\mathbb{R}^{n}, s \in(0, \infty), m \in \mathbb{Z}$ and $f \in \mathcal{D}^{\prime}(\Omega)$. For any $i \in \mathbb{Z}$, let

$$
\Omega_{i}:=\left\{x \in \Omega: f_{s, \Omega}^{*}(x)>2^{i}\right\}
$$

where $f_{s, \Omega}^{*}$ is as in (2.1.2). Then the following statements hold true.

1. For any $i \in \mathbb{Z}$,

$$
\Omega_{i}=\cup_{Q \in W\left(\Omega_{i}\right)} Q \quad \text { and } \quad \Omega=\cup_{Q \in W(\Omega)} Q
$$

Moreover, there exists a constant $c$ depending only on $n$ such that for any $i \in \mathbb{Z}, \sum_{Q \in W\left(\Omega_{i}\right)} \chi_{10 Q} \leq$ c.
2. There exist functions $\left\{h_{I}^{i}\right\}_{i \in \mathbb{Z}, I \in \mathbf{Q}}$ and $\left\{h_{Q}^{\Omega}\right\}_{Q \in \mathbf{Q}}$ such that

$$
f=\sum_{i \in \mathbb{Z}, I \in \mathbf{Q}} h_{I}^{i}+\sum_{Q \in \mathbf{Q}} h_{Q}^{\Omega} \quad \text { in } \quad \mathcal{D}^{\prime}(\Omega)
$$

3. For any $i \in \mathbb{Z}$ and $I \in \mathbf{Q}$, if $I \notin W\left(\Omega_{i-1}\right)$, then $h_{I}^{i}=0$; if $I \in W\left(\Omega_{i-1}\right)$, then $h_{I}^{i} \in L^{\infty}(\Omega)$, supp $h_{I}^{i} \subset 9 I,<h_{I}^{i}, P>=0$ for all $P \in \mathbb{P}_{m}$, and there exists a positive constant $C$ depending only on $n$, s and $m$, such that $\left\|h_{I}^{i}\right\|_{L^{\infty}(\Omega)} \leq C 2^{i}$.
4. For any $Q \in \mathbf{Q}$, if $Q \in W(\Omega)$, then $h_{Q}^{\Omega}=0$; if $Q \in W(\Omega)$, then $h_{Q}^{\Omega} \in L^{\infty}(\Omega)$, $\operatorname{supp} h_{Q}^{\Omega} \subset 9 Q$ and there exists a positive constant $C$, depending only on $n$, s and $m$, such that $\left\|h_{Q}^{\Omega}\right\|_{L^{\infty}(\Omega)} \leq$ $C \inf _{x \in 9 Q} f_{s, \Omega}^{*}(x)$.

The next lemma plays a crucial role in the proof of the atomic decomposition. It is a slight variant of [39, Lemma 5.4]. It can be seen as the extension of [81, lemma 4.1] to the weighted case. For a general version and for the proof see lemma 3.1.5.

Lemma 2.2.4. Let $p(\cdot) \in \mathcal{P}(\Omega), w \in W_{p(\cdot)}, r \in\left(0, s_{w}^{-1}\right)$ and $q \in\left(\left(k_{w}^{s_{w}}\right)^{\prime}, \infty\right)$. Then there exists a positive constant $C$ such that for any sequence $\left\{B_{j}\right\}_{j \in \mathbb{N}}$ of balls in $\mathbb{R}^{n},\left\{\lambda_{j}\right\}_{j \in \mathbb{N}} \subset \mathbb{C}$ and functions $\left\{a_{j}\right\}_{j \in \mathbb{N}}$ satisfying that for any $j \in \mathbb{N}$, Supp $a_{j} \subset B_{j}$ and $\left\|a_{j}\right\|_{L^{q}\left(\mathbb{R}^{n}\right)} \leq\left|B_{j}\right|^{1 / q}$,

$$
\left\|\left(\sum_{j=1}^{\infty}\left|\lambda_{j} a_{j}\right|^{r}\right)^{\frac{1}{r}}\right\|_{L_{w}^{p(\cdot)}(\Omega)} \leq C\left\|\left(\sum_{j=1}^{\infty}\left|\lambda_{j} \chi_{B_{j}}\right|^{r}\right)^{\frac{1}{r}}\right\|_{L_{w}^{p(\cdot)}(\Omega)} .
$$

The next lemma plays a key role in the proof of theorem 2.2.2.
Lemma 2.2.5. Let $p(\cdot) \in \mathcal{P}(\Omega), q \in(1, \infty]$. Then there exists a positive constant $C$ such that, for any $j \in \mathbb{N}$, type $(a)(p(\cdot), q, w)$-atom $a_{j}$ and $x \notin 2 \sqrt{n} Q_{j}$,

$$
\begin{equation*}
M_{\phi, \Omega}^{+}\left(a_{j}\right)(x) \leq C \frac{\left[M\left(\chi_{Q_{j}}\right)(x)\right]^{\frac{n+d_{w}+1}{n}}}{\left\|\chi_{Q_{j}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}} \tag{2.2.1}
\end{equation*}
$$

where, $Q_{j}:=Q\left(c_{Q j}, \ell\left(Q_{j}\right)\right.$.
Proof. Let $x \notin 2 \sqrt{n} Q_{j}$ and for any $k \in \mathbb{Z}$, we set $\phi_{k}(x-\cdot):=2^{-k n} \phi\left(2^{-k}(x-\cdot)\right)$.
Since $\operatorname{supp}\left(\phi_{k}(x-\cdot) \subset Q\left(x ; 2^{1-k}\right)\right.$, it follows, if $\operatorname{supp}\left(\phi_{k}(x-\cdot) \cap Q_{j}=\varnothing\right.$, then we have, $<a_{j}, \phi_{k}(x-\cdot)>=0$. Moreover, if $\operatorname{supp}\left(\phi_{k}(x-\cdot) \cap Q_{j}=\varnothing\right.$, then $\ell\left(Q_{j}\right) \leq 2^{1-k}$ and $\left|x-c_{Q_{j}}\right| \leq$ $2^{2-k}$.

Let $q_{x, k}$ be the Taylor polynomial of degree $d_{w}$. For any $y \in Q_{j}$, we have

$$
\begin{aligned}
\left|\phi_{k}(x-y)-q_{x, k}\right| & \lesssim 2^{k\left(n+d_{w}+1\right)}\left|y-c_{Q_{j}}\right|^{d_{w}+1} \\
& \lesssim \frac{\left(\ell\left(Q_{j}\right)\right)^{d_{w}+1}}{\left(\ell\left(Q_{j}\right)\right)^{n+d_{w}+1}+\left|x-c_{Q_{j}}\right|^{n+d_{w}+1}} .
\end{aligned}
$$

The fact that for any $j \in \mathbb{N}$, type $(a)(p(\cdot), q, w)$-atom $a_{j}$ and the Hölder inequality inequality leads us to

$$
\begin{aligned}
\left|<a_{j}, \phi_{k}(x-\cdot)>\right| & =\mid \int_{Q_{j}}\left[\phi_{k}(x-y)-q(y)\right] a_{j}(y) d y \\
& \lesssim \frac{\left(\ell\left(Q_{j}\right)\right)^{d_{w}+1}}{\left(\ell\left(Q_{j}\right)\right)^{n+d_{w}+1}+\left|x-c_{Q_{j}}\right|^{n+d_{w}+1}} \int_{Q_{j}}\left|a_{j}(y)\right| d y \\
& \lesssim \frac{\left(\ell\left(Q_{j}\right)\right)^{d_{w}+1}}{\left(\ell\left(Q_{j}\right)\right)^{n+d_{w}+1}+\left|x-c_{Q_{j}}\right|^{n+d_{w}+1}}\left|Q_{j}\right|^{1-\frac{1}{q}}\left\|a_{j}\right\|_{L^{q}(\Omega)} \\
& \lesssim \frac{\left(\ell\left(Q_{j}\right)\right)^{n+d_{w w}+1}}{\left(\ell\left(Q_{j}\right)\right)^{n+d_{w}+1}+\left|x-c_{Q_{j}}\right|^{n+d_{w}+1}}\left\|\chi_{Q_{j}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{-1} \\
& \lesssim \frac{\left[M\left(\chi_{Q_{j}}\right)(x)\right]^{\frac{n+d_{w}+1}{n}}}{\left\|\chi_{Q_{j}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}},
\end{aligned}
$$

which gives the desired result.
Proof of theorem 2.2.2. We start by the first inclusion $H_{w, \text { atom }}^{p(\cdot)}(\Omega) \hookrightarrow H_{w}^{p(\cdot)}(\Omega)$.
Let $f \in H_{w, \text { atom }}^{p(\cdot)}(\Omega)$, by definition we conclude that there exist $\left\{\lambda_{i}\right\}_{i \geq 1},\left\{\kappa_{i}\right\}_{i \geq 1} \subset \mathbb{C}$ and two sequences $\left\{a_{i}\right\}_{i \geq 1},\left\{b_{i}\right\}_{i \geq 1}$ of type $(a)-(p(\cdot), q, w)_{\Omega}$ atoms and $(b)-(p(\cdot), q, w)_{\Omega}$ atoms ,respectively, such that

$$
f=\sum_{i \geq 1} \lambda_{i} a_{i}+\sum_{i \geq 1} \kappa_{i} b_{i}, \quad \text { in } \mathcal{D}^{\prime}(\Omega)
$$

By Remark 1.2.8, we have

$$
\begin{aligned}
\|f\|_{H_{w}^{p(\cdot)}(\Omega)}^{p} & =\left\|M_{\phi, \Omega}^{+}(f)\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{p} \\
& \leq\left\|\sum_{i \in \mathbb{N}}\left|\lambda_{i}\right| M_{\phi, \Omega}^{+}\left(a_{i}\right)\right\|_{L_{w}^{p, \cdot}(\Omega)}^{\underline{p}}+\left\|\sum_{i \in \mathbb{N}}\left|\kappa_{i}\right| M_{\phi, \Omega}^{+}\left(a_{i}\right)\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{p},
\end{aligned}
$$

thus

$$
\begin{aligned}
\|f\|_{H_{w}^{p(\cdot)}(\Omega)}^{p} \leq & \left\|\sum_{i \in \mathbb{N}}\left|\lambda_{i}\right| \chi_{2 \sqrt{n} Q_{i j}} M_{\phi, \Omega}^{+}\left(a_{i}\right)\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{\underline{p}} \\
& +\left\|\sum_{i \in \mathbb{N}}\left|\lambda_{i}\right| \chi_{\left(2 \sqrt{n} Q_{i j}\right)} M_{\phi, \Omega}^{+}\left(a_{i}\right)\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{\|^{p}} \\
+ & \left\|\sum_{i \in \mathbb{N}}\left|\kappa_{i}\right| \chi_{8 \widetilde{Q_{i j}}} M_{\phi, \Omega}^{+}\left(a_{i}\right)\right\|_{L_{w o}^{p(\cdot)}(\Omega)}^{\underline{p}} \\
& +\left\|\sum_{i \in \mathbb{N}}\left|\kappa_{i}\right| \chi_{\left(8 \widetilde{Q_{i j}}\right) \mathrm{c}} M_{\phi, \Omega}^{+}\left(a_{i}\right)\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{\underline{p}} \\
:= & \mathbb{I}_{1}+\mathbb{I}_{2}+\mathbb{I}_{3}+\mathbb{I}_{4} .
\end{aligned}
$$

Note that for all $i \in \mathbb{N}, M_{\phi, \Omega}^{+}\left(a_{i}\right)(x) \leq M\left(a_{i}\right)(x)$ for any $x \in 2 \sqrt{n} Q_{i j}$, then by Lemma 2.2.4, we have

$$
\begin{align*}
\mathbb{I}_{1} & \lesssim\left\|\sum_{i \in \mathbb{N}}\left|\lambda_{i}\right| \chi_{2 \sqrt{n} Q_{i j}} M\left(a_{i}\right)\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{\underline{p}} \\
& \lesssim\left\|\left[\sum_{i \in \mathbb{N}}\left(\left|\lambda_{i}\right| \chi_{2 \sqrt{n} Q_{i j}} M\left(a_{i}\right)\right)^{\theta}\right]^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{\underline{p}}  \tag{2.2.2}\\
& \lesssim\left[\mathcal{A}\left(\left\{\lambda_{i}\right\}_{i \geq 1},\left\{Q_{i}\right\}_{i \geq 1}\right)\right]^{\underline{p}} .
\end{align*}
$$

We pass to deal with $\mathbb{I}_{2}$. According to lemma 2.2.5, for $x \notin 2 \sqrt{n}\left(Q_{i}\right)_{i \in \mathbb{N}}$, we have

$$
M_{\phi, \Omega}^{+}\left(a_{j}\right)(x) \lesssim \frac{\left[M\left(\chi_{Q_{i}}\right)(x)\right]^{s}}{\left\|\chi_{Q_{i}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}
$$

where $s=\frac{n+d_{w}+1}{n}$. Thus

$$
\mathbb{I}_{2} \lesssim\left\|\sum_{i \in \mathbb{N}} \frac{\left|\lambda_{i}\right|}{\left\|\chi_{Q_{i}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\left[M\left(\chi_{Q_{i}}\right)\right]^{s}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{\underline{p}}
$$

By Remark 1.2.8, we have

$$
\mathbb{I}_{2} \lesssim\left\|\left[\sum_{i \in \mathbb{N}} \frac{\left|\lambda_{i}\right|}{\left\|\chi_{Q_{i}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\left[M\left(\chi_{Q_{i}}\right)\right]^{s}\right]^{1 / s}\right\|_{L_{w w^{1 / s}}^{p(\cdot)}(\Omega)}^{s p}
$$

Since $s \geq s_{w}$ then from theorem1.2.10 and Remark 2.1.5, we obtain

$$
\begin{align*}
\mathbb{I}_{2} & \lesssim\left\|\left[\sum_{i \in \mathbb{N}} \frac{\left|\lambda_{i}\right|}{\left\|\chi_{Q_{i}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\left(\chi_{Q_{i}}\right)^{s}\right]^{1 / s}\right\|_{L_{w w^{p / s}}^{p(\cdot)}(\Omega)}^{s p}  \tag{2.2.3}\\
& \lesssim \| \sum_{i \in \mathbb{N}} \frac{\left|\lambda_{i}\right| \chi_{Q_{i}}}{\left\|\chi_{Q_{i}}\right\|_{L_{w o}^{p(\cdot)}(\Omega)} \|_{L_{w}^{p(\cdot)}(\Omega)}^{\underline{p}}} \\
& \lesssim\left[\mathcal{A}^{*}\left(\left\{\lambda_{i}\right\}_{i \geq 1},\left\{Q_{i}\right\}_{i \geq 1}\right)\right]^{\underline{p}} \\
& \lesssim\left[\mathcal{A}\left(\left\{\lambda_{i}\right\}_{i \geq 1},\left\{Q_{i}\right\}_{i \geq 1}\right)\right]^{\underline{p}}
\end{align*}
$$

In a similar manner to $\mathbb{I}_{1}$, we get

$$
\begin{align*}
\mathbb{I}_{3} & \lesssim\left\|\sum_{i \in \mathbb{N}}\left|\lambda_{i}\right| \chi_{8 \widetilde{\mathbb{Q}_{i j}}} M_{\phi, \Omega}^{+}\left(b_{i}\right)\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{\underline{p}}  \tag{2.2.4}\\
& \lesssim\left\|\left[\sum_{i \in \mathbb{N}}\left(\left|\kappa_{i}\right| \chi_{8 \widetilde{Q_{i j}}} M_{\phi, \Omega}^{+}\left(b_{i}\right)\right)^{\theta}\right]^{1 / \theta}\right\|_{L_{w v}^{p(\cdot)}(\Omega)}^{\underline{p}} \\
& \lesssim\left[\mathcal{B}\left(\left\{\kappa_{i}\right\}_{i \geq 1,}\left\{\widetilde{Q}_{i}\right\}_{i \geq 1}\right)\right]^{\underline{p}} .
\end{align*}
$$

For any $\left(x \in\left(8 \widetilde{Q_{i j}}\right)^{c}\right)$, we have

$$
M_{\phi, \Omega}^{+}\left(b_{i}\right)(x)=\sup _{\left\{k \in \mathbb{Z}: 0<2^{k}<\operatorname{dist}\left(x, \Omega^{c}\right) / 2\right\}}\left|\int_{\widetilde{Q}_{i}} 2^{-k n} \phi\left(2^{-k}(x-y)\right) b_{i}(y) d y\right|,
$$

consequently,

$$
\begin{aligned}
M_{\phi, \Omega}^{+}\left(b_{i}\right)(x) & \lesssim \sup _{\left\{k \in \mathbb{Z}: 0<2^{k}<7 \ell\left(\widetilde{Q}_{i}\right)\right\}}\left|\int_{\widetilde{Q}_{i}} 2^{-k n} \phi\left(2^{-k}(x-y)\right) b_{i}(y) d y\right| \\
& +\sup _{\left\{k \in \mathbb{Z}: 7 \ell\left(\widetilde{Q}_{i}\right)<2^{k}<\operatorname{dist}\left(x, \Omega^{\mathrm{c}}\right) / 2\right\}}\left|\int_{\widetilde{Q}_{i}} 2^{-k n} \phi\left(2^{-k}(x-y)\right) b_{i}(y) d y\right| \\
& :=\mathfrak{A}_{1}+\mathfrak{A}_{2} .
\end{aligned}
$$

For any $x \in\left(8 \widetilde{Q_{i j}}\right)^{\mathbf{c}}$ and $y \in \widetilde{Q}_{i j}$ we have $|x-y|>7 \ell\left(\widetilde{Q}_{i}\right)>2^{k}$ then $2^{-k}|x-y|>1$. Since $\phi \in \mathcal{D}\left(B\left(0_{\mathbb{R}^{n}}, 1\right)\right)$, we conclude that $\mathfrak{A}_{1}=0$. We note that $\mathfrak{A}_{2}=0$ if $\operatorname{dist}\left(x, \Omega^{\mathfrak{c}}\right) \leq 14 \ell\left(\widetilde{Q}_{i}\right)$. We suppose that $\operatorname{dist}\left(x, \Omega^{c}\right)>14 \ell\left(\widetilde{Q}_{i}\right)$, we have

$$
|x-y|>\operatorname{dist}\left(x, \Omega^{\mathfrak{c}}\right)-\operatorname{dist}\left(y, \Omega^{\mathfrak{c}}\right)>\operatorname{dist}\left(x, \Omega^{\mathfrak{c}}\right)-\operatorname{dist}\left(y, \Omega^{\mathbf{c}}\right) .
$$

As $\widetilde{Q}_{i}$ is type $(b)$-cube then $4 \widetilde{Q}_{i} \cap \Omega^{\mathfrak{c}} \neq \varnothing$, consequently $\operatorname{dist}\left(y, \Omega^{\mathfrak{c}}\right)<4 \ell\left(\widetilde{Q}_{i j}\right)$ for any $y \in \widetilde{Q}_{i j}$. Hence for any $y \in \widetilde{Q}_{i}$ and $x \in\left(8 \widetilde{Q}_{i j}\right)^{\mathfrak{c}} \cap \Omega$,

$$
\begin{aligned}
|x-y| & >\operatorname{dist}\left(x, \Omega^{\mathbf{c}}\right)-\operatorname{dist}\left(y, \Omega^{\mathbf{c}}\right)>\operatorname{dist}\left(x, \Omega^{\mathbf{c}}\right)-4 \ell\left(\widetilde{Q}_{i}\right) \\
& >\operatorname{dist}\left(x, \Omega^{\mathbf{c}}\right)-\frac{2}{7} \operatorname{dist}\left(x, \Omega^{\mathbf{c}}\right)>\operatorname{dist}\left(x, \Omega^{\mathbf{c}}\right) / 2 \\
& >2^{k} .
\end{aligned}
$$

Thus, we conclude that $\mathbb{I}_{4}=0$. Putting (2.2.2), (4.3.6) and (4.3.7) together we find out that $f \in H_{w}^{p(\cdot)}(\Omega)$ with $\|f\|_{H_{w}^{p(\cdot)}(\Omega)} \lesssim\|f\|_{H_{w, \text { atom }}^{p(\cdot), q}(\Omega)}$.

Let us now prove the reverse inclusion, let $f \in H_{w}^{p(\cdot)}(\Omega)$ and

$$
\Omega_{j}:=\left\{x \in \Omega: f_{s, b}^{* \Omega}(x)>2^{j}\right\}, \quad \forall j \in \mathbb{Z}
$$

It is easy to check that $\Omega_{j+1} \subset \Omega_{j}$ for any $k \in \mathbb{Z}$. In view of Lemma 2.2.3, there exists two functions $\left\{h_{J}^{j}\right\}_{j \in \mathbb{Z}, J \in \mathbf{Q}}$ and $\left\{h_{Q}^{\Omega}\right\}_{Q \in \mathbf{Q}}$ such that

$$
f=\sum_{j \in \mathbb{Z}, J \in \mathbf{Q}} h_{J}^{j}+\sum_{Q \in \mathbf{Q}} h_{Q}^{\Omega} \text { in } \mathcal{D}^{\prime}(\Omega) .
$$

Let $\lambda_{J}^{j}:=\left\|h_{J}^{j}\right\|_{L^{\infty}(\Omega)}\left\|\chi_{9 J}\right\|_{L_{w}^{p(\cdot)}(\Omega)}$. According to (3) of Lemma 2.2.3, we can see that $\left(\lambda_{J}^{j}\right)^{-1} h_{J}^{j}$
is $(a)-(p(\cdot), \infty)_{\Omega, w}-$ atom. By (1) and (3) of Lemma 2.2.3, we find out

$$
\begin{aligned}
\mathcal{A}\left(\left\{\lambda_{J}^{j}\right\}_{j \in \mathbb{Z}, J \in \mathbf{Q}},\right. & \left.\{9 J\}_{J \in \mathbf{Q}}\right) \\
& =\left\|\left(\sum_{j \in \mathbb{Z}, J \in W\left(\Omega_{j-1}\right)}\left[\lambda_{J}^{j} \chi_{9 J}\left\|\chi_{9 J}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{-1}\right]^{\theta}\right)^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{1 / \theta} \\
& \lesssim\left\|\left(\sum_{j \in \mathbb{Z}}\left[2^{j} \chi_{\Omega_{j-1}}\right]^{\theta}\right)^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{1 / \theta} \cdot
\end{aligned}
$$

On the other hand we have

$$
\sum_{j \in \mathbb{Z}}\left(2^{j} \chi_{\Omega_{j-1}}\right)^{\theta} \sim\left(\sum_{j \in \mathbb{Z}} 2^{j} \chi_{\Omega_{j-1}}\right)^{\theta} \sim\left(\sum_{j \in \mathbb{Z}} 2^{j} \chi_{\Omega_{j-1} \backslash \Omega_{j}}\right)^{\theta}
$$

Consequently,

$$
\begin{aligned}
\mathcal{A}\left(\left\{\lambda_{J}^{j}\right\}_{j \in \mathbb{Z}, J \in \mathbf{Q}},\{9 J\}_{J \in \mathbf{Q}}\right) & \leq\left\|\sum_{j \in \mathbb{Z}} 2^{j} \chi_{\Omega_{j-1} \backslash \Omega_{j}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{1 / \theta} \\
& \lesssim\left\|f_{s, b}^{* \Omega}\right\|_{L_{w}^{p(\cdot)}(\Omega)} \\
& \lesssim\|f\|_{H_{w}^{p(\cdot)}(\Omega)}
\end{aligned}
$$

Let $\kappa_{Q}:=\left\|h_{Q}^{\Omega}\right\|_{L^{\infty}}\left\|\chi_{9 Q}\right\|_{L_{w}^{p(\cdot)}(\Omega)}$, it is easy to see that $\left(\kappa_{Q}\right)^{-1} h_{Q}^{\Omega}$ is a $(b)-(p(\cdot), \infty)_{\Omega, w}$-atom. Then we obtain

$$
\begin{aligned}
\mathcal{B}\left(\left\{\kappa_{Q}\right\}_{Q \in \mathbf{Q}},\{9 Q\}_{Q \in \mathbf{Q}}\right) & =\left\|\left[\sum_{Q \in W(\Omega)}\left(\kappa_{Q} \chi_{9 Q}\left\|\chi_{9 j}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{-1}\right)^{\theta}\right]^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{1 / \theta} \\
& \lesssim\left\|\left[\sum_{Q \in W(\Omega)}\left(\inf _{x \in 9 Q} f_{s, b}^{* \Omega} \chi_{9 Q}\right)^{\theta}\right]^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}(\Omega)}^{1 / \theta} \\
& \lesssim\left\|f_{s, b}^{* \Omega}\right\|_{L_{w}^{p(\cdot)}(\Omega)} \\
& \lesssim\|f\|_{H_{w}^{p(\cdot)}(\Omega)}
\end{aligned}
$$

Summing up the above estimates, we conclude that $f \in H_{w, \text { atom }}^{p(\cdot), q}(\Omega)$ and $\|f\|_{H_{w, a t o m}^{p(\cdot), q}(\Omega)} \lesssim$ $\|f\|_{H_{w}^{p(\cdot)}(\Omega)}$.

