RÉPUBLIQUE ALGERIENNE DÉMOCRATIQUE ET POPULAIRE
MINISTÉRE DE L'ENSEIGNEMENT SUPÉRIEURE
ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITÉ MOSTAPHA BEN BOULAID BATNA 2
FACULTÉ DES MATHÉMATIQUES ET D'INFORMATIQUE
DÉPARTEMENT DE MATHÉMATIQUES
LABORATOIRE DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES ET APPLICATIONS, LEDPA

THÈSE

PRÉSENTÉE EN VUE D'OBTENIR LE DIPLÔME DE
DOCTORAT EN SCIENCES
SPÉCIALITÉ MATHÉMATIQUES

Par
ADALET HANACHI

THÈME

ANALYSE MATHÉMATIQUE DES SOLUTIONS DE QUELQUES MODÈLES ISSUS DE LA MÉCANIQUE DES FLUIDES

Soutenue le: 16/03/2022
Devant le jury composé de:

Mokrane Ahmed Zerrouk	M.C.A	Université de Batna 2	Président
Mohamed Zerguine	M.C.A	Université de Batna 2	Rapporteur
Amar Youkana	PR	Université de Batna 2	Invité
Khaled Melkemi	PR	Université de Batna 2	Examinateur
Taoufik Hmidi	M.C. HDR	Université de Rennes 1	Examinateur
Salim Mesbahi	M.C.A	Université de Setif 2	Examinateur
Amrane Houas	M.C.A	Université de Biskra	Examinateur

Année Universitaire 2021-2022

Acknowledgements

Foremost, El hamdou lillah who gave me the strength to complete this work and crowned my patience with this success. El hamdou liallah.

With many thanks to my supervisor Dr. Mohamed Zerguine, associate professor, for the continuous support of my research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I fell very fortunate to have an opportunity work under his patient supervisor.

A great thanks to Dr. Ahmed Zerrouk Mokrane, associate professor, for giving me the honor to preside my jury. I am very indebted for the knowledge that he taught us throughout many years of academic study.

I am also very honored that Professor Khaled Melkemi, Dr. Salim Mesbahi, associate professor, Dr. Amrane Houas, associate professor, Dr. Taoufik Hmidi, associate professor, HDR, have agreed to be part of the jury and I would like to express my recognition and my thanks to all of them.

I would like to express my gratitude to Professor Amar Youkana for accepting the invitation and being our guest. I would also like to thank my collaborator, Dr. Haroune Houamed, for his dedicated cooperation, valuable discussions, and good comments.

Particular appreciation to Dr. Taoufik Hmidi, associate professor, for his exceptional scientific and moral support and continuous encouragement. I am grateful for his rich and fruitful scientific discussions, despite his preoccupations.

Thanks are due to all the professors of the Department of Mathematics, especially Professor Lakhdar Meziani and Professor Abdullah Benaissa. Also can't fail to thank all my colleagues for their support, and they spared no effort to help me.

I hold an attitude of greatness and respect towards my mother, may God protect her and take care of her, and I thank her for her prayers for my success.

Finally, I would like to thank with great emotion all my family for their irreplaceable and unconditional assistance and their unfailing support. Especially my little family, my husband and my children, and I ask them to excuse me for my failures towards them because of my studies.

To all those and others who did not mention my sincere thanks.

Adalet HANACHI

I dedicate this work to my father, may God wrap him with his great mercy

Contents

Notation

$t \in \mathbb{R}_+$ or $(0, \infty)$:	The time variable.
$x = (x_1, x_2, x_3) \in \mathbb{R}^3$:	The spacial variable.
$v(t, x) \in \mathbb{R}^3$:	The distribution of the fluid velocity localized in $x \in \mathbb{R}^3$, at time $t > 0$.
$\rho(t, x) \in \mathbb{R}$:	The mass density of the fluid.
$p(t, x) \in \mathbb{R}$:	The force of internal pressure.
$\rho \vec{e}_z$:	The influence of the buoyancy force in the fluid motion in the vertical direction $\vec{e}_z = (0, 0, 1)$.
$\mu > 0$:	The viscosity of the fluid.
$\kappa > 0$:	The molecular diffusion of the fluid.
$\mathrm{div} v = 0$:	The fluid is incompressible.
$(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$:	The cylindrical base.
$x = (r, z, \theta)$:	The components of x in cylindrical coordinates.
Ω:	$\{(r, z) \in \mathbb{R}^2 : r > 0\}$
$v(t, r, \theta, z) = (v^r(t, r, z), 0, v^z(t, r, z))$:	Axisymmetric flow without swirl.
$\omega_\theta = \partial_z v^r - \partial_r v^z$:	The component of the vorticity along $\vec{e}_\theta$.
$\frac{d}{dt}$:	$\partial_t + v \cdot \nabla$ The convective derivative.
$v \cdot \nabla$:	$v^r \partial_r + v^z \partial_z$ in cylindrical case.
Δ:	$\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$ in cylindrical case.
$B^s_{p,r}$:	$\{u \in \mathcal{S}'; \|u\|_{B^s_{p,r}} = \big(\sum_{j \geq -1} 2^{rjs} \|\Delta_j u\|^r_{L^p}\big)^{1/r} < \infty\}$ with $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$.
$\dot{B}^s_{p,r}$:	$\{u \in \mathcal{S}'; \|u\|_{\dot{B}^s_{p,r}} = \big(\sum_{j \in \mathbb{Z}} 2^{rjs} \|\dot{\Delta}_j u\|^r_{L^p}\big)^{1/r} < \infty\}$ with $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$.

$B^{s,t}_{p,q}(\mathbb{R}^3)$: $\{u\in\mathcal{S}'; \|u\|_{B^{s,t}_{p,q}} = \big(\sum_{j,k\geq -1} 2^{jsq}2^{ktq}\|\Delta^h_j\Delta^v_k u\|^q_{L^p}\big)^{1/q}<\infty\}$ with $(s,t,p,q)\in\mathbb{R}^2\times[1,\infty]^2$? Anisotropic Besov spaces.

$H^{s,t}$: $B^{s,t}_{2,2}$.

$H^s(\mathbb{R}^3)$: $\{u\in\mathcal{S}'/\widehat{u}\in L^2_{Loc}(\mathbb{R}^d) \quad and \quad \|u\|^2_{H^s}=\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\widehat{u}(\xi)|^2d\xi<\infty\}$.

$\dot{H}^s(\mathbb{R}^3)$: $\{u\in\mathcal{S}'/\widehat{u}\in L^1_{Loc}(\mathbb{R}^d) \quad and \quad \|u\|^2_{\dot{H}^s}=\int_{\mathbb{R}^d}|\xi|^{2s}|\widehat{u}(\xi)|^2d\xi<\infty\}$.

$\mathcal{M}^p_q(\mathbb{R}^3)$: $\{f\in L^q_{Loc}/\sup R^{\frac{3}{p}}\big(\frac{1}{|B|}\int_B|f|^q\big)^{\frac{1}{q}}<\infty\}$ with $1<q\leq p<\infty$ the Morrey space on $\mathbb{R}^3$.

$\mathscr{M}(\mathbb{R}^3)$: $\{\mu:\mathcal{B}(\mathbb{R}^3)\to[0,\infty]/\mu(K)<\infty\}$ with K is a compact set.

$C_0(\mathbb{R}^3)$: The continuous functions space on $\mathbb{R}^3$.

$\langle\cdot,\cdot\rangle$: The pairing between $\mathscr{M}(\mathbb{R}^3)$ and $C_0(\mathbb{R}^3)$.

$\mu=\mu_{ac}+\mu_{sc}+\mu_{pp}$: The Lebesgue decomposition of μ.

$\mu_{ac}\perp\mu_{sc}\perp\mu_{pp}$: μ_{ac},μ_{sc} and μ_{pp} are concentrated on pairwise disjoint sets.

μ_{ac}: Absolutely continuous part of μ.

μ_{sc}: Singular continuous part of μ which has no atom.

μ_{pp}: Punctual part of μ.

$\|\mu\|_{\mathscr{M}(\mathbb{R}^3)}$: Total variation of μ.

$\rightharpoonup$: Weak convergence in $\mathscr{M}(\mathbb{R}^3)$.

1 General introduction

The current thesis occupies the study of some problems raised by the incompressible three-dimensional fluid mechanics. It mainly deals with the Boussinesq system. Through it, we will present and comment on the results obtained in our research concerning the well-posedness of the studied system. It comprises five chapters. We embark on a general introduction, where we state the system in question in the axisymmetric framework and a brief concise for its derivation. To simplify our presentation we start, in particular with the Navier-Stokes system, which is a special case of this system, where the density is constant. Then we present the most important results achieved by previous researchers, then we focus on the most recent ones that are directly related to our research.

In the second Chapter, we provide the basic tools freely used throughout this thesis, which allow us to understand the rest Chapters. We begin by some vocabulary of fluid mechanics like the vortex and its explanation, illustrated by many examples, Biot-Savart law which can be viewed as the source to recuperate the velocity through the vorticity, a brief reminder on the theory semi-groups and some applications to solve evolution problems in the general case. We end with some phrases on the measure theory.

The third Chapter mainly deals with the global well-posedness of the axisymmetric Boussinesq system in critical Lebesgue spaces. Such spaces contribute some difficulties like the velocity in this situation do not belong to the energy space. To remedy this problem and derive the local existence in time, we handle with the equivalent Duhamel's formula for vorticity -density equations and we apply the fixed point formalism in some appropriate function spaces combined with the axisymmetric Biot-Savart law. Even so, we find another difficulty arises from vorticity and density which are defined in different spaces endowed with different norms. For this purpose, we introduce a new unknown which satisfying the same equation than the vorticity. To deduce the global well-posedness we combine the Bootstrap method in Lebesgue spaces with some a priori estimates for vorticity and density though the coupled function.

The fourth Chapter is dedicated also to studying the global well-posedness of the axisymmetric Boussinesq system with a finite measure as initial data provided that the atomic part is small enough. More precisely, we shall extend the results already obtained in the previous chapter, but the situation, in this case, is very hard to deal with it, in other words, the question is how to give a rigorous sense to some quantity if the initial density is a finite measure. To surmount this problem, we shall introduce some terminology about the measure theory, in particular, the push forward of a measure with a specific function and we state a new concept called an axisymmetric measure. To establish the local well-posedness, we employ the fixed point method combined with the smallness of the atomic part and some properties of the associated semi-groups, while the global well-posedness is also derived from the Bootstrap argument and some asymptotic estimates.

We conclude this thesis with an appendix in which we shed light on the derivation of the system in question for better understanding. It is dedicated to the physical side and is considered a new aspect of knowledge for us from the physical aspect, where we pass from pure mathematics to fundamental mathematics dedicated to physics.

1.1 Boussinesq system

The environmental concerns raised by the potential impact of industrial activities on the climate and the accompanying changes in the atmosphere and oceans are the decisive issue of our time, and this has led many researchers in various branches of science to try understanding the phenomena of geophysical fluid dynamics, especially in light of the tremendous numerical progress, counting on the modelling of these phenomena in simple ways to give global previsions as numerical solutions. Despite the efforts carried out by researchers for the purpose of geophysical fluid dynamics, the models presented are still very complex. We try to highlight one of these models which is called the Boussinesq approximation.

In geophysical fluid dynamics, density variations may arise at low speeds due to the changes in temperature or humidity like in atmosphere, or salinity as in oceans which give rise to buoyancy forces. The effect of these density changes can be expressive even if the fractional change in density is small. The Boussinesq approximation retains density variations in gravity term responsible for the buoyancy effect but disregards them in the inertial term. The outcome of this analysis in the space is the following full viscous Boussinesq system

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = \rho\vec{e}_z & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ \partial_t\rho + v\cdot\nabla\rho - \kappa\Delta\rho = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ \mathrm{div} v = 0, \\ (v,\rho)_{|t=0} = (v_0,\rho_0). \end{cases} \tag{$\mathrm{B}_{\mu,\kappa}$}$$

Usually, $v(t,x)$ refers to the distribution of the fluid velocity localized in $x \in \mathbb{R}^3$ at a time $t \in (0,\infty)$ with free-divergence, the scalar function $\rho(t,x)$ designates the mass density in the modelling of geophysical fluids and $p(t,x)$ is the force of internal pressure. The non-negative parameters μ and κ represent respectively the kinematic viscosity and molecular diffusivity of the fluid which can be seen as the inverse of Reynolds numbers and $\rho\vec{e}_z$ models the influence of the buoyancy force in the fluid motion in the vertical direction $\vec{e}_z = (0,0,1)$.

Notice that the system ($\mathrm{B}_{\mu,\kappa}$) seriously omnipresent in the mathematics community either theoretically or experimentally because that arises in many phenomena like thermal convection, dynamic of geophysical fluids, and optimal mass transport topic, see, e.g. [13, 74]. Also, we mention that for 2D and $\mu = \kappa = 0$, ($\mathrm{B}_{\mu,\kappa}$) has a close resemblance with three-dimensional Euler axisymmetric swirling flows.

When the initial density is constant, the Boussinesq system ($\mathrm{B}_{\mu,\kappa}$) can be reduced to the classical Navier-Stokes equations. This provides us the following system

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = 0 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \mathrm{div} v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{NS_μ}$$

The first successful attempt goes back to J. Leray's paper in the thirties of the last century, where he established the global existence of weak solutions in energy space for any dimension. Nevertheless, the uniqueness of such solutions has been till now an open question, unless for the two-dimensional case. Lately, the local well-posedness issue in the setting of mild solutions for (NS_μ) was done by H. Fujita and T. Kato [32] for initial data belonging to the critical Sobolev space $\dot{H}^{\frac{1}{2}}$ in the sense of *scale invariance* which means that the (NS_μ) is the fact that it is invariant under the transformation

$$v(t,x) \mapsto v_\lambda(t,x) \triangleq \lambda v(\lambda^2 t, \lambda x). \tag{1.1}$$

In other words, if v is a solution of (NS_μ) on $[0,T]$ with initial data v_0, then v_λ is a solution on $[0,\lambda^{-2}T]$ with intial data $v_{0_\lambda} \triangleq \lambda v_0(\lambda\cdot)$. More similar results are established in several functional spaces like L^3, $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}$ and BMO^{-1}. It should be noted that these types of solutions are globally well-posed in time for initial data sufficiently small with respect to the viscosity, except in two-dimension, see [63, 76].

We point out that the topic of blow-up in finite time of smooth solutions with large initial data of (NS_μ) is still now not known, except in some partial situations. Chemin et *all.* Investigated in a series of references [22, 23] that (NS_μ) is, in fact, global in time where the initial data which are not small in any critical space but satisfies some structure like oscillations or slow variations in one direction. For another connected subject, we refer to [16, 48, 69, 83] and the references therein.

About the system ($B_{\mu,\kappa}$) we will give a brief overview of the existing results. K. Moffat is among the first to raise the global existence problem of solutions for the Boussinesq model in his XXI Century Problem 3, see [75].

≪*Suppose that the velocity field v is itself " driven " by in-homogeneity of the ρ-field, according to some well-defined dynamical prescription (e.g. ρ could represent temperature variation in a gravity field, the flow being driven by the buoyancy force in the Boussinesq approximation). The problem is to examine the evolution of the ρ-field in the neighborhood of its saddle-points, to determine whether singularities of $\nabla\rho$ can develop, and to examine the influence of weak molecular diffusivity κ in controlling the approach to such singularities.*≫

Lately, D. Cordoba, C. Fefferman and R. De La Llave [24] presented a kinematic argument which shows that if a volume preserving field has these singularities, then some integrals related to the vector field must diverge. They also showed that if the vector fields satisfy certain partial differential equations (Navier-Stokes, Boussinesq), then the integrals must be finite. These singularities are therefore absent in the solutions of the above equations. In dimension two of spaces, the system ($B_{\mu,\kappa}$) was tackled by lot of authors in various functional spaces and different values for the parameters κ and μ. For a details literature, we refer to some selected references [49, 51, 52, 55, 62, 70, 82].