### 2.3 Duality result

In this section, we figure out the dual space of the weighted variable Hardy space $H_{w}^{p(\cdot)}(\Omega)$. We begin by introducing the definition of the weighted variable Hölder space.

Definition 2.3.1. Let $\Omega$ be a proper open subset of $\mathbb{R}^{n}, p(\cdot) \in \mathcal{P}(\Omega), w \in W_{p(\cdot)}(\Omega), q \in[1, \infty)$ and $d \in \mathbb{Z}_{+}$. A function $f \in L_{\text {loc }}^{1}(\Omega)$ is said to belong to the weighted variable Hölder space $\Lambda_{w}^{p(\cdot), q, d}(\Omega)$, if

$$
\begin{aligned}
&\|f\|_{\Lambda_{w}^{p(\cdot), q, d}(\Omega)} \\
&:=\sup _{Q: \operatorname{type}(a) \text { cubes }}\left\{\inf _{P \in \mathbb{P}_{d}}\right.\left.\left(\frac{|Q|^{1 / q^{\prime}}}{\left\|\chi_{Q}\right\|_{L_{w}^{p(\cdot)}(\Omega)}} \int_{Q}|f(x)-P(x)|^{q} d x\right)^{1 / q}\right\} \\
&+\sup _{Q: \operatorname{type}(b) \text { cubes }}\left(\frac{|Q|^{1 / q^{\prime}}}{\left\|\chi_{Q}\right\|_{L_{w}^{p(\cdot)}(\Omega)}} \int_{Q}|f(x)| d x\right)<\infty,
\end{aligned}
$$

where $q^{\prime}$ denotes the conjugate of $q$.
Theorem 2.3.2. Let $\Omega$ be a proper open subset of $\mathbb{R}^{n}, p(\cdot) \in \mathcal{P}(\Omega)$, w $\in W_{p(\cdot)}(\Omega)$ with $0<$ $p_{-} \leq p_{+} \leq 1, q \in[1, \infty)$ and $d \in \mathbb{Z}_{+}$with $q \geq d_{w}$. Then, the dual space of $H_{w}^{p(\cdot)}(\Omega)$, denoted by $\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}$, is $\Lambda_{w}^{p(\cdot), q, d}(\Omega)$ in the following sense

1. Let $g \in \Lambda_{w}^{p(\cdot), q, d}(\Omega)$. Then the linear functional

$$
\begin{equation*}
L: f \rightarrow L(f):=\int_{\Omega} f(x) g(x) d x \tag{2.3.1}
\end{equation*}
$$

initially defined for $H_{w, \text { atom,fin }}^{p(\cdot), q^{\prime}}(\Omega)$ has a bounded extension to $H_{w}^{p(\cdot)}(\Omega)$.
2. Conversely, if $L \in\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}$, then there exists a unique extension $g \in \Lambda_{w}^{p(\cdot), q, d}(\Omega)$ such (2.3.1) holds true.

Before proving the above theorem, we introduce the following result which can be proved by using the argument used for [100, Lemma 5.9] mutatis mutandis.

Lemma 2.3.3. Let $p_{+} \in(0,1], \alpha \in\left[p_{+}, 1\right]$. Then there exists a positive constant $C$ such that, for all $\left\{\lambda_{i}\right\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and any sequence $\left\{Q_{i}\right\}_{i \in \mathbb{N}}$ of cubes in $\Omega$,

$$
\left(\sum_{i \in \mathbb{N}}\left|\lambda_{i}\right|^{\alpha}\right)^{\frac{1}{\alpha}} \leq \mathcal{A}\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{Q_{i}\right\}_{i \in \mathbb{N}}\right)
$$

Proof of theorem 2.3.2. Let $j_{0} \in \mathbb{N}, f:=\sum_{i=1}^{j_{0}} \lambda_{i} a_{i}+\sum_{i=1}^{i_{0}} \kappa_{i} b_{i} \in H_{w, \text { atom,fin }}^{p(\cdot), q^{\prime}}(\Omega)$ and $P$ be a Taylor expansion of $g$ and of order $d_{w}$, we use the vanishing moment condition fulfills by $a_{i}$ we obtain

$$
\begin{aligned}
&\left|\int_{\Omega} f(x) g(x) d x\right|= \mid\left(\sum_{i=1}^{k_{0}} \lambda_{i} a_{i}(x)\right. \\
&\left.+\sum_{i=1}^{k_{0}} \kappa_{i} b_{i}(x)\right) g(x) d x \mid \\
& \leq \sum_{i=1}^{k_{0}}\left|\lambda_{i}\right| \int_{Q_{i}}|g(x)-P(x)|\left|a_{i}(x)\right| d x \\
& \quad+\sum_{i=1}^{k_{0}}\left|\kappa_{i}\right| \int_{\widetilde{Q}_{i}}|g(x)|\left|b_{i}(x)\right| d x .
\end{aligned}
$$

By the Hölder inequality, we get

$$
\begin{aligned}
\left|\int_{\Omega} f(x) g(x) d x\right| \leq \sum_{i=1}^{k_{0}}\left|\lambda_{i}\right| \|(g-P) & \chi_{Q_{i}}\left\|_{L^{q}(\Omega)}\right\| a_{i} \chi_{Q_{i}} \|_{L^{q^{\prime}}(\Omega)} \\
& +\sum_{i=1}^{k_{0}}\left|\kappa_{i}\right|\left\|g(x) \chi_{\widetilde{Q}_{i}}\right\|_{L^{q}(\Omega)}\left\|b_{i} \chi_{\widetilde{Q}_{i}}\right\|_{L^{q^{\prime}}(\Omega)}
\end{aligned}
$$

Since $a_{i}$ and $b_{i}$ are $(a)\left(p(\cdot), q^{\prime}, w\right)_{\Omega}$-atom and $(b)\left(p(\cdot), q^{\prime}, w\right)_{\Omega}$-atom respectively, we obtain

$$
\begin{aligned}
\left|\int_{\Omega} f(x) g(x) d x\right| \leq \sum_{i=1}^{k_{0}}\left|\lambda_{i}\right| & \frac{\left|Q_{i}\right|^{1 / q^{\prime}}}{\left\|\chi_{Q_{i}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\left\|(g-P) \chi_{Q_{i}}\right\|_{L^{q}(\Omega)} \\
& +\sum_{i=1}^{k_{0}}\left|\kappa_{i}\right| \frac{\left|\widetilde{Q}_{i}\right|^{1 / q^{\prime}}}{\left\|\chi_{\widetilde{Q}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\left\|g(x) \chi_{\widetilde{Q}_{i}}\right\|_{L^{q}(\Omega)} .
\end{aligned}
$$

According to the definition of the space $\Lambda_{w}^{p(\cdot), d}\left(\mathbb{R}^{n}\right)$, we get

$$
\left|\int_{\Omega} f(x) g(x) d x\right| \leq\|g\|_{\Lambda_{w}^{p(\cdot), d}\left(\mathbb{R}^{n}\right)} \times\left(\sum_{i=1}^{k_{0}}\left(\left|\lambda_{i}\right|+\left|\kappa_{i}\right|\right)\right)
$$

By Lemma 2.3.3, we conclude that

$$
\begin{aligned}
& \left|\int_{\Omega} f(x) g(x) d x\right| \\
& \quad \leq\|g\|_{\Lambda_{w}^{p(\cdot), d}\left(\mathbb{R}^{n}\right)} \times\left(\mathcal{A}\left(\left\{\lambda_{i}\right\}_{i \geq 1},\left\{Q_{i}\right\}_{i \geq 1}\right)+\mathcal{B}\left(\left\{\kappa_{i}\right\}_{i \geq 1},\left\{\widetilde{Q}_{i}\right\}_{i \geq 1}\right)\right) .
\end{aligned}
$$

On the other hand it is easy to see that the space $H_{w, \text { atom, fin }}^{p(\cdot), q^{\prime}}(\Omega)$ is dense in $H_{w, \text { atom }}^{p(\cdot), q^{\prime}}(\Omega)$, thus we conclude that the linear functional $\ell_{g}$ is bounded on $H_{w}^{p(\cdot)}(\Omega)$.

Conversely, let $\ell \in\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}$, we have $L^{q^{\prime}}(Q) \hookrightarrow H_{w}^{p(\cdot)}(\Omega)$, where $Q$ is a cube of $\Omega$. Thus $\ell$ defines a functional on $L^{q^{\prime}}(Q)$. From of the Riesz representation theorem, there exists $g_{Q} \in L^{q}(Q)$ such that $\ell(f)=\int_{Q} g_{Q}(x) f(x) d x$ for every $f \in L^{q^{\prime}}(Q)$. According to the HahnBanach theorem we deduce that there exists a unique measurable function $g \in L^{q}(\Omega)$ and

$$
\ell(f)=\int_{\Omega} g(x) f(x) d x, \quad \forall f \in L^{q^{\prime}}(\Omega)
$$

Our next step is to show that $g \in \Lambda_{w}^{p(\cdot), d}\left(\mathbb{R}^{n}\right)$. By taking the supremum of $\ell(f)$ over all $(a)\left(p(\cdot), q^{\prime}\right)_{\Omega}$-atoms $a$ and $(b)\left(p(\cdot), q^{\prime}, w\right)_{\Omega}$-atoms $b$ we obtain

$$
|\ell(a)|+|\ell(b)| \lesssim\|\ell\|_{\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}}
$$

where

$$
\|\ell\|_{\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}}:=\sup _{f \in H_{w}^{p(\cdot)}(\Omega), f \neq 0}\left\{\frac{\ell(f)}{\|f\|_{H_{w}^{p(\cdot)}(\Omega)}}\right\}
$$

Let $Q \subset \Omega$ be of type $(a)$ cube and $f \in L^{q^{\prime}}(X)$ with $\|f\|_{L^{q^{\prime}}(X)} \leq 1$. We set

$$
\begin{equation*}
a:=\frac{|Q|^{1 / q^{\prime}}(f-f P) \chi_{Q}}{\left\|\chi_{Q}\right\|_{L_{w}^{p(\cdot)}(\Omega)}\|f\|_{L^{\prime}(\Omega)}} . \tag{2.3.2}
\end{equation*}
$$

We observe that $a$ is a $(a)\left(p(\cdot), q^{\prime}, w\right)_{\Omega}$-atom. Combining (2.3), supp $a \subset Q$ and the vanishing moment condition, we find out

$$
\begin{equation*}
|\ell(a)|=\left|\int_{Q} a(x)(g(x)-P(x)) d x\right| \lesssim\|\ell\|_{\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}} \tag{2.3.3}
\end{equation*}
$$

In view of (2.3.2) and (2.3.3) we conclude,

$$
\frac{|Q|^{1 / q^{\prime}}}{\left\|\chi_{Q}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\left|\int_{Q} f(x)(g(x)-P(x)) d x\right| \lesssim\|\ell\|_{\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}}
$$

Consequently,

$$
\begin{equation*}
\sup _{Q: \text { type }(a)}\left[\inf _{P \in \mathbb{P}_{d_{w}}}\left(\frac{|Q|^{1 / q^{\prime}}}{\left\|\chi_{Q}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\left(\int_{Q}|g(x)-P(x)|^{q} d x\right)^{\frac{1}{q}}\right)\right] \lesssim\|\ell\|_{\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}} \tag{2.3.4}
\end{equation*}
$$

In a similar way, we set

$$
b:=\frac{|\widetilde{Q}|^{1 / q^{\prime}} f \chi_{\widetilde{Q}}}{\left\|\chi_{Q}\right\|_{L_{w}^{p(\cdot)}(\Omega)}\|f\|_{L^{q^{\prime}}(\Omega)}} .
$$

We note that $b$ is a type $(b)\left(p(\cdot), q^{\prime}, w\right)_{\Omega}$-atom. From (2.3), we obtain

$$
|\ell(b)|=\left|\int_{\widetilde{Q}} b(x) g(x) d x\right| \lesssim\|\ell\|_{\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*} .}
$$

Hence

$$
\frac{|\widetilde{Q}|^{1 / q^{\prime}}}{\left\|\chi_{\widetilde{Q}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\left|\int_{\widetilde{Q}} f(x) g(x) d x\right| \lesssim\|\ell\|_{\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}}
$$

Taking the supremum over all the cubes of type (b), we conclude

$$
\begin{equation*}
\sup _{\widetilde{Q}: t y p e(b)}\left(\frac{|\widetilde{Q}|^{1 / q^{\prime}}}{\left\|\chi_{\widetilde{Q}}\right\|_{L_{w}^{p(\cdot)}(\Omega)}}\left(\int_{\widetilde{Q}}|g(x)|^{q} d x\right)^{\frac{1}{q}}\right) \lesssim\|\ell\|_{\left(H_{w}^{p(\cdot)}(\Omega)\right)^{*}} \tag{2.3.5}
\end{equation*}
$$

Putting (2.3.4) and (2.3.5) together we get the desired result.

## CHAPTER 3

## WEIGHTED VARIABLE HARDY SPACES ASSOCIATED WITH OPERATORS

This chapter is devoted to the study of the weighted variable Hardy space $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ associated with operator $L$ which has a bounded holomorphic functional calculus and fulfills the Davies-Gaffney estimates. In particular, we show the molecular characterization of $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ via the atomic decomposition of weighted tent spaces, and we establish a duality relation between $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ and $B M O_{L^{*}, w}^{p(\cdot), M}$.

### 3.1 Preparation and helpful results

Let $\mathbb{R}_{+}^{n+1}:=\mathbb{R}^{n} \times(0, \infty)$. For any $\alpha \in(0, \infty)$ and $x \in \mathbb{R}^{n}$, define

$$
\Gamma_{\alpha}(x):=\left\{(y, t) \in \mathbb{R}_{+}^{n+1}:|y-x|<\alpha t\right\} .
$$

If $\alpha=1$, for the sake of simplicity, we write $\Gamma(x)$ instead of $\Gamma_{\alpha}(x)$.
For any ball $B:=B\left(x_{B}, r_{B}\right) \subset \mathbb{R}^{n}$ with $x_{B} \in \mathbb{R}^{n}$ and $r_{B} \in(0, \infty), \lambda \in(0, \infty)$ and $j \in \mathbb{N}$. Let $\lambda B:=B\left(x_{B}, \lambda r_{B}\right)$,

$$
\widehat{B}=\left\{(y, t) \in \mathbb{R}_{+}^{n+1}, \operatorname{dist}\left(y, B^{c}\right) \geq t\right\}
$$

For any ball $B(x, r)$, with $x \in \mathbb{R}^{n}$ and $r \in(0, \infty)$ there exists a positive constant $C$ such that

$$
|B(x, 2 r)| \leq C|B(x, r)|, \quad|B(x, 2 r)|=2^{n}|B(x, r)|
$$

and for $\lambda \in[1, \infty)$

$$
|B(x, \lambda r)| \leq C \lambda^{n}|B(x, r)| .
$$

There exists a positive constant $C$ such that

$$
\begin{equation*}
|B(y, r)| \leq C\left[1+\frac{d(x, y)}{r}\right]^{n}|B(x, r)| . \tag{3.1.1}
\end{equation*}
$$

Lemma 3.1.1. [94]
There exist a collection of open sets $\left\{Q_{\alpha, k} \subset \mathbb{R}^{n}: k \in \mathbb{Z}, \alpha \in I_{k}\right\}$, and $\delta \in(0,1), a_{0}>0$ and $\widetilde{C} \in$ $(0, \infty)$ such that

1. for any $k \in \mathbb{Z},\left|\mathbb{R}^{n} \backslash \underset{\alpha \in I_{k}}{\cup} Q_{\alpha, k}\right|=0$;
2. if $i \geq k$, then either $Q_{\alpha, i} \subset Q_{\beta, k}$ or $Q_{\alpha, i} \cap Q_{\beta, k}=\varnothing$;
3. for any fixed $k \in \mathbb{Z}, \alpha \in I_{k}$ and $i<k$, there exists a unique $\beta \in I_{i}$ such that $Q_{\alpha, k} \subset Q_{\beta, i}$;
4. for any $k \in \mathbb{Z}, \alpha \in I_{k}$, the diameter of $Q_{\alpha, k}$ does not exceed $\widetilde{C} \delta^{k}$;
5. for any $k \in \mathbb{Z}, \alpha \in I_{k}$, there exists a ball $B\left(z_{\alpha, k}, a_{0} \delta^{k}\right) \subset Q_{\alpha, k}$.

The square function $S_{L}$ associated with $L$ is defined by setting, for any $f \in L^{2}\left(\mathbb{R}^{n}\right)$ and $x \in \mathbb{R}^{n}$ by,

$$
\begin{equation*}
S_{L}(f)(x):=\left[\int_{0}^{\infty} \int_{B(x, t)}\left|t^{2} L e^{-t^{2} L}(f)(y)\right|^{2} \frac{d y d t}{t^{n+1}}\right]^{1 / 2} \tag{SL}
\end{equation*}
$$

Now, we recall some notions of bounded holomorphic calculi which was introduced by McIntosh [65].

Let $0 \leq \eta<\pi$. The closed sector in the complex plane $\mathbb{C}$ is defined as follows,

$$
\begin{equation*}
S_{\eta}=\{z \in \mathbb{C}:|\arg z| \leq \eta\} \cup\{0\} \tag{3.1.2}
\end{equation*}
$$

and its interior is denoted by $S_{\eta}^{0}$, i.e.

$$
\begin{equation*}
S_{\eta}^{0}=\{z \in \mathbb{C} \backslash\{0\}:|\arg , z|<\eta\} . \tag{3.1.3}
\end{equation*}
$$

Denote the set of all holomorphic functions on $S_{\eta}^{0}$ by $H\left(S_{\eta}^{0}\right)$ and for any $b \in H\left(S_{\eta}^{0}\right)$, we define $\|b\|_{\infty}$ by

$$
\begin{equation*}
\|b\|_{\infty}=\sup \left\{|b(z)|: z \in S_{\eta}^{0}\right\} \tag{3.1.4}
\end{equation*}
$$

The set of all $b \in H\left(S_{\eta}^{0}\right)$ satisfying $\|b\|_{\infty}<\infty$ is denoted by $H_{\infty}\left(S_{\eta}^{0}\right)$ and define the set $\Psi\left(S_{\eta}^{0}\right)$ by

$$
\begin{equation*}
\Psi\left(S_{\eta}^{0}\right)=\left\{\psi \in H_{\infty}\left(S_{\eta}^{0}\right): \exists v, C>0:|\psi(z)| \leq \frac{c|z|^{v}}{1+|z|^{2 v}}, \forall z \in S_{\eta}^{0}\right\} \tag{3.1.5}
\end{equation*}
$$

Let $\eta \in[0, \pi)$ and denote the spectre of $L$ by $\sigma(L)$. Then, we say that the closed operator $L$ on $L^{2}\left(\mathbb{R}^{n}\right)$ is of type $\eta$ if

1. $\sigma(L)$ is a subset of $S_{\eta}$,
2. for any $v \in(\eta, \pi)$, there exists a positive constant $C_{v}$ such that for all $\lambda \notin S_{v}$,

$$
\begin{equation*}
\left\|(L-\lambda I)^{-1}\right\|_{\mathcal{L}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)} \leq C_{v}|\lambda|^{-1} \tag{3.1.6}
\end{equation*}
$$

where $\mathcal{L}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)$ denotes the set of all linear continuous operators from $L^{2}\left(\mathbb{R}^{n}\right)$ to itself and for any operator $T \in \mathcal{L}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)$, its norm is denoted by $\|T\|_{\mathcal{L}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)}$.
Let $\eta \in[0, \pi)$, L be a one-to-one operator of type $\eta$ in $L^{2}\left(\mathbb{R}^{n}\right), v \in(\eta, \pi)$ and $\psi \in \Psi\left(S_{v}^{0}\right)$. The operator $\psi(L)$ is defined as follows

$$
\begin{equation*}
\psi(L)=\frac{1}{2 \pi i} \int_{\Theta} \psi(\lambda)(\lambda I-L)^{-1} d \lambda \tag{3.1.7}
\end{equation*}
$$

where $\Theta:=\left\{r e^{+i v}: r \in(0, \infty)\right\} \cup\left\{r e^{-i v}: r \in(0, \infty)\right\}, v \in(\eta, v)$ is the curve consisting of two rays parameterized anti-clockwise. It is well-known that the integral in (3.1.7) is absolutely convergent in $L^{2}\left(\mathbb{R}^{n}\right)$ (see [35, 65] for more details) and $\psi(L)$ does not depend on the choice of $v$ (see for instance [3, Lecture 2]). By a limiting procedure we can extend the above holomorphic functional calculus on $\Psi\left(S_{v}^{0}\right)$ to $H_{\infty}\left(S_{v}^{0}\right)$ (the reader is referred to [65] for more details). Let $0<v<\pi$, we say that the operator $L$ has a bounded $H_{\infty}\left(S_{v}^{0}\right)$-calculus in $L^{2}\left(\mathbb{R}^{n}\right)$ if there exists a positive constant $C$, such that for all $\psi \in H_{\infty}\left(S_{v}^{0}\right)$,

$$
\begin{equation*}
\|\psi(L)\|_{\mathcal{L}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)} \leq C\|\psi\|_{L^{\infty}\left(S_{v}^{0}\right)} \tag{3.1.8}
\end{equation*}
$$

Remark 3.1.2. Let $\eta \in\left[0, \frac{\pi}{2}\right)$. If $L$ is an operator of type $\eta$ in $L^{2}\left(\mathbb{R}^{n}\right)$, then $L$ generates a bounded holomorphic semigroup $\left\{e^{-z L}\right\}_{z \in S_{\frac{\pi}{2}-\eta}^{0}}$ on the open sector $S_{\frac{\pi}{2}-\eta}^{0}$ (see [35, Proposition 7.1.1]).