Before sketching some theoretical keystone results on the well-posedness topic for the three-dimensional viscous Boussinesq system ($B_{\mu,\kappa}$), first, let us point out again that the topic of global existence and uniqueness for (NS_μ) in the general case is till now an open problem in PDEs. It is therefore incumbent upon us to seek out a subclass of vector fields which in turn leads to some conservation quantities, and so the global well-posedness result. Such subclass involves rewriting (NS_μ) under vorticity formulation by applying the "curl" operator to the momentum equation, which is defined by $\omega = \nabla\times v$. Thus, we get:

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega - \mu\Delta\omega = \omega\cdot\nabla v & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ \omega_{|t=0} = \omega_0. \end{cases} \tag{1.2}$$

According to Beale-Kato-Majda criterion in [8], we have the following blow-up criterion

$$T^\star < \infty \Leftrightarrow \int_0^{T^\star} \|\omega(\tau)\|_{L^\infty} d\tau = +\infty. \tag{1.3}$$

So, the control ω in $L^1_t L^\infty$ is a key step for the global well-posedness of solutions for (NS_μ). Notice that for the 2D case, we have $\omega\cdot\nabla v\equiv 0$, then we immediately deduce for $t\geq 0$ that $\|\omega(t)\|_{L^p}\leq\|\omega_0\|_{L^p}$ for all $p\in[1,\infty]$. According to (1.3), this latter boundedness is the main tool to achieve the global well-posedness. However, for three-dimensional flow the situation is more complicated due to the presence of stretching term $\omega\cdot\nabla v$, which contributes additional drawbacks for the fluid motion. In the class of axisymmetric flows without swirl, the velocity vector field v can be decomposed in cylindrical coordinates $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ as follows:

$$v(t, r, \theta, z) = v^r(t, r, z)\vec{e}_r + v^z(t, r, z)\vec{e}_z,$$

where for every $x = (x_1, x_2, z) \in \mathbb{R}^3$ we have

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad r > 0, \quad 0 \le \theta < 2\pi.$$

Above, the triplet $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ represents the usual frame of unit vectors in the radial, azimuthal and vertical directions with the notation

$$\vec{e}_r = \Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big), \quad \vec{e}_\theta = \Big(-\frac{x_2}{r}, \frac{x_1}{r}, 0\Big), \quad \vec{e}_z = (0, 0, 1).$$

For these flows the vorticity ω takes the form $\omega \triangleq \omega_\theta \vec{e}_\theta$, with

$$\omega_\theta = \partial_z v^r - \partial_r v^z. \tag{1.4}$$

An elementary calculus claims that $\omega \cdot \nabla v$ close to $\frac{v^r}{r}\omega$ and ω_θ obeys the following inhomogeneous transport-diffusion equation

$$\begin{cases} \partial_t \omega_\theta + v \cdot \nabla \omega_\theta - \mu\big(\Delta - \frac{1}{r^2}\big)\omega_\theta = \frac{v^r}{r}\omega_\theta \\ \omega_{\theta|t=0} = \omega_0. \end{cases} \tag{1.5}$$

By taking $\Pi = \frac{\omega_\theta}{r}$, we check obviously that Π satisfies

$$\begin{cases} \partial_t \Pi + v \cdot \nabla \Pi - \mu\big(\Delta + \frac{2}{r}\partial_r\big)\Pi = 0 \\ \Pi_{|t=0} = \Pi_0, \end{cases} \tag{1.6}$$

with homogeneous Neumann condition at the boundary $r = 0$. The fact that $\mathrm{div} v = 0$ and $\Delta + \frac{2}{r}\partial_r$ has a good sign enable us for $t \ge 0$ to write

$$\|\Pi(t)\|_{L^p} \le \|\Pi_0\|_{L^p}, \quad p \in [1, \infty]. \tag{1.7}$$

This latter boundedness offers a good setting to Ladyzhenskaya [61], independently Ukhovksii and Yudovich [80] fifty years ago to establish that (NS_μ) is global well-posed in time as soon as $v_0 \in H^1$ and $\omega_0, \frac{\omega_0}{r} \in L^2 \cap L^\infty$. Lately, this result was relaxed by S. Leonardi, J. Màlek, J. Necăs and M. Pokornýin [64] for initial velocity v_0 in H^2 . But it is possible to obtain the same result under the weaker hypothesis $H^{\frac{1}{2}}$.

Now, we shed light on the $2D$ Navier Stokes system under rough initial data. We recall that the study of this system in vorticity formulation with singular initial data has attracted by many authors. Worth motioning that Ben Artzi in [5] proved that (NS_μ) is well-posed for initial vorticity laying in $L^1(\mathbb{R}^2)$. His approach is based entirely on elementary comparison principles for linear parabolic equations. In the context of finite measure, Cottet in [25], independently Giga, Miyakawa, and Osada in [41] have granted a global result when the initial vorticity ω_0 belongs to $\mathscr{M}(\mathbb{R}^2)$ (where $\mathscr{M}(\mathbb{R}^2)$ is the space of Radon measures with finite mass). The uniqueness issue in this situation seems very hard for an arbitrary initial data in $\mathscr{M}(\mathbb{R}^2)$. For this purpose, Giga, Miyakawa and Osada imposed that the atomic part of ω_0 is small enough. The interpretation of [41] that the size requirement only entails the atomic part of the measure coming from the axial estimate

$$\limsup_{t\uparrow 0} t^{1-\frac{1}{p}}\|e^{t\Delta}\mu\|_{L^p} \leq C_p\|\mu_{pp}\|_{\mathscr{M}(\mathbb{R}^2)}, \quad p\in(1,\infty],$$

with $\|\mu_{pp}\|_{\mathscr{M}(\mathbb{R}^2)}$ refers to the total variation of the atomic part of $\mu\in\mathscr{M}(\mathbb{R}^2)$. The case of a large Dirac mass was solved recently by C. E. Wayne and Th. Gallay in [35] by using a completely different approach where it is based on entropy estimates. This latter's result was early enhanced by Gallagher and Gallay in [33], where they established that if $\omega_0\in\mathscr{M}(\mathbb{R}^2)$, there exists a unique solution $\omega\in C\big((0,\infty);L^1\cap L^\infty\big)$ and so we have $\|\omega(t,\cdot)\|_{L^1}\leq\|\omega_0\|_{\mathscr{M}(\mathbb{R}^2)}$. In addition, they demonstrated that such solution is continuously dependent on initial data, deducing that the Navier-Stokes equations are globally well-posed in 2D case. For large literature, we refer the reader to [39].

The global regularity topic of ($\mathrm{B}_{\mu,\kappa}$) in dimension three of spaces has received considerable attention. As shown in [27], for $\kappa=0$ R. Danchin and M. Paicu investigated that ($\mathrm{B}_{\mu,\kappa}$) is locally well-posed in time in any dimension in the framework of Fujita-Kato's and Leray's solutions. Next, in axisymmetric case H. Abidi, T. Hmidi and S. Keraani proved in [3] that ($\mathrm{B}_{\mu,\kappa}$) is globally well-posed by rewriting it under vorticity-density formulation:

$$\begin{cases} \partial_t\omega_\theta+v\cdot\nabla\omega_\theta-\frac{v^r}{r}\omega_\theta=\big(\Delta-\frac{1}{r^2}\big)\omega_\theta-\partial_r\rho & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3\\ \partial_t\rho+v\cdot\nabla\rho=0,\\ (\omega_\theta,\rho)_{|t=0}=(\omega_0,\rho_0), \end{cases}\tag{1.8}$$

with, the notation $v\cdot\nabla=v^r\partial_r+v^z\partial_z$ and $\Delta=\partial_r^2+\frac{1}{r}\partial_r+\partial_z^2$. Consequently, the quantity $\Pi=\frac{\omega_\theta}{r}$ solves the equation

$$\partial_t\Pi+v\cdot\nabla\Pi-(\Delta+\frac{2}{r}\partial_r)\Pi=-\frac{\partial_r\rho}{r}.\tag{1.9}$$

They assumed that $v_0\in H^1(\mathbb{R}^3)$, $\Pi_0\in L^2(\mathbb{R}^3)$, $\rho_0\in L^2\cap L^\infty$ with supp $\rho_0\cap(Oz)=\emptyset$ and P_z(supp ρ_0) is a compact set in $\mathbb{R}^3$, especially to remove the violent singularity of $\frac{\partial_r\rho}{r}$, with P_z being the orthogonal projector over (Oz). These results are improved later by T. Hmidi and F. Rousset in [53] for $\kappa\geq0$ by dropping the assumption on the support of the density. Their paradigm is heavily based on the coupling between the two equations of the system (1.10) by introducing a new unknown which is called *coupled function*. In the same way, C. Miao and X. Zheng have succeeded in [71] to recover the system (1.8) globally in time, where they replaced the full dissipation by a horizontal one by keeping the same conditions as in [3] expect the initial density $\rho_0\in H^1\cap L^\infty$ and $\partial_z\omega_0\in L^2$. Taking advantage of the coupled function introduced in [53], another result was established later by the same authors [72] with the presence of horizontal dissipation in both equations for initial data $(v_0,\rho_0)\in H^1(\mathbb{R}^3)\times H^{1,0}(\mathbb{R}^3)$ with $\Pi_0\in L^2(\mathbb{R}^3)$ and $\partial_z\omega_0\in L^2(\mathbb{R}^3)$, with $H^{1,0}(\mathbb{R}^3)=B^{0,1}_{2,2}$ anisotropic Besov space. More recently, P. Dreyfuss and H. Houamed treated in [29] the uniqueness issue of the Boussinesq equations with horizontal dissipation, and they proved the global well-posedness if the axisymmetric initial data (v_0, Π_0, ρ_0) lies in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, improving some conditions to that of [72]. In the same fashion, in [47] H. Houamed and M. Zerguine demonstrated that ($\mathrm{B}_{\mu,\kappa}$) is globally well-posed in time for $\kappa = 0$ and axisymmetric initial data $(v_0, \rho_0) \in \big(H^{\frac{1}{2}} \cap \dot{B}^0_{3,1}\big)(\mathbb{R}^3) \times \big(L^2 \cap \dot{B}^0_{3,1}\big)(\mathbb{R}^3)$. Their idea is inspired from [1, 27, 53].

1.1.1 Aims

The objective of this thesis is to derive the same results recently obtained by Gallay and Sverák in [36] for the viscous axisymmetric Boussinesq system ($\mathrm{B}_{\mu,\kappa}$) expressed by the following vorticity-density formulation.

$$\begin{cases} \partial_t \omega_\theta + v \cdot \nabla \omega_\theta - \frac{v^r}{r} \omega_\theta = \big(\Delta - \frac{1}{r^2}\big)\omega_\theta - \partial_r \rho \\ \partial_t \rho + v \cdot \nabla \rho - \Delta \rho = 0 \\ (\omega_\theta, \rho)_{|t=0} = (\omega_0, \rho_0). \end{cases} \tag{1.10}$$

The significant remark is that the axisymmetric flows without swirl constitutes a special class relatively simple to study compared to other three-dimensional incompressible flows, although it still contains interesting examples, such as circular vortex filaments or toroidal vortex rings. Notice that the study of these flows is due to the similarity between this category and the 2D flows for which it has a huge amount of studies and results. The importance of the 2D flows arises in the observation that the fluid's motion is essentially flat, meaning that the velocity of the fluid in some characteristic spatial directions is negligible compared to the velocity in the perpendicular plane. This situation often occurs for liquids in thickness layers, or for rapidly rotating fluids where the Coriolis force strongly alternates displacements along the axis of rotation. Typical examples are geophysical flows, for which are agreed upon by the geometry of the field (atmosphere or ocean) and the effect of earth's rotation to make the 2D approximation accurate and effective.

1.2 Main results

In this section, we will discuss the results obtained for the global well-posedness for the $3D-$axisymmetric Boussinesq system ($\mathrm{B}_{\mu,\kappa}$). In particular, it is shown that this system is generally well-posed for optimal regularities, also called critical regularities.

This study is motivated by the work published recently by Gallay and Sverák in [36] for the axisymmetric three-dimensional Navier-Stokes system (NS_μ). In [36] the authors studied this system with initial data in $L^1(\Omega)$, where they proved the global existence of an infinite energy solution. More precisely, they proved the following theorem

Theorem 1.2.1. *For any initial data $\omega_0 \in L^1(\Omega)$, with $\Omega = \{(r,z) \in \mathbb{R}^2, r > 0\}$ endowed with the measure $drdz$, the axisymmetric vorticity equation* (1.5) *for $\mu = 1$ admits a unique global mild solution*

$$\omega_\theta \in C^0\big([0,\infty), L^1(\Omega)\big) \cap C^0\big((0,\infty), L^\infty(\Omega)\big).$$

The solution satisfies $\|\omega_\theta(t)\|_{L^1(\Omega)} \leq \|\omega_0\|_{L^1(\Omega)}$ for all $t > 0$, and

$$\begin{cases} \lim_{t\to 0} t^{1-\frac{1}{p}} \|\omega_\theta(t)\|_{L^p(\Omega)} = 0, & \text{for} \quad 1 < p \leq \infty, \\ \\ \lim_{t\to\infty} t^{1-\frac{1}{p}} \|\omega_\theta(t)\|_{L^p(\Omega)} = 0, & \text{for} \quad 1 \leq p \leq \infty. \end{cases}$$

If in addition, the axisymmetric vorticity is non-negative and has finite impulse

$$\mathscr{I} = \int_\Omega r^2 \omega_0(r,z) drdz < \infty,$$

then

$$\lim_{t\to\infty} t^2 \omega_\theta(r\sqrt{t}, z\sqrt{t}, t) = \frac{\mathscr{I}}{16\pi} r e^{-\frac{r^2+z^2}{4}}, \quad (r,z) \in \Omega,$$

where, the convergence holds in $L^p(\Omega)$ for $1 \leq p \leq \infty$. In particular $\|\omega_\theta(t)\|_{L^p(\Omega)} = \theta(t^{-2+\frac{1}{p}})$ as $t \to \infty$ in that case.

Remark 1.2.2. The local well-posedness claim was expected, because the class of initial data considered by the authors covers by at least two existence results in the literature, like $\omega_\theta \in L^1(\Omega)$, so $\omega = \omega_\theta \vec{e}_\theta$ belongs to the Morrey space $M^{3/2}(\mathbb{R}^3)$. From Giga and Miyakawa's result in [40], the Navier-Stokes system in $\mathbb{R}^3$ admits a unique local solution. On the other hand, under the same assumption on the vorticity, the velocity field v given by the Biot-Savart law in $\mathbb{R}^3$ belongs to the space $BMO^{-1}(\mathbb{R}^3)$. So, thanks to the result of Koch and Tataru [48] the local existence is guaranteed.

The second result due to Gallay and Sverák is the generalization of the previous result, where they assumed that the initial vorticity is a finite measure whose atomic part is sufficiently small with respect to the viscosity. Their result is structured in the following theorem.

Theorem 1.2.3. *There exist positive constants ε and C such that, for any initial data $\omega_0 \in \mathscr{M}(\Omega)$, with $\|\omega_{0,pp}\|_{tv} \leq \varepsilon$, the axisymmetric vorticity equation* (1.5) *for $\mu = 1$ has a unique global mild solution*

$$\omega_\theta \in C^0\big((0,\infty), L^1(\Omega) \cap L^\infty(\Omega)\big)$$

satisfying

$$\limsup_{t\to 0} \|\omega_\theta(t)\|_{L^1(\Omega)} < \infty, \quad \limsup_{t\to 0} t^{\frac{1}{4}} \|\omega_\theta(t)\|_{L^{\frac{4}{3}}(\Omega)} \leq C\varepsilon$$

and such that $\omega_\theta(t) \rightharpoonup \omega_0$ as $t \to 0$. Moreover, the asymptotic estimates for $t \to \infty$ given in Theorem 1.2.1 hold without change.

Let us briefly give outlines of the proof of the previous theorems. For the local existence issue, they rewrite the equation (1.5) for $\mu = 1$ as follows

$$\partial_t \omega_\theta + \mathrm{div}_\star(v\omega_\theta) - \big(\Delta - \frac{1}{r^2}\big)\omega_\theta = 0. \tag{1.11}$$

Thereafter, they associate Duhamel's formula to the last equation (1.11) in the following way.

$$\omega_\theta(t) = \mathbb{S}(t)\omega_0 + \int_0^t \mathbb{S}(t-\tau)\mathrm{div}_\star(v\omega_\theta)(\tau)d\tau, \tag{1.12}$$

with $\mathrm{div}_\star = \partial_r + \partial_z, \Delta = \partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r$ in the axisymmetric setting and $(\mathbb{S}(t))_{t\geq 0}$ refers to the semi-group family associated to the differential operator $\Delta - \frac{1}{r^2}$. Afterward, they applied the fixed-point formalism to (1.12) in an appropriate functional space, so, by using Biot-Savart's law and the asymptotic properties of the semigroup, they derived local existence. For uniqueness, they explored a technique due to Brezis, while the globality and the asymptotic behavior of the solution in the neighborhood of infinity are guaranteed by a priori estimations in appropriate functional spaces.

Our contribution in this field is divided into three parts. The first one deals with the global well-posedness of axisymmetric viscous Boussinesq system in critical Lebesgue space $L^1(\Omega)\times L^1(\mathbb{R}^3)$. The second one treats a partial extension of the first result when the initial vorticity is a finite measure, in other words, $(\omega_0, \rho_0) \in \mathscr{M}(\Omega)\times L^1(\mathbb{R}^3)$, where $\mathscr{M}(\Omega)$ is the set of finite measure. In the last part, we give a global extension where all the initial data are finite measure $(\omega_0, \rho_0) \in \mathscr{M}(\Omega)\times \mathscr{M}(\mathbb{R}^3)$.