We assume that $L$ is an operator that fulfills the following assumptions.
Assumption $(A)$. $L$ is one-to-one operator of type $\eta$ in $L^{2}\left(\mathbb{R}^{\eta}\right)$ with $\eta \in\left[0, \frac{\pi}{2}\right)$ and has a bounded holomorphic functional calculus.
Assumption $(B)$. The semigroup $\left\{e^{-t L}\right\}_{t>0}$ generated by $L$ satisfies the Davies-Gaffney estimates, i.e. there exist a positive constants $c_{1}$ and $c_{2}$ such that, for any function $f$ in $L^{2}\left(\mathbb{R}^{n}\right)$ and closed sets $E$ and $F$ of $\mathbb{R}^{n}$ with $\operatorname{supp} f \subset E$,

$$
\begin{equation*}
\left\|e^{-t L}(f)\right\|_{L^{2}(F)} \leq c_{1} e^{-c_{2} \frac{[\operatorname{dist}(E, F)]^{2}}{t}}\|f\|_{L^{2}(E)} \tag{3.1.9}
\end{equation*}
$$

where $\operatorname{dist}(E, F):=\inf \{|x-y|: x \in E, y \in F\}$.
Remark 3.1.3. Let $L$ be an operator that fulfills Assumption $(A)$ and $\operatorname{Assumption}(B)$. Then,
(1) for any $i \in \mathbb{Z}_{+}$, the family of operators $\left\{(t L)^{i} e^{-t L}\right\}_{t>0}$ satisfies the Davies-Gaffney estimates (see Remark 2.5(i) in [92]).
(2) we can easily prove that the operator $S_{L}$ defined in (SL) is bounded on $L^{2}\left(\mathbb{R}^{n}\right)$ by means of Fubini's theorem and [40, (4.1)].

Remark 3.1.4. Here we give some operators which fulfill Assumption $(A)$ and Assumption $(B)$ :
(a) The one-to-one non-negative self-adjoint operator $L$ satisfying Gaussian upper bounds.
(b) The second order divergence form elliptic operators with complex bounded coefficients.
(c) The Schrödinger operator $-\triangle+V$ on $\mathbb{R}^{n}$ with the non-negative potential $V \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$.

We end this section by giving a useful result which play an important role in the atomic decomposition part,

Lemma 3.1.5. Let $p(\cdot): \mathbb{R}^{n} \rightarrow(0, \infty)$ be a Lebesgue measurable function with $0 \leq p_{-} \leq p_{+}<\infty$ and $w \in W_{p(\cdot)}$. Let $s \in S_{w}$ and $\left\{\lambda_{k}\right\}_{k \in \mathbb{N}}$ be a sequence of scalars. For any $r>\left(k_{w}^{s}\right)^{\prime}$, and $\left\{b_{k}\right\}_{k \in \mathbb{N}} \in$ $L^{r}\left(\mathbb{R}^{n}\right)$ with suppb $_{k} \subset Q_{k} \subset Q$, such that

$$
\begin{equation*}
\left\|b_{k}\right\|_{L^{r}\left(\mathbb{R}^{n}\right)} \leq A_{k}\left|Q_{k}\right|^{1 / r} \tag{3.1.10}
\end{equation*}
$$

where $A_{k}>0$, for all $k \in \mathbb{N}$, we have

$$
\begin{equation*}
\left\|\sum_{k \in \mathbb{N}} \lambda_{k} b_{k}\right\|_{L_{w^{1 / s}}^{s p(\cdot)}} \leq C\left\|\sum_{k \in \mathbb{N}} A_{k}\left|\lambda_{k}\right| \chi_{Q_{k}}\right\|_{L_{w^{1 / s}}^{s p(\cdot)}} \tag{3.1.11}
\end{equation*}
$$

where $C$ is constant independent of $\left\{A_{k}\right\}_{k \in \mathbb{N}},\left\{\lambda_{k}\right\}_{k \in \mathbb{N}}$, and $\left\{b_{k}\right\}_{k \in \mathbb{N}}$.
Proof. Fix $s \in \mathbb{S}_{w}$. For any $g \in L_{w^{-1 / s}}^{(s p(\cdot))^{\prime}}\left(\mathbb{R}^{n}\right)$ with $\|g\|_{L_{w-1 / s}^{(s p(\cdot))^{\prime}}\left(\mathbb{R}^{n}\right)} \leq 1$, we get

$$
\int_{\mathbb{R}^{n}} b_{k}(x) g(x) d x \leq\left\|b_{k}\right\|_{L^{r}\left(\mathbb{R}^{n}\right)}\left\|\chi_{Q_{k} g}\right\|_{L^{r^{\prime}}\left(\mathbb{R}^{n}\right)}
$$

From (3.1.10) we deduce that

$$
\int_{\mathbb{R}^{n}} b_{k}(x) g(x) d x \leq A_{k}\left|Q_{k}\right|^{1 / r}\left(\int_{Q_{k}}|g(x)|^{r^{\prime}} d x\right)^{1 / r^{\prime}}
$$

Hence,

$$
\begin{aligned}
\left|\int_{\mathbb{R}^{n}} b_{k}(x) g(x) d x\right| & \leq A_{k}\left|Q_{k}\right|\left(\frac{1}{\left|Q_{k}\right|} \int_{Q_{k}}|g(x)|^{r^{\prime}} d x\right)^{1 / r^{\prime}} \\
& \leq C A_{k}\left|Q_{k}\right| \inf _{x \in Q_{k}}\left(M\left(|g|^{r^{\prime}}\right)(x)\right)^{1 / r^{\prime}} \\
& \leq C A_{k} \int_{Q_{k}}\left(M\left(|g|^{r^{\prime}}\right)(x)\right)^{1 / r^{\prime}} d x
\end{aligned}
$$

for some $C>0$. By virtue of Lemma 1.2.12, we find out

$$
\begin{aligned}
\left|\int_{\mathbb{R}^{n}}\left(\sum_{k \in \mathbb{N}} \lambda_{k} b_{k}(x)\right) d x\right| & \lesssim \sum_{k \in \mathbb{N}} A_{k}\left|\lambda_{k}\right| \int_{Q_{k}}\left(M\left(|g|^{r^{\prime}}\right)(x)\right)^{1 / r^{\prime}} d x \\
& \lesssim \int_{\mathbb{R}^{n}}\left(\sum_{k \in \mathbb{N}} A_{k}\left|\lambda_{k}\right| \chi_{Q_{k}}(x)\right)\left(M\left(|g|^{r^{\prime}}\right)(x)\right)^{1 / r^{\prime}} d x
\end{aligned}
$$

By using Hölder inequality, we obtain,

$$
\begin{aligned}
\left|\int_{\mathbb{R}^{n}}\left(\sum_{k \in \mathbb{N}} \lambda_{k} b_{k}(x)\right) d x\right| & \lesssim\left\|\sum_{k \in \mathbb{N}} A_{k}\left|\lambda_{k}\right| \chi_{Q_{k}}\right\|_{L_{w^{1 / s}}^{s p(\cdot)}\left(\mathbb{R}^{n}\right)} \|\left(M\left(|g|^{r^{\prime}}\right)^{1 / r^{\prime}} \|_{L^{(s p(\cdot))^{\prime}}\left(\mathbb{R}^{n}\right)}\right. \\
& \lesssim\left\|\sum_{k \in \mathbb{N}} A_{k}\left|\lambda_{k}\right| \chi_{Q_{k}}\right\|_{L_{w^{1 / s}}^{s p(\cdot)}\left(\mathbb{R}^{n}\right)} \|\left(M\left(|g|^{r^{\prime}}\right) \|_{\substack{1 / r^{\prime} \\
L^{-}-r^{\prime} / s}}^{1 / \cdot)^{\prime} / r^{\prime}}\left(\mathbb{R}^{n}\right)^{n}\right.
\end{aligned}
$$

Since $r^{\prime}<k_{w}^{s}$, the definition of $k_{w}^{s}$ ensures that there exists $r^{\prime}<k<k_{w}^{s}$ such that the operator $M$ is bounded on $L_{w^{-k / s}}^{(s p(\cdot))^{\prime} / k}\left(\mathbb{R}^{n}\right)$, on the other hand from the following inequality,
for all $r \in(1, \infty)$

$$
(M f)^{r} \leq M\left(|f|^{r}\right)
$$

Hence, the operator $M$ is bounded on $L_{w^{-r^{\prime} / s}}^{(s p(\cdot))^{\prime} / r^{\prime}}\left(\mathbb{R}^{n}\right)$. In view of Proposition 1.2 .13 we get (3.1.11) which is the desired result.

### 3.2 Weighted variable Hardy spaces associated with operators satisfying Davies-Gaffney estimates

In this section, we establish the molecular characterization of the weighted variable Hardy space via the atomic decomposition of the weighted variable tent spaces given in this section.

For all measurable functions $f$ on $\mathbb{R}_{+}^{n+1}$ and for any $x \in \mathbb{R}^{n}$, we define the Lusin-area function by

$$
\begin{equation*}
\mathcal{A}(f)(x):=\left(\iint_{\Gamma(x)}|f(y, t)|^{2} \frac{d y d t}{t^{n+1}}\right)^{1 / 2} \tag{3.2.1}
\end{equation*}
$$

Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w: \mathbb{R}^{n} \rightarrow(0, \infty)$. The tent space $T^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$ is the space of all measurable functions $f$ on $\mathbb{R}_{+}^{n+1}$ such that

$$
\begin{equation*}
\|f\|_{T_{2}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)}=\|\mathcal{A}(f)\|_{L^{p(\cdot)}\left(\mathbb{R}^{n}\right)}<\infty . \tag{3.2.2}
\end{equation*}
$$

The weighted variable tent space $T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$ is defined to be the space of all measurable functions $f$ on $\mathbb{R}_{+}^{n+1}$ such that $\mathcal{A}(f) \in L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, with the norm

$$
\begin{equation*}
\|f\|_{T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)}=\|\mathcal{A}(f)\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} . \tag{3.2.3}
\end{equation*}
$$

Remark 3.2.1. If $f \in T_{2}^{2}\left(\mathbb{R}_{+}^{n+1}\right)$, then

$$
\|f\|_{T_{2}^{2}\left(\mathbb{R}_{+}^{n+1}\right)}=\left(\iint_{\mathbb{R}_{+}^{n+1}}|f(x, t)|^{2} \frac{d x d t}{t}\right)^{1 / 2}
$$

Let $F$ be a closed set in $\mathbb{R}^{n}$ and $O \equiv F^{\complement}$, we denote by $\widehat{O}$ the tent over $O$ which is the set

$$
\begin{equation*}
\widehat{O}:=\left\{(x, t) \in \mathbb{R}_{+}^{n+1}: \operatorname{dist}(x, F) \geq t\right\} . \tag{3.2.4}
\end{equation*}
$$

Next, we give the definition of $(p(\cdot), w, \infty)$-atoms.
Definition 3.2.2. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right), w: \mathbb{R}^{n} \rightarrow(0, \infty)$ be a Lebesgue measurable function and $r \in(1, \infty)$. A function $a$ on $\mathbb{R}_{+}^{n+1}$ is called a $(p(\cdot), w, \infty)$-atom if
(i) there exists a ball $B \subset \mathbb{R}^{n}$ such that supp $a \subset \widehat{B}$;
(ii) $\|a\|_{T_{2}^{r}\left(\mathbb{R}_{+}^{n+1}\right)} \leq|B|^{1 / r}\left\|\chi_{B}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{-1}$.

The following Theorem presents the atomic characterization of the weighted variable tent space $T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$.

Theorem 3.2.3. Let $p(\cdot): \mathbb{R}^{n} \rightarrow(0, \infty)$, $w \in W_{p(\cdot)}$. Then for $f \in T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$, there exists $(p(\cdot), w, \infty)$-atoms $\left\{a_{i}\right\}_{i \in \mathbb{N}}$ associated with the balls $\left\{B_{i}\right\}_{i \in \mathbb{N}}$, respectively, and numbers $\left\{\lambda_{i}\right\}_{i \in \mathbb{N}} \subset$ $\mathbb{C}$ such that for almost every $(x, t) \in \mathbb{R}_{+}^{n+1}$,

$$
\begin{equation*}
f(x, t)=\sum_{i \in \mathbb{N}} \lambda_{i} a_{i}(x, t) \tag{3.2.5}
\end{equation*}
$$

Moreover, there exists a positive constant $C$ such that, for all $f \in T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$,

$$
\begin{equation*}
\Lambda\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{B_{i}\right\}_{i \in \mathbb{N}}\right) \leq C\|f\|_{T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)^{\prime}} \tag{3.2.6}
\end{equation*}
$$

where for any sequence of numbers $\left\{\lambda_{i}\right\}_{i \in \mathbb{N}} \in \mathbb{C}$ and sequence of balls $\left\{B_{i}\right\}_{i \in \mathbb{N}}$

$$
\begin{equation*}
\Lambda\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{B_{i}\right\}_{i \in \mathbb{N}}\right):=\left\|\left(\sum_{i \in \mathbb{N}}\left[\frac{\left|\lambda_{i}\right| \chi_{B_{i}}}{\left\|\chi_{B_{i}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}}\right]^{\theta}\right)^{\frac{1}{\theta}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}, \tag{3.2.7}
\end{equation*}
$$

where $\theta \in\left(0, s_{w}^{-1}\right)$.
Proof. Let $f \in T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$. For any $i \in \mathbb{Z}$, let

$$
\Omega_{i}=\left\{x \in \mathbb{R}^{n}: \mathcal{A}(f)(x)>2^{i}\right\} .
$$

Since $f \in T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$, it is easy to check that $\Omega_{i}$ is a proper open set and $\left|\Omega_{i}\right|<\infty$ for each $i \in \mathbb{Z}$. By a similar argument used in the proof of [42, Theorem 3.2], we can show that
$\operatorname{supp} f \subset\left[\left(\cup_{i \in \mathbb{Z}} \widehat{\Omega_{i}^{*}}\right) \cup E\right]$, where $E \subset \mathbb{R}_{+}^{n+1}$ satisfying $\int_{E} \frac{d y d t}{t}=0$. Thus, for each $i \in \mathbb{Z}$, by applying the Whitney decomposition (see [85, p. 167]) to $\Omega_{i}^{*}$, we get a sequence $\left\{Q_{i, j}\right\}_{j \in \mathbb{N}}$ of disjoint cubes such that

1. $\bigcup_{j \in \mathbb{N}} Q_{i, j}=\Omega_{i}^{*}$ and $\left\{Q_{i, j}\right\}_{j \in \mathbb{N}}$ have disjoint interiors,
2. for all $j \in \mathbb{N}$,

$$
\begin{equation*}
c_{1} \sqrt{n} l_{Q_{i, j}} \leq \operatorname{dist}\left(Q_{i, j}\left(\Omega_{i}^{*}\right)^{\complement}\right) \leq c_{2} \sqrt{n} l_{Q_{i, j}} \tag{3.2.8}
\end{equation*}
$$

where $l_{Q_{i, j}}$ denotes the side-length of the cube $Q_{i, j} \operatorname{dist}\left(Q_{i, j},\left(\Omega_{i}^{*}\right)^{\complement}\right):=\inf \{|x-y|: x \in$ $\left.Q_{i, j}, y \in\left(\Omega_{i}^{*}\right)^{\complement}\right\}$.

For each $j \in \mathbb{N}$, choose a ball $B_{i, j}$ with the same center with $Q_{i, j}$ and with radius $\frac{11}{2} \sqrt{n} l\left(Q_{i, j}\right)$. We define

$$
\begin{gather*}
A_{i, j}=\widehat{B}_{i, j} \cap\left(Q_{i, j} \times(0, \infty)\right) \cap\left(\widehat{\Omega_{i}^{*}} \backslash \widehat{\Omega_{i+1}^{*}}\right), \\
a_{i, j}=2^{-i}\left\|\chi_{B_{i, j}}\right\|_{L_{w}^{p(-)}\left(\mathbb{R}^{n}\right)}^{-1} f \chi_{A_{i, j}} \text { and } \quad \lambda_{i, j}=2^{i}\left\|\chi_{B_{i, j}}\right\|_{L_{w}^{p(-)}\left(\mathbb{R}^{n}\right)^{n}} . \tag{3.2.9}
\end{gather*}
$$

Notice that $\left\{\left(Q_{i, j} \times(0, \infty)\right) \cap\left(\widehat{\Omega_{i}^{*}} \backslash \widehat{\Omega_{i+1}^{*}}\right)\right\} \subset \widehat{B}_{i, j}$. Following the proof used in [101, Theorem 2.16], we can show that $a_{i, j}$ is a $(p(\cdot), w, \infty)$-atom associated to the ball $B_{i, j}$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$. We obtain that $f=\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{i, j} a_{i, j}$ almost everywhere. Then, it remains to show that $\Lambda\left(\left\{\lambda_{i}\right\}_{i \in \mathbb{N}},\left\{B_{i}\right\}_{i \in \mathbb{N}}\right) \leq C\|f\|_{T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)}$. Indeed, by the definition of $\lambda_{i, j}$ (3.2.9), we get

$$
\begin{aligned}
\Lambda\left(\left\{\lambda_{i, j}\right\}_{i \in \mathbb{Z}, j \in \mathbb{N}},\left\{B_{i, j}\right\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\right) & =\sum_{i \in \mathbb{Z}}\left\|\left(\sum_{j \in \mathbb{N}}\left(2^{i} \chi_{B_{i, j}}\right)^{\theta}\right)^{\frac{1}{\theta}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim \sum_{i \in \mathbb{Z}} 2^{i}\left\|\left(\sum_{j \in \mathbb{N}}\left(\chi_{B_{i, j}}\right)^{\theta}\right)^{\frac{1}{\theta}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)^{p}}
\end{aligned}
$$

From the fact that,

$$
\sum_{j \in \mathbb{N}} \chi_{B_{i j}} \lesssim \chi_{\Omega_{i}^{*}} \lesssim 1,
$$

we deduce that

$$
\begin{aligned}
\Lambda\left(\left\{\lambda_{i, j}\right\}_{i \in \mathbb{Z}, j \in \mathbb{N}},\left\{B_{i, j}\right\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\right) & \lesssim \sum_{i \in \mathbb{Z}} 2^{i}\left\|\chi_{\Omega_{i}^{*}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& =\sum_{i \in \mathbb{Z}} 2^{i}\left\|\chi_{\Omega_{i}^{*}}\right\|_{L_{w}^{p(j) / r}\left(\mathbb{R}^{n}\right)}^{1 / r} .
\end{aligned}
$$

Since $r<s_{w}^{-1}$, we use Theorem 1.2.10 and the fact $\chi_{\Omega_{i}^{*}} \lesssim\left[M\left(\chi_{\Omega_{i}}\right)\right]$, we find

$$
\begin{aligned}
\Lambda\left(\left\{\lambda_{i, j}\right\}_{i \in \mathbb{Z}, j \in \mathbb{N}},\left\{B_{i, j}\right\}_{i \in \mathbb{Z}, j \in \mathbb{N}}\right) & \lesssim \sum_{i \in \mathbb{Z}} 2^{i}\left\|M\left(\chi_{\Omega_{i}}\right)\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\|\mathcal{A}(f)\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}=\|f\|_{T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)} .
\end{aligned}
$$

Then, the proof is completed.
Proposition 3.2.4. If $f \in T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$, then $f=\sum_{i} \sum_{j} \lambda_{i j} a_{i j}$ converges in $T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$.
Proof. Let $f \in T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$ and $r \in\left(0, s_{w}^{-1}\right)$, then from Lemma 3.1.5, we find that, for any $N \in \mathbb{N}$,

$$
\begin{aligned}
\left\|\mathcal{A}\left(f-\sum_{|i|+|j| \leq N} \lambda_{i j} a_{i j}\right)\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} & =\left\|\mathcal{A}\left(\sum_{|i|+|j|>N} \lambda_{i j} a_{i j}\right)\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \leq\left\|\sum_{|i|+|j|>N}\left|\lambda_{i j}\right| \mathcal{A}\left(a_{i j}\right)\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \leq\left\|\left\{\sum_{|i|+|j|>N}\left[\left|\lambda_{i j}\right| \mathcal{A}\left(a_{i j}\right)\right]^{r}\right\}^{1 / r}\right\| \|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \leq \|\left\{\sum _ { | i | + | j | > N } \left[\frac{\left|\lambda_{i j}\right| \chi_{B_{i j}}}{\left.\left.\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}}\right]^{r}\right\}^{1 / r}\| \|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} .} .\right.\right.
\end{aligned}
$$

Putting together the last inequality with (3.2.6) and the dominated convergence Theorem, we get

$$
\begin{aligned}
& \lim _{N \rightarrow \infty}\left\|\mathcal{A}\left(f-\sum_{|i|+|j| \leq N} \lambda_{i j} a_{i j}\right)\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \leq \| \lim _{N \rightarrow \infty}\left\{\sum _ { | i | + | j | > N } \left[\frac{\left|\lambda_{i j}\right| \chi_{B_{i j}}}{\left.\left.\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}}\right]^{r}\right\}^{1 / r} \|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}=0 .}\right.\right.
\end{aligned}
$$

Hence, $f=\sum_{i} \sum_{j} \lambda_{i j} a_{i j}$ converge in $T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$. Thus the proof is achieved.

We denote by $T_{c, w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$ and $T_{2, c}^{2}\left(\mathbb{R}_{+}^{n+1}\right)$ respectively, the set of all functions in $T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$ and $T_{2}^{2}\left(\mathbb{R}_{+}^{n+1}\right)$ which have compact supports.
Proposition 3.2.5. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right), w \in W_{p(\cdot)}$. Then $T_{c, w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right) \subset T_{2, c}^{2}\left(\mathbb{R}_{+}^{n+1}\right)$ in the meaning of sets.

Proof. By [34, Lemma 3.3(i)], we know that, for any $q \in(0, \infty), T_{2, c}^{q}\left(\mathbb{R}_{+}^{n+1}\right) \subset T_{2, c}^{2}\left(\mathbb{R}_{+}^{n+1}\right)$. Then it suffices to show that $T_{c, w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right) \subset T_{2, c}^{q^{\prime}}\left(\mathbb{R}_{+}^{n+1}\right)$, for some $q^{\prime} \in(0, \infty)$.