1.2.1 When all initial data is Lebesgue integrable

This subsection addresses to state the first main result for initial data (ω_0, ρ_0) in $L^1(\Omega)\times L^1(\mathbb{R}^3)$, where $L^p(\Omega)$, $L^p(\mathbb{R}^3)$ respectively equipped by the following norms:

$$\|\omega_\theta\|_{L^p(\Omega)} = \begin{cases} \Big(\int_\Omega |\omega_\theta(r,z)|^p drdz\Big)^{\frac{1}{p}} & \text{if } p\in[1,\infty), \\ \mathrm{essup}_{(r,z)\in\Omega}|\omega_\theta(r,z)| & \text{if } p=\infty. \end{cases}$$

and

$$\|\Pi\|_{L^p(\mathbb{R}^3)} = \begin{cases} \Big(\int_\Omega |\Pi(r,z)|^p rdrdz\Big)^{\frac{1}{p}} & \text{if } p\in[1,\infty), \\ \mathrm{essup}_{(r,z)\in\Omega}|\Pi(r,z)| & \text{if } p=\infty. \end{cases}$$

Let us denote that the spaces $L^1(\Omega)$ and $L^1(\mathbb{R}^3)$ are scale invariant, in the sense

$$\|\lambda^2\omega_0(\lambda\cdot)\|_{L^1(\Omega)} = \|\omega_0\|_{L^1(\Omega)}, \quad \|\lambda^3\rho_0(\lambda\cdot)\|_{L^1(\mathbb{R}^3)} = \|\rho_0\|_{L^1(\mathbb{R}^3)}.$$

The emergence of the term $\partial_r\rho$ in the first equation of (1.10) leads to complications in the computations, in particular when we deal in a critical space, which prompted us to define a new function named $\widetilde{\rho} \triangleq r\rho$ which solves

$$\partial_t \widetilde{\rho} + \mathrm{div}_\star(v\widetilde{\rho}) = \Big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big)\widetilde{\rho} - 2\partial_r \rho.$$

We can easily see that $\widetilde{\rho}$ satisfy the same equation as ω_θ. To achieve our aim we will handle with the following system.

$$\begin{cases} \partial_t \omega_\theta + v\cdot\nabla\omega_\theta - \frac{v^r}{r}\omega_\theta = \big(\Delta - \frac{1}{r^2}\big)\omega_\theta - \partial_r\rho, \quad (t,r,z)\in\mathbb{R}_+\times\Omega \\ \partial_t\widetilde{\rho} + \mathrm{div}_\star(v\widetilde{\rho}) = \Big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big)\widetilde{\rho} - 2\partial_r\rho, , \quad (t,r,z)\in\mathbb{R}_+\times\Omega \\ \partial_t\rho + v\cdot\nabla\rho - \Delta\rho = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3 \\ (\omega_\theta,\rho)_{|t=0} = (\omega_0,\rho_0). \end{cases}, \tag{1.13}$$

At this stage we are ready to show that the system (1.13) is globally well-posed. More precisely, our first main result is the following.

Theorem 1.2.4. *Let $(\omega_0,\rho_0)\in L^1(\Omega)\times L^1(\mathbb{R}^3)$ be axisymmetric initial data, then the system (1.10) admits a unique global mild solution. More precisely we have:*

$$(\omega_\theta, r\rho) \in \Big(C^0\big([0,\infty);L^1(\Omega)\big)\cap C^0\big((0,\infty);L^\infty(\Omega)\big)\Big)^2$$

$$\rho\in C^0\big([0,\infty);L^1(\mathbb{R}^3)\big)\cap C^0\big((0,\infty);L^\infty(\mathbb{R}^3)\big)$$

Furthermore, for every $p\in[1,\infty]$, there exists some constant $K_p(D_0)>0$, for which, and for all $t>0$ the following statements hold

$$\|(\omega_\theta(t), r\rho(t))\|_{L^p(\Omega)\times L^p(\Omega)} \le t^{-(1-\frac{1}{p})}K_p(D_0).$$

$$\|\rho(t)\|_{L^p(\mathbb{R}^3)} \le t^{-\frac{3}{2}(1-\frac{1}{p})}K_p(D_0),$$

where

$$D_0 \triangleq \|(\omega_0,\rho_0)\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)}.$$

Remark 1.2.5. First, we note that the assumption $\omega_0\in L^1(\Omega)$ doesn't imply in general that the associated velocity v is in $L^2(\Omega)$ space. Indeed if $v\in(L^2(\mathbb{R}^2))^2$ and if $\omega\in L^1(\mathbb{R}^2)$, we always have $\int_{\mathbb{R}^2}\omega dx = 0$. So, the obtained solution is never of finite energy. This mismatch of functional spaces for the vorticity and the velocity field is specific to dimension two. Consequently, the classical energy estimate is not available to derive a uniform bound for the velocity. In dimension three, if $\omega(t,x)$ is a solution of the vorticity equation with $\omega_0\in(L^{\frac{3}{2}}(\mathbb{R}^3))^3)$, the velocity field obtained by Biot-Savart's law is indeed a solution for the velocity equation in $(L^3(\mathbb{R}^3))^3$. In this case, we do not gain anything in generality by studying the equation for the vorticity.

Theorem 1.2.4 is proved by rewriting the system (1.13) in terms of a fixed point problem for a functional constructed using an appropriate pair of semi-group. Under adequate smallness assumptions, the Picard fixed point theorem then allows to obtain a unique local solution. This approach is particularly efficient if the functional space used is critical. For this we define the Banach space of contraction $X_T \times X_T \times Z_T$ where the X_T and Z_T are defined by:

$$X_T = \Big\{ f \in C^0\big((0,T], L^{4/3}(\Omega)\big) : \|f\|_{X_T} < \infty \Big\},$$

$$Z_T = \Big\{ h \in C^0\big((0,T], L^{4/3}(\mathbb{R}^3)\big) : \|h\|_{Z_T} < \infty \Big\},$$

equipped with the following norms

$$\|f\|_{X_T} \quad = \quad \sup_{0<t\leq T} t^{1/4}\|f(t)\|_{L^{4/3}(\Omega)}, \ \|h\|_{Z_T} = \sup_{0<t\leq T} t^{3/8}\|h(t)\|_{L^{4/3}(\mathbb{R}^3)}.$$

As we mentioned before the main difficulty in the proof is the estimation of the term $\partial_r \rho$, for this we defined the function $\widetilde{\rho}$. We take advantage of the fact that it has the same semi-group as ω_θ, and it helps us with weighted estimates to estimate the nonlinear term in $\rho-$equation. We showed that the local solution often constructed can be extended to the global one by introducing two new unknown functions $\Gamma = \Pi - \frac{\rho}{2}$ and $\widetilde{\Gamma}$ where $\widetilde{\Gamma} = r\Gamma$ following [53] with $\Pi = \frac{\omega_\theta}{r}$, which solve respectively

$$\begin{cases} \partial_t \Gamma + v\cdot\nabla\Gamma - (\Delta + \frac{2}{r}\partial_r)\Gamma = 0 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \Gamma_{|t=0} = \Gamma_0. \end{cases},$$

and

$$\begin{cases} \partial_t \widetilde{\Gamma} + \mathrm{div}_*(v\widetilde{\Gamma}) - (\Delta - \frac{1}{r^2})\widetilde{\Gamma} = 0 & \text{if } (t,x) \in \mathbb{R}_+ \times \Omega, \\ \widetilde{\Gamma}_{|t=0} = \widetilde{\Gamma}_0. \end{cases}$$

The previous functions Γ and $\widetilde{\Gamma}$ play the same role in a priori estimates for (1.10) as the function Π with respect to the Navier-Stokes system, where we do such estimates for $(\Gamma, \widetilde{\Gamma}, \omega_\theta, \rho, \widetilde{\rho})$ in Lebesgue spaces.

1.2.2 When the initial data is finite measure

It is natural to ask whether the results obtained in Theorem 1.2.4 remain valid in the space of finite measures, especially since the latter contains the Lebesgue space L^1? We have succeed to answer this question in the affirmative in two steps, where we have restricted in a first step the fact that the vorticity ω_0 is only a finite measure, taking advantage of the results obtained by [36]. Thus, the local existence of the solution cost us to impose the smallness condition on the punctual part of the initial vorticity $\omega_{0,pp}$.

To streamline the above topic, we fix some notations freely used in the last chapters. Let us start by recalling that $C_0(\Omega)$ (resp $C_0(\mathbb{R}^3)$) the set of all continuous functions over Ω (resp $\mathbb{R}^3$) that vanish at infinity and on the boundary $\partial\Omega$

and $\mathscr{M}(\Omega)$ (resp $\mathscr{M}(\mathbb{R}^3)$) the class of all real-valued finite measures over the half-plane Ω (resp $\mathbb{R}^3$). Notice that the pairing between $\mathscr{M}(\Omega)$ and $C_0(\Omega)$ is defined by $\langle \mu, \varphi\rangle = \int_\Omega \varphi(x)d\mu(x)$. Consequently, $\mathscr{M}(\Omega)$ endowed with the norm

$$\|\mu\|_{\mathscr{M}(\Omega)} \triangleq \sup_{\{\varphi\in C_0(\Omega), \|\varphi\|_{L^\infty(\Omega)}\leq 1\}} |\langle \mu, \varphi\rangle|, \quad \mu\in\mathscr{M}(\Omega).$$

is a Banach space. For $f\in L^1(\Omega)$ define the measure μ_f by $\langle \mu_f, \varphi\rangle = \int_\Omega f(x)\varphi(x)dx$, where dx designates the Lebesgue measure. It can easily be seen that $\mu_f\in\mathscr{M}(\Omega)$, thus we deduce that $L^1(\Omega)$ can be identified as a closed subspace of $\mathscr{M}(\Omega)$ and $\|\mu_f\|_{\mathscr{M}(\Omega)} = \|f\|_{L^1(\Omega)}$. Each $\mu\in\mathscr{M}(\Omega)$ can be decomposed in unique way as

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}, \quad \mu_{ac}\perp\mu_{sc}\perp\mu_{pp}$$

and

$$\|\mu\|_{\mathscr{M}(\Omega)} = \|\mu_{ac}\|_{\mathscr{M}(\Omega)} + \|\mu_{sc}\|_{\mathscr{M}(\Omega)} + \|\mu_{pp}\|_{\mathscr{M}(\Omega)}.$$

where, in the sequel we denote by:
• μ_{ac} is a measure which is absolutely continuous with respect to Lebesgue measure, that is $\frac{d\mu_{ac}}{dx} = f$ for some $f\in L^1(\Omega)$.
• μ_{sc} is a singular continuous measure which has no atom but is supported on a set of zero Lebesgue measure.
• μ_{pp} is punctual part (an atomic measure), $\mu_{pp} = \sum_{n\geq 1}\lambda_n\delta_{a_n}$, $(\lambda_n)\subset\mathbb{R}$, $(a_n)\subset\Omega$, with δ_{a_n} stands to be the Dirac measure supported at $a_n\in\Omega$.
We will explore the second main result which treats principally the global well-posedness issue for the system (1.10) whenever the initial data $(\omega_0, \rho_0)\in\mathscr{M}(\Omega)\times L^1(\mathbb{R}^3)$ and some convergence. More precisely the first result in this framework is given by the following theorem

Theorem 1.2.6. *There exist non negative constants ε and C such that the following hold. Let $(\omega_0, \rho_0)\in\mathscr{M}(\Omega)\times L^1(\mathbb{R}^3)$ with ρ_0 axisymmetric and $\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)}\leq\varepsilon$, then, the Boussinesq system (1.10) admits a unique global axisymmetric mild solution (ω_θ, ρ) satisfying*

$$(\omega_\theta, \rho)\in C^0\big((0,\infty); L^1(\Omega)\cap L^\infty(\Omega)\big)\times C^0\big([0,\infty); L^1(\mathbb{R}^3)\big)\cap C^0\big((0,\infty); L^\infty(\mathbb{R}^3)\big),$$
$$r\rho\in C^0\big([0,\infty); L^1(\Omega)\big)\cap C^0\big(L^\infty(0,\infty); L^\infty(\Omega)\big).$$

Furthermore, for every $p\in(1,\infty]$, we have

$$\limsup_{t\to 0}\ t^{\frac{3}{2}(1-\frac{1}{p})}\|\rho(t)\|_{L^p(\mathbb{R}^3)} = \limsup_{t\to 0}\ t^{1-\frac{1}{p}}\|r\rho(t)\|_{L^p(\Omega)} = 0$$

and

$$\limsup_{t\to 0}\ t^{1-\frac{1}{p}}\|\omega_\theta(t)\|_{L^p(\Omega)}\leq C\varepsilon.$$

Moreover, we have

$$\limsup_{t\to 0}\ \|\omega_\theta(t)\|_{L^1(\Omega)} < \infty, \quad \lim_{t\to 0}\|\rho(t) - \rho_0\|_{L^1(\mathbb{R}^3)} = 0$$

and $\omega_\theta(t)\rightharpoonup\omega_0$ as $t\to 0$.

The proof of local existence in Theorem 1.2.6 is almost the same as the proof of Theorem 1.2.4 where the initial data is L^1-integrable, with minor changes due to the fact that we are not act with the same data. We can say that all the difficulties appeared in the proof of weak convergence of ω_θ to ω_0, where we combined the idea cited in [36, 34] and the coupled function $\widetilde{\Gamma}$.
For the global existence, since we are concerned with the behavior of the solution when t is big enough and the fact that the solution is Lebesgue's integrable when $t > 0$, then the global existence and all the a priori estimates which proved in the first Theorem 1.2.4 remain valid.

The study in the case where all the initial data laying in $\mathscr{M}(\Omega) \times \mathscr{M}(\mathbb{R}^3)$ is more delicate, because we do not have an explicit relation between $\widetilde{\rho}_0 \in \mathscr{M}(\Omega)$ and $\rho_0 \in \mathscr{M}(\mathbb{R}^3)$ - especially as we are forced to use the equation of $\widetilde{\rho}$-. In other words, how to define a measure on Ω from a measure on $\mathbb{R}^3$, in addition, does the constructed measure keep the property of axisymmetry? Furthermore, the constructed measure must not disturb the weak continuity of the solution near $t = 0$. We overcame these difficulties firstly by using the concept of "push-forward measure" which is in fact an image of a measure by a measurable function. Thanks to this concept we were able to define an axisymmetric measure, then define a measure that answers our questions.
The last main result is presented in the following theorem.

Theorem 1.2.7. *There exists a non negative constant $\varepsilon > 0$ such that the following hold. Let $(\omega_0, \rho_0) \in \mathscr{M}(\Omega) \times \mathscr{M}(\mathbb{R}^3)$ with ρ_0 being axisymmetric in the sense of Definition 4.2.3 and*

$$\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)} + \|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)} \leq \varepsilon, \tag{1.14}$$

then, the Boussinesq system (1.10) admits a unique global mild axisymmetric solution (ω_θ, ρ) such that

$$(\omega_\theta, \rho) \in C^0\big((0,\infty); L^1(\Omega) \cap L^\infty(\Omega)\big) \times C^0\big((0,\infty); L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\big),$$

$$r\rho \in C^0\big((0,\infty); L^1(\Omega) \cap L^\infty(\Omega)\big).$$

Furthermore, for every $p \in (1, \infty]$, we have

$$\limsup_{t\to 0} \; t^{\frac{3}{2}(1-\frac{1}{p})} \|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq C\varepsilon, \qquad \limsup_{t\to 0} \; t^{1-\frac{1}{p}} \|(\omega_\theta(t), r\rho(t))\|_{L^p(\Omega)\times L^p(\Omega)} \leq C\varepsilon$$

and

$$\limsup_{t\to 0} \; \|(\omega_\theta(t), \rho(t))\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)} < \infty.$$

Moreover, we have that $(\omega_\theta(t), \rho(t)) \rightharpoonup (\omega_0, \rho_0)$ as $t \to 0$.

Let us briefly sketch the proof of the Theorem 1.2.7. It is essentially based on the proof of the weak convergence of the solution towards the initial data. After having indicated the existence of such solution by an argument of the fixed point under a condition of smallness to the punctual part of the initial data.
The major difficulty encountered is the convergence of the nonlinear term in

Duhamel's formula associated to the equation of ω_θ or $\widetilde{\rho}$ (almost the same equation) to 0, because the differential operators ∂_r and div do not commute with $\mathbb{S}_1$. For this we have rewritten the equation of $\widetilde{\rho}$ in terms of $\mathbb{S}_2$. By this remedy we have succeeded in proving the convergence. Concerning the globality of solution is deduced by the same way from the second theorem because the solution for $t > 0$ is more regular more precisely, it is in the Lebesgue spaces.

2 Preliminary Chapter

In this preliminary chapter, we gather the basic ingredients freely explored in the course of this work. We start with the definition of critical spaces based on scale invariance. As our research is devoted to axisymmetric incompressible fluids based on the study of the vorticity equation combined with the density one, we thought it best to present the vortex as a physical ingredient. To simplify the presentation, we have focused on the two-dimensional Navier-Stokes equation. We embark this paragraph by defining this ingredient and give some illustrative examples, to conclude with the famous Biot-Savart Law, which allows to recover velocity from the vortex. Before concluding this chapter with some concepts related to positive measures, we mention the definition of semi-groups and some properties

2.1 Scaling invariance

The definition of critical spaces is based on the properties of invariance by dilatation and translation of the solutions of a partial differential evolution equation (PDE). Let's be more explicit we have the following definition.