Indeed, let $f \in T_{c, w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$ such that supp $f \subset E$, where $E$ is a compact set of $\mathbb{R}_{+}^{n+1}$. Let $B$ be a ball of $\mathbb{R}_{+}^{n+1}$ such that $E \subset \widehat{B}$. Then, $\operatorname{supp} \mathcal{A} f \subset B$

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}[\mathcal{A} f(x)]^{\underline{p}} d x & =\int_{\{x \in B: \mathcal{A} f(x) \leq 1\}}[\mathcal{A} f(x)]^{\underline{p}} d x+\int_{\{x \in B: \mathcal{A} f(x)>1\}}[\mathcal{A} f(x)]^{\underline{p}} d x \\
& \leq|B|+\int_{\mathbb{R}^{n}}[\mathcal{A} f(x)]^{\underline{\underline{p}}} \chi_{B} d x \\
& \leq|B|+\left\|\mathcal{A} f \chi_{B}\right\|_{\underline{L}^{\underline{p}}} \\
& \leq|B|+\left\|(\mathcal{A} f)^{\underline{p}} \chi_{B}\right\|_{L^{1}} \\
& \leq|B|+\left\|(\mathcal{A} f)^{\underline{p} \|_{L^{p}}^{p(\cdot) / p}}\right\| \chi_{B} \|_{\left.L_{w^{(p p}(\cdot) / p}^{p}\right)^{\prime}} \\
& \lesssim|B|+\|\mathcal{A} f\|_{L_{w}^{p(\cdot)}}^{\underline{p}}
\end{aligned}
$$

this ends the proof.
Next, we establish the molecular characterization of the weighted variable Hardy spaces associated with operators satisfying the Davies-Gaffney estimates. These spaces are denoted by $H_{L, w}^{p(\cdot)}$. We begin with some definitions.

Definition 3.2.6. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w: \mathbb{R}^{n} \rightarrow(0, \infty)$ be a Lebesgue measurable function. Let $L$ be an operator satisfying Assumption $(A)$ and Assumption $(B)$. The weighted variable Hardy space $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ is defined as the completion of the space $\widetilde{H}_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$,

$$
\widetilde{H}_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right):=\left\{f \in L^{2}\left(\mathbb{R}^{n}\right):\left\|S_{L}(f)\right\|_{L_{w}^{p(\cdot)}}<\infty\right\}
$$

with respect to the quasi-norm

$$
\|f\|_{H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}=\left\|S_{L}(f)\right\|_{L_{w}^{p(\cdot)}}=\inf \left\{\lambda>0: \rho_{p(\cdot), w}\left(\frac{S_{L}(f)}{\lambda}\right) \leq 1\right\} .
$$

To introduce the molecular weighted variable Hardy spaces $H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$, we give the definition of a $(p(\cdot), w, M, \epsilon)_{L}$ molecule.
Definition 3.2.7. Let $L$ be an operator satisfying $\operatorname{Assumption}(A)$ and $\operatorname{Assumption}(B)$ and $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w: \mathbb{R}^{n} \rightarrow(0, \infty)$ be a Lebesgue measurable function. Assume that $M \in \mathbb{N}$ and $\epsilon \in(0, \infty)$. A function $m \in L^{2}\left(\mathbb{R}^{n}\right)$ is called $(p(\cdot), w, M, \epsilon)_{L}$ molecule, if $m \in R\left(L^{M}\right)$ (the range of $L^{M}$ ) and there exists a ball $B:=B\left(x_{B}, r_{B}\right) \subset \mathbb{R}^{n}$ with $x_{B} \in \mathbb{R}^{n}$ and $r_{B}>0$ such that, for every $k=0, \cdots, M$ and $j \in \mathbb{Z}_{+}$

$$
\begin{equation*}
\left\|\left(r_{B}^{-2} L^{-1}\right)^{k} m\right\|_{L^{2}\left(U_{j}(B)\right)} \leq 2^{-\epsilon j}\left|2^{j} B\right|^{1 / 2}\left\|\chi_{B}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)^{\prime}}^{-1} \tag{3.2.10}
\end{equation*}
$$

where for $j \in \mathbb{Z}_{+}$,

$$
U_{j}(B):=B\left(x_{B}, 2^{j} r_{B}\right) \backslash B\left(x_{B}, 2^{j-1} r_{B}\right)
$$

Remark 3.2.8. If $\epsilon \in\left(\frac{n}{2}, \infty\right)$, then for any $k \in\{0, \cdots, M\}$,

$$
\left\|\left(r_{B}^{-2} L^{-1}\right)^{k} m\right\|_{L^{2}\left(U_{j}(B)\right)} \leq C|B|^{1 / 2}\left\|\chi_{B}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)^{\prime}}^{-1}
$$

where $C$ is a positive constant independent of $m, k$ and $B$.
Definition 3.2.9. Let $L$ be an operator satisfying Assumption $(A)$ and Assumption(B). Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w \in W_{p(\cdot)}$. Assume that $M \in \mathbb{N}$ and $\epsilon \in(0, \infty)$. For a measurable function $f$ on $\mathbb{R}^{n}$,

$$
\begin{equation*}
f=\sum_{j=1}^{\infty} \lambda_{j} m_{j} \tag{3.2.11}
\end{equation*}
$$

is called molecular $(p(\cdot), w, M, \epsilon)$ - representation of $f$ if $\left\{m_{j}\right\}_{j \in \mathbb{N}}$ is a family of $(p(\cdot), w, M, \epsilon)_{L}$ molecules, the sum converges in $L^{2}\left(\mathbb{R}^{n}\right)$ and $\left\{\lambda_{j}\right\}_{j \in \mathbb{N}} \subset \mathbb{C}$ satisfies that

$$
\Lambda\left(\left\{\lambda_{j}\right\}_{j \in \mathbb{N}},\left\{B_{j}\right\}_{j \in \mathbb{N}}\right)<\infty
$$

where for any $j \in \mathbb{N}, B_{j}$ is the ball associated with $m_{j}$.
The space $\widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ is defined to be the set of all functions $f \in L^{2}\left(\mathbb{R}^{n}\right)$ which has a molecular $(p(\cdot), w, M, \epsilon)$-representation.

The molecular weighted variable Hardy spaces $H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ is the completion of $\widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ with respect to the quasi-norm

$$
\|f\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}:=\inf \left\{\Lambda\left(\left\{\lambda_{j}\right\}_{j \in \mathbb{N}},\left\{B_{j}\right\}_{j \in \mathbb{N}}\right)\right\}
$$

where the infimum is taken over all decompositions of $f$ as (3.2.11).
To establish the molecular characterization of $H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ we need the following technical Lemmas. Let $L$ be an operator satisfying Assumption (a) and (b) and $M \in \mathbb{N}$. The next lemma is a slight variant of [92, Proposition 3.10].

Lemma 3.2.10. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w \in W_{p(\cdot)}$. There exists constant $C$ and $\sigma \in\left(0, n s_{w}\right)$ such that, for any $j \in \mathbb{Z}_{+}$and $(p(\cdot), w, M, \epsilon)_{L}$ molecule $m$, associated with ball $B:=B\left(x_{R}, r_{B}\right) \subset \mathbb{R}^{n}$ with $x_{B} \in \mathbb{R}^{n}$ and $r_{b}>0$,

$$
\left\|S_{L}(m)\right\|_{L^{2}\left(U_{j}(B)\right)} \leq C 2^{-j \theta}\left|2^{j} B\right|^{1 / 2}\left\|\chi_{B}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{-1}
$$

The following lemma can be found in [92] or [9, Proposition 4.5(i)].
Lemma 3.2.11. For any $G \in T^{2}\left(\mathbb{R}_{+}^{n+1}\right)$ and $x \in \mathbb{R}^{n}$, the operator defined as

$$
\pi_{M, L}(G)(x):=\int_{0}^{\infty}\left(t^{2} L\right)^{M+1} e^{-t^{2} L}(G(\cdot, t))(x) \frac{d t}{t}
$$

is bounded from $T_{c}^{2}\left(\mathbb{R}_{+}^{n+1}\right)$ to $L^{2}\left(\mathbb{R}^{n}\right)$.
For the next lemma we refer the reader to [92].
Lemma 3.2.12. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right), w \in W_{p(\cdot)}$. Assume that $a$ is $a(p(\cdot), w, \infty)$-atom associated with ball $B \subset \mathbb{R}^{n}$. Then for any $M \in \mathbb{N}, \epsilon \in(0, \infty)$, there exists a positive constant $C_{M, \epsilon}$ depending only on $M$ and $\epsilon$ such that $C_{M, \epsilon} \pi_{M, L}(a)$ is a $(p(\cdot), w, M, \epsilon)_{L}$ molecule associated with the ball $B$.

Proposition 3.2.13. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$, $w \in W_{p(\cdot)}$. Let $M \in \mathbb{N}$ and $\epsilon \in(0, \infty)$. Then the set of all finite linear combinations of $(p(\cdot), w, M, \epsilon)_{L}$ molecule noted by $H_{L, w, f i n}^{p(\cdot), \epsilon, \epsilon}\left(\mathbb{R}^{n}\right)$ is dense in $H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ with respect to the quasi-norm $\|\cdot\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}$.

Proof. Let $g \in H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$. Then, by definition we know that for any $\delta \in(0, \infty)$ there exists a function $f \in \widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ such that

$$
\|g-f\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)} \leq \frac{\delta}{2},
$$

by the definition of $\widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ we conclude that, there exists $\left\{\lambda_{j}\right\}_{j \in \mathbb{N}} \subset \mathbb{C}$ and a family $\left\{m_{j}\right\}_{j \in \mathbb{N}}$ of $(p(\cdot), w, M, \epsilon)_{L}$ molecules, associated with balls $\left\{B_{j}\right\}_{j \in \mathbb{N}}$ of $\mathbb{R}^{n}$, such that $f=\sum_{j=1}^{\infty} \lambda_{j} m_{j}$ in $L^{2}\left(\mathbb{R}^{n}\right)$ and $\Lambda\left(\left\{\lambda_{j}\right\}_{j \in \mathbb{N}},\left\{B_{j}\right\}_{j \in \mathbb{N}}\right)<\infty$.

For any $N \in \mathbb{N}$, let $f_{N}=\sum_{j=1}^{N} \lambda_{j} m_{j}$, then we have

$$
\begin{aligned}
\left\|f-f_{N}\right\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)} & =\left\|\sum_{j=N+1}^{\infty} \lambda_{j} m_{j}\right\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)} \\
& \leq \Lambda\left(\left\{\lambda_{j}\right\}_{j=N+1}^{\infty}\left\{B_{j}\right\}_{j=N+1}^{\infty}\right) \\
& =\|\left\{\sum _ { j = N + 1 } ^ { \infty } \left[\frac{\left|\lambda_{j}\right| \chi_{B_{j}}}{\left.\left.\left\|\chi_{B_{j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}\right]^{\theta}\right\}^{1 / \theta} \|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}}\right.\right. \\
& =\| \sum_{j=N+1}^{\infty}\left[\frac{\left|\lambda_{j}\right| \chi_{B_{j}}}{\left.\left\|\chi_{B_{j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}\right]^{\theta} \|_{L_{w}^{p(\cdot) / \theta}\left(\mathbb{R}^{n}\right)}^{1 / \theta},}\right.
\end{aligned}
$$

where $\theta \in\left(0, s_{w}^{-1}\right)$. Since

$$
\Lambda\left(\left\{\lambda_{j}\right\}_{j=N+1}^{\infty},\left\{B_{j}\right\}_{j=N+1}^{\infty}\right)=\left\|\sum_{j=N+1}^{\infty}\left[\frac{\left|\lambda_{j}\right| \chi_{B_{j}}}{\left\|\chi_{B_{j}}\right\|_{L_{w}^{p(\cdot)}}\left(\mathbb{R}^{n}\right)}\right]^{\theta}\right\|_{\substack{L^{p(\cdot) / \theta} \\ w \\\left(\mathbb{R}^{n}\right)}}^{1 / \theta}<\infty
$$

it follows that, for almost every $x \in \mathbb{R}^{n}$,

$$
\lim _{N \rightarrow \infty} \sum_{j=N+1}^{\infty}\left[\frac{\left|\lambda_{j}\right| \chi_{B_{j}}}{\left\|\chi_{B_{j}}\right\|_{L^{p(\cdot)}\left(\mathbb{R}^{n}\right)}}\right]^{\theta}=0 .
$$

Combining this and the dominated convergence Theorem, we obtain

$$
\lim _{N \rightarrow \infty} \| \sum_{j=N+1}^{\infty}\left[\frac{\left|\lambda_{j}\right| \chi_{B_{j}}}{\left.\left.\left\|\chi_{B_{j}}\right\|_{L^{p(\cdot)}\left(\mathbb{R}^{n}\right)}\right]^{\theta} \|_{\substack{L^{p(\cdot) / \theta} \\ w w^{\theta}}}^{1 / \theta}=0, ~ \mathbb{R}^{n}\right)}=0\right.
$$

we conclude that,

$$
\lim _{N \rightarrow \infty}\left\|f-f_{N}\right\|_{H_{L, w}^{p, \cdot)}, M, \epsilon}\left(\mathbb{R}^{n}\right)=0
$$

Hence, we find that, for any $\delta \in(0, \infty)$, there exists some $N_{0} \in \mathbb{N}$ such that, for any $N>N_{0}$

$$
\left\|f-f_{N}\right\|_{H_{L, v}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}<\frac{\delta}{2} .
$$

Obviously, for any $N \in \mathbb{N}, f_{N} \in H_{L, w, \text { fin }}^{p(\cdot), \epsilon}\left(\mathbb{R}^{n}\right)$. Then for any $\delta \in(0, \infty)$ when $N>N_{0}$

$$
\left\|g-f_{N}\right\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)} \lesssim\|g-f\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}+\left\|f-f_{N}\right\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)} \lesssim \delta .
$$

Thus, $H_{L, w, f i n}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ is dense in $H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ with respect to the quasi-norm $\|\cdot\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}$.

The following Theorem deals with the molecular characterization of $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$
Theorem 3.2.14. Let $L$ be an operator satisfying Assumption $(A)$ and Assumption $(B)$. Let $p(\cdot) \in$ $\mathcal{P}\left(\mathbb{R}^{n}\right), w \in W_{p(\cdot)}$. Let $M \in\left(\frac{n}{2}\left[s_{w}-\frac{1}{2}\right], \infty\right) \cap \mathbb{N}$ and let $\epsilon \in\left(n s_{w}, \infty\right)$. Then $H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ and $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ coincide with equivalent quasi-norm.

To prove this theorem, we first show the following inclusion, $\widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right) \subset\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right) \cap\right.$ $\left.L^{2}\left(\mathbb{R}^{n}\right)\right]$.

Proposition 3.2.15. Let L be an operator satisfying Assumption $(A)$ and ( $B$ ). Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$, w $\in$ $W_{p(\cdot)}$. Let $M \in\left(\frac{n}{2}\left[s_{w}-\frac{1}{2}\right], \infty\right) \cap \mathbb{N}$ and let $\epsilon \in\left(n s_{w}, \infty\right)$. Then there exists a positive constant $C$ such that, for any $f \in \widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$,

$$
\|f\|_{H_{L, v}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{\widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}
$$

Proof. Let $f \in \widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$. Then by definition, we know that there exists $\left\{\lambda_{j}\right\}_{j \in \mathbb{N}} \subset \mathbb{C}$ and a family $\left\{m_{j}\right\}_{j \in \mathbb{N}}$ of $(p(\cdot), w, M, \epsilon)_{L}$ molecules associated with balls $\left\{B_{j}\right\}_{j \in \mathbb{N}}$ of $\mathbb{R}^{n}$, such that $f=\sum_{j=1}^{\infty} \lambda_{j} m_{j}$ in $L^{2}\left(\mathbb{R}^{n}\right)$ and

$$
\begin{equation*}
\|f\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)} \sim \Lambda\left(\left\{\lambda_{j}\right\}_{j \in \mathbb{N}},\left\{B_{j}\right\}_{j \in \mathbb{N}}\right) \tag{3.2.12}
\end{equation*}
$$

Since the operator $S_{L}$ is bounded on $L^{2}\left(\mathbb{R}^{n}\right)$, we find that

$$
\left\|S_{L}(f)-S_{L}\left(\sum_{j=1}^{N} \lambda_{j} m_{j}\right)\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \underset{N \rightarrow \infty}{\rightarrow} 0
$$

Hence, there exists a subsequence $\left\{S_{L}\left(\sum_{j=1}^{N_{k}} \lambda_{j} m_{j}\right)\right\}_{k \in \mathbb{N}}$ such that, for almost every $x \in \mathbb{R}^{n}$

$$
\lim _{k \rightarrow \infty} S_{L}\left(\sum_{j=1}^{N_{k}} \lambda_{j} m_{j}\right)(x)=S_{L}(f)(x) .
$$

Thus, for almost every $x \in \mathbb{R}^{n}$, we have

$$
\begin{aligned}
S_{L}(f)(x) & \leq \sum_{j=1}^{\infty}\left|\lambda_{j}\right| S_{L}\left(m_{j}\right)(x) \\
& =\sum_{j=1}^{\infty} \sum_{i=0}^{\infty}\left|\lambda_{j}\right| S_{L}\left(m_{j}\right)(x) \chi_{U_{i}\left(B_{j}\right)}(x),
\end{aligned}
$$

where, for each $j \in \mathbb{N}$ and $i \in \mathbb{Z}_{+}$,

$$
U_{j}(B):=B\left(x_{B}, 2^{j} r_{B}\right) \backslash B\left(x_{B}, 2^{j-1} r_{B}\right) .
$$

Thus

$$
\begin{align*}
\left\|S_{L}(f)\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{\theta} & =\left\|\left[S_{L}(f)\right]^{\theta}\right\|_{L_{w}^{p(\cdot) / \theta}}\left(\mathbb{R}^{n}\right)  \tag{3.2.13}\\
& \leq \sum_{i=0}^{\infty}\left\|\sum_{j=1}^{\infty}\left|\lambda_{j}\right|^{\theta}\left[S_{L}\left(m_{j}\right) \chi_{U_{i}\left(B_{j}\right)}\right]^{\theta}\right\|_{L^{p(\cdot) / \theta}\left(\mathbb{R}^{n}\right)} \\
& =\sum_{i=0}^{\infty}\left\|\left[\sum_{j=1}^{\infty}\left|\lambda_{j}\right|^{\theta}\left[S_{L}\left(m_{j}\right) \chi_{U_{i}\left(B_{j}\right)}\right]^{\theta}\right]^{1 / \theta}\right\| \|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{\theta} .
\end{align*}
$$

By virtue of Lemma 3.2.10, we find that, for any $j \in \mathbb{N}$ and $i \in \mathbb{Z}_{+}$,

$$
\begin{equation*}
\left\|2^{i \sigma}\right\| \chi_{B_{j}}\left\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} S_{L}\left(m_{j}\right) \chi_{U_{i}\left(B_{j}\right)}\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \lesssim\left|2^{i} B_{j}\right|^{1 / 2}, \tag{3.2.14}
\end{equation*}
$$

where $\sigma \in\left(n s_{w}, \infty\right)$. According to above estimate, we apply Lemma 3.1.5, to conclude that

$$
\left\|\left\{\sum_{j=1}^{\infty}\left|\lambda_{j}\right|^{\theta}\left[S_{L}\left(m_{j}\right) \chi_{U_{i}\left(B_{j}\right)}\right]^{\theta}\right\}^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \lesssim\left\|\left\{\sum_{j=1}^{\infty}\left[2^{-i \sigma}\left\|\chi_{B_{j}}\right\|_{L_{w}^{p^{p}}(-)\left(\mathbb{R}^{n}\right)}^{-1}\left|\lambda_{j}\right| \chi_{2^{i} B_{j}}\right]^{\theta}\right\}^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} .
$$

From the fact that

$$
\chi_{2^{i} B_{j}}(x) \leq 2^{i n} M\left(\chi_{B_{j}}\right)(x)
$$

we deduce that

$$
\left\|\left\{\sum_{j=1}^{\infty}\left|\lambda_{j}\right|^{\theta}\left[S_{L}\left(m_{j}\right) \chi_{U_{i}\left(B_{j}\right)}\right]^{\theta}\right\}^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \lesssim\left\|\left\{\sum_{j=1}^{\infty}\left[2^{-i \sigma}\left\|\chi_{B_{j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{-1}\left|\lambda_{j}\right| 2^{i n} M\left(\chi_{B_{j}}\right)\right]^{\theta}\right\}^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}
$$

Combining Remark 1.2.8 and Theorem 1.2.10, we find out, for $r \in\left(0, s_{w}^{-1}\right)$

$$
\begin{aligned}
& \|\left\{\sum_{j=1}^{\infty}\left|\lambda_{j}\right|^{\theta}\left[S_{L}\left(m_{j}\right) \chi_{U_{i}\left(B_{j}\right)}\right]^{\theta}\right\}^{1 / \theta} \|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|\left\{\sum_{j=1}^{\infty}\left[\left(2^{-i \sigma}\right)^{r}\left(2^{i n}\right)^{r}\left\|\chi_{B_{j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{-r}\left|\lambda_{j}\right|^{r} M\left(\chi_{B_{j}}\right)\right]^{\theta / r}\right\}^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim 2^{-i\left(\sigma-\frac{n}{r}\right)}\left\|\left\{\sum_{j=1}^{\infty}\left[M\left(\frac{\left|\lambda_{j}\right|^{r}}{\left\|\chi_{B_{j}}\right\|_{L_{w}^{r}}^{r p(\cdot)}\left(\mathbb{R}^{n}\right)} \chi_{B_{j}}\right)\right]^{\theta / r}\right\}^{r / \theta}\right\|_{L_{L_{w}}^{p(\cdot) / r}\left(\mathbb{R}^{n}\right)}^{1 / r} \\
& \lesssim 2^{-i\left(\sigma-\frac{n}{r}\right)}\left\|\left\{\sum_{j=1}^{\infty}\left[\frac{\left|\lambda_{j}\right|^{r}}{\left\|\chi_{B_{j}}\right\|_{L_{w}^{r}}^{r(\cdot)}\left(\mathbb{R}^{n}\right)} \chi_{B_{j}}\right]^{\theta / r}\right\}^{r / \theta}\right\|_{L_{w^{r}}^{p(\cdot) / r}\left(\mathbb{R}^{n}\right)}^{1 / r} \\
& \lesssim 2^{-i\left(\sigma-\frac{n}{r}\right)}\left\|\left\{\sum_{j=1}^{\infty}\left[\frac{\left|\lambda_{j}\right|}{\left\|\chi_{B_{j}}\right\|_{L_{w} p(\cdot)}\left(\mathbb{R}^{n}\right)} \chi_{B_{j}}\right]^{\theta}\right\}^{1 / \theta}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim 2^{-i\left(\sigma-\frac{n}{r}\right)} \Lambda\left(\left\{\lambda_{j}\right\}_{j \in \mathbb{N},}\left\{B_{j}\right\}_{j \in \mathbb{N}}\right) \sim 2^{-i\left(\sigma-\frac{n}{r}\right)}\|f\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}
\end{aligned}
$$

From the above inequality, (3.2.12) and (3.2.13), we infer that, for any $f \in \widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$,

$$
\begin{aligned}
\|f\|_{\left.H_{L, w}^{p \cdot( }\right)\left(\mathbb{R}^{n}\right)} & =\left\|S_{L}(f)\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\{\sum_{i=0}^{\infty} 2^{-i\left(\sigma-\frac{n}{r}\right)}\right\}^{1 / \theta}\|f\|_{H_{L, w}^{p(\cdot), M, \epsilon}}\left(\mathbb{R}^{n}\right) \\
& \sim\|f\|_{H_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)^{\prime}}
\end{aligned}
$$

which is the desired result.

The following proposition shows that $\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)\right]$ is a subset of $\widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$.
Proposition 3.2.16. Let L be an operator satisfying Assumption $(A)$ and (B). Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right), w \in$ $W_{p(\cdot)}$. Let $M \in \mathbb{N}$ and let $\epsilon \in(0, \infty)$. Then for any $f \in\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)\right]$, there exists $\left\{\lambda_{j}\right\}_{j \in \mathbb{N}} \subset \mathbb{C}$ and a family $\left\{m_{j}\right\}_{j \in \mathbb{N}}$ of $(p(\cdot), w, M, \epsilon)_{L}$ molecules, associated with balls $\left\{B_{j}\right\}_{j \in \mathbb{N}}$ of $\mathbb{R}^{n}$, such that $f=\sum_{j=1}^{\infty} \lambda_{j} m_{j}$ in $L^{2}\left(\mathbb{R}^{n}\right)$,

$$
\Lambda\left(\left\{\lambda_{j}\right\}_{j \in \mathbb{N}},\left\{B_{j}\right\}_{j \in \mathbb{N}}\right) \leq C\|f\|_{H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}
$$

Proof. Let $f \in H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)$ and let,

$$
F(x, t):=t^{2} L e^{-t^{2} L} f(x), \quad \text { for } \operatorname{all}(x, t) \in \mathbb{R}_{+}^{n+1}
$$

Then $F \in T_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)$. Hence, by Theorem 3.2.3 there exists $(p(\cdot), w, \infty)$-atoms $\left\{a_{j}\right\}_{j \in \mathbb{N}}$ associated with the balls $\left\{B_{j}\right\}_{j \in \mathbb{N}}$ respectively and numbers $\left\{\lambda_{j}\right\}_{j \in \mathbb{N}} \subset \mathbb{C}$ such that for $\operatorname{almost}(x, t) \in \mathbb{R}_{+}^{n+1}$,

$$
F(x, t)=\sum_{j \in \mathbb{N}} \lambda_{j} a_{j}(x, t), \text { in } T_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)
$$

By the $H_{\infty}$-calculi of $L$, we know that

$$
f=C_{M} \int_{0}^{\infty}\left(t^{2} L\right)^{M+1} e^{-t^{2} L}\left(t^{2} L e^{-t^{2} L}(f)\right) \frac{d t}{t}=\pi_{M, L}(F), \text { in } L^{2}\left(\mathbb{R}^{n}\right)
$$

where $C_{M} \int_{0}^{\infty} t^{2(M+2)} e^{-t^{2}} \frac{d t}{t}=1$. The fact that $\pi_{M, L}$ is bounded from $T^{2}\left(\mathbb{R}_{+}^{n+1}\right)$ to $L^{2}\left(\mathbb{R}^{n}\right)$, it follows that

$$
f=C_{M} \times \pi_{M, L}\left(\sum_{j \in \mathbb{N}} \lambda_{j} a_{j}\right)=C_{M}\left(\sum_{j \in \mathbb{N}} \lambda_{j} \pi_{M, L}\left(a_{j}\right)\right), \text { in } L^{2}\left(\mathbb{R}^{n}\right)
$$

From Lemma 3.2.11 we know that $m_{j}=\pi_{M, L}\left(a_{j}\right)$ is a multiple of a $(p(\cdot), w, M, \epsilon)_{L}$ molecule adopted to $B_{j}$, which implies the desired result.

Proof of Theorem 3.2.14. From the Proposition 3.2.15, Proposition 3.2.16 and a density argument we have, for $M \in\left(\frac{n}{2}\left[s_{w}-\frac{1}{2}\right], \infty\right) \cap \mathbb{N}$ and each $\epsilon \in\left(n s_{w}, \infty\right)$, then

$$
\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)\right]=\widetilde{H}_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)
$$

with equivalent quasi-norm.

### 3.3 Dual space

In this section, we study the duality of $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$. Here and hereafter, we denote by $L^{*}$ the adjoint operator of $L$ in $L^{2}\left(\mathbb{R}^{n}\right)$. Let us first recall some basic notions and definitions.

Definition 3.3.1. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w: \mathbb{R}^{n} \rightarrow(0, \infty)$ be a Lebesgue measurable function. Let $L$ be an operator satisfying Assumption $(A)$ and Assumption (B). Then for any $M \in \mathbb{N}$ and $\epsilon \in(0, \infty)$ define

$$
M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right):=\left\{f=L^{M}(g) \in L^{2}\left(\mathbb{R}^{n}\right): g \in \mathcal{D}\left(L^{M}\right),\|f\|_{M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}<\infty\right\}
$$

where $\mathcal{D}\left(L^{M}\right)$ denote the domain of the operator $L^{M}$ and

$$
\begin{equation*}
\|f\|_{M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}:=\sup _{j \in \mathbb{Z}_{+}}\left\{2^{j\left(\epsilon-\frac{n}{2}\right)}\left\|\chi_{B(0,1)}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \sum_{k=0}^{M}\left\|L^{-k}(f)\right\|_{L^{2}\left(U_{j}(B(0,1))\right.}\right\} . \tag{3.3.1}
\end{equation*}
$$

The dual space of $M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ is defined as the set of all the bounded linear functionals on $M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$, and denoted $\left[M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)\right]^{*}$. Then for for any $f \in\left[M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)\right]^{*}$ and $g \in M_{L,}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$, the duality between $\left[M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)\right]^{*}$ and $M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ denoted by $\langle f, g\rangle_{M}$. Let $M_{L, w}^{p(\cdot), M, *}\left(\mathbb{R}^{n}\right)=\cap_{\epsilon \in(0, \infty)}\left[M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)\right]^{*}$.
Definition 3.3.2. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w: \mathbb{R}^{n} \rightarrow(0, \infty)$ be a Lebesgue measurable function. Let $M \in \mathbb{N}$ and $L$ be an operator satisfying Assumption $(A)$ and Assumption (B). We say that an element $f \in M_{L, w}^{p(\cdot), M, *}\left(\mathbb{R}^{n}\right)$ is in $B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$ if
where the supremum is taken over all balls of $\mathbb{R}^{n}$.
The following result can be seen as an extention of [92, Proposition 4.3] to the weighted $M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$.

Proposition 3.3.3. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w \in W_{p(\cdot)}$. Let $M \in \mathbb{N}$ and $\epsilon \in(0, \infty)$.
If $f \in M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$. Then $f$ is a harmless positive constant multiple of a $(p(\cdot), w, M, \epsilon)_{L}$ molecule associated with ball $B(0,1)$. Conversely, if $m$ is $a(p(\cdot), w, M, \epsilon)_{L}$ molecule associated with ball $B \subset \mathbb{R}^{n}$, then $m \in M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$.

The following three estimates play an important roles in the proof of our main results in this section. The proof of next lemma can be done with similar arguments of [92, Lemma 9].

Lemma 3.3.4. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right), w \in W_{p(\cdot)}$ and $M \in \mathbb{N}$. Then $f \in B M O_{L, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$ is equivalent to that

$$
\|f\|_{B M O_{L, w}^{p(\cdot), M, r e s}\left(\mathbb{R}^{n}\right)}:=\sup _{B \subset \mathbb{R}^{n}} \frac{|B|^{1 / 2}}{\left\|\chi_{B}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}}\left[\int_{B}\left|\left(I-\left(I+r_{B}^{2} L\right)^{-1}\right)^{M}(f)(x)\right|^{2} d x\right]^{1 / 2}<\infty,
$$

where the supremum is taken over all balls of $\mathbb{R}^{n}$. Moreover, there exists a positive constant $C$ such that, for any $f \in B M O_{L, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$, we have

$$
C^{-1}\|f\|_{B M O_{L, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)} \leq\|f\|_{B M O_{L, w}^{p(\cdot), M, r e s}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{B M O_{L, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)}
$$

The next lemma play a key role in the proof of the main results of this section, for more details we refer the reader to [92, Lemma 4.5.].

Lemma 3.3.5. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w \in W_{p(\cdot)}$. Let $\epsilon, \widetilde{\epsilon} \in(0, \infty)$ and $M \in \mathbb{N}$ and $\widetilde{M}>M+\widetilde{\epsilon}+\frac{n}{4}$. Suppose that $f \in M_{L, w}^{p(\cdot), M, *}\left(\mathbb{R}^{n}\right)$ satisfies

$$
\begin{equation*}
\int_{\mathbb{R}^{n}} \frac{\left|\left[I-\left(I+L^{*}\right)^{-1}\right]^{M}(f)(x)\right|^{2}}{1+|x|^{n+\widetilde{\epsilon}}} d x<\infty \tag{3.3.2}
\end{equation*}
$$

Then, for any $(p(\cdot), w, \widetilde{M}, \epsilon)_{L}$ molecule $m$, it hold true that,

$$
\langle f, m\rangle_{M}=C_{M} \iint_{\mathbb{R}_{+}^{n+1}}\left(t^{2} L^{*}\right)^{M} e^{-t^{2} L^{*}}(f)(x) \overline{t^{2} L e^{-t^{2} L}(m)(x)} \frac{d x d t}{t}
$$

where $C_{M}$ is a positive constant, depending on $M$, which satisfies $C_{M} \int_{0}^{\infty} t^{M+1} e^{2 t^{2}} \frac{d t}{t}=1$.
The proof of the following lemma is similar to that of [41, Lemma 8.3] and [92, Lemma 4.7.].

Lemma 3.3.6. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w \in W_{p(\cdot)}$ and $M \in \mathbb{N}$. Then there exists a positive constant $C$ such that, for any $f \in B M O_{L, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$,

$$
\sup _{B \subset \mathbb{R}^{n}} \frac{|B|^{1 / 2}}{\left\|\chi_{B}\right\|_{L_{w}^{p(\cdot)}}\left(\mathbb{R}^{n}\right)}\left[\iint_{\widehat{B}}\left|\left(t^{2} L\right)^{M} e^{-t^{2} L}(f)(x)\right|^{2} \frac{d x d t}{t}\right]^{1 / 2}<\|f\|_{B M O_{L, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)^{\prime}}
$$

where the supremum is taken over all balls of $\mathbb{R}^{n}$.
Proposition 3.3.7. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w \in W_{p(\cdot)}$. For any $\widetilde{\epsilon} \in\left(2 n s_{w}, \infty\right), M \in \mathbb{N}$ and $f \in$ $B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$. Then $f$ satisfies (3.3.2).
Proof. Let $f \in B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$, then by Lemma 3.3.4, we have

$$
\sup _{B \subset \mathbb{R}^{n}} \frac{|B|^{1 / 2}}{\left\|\chi_{B}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}}\left[\int_{B}\left|\left[I-\left(I+r_{B}^{2} L^{*}\right)^{-1}\right]^{M}(f)(x)\right|^{2} d x\right]^{1 / 2}<\infty .
$$

We write

$$
\begin{aligned}
J & =\int_{\mathbb{R}^{n}} \frac{\left|\left[I-\left(I+L^{*}\right)^{-1}\right]^{M}(f)(x)\right|^{2}}{1+|x|^{n+\widetilde{\epsilon}}} d x \\
& =\sum_{j=0}^{\infty} \int_{U_{j}(B(0,1))} \frac{\left|\left[I-\left(I+L^{*}\right)^{-1}\right]^{M}(f)(x)\right|^{2}}{1+|x|^{n+\widetilde{\epsilon}}} d x \\
& \leq \sum_{j=0}^{\infty} 2^{-j(n+\widetilde{\epsilon})} \int_{U_{j}(B(0,1))}\left|\left[I-\left(I+L^{*}\right)^{-1}\right]^{M}(f)(x)\right|^{2} d x .
\end{aligned}
$$

For any $j \in \mathbb{Z}_{+}$, choose $k_{j} \in \mathbb{Z}$ such that $\widetilde{C} \delta^{k_{j}} \leq 2^{j}<\widetilde{C} \delta^{k_{j}-1}$. Let

$$
M_{j}:=\left\{\beta \in I_{k_{0}}: Q_{\beta, k_{0}} \cap B\left(0, \widetilde{C} \delta^{k_{j}-1}\right)\right\} \neq \varnothing
$$

By Lemma 3.1.1(1), we find for any $j \in \mathbb{Z}_{+}$

$$
U_{j}(B(0,1)) \subset B\left(0, \widetilde{C} \delta^{k_{j}-1}\right) \subset \cup_{\beta \in M_{j}} Q_{\beta, k_{0}}
$$

Moreover, by (4),(5) of Lemma 3.1.1 and the fact that $\widetilde{C} \delta^{k_{0}} \leq 1$, we know that, for any $\beta \in M_{j}$, there exists some $z_{\beta, k_{0}} \in Q_{\beta, k_{0}}$ such that,

$$
B\left(z_{\beta, k_{0}} ; a_{0} \delta^{k_{0}}\right) \subset Q_{\beta, k_{0}} \subset B\left(z_{\beta, k_{0}} ; \widetilde{C} \delta^{k_{0}}\right) \subset B\left(z_{\beta, k_{0}}, 1\right)
$$

From Lemma 3.3.4, we infer that

$$
\begin{aligned}
J & \lesssim \sum_{j=0}^{\infty} 2^{-j(n+\widetilde{\epsilon})}\left\{\sum_{\beta \in M_{j}} \int_{\left.B\left(z_{\beta, k_{0}}, 1\right)\right)}\left|\left[I-\left(I+L^{*}\right)^{-1}\right]^{M}(f)(x)\right|^{2} d x\right\} \\
& \lesssim \sum_{j=0}^{\infty} 2^{-j(n+\widetilde{\epsilon})}\left\{\sum_{\beta \in M_{j}}\left\|\chi_{B\left(z_{\beta, k_{0}}, 1\right)}\right\|_{L_{w}^{p(\cdot)}}^{2}\left|B\left(z_{\beta, k_{0}}, 1\right)\right|^{-1}\right\}\|f\|_{B M O_{L^{*}, z v}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)}^{2}
\end{aligned}
$$

According to Lemma 3.1.1(4) with $k=k_{0}$, we conclude that,

$$
B\left(z_{\beta, k_{0}}, 1\right) \subset B\left(0,1+1+\widetilde{C} \delta^{k_{j}-1}\right) \subset B\left(0,3 \widetilde{C} \delta^{k_{j}-1}\right) \subset B\left(0,3 \delta^{-1} 2^{j}\right)
$$

The fact that, for any $r \in\left(0, s_{w}^{-1}\right), \chi_{B\left(0,3 \delta^{-1} 2^{j}\right)} \lesssim 2^{j \frac{n}{r}}\left[M\left(\chi_{\chi_{B(0,1)}}\right)\right]^{1 / r}$ and Theorem 1.2.10, implies that

$$
\begin{aligned}
\left\|\chi_{B\left(z_{\beta, k_{0}}, 1\right)}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} & \lesssim\left\|\chi_{B\left(0,3 \delta^{-1} 2^{j}\right)}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim 2^{j \frac{n}{r}}\left\|\left[M\left(\chi_{B(0,1)}\right)\right]^{1 / r}\right\|_{L_{w}^{p(\cdot)}}\left(\mathbb{R}^{n}\right) \\
& \lesssim 2^{j \frac{n}{r}}\left\|M\left(\chi_{B(0,1)}\right)\right\|_{L_{w p^{p}}^{p(\cdot) / r}}^{\left.1 / \mathbb{R}^{n}\right)} \\
& \lesssim 2^{j \frac{n}{r}}\left\|\chi_{B(0,1)}\right\|_{L_{w r^{r}}^{1 / r}\left(\mathbb{R}^{p}\right) / r}^{\left.1 / \mathbb{R}^{n}\right)} \\
& \lesssim 2^{j \frac{n}{r}}\left\|\chi_{B(0,1)}\right\|_{L_{w}^{p \cdot()}\left(\mathbb{R}^{n}\right)^{\prime}} .
\end{aligned}
$$

On the other hand, from (3.1.1), we have,

$$
\left|B\left(z_{\beta, k_{0}}, 1\right)\right|^{-1} \lesssim 2^{j n}|B(0,1)|^{-1}
$$

Since $\widetilde{\epsilon} \in\left(2 n s_{w}, \infty\right)$, it follows that there exists $r \in\left(0, s_{w}^{-1}\right)$ such that $\widetilde{\epsilon} \in\left(\frac{2 n}{r}, \infty\right)$, we obtain

$$
\begin{aligned}
J & \lesssim \sum_{j=0}^{\infty} 2^{-j(n+\widetilde{\epsilon})}\left\{\sum_{\beta \in M_{j}} 2^{j \frac{n}{r}}\left\|\chi_{B(0,1)}\right\|_{L_{w}^{p(\cdot)}}^{2} 2^{j n}|B(0,1)|^{-1}\right\}\|f\|_{B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)}^{2} \\
& \lesssim \sum_{j=0}^{\infty} 2^{-j\left(\widetilde{\epsilon}-\frac{2 n}{r}\right)}\|f\|_{B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)}^{2}\left\|\chi_{B(0,1)}\right\|_{L_{w}^{p(\cdot)}}^{2}|B(0,1)|^{-1} \\
& \sim\|f\|_{B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)}^{2}\left\|\chi_{B(0,1)}\right\|_{L_{w}^{p(\cdot)}}^{2}|B(0,1)|^{-1}<\infty .
\end{aligned}
$$

Hence $f$ satisfies (3.3.2).
In the following result, we prove the duality of the space $H_{L^{*}, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$.
Theorem 3.3.8. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $w \in W_{p(\cdot)}$. Let $M \in\left(\frac{n}{2}\left[s_{w}-\frac{1}{2}\right], \infty\right) \cap \mathbb{N}$ and $\epsilon \in\left(n s_{w}, \infty\right)$ and $\tilde{M}>M+2 n s_{w}+\frac{n}{4}$. Then, we have,

$$
\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)\right]^{*}=B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)
$$

with equivalent norms. More precisely,
(i) Let $g \in\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)\right]^{*}$. Then $g \in B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$, for any $f \in H_{L, w, f i n}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$, it hold true $g(f)=\langle g, f\rangle_{M}$ and

$$
\|g\|_{B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)} \leq C\|g\|_{\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)\right]^{*} .}
$$

(ii) Conversely, let $g \in B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$. Then, for any $f \in H_{L, w, f i n}^{p(\cdot), \widetilde{M}, \epsilon}\left(\mathbb{R}^{n}\right)$, the linear functional $\ell_{g}$ defined by $\ell_{g}(f)=\langle g, f\rangle_{M}$, has a unique bounded extension to $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ and there exists a positive $C$ such that, for any $g \in B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$

$$
\left\|\ell_{g}\right\|_{\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)\right]^{*}} \leq C\|g\|_{B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)} .
$$

Proof. We first show $(i)$. Let $g \in\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)\right]^{*}$. Then, for any $f \in H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, we have

$$
|g(f)| \leq\|g\|_{\left[H_{L, w}^{p \cdot()}\left(\mathbb{R}^{n}\right)\right]^{*}\|f\|_{H_{L, w}^{p \cdot()}\left(\mathbb{R}^{n}\right)^{\prime}}, ~}
$$

we know that, for any $(p(\cdot), w, M, \epsilon)_{L}$ molecule $m$,

$$
\|m\|_{H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \lesssim 1
$$

thus

$$
\begin{equation*}
|g(m)| \leq\|g\|_{\left[H_{L, v}^{p(\cdot)}\left(\mathbb{R}^{n}\right)\right]^{*}} \tag{3.3.3}
\end{equation*}
$$

On the other hand, by Proposition 3.3.3, we find that for any $h \in M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$ with
$\|h\|_{M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)}=1$, where $h$ is harmless positive constant multiple of a $(p(\cdot), w, M, \epsilon)_{L}$ molecule associated with the ball $B(0,1)$. From (3.3.3), we find that for any $\epsilon \in(0, \infty), g \in$ $\left[M_{L, w}^{p(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)\right]^{*}$. Hence, $g \in M_{L, w}^{p(\cdot), M, *}\left(\mathbb{R}^{n}\right)$ and for any $h \in M_{L, w}^{p \cdot(\cdot), M, \epsilon}\left(\mathbb{R}^{n}\right)$,

$$
\langle g, h\rangle_{M}=g(h) .
$$

Next we show that

$$
\|g\|_{B M O_{L, t w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)} \leq C\|g\|_{\left[H_{L, \psi}^{p(\cdot)}\left(\mathbb{R}^{n}\right)\right]^{*}} .
$$

We take a ball $B \subset \mathbb{R}^{n}, h \in L^{2}(B)$ with $\|h\|_{L^{2}(B)}=1$. Following the argument used in [94], we know that $\frac{|B|^{1 / 2}}{\|x\|_{L_{w}^{p} \cdot(\cdot)}^{\left(\mathbb{R}^{n}\right)}}\left(I-e^{r_{B}^{2} L}\right)^{M}(h)$ is a harmless positive constant multiple of a $(p(\cdot), w, M, \epsilon)_{L}$ molecule. Therefore,

$$
\begin{aligned}
\left|\frac{|B|^{1 / 2}}{\|\chi\|_{L_{w}^{p(\cdot)}}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \int_{B}\left(I-e^{r_{B}^{2} L^{*}}\right)^{M}(g)(x) h(x) d x\right| & =\left|\left\langle g, \frac{|B|^{1 / 2}}{\|\chi\|_{L_{w}^{p()}()}\left(\mathbb{R}^{n}\right)}\left(I-r^{r_{B}^{2} L}\right)^{M}(h)\right\rangle_{M}\right| \\
& \lesssim\|g\|_{\left[H_{L, w}^{p(.)}\left(\mathbb{R}^{n}\right)\right]^{*,}}
\end{aligned}
$$

which implies that, for any ball $B \subset \mathbb{R}^{n}$

$$
\left.\frac{|B|^{1 / 2}}{\|\chi\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}}\left\{\int_{B}\left|\left(I-e^{r^{2} L^{*}}\right)^{M}(g)(x)\right|^{2} d x\right\}^{1 / 2} \lesssim\|g\|_{\left[H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)\right.}\right]^{*} .
$$

Hence we get the desired result. Now we turn to prove (ii). Let $g \in B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)$. For $f \in H_{L, w, f i n}^{p(\cdot), \tilde{M}, \epsilon}\left(\mathbb{R}^{n}\right)$, we define

$$
\ell_{g}(f):=\int_{\mathbb{R}^{n}} f(x) g(x) d x .
$$

Since $f \in H_{L, w, f i n}^{p(\cdot), \tilde{M}, \epsilon}\left(\mathbb{R}^{n}\right) \subset H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, we have that $t^{2} L e^{-t^{2} L} f \in T_{w}^{p(\cdot)}\left(\mathbb{R}_{+}^{n+1}\right)$. Then by Theorem 3.2.3, we can assume that $t^{2} L e^{-t^{2} L} f=\sum_{j \in \mathbb{N}} \lambda_{j} a_{j}$, where $\left\{a_{j}\right\}_{j \in \mathbb{N}}$ is a sequence of $(p(\cdot), w, \infty)$-atoms supported of $\left\{B_{j}\right\}_{j \in \mathbb{N}}$. By Proposition 3.3.7 we know that $g$ satisfies inequality (3.3.2) for $\tilde{\epsilon}>2 n s_{w}$. Thus, it follows from Lemma 3.3.5, the Hölder inequality and Lemma 3.3.6,

$$
\begin{aligned}
\ell_{g}(f) & =\left|C_{M} \iint_{\mathbb{R}_{+}^{n+1}}\left(t^{2} L^{*}\right)^{M} e^{-t^{2} L^{*}}(g)(x) \overline{t^{2} L e^{-t^{2} L}(f)(x)} \frac{d x d t}{t}\right| \\
& \lesssim \sum_{j=1}^{\infty}\left|\lambda_{j}\right| \iint_{\mathbb{R}_{+}^{n+1}}\left|\left(t^{2} L^{*}\right)^{M} e^{-t^{2} L^{*}}(g)(x)\right|\left|a_{j}(x, t)\right| \frac{d x d t}{t} \\
& \lesssim \sum_{j=1}^{\infty}\left|\lambda_{j}\right|\left[\iint_{\widehat{B}_{j}}\left|\left(t^{2} L^{*}\right)^{M} e^{-t^{2} L^{*}}(g)(x)\right|^{2} \frac{d x d t}{t}\right]^{1 / 2}\left[\iint_{\widehat{B_{j}}}\left|a_{j}(x, t)\right|^{2} \frac{d x d t}{t}\right]^{1 / 2} \\
& \lesssim \sum_{j=1}^{\infty}\left|\lambda_{j}\right|\|g\|_{B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)} \\
& \lesssim \Lambda\left(\left\{\lambda_{j}\right\}_{j \in \mathbb{N}},\left\{B_{j}\right\}_{j \in \mathbb{N}}\right)\|g\|_{B M O_{L^{*}, w}^{p(\cdot), M}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\|f\|_{H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}\|g\|_{B M O_{L^{*}, w}^{p(\cdot), M}} \mathbb{R}_{\left.\mathbb{R}^{n}\right)} .
\end{aligned}
$$

Hence, the proof of Theorem 3.3.8 is achieved.