Definition 2.1.1. A partial differential evolution equation is said to be scale invariant if there are two parameters α and β such that for any $\lambda > 0$, we have : if the initial vector field $v_0(x)$ generates a solution $v(t,x)$, then the solution associated with the rescaled data $\lambda^\alpha v_0(\lambda x)$ is $\lambda^\alpha v(\lambda^\beta t, \lambda x)$.

Example 2.1.2. - The Boussinesq system

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = \rho\vec{e}_z & x\in\mathbb{R}^3, \quad t\in(0,\infty),\\ \partial_t\rho + v\cdot\nabla\rho - \kappa\Delta\rho = 0 & x\in\mathbb{R}^3, \quad t\in(0,\infty),\\ \mathrm{div} v = 0, \\ (v,\rho)_{|t=0} = (v_0,\rho_0). \end{cases} \tag{B$_{\mu,\kappa}$}$$

is invariant by the transformation

$$v(t,x)\mapsto \lambda v(\lambda^2 t, \lambda x); \quad \rho(t,x)\mapsto \lambda^3\rho(\lambda^2 t,\lambda x)$$

- Navier Stokes system

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = 0 \ \text{ if } x\in\mathbb{R}^3, \quad t\in(0,\infty),\\ \mathrm{div} v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{NS$_\mu$}$$

is invariant by the transformation

$$v(t,x) \mapsto \lambda v(\lambda^2 t, \lambda x)$$

Remark 2.1.3. If (v,ρ) is a solution of ($B_{\mu,\kappa}$) associated with the initial condition (v_0,ρ_0) on $[0,T]\times\mathbb{R}^d$, then for all $\lambda>0$ and $x_0\in\mathbb{R}^d$, $(v_\lambda(t,x),\rho_\lambda(t,x)) = (\lambda v(\lambda^2 t,\lambda x), \lambda^3\rho(\lambda^2 t,\lambda x))$ is a solution of ($B_{\mu,\kappa}$) associated with the initial condition $(\lambda v_0(\lambda(x-x_0)), \lambda^3\rho_0(\lambda(x-x_0)))$ on $[0,\lambda^{-2}T]\times\mathbb{R}^d$.

Definition 2.1.4. A Banach space $X \hookrightarrow \mathcal{S}'$ is said to be critical for the initial conditions of ($B_{\mu,\kappa}$) if its norm satisfies

$$\forall\lambda>0, \forall x_0\in\mathbb{R}^d, \quad \|v_0\|_X = \lambda\|v_0\big(\lambda(\cdot - x_0)\big)\|_X, \quad \text{and} \quad \|\rho_0\|_X = \lambda^3\|\rho_0\big(\lambda(\cdot - x_0)\big)\|_X$$

Remark 2.1.5. [27] The critical spaces for the velocity of the Boussinesq system ($B_{\mu,\kappa}$) are the same as for the Navier-Stokes system (NS_μ), As for density, it requires two derivatives more than the velocity.

Example 2.1.6. Let us consider the Navier Stokes system (NS_μ), according to the above definition the following spaces are critical

$$L^q(\mathbb{R}^+, L^r(\mathbb{R}^d)); \quad \text{with} \quad \frac{2}{q}+\frac{d}{r}=1.$$

Remark 2.1.7. [39]

1- Any critical space X for (NS_μ) verifies $X \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$.

2- We have the following chain of inclusions:

$$L^d \hookrightarrow \dot{B}^{-1+\frac{d}{p}}_{p,q} \hookrightarrow \dot{B}^{-1+\frac{d}{\tilde{p}}}_{\tilde{p},\tilde{q}} \hookrightarrow \partial BMO \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$$

with $2\le p\le\tilde{p}<\infty$ and $2\le q\le\tilde{q}<\infty$.

Definition 2.1.8. We say that a space X is critical for the vorticity if its norm satisfies

$$\forall\lambda>0, \forall x_0\in\mathbb{R}^d, \quad \|\omega\|_X = \lambda^2\|\omega\big(\lambda(\cdot - x_0)\big)\|_X$$

Example 2.1.9. $L^1(\mathbb{R}^d)$ and $\mathscr{M}(\mathbb{R}^d)$ are critical spaces for the vorticity.

2.2 Vortex

The solution of the momentum equations is often complicated by the presence of pressure terms. However, in some circumstances, it is possible to get rid of them, by relying on other parameters of motion. Among these parameters, there is a quantity that plays a fundamental role in the description of the movement of a fluid: it is the "vortex".

Definition 2.2.1. The vortex is defined as half of the curl of the velocity: $\boldsymbol{\omega} = \frac{1}{2}\nabla \times v$, which is a vector. The coefficient $\frac{1}{2}$ ensures that the magnitude is exactly equal to the local angular velocity ω_3 of a fluid particle. The magnitude $\omega^2/2$ is called the vortex intensity or the vorticity.

In other words, in a fluid in motion with a local velocity $v = (v_1, v_2, v_3)$, the vortex in cartesian coordinates is the vector where its components are defined by the expression: $\boldsymbol{\omega}= (\omega_1, \omega_2, \omega_3)$ are related to those of the velocity by the relations:

$$\omega_1 = \frac{1}{2}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right), \quad \omega_2 = \frac{1}{2}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right), \quad \omega_3 = \frac{1}{2}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

2.2.1 Interpretation and Examples

To understand and illustrate the information carried by this vortex vector, let us consider the differences between the displacements of two very close points P and Q within a small fluid particle, which we assume to be two-dimensional.
Let δx and δy the components of the vector $\mathbf{PQ}$. We have

$$\delta v = v(Q) - v(P) = \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y,$$

which we can write it as:

$$\begin{aligned}
\delta v_1 &= \frac{\partial v_1}{\partial x}\delta x + \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right)\delta y + \frac{1}{2}\left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\right)\delta y, \\
\delta v_2 &= \underbrace{\frac{\partial v_2}{\partial y}\delta y}_{\text{dilatation}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}\right)\delta x}_{\text{pure deformation}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\delta x}_{\text{rotation}}.
\end{aligned}$$

These expressions show that, in a very general way, the displacement $\mathbf{PQ}$ has three distinct contributions. The first one represents the dilatation of the fluid particle, which is identically zero in the case of a fluid with invariant density (incompressible) where $\dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} = 0$. The second one represents the pure deformation and has the characteristic of not involving any block rotation of the particle, but a simple shear. As for the third one, it represents the block rotation of the fluid particle around an axis perpendicular to the plane of the figure passing through the point P, with the angular velocity $\omega_3 = \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y}\right)$, without any shear, as shown in the figure (2.1)

Let us now give some examples of vortex structures, represented in figure (2.2).

On the example of figure (2.2(a)), we can notice that the vortex is concentrated in the center of the structure, where the velocity distribution is close to that of a

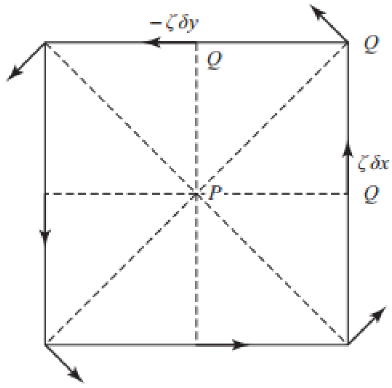

Figure 2.1: *Illustration of the displacements $\delta v_1 = -\omega_3 \delta y$ and $\delta v_2 = \omega_3 \delta x$ where $\omega_3 = \zeta$ of various points Q of a particle supposed to be square, showing its rotation in block with the angular velocity ω_3.*

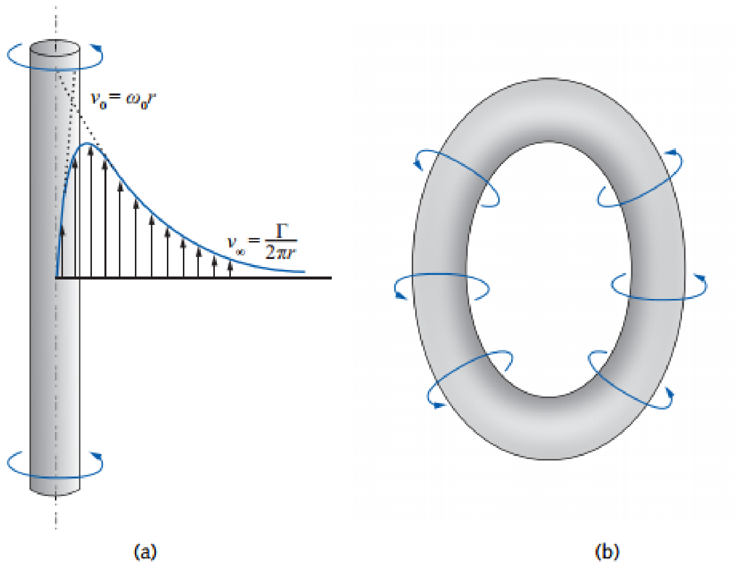

Figure 2.2: *Simple examples of vortex structures: (a) rectilinear vortex schematizing a tornado , (b) toric vortex schematizing a round of smoke.*

block rotation, of the form ωr. Otherwise, the velocity is annulled out quite quickly as soon as we leave this structure, since the velocity distribution then becomes proportional to $1/r$ and keeps the quantity $\Gamma = 2\pi r v_2$ invariant, which is nothing other than the circulation of the velocity on the circle of radius r, or even the flow of the vortex vector through the portion of flat surface limited by this circle. This often leads to schematize the vortex structures as vortex lines, by attributing to each the circulation of the velocity:

$$\Gamma = \oint_C v \cdot ds = 2 \int_S \boldsymbol{\omega} \cdot \mathbf{n} dS \tag{2.1}$$

The second equality of (2.1), deduced from the Stokes formula, shows that the circulation of the velocity Γ is necessarily equal to the double of the flow of the vortex through the surface S limited by the contour C. Note that, since the vortex vector is a pure rotational , its divergence is zero. Thus, this vector field is part of the conservative fields, like the magnetic field and the density of electric current. Its flow Γ through a portion of surface S therefore constitutes an invariant of a vortex tube, just like the flow which circulates in a pipe, or the electric current which passes through a conductive fill of electricity. Since this flow cannot vary from end to end, a vortex tube must either close in on itself, as in the case of the smoke ring (2.2 (b)), or go from one wall to another.

The vortex equation

To simplify, let us limit ourselves to the case of an incompressible fluid, whose motion satisfies the Navier-Stokes equation.

$$\partial_t v + v \cdot \nabla v + \nabla p = \nu \Delta v \tag{2.2}$$

To know the dynamics of the vortex, we take the rotational of the Navier-Stokes equation (2.2), taking into account div$v = 0$, we acquire

$$\partial_t \boldsymbol{\omega} + (v \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) v + \nu \Delta \boldsymbol{\omega}. \tag{2.3}$$

The interpretation of equation(2.3) is: For the first member, which is the particle derivative of the vortex vector, as well as for the last term, which represents the diffusion of the vortex by viscosity. The question then arises, what is the meaning of the term $(\boldsymbol{\omega} \cdot \nabla) v = \omega \dfrac{\partial v}{\partial s}$, especially since it has no equivalent in the Navier-Stokes equation?

To answer this question, consider a small section $\mathbf{MN} = \delta s$ of a vortex line, as shown in figure (2.3) . Neglecting the viscous term, generally which is the case of air and water, the vortex equation (2.3) can be written

$$\frac{1}{\omega}\frac{d\boldsymbol{\omega}}{dt} = \frac{\partial v}{\partial s} \tag{2.4}$$

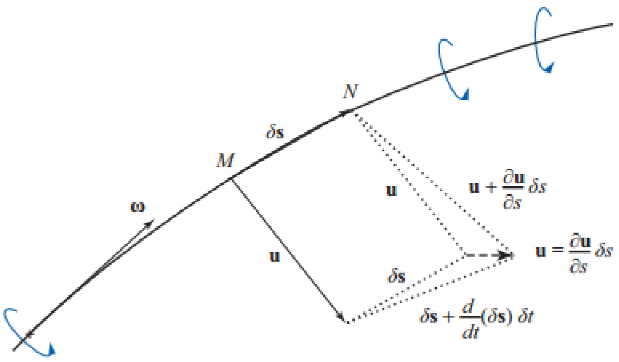

Figure 2.3: figure 3. Illustration of the variation of the vector $\mathbf{MN} = \delta s$ linked to the difference to in velocity between its ends M and N.

Moreover, according to the construction performed on figure (2.3), the relative variation of the vector δs is given by

$$\frac{1}{\delta s}\frac{d}{dt}(\delta s) = \frac{\partial v}{\partial s} \tag{2.5}$$

These two equations (2.4) and (2.5) show that the two vectors $\boldsymbol{\omega}$ and δs have identical dimensional behaviors: their relative elongation and rotation coincide exactly. Fluid mechanics are thus accustomed to calling $(\boldsymbol{\omega} \cdot \nabla) v$ the term for the production of vorticity by stretching of the vortex lines.

2.2.2 Biot-Savart Law

We also recall that for the Navier Stokes equations we can recover the velocity v from the vorticity by a non-local operator. To achieve this, it may be advantageous to introduce, in addition to ω, a second function of v_1 and v_2, the latter is called the *stream function* Ψ.
Its composition goes back to the well-known relation $\text{div}(\text{curl}A) = 0$ for any vector $A = (A_x, A_y, A_z)$, then we can choose the vector A such that $v = \text{curl}A$, thanks to the definition of the rotational, plus that the flow is in the plane (xoy), we can find the following relations:

$$\begin{cases} v_1 = -\dfrac{\partial A_z}{\partial y} \\ v_2 = \dfrac{\partial A_z}{\partial x} \end{cases} \tag{2.6}$$

Let's pose now $\Psi = A_z(x, y)$, then

$$v = (v_1, v_2) = \nabla^{\perp}\Psi := (-\partial_2\Psi, \partial_1\Psi) \tag{2.7}$$

This function satisfies the continuity equation, moreover the vorticity ω [1] is easily expressed as a function of Ψ

$$\omega = \partial_1 v_2 - \partial_2 v_1 = \partial_1^2 \Psi + \partial_2^2 \Psi = \Delta \Psi$$

The solution to this equation is given by a convolution with the Newtonian potential with ω as shown by the following lemma

Lemma 2.2.2. *[68] Suppose that $f \in L^1(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$ and $\int_{|x|\geq 1} |f(x)| \log |x| dx < \infty$, for $d = 2$. Then there exists a solution $v \in C^2(\mathbb{R}^d)$ to the Poisson equation*

$$-\Delta v = f$$

given by the convolution

$$v(x) = \int_{\mathbb{R}^d} N(x-y) f(y) dy$$

where $N(\cdot)$ is given by

$$N(x) = \begin{cases} -\frac{1}{2\pi} \log |x|; & d = 2 \\ -\frac{|x|^{2-d}}{(2-d)\omega_d}; & d > 2. \end{cases} \tag{2.8}$$

with ω_d is the area of unit sphere in $\mathbb{R}^d$ given by $\omega_d = \dfrac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$. This solution can be differentiated under the integral to yield

$$\nabla v(x) = \frac{1}{\omega_d} \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} f(y) dy.$$

Then

$$\Psi(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \omega(t,y) dy$$

Because we can compute the gradient of Ψ by differentiating under the integral, the velocity v can be recovered from Ψ by

$$v(t,x) = \int_{\mathbb{R}^2} K_2(x-y)\omega(t,y) dy \tag{2.9}$$

where the kernel $K_2(\cdot)$ is defined by

$$K_2 = \frac{1}{2\pi}\Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\Big)^t \tag{2.10}$$

Equation (2.9) is the famous law of Biot Savart

[1] The word vorticity refers to the rate of rotation of the vortex, which is a scalar quantity like kinetic energy. Its local value within a mass of fluid rotating with angular velocity ω_3 is $\omega^2/2$. The rotating mass of fluid represents what is typically called an eddy, which, during its lifetime, may be considered as a single object

2.3 Semigroups of linear operators

In this section, we present some of the main points of the theory of semigroups and evolution equations. The study of the linear part of semigroup theory began in the 1930s with the work of E. Hille, Y. Yosida and R. Phillips on semigroups of linear operators in Banach spaces. The initial idea of this theory came from an article by G. Peano of 1887 where he wrote the system of differential equations in a matrix form. For more details see [[4] [30],[73]]

$$\begin{cases} \dfrac{du_1}{dt} = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n + f_1(t) \\ \vdots \\ \dfrac{du_n}{dt} = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n + f_n(t) \end{cases} \tag{2.11}$$

in a matrix form as

$$u'(t) = Au(t) + f(t) \tag{2.12}$$

where $u(t) = (u_1(t), u_2(t), \cdots, u_n(t))^t$, $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))^t$ and $A = (a_{ij})$ and solved it by means of the explicit formula

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A}f(s)ds \tag{2.13}$$

where $e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!}t^k A^k$. So he transformed a complicated one-dimensional problem to a formally simpler one in a higher dimension.

2.3.1 Definitions and basic properties

Let X be a Banach space

Definition 2.3.1. A one parameter family $T(t)$ with $0 \leq t < \infty$ of bounded linear operators from X into X is a semigroup of bounded linear operator on X if

i $T(0) = I$, (I is the identity operator on X.)

ii $T(s+t) = T(s)T(t)$ for every $s, t \geq 0$ (the semigroup property.)

A semigroup of bounded linear operators $T(t)$ is uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0$$

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \quad \text{exists}\}$$

and

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}|_{t=0}, \quad for \quad x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$. $D(A)$ is the domain of A.

Remark 2.3.2. If $T(t)$ is uniformly continuous semigroup of bounded linear operators then

$$\lim_{s \to t} \|T(s) - T(t)\| = 0$$

Corollary 2.3.3. *[73] Let $T(t)$ be uniformly continuous of bounded linear operators, then*

i *There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.*

ii *There exists a unique bounded linear operator A such that $T(t) = e^{tA}$.*

iii *The operator A is the infinitesimal generator of $T(t)$.*

iv *$t \mapsto T(t)$ is differentiable and*

$$\frac{dT(t)}{dt} = AT(t) = T(t)A$$

.

2.3.2 Strongly continuous semigroup of bounded linear operators

Definition 2.3.4. A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \to 0} T(t)x = x, \quad \forall x \in X \tag{2.14}$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or C_0-semigroup.

Theorem 2.3.5. *[73] Let $T(t)$ be C_0-semigroup. There exist constants $\omega \geq 0$ and $M \geq 0$ such that*

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for} \quad 0 \leq t < \infty. \tag{2.15}$$

Remark 2.3.6. If $\omega = 0$, $T(t)$ is called uniformly bounded and if moreover $M = 1$ it is called C_0-semigroup of contraction.

Lemma 2.3.7. *[30] For the generator $(A, D(A))$ of a C_0-semigroup $T(t)$, $t \geq 0$, the following properties hold*

i *$A : D(A) \subseteq X \to X$ is a linear operator.*

ii *If $x \in D(A)$, then $T(t)x \in D(A)$ and*

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x, \quad \text{for every} \quad t \geq 0.$$

iii *For every $t \geq 0$ and $x \in X$, one has*

$$\begin{aligned} T(t)x - x &= A\int_0^t T(s)xds, \quad \text{if} \quad x \in X \\ &= \int_0^t T(s)Axds, \quad \text{if} \quad x \in D(A). \end{aligned}$$

2.3.3 The Hille-Yosida Theorem

In the previous subsection we saw that for any semigroup $T(t)$, $0 \leq t < \infty$, there is an associated infinitesimal generator operator, so it is natural to ask whether an unbounded operator on X is the infinitesimal generator of a C_0-semigroup? The Hille-Yosida theorem is central in the theory of semigroups, providing a clear answer to this question.

Theorem 2.3.8 (Hille-Yosida)**.** *[73] A linear (unbounded) operator A is the infinitesimal generator of C_0-semigroup of contraction if and only if*

i *A is closed and and $\overline{D(A)} = X$.*

ii *The resolvant set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$, $\|R(\lambda : A)\| \leq \frac{1}{\lambda}$.*

Where the resolvant set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is inversible .i.e $(\lambda I - A)^{-1}$ is bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1}; \quad \lambda \in \rho(A)$ of bounded linear operators is called the resolvant of A.

Definition 2.3.9 (differentiable Semigroup)**.** A C_0-semigroup $T(t)$ is called differentiable for $t > t_0$ if for all $x \in X$ the map $t \mapsto T(t)x$ is differentiable for $t > t_0$.

2.4 Well-Posedness for Evolution Equations

One of the fundamental problems of operator semigroup theory is how to express the solution of an evolution problem using the semigroup generated by the differential operator. Now, since we want to apply this abstract theory to solve partial differential equations, we must work in infinite dimensions. So our main object will be the study of evolution problems of the form:

$$\begin{cases} u'(t) + Au(t) = f(t), & \text{on} \quad (0,T) \\ u(0) = x. \end{cases} \tag{2.16}$$

where the independent variable t represents time, $u(\cdot)$ is a function with values in a Banach space X, $f : (0,T) \to X$, $A : D(A) \to X$ is a linear operator and $x \in X$.

2.4.1 The abstract Cauchy problem

The homogeneous initial value problem

Definition 2.4.1. The initial value problem

$$(ACP)\begin{cases} u'(t) = Au(t), & \text{on} \quad t \geq 0 \\ u(0) = x. \end{cases} \tag{2.17}$$

is called the abstract Cauchy problem associated to $(A, D(A))$ and the initial value x.

Definition 2.4.2. A function $u : \mathbb{R}^+ \to X$ is called a classical solution of (ACP) if is continuously differentiable with respect to X, $u(t) \in D(A)$ for all $t \geq 0$ and (ACP) holds.

By using Lemma 2.3.7 and Theorem 1.24 in [73] we get, if A is the generator of a strongly continuous semigroup, then the semigroup yields solutions of the associated abstract Cauchy problem. This illustrates by the following result.

Proposition 2.4.3. *[30] Let $(A, D(A))$ be the generator of the C_0-semigroup $T(t)$, $t \geq 0$. Then for every $x \in D(A)$, the function*

$$u : t \mapsto u(t) = T(t)x$$

is the unique classical solution of (ACP).

Theorem 2.4.4. *[73] If A is the infinitesimal generator of a differentiable semigroup, then for every $x \in X$ the initial value problem (APC) has a unique solution.*

Remarks 2.4.5. 1- If the C_0-semigroup generated by the infinitesimal operator A is not differentiable, then in general, if $x \notin D(A)$, the problem of initial value

$$(APCh)\begin{cases} u'(t) = Au(t), & \text{on} \quad t > 0 \\ u(0) = x. \end{cases} \tag{2.18}$$

does not have a solution.

2- The function $t \mapsto T(t)x$ is a generalized solution of the problem (2.18) wich called a mild solution.

The inhomogeneous initial value problem

In this case the problem is given by

$$\begin{cases} u'(t) + Au(t) = f(t), & \text{on} \quad (0, T) \\ u(0) = x. \end{cases} \tag{2.19}$$

where $f : (0, T) \to X$, A is the infinitesimal generator of C_0-semigroup $T(t)$.

Definition 2.4.6. A function $u : [0, T) \to X$ is a classical solution of (2.19) on $[0, T)$ if u is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $u(t) \in D(A)$ for $0 < t < T$ and (2.19) is satisfied on $[0, T)$.

Let $T(t)$ be the C_0-semigroup generated by A and u the solution of (2.19). Then the function $g : [0, T) \to X$ defined by $g(s) = T(t - s)u(s)$ is differentiable and

$$g'(s) = T(t - s)f(s) \tag{2.20}$$

By integration from 0 to t of (2.20), we obtain

$$u(t) = T(t)x + \int_0^t T(t - s)f(s)ds \tag{2.21}$$

Definition 2.4.7. Let A be the infinitesimal generator of C_0-semigroup $T(t)$. Let $x \in X$ and $f \in L^1\big([0, T); X\big)$. The function $u \in C\big([0, T); X\big)$ given by

$$u(t) = T(t)x + \int_0^t T(t - s)f(s)ds, \quad 0 \leq t \leq T. \tag{2.22}$$

is the mild solution of the infinitesimal value problem (2.19) on $[0, T)$.

Remark 2.4.8. In general, the continuity of the function f is not sufficient to guarantee the existence of the classical solution of problem (2.19) for $x \in D(A)$. The following example illustrates this remark.

Example 2.4.9. Let A be the infinitesimal generator of C_0-semigroup $T(t)$ and let $x \in X$ be such that $T(t)x \notin D(A)$ for all $t \geq 0$. Let $f(t) = T(t)x$, so the function f is continuous for all $t \geq 0$. Now consider the following problem

$$\begin{cases} u'(t) = Au(t) + T(t)x \\ u(0) = 0 \end{cases} \tag{2.23}$$

Note that the problem (2.23) has no solution even though $u(0) = 0 \in D(A)$, because the mild solution of (2.23) is

$$u(t) = \int_0^t T(t - s)T(s)x = tT(t)x$$

is not differentiable for $t > 0$, and therefore cannot be the solution of (2.23).

The following theorem gives the conditions which guarantee the existence of a classical solution.

Theorem 2.4.10. *[73] Let A be the infinitesimal generator of C_0-semigroup $T(t)$. Let $f \in L^1\big([0,T);X\big)$ be continuous on $(0,T)$ and let*

$$v(t) = \int_0^t T(t-s)f(s)ds, \qquad 0 \leq t \leq T.$$

The problem (2.19) has a solution u on $(0,T)$ for every $x \in D(A)$ if the following conditions is satisfied

i *$v(t)$ is continuously differentiable on $(0,T)$.*

ii *$v(t) \in D(A)$ for $0 < t < T$ and $Av(t)$ is continuous on $(0,T)$.*

If (2.19) has a solution u for some $x \in D(A)$, then v satisfied both (i) and (ii).

2.5 About Measure Theory and Integration

This section is devoted to a reminder on the measure theory and integration we start with the definitions and properties of a few kinds of measures, then we give the main theorems used in this study. Before presenting these concepts, we ask what is Measure Theory. And what is Integration Theory? In short, Measure theory is concerned with the distribution of mass over a set X, while integration theory is the theory of weighted sums of functions over a set X when the weights are specified by a mass distribution μ.

–Measurable spaces

Definition 2.5.1. A collection of sets $\mathscr{M} \subset \mathscr{P}(X)$ is called an $\sigma-$algebra if

i $X \in \mathscr{M}$.

ii $A \in \mathscr{M} \Rightarrow A^c \in \mathscr{M}$.

iii If $\{A_n, n \geq 1\} \subset \mathscr{M}$ then $\cup_{n\in\mathbb{N}^*} A_n \in \mathscr{M}$.

The elements of $\mathscr{M}$ are called the measurable parts of X. We say that $(X, \mathscr{M})$ is a measurable space.

–Definition of a measure

Definition 2.5.2. Let $(X, \mathscr{M})$ be a measurable space. Then a set function μ on $\mathscr{M}$ is called a measure if

i $\mu(A) \in [0, \infty]$ for all $A \in \mathscr{M}$.

ii $\mu(\emptyset) = 0$.

iii For any disjoint collection of sets $A_1, A_2, ... \in \mathscr{M}$, then $\mu(\cup_{n\geq 1} A_n) = \sum_{n\geq 1} \mu(A_n)$.

We say that $(X, \mathscr{M}, \mu)$ is a *measured space*.
We now present some special measures that we need during our studies.

–Absolutely continuous measure

Definition 2.5.3. Let $(X, \mathscr{M})$ be a measurable space and let μ and ν be two measures on $(X, \mathscr{M})$. The measure μ is said to be dominated by ν or absolutely continuous with respect with ν and written as $\mu \ll \nu$ if

$$\nu(A) = 0 \Rightarrow \mu(A) = 0; \quad \forall A \in \mathscr{M}$$

Proposition 2.5.4. *The property $\mu \ll \nu$ is equivalent to the following statement for any $\varepsilon > 0$ there exists $\delta > 0$, such that $\mu(A) < \varepsilon$ for every A with $\nu(A) < \delta$.*

Example 2.5.5.
- The measure $\mu(A) = \int_A |x| dx$ on the real line is absolutely continuous with respect to the Lebesgue measure.
- The measure supported at 0 ($\mu(A) = 1 \quad \text{iff} \quad 0 \in A$) is not absolutely continuous with respect to the Lebesgue measure, and is a singular measure.

–Density of measure

Definition 2.5.6. Let $(X, \mathscr{M}, \mu)$ be a measured space, and let $f : X \to [0, \infty]$ a measurable function. We define an application $\nu : \mathscr{M} \to [0, \infty]$ by

$$\nu(A) = \int_A f d\mu = \int f 1_A d\mu. \tag{2.24}$$

Then ν is a measure on $(X, \mathscr{M})$ called a measure of density f with respect to μ.

Remark 2.5.7. Note that the measure ν defined by (2.24) is absolutely continuous with respect to μ because if $A \in \mathscr{M}$ verifies that $\mu(A) = 0$, then $\nu(A) = 0$.

–Singular measure

Definition 2.5.8. Let $(X, \mathscr{M})$ be a measurable space and let μ and ν be two measures on $(X, \mathscr{M})$. The measure μ is called singular with respect to ν and written as $\mu \perp \nu$ if there exists a set $B \in \mathscr{M}$ such that

$$\mu(B) = 0 \quad \textit{and} \quad \nu(B^c) = 0$$

.

Example 2.5.9. The Dirac measure δ_0 is not absolutely continuous with respect to Lebesgue measure.

Remark 2.5.10. Note that if μ is singular w.r.t. ν implies that ν is singular w.r.t. μ. Therefore, the notion of singularity between two measures μ and ν is symmetric but that of absolutely continuity is not.

–Borel and Radon Measures

Definition 2.5.11. Let X a topological space.

1- A measure on the Borel σ-algebra $\mathscr{B}(X)$ is called a Borel measure on X.

2- A Borel measure μ on X is called a Radon measure if for every B in $\mathscr{B}(X)$ and $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset B$ such that

$$|\mu|(B \setminus K_\varepsilon) < \varepsilon$$

where $|\mu| = \mu^+ + \mu^-$ the total variation of μ and μ^+, μ^- are the positive and negative parts of μ.

2.5.1 The Lebesgue-Radon-Nikodym theorem

The following theorem is known as the Radon Nikodym Theorem, it considered as one of the key facts of the theory of measure

Theorem 2.5.12. *[10] Let μ and ν be two finite measures on a space $(X, \mathscr{M})$. The measure μ is absolutely continuous with respect to the measure ν precisely when there exists a $\nu-$integrable function f such that μ is given by*

$$\mu(A) = \int_A f d\nu; \quad \forall A \in \mathscr{M}$$

We denote μ by $f.\nu$. The function f is called the density of measure μ with respect to ν (or the Radon - Nikodym density) and is denoted by $\frac{d\mu}{d\nu}$.
Among the results of Radon Nikodym's theorem, Lebesgue's decomposition

Theorem 2.5.13. *[10] Let μ and ν be two finite measures on a $\sigma-$algebra $\mathscr{M}$. Then, there exists a measure ν_0 on $\mathscr{M}$ and a $\nu-$integrable function f such that*

$$\mu = f.\nu + \nu_0, \quad \nu_0 \perp \nu$$

For the proof of the two previous theorems, you can consult the following two references [[6] and [10]]

Remark 2.5.14. The decomposition of the singular part of a regular Borel measure on real line can be refined as follows (see [45])

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$$

with

- μ_{ac} is the absolutely continuous part.
- μ_{sc} is the singular continuous part.
- μ_{pp} is the pure ponctuel part.

Example 2.5.15. The Cantor measure (the probability measure on the real line whose cumulative distribution function is the Cantor function) is an exemple of a singular contiuous measure.

–Lebesgue' s dominated convergence theorem (DCT)

Theorem 2.5.16. *[6] Let $(f_n)_{n\geq 1}$ be a sequence of measurable function from a measure space $(X, \mathscr{M}, \mu)$ to $\mathbb{R}$ and let g be a measurable nonnegative function on $(X, \mathscr{M}, \mu)$. Suppose that for each $x \in X$*

1- $|f_n(x)| \leq g(x)$ *for all* $n \geq 1$.

2- $\lim_{n\to\infty} f_n(x) = f(x)$.

Then, f is integrable and

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu = \int \lim_{n\to\infty} f_n d\mu$$

.

3 On the global well-posedness of axisymmetric Boussinesq system in critical Lebesgue spaces

This chapter is the subject of the following publication:

Adalet Hanachi, Haroune Houamed and M. Zerguine: *On the global well-posedness of axisymmetric Boussinesq system in critical Lebesgue spaces*. Discrete and Continuous Dynamical Systems - A, 2020, 40 (11) : 6473-6506.