## CHAPTER 4

## WEIGHTED HARDY-LORENTZ SPACES WITH VARIABLE EXPONENTS

In this chapter, we are interested essentially with the weighted Hardy-Lorentz spaces with variable exponents. First, we define the variable weighted Lorentz spaces, and we prove the boundedness of the maximal operator on $L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$ for $1<p_{-} \leq p_{+}<\infty$ and $w \in$ $W_{p(\cdot)}\left(\mathbb{R}^{n}\right)$, then we introduce the weighted Hardy-Lorentz spaces with variable exponent and we establish its atomic decomposition.

### 4.1 Preparation and helpful results

For $N \in \mathbb{N}$, let

$$
\begin{equation*}
\mathcal{F}_{N}\left(\mathbb{R}^{n}\right):=\left\{\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right): \sum_{\beta \in \mathbb{Z}_{+}^{n},|\beta| \leq N} \sup _{x \in \mathbb{R}^{n}}\left[(1+|x|)^{N}\left|D^{\beta} \psi(x)\right|\right] \leq 1\right\}, \tag{4.1.1}
\end{equation*}
$$

where, for any $\beta:=\left(\beta_{1}, \ldots, \beta_{n}\right) \in \mathbb{Z}_{+}^{n},|\beta|=\beta_{1}+\ldots+\beta_{n}$ and $D^{\beta}:=\left(\frac{\partial}{\partial x_{1}}\right)^{\beta_{1}} \ldots\left(\frac{\partial}{\partial x_{n}}\right)^{\beta_{n}}$.
For all $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$, define radial grand maximal function $f_{N,+}^{*}$ of $f$ by

$$
\begin{equation*}
f_{N,+}^{*}:=\sup \left\{\left|f * \psi_{t}(x)\right|: t \in(0, \infty) \quad \text { and } \quad \psi \in \mathcal{F}_{N}\left(\mathbb{R}^{n}\right)\right\} \tag{4.1.2}
\end{equation*}
$$

where, for all $t \in(0, \infty)$ and $\xi \in \mathbb{R}^{n}, \psi_{t}:=t^{-n} \psi(\xi / t)$.

Definition 4.1.1. Let $p(\cdot) \in C^{\log }\left(\mathbb{R}^{n}\right)$ and $N \in\left(\frac{n}{p_{-}}+n+1, \infty\right)$ be a positive integer. The weighted Hardy-Lorentz space $H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$ is defined to be the set of all $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ such that $f_{N,+}^{*} \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, equipped with the quasi-norm

$$
\begin{equation*}
\|f\|_{H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}=\left\|f_{N,+}^{*}\right\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \tag{4.1.3}
\end{equation*}
$$

Next, we introduce the definition of the atomic weighted Hardy-Lorentz space, and the $(p(\cdot), r, s)$-atom is given in the following definition.

Definition 4.1.2. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$ and $r>1$. Fix an integer $d_{w}=n\left(s_{w}-1\right)$. A measurable function $a$ on $\mathbb{R}^{n}$ is called a $(p(\cdot), r, s)$-atom if there exists a ball $B$ such that
(1) $\operatorname{supp} a \subset B$;
(2) $\|a\|_{L_{w}^{r}\left(\mathbb{R}^{n}\right)} \leq \frac{|B|^{1 / r}}{\left\|\chi_{B}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}}$;
(3) there exist $s \geq d_{w}$ such that $\int_{\mathbb{R}^{n}} a(x) x^{\alpha} d x=0$ for all $\alpha \in \mathbb{Z}_{+}$with $|\alpha| \leq s$.

Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right),\left\{\lambda_{k, j}\right\}_{k \in \mathbb{Z}, j \in \mathbb{N}}$ be a sequence of numbers in $\mathbb{C}$ and $\left\{B_{k, j}\right\}_{k \in \mathbb{Z}, j \in \mathbb{N}}$ be a sequence of balls in $\mathbb{R}^{n}$. Define

$$
\begin{equation*}
\mathcal{A}\left(\left\{\lambda_{k, j}\right\}_{k \in \mathbb{Z}, j \in \mathbb{N}},\left\{B_{k, j}\right\}_{k \in \mathbb{Z}, j \in \mathbb{N}}\right):=\left(\sum_{k \in \mathbb{Z}}\left\|\left\{\sum_{j \in \mathbb{N}}\left[\frac{\left|\lambda_{k, j}\right| \chi_{B_{k, j}}}{\left\|\chi_{B_{k, j}}\right\|_{L^{p(\cdot)}\left(\mathbb{R}^{n}\right)}}\right]^{\theta}\right\}^{\frac{1}{\theta}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{\frac{1}{q}}, \tag{4.1.4}
\end{equation*}
$$

here and hereafter $\theta \in\left(0, s_{w}^{-1}\right)$.
Definition 4.1.3. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right), r \in(1, \infty]$ and $s$ as in Definition 4.1.2. The weighted variable atomic Hardy-Lorentz space $H_{w, \text { atom, }, \mathrm{S}}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$ is defined to be the space of all functions $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ which can be decomposed as

$$
\begin{equation*}
f=\sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{k, j} a_{k, j} \quad \text { in } \quad \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right) \tag{4.1.5}
\end{equation*}
$$

where $\left\{a_{k, j}\right\}_{k \in \mathbb{Z}, j \in \mathbb{N}}$ is a sequence of $(p(\cdot), r, s)$-atoms, associated with balls $\left\{B_{k, j}\right\}_{k \in \mathbb{Z}, j \in \mathbb{N}}$, satisfying that, for all $x \in \mathbb{R}^{n}$ and $k \in \mathbb{Z}, \sum_{j \in \mathbb{N}} \chi_{B_{k, j}}(x) \leq \widehat{A}$ with $\widehat{A}$ being a positive constant independent of $x$ and $k$, and for all $k \in \mathbb{Z}$ and $j \in \mathbb{N}, \lambda_{k, j}=C 2^{k}\left\|\chi_{B_{k, j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}$ with $C$ being a positive constant independent of $k$ and $j$. Moreover, for any $f \in H_{w, \text { atom,r,s }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, we define

$$
\begin{equation*}
\|f\|_{H_{w, \text { atom }, \text {,s,s }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}:=\inf \mathcal{A}\left(\left\{\lambda_{k, j}\right\}_{k \in \mathbb{Z}, j \in \mathbb{N}},\left\{B_{k, j}\right\}_{k \in \mathbb{Z}, j \in \mathbb{N}}\right), \tag{4.1.6}
\end{equation*}
$$

where the infimum is taken over all the decompositions of $f$ as (4.1.5).
Now, we give a characterization for the weighted variable Hardy-Lorentz spaces via the radial or non-tangential maximal functions.

Definition 4.1.4. Let $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ such that $\int_{\mathbb{R}^{n}} \psi(x) d x \neq 0$. Let $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$. The radial maximal function of $f$ associated to the function $\psi$ is defined by setting, for all $x \in \mathbb{R}^{n}$,

$$
\psi_{+}^{*}(f)(x):=\sup _{t \in(0, \infty)}\left|f \star \psi_{t}(x)\right|
$$

and for any $a \in(0, \infty)$, the non-tangential maximal function of $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ associated to $\psi$ is defined by setting, for all $x \in \mathbb{R}^{n}$,

$$
\psi_{\nabla, a}^{*}(f)(x):=\sup _{t \in(0, \infty)}\left|f \star \psi_{t}(x)\right| .
$$

When $a=1$, we use the notation $\psi_{\nabla}^{*}(f)(x)$ instead of $\psi_{\nabla, a}^{*}(f)(x)$. For any $N \in \mathbb{N}$ and $0<a<\infty$, the non-tangential grand maximal function of $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ is defined by setting, for all $x \in \mathbb{R}^{n}$,

$$
f_{N, \nabla, a}^{*}(x):=\sup _{\psi \in \mathcal{F}_{N}} \sup _{t \in(0, \infty),|y-x|<a t}\left|f \star \psi_{t}(y)\right| .
$$

When $a=1$, we use the notation $f_{N, \nabla}^{*}$ instead of $f_{N, \nabla, a}^{*}$. A distribution $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ is called a bounded distribution if, for all $\phi \in \mathcal{S}\left(\mathbb{R}^{n}\right), f \star \phi \in L^{\infty}\left(\mathbb{R}^{n}\right)$. For a bounded distribution $f$, its the non-tangential maximal function, with respect to Poisson kernels $\left\{\mathbf{P}_{t}\right\}_{t>0}$, is defined by setting for all $x \in \mathbb{R}^{n}$,

$$
\mathcal{N}(f)(x):=\sup _{t \in(0, \infty),|y-x|<t}\left|f \star \mathbf{P}_{t}(y)\right|
$$

where, for all $x \in \mathbb{R}^{n}$, and $t \in(0, \infty)$,

$$
\mathbf{P}_{t}(y):=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{\left(t^{2}+|x|^{2}\right)^{\frac{n+1}{2}}}
$$

where $\Gamma$ denotes the Gamma function.
For $s \in \mathbb{Z}$, let $\mathbb{P}^{s}\left(\mathbb{R}^{n}\right)$ be the set of all polynomials having degree at most $s$. Denote by Q the set of all cubes whose edges are parallel to the coordinate axis. For locally function
integrable function $f$, a cubes $Q \in Q$ and a nonnegative integer $s$, there exists a unique polynomial $\mathbf{P}$ such that for any polynomial $\mathbf{R} \in \mathbb{P}^{s}\left(\mathbb{R}^{n}\right)$,

$$
\int_{Q}(f(x)-\mathbf{P}(x)) \mathbf{R}(x) d x=0
$$

Denote this unique polynomial $\mathbf{P}$ by $\mathbf{P}_{Q}^{s} f$. For $p(\cdot) \in \mathcal{P}$, the space $B M O_{w, p(\cdot), 1}\left(\mathbb{R}^{n}\right)$ is defined as

$$
B M O_{w, p(\cdot), 1}\left(\mathbb{R}^{n}\right):=\left\{f \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{n}\right):\|f\|_{B M O_{w, p(\cdot), 1}\left(\mathbb{R}^{n}\right)}<\infty\right\},
$$

where

$$
\|f\|_{B M O_{p(\cdot), 1}\left(\mathbb{R}^{n}\right)}:=\sup _{Q \in \mathrm{Q}} \frac{1}{\left\|\chi_{Q}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}} \int_{Q}\left|f(x)-\mathbf{P}_{Q}^{s} f(x)\right| d x .
$$

The following lemma is a slight variant of [101, Lemma 2.8].
Lemma 4.1.5. Let $p \in \mathcal{P}\left(\mathbb{R}^{n}\right)$, If $f \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ then $f \in B M O_{w, p(\cdot), 1}$.
The next lemma is the Calderón formula, we refer to the reference [10, p.219].
Lemma 4.1.6. Let $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ be such that $\operatorname{supp} \psi \subset B(0,1)$ and let $\int_{\mathbb{R}^{n}} \psi(x) d x=0$. Then, there exists a function $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ such that its Fourier transform $\widehat{\phi}$ has a compact support away from the origin and, for all $x \in \mathbb{R}^{n} \backslash\{0\}$.

$$
\int_{0}^{\infty} \widehat{\psi}(t x) \widehat{\phi}(t x) \frac{d t}{t}=1
$$

We finish this section by the following useful lemma.
Lemma 4.1.7. Let $(i, j) \in \mathbb{Z} \times \mathbb{N}$ and $B_{i j}:=B_{i j}\left(x_{i j}, r_{i j}\right)$ for some $x_{i j} \in \mathbb{R}^{n}$ and $r_{i j} \in(0, \infty)$. Then, for any $x \in\left(2 B_{i j}\right)^{\complement}$, we have

$$
\begin{equation*}
\left(a_{i j}\right)^{*}(x) \lesssim \frac{\left[M\left(\chi_{B_{i j}}(x)\right]^{\frac{n+d_{w}+1}{n}}\right.}{\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}} \tag{4.1.7}
\end{equation*}
$$

Proof. Let $\phi \in \mathcal{F}_{N}$. Combining the vanishing moment condition of $a_{i j}$, the Taylor remainder theorem and the Hölder inequality together, we find out, for $i \in \mathbb{Z} \cap\left[i_{0}, \infty\right), j \in \mathbb{N}$ and $t \in$ $(0, \infty)$,

$$
\begin{aligned}
\left|a_{i j} \star \phi_{t}(x)\right| & =\left\lvert\, \int_{B_{i j}} a_{i j}(y)\left[\left.\phi\left(\frac{x-y}{t}\right)-\sum_{|\beta| \leq s} \frac{D^{\beta} \phi\left(\frac{x-x_{i j}}{t}\right)}{\beta!}\left(\frac{x_{i j}-y}{t}\right)^{\beta} \frac{d t}{t^{n}} \right\rvert\,\right.\right. \\
& \lesssim \int_{B_{i j}}\left|a_{i j}(y)\right| \frac{\left|y-x_{i j}\right|^{d_{w}+1}}{\left|x-x_{i j}\right|^{n+d_{w}+1}} d y \\
& \lesssim \frac{r_{i j}^{d_{w j}+1}}{\mid x-x_{i j \mid n+d_{w}+1}}\left(\int_{B_{i j}}\left|a_{i j}(y)\right|^{q} d y\right)^{1 / q}\left(\int_{B_{i j}} d y\right)^{1 / q^{\prime}} \\
& \lesssim\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{-1}\left(\frac{r_{i j}}{\left|x-x_{i j}\right|}\right)^{n+d_{w}+1} .
\end{aligned}
$$

Hence, for any $x \in\left(2 B_{i j}\right)^{\complement}$,

$$
\begin{aligned}
\left(a_{i j}\right) *(x) & \lesssim\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{-1}\left(\frac{r_{i j}}{\left|x-x_{i j}\right|}\right)^{n+d_{w}+1} \\
& \lesssim \frac{\left[M\left(\chi_{B_{i j}}(x)\right]^{\frac{n+d_{w}+1}{n}}\right.}{\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}} .
\end{aligned}
$$

### 4.2 The Hardy-Littlewood maximal Operator on the variable weighted Lorentz spaces

The interpolation theorem is crucial in the proof of the boundedness of the Hardy-Littlewood maximal operator on the weighted Lorentz spaces with variable exponents.

Theorem 4.2.1. Let $p(\cdot) \in \mathcal{P}\left(\mathbb{R}^{n}\right)$, let $p_{1} \in(0,1)$ and $p_{2} \in(1, \infty)$ and $q \in(0, \infty]$. Assume that $T$ is a sublinear operator defined on $L^{p_{1} p(\cdot)}\left(\mathbb{R}^{n}\right)+L^{p_{2} p(\cdot)}\left(\mathbb{R}^{n}\right)$ satisfying that there exist a positive constants $C_{1}$ and $C_{2}$ such that, for all $i=1,2, f \in L_{w_{i}}^{p_{i} p(\cdot)}\left(\mathbb{R}^{n}\right)$, with $w_{i}(x)=w^{\frac{1}{p_{i}}}(x)$ and $\lambda \in(0, \infty)$,

$$
\begin{equation*}
\lambda\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|T(f)>\lambda|\right\}}\right\|_{L_{w_{i}\left(\mathbb{R}^{n}\right)}^{p_{i} p(\cdot)}} \leq C_{i}\|f\|_{L_{w_{i}\left(\mathbb{R}^{n}\right)}^{p_{i} p(\cdot)}} . \tag{4.2.1}
\end{equation*}
$$

Then the operator $T$ is bounded on $L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, moreover, there exists a positive constant $C$ depending only on $p(\cdot)$ and $q$ such that, for all $f$ in $L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$,

$$
\|T f\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} .
$$

Proof. Let $f \epsilon_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, combining Lemma 1.3.5, Lemma 1.3.6 and (4.2.1) together, we conclude that

$$
\begin{aligned}
& \|T f\|_{L_{w}^{p(\cdot),}\left(\mathbb{R}^{n}\right)}^{q} \approx \sum_{k \in \mathbb{Z}} 2^{k q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|T f(x)|>2^{k}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \\
& \lesssim \sum_{k \in \mathbb{Z}} 2^{k q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:\left|T f_{k, 1}(x)\right|>2^{k}\right\}}\right\|_{L_{w}^{L_{n}^{p .-}}}^{q}\left(\mathbb{R}^{n}\right), \sum_{k \in \mathbb{Z}} 2^{k q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:\left|T f_{k, 1}(x)\right|>2^{k}\right\}}\right\|_{L_{w}^{p \cdot()}\left(\mathbb{R}^{n}\right)}^{q}
\end{aligned}
$$

$$
\begin{aligned}
& :=\mathbb{A}_{1}+\mathbb{A}_{2} \text {, }
\end{aligned}
$$

where $f_{k, 1}:=f \chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>2^{k}\right\}}$ and $f_{k, 2}:=f \chi_{\left\{x \in \mathbb{R}^{n}:|f(x)| \leq 2^{k}\right\}}$.
For the first term $\mathbb{A}_{1}$, we have

$$
\begin{aligned}
& \mathbb{A}_{1}=\sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{1}\right)}\left\|f \chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>2^{k}\right\}}\right\|_{L_{w^{1} / p_{1}}^{p_{p} q}\left(\mathbb{R}^{n}\right)} \\
& =\sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{1}\right)}\left\|f \sum_{j=k}^{\infty} \chi_{\left\{x \in \mathbb{R}^{n}: 2 j<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w^{1}}^{p_{1} p(\cdot)}\left(\mathbb{R}^{n}\right)}^{p_{1} q} \\
& \lesssim \sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{1}\right)}\left\|\sum_{j=k}^{\infty} 2^{j p_{1}} \chi_{\left\{x \in \mathbb{R}^{n}: j^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} .
\end{aligned}
$$

From Remark 1.2.9, we infer that

$$
\mathbb{A}_{1} \lesssim \sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{1}\right)}\left(\left\|\sum_{j=k}^{\infty} 2^{j p_{1}} \chi_{\left\{x \in \mathbb{R}^{n}: 2 j<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p \cdot()}\left(\mathbb{R}^{n}\right)}^{\|^{p}}\right)^{q / \underline{p}} .
$$

Since $q \in(0, \infty)$, then we will estimate the term $\mathbb{A}_{1}$ in two cases.
First case. When $q \in(0, \underline{p}]$. We have,

$$
\begin{aligned}
\mathbb{A}_{1} & \lesssim \sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{1}\right)} \sum_{j=k}^{\infty} 2^{j p_{1} q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \\
& =\sum_{j \in \mathbb{Z}} 2^{j p_{1} q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\left(\sum_{k=-\infty}^{j} 2^{k q\left(1-p_{1}\right)}\right) \\
& \lesssim \sum_{j \in \mathbb{Z}} 2^{j q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>2 j\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \\
& \approx\|f\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}^{q} .
\end{aligned}
$$

Second case. When $q \in(\underline{p}, \infty)$, take $\varepsilon_{1} \in\left(p_{1}, 1\right)$. In view of the Hölder inequality, we find out

$$
\mathbb{A}_{1} \lesssim \sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{1}\right)}\left(\sum_{j=k}^{\infty} 2^{j \underline{p}\left(p_{1}-\varepsilon_{1}\right)} 2^{j \underline{j} \varepsilon_{1}}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\| \|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{p}\right)^{\underline{\underline{p}}}
$$

$$
\begin{aligned}
\mathbb{A}_{1} & \lesssim 2^{k q\left(1-p_{1}\right)}\left\{\left(\sum_{j=k}^{\infty} 2^{j \underline{p}\left(p_{1}-\varepsilon_{1}\right)(q / q-\underline{p})}\right)^{(q-\underline{p}) / q}\left(\sum_{j=k}^{\infty} 2^{j q \varepsilon_{1}}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{\underline{p} / q}\right\}^{q / \underline{p}} \\
& \lesssim \sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{1}\right)} 2^{k q\left(p_{1}-\varepsilon_{1}\right)} \sum_{j=k}^{\infty} 2^{j q \varepsilon_{1}}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w v}^{p \cdot()}\left(\mathbb{R}^{n}\right)}^{q}
\end{aligned}
$$

Since $\varepsilon<1$. Then, we obtain

$$
\begin{aligned}
\mathbb{A}_{1} & \lesssim \sum_{j=k}^{\infty} 2^{j q \varepsilon_{1}}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \sum_{k=-\infty}^{j} 2^{k q\left(1-\varepsilon_{1}\right)} \\
& \lesssim \sum_{j \in \mathbb{Z}} 2^{j q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>2 j\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \approx\|f\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}^{q}
\end{aligned}
$$

The estimate of the term $\mathbb{A}_{2}$ is analogous to the first term $\mathbb{A}_{1}$, here we show the Second case corresponding to the case $q \in(\underline{p}, \infty)$. Let $\varepsilon_{2} \in\left(1, p_{2}\right)$. In view of the Hölder inequality, we find out

$$
\begin{aligned}
\mathbb{A}_{2} & \lesssim \sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{2}\right)}\left(\sum_{j=-\infty}^{k-1} 2^{j \underline{p} p_{2}}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}\right)^{\frac{p}{\underline{p}}} \\
& \lesssim \sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{2}\right)}\left\{\left(\sum_{j=-\infty}^{k-1} 2^{j \underline{p}\left(p_{2}-\varepsilon_{2}\right)(q / q-\underline{p})}\right)^{q-\underline{p} / q}\left(\sum_{j=-\infty}^{k-1} 2^{j q \varepsilon_{2}}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{\underline{p} q}\right\}^{q / \underline{p}} \\
& \lesssim \sum_{k \in \mathbb{Z}} 2^{k q\left(1-p_{2}\right)} 2^{k q\left(p_{2}-\varepsilon_{2}\right)} \sum_{j=-\infty}^{k-1} 2^{j q \varepsilon_{2}}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} .
\end{aligned}
$$

As $\varepsilon_{2}>1$, we deduce that

$$
\begin{aligned}
\mathbb{A}_{2} & \lesssim \sum_{j \in \mathbb{Z}} 2^{j q \varepsilon_{2}}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: 2^{j}<|f(x)| \leq 2^{j+1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \sum_{k=j+1}^{\infty} 2^{k q\left(1-\varepsilon_{2}\right)} \\
& \lesssim \sum_{j \in \mathbb{Z}} 2^{j q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:|f(x)|>2^{j}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \approx\|f\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}^{q}
\end{aligned}
$$

As an immediate result of the above theorem, we have the following conclusion.
Proposition 4.2.2. Let $p(\cdot) \in C^{\log }\left(\mathbb{R}^{n}\right)$ with $1<p_{-} \leq p_{+}<\infty$ and let $q \in(0, \infty]$. Then the HardyLittelwood maximal operator $M$ is bounded in the weighted variable Lorentz spaces $L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$.