3.1 Introduction

The contribution of this chapter will be focused on the global existence and uniqueness topic in three-dimensional case of the axisymmetric viscous Boussinesq system in critical Lebesgue spaces. We aim at deriving analogous results for the classical two-dimensional and three-dimensional axisymmetric Navier-Stokes equations recently obtained in [34, 36]. Roughly speaking, we show essentially that if the initial data (v_0, ρ_0) is axisymmetric and (ω_0, ρ_0) belongs to the critical space $L^1(\Omega)\times L^1(\mathbb{R}^3)$, with ω_0 is the initial vorticity associated to v_0 and $\Omega = \{(r, z) \in \mathbb{R}^2 : r > 0\}$, then the viscous Boussinesq system has a unique global solution.
T. Gallay and V. Sverák in [36] essentially used the fixed-point argument in an appropriate functional space coupled with estimations based mainly on the properties of the semigroup. Since our system is given by a coupling between the velocity field $v(t, x)$ and the density $\rho(t, x)$ according to the equations:

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = \rho\vec{e}_z, & \text{if} \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ \partial_t\rho + v\cdot\nabla\rho - \kappa\Delta\rho = 0, & \text{if} \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ \mathrm{div} v = 0, & \\ (v,\rho)_{|t=0} = (v_0,\rho_0), & \end{cases} \tag{$\mathrm{B}_{\mu,\kappa}$}$$

and the first equation of ($\mathrm{B}_{\mu,\kappa}$) is a non-homogeneous Navier-Stokes equation, so under vorticity-density formulation for $\mu = \kappa = 1$

$$\begin{cases} \partial_t \omega_\theta + v \cdot \nabla \omega_\theta - \frac{v^r}{r}\omega_\theta = \big(\Delta - \frac{1}{r^2}\big)\omega_\theta - \partial_r \rho, & \text{if} \quad (t, r, z) \in \mathbb{R}_+ \times \Omega, \\ \partial_t \rho + v \cdot \nabla \rho - \Delta \rho = 0, & \text{if} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ (\omega_\theta, \rho)_{|t=0} = (\omega_0, \rho_0), \end{cases} \tag{3.1}$$

where $v \cdot \nabla = v^r \partial_r + v^z \partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$, we can check that the linear vorticity equation keeps the same semigroup, that is the reason why we try to inspire a couple of semigroups that achieve the same goals as their semigroup. Although we succeeded in forming this pair, we only could estimate the solution in the functional space after creating a new function $r\rho$ so a new equation. Fortunately it admits the same semigroup as the vorticity equation. Thanks to this new function, we could estimate the term $\partial_r \rho$, which was considered as an obstacle in the estimations.

As we pointed out in the introduction the global regularity of ($\mathrm{B}_{\mu,\kappa}$) in dimension three of spaces has received a considerable attention. Let us mention in particular R. Danchin and M. Paicu [27], H. Abidi, T. Hmidi and S. Keraani [3], T. Hmidi and F. Rousset [53], C. Miao and X. Zheng [[71], [72]], P. Dreyfuss and H. Houamed [29], H. Houamed and M. Zerguine[47].

In the present chapter, we are interested to study the global well-posedness of axisymmetric Boussinesq system (3.1) in critical Lebesgue space in other words, $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$.

During this work, we will fix some notations which be useful in the sequel. First, we endow the half-space $\Omega = \{(r, z) \in \mathbb{R}^2 : r > 0\}$ with the two-dimensional measure $drdz$ and define for $p \in [1, \infty]$ the Lebesgue space $L^p(\Omega)$ as the set of measurable functions $\omega_\theta : \Omega \to \mathbb{R}$ such that the norm

$$\|\omega_\theta\|_{L^p(\Omega)} = \begin{cases} \Big(\int_\Omega |\omega_\theta(r, z)|^p drdz\Big)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \mathrm{essup}_{(r,z)\in\Omega} |\omega_\theta(r, z)| & \text{if } p = \infty. \end{cases}$$

and

$$\|\Pi\|_{L^p(\mathbb{R}^3)} = \begin{cases} \Big(\int_\Omega |\Pi(r, z)|^p rdrdz\Big)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \mathrm{essup}_{(r,z)\in\Omega} |\Pi(r, z)| & \text{if } p = \infty. \end{cases}$$

For the reader's convenience, we provide a brief headline of this chapter. In section 2, we briefly depict the framework that exists regarding the axisymmetric Biot-Savart law. Many results could be spent in explaining this framework in detail, in particular, the relation between the velocity vector field and its vorticity by means of stream function. Along the way, we recall some weighted estimates which will be crucial in the sequel. Afterwards, we focus in the linear equation of (3.1) and some characterization of their associated semigroup, in particular the $L^p \to L^q$ estimate as in two-dimension space. Section 3, mainly treats the well-posedness topic for the system (3.1). The main tool in local well-posedness is the fixed point argument on the product space combined with a few techniques about the semigroup estimates. We conclude this section by investigating some global a priori estimates through coupling the system (3.1) and introducing the new unknowns Γ and $\widetilde{\Gamma}$. Considering these latest quantities will be a helpful to derive the global existence for the equivalent system (3.1) and consequently the system $(\mathrm{B}_{\mu,\kappa})$.

3.2 Setup and preliminary results

In this section we recall some basic tools which will be employed in the subsequent sections. In particular, we develop the Biot-Savart law in the framework of axisymmetric vector fields, and we study the linear equation associated to the system (3.1), usually specialized to the local existence.

3.2.1 The tool box of Axisymmetric Biot-Savart law

Recalling that in the cylindrical coordinates and in the class of axisymmetric vector fields without swirl the velocity is given by $v = (v^r, 0, v^z)$ with v^r and v^z are independently of $\theta-$variable, ω_θ its vorticity defined from Ω into $\mathbb{R}$ by $\omega_\theta = \partial_z v^r - \partial_r v^z$ and the divergence-free condition $\mathrm{div} v = 0$ under homogeneous boundary conditions $v^r = \partial_r v^z = 0$, turns out to be

$$\partial_r(rv^r) + \partial_z(rv^z) = 0.$$

In this case, it is not difficult to build a scalar function $\Omega \ni (r,z) \mapsto \psi(r,z) \in \mathbb{R}$ which called *axisymmetric stream function* and satisfying

$$v^r = -\frac{1}{r}\partial_z\psi, \quad v^z = \frac{1}{r}\partial_r\psi. \tag{3.2}$$

Consequently, one obtains that ψ evolves the following linear elliptic inhomogeneous equation

$$-\frac{1}{r}\partial_r^2\psi + \frac{1}{r^2}\partial_r\Psi - \frac{1}{r}\partial_z^2\psi = \omega_\theta,$$

with the boundary conditions $\psi(0,z) = \partial_r\psi(0,z) = 0$. By setting $\mathcal{L} = -\frac{1}{r}\partial_r^2 + \frac{1}{r^2}\partial_r - \frac{1}{r}\partial_z^2$, one finds the following boundary value problem

$$\begin{cases} \mathcal{L}\psi(r,z) = \omega_\theta(r,z) & \text{if } (r,z) \in \Omega \\ \psi(r,z) = \partial_r\psi(r,z) = 0 & \text{if } (r,z) \in \partial\Omega, \end{cases} \tag{3.3}$$

where $\partial\Omega = \{(r,z) \in \mathbb{R}^2 : r = 0\}$. It is evident that $\mathcal{L}$ is an elliptic operator of second order, then according to [79], $\mathcal{L}$ is invertible with an inverse $\mathcal{L}^{-1}$ (For more details see Appendix). Consequently, the boundary value problem (3.3) admits a unique solution given by

$$\Psi(r,z) \triangleq \mathcal{L}^{-1}\omega_\theta(r,z) = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\sqrt{\widetilde{r}r}}{2\pi} F\left(\frac{(r-\widetilde{r})^2 + (z-\widetilde{z})^2}{\widetilde{r}r}\right)\omega_\theta(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}, \tag{3.4}$$

where the function $F :]0,\infty[\to \mathbb{R}$ is expressed as follows:

$$F(s) = \int_0^\pi \frac{\cos\alpha d\alpha}{\big(2(1-\cos\alpha)+s\big)^{1/2}}. \tag{3.5}$$

Remark 3.2.1. Although F in (3.5) cannot be expressed as an elementary functions, but it has nice asymptotic properties near $s=0$ and $s=\infty$ listed in the following proposition. For more details about the proof, see [31, 79].

Proposition 3.2.2. *Let F be the function defined in (3.5), then the following assertions are hold.*

i $F(s) = \frac{1}{2}\log\frac{1}{s} + \log 8 - 2 + O\Big(s\log\frac{1}{s}\Big)$ *and* $F'(s) = -\frac{1}{2s} + O\Big(\log\frac{1}{s}\Big)$ *as* $s \to 0^+$.

ii $F(s) = \frac{\pi}{2s^{3/2}} + O\Big(\frac{1}{s^{5/2}}\Big)$ *and* $F'(s) = -\frac{3\pi}{4s^{5/2}} + O\Big(\frac{1}{s^{7/2}}\Big)$ *as* $s \to \infty$.

iii *For every $k \in \mathbb{N}^\star$, we have*

$$|F(s)| \lesssim \min\bigg(\Big(\frac{1}{s}\Big)^{\epsilon}, \Big(\frac{1}{s}\Big)^{\frac{1}{2}}, \Big(\frac{1}{s}\Big)^{\frac{3}{2}}\bigg), \quad \epsilon \in]0, \frac{1}{2}[.$$

and

$$|F^{(k)}(s)| \lesssim \min\bigg(\Big(\frac{1}{s}\Big)^{k}, \Big(\frac{1}{s}\Big)^{k+\frac{1}{2}}, \Big(\frac{1}{s}\Big)^{k+\frac{3}{2}}\bigg), \quad s \in]0,\infty[.$$

iv *The maps $s \mapsto s^\alpha F(s)$ and $s \mapsto s^\beta F'(s)$ are bounded for $0 < \alpha \le \frac{3}{2}$ and $1 \le \beta \le \frac{5}{2}$ respectively.*

Now, let

$$\mathcal{K}(r,z,\widetilde{r},\widetilde{z}) = \frac{\sqrt{\widetilde{r}r}}{2\pi} F\bigg(\frac{(r-\widetilde{r})^2 + (z-\widetilde{z})^2}{\widetilde{r}r}\bigg). \tag{3.6}$$

Thus in view of (3.4), Ψ takes the form

$$\Psi(r,z) = \int_{-\infty}^{\infty}\int_0^\infty \mathcal{K}(r,z,\widetilde{r},\widetilde{z})\omega_\theta(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z},$$

with $\mathcal{K}$ can be seen as the kernel of the last integral representation. The last estimate combined with (3.2) claim that there exists a genuine connection between the velocity and its vorticity, namely, *axisymmatric Biot-Savart*law which reads as follows

$$v^r(r,z) = \int_{-\infty}^{\infty}\int_0^\infty \mathcal{K}_r(r,z,\widetilde{r},\widetilde{z})\omega_\theta(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}, \quad v^z(r,z) = \int_{-\infty}^{\infty}\int_0^\infty \mathcal{K}_z(r,z,\widetilde{r},\widetilde{z})\omega_\theta(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}. \tag{3.7}$$

Remark 3.2.3. At first glance, the comparison between the usual Biot-Savart law in $\mathbb{R}^3$

$$v(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}\times\omega_\theta(y)dy$$

and the axisymmetric biot-savart law (3.7) seems that the latter is more complicated and have no advantage. But (3.7) indeed capture certain characteristics of axisymmetric fields without swirl.

Here, with the notation $\xi^2 = \frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{\widetilde{r}r}$ we have

$$\mathcal{K}_r(r,z,\widetilde{r},\widetilde{z}) = -\frac{1}{\pi}\frac{z-\widetilde{z}}{r^{3/2}\widetilde{r}^{1/2}}F'(\xi^2) \tag{3.8}$$

and

$$\mathcal{K}_z(r,z,\widetilde{r},\widetilde{z}) = \frac{1}{\pi}\frac{r-\widetilde{r}}{r^{3/2}\widetilde{r}^{1/2}}F'(\xi^2) + \frac{1}{4\pi}\frac{\widetilde{r}^{1/2}}{r^{3/2}}\big(F(\xi^2)-2\xi^2F'(\xi^2)\big). \tag{3.9}$$

A worthwhile property of the kernels $\mathcal{K}_r$ and $\mathcal{K}_z$ are given in the following result. For more details about the proof, see [36].

Proposition 3.2.4. *Let $(r,z,\widetilde{r},\widetilde{z})\in\Omega\times\Omega$, then we have*

$$|\mathcal{K}_r(r,z,\widetilde{r},\widetilde{z})| + |\mathcal{K}_z(r,z,\widetilde{r},\widetilde{z})| \leq \frac{C}{\big((r-\widetilde{r})^2+(z-\widetilde{z})^2\big)^{1/2}}. \tag{3.10}$$

Now, we state the first consequence of the above result, in particular the $L^p\to L^q$ between the velocity and its vorticity. Specifically we have:

Proposition 3.2.5. *Let v be an axisymmetric velocity vector associated to the vorticity ω_θ via the axisymmetric Biot-Savart law (3.7). Then the following assertions are hold.*

(i) *Let $(p,q)\in]1,2[\times]2,\infty[$, with $p<q$ and $\frac{1}{p}-\frac{1}{q}=\frac{1}{2}$. For $\omega_\theta\in L^p(\Omega)$, then $v\in(L^q(\Omega))^2$ and*

$$\|v\|_{L^q(\Omega)} \leq C\|\omega_\theta\|_{L^p(\Omega)}. \tag{3.11}$$

(ii) *Let $(p,q)\in[1,2[\times]2,\infty]$, with $p<q$, and define $\sigma\in]0,1[$ by $\frac{1}{2}=\frac{\sigma}{p}+\frac{1-\sigma}{q}$. Then for $\omega_\theta\in L^p(\Omega)\cap L^q(\Omega)$, we have $v\in(L^\infty(\Omega))^2$ and*

$$\|v\|_{L^\infty(\Omega)} \leq C\|\omega_\theta\|^{\sigma}_{L^p(\Omega)}\|\omega_\theta\|^{1-\sigma}_{L^q(\Omega)}. \tag{3.12}$$

Proof. **(i)** Combining (3.7) and (3.10), we get

$$|v^r(r,z)| \leq C\int_{-\infty}^{\infty}\int_0^{\infty}\frac{|\omega_\theta(\widetilde{r},\widetilde{z})|}{\big((r-\widetilde{r})^2+(z-\widetilde{z})^2\big)^{1/2}}d\widetilde{r}d\widetilde{z},$$

and

$$|v^z(r,z)| \le C \int_{-\infty}^{\infty}\int_0^{\infty} \frac{|\omega_\theta(\widetilde{r},\widetilde{z})|}{\big((r-\widetilde{r})^2+(z-\widetilde{z})^2\big)^{1/2}} d\widetilde{r}d\widetilde{z}.$$

The last two integrals of the right hand side are be seen as a singular integral. So, by hypothesis $\frac{1}{p}-\frac{1}{q}=\frac{1}{2}$, Hardy-Littlewood-Sobolev theorem, see e.g. [19, Theorem 6.1.3] yields the desired estimate.
(ii) Let $R>0$, then in view of (3.10) we have

$$|v(r,z)| \lesssim \int_{\Omega_R} \frac{|\omega_\theta(r-\widetilde{r},z-\widetilde{z})|}{(\widetilde{r}^2+\widetilde{z}^2)^{\frac12}} d\widetilde{r}d\widetilde{z} + \int_{\Omega\setminus\Omega_R} \frac{|\omega_\theta(r-\widetilde{r},z-\widetilde{z})|}{(\widetilde{r}^2+\widetilde{z}^2)^{\frac12}} d\widetilde{r}d\widetilde{z},$$

where $\Omega_R=\{(r,z)\in\Omega : 0<r\le R, -R\le z\le R\}$. Thus, Hölder's inequality implies

$$|v(r,z)| \lesssim \|\omega_\theta\|_{L^q(\Omega)} R^{1-\frac{2}{q}} + \|\omega_\theta\|_{L^p(\Omega)} \frac{1}{R^{\frac{2}{p}-1}}.$$

It is enough to take $R=\big(\|\omega_\theta\|_{L^p(\Omega)}/\|\omega_\theta\|_{L^q(\Omega)}\big)^{\ell}$, with $\ell=\frac{\sigma}{1-2/q}=\frac{1-\sigma}{2/p-1}$. Then by an easy computations we achieve the estimate. $\square$

In the axisymmetric case the weighted estimates practice a decisive role to bound some quantities like $r^\alpha v$ in Lebesgue spaces for some α. Now, we state some of them which their proofs can be found in [31, 36].

Proposition 3.2.6. *Let $\alpha,\beta\in[0,2]$ be such that $\beta-\alpha\in[0,1[$, and assume that $p,q\in]1,\infty[$ satisfy*

$$\frac{1}{p}-\frac{1}{q}=\frac{1+\alpha-\beta}{2}.$$

Assume that $r^\beta\omega_\theta\in L^p(\Omega)$, then $r^\alpha v\in(L^q(\Omega))^2$ and the following bound holds true.

$$\|r^\alpha v\|_{L^q(\Omega)} \le C\|r^\beta\omega_\theta\|_{L^p(\Omega)}. \tag{3.13}$$

Proposition 3.2.7. *Let v be the axisymmetric velocity vector associated to the vorticity ω_θ via the axisymmetric Biot-Savart law (3.7). Then the following weighted estimates are hold.*

$$\|v\|_{L^\infty(\Omega)} \le C\|r\omega_\theta\|_{L^1(\Omega)}^{1/2}\|\omega_\theta/r\|_{L^\infty(\Omega)}^{1/2}, \tag{3.14}$$

and

$$\Big\|\frac{v^r}{r}\Big\|_{L^\infty(\Omega)} \le C\|\omega_\theta\|_{L^1(\Omega)}^{1/3}\|\omega_\theta/r\|_{L^\infty(\Omega)}^{2/3}. \tag{3.15}$$

3.2.2 Characterizations of semi-groups associated with the linear equation

We focus on studying the linearized boundary initial value problem associated to the system (3.1) and we state some properties of their semipgroups. Specifically, we consider

$$\begin{cases} \partial_t \omega_\theta - \Big(\Delta - \frac{1}{r^2}\Big)\omega_\theta = 0, \\ \partial_t \rho - \Delta\rho = 0, \\ (\omega_\theta, \rho)_{|t=0} = (\omega_0, \rho_0) \end{cases} \tag{3.16}$$

in the product space $\Omega \times \mathbb{R}^3$, with $\Omega = \{(r,z) \in \mathbb{R}^2 : r > 0\}$ is the half-space by prescribing the homogeneous Dirichlet conditions at the boundary $r = 0$ for ω_θ variable. For $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$, the solution of (3.16) is given explicitly by

$$\begin{cases} \omega_\theta(t) = \mathbb{S}_1(t)\omega_0, \\ \rho(t) = \mathbb{S}_2(t)\rho_0, \end{cases}$$

where $(\mathbb{S}_1(t))_{t\geq 0}$ and $(\mathbb{S}_2(t))_{t\geq 0}$ being respectively the semigroups or evolution operators associated to the dissipative operators $(\Delta - \frac{1}{r^2})$ and Δ.

Such are characterized by the following explicit formula, namely we have:

Proposition 3.2.8. *The family $(\mathbb{S}_1(t), \mathbb{S}_2(t))_{t\geq 0}$ associated to (3.16) is expressed by the following*

$$\begin{cases} (\mathbb{S}_1(t)\omega_0)(r,z) = \frac{1}{4\pi t}\int_\Omega \frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\omega_0(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}, \\ \\ (\mathbb{S}_2(t)\rho_0)(r,z) = \frac{1}{4\pi t}\int_\Omega \frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\rho_0(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}, \end{cases} \tag{3.17}$$

where the functions $]0,+\infty[\ni t \mapsto \mathscr{N}_1(t), \mathscr{N}_2(t) \in \mathbb{R}$ are defined for every $t > 0$ by

$$\begin{cases} \mathscr{N}_1(t) = \frac{1}{\sqrt{\pi t}}\int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2\alpha}{t}}\cos(2\alpha)d\alpha, \\ \\ \mathscr{N}_2(t) = \frac{1}{\sqrt{\pi t}}\int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2\alpha}{t}}d\alpha. \end{cases} \tag{3.18}$$

Proof. We assume that (ω_θ, ρ) solving (3.16), then a straigthforward computations claim that (ω, ρ), with $\omega = \omega_\theta \vec{e}_\theta$ satisfying the usual heat equation $\partial_t\omega - \Delta\omega = 0$ and $\partial_t\rho - \Delta\rho = 0$ in $\mathbb{R}^3$ with initial data $(\omega(0,\cdot), \rho(0,\cdot))$. Therefore, for every $t > 0$ we have

$$\begin{cases} \omega(t,x) = \frac{1}{(4\pi t)^{3/2}}\int_{\mathbb{R}^3} e^{-\frac{|x-\widetilde{x}|^2}{4t}}\omega(0,\widetilde{x})d\widetilde{x}, \\ \\ \rho(t,x) = \frac{1}{(4\pi t)^{3/2}}\int_{\mathbb{R}^3} e^{-\frac{|x-\widetilde{x}|^2}{4t}}\rho(0,\widetilde{x})d\widetilde{x}. \end{cases} \tag{3.19}$$

In the cylindrical basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$, we write $x = (r\cos\theta, r\sin\theta, z)$ and $\widetilde{x} = (\widetilde{r}\cos\widetilde{\theta}, \widetilde{r}\sin\widetilde{\theta}, \widetilde{z})$, hence the first equation of (3.19) takes the form

$$\begin{aligned} \omega_\theta(t,r,z)\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} &= \frac{1}{(4\pi t)^{3/2}}\int_0^\infty\int_{\mathbb{R}}\int_{-\pi}^{\pi} e^{-\frac{|x-\widetilde{x}|^2}{4t}}\omega_0(\widetilde{r},\widetilde{z})\begin{pmatrix} -\sin\widetilde{\theta} \\ \cos\widetilde{\theta} \\ 0 \end{pmatrix}\widetilde{r}d\widetilde{\theta}d\widetilde{z}d\widetilde{r} \\ &= \mathrm{I}_1. \end{aligned} \tag{3.20}$$

Since, $|x-\widetilde{x}|^2 = (r-\widetilde{r})^2 + (z-\widetilde{z})^2 + 4r\widetilde{r}\sin^2\left(\frac{\theta-\widetilde{\theta}}{2}\right)$, thus we have

$$\mathrm{I}_1 = \frac{1}{(4\pi t)}\int_0^\infty\int_{\mathbb{R}}\left(\frac{1}{(4\pi t)^{1/2}}\int_{-\pi}^{\pi} e^{\frac{-r\widetilde{r}\sin^2\frac{\theta-\widetilde{\theta}}{2}}{t}}\begin{pmatrix}-\sin\widetilde{\theta}\\ \cos\widetilde{\theta}\\ 0\end{pmatrix}\widetilde{r}d\widetilde{\theta}\right)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\omega_0(\widetilde{r},\widetilde{z})d\widetilde{z}d\widetilde{r}. \tag{3.21}$$

To treat I_1, we set $\alpha = \frac{\theta-\widetilde{\theta}}{2}$ then we have

$$\begin{aligned}\frac{1}{\sqrt{4\pi t}}\int_{-\pi}^{\pi} e^{\frac{-r\widetilde{r}\sin^2\frac{\theta-\widetilde{\theta}}{2}}{t}}(-\sin\widetilde{\theta})\widetilde{r}d\widetilde{\theta} &= -\frac{1}{\sqrt{\pi t}}\int_{\theta/2-\pi/2}^{\theta/2+\pi/2} e^{\frac{-r\widetilde{r}\sin^2\alpha}{t}}\big(\sin\theta\cos 2\alpha-\cos\theta\sin 2\alpha\big)\widetilde{r}d\alpha\\ &= -\frac{1}{\sqrt{\pi t}}\int_{-\pi/2}^{+\pi/2} e^{\frac{-r\widetilde{r}\sin^2\alpha}{t}}\big(\sin\theta\cos 2\alpha\big)\widetilde{r}d\alpha\\ &\quad -\frac{1}{\sqrt{\pi t}}\int_{-\pi/2}^{+\pi/2} e^{\frac{-r\widetilde{r}\sin^2\alpha}{t}}\big(\cos\theta\sin 2\alpha\big)\widetilde{r}d\alpha.\end{aligned}$$

For $t\in]0,\infty[$, define

$$\mathscr{N}_1(t) = \frac{1}{\sqrt{\pi t}}\int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2\alpha}{t}}\cos(2\alpha)d\alpha.$$

Then the last estimate becomes

$$\frac{1}{(4\pi t)^{\frac{1}{2}}}\int_{-\pi}^{\pi} e^{\frac{-r\widetilde{r}\sin^2\frac{\theta-\widetilde{\theta}}{2}}{t}}(-\sin\widetilde{\theta})\widetilde{r}d\widetilde{\theta} = \frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)(-\sin\theta).$$

Similarly,

$$\frac{1}{(4\pi t)^{1/2}}\int_{-\pi}^{\pi} e^{\frac{-r\widetilde{r}\sin^2\frac{\theta-\widetilde{\theta}}{2}}{t}}\cos\widetilde{\theta}\widetilde{r}d\widetilde{\theta} = \frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)\cos\theta.$$

Combining the last two estimates and plug them in I_1 we reach the desired estimate.

For the second equation in (3.19) we express the density formula in $(\vec{e}_r,\vec{e}_\theta,\vec{e}_z)$ basis

$$\rho(t,r,z) = \frac{1}{4\pi t}\int_\Omega\left(\frac{1}{2\sqrt{\pi t}}\int_{-\pi}^{\pi} e^{-\frac{r\widetilde{r}\sin^2(\frac{\theta-\widetilde{\theta}}{2})}{t}}\widetilde{r}d\widetilde{\theta}\right)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\rho(0,\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}. \tag{3.22}$$

Setting

$$\mathrm{I}_2 = \frac{1}{2\sqrt{\pi t}}\int_{-\pi}^{\pi} e^{-\frac{r\widetilde{r}\sin^2\left(\frac{\theta-\widetilde{\theta}}{2}\right)}{t}}\widetilde{r}d\widetilde{\theta}.$$

The same variable $\alpha = \frac{\theta-\widetilde{\theta}}{2}$ allows us to write

$$\mathrm{I}_2 = \frac{1}{\sqrt{\pi t}}\int_{-\pi/2}^{\pi/2} e^{-\frac{r\widetilde{r}\sin^2\alpha}{t}}\widetilde{r}d\alpha = \sqrt{\frac{\widetilde{r}}{r}}\mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big),$$

with $\mathscr{N}_2$ is defined for $t>0$ by

$$\mathscr{N}_2(t) = \frac{1}{\sqrt{\pi t}}\int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2\alpha}{t}}d\alpha.$$

Plug I_2 in (3.22), we get the result. This reached the proof of the Proposition. $\square$

The following Proposition provides some asymptotic behavior of the functions $\mathscr{N}_1$ and $\mathscr{N}_2$ near 0 and ∞, will be fundamental in the sequel.

Proposition 3.2.9. *Let $\mathscr{N}_1, \mathscr{N}_2 :]0, \infty[\to \mathbb{R}$ be the functions defined in (3.18). Then the following statements are hold.*

(i) $\mathscr{N}_1(t) = 1 - \frac{3t}{4} + O(t^2)$ *and* $\mathscr{N}_1'(t) = -\frac{3}{4} + O(t)$ *when* $t \downarrow 0$*;*

(ii) $\mathscr{N}_1(t) = \frac{\pi^{1/2}}{4t^{3/2}} + O\Big(\frac{1}{t^{5/2}}\Big)$ *and* $\mathscr{N}_1'(t) = -\frac{3\pi^{1/2}}{8t^{5/2}} + O\Big(\frac{1}{t^{7/2}}\Big)$ *when* $t \uparrow \infty$*;*

(iii) $\mathscr{N}_2(t) = 1 - \frac{t}{4} + O(t^2)$ *and* $\mathscr{N}_2'(t) = -\frac{1}{4} + O(t)$ *when* $t \downarrow 0$*;*

(iv) $\mathscr{N}_2(t) = \frac{\pi^{1/2}}{t^{1/2}} - \frac{\pi^{5/2}}{12t^{3/2}} + O\Big(\frac{1}{t^{5/2}}\Big)$ *and* $\mathscr{N}_2'(t) = -\frac{\pi^{1/2}}{12t^{3/2}} - \frac{\pi^{5/2}}{8t^{5/2}} + O\Big(\frac{1}{t^{7/2}}\Big)$ *when* $t \uparrow \infty$*.*

Proof. **(i)** Substituting $\zeta = \frac{\sin\alpha}{\sqrt{t}}$ in $\mathscr{N}_1$, we shall have

$$\begin{aligned}
\mathscr{N}_1(\zeta) &= \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{\sqrt{t}}}^{\frac{1}{\sqrt{t}}} e^{-\zeta^2} \frac{1-2t\zeta^2}{\sqrt{1-t\zeta^2}} d\zeta \\
&= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{t}}} e^{-\zeta^2} \frac{1-2t\zeta^2}{\sqrt{1-t\zeta^2}} d\zeta \\
&= \frac{2}{\sqrt{\pi}} \left(\int_0^{\frac{1}{2\sqrt{t}}} e^{-\zeta^2} \frac{1-2t\zeta^2}{\sqrt{1-t\zeta^2}} d\zeta + \int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}} e^{-\zeta^2} \frac{1-2t\zeta^2}{\sqrt{1-t\zeta^2}} d\zeta \right) \\
&= \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}$$

Note that $\lim_{t\downarrow 0} \mathrm{II}_2 = 0$, so the behavior of $\mathscr{N}_1$ near 0 comes from II_1. Hence, let us deal with II_1, we insert the Taylor expansion of the function $\zeta \to \frac{1}{\sqrt{1-t\zeta^2}}$ in the integral of II_1 to obtain

$$\mathrm{II}_1 = \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{2\sqrt{t}}} e^{-\zeta^2} (1 - \frac{3}{2} t\zeta^2 - t^2\zeta^4) d\zeta + O(t^3).$$

It is straightforward to show that

$$\int_0^\infty e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{2}, \quad \int_0^\infty \zeta^2 e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{4}, \quad \int_0^\infty \zeta^4 e^{-\zeta^2} d\zeta = -3\frac{\sqrt{\pi}}{8}.$$

Consequently, $\lim_{t\downarrow 0} \mathrm{II}_1 = 1$. Combining all the previous quantities, we find the asymptotic behavior of II_1 near 0, that is,

$$\mathrm{II}_1 = 1 - \frac{3}{4} t + O(t^2).$$

By derivation of II_1, we find the behavior of $\mathscr{N}_1'$.
(ii) The Mac Laurin's expansion of the function $\alpha \mapsto e^{-\frac{\sin^2\alpha}{t}}$ at 0 is given by

$$e^{-\frac{\sin^2\alpha}{t}} = 1 - \frac{\alpha^2}{t} + O\Big(\frac{1}{t^2}\Big).$$

Thus we get

$$\mathscr{N}_1(t) = \frac{1}{\sqrt{\pi t}}\int_{-\pi/2}^{\pi/2}\big(1-\frac{\alpha^2}{t}\big)\cos 2\alpha d\alpha + O\Big(\frac{1}{t^{\frac{5}{2}}}\Big).$$

After an easy computations we achieve the estimate.
(iii) To prove this assertion, setting $y = \frac{\sin\alpha}{\sqrt{t}}$ in $\mathscr{N}_2$ and we split the integral into two parts, one has

$$\mathscr{N}_2(t) = \frac{2}{\sqrt{\pi}}\bigg(\int_0^{\frac{1}{2\sqrt{t}}}\frac{e^{-y^2}}{\sqrt{1-ty^2}}dy + \int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}}\frac{e^{-y^2}}{\sqrt{1-ty^2}}dy\bigg).$$

We follow the same steps as $\mathscr{N}_1$. For the second integral in right-hand side, we have

$$\begin{aligned}\frac{2}{\sqrt{\pi}}\int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}}\frac{e^{-y^2}}{\sqrt{1-ty^2}}dy &= \frac{2}{\sqrt{\pi}}\int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}}\frac{e^{-y^2}}{(\sqrt{1-\sqrt{t}y})(\sqrt{1+\sqrt{t}y})}dy\\ &\leq Ce^{-\frac{1}{4t}}\int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}}\frac{1}{\sqrt{1-\sqrt{t}y}}dy.\end{aligned}$$

Let us observe that the last estimate goes to 0 as $t\downarrow 0$, so the asymptotic behavior of $\mathscr{N}_2$ near 0 comes only from the first integral. To be precise, it is clear that $t\mapsto \frac{1}{\sqrt{1-ty^2}}$ is bounded function whenever $0<y<\frac{1}{2\sqrt{t}}$ and

$$\lim_{t\downarrow 0}\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2\sqrt{t}}} e^{-y^2}dy \approx 1.$$

Thus, the expansion of the function $x\mapsto (1-x)^{-\frac{1}{2}}$ for $x = ty^2$ enuble us to write

$$\begin{aligned}\mathscr{N}_2(t) &= \frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2\sqrt{t}}} e^{-y^2}\Big(1+\frac{ty^2}{2}\Big)dy + O(t^2)\\ &= 1-\frac{t}{4}+O(t^2).\end{aligned}$$

(iv) Using the fact $\sin\alpha \simeq \alpha$ near 0, then we get

$$\mathscr{N}_2(t) = \frac{1}{\sqrt{\pi t}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\frac{\alpha^2}{t}}d\alpha.$$

We set $y = \frac{\alpha}{\sqrt{t}}$, clearly that $y\downarrow 0$ as $t\uparrow\infty$ and the power expansion of the function e^y near 0 yields the asymptotic expansion, whereas $\mathscr{N}_2'$ is a direct derivative of $\mathscr{N}_2$ expansion.

□

Remark 3.2.10. **(i)** It should be noted that the functions $t \mapsto \mathscr{N}_1(t)$ and $t \mapsto \mathscr{N}_2(t)$ are decreasing over $]0,\infty[$, but the proof seems very hard.

(ii) The functions $t \mapsto t^\alpha \mathscr{N}_1(t), t \mapsto t^\alpha \mathscr{N}_2(t)$ and $t \mapsto t^\beta \mathscr{N}_1'(t), t \mapsto t^\beta \mathscr{N}_2'(t)$ are bounded for $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta \leq \frac{3}{2}$.