The next theorem play an important role in the proof of the atomic decomposition.
Theorem 4.2.3. Let $p(\cdot) \in C^{\log }\left(\mathbb{R}^{n}\right)$ and let $q \in(0, \infty]$. Suppose that $N \in\left(\frac{n}{\underline{p}}+n+1, \infty\right)$. Then the following items are equivalent:
(i) $f \in H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, i.e. $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and $f^{*} \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$.
(ii) $f$ is bounded distribution and $\mathcal{N}(f) \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$.
(iii) $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and there exists $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ with $\int_{\mathbb{R}^{n}} \psi(x) d x=1$ such that $\psi_{+}^{*}(f) \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$. Moreover, for any $f \in H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, it holds true that

$$
\left\|f^{*}\right\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \approx\|\mathcal{N}(f)\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \approx\left\|\psi_{+}^{*}(f)\right\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}
$$

Proof. We start by the first implication $(i) \Rightarrow(i i)$.
Let $f \in H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, then we have $B(x, 1) \subset\left\{y \in \mathbb{R}^{n}: f_{N,+}^{*} \geq C_{N}|f \star \phi(x)|\right\}:=\Omega_{f, x}$. (See [90], pp.20). By virtue of the definition of $W_{p(\cdot)}$ and the Hölder inequality, yields

$$
\begin{aligned}
\min \left\{|f \star \phi(x)|^{p_{-}},|f \star \phi(x)|^{p_{+}}\right\} & \leq \min \left\{|f \star \phi(x)|^{p_{-}},|f \star \phi(x)|^{p_{+}}\right\} \frac{1}{|B(x, 1)|} \int_{\mathbb{R}^{n}} \chi_{B_{(x, 1)}}^{2}(y) d y \\
& \leq \min \left\{|f \star \phi(x)|^{p_{-}},|f \star \phi(x)|^{p_{+}}\right\} \frac{1}{|B(x, 1)|} \\
& \times \int_{\mathbb{R}^{n}} \chi_{B_{(x, 1)}} w^{-\underline{p}} \chi_{\Omega_{f, x}}(y)(y) w^{\underline{p}}(y) d y \\
& \lesssim \min \left\{|f \star \phi(x)|^{p_{-}},|f \star \phi(x)|^{p_{+}}\right\} \\
& \times \max \left\{\left\|\chi_{\Omega_{f, x} \|_{L_{w}}^{p_{-}}(\cdot)\left(\mathbb{R}^{n}\right)},\right\| \chi_{\Omega_{f, x}} \|_{L_{w}^{p}}^{p_{+}(\cdot)}\left(\mathbb{R}^{n}\right)\right. \\
& \lesssim \max \left\{\left\|f^{*}\right\|_{L_{w}^{p-(\cdot), \infty}\left(\mathbb{R}^{n}\right)}^{p_{-}},\left\|f^{*}\right\|_{L_{w}^{p(\cdot), \infty}\left(\mathbb{R}^{n}\right)}^{p_{+}}\right\} .
\end{aligned}
$$

By using the embedding $L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right) \hookrightarrow L_{w}^{p(\cdot), \infty}\left(\mathbb{R}^{n}\right)$, for $0<q<\infty$, we find out

$$
\left.\begin{array}{rl}
\min \left\{|f \star \phi(x)|^{p_{-}},|f \star \phi(x)|^{p_{+}}\right\} & \lesssim \max \left\{\left\|f^{*}\right\|_{L_{w}^{p} \cdot(\cdot), \infty}^{p_{-}}\left(\mathbb{R}^{n}\right)\right. \\
& \lesssim \max \left\{\left\|f^{*}\right\|_{L_{w}^{p-(\cdot), q}\left(\mathbb{R}^{n}\right)}^{p_{-}},\left\|f_{L_{w}^{p}}^{p_{+}^{p(\cdot), \infty}\left(\mathbb{R}^{n}\right)}\right\|_{L_{w}^{p} \cdot(\cdot), q}^{p^{p}\left(\mathbb{R}^{n}\right)}\right\}
\end{array}\right\} .
$$

Hence, $f \star \phi \in L^{\infty}\left(\mathbb{R}^{n}\right)$ and $f$ is bounded distribution. From the argument used in ([84] pp.98), we have

$$
\mathcal{N}(f) \leq \sum_{k=0}^{\infty} 2^{-k}\left(\psi_{k}\right)_{\nabla}^{*}(f)(x), \quad x \in \mathbb{R}^{n}
$$

where $\left\{\psi_{k}\right\}_{k \in \mathbb{N}} \subset \mathcal{S}\left(\mathbb{R}^{n}\right)$ have uniformly bounded seminorms in $\mathcal{S}\left(\mathbb{R}^{n}\right)$. Proposition 3.10 in [8] ensures that $f^{*}(x) \approx f_{N, \nabla}^{*}(x)$ for all $x \in \mathbb{R}^{n}$. Combining, Remark 2.7 in [32], Fatou's Lemma and the fact that $\psi_{\nabla}^{*}(f)(x) \lesssim f_{N, \nabla}^{*}(x)$ for all $x \in \mathbb{R}^{n}$, we conclude that

$$
\begin{aligned}
\|\mathcal{N}(f)\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}^{\tau} & \leq\left\|\sum_{k \in \mathbb{N}} 2^{-k}\left(\psi_{k}\right)_{\nabla}^{*}(f)\right\|_{L_{w}^{p(\cdot), q}}^{\tau}\left(\mathbb{R}^{n}\right) \\
& \lesssim \sum_{k \in \mathbb{N}} 2^{-k \tau}\left\|\left(\psi_{k}\right)_{\nabla}^{*}(f)\right\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}^{\tau} \\
& \lesssim\left\|f^{*}\right\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}^{\tau} .
\end{aligned}
$$

Therefore $\mathcal{N}(f) \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$.
Now we turn to the second implication, $(i i) \Rightarrow(i i i)$. If $f$ is a bounded distribution and $\mathcal{N}(f) \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$. From estimate in ([84] p.99), there exists a function $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ such that $\int_{\mathbb{R}^{n}} \psi(x) d x=1$ and $\psi_{+}^{*}(f)(x) \lesssim \mathcal{N}(f)(x)$, which implies that (iii) holds true.

It remains to prove $(i i i) \Rightarrow(i)$, from corollary 4.2 .2 and by using an analogous argument used in [90, p.2842-2845] we get the desired result and we omit the details.

The next result is an immediate consequence from the above theorem.
Corollary 4.2.4. Let $p(\cdot) \in C^{\log }\left(\mathbb{R}^{n}\right)$, let $q \in(0, \infty], a \in(0, \infty)$ and let $N$ be as in the above theorem. Then $f \in H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$ if and only if on of the following items holds true
(i) $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and there exists $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ with $\int_{\mathbb{R}^{n}} \psi(x) d x=1$ such that $\psi_{\nabla, a}^{*}(f) \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$
(i) $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and $f_{N, \nabla}^{*} \in L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$.

Moreover, for any $f \in H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, it holds true that

$$
\|f\|_{H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \approx\left\|f_{N, \nabla}^{*}\right\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \approx\left\|\psi_{\nabla, a}^{*}\right\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)^{\prime}}
$$

where the implicit equivalent positive constants independent of $f$.

### 4.3 Atomic decomposition

In this section, we deals with the atomic decomposition of the weighted Hardy-Lorentz spaces. More precisely, we have the following result

Theorem 4.3.1. Let $p(\cdot) \in C^{\log }\left(\mathbb{R}^{n}\right), q \in(0, \infty]$ and $r \in\left(\left(k_{w}^{1 / b}\right)^{\prime}, \infty\right]$ such that $\frac{1}{b} \in \mathrm{~S}_{w}$ and let $d_{w}=n\left(s_{w}-1\right)$. Let $w \in W_{p(\cdot)}$. Then $H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)=H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$ with equivalent quasi-norms.
Proof. First, we prove the following embedding $H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right) \subset H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$.
Since each $(p(\cdot), \infty, s)$-atom is also a $(p(\cdot), r, s)$-atom, then it suffices to prove $H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right) \subset$ $H_{w, \text { atom }}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$. Let $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ be such that $\operatorname{supp} \psi \subset B(0,1), \int_{\mathbb{R}^{n}} \psi(x) x^{\gamma} d x=0$ for all $\gamma \in \mathbb{Z}_{+}^{n}$ with $|\gamma| \leq s$. Then, from Lemma 4.1.6, there exists a function $\psi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ such that its Fourier transform $\widehat{\phi}$ has a compact support away from the origin and, for all $x \in \mathbb{R}^{n} \backslash\{0\}$.

$$
\int_{0}^{\infty} \widehat{\psi}(t x) \widehat{\phi}(t x) \frac{d t}{t}=1
$$

We define a function $\eta$ on $\mathbb{R}^{n}$ as follows, for all $x \in \mathbb{R}^{n} \backslash 0$.

$$
\widehat{\eta}(x):=\left\{\begin{array}{l}
\int_{1}^{\infty} \widehat{\psi}(t x) \widehat{\phi}(t x) \frac{d t}{t} \quad \text { if } \quad x \neq 0 \\
1 \quad \text { if } \quad x=0
\end{array}\right.
$$

By [10, p.219], we have $\eta$ is infinitely differentiable, has compact support and equals to 1 near the origin. Let $x_{0}:=\{2, \cdots, 2\} \in \mathbb{R}^{n}$ and $f \in H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$. For any $x \in \mathbb{R}^{n}$, set

$$
\widetilde{\phi}(x):=\phi\left(x-x_{0}\right), \quad \widetilde{\psi}(x):=\phi\left(x+x_{0}\right) .
$$

Given $f \in \mathcal{S}\left(\mathbb{R}^{n}\right)$, for all $a \in \mathbb{R}^{n}$ and $t \in(0, \infty)$. Consider the functions

$$
F(x, t):=f \star \widetilde{\phi}_{t}(x) \quad \text { and } \quad G(x, t):=f \star \eta_{t}(x) .
$$

Thus, by [10, p.220],

$$
f(\cdot):=\int_{0}^{\infty} \int_{\mathbb{R}^{n}} F(y, t) \widetilde{\psi}_{t}(\cdot-y) \frac{d y d t}{t} \quad \text { in } \quad \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)
$$

Let

$$
\begin{equation*}
M_{\nabla}(f)(x):=\sup _{t \in(0, \infty),|y-x| \leq 3 t\left(\left|x_{0}\right|+1\right)}(|F(y, t)|+|G(y, t)|) \tag{4.3.1}
\end{equation*}
$$

and

$$
\Omega_{i}:=\left\{x \in \mathbb{R}^{n}: M_{\nabla}(f)(x)>2^{i}\right\} .
$$

Then $M_{\nabla}(f)(x)$ is semi-continuous which implies that the set $\Omega_{i}$ is open, and from corollary 4.2.4, we have

$$
\left\|M_{\nabla}(f)\right\|_{L_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \lesssim\|f\|_{H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} .
$$

Since $\Omega_{i}$ is a proper open set, by means of the whitney decomposition [85, p. 167], there exists a sequence $\left\{Q_{i j}\right\}_{j \in \mathbb{N}}$ of cubes such that, for all $i \in \mathbb{Z}_{+}$,
(1) $\cup_{j \in \mathbb{N}} Q_{i j}=\Omega_{i}$ and $\left\{Q_{i j}\right\}_{j \in \mathbb{N}}$ have disjoint interiors
(2) for all $j \in \mathbb{N}, c_{1} \sqrt{n} \ell_{Q_{i j}} \leq \operatorname{dist}\left(Q_{i j}, \Omega_{i}^{\complement}\right) \leq c_{1} \sqrt{n} \ell_{Q_{i j}}$, where $\ell_{Q_{i j}}$ represents the length of the cube $Q_{i j}, \operatorname{dist}\left(Q_{i j}, \Omega_{i}^{\complement}\right):=\inf \left\{|x-y|: x \in Q_{i j}, y \in \Omega_{i}^{\complement}\right\}$ and $c_{1}, c_{2}$ are two positive constants.

Now, for any $\varepsilon \in(0, \infty), i \in \mathbb{Z}, j \in \mathbb{N}$ and $x \in \mathbb{R}^{n}$, let

$$
\begin{aligned}
\operatorname{dist}\left(x, \Omega_{i}^{\complement}\right) & :=\inf \left\{|x-y|: y \in \Omega_{i}^{\complement}\right\} \\
\widetilde{\Omega}_{i} & :=\left\{(x, t) \in \mathbb{R}_{+}^{n+1}: 0<2 t\left(\left|x_{0}\right|+1\right)<\operatorname{dist}\left(x, \Omega_{i}^{\complement}\right)\right\} \\
\widetilde{Q}_{i j} & :=\left\{(x, t) \in \mathbb{R}_{+}^{n+1}: x \in Q_{i j},(x, t) \in \widetilde{\Omega}_{i} \backslash \widetilde{\Omega}_{i+1}\right\}
\end{aligned}
$$

and we define the function $b_{i j}^{\varepsilon}$ as follows

$$
b_{i j}^{\varepsilon}:=\int_{\varepsilon}^{\infty} \int_{\mathbb{R}^{n}} \chi_{\widetilde{Q}_{i j}}(y, t) F(y, t) \widetilde{\psi}_{t}(x-y) \frac{d y d t}{t}
$$

Analogous to [10, p.221-222], there exists constants $C_{1}$ and $C_{2}$ such that, for all $\varepsilon \in$ $(0, \infty), i \in \mathbb{Z}$ and $j \in \mathbb{N}, \operatorname{supp} b_{i j}^{\varepsilon} \subset B\left(x_{Q_{i j}} \frac{5}{2} \sqrt{n} \ell_{Q_{i j}}\right) \subset C_{1} Q_{i j},\left\|b_{i j}^{\varepsilon}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq C_{2} 2^{i}, \int_{\mathbb{R}}^{n} b_{i j}^{\varepsilon}(x) x^{\gamma} d x=$ 0 for all $\gamma \in \mathbb{Z}_{+}^{n}$ obeys $|\gamma| \leq s$ and

$$
f=\lim _{\varepsilon \rightarrow 0} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} b_{i j}^{\varepsilon}(x) \quad \text { in } \quad \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)
$$

From [32, Remark 5.5], we have, for these balls $B\left(x_{Q_{i j}}, \frac{5}{2} \sqrt{n} \ell_{Q_{i j}}\right)$,

$$
\begin{equation*}
\sum_{j \in \mathbb{N}} B\left(x_{Q_{i j}} \frac{5}{2} \sqrt{n} \ell_{Q_{i j}}\right) \leq A \chi_{\Omega_{i}} \tag{4.3.2}
\end{equation*}
$$

By the Alaoglu theorem [78, Theorem 3. 17] and diagonal rule, we conclude that there exists a sequence $\left\{b_{i j}\right\}_{(i, j) \in \mathbb{Z} \times \mathbb{N}} \subset L^{\infty}\left(\mathbb{R}^{n}\right)$ and a sequence $\left\{\varepsilon_{k}\right\}_{k \in \mathbb{N}}$ such that $\varepsilon_{k} \rightarrow 0$ as $k \rightarrow \infty$ and for any $g \in L^{1}\left(\mathbb{R}^{n}\right)$,

$$
\lim _{k \rightarrow \infty}<b_{i j}^{\varepsilon_{k}}, g>=<b_{i j}, g>
$$

with supp $b_{i j} \subset C_{1} Q_{i j},\left\|b_{i j}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \lesssim 2^{i}$. For all $\gamma \in \mathbb{Z}_{+}^{n}$ satisfying $|\gamma| \leq s$,

$$
\int_{\mathbb{R}^{n}} b_{i j}(x) x^{\gamma} d x=<b_{i j}, x^{\gamma} \chi_{C_{1} \mathrm{Q}_{i j}}>=\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{n}} b_{i j}^{\varepsilon_{k}}(x) x^{\gamma} d x=0 .
$$

At this stage, we show that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} b_{i j}^{\varepsilon_{k}}(x)=\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} b_{i j}(x) \tag{4.3.3}
\end{equation*}
$$

Let $\zeta$ be a function belongs to $\mathcal{S}\left(\mathbb{R}^{n}\right)$. Lemma 4.1.5 ensures that $\|\zeta\|_{B M O_{w, p(\cdot), 1}}<\infty$.

On the other hand, we have

$$
\begin{aligned}
& \sum_{|i|>N} \sum_{j \in \mathbb{N}}\left(\left|<b_{i j}^{\varepsilon_{k}} \zeta>\left|+\left|<b_{i j}, \zeta>\right|\right)\right.\right. \\
&=\sum_{i=-\infty}^{-N-1} \sum_{j \in \mathbb{N}}\left(\left|<b_{i j}^{\varepsilon_{k}}, \zeta>\left|+\left|<b_{i j}, \zeta>\right|\right)+\sum_{i=N+1}^{\infty} \sum_{j \in \mathbb{N}}\left\{\left|\int_{C_{1} Q_{i j}} b_{i j}^{\varepsilon_{k}}(x)\left[\zeta(x)-\mathbf{P}_{C_{1} Q_{i j}}^{s} \zeta(x)\right] d x\right|\right.\right.\right. \\
&\left.+\left|\int_{C_{1} Q_{i j}} b_{i j}(x)\left[\zeta(x)-\mathbf{P}_{C_{1} Q_{i j}}^{s} \zeta(x)\right] d x\right|\right\} \\
& \lesssim \sum_{i=-\infty}^{-N-1} 2^{i} \int_{\mathbb{R}^{n}}|\zeta| d x+\sum_{i=N+1}^{\infty} \sum_{j \in \mathbb{N}} 2^{i} \int_{C_{1} Q_{i j}}\left|\zeta(x)-\mathbf{P}_{C_{1} Q_{i j}}^{s} \zeta(x)\right| d x \\
& \lesssim 2^{-N}\|\zeta\|_{L^{1}}+\sum_{i=N+1}^{\infty} \sum_{j \in \mathbb{N}} 2^{i}\left\|\chi_{Q_{i j}}\right\|_{L_{w r}^{p r}}^{p(\cdot) / r}\|\zeta\|_{B M O_{w w^{r}, p(\cdot) / r, 1}\left(\mathbb{R}^{n}\right)} \\
& \quad \lesssim 2^{-N}\|\zeta\|_{L^{1}\left(\mathbb{R}^{n}\right)}+\|\zeta\|_{B M O_{w^{r}, p(\cdot) / r, 1}\left(\mathbb{R}^{n}\right)} \sum_{i=N+1}^{\infty} 2^{i}\left\|\chi_{\Omega_{i}}\right\|_{L_{w r^{\prime}}^{p(\cdot) / r}\left(\mathbb{R}^{n}\right)} \\
& \quad \lesssim 2^{-N}\|\zeta\|_{L^{1}\left(\mathbb{R}^{n}\right)}+2^{-N(r-1)}\|\zeta\|_{B M O_{w w, p(\cdot) / r, 1}\left(\mathbb{R}^{n}\right)}\|f\|_{H_{w}^{p(\cdot), \infty}\left(\mathbb{R}^{n}\right)}^{r}
\end{aligned}
$$

From Lemma 1.3.7, we conclude that
$\sum_{|i|>N} \sum_{j \in \mathbb{N}}\left(\left|<b_{i j}^{\varepsilon_{k}}, \zeta>\left|+\left|<b_{i j}, \zeta>\right|\right) \lesssim 2^{-N}\|\zeta\|_{L^{1}\left(\mathbb{R}^{n}\right)}+2^{-N(r-1)}\|\zeta\|_{B M O_{w, p(\cdot) / r, 1}\left(\mathbb{R}^{n}\right)}\|f\|_{H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)^{\prime}}^{r}\right.\right.$
which tends to 0 as $N$ tends to $\infty$, the constant $r$ is chosen such that $r \in\left(\max \left\{1, p_{+}, \infty\right\}\right)$. In a similar way, it hold true that

$$
\sum_{|i| \leq N} \sum_{j \in \mathbb{N}}\left(\left|<b_{i j}^{\varepsilon_{k}}, \zeta>\left|+\left|<b_{i j}, \zeta>\right|\right)<\infty\right.\right.
$$

Finally, by the same argument used in [56, p. 651] we obtain (4.3.3). For $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, let $B_{i j}$ be the ball having the same center as $Q_{i j}$ with the radius $\frac{5}{2} \sqrt{n} \ell_{Q_{i j}}$,

$$
a_{i j}:=\frac{b_{i j}}{C_{2} 2^{i}\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}} \quad \text { and } \quad \lambda_{i j}:=C_{2} 2^{i}\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}
$$

Thus, $a_{i j}$ is a $(p(\cdot), \infty, s)$-atom, which is also a $(p(\cdot), r, s)$-atom,

$$
f=\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{i j} a_{i j} \quad \text { in } \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)
$$

Furthermore, we have

$$
\begin{aligned}
\left\|\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} b_{i, j}\right\|_{H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} & \approx\left[\sum_{i \in \mathbb{Z}} 2^{i q}\left\|\left(\sum_{j \in \mathbb{N}} \chi_{B_{i, j}}\right)^{1 / \underline{p}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right]^{\frac{1}{q}} \lesssim\left(\sum_{i \in \mathbb{Z}} 2^{i q}\left\|\chi_{\Omega_{i}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{\frac{1}{q}} \\
& \lesssim\|f\|_{H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}
\end{aligned}
$$

By taking the infimum over all decompositions, we deduce that

$$
\|f\|_{H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \lesssim\|f\|_{H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}
$$

and so we find that $H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right) \subset H_{w, \text { atom }}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$. Now, we turn out to the second inclusion. Let $f \in H_{w, \text { atom }}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$, there exists a sequence $\left\{a_{i, j}\right\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ of $(p(\cdot), r, s)$-atoms associated with the balls $\left\{B_{i, j}\right\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ and a sequence of complex numbers $\left\{\lambda_{i, j}\right\}_{i \in \mathbb{Z}, j \in \mathbb{N}}$ such that $f$ has a decomposition as in [32, (5.2)] with $\lambda_{i, j}:=\widehat{A} 2^{i}\left\|\chi_{B_{i, j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}$. In view of Lemma 1.3.6 it suffices to prove that

$$
\left[\sum_{i \in \mathbb{Z}} 2^{i q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}: f^{*}(x)>2^{i}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right]^{\frac{1}{q}} \lesssim\|f\|_{H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} .
$$

For $k \in \mathbb{Z}$, we set

$$
f=\sum_{i=-\infty}^{k-1} \sum_{j \in \mathbb{N}} \lambda_{i, j} a_{i, j}+\sum_{i=k}^{\infty} \sum_{j \in \mathbb{N}} \lambda_{i, j} a_{i, j}:=f_{1}+f_{2}
$$

Remark 1.2.9 leads to

$$
\begin{aligned}
\left\|\chi_{\left\{x \in \mathbb{R}^{n}: f^{*}(x)>2^{k}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} & \lesssim\left\|\chi_{\left\{x \in \mathbb{R}^{n}: f_{1}^{*}(x)>2^{k-1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}+\left\|\chi_{\left\{x \in A_{k}: f_{2}^{*}(x)>2^{k-1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& +\left\|\chi_{\left\{x \in A_{k}^{C}: f_{2}^{*}(x)>2^{k-1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& :=\mathbb{I}_{1}+\mathbb{I}_{2}+\mathbb{I}_{3}
\end{aligned}
$$

where $A_{k}:=\cup_{i=k}^{\infty} \cup_{j \in \mathbb{N}} 2 B_{i, j}$. We start by $\mathbb{I}_{1}$, we have

$$
\begin{aligned}
\mathbb{I}_{1} & \lesssim\left\|\chi_{\left\{x \in \mathbb{R}^{n}: \sum_{i=-\infty}^{k-1} \sum_{j \in \mathbb{N}} \lambda_{i, j}\left(a_{i, j}\right)^{*} \chi_{2 B_{i, j}}(x)>2^{k-2}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& +\left\|\chi_{\left\{x \in \mathbb{R}^{n}: \Sigma_{i=-\infty}^{k-1} \sum_{j \in \mathbb{N}} \lambda_{i, j}\left(a_{i, j}\right)^{*} \chi_{2 B_{i, j}^{C}}(x)>2^{k-2}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& :=\mathbb{I}_{11}+\mathbb{I}_{12} .
\end{aligned}
$$

Let $\frac{1}{b} \in \mathrm{~S}_{w}$ and $\widetilde{q} \in\left(1, \min \left\{\frac{q}{\left(k_{w}^{1 / b}\right)^{\prime}}, \frac{1}{b}\right\}\right)$ and $a \in\left(0,1-\frac{1}{\widetilde{q}}\right)$. Then Hölder inequality gives

$$
\begin{aligned}
& \sum_{i=-\infty}^{k-1} \sum_{j \in \mathbb{N}} \lambda_{i j}\left(a_{i j}\right)^{*}(x) \chi_{2 B_{i j}}(x) \leq\left(\sum_{i=-\infty}^{k-1} 2^{-i a \widetilde{q}^{\prime}}\right)^{1 / \widetilde{q}^{\prime}} \\
& \times\left[\sum_{i=-\infty}^{k-1} 2^{-i a \widetilde{q}}\left(\sum_{j \in \mathbb{N}} \lambda_{i j}\left(a_{i j}\right)^{*}(x) \chi_{2 B_{i j}}(x)\right)^{\widetilde{q}}\right]^{1 / \widetilde{q}}
\end{aligned}
$$

$$
\begin{aligned}
& :=\Phi(x) \text {, }
\end{aligned}
$$

where, $1 / \widetilde{q}+1 / \widetilde{q}=1$. We use the fact that $\widetilde{q} b<1$ and $a_{i j}^{*}(x) \lesssim M a_{i j}(x)$, for all $x \in \mathbb{R}^{n},[90$, Remark 2.1(i)] and [17, theorem 2.61], we deduce that,

$$
\begin{aligned}
\mathbb{I}_{11} & \leq\left\|\chi_{\left\{x \in \mathbb{R}^{n}: \Phi(x)>2^{k-2}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim 2^{-k \widetilde{q}(1-a)}\left\|\sum_{i=-\infty}^{k-1} 2^{-i a \widetilde{q}}\left[\sum_{j \in \mathbb{N}} \lambda_{i j}\left(a_{i j}\right)^{*} \chi_{2 B_{i j}}\right]^{\widetilde{q}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim 2^{-k \widetilde{q}(1-a)}\left\|\sum_{i=-\infty}^{k-1} 2^{-i a \widetilde{q}} \sum_{j \in \mathbb{N}}\left[\left(a_{i j}\right)^{*} \chi_{2 B_{i j}}\right]^{\widetilde{q}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim 2^{-k \widetilde{q}(1-a)}\left\|\sum_{i=-\infty}^{k-1} 2^{-i(1-a) b \widetilde{q}} \sum_{j \in \mathbb{N}}\left[\left\|\chi_{B_{i, j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} M a_{i j} \chi_{2 B_{i j}}\right]^{\widetilde{q} b}\right\|_{L_{w}^{p(\cdot) / b}\left(\mathbb{R}^{n}\right)}^{1 / b} \\
& \lesssim 2^{-k \widetilde{q}(1-a)}\left[\sum_{i=-\infty}^{k-1} 2^{-i(1-a) b \widetilde{q}}\left\|\sum_{j \in \mathbb{N}}\left[\left\|\chi_{B_{i, j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} M a_{i j} \chi_{2 B_{i j}}\right]^{\widetilde{q} b}\right\|_{L_{w b}^{p(\cdot) / b}\left(\mathbb{R}^{n}\right)}\right]^{\frac{1}{b}} .
\end{aligned}
$$

Let $r:=q / \widetilde{q}>\left(k_{w}^{1 / b}\right)^{\prime}$, since the operator $M$ is bounded on $L^{r}$, we find out, for all $i \in \mathbb{Z}$ and
$j \in \mathbb{N}$,

$$
\begin{aligned}
\left\|\left(\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} M\left(a_{i j}\right) \chi_{B_{i j}}\right)^{\widetilde{q}}\right\|_{L^{r}\left(\mathbb{R}^{n}\right)} & \lesssim\left\|\chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{\widetilde{q}}\left\|M\left(a_{i j} \chi_{B_{i j}}\right)\right\|_{L^{r}\left(\mathbb{R}^{n}\right)}^{\widetilde{q}} \\
& \lesssim\left|B_{i j}\right|^{1 / r}
\end{aligned}
$$

It follows from Lemma 3.1.5

$$
\mathbb{I}_{11} \lesssim 2^{-k \widetilde{q}(1-a)}\left[\sum_{i=-\infty}^{k-1} 2^{-i(1-a) b \widetilde{q}}\left\|\sum_{j \in \mathbb{N}} \chi_{2 B_{i j}}\right\|_{L_{w b}^{p(\cdot) / b}\left(\mathbb{R}^{n}\right)}\right]^{\frac{1}{b}}
$$

From Remark1.2.11, we find

$$
\mathbb{I}_{11} \lesssim 2^{-k \widetilde{q}(1-a)}\left[\sum_{i=-\infty}^{k-1} 2^{-i(1-a) b \widetilde{q}}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w^{b}}^{p(\cdot) / b}\left(\mathbb{R}^{n}\right)}\right]^{\frac{1}{b}}
$$

Let $\epsilon_{1} \in(1,(1-a) \widetilde{q})$. According to Hölder inequality, we deduce that

$$
\begin{aligned}
\mathbb{I}_{11} & \lesssim 2^{-k \widetilde{q}(1-a)}\left[\sum_{i=-\infty}^{k-1} 2^{-i b\left[(1-a) \widetilde{q}-\epsilon_{1}\right.} 2^{i b \epsilon_{1}}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\| \|_{L_{w b}^{p(\cdot) / b}\left(\mathbb{R}^{n}\right)}\right]^{\frac{1}{b}} \\
& \lesssim 2^{-k \widetilde{q}(1-a)}\left[\sum_{i=-\infty}^{k-1} 2^{-i b\left[(1-a) \widetilde{q}-\epsilon_{1}\right.} 2^{i b \epsilon_{1}}\left\|\left(\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right)^{1 / b}\right\|_{L_{w w}^{p \cdot()}\left(\mathbb{R}^{n}\right)}^{b}\right]^{\frac{1}{b}} \\
& \lesssim 2^{-k \widetilde{q}(1-a)}\left\{\sum_{=-\infty}^{k-1} 2^{i b \frac{q}{q-b}\left[(1-a) \widetilde{q}-\epsilon_{1}\right.}\right\}^{\frac{q-b}{b q}}\left\{\sum_{i=-\infty}^{k-1} 2^{i q \epsilon_{1}}\left\|\left(\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right)^{1 / b}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right\}^{1 / q} \\
& \lesssim 2^{-k \epsilon_{1}}\left(\sum_{i=-\infty}^{k-1} 2^{i q \epsilon_{1}}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{1 / q} .
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\sum_{k=\infty}^{\infty}\left(2^{k} \mathbb{I}_{11}\right)^{q} & =\sum_{k=-\infty}^{\infty} 2^{k q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:\left(f_{1}\right)^{*} \chi_{2 B_{i, j}}(x)>2^{k}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim \sum_{k=-\infty}^{\infty} 2^{k q} 2^{-k \epsilon_{1} q} \sum_{i=-\infty}^{k-1} 2^{i q \epsilon_{1}}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}
\end{aligned}
$$

$$
\begin{align*}
& =\sum_{i=-\infty}^{\infty} 2^{i q \epsilon_{1}}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \sum_{k=i+1}^{\infty} 2^{k q\left(1-\epsilon_{1}\right)} \\
& \lesssim \sum_{i \in \mathbb{Z}} 2^{i q}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \lesssim\|f\|_{H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}^{q} . \tag{4.3.4}
\end{align*}
$$

Now, we turn out to estimate $\mathbb{I}_{12}$. For any $b \in\left(0, \frac{n}{n+d_{w}+1}\right), q_{1} \in\left(\frac{n}{\left(n+d_{w}+1\right) b}, \frac{1}{b}\right)$ and $a \in(0,1-$ $\left.\frac{1}{q_{1}}\right)$, we have

$$
\mathbb{I}_{12} \lesssim\left\|\chi_{\left\{x \in \mathbb{R}^{n}: \Phi_{1}(x)>2^{k-1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)^{\prime}}
$$

where $\Phi_{1}(x):=\frac{2^{k a}}{\left(2^{a q_{1}^{\prime}-1}\right)^{1 / q_{1}^{\prime}}}\left[\sum_{i=-\infty}^{k-1} 2^{-i a q_{1}}\left(\sum_{j \in \mathbb{N}} \lambda_{i j}\left(a_{i j}\right)^{*}(x) \chi_{\left(2 B_{i j}\right)}(x)\right)^{q_{1}}\right]^{1 / q_{1}}$, with $\frac{1}{q_{1}}+\frac{1}{q_{1}^{\prime}}=$ 1.

From the estimate (4.1.7), we get

$$
\begin{aligned}
\mathbb{I}_{12} & \lesssim 2^{-k q_{1}(1-a)} \| \sum_{i=-\infty}^{k-1} 2^{-i a q_{1}}\left[\sum _ { j \in \mathbb { N } } \lambda _ { i j } \left(a_{i j} \chi_{\left.\left(2 B_{i j}\right)^{\complement}\right]^{q_{1}}}^{\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}} \begin{array}{l} 
\\
\\
\lesssim 2^{-k q_{1}(1-a)}\left\{\sum_{i=-\infty}^{k-1} 2^{-i(1-a) q_{1} b}\left\|\sum_{j \in \mathbb{N}}\left[M\left(\chi_{B_{i j}}\right)\right]^{\frac{n+d_{w}+1}{n} q_{1} b}\right\|_{L_{w^{b}}^{p(\cdot) / b}\left(\mathbb{R}^{n}\right)}\right\}^{1 / b} .
\end{array} . . . \begin{array}{l}
\end{array} .\right.\right.
\end{aligned}
$$

Since $\frac{1}{b}>s_{w}$ we apply theorem1.2.10 to obtain

$$
\begin{aligned}
\mathbb{I}_{12} & \lesssim 2^{-k q_{1}(1-a)}\left\{\sum_{i=-\infty}^{k-1} 2^{-i(1-a) q_{1} b}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot) / b}\left(\mathbb{R}^{n}\right)}\right\}^{1 / b} \\
& \lesssim 2^{-k q_{1}(1-a)}\left\{\sum_{i=-\infty}^{k-1} 2^{-i(1-a) q_{1} b}\left\|\left(\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right)^{1 / b}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{b}\right\}^{1 / b} .
\end{aligned}
$$

Let $\epsilon_{2} \in\left(1,(1-a) q_{1}\right)$. Hölder inequality ensures that,

$$
\begin{aligned}
\mathbb{I}_{12} & \lesssim 2^{-k q_{1}(1-a)}\left\{\sum_{i=-\infty}^{k-1} 2^{-i b \epsilon_{2}\left[(1-a) q_{1}-\epsilon_{2}\right]} 2^{i b \epsilon_{2}}\left\|\left(\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right)^{1 / b}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{b}\right\}^{1 / b} \\
& \lesssim 2^{-k q_{1}(1-a)}\left(\sum_{i=-\infty}^{k-1} 2^{\left[(1-a) q_{1}-\epsilon_{2}\right] i b \frac{q}{q-b}}\right)^{\frac{q-b}{b q}}\left(\sum_{i=-\infty}^{k-1} 2^{i q \epsilon_{2}}\left\|\left(\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right)^{1 / b}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}\right)^{1 / q} \\
& \lesssim 2^{-k \epsilon_{2}}\left(\sum_{i=-\infty}^{k-1} 2^{i q \epsilon_{2}}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}\right)^{1 / q} .
\end{aligned}
$$

Therefore, we obtain

$$
\begin{align*}
\sum_{k=-\infty}^{\infty}\left(2^{k} \mathbb{I}_{12}\right)^{q} & =\sum_{k \in \mathbb{Z}} 2^{k q}\left\|\chi_{\left\{x \in \mathbb{R}^{n}:\left(f_{1}\right)^{*} \chi_{\left(2 B_{i j}\right)^{( }}(x)>2^{k}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \\
& \lesssim \sum_{k \in \mathbb{Z}} 2^{k q} 2^{-k \epsilon_{2} q} \sum_{i=-\infty}^{k-1} 2^{i q \epsilon_{2}}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \\
& \lesssim \sum_{i \in \mathbb{Z}} 2^{i q \epsilon_{2}}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \sum_{k=i+1}^{\infty} 2^{k q\left(1-\epsilon_{2}\right)} \\
& \lesssim \sum_{i \in \mathbb{Z}} 2^{i q}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \lesssim\|f\|_{H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} . \tag{4.3.5}
\end{align*}
$$

To estimate $\mathbb{I}_{2}$, we choose $r_{1} \in(0, \min (\underline{p}, q))$. Then from definition and Remark1.2.11, we obtain

$$
\begin{aligned}
\mathbb{I}_{2} & \leq\left\|\chi_{A_{k}}\right\|_{L_{w}^{p(\cdot)}}\left(\mathbb{R}^{n}\right) \\
& \lesssim\left\|\sum_{i=k}^{\infty} \sum_{j \in \mathbb{Z}} \chi_{2 B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|\sum_{i=k}^{\infty} \sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \lesssim\left[\sum_{i=k}^{\infty}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{r_{1}}\right]^{1 / r_{1}} .
\end{aligned}
$$

Let $\epsilon_{3} \in(0,1)$. According to Hölder inequality, we deduce

$$
\begin{aligned}
\mathbb{I}_{2} & \lesssim\left[\sum_{i=k}^{\infty} 2^{-i \epsilon_{3} r_{1}} 2^{i \epsilon_{3} r_{1}}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{r_{1}}\right]^{1 / r_{1}} \\
& \lesssim\left(\sum_{i=k}^{\infty} 2^{-i \epsilon_{3} r_{1} \frac{q}{q-r_{1}}}\right)^{\frac{q-r_{1}}{q r_{1}}}\left(\sum_{i=k}^{\infty} 2^{i \epsilon_{3} q}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w o}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{1 / q} \\
& \lesssim 2^{-i \epsilon_{3}}\left(\sum_{i=k}^{\infty} 2^{i \epsilon_{3} q}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{1 / q} .
\end{aligned}
$$

Thus,

$$
\begin{align*}
\sum_{k \in \mathbb{Z}}\left(2^{k} \mathbb{I}_{2}\right)^{q} & =\sum_{k \in \mathbb{Z}} 2^{k q}\left\|\chi_{\left\{x \in A_{k}:\left(f_{2}\right)^{*}(x)>2^{k}\right\}}\right\|_{L_{w}^{p(\cdot)}}^{q}\left(\mathbb{R}^{n}\right) \\
& \lesssim \sum_{k \in \mathbb{Z}} 2^{k q_{2}} 2^{-k \epsilon_{3}}\left[\sum_{i=k}^{\infty} 2^{i q \epsilon_{3}}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right]^{1 / q} \\
& \lesssim \sum_{i=k}^{\infty} 2^{i q \epsilon_{3}}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \sum_{k=-\infty}^{i} 2^{k q\left(1-\epsilon_{3}\right)} \\
& \lesssim \sum_{i \in \mathbb{Z}} 2^{i q}\left\|\sum_{j \in \mathbb{Z}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \lesssim\|f\|_{H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)}^{q} \tag{4.3.6}
\end{align*}
$$

It remains to estimate $\mathbb{I}_{3}$. Since $\underline{p} \in\left(\frac{n s_{w}}{n s_{w}+1}, 1\right)$, there exists $r_{2} \in(0, \infty)$ such that $r_{2} \in\left(\frac{n s_{w}}{\underline{p}\left(n s_{w}+1\right)}, 1\right)$. By taking $b_{1} \in\left(0, \min \left\{\frac{n}{r_{2}\left(n+d_{w}+1\right)}, q\right\}\right)$. Then, we have

$$
\begin{aligned}
\mathbb{I}_{3} & =\left\|\chi_{\left\{x \in\left(A_{k}\right)^{\complement}:\left(f_{2}\right)^{*}(x)>2^{k-1}\right\}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim 2^{-k r_{2}}\left\|\sum_{i=k}^{\infty} \sum_{j \in \mathbb{N}}\left(\lambda_{i j}\left(a_{i j}\right)^{*}\right)^{r_{2}} \chi_{\left(A_{k}\right)}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \\
& \lesssim 2^{-k r_{2}}\left[\sum_{i=k}^{\infty}\left\|\left(\sum_{j \in \mathbb{N}}\left(\lambda_{i j}\left(a_{i j}\right)^{*}\right)^{r_{2}} \chi_{\left(A_{k}\right)^{\complement}}\right)^{b_{1}}\right\|_{L_{w b^{p}}^{p(\cdot) / b_{1}}\left(\mathbb{R}^{n}\right)}\right]^{1 / b_{1}} .
\end{aligned}
$$

We observe that $1 / b_{1}>s_{w} / \underline{p}$. Then, by theorem1.2.10 and (4.1.7) we conclude that,

$$
\begin{aligned}
\mathbb{I}_{3} & \lesssim 2^{-k r_{2}}\left[\sum_{i=k}^{\infty} 2^{i r_{2} b_{1}} \|\left(\sum_{j \in \mathbb{N}}\left(M\left(\chi_{B_{i j}}\right)^{\frac{1}{b_{1}}}\right)^{b_{1}} \|_{L_{w^{p}(\cdot)}^{p\left(\cdot b_{1}\right.}\left(\mathbb{R}^{n}\right)}\right]^{1 / b_{1}}\right. \\
& \lesssim 2^{-k r_{2}}\left[\sum_{i=k}^{\infty}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{b_{1}}\right]^{1 / b_{1}} .
\end{aligned}
$$

Let $\epsilon>1$. Thanks to the Hölder inequality, we have

$$
\begin{aligned}
\mathbb{I}_{3} & \lesssim 2^{-k r_{2}}\left(\sum_{i=k}^{\infty} 2^{i b_{1} r_{2}-i \epsilon b_{1}} 2^{i \epsilon b_{1}}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{b_{1}}\right)^{1 / b_{1}} \\
& \lesssim 2^{-k r_{2}}\left(\sum_{i=k}^{\infty} 2^{\left(i b_{1} r_{2}-i \epsilon b_{1}\right) \frac{q}{q-b_{1}}}\right)^{\frac{q-b_{1}}{b_{1} q}}\left(\sum_{i=k}^{\infty} 2^{i \epsilon q}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{1 / q} \\
& \lesssim 2^{-k \epsilon}\left(\sum_{i=k}^{\infty} 2^{i \epsilon q}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q}\right)^{1 / q} .
\end{aligned}
$$

Then, we have

$$
\begin{align*}
\sum_{k \in \mathbb{Z}}\left(2^{k} \mathbb{I}_{3}\right)^{q} & \lesssim \sum_{k \in \mathbb{Z}} 2^{k q} 2^{-k \epsilon q} \sum_{i=k}^{\infty} 2^{i q \epsilon}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \\
& \lesssim \sum_{k \in \mathbb{Z}} 2^{i q \epsilon}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)}^{q} \sum_{k=-\infty}^{i} 2^{k q \epsilon} \\
& \lesssim \sum_{i \in \mathbb{Z}} 2^{i q}\left\|\sum_{j \in \mathbb{N}} \chi_{B_{i j}}\right\|_{L_{w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)} \lesssim\|f\|_{H_{w, a t o m}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} . \tag{4.3.7}
\end{align*}
$$

Putting (4.3.4),(4.3.5),(4.3.6) and (4.3.7) together we conclude that

$$
\|f\|_{H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)} \lesssim\|f\|_{H_{w, \text { atom }}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)^{\prime}}
$$

consequently $f \in H_{w}^{p(\cdot), q}\left(\mathbb{R}^{n}\right)$, this ends the proof.

## Conclusion and perspectives.

## Conclusion

The theory of function spaces with variable exponents is a classical branch of the harmonic analysis, and since these spaces are widely used in the mathematical modeling in the real-world phenomena such as, modeling of electrorheological fluids, thermorheological fluids...etc. Therefore, it is of interest to explore and studying this kind of spaces as the Hardy spaces with variable exponents in different situations. Motivated by the works [39, 57, $69,68,74,23,92,90,32]$. In this thesis, we have established some results related to variable Hardy spaces. The first one, is the atomic decomposition of the weighted variable Hardy spaces on domains and then we have studied the duality result of these spaces. Secondly, we have investigated the weighted variable Hardy spaces associated with operators satisfying the Davies-Gaffney estimates, where, we have established the molecular characterization of $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ and we have proved also a duality relation between $H_{L, w}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ and $B M O_{L^{*}, w}^{p(\cdot), M}$. Finally, we established an interpolation theorem, then we showed Feffer-man-Stein vector-valued inequality on the weighted variable Lorentz space and we have showed the atomic characterization of weighted Hardy-Lorentz spaces with variable exponents.

## Perspectives

In the future, we hope to continue in this axis of research, as we will try to understand and explore some results for the variable Hardy-Lorentz spaces on domains. Our next goal is to find some applications to the atomic characterization of the weighted variable HardyLorentz spaces established in chapter 4 such as the duality result for this space.

## List Of Publications.

The main results of this thesis are the series of the following papers [66, 54, 67].

- O. Melkemi, K. Saibi, and Z. Mokhtari, : Weighted variable Hardy spaces on domains. Adv. Oper. Theory 6, 56 (2021). https/ doi.org/10.1007/s43036-021-00151-4
- B. Laadjal, K. Saibi, O. Melkemi and Z. Mokhtari : Weighted variable Hardy associated with operators satisfying Davies-Gaffney estimates. Submitted.
- O. Melkemi, Z. Mokhtari, and K. Saibi, : Atomic characterization of the weighted Hardy-Lorentz spaces with variable exponent. Submitted.


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