Other nice properties of $(\mathbb{S}_i(t))_{t\geq 0}$, with $i = 1, 2$, in particular the estimate $L^p \to L^q$ are given in the following result.

Proposition 3.2.11. *The family $((\mathbb{S}_1(t), \mathbb{S}_2(t))_{t\geq 0}$ associated to (3.16) is a strongly continuous semigroup of bounded linear operators in $L^p(\Omega) \times L^p(\mathbb{R}^3)$ for any $p \in [1,\infty)$. Furthermore, for $1 \leq p \leq q \leq \infty$ the following assertions are hold.*

(i) *For $(\omega_0, \rho_0) \in L^p(\Omega) \times L^p(\mathbb{R}^3)$, we have for every $t > 0$*

$$\|(\mathbb{S}_1(t)\omega_0, \mathbb{S}_2(t)\rho_0)\|_{L^q(\Omega)\times L^q(\mathbb{R}^3)} \leq \frac{C}{t^{\frac{1}{p}-\frac{1}{q}}} \|(\omega_0, \rho_0)\|_{L^p(\Omega)\times L^p(\mathbb{R}^3)}. \tag{3.23}$$

(ii) *For $f = (f^r, f^z) \in L^p(\Omega) \times L^p(\Omega)$, we have for every $t > 0$*

$$\|\mathbb{S}_1(t)\mathrm{div}_\star f\|_{L^q(\Omega)} \leq \frac{C}{t^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}} \|f\|_{L^p(\Omega)}. \tag{3.24}$$

(iii) *For $f = (f^r, f^z) \in L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$, we have every $t > 0$*

$$\|\mathbb{S}_2(t)\mathrm{div} f\|_{L^q(\mathbb{R}^3)} \leq \frac{C}{t^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}} \|f\|_{L^p(\mathbb{R}^3)} \tag{3.25}$$

Here, $\mathrm{div}_\star f = \partial_r f^r + \partial_z f^z$ (resp. $\mathrm{div} f = \partial_r f^r + \partial_z f^z + \frac{f^r}{r}$) stands the divergence operator over $\mathbb{R}^2$ (resp. the divergence operator over $\mathbb{R}^3$ in the axisymmetric case).

Proof. **(i)** We follow the proof of [36] with minor modifications, for this aim let $(r,z), (\widetilde{r}, \widetilde{z}) \in \Omega$, we will prove the following worth while estimates

$$\begin{cases} \frac{1}{4\pi t} \frac{\widetilde{r}^{1/2}}{r^{1/2}} \mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big) e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}} \leq \frac{C}{t} e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{5t}}, \\ \\ \frac{1}{4\pi t} \frac{\widetilde{r}^{1/2}}{r^{1/2}} \mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big) e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}} \leq \frac{C}{t} e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{5t}}. \end{cases} \tag{3.26}$$

We distinguish two cases $\widetilde{r} \leq 2r$ and $\widetilde{r} > 2r$.
• $\widetilde{r} \leq 2r$. Employing the fact $t \mapsto (t^\alpha \mathscr{N}_1(t), t^\alpha \mathscr{N}_2(t))$ is bounded for $\alpha \in [0, \frac{1}{2}]$, see, **(ii)**-Remark 3.2.10 and $t \mapsto e^{-t}$ is decreasing, we get the result.

• $\widetilde{r}>2r$. The remark $\widetilde{r}\leq 2\big((r-\widetilde{r})^2+(z-\widetilde{z})^2\big)^{\frac12}$, a new use of $t\mapsto (t^\alpha\mathscr{N}_1(t),t^\alpha\mathscr{N}_2(t))$ is bounded for $\alpha\in[0,\frac12]$ and $te^{-\frac{t^2}{4}}\leq Ce^{-\frac{t^2}{5}}$ for $t\geq 0$ leading to

$$\begin{aligned}\frac{1}{4\pi t}\frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_i\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}} &\leq \frac{C}{t}\bigg(\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}\bigg)^{\frac12}e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\\ &\leq \frac{C}{t}e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{5t}},\quad i\in\{1,2\}.\end{aligned}$$

Next, from (3.26) and the last estimate we write

$$\begin{aligned}|\mathbb{S}_1(t)\omega_0|+|\mathbb{S}_2(t)\rho_0| &\leq \frac{1}{4\pi t}\int_\Omega\bigg|\frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\omega_0(\widetilde{r},\widetilde{z})\bigg|d\widetilde{r}d\widetilde{z}\\ &\quad+\frac{1}{4\pi t}\int_\Omega\bigg|\frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\rho_0(\widetilde{r},\widetilde{z})\bigg|d\widetilde{r}d\widetilde{z}\\ &\leq \frac{C}{t}\int_\Omega e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{5t}}\big(|\omega_0(\widetilde{r},\widetilde{z})|+|\rho_0(\widetilde{r},\widetilde{z})|\big)d\widetilde{r}d\widetilde{z}.\end{aligned}$$

The last line can be seen as a convolution product, then Young's inequality gives the desired estimate.
(ii) By definition for every $(r,z)\in\Omega$, we have

$$\begin{aligned}\big(\mathbb{S}_1(t)\mathrm{div}_\star f\big)(r,z) &= \frac{1}{4\pi t}\int_\Omega\frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}(\partial_{\widetilde{r}}f^r(\widetilde{r},\widetilde{z})+\partial_{\widetilde{z}}f^z(\widetilde{r},\widetilde{z}))d\widetilde{r}d\widetilde{z}\\ &= \frac{1}{4\pi t}\int_\Omega\frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\partial_{\widetilde{r}}f^r(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}\\ &+ \frac{1}{4\pi t}\int_\Omega\frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\partial_{\widetilde{z}}f^z(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}\\ &= \mathrm{II}_1+\mathrm{II}_2.\end{aligned}$$

After an integration by parts, it happens

$$\mathrm{II}_1=\frac{1}{4\pi t}\int_\Omega\frac{\widetilde{r}^{1/2}}{r^{1/2}}\bigg(\frac{t}{r\widetilde{r}^2}\mathscr{N}_1'\Big(\frac{t}{r\widetilde{r}}\Big)-\Big(\frac{1}{2\widetilde{r}}+\Big(\frac{r-\widetilde{r}}{2t}\Big)\Big)\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)\bigg)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}f^r(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}$$

and

$$\mathrm{II}_2=-\frac{1}{4\pi t}\int_\Omega\frac{\widetilde{r}^{1/2}}{r^{1/2}}\Big(\frac{z-\widetilde{z}}{2t}\Big)\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}f^z(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}.$$

We proceed by the same manner as above, that is to say, the fact that the functions $\mathscr{N}_1,\mathscr{N}_1'$ and $t\mapsto t^\alpha\mathscr{N}_1(t),t\mapsto t^\alpha\mathscr{N}_1'(t)$ are bounded, see Remark 3.2.10, one finds

$$|\mathrm{II}_1|\leq\frac{C}{t^{\frac32}}\int_\Omega e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{5t}}|f^r(\widetilde{r},\widetilde{z})|d\widetilde{r}d\widetilde{z},$$

and

$$|\mathrm{II}_2|\leq\frac{C}{t^{\frac32}}\int_\Omega e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{5t}}|f^z(\widetilde{r},\widetilde{z})|d\widetilde{r}d\widetilde{z}.$$

Together with Young's inequality, we obtain (3.24).
(iii) Let $(r,z)\in\Omega$, then we have

$$\begin{aligned}
\mathbb{S}_2(t)\mathrm{div}f(r,z) &= \frac{1}{4\pi t}\int_\Omega \frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\Big(\partial_{\widetilde{r}}f^r(\widetilde{r},\widetilde{z})+\partial_{\widetilde{r}}f^z(\widetilde{r},\widetilde{z})+\frac{1}{\widetilde{r}}f^r(\widetilde{r},\widetilde{z})\Big)d\widetilde{r}d\widetilde{z}\\
&= \frac{1}{4\pi t}\int_\Omega \frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\partial_{\widetilde{r}}f^r(\widetilde{r},\widetilde{z})\widetilde{r}d\widetilde{z}\\
&\quad+\frac{1}{4\pi t}\int_\Omega \frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\partial_{\widetilde{z}}f^z(\widetilde{r},\widetilde{z})\widetilde{r}d\widetilde{z}\\
&\quad+\frac{1}{4\pi t}\int_\Omega \frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\frac{1}{\widetilde{r}}f^r(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}\\
&= \mathrm{III}_3+\mathrm{III}_4+\mathrm{III}_5.
\end{aligned}\tag{3.27}$$

The two terms III_3 and III_4 ensue by the same argument as in **(ii)**. It remains to treat the term III_5 in the following way

$$\begin{aligned}
\mathrm{III}_5 &= \frac{1}{4\pi t}\int_\Omega \frac{\widetilde{r}^{1/2}}{r^{1/2}}\mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}\frac{1}{\widetilde{r}}f^r(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}\\
&= \frac{1}{4\pi t}\int_\Omega \frac{1}{(r\widetilde{r})^{1/2}}\mathscr{N}_2\Big(\frac{t}{r\widetilde{r}}\Big)e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}f^r(\widetilde{r},\widetilde{z})d\widetilde{r}d\widetilde{z}.
\end{aligned}\tag{3.28}$$

The fact that $(\cdot/r\widetilde{r})^{1/2}\mathscr{N}_2(\cdot/r\widetilde{r})$ is bounded guided to

$$|\mathrm{III}_5|\leq\frac{C}{t^{3/2}}\int_\Omega e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{4t}}|f^r(\widetilde{r},\widetilde{z})|d\widetilde{r}d\widetilde{z}.$$

By pluging the last estimate in (3.28) and combine it with (3.27), it follows

$$|\mathbb{S}_2(t)\mathrm{div}f|\leq\frac{C}{t^{3/2}}\int_\Omega e^{-\frac{(r-\widetilde{r})^2+(z-\widetilde{z})^2}{5t}}\big(|f^r(\widetilde{r},\widetilde{z})|+|f^z(\widetilde{r},\widetilde{z})|\big)d\widetilde{r}d\widetilde{z}.$$

Then a new use of Young's inequality leading to the result.

To close our claim, it remains to establish that $\mathbb{R}_+\ni t\mapsto\mathbb{S}_1(t)$ (resp. $\mathbb{R}_+\ni t\mapsto\mathbb{S}_2(t)$) is continuous on $L^p(\Omega)$ (resp. on $L^p(\mathbb{R}^3)$). We restrict ourselves only for $(\mathbb{S}_1(t))_{t\geq0}$. Let $\omega_0\in L^p(\Omega)$ and define its extension on $\mathbb{R}^2$ by $\widetilde{\omega}_0$ which equal to 0 outside of Ω. Thus, in view the change of variables $\widetilde{r}=r+\sqrt{t}\vartheta$ and $\widetilde{z}=z+\sqrt{t}\gamma$, the statement (3.17) takes the form

$$(\mathbb{S}_1(t)\omega_0)(r,z)=\frac{1}{4\pi}\int_{\mathbb{R}^2}\Big(1+\frac{\sqrt{t}\vartheta}{r}\Big)^{1/2}\mathscr{N}_1\Big(\frac{t}{r(r+\sqrt{t}\vartheta)}\Big)e^{-\frac{\beta^2+\gamma^2}{4}}\widetilde{\omega}_0(r+\sqrt{t}\vartheta,r+\sqrt{t}\gamma)d\vartheta d\gamma.$$

The fact

$$\frac{1}{4\pi}\int_{\mathbb{R}^2}e^{-\frac{\vartheta^2+\gamma^2}{4}}d\vartheta d\gamma=1$$

leading to

$$(\mathbb{S}_1(t)\omega_0)(r,z)-\omega_0(r,z)=\frac{1}{4\pi}\int_{\mathbb{R}^2}e^{-\frac{\vartheta^2+\gamma^2}{4}}\Upsilon(t,r,z,\vartheta,\gamma)d\vartheta d\gamma,\tag{3.29}$$

where

$$\Upsilon(t,r,z,\vartheta,\gamma)=\Big(1+\frac{\sqrt{t}\vartheta}{r}\Big)^{1/2}\mathscr{N}_1\Big(\frac{t}{r(r+\sqrt{t}\vartheta)}\Big)\widetilde{\omega}_0(r+\sqrt{t}\vartheta,r+\sqrt{t}\gamma)-\widetilde{\omega}_0(r,z).$$

Taking the L^p-estimate of (3.29), then with the aid of the following Minkowski's integral formula in general case

$$\Big(\int_{X_1}\Big(\int_{X_2}F(x_1,x_2)d\lambda_2(x_2)\Big)^pd\lambda_1(x_1)\Big)^{1/p}\le\int_{X_2}\Big(\int_{X_1}F(x_1,x_2)^pd\lambda_1(x_1)\Big)^{1/p}d\lambda_2(x_2),$$

one obtains for $p\in[1,\infty)$ that

$$\|(\mathbb{S}_1(t)\omega_0)(r,z)-\omega_0(r,z)\|_{L^p(\Omega)}\le\frac{1}{4\pi}\int_{\mathbb{R}^2}e^{-\frac{\vartheta^2+\gamma^2}{4}}\|\Upsilon(t,r,z,\vartheta,\gamma)\|_{L^p(\Omega)}d\vartheta d\gamma.$$

Now, we must establish that $\|\Upsilon(t,r,z,\vartheta,\gamma)\|_{L^p(\Omega)}\to 0$ as $t\to 0$. To do this, let $r>0$ and $r+\sqrt{t}\vartheta>0$. Writting

$$\begin{aligned}\Big(1+\sqrt{t}\frac{\vartheta}{r}\Big)^{1/2}\mathscr{N}_1\Big(\frac{t}{r(r+\sqrt{t}\vartheta)}\Big)&=\Big(\frac{\widetilde{r}}{r}\Big)^{1/2}\mathscr{N}_1\Big(\frac{t}{r\widetilde{r}}\Big)\\&\le C\frac{|r-\widetilde{r}|}{\sqrt{t}}\le C(1+|\vartheta|).\end{aligned}$$

Therefore

$$\begin{aligned}\|\Upsilon(t,\cdot,\cdot,\vartheta,\gamma)\|_{L^p(\Omega)}&\le C(1+|\vartheta|)\big(\|\omega_0(\cdot+\sqrt{t}\vartheta,\cdot+\sqrt{t}\gamma)\|_{L^p(\Omega)}+\|\omega_0\|_{L^p(\Omega)}\big)\\&\le C(1+|\vartheta|)\|\omega_0\|_{L^p(\Omega)}.\end{aligned}$$

On the other hand, it is clear to verify that $\Big(1+\frac{\sqrt{t}\vartheta}{r}\Big)^{1/2}\mathscr{N}_1\Big(\frac{t}{r(r+\sqrt{t}\vartheta)}\Big)$ goes to 1 as $t\to 0$. Thus Lebesgue's dominated convergence asserts for $(\vartheta,\gamma)\in\mathbb{R}^2$ that $\|\Upsilon(t,r,z,\vartheta,\gamma)\|_{L^p(\Omega)}\to 0$ when $t\to 0$. A new use of Lebesgue's dominated convergence, we finally deduce

$$\|(\mathbb{S}_1(t)\omega_0)(r,z)-\omega_0(r,z)\|_{L^p(\Omega)}\to 0,\quad t\to 0,\tag{3.30}$$

which accomplished the proof. □

In the spirit of Proposition 3.5 in [36], another weighted estimates for the linear semigroup (3.17) is shown in the following proposition, the proof of which can be done by the same reasoning as in the previous proposition.

Proposition 3.2.12. *Let $1\le p\le q\le\infty$, $i\in\{1,2\}$ and $(\alpha,\beta)\in[-1,2]$, with $\alpha\le\beta$. Assume that $r^\beta f\in L^p(\Omega)$, then*

$$\|r^\alpha\mathbb{S}_i(t)f\|_{L^q(\Omega)}\le\frac{C}{t^{\frac{1}{p}-\frac{1}{q}+\frac{(\beta-\alpha)}{2}}}\|r^\beta f\|_{L^p(\Omega)}.\tag{3.31}$$

In addition, if $(\alpha,\beta)\in[-1,1]$, $\alpha\le\beta$ and $r^\beta f\in L^p(\Omega)$, then

$$\|r^\alpha\mathbb{S}_i(t)\mathrm{div}_\star f\|_{L^q(\Omega)}\le\frac{C}{t^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}+\frac{(\beta-\alpha)}{2}}}\|r^\beta f\|_{L^p(\Omega)}.\tag{3.32}$$

We end this section by recalling the following classical estimate on the heat kernel in dimension three, the proof of which is left to the reader.

Proposition 3.2.13. *Let $1 \le p \le q \le \infty$. Assume that $f \in L^p(\mathbb{R}^3)$, then*

$$\|\mathbb{S}_2(t)f\|_{L^q(\mathbb{R}^3)} \le \frac{C}{t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}}\|f\|_{L^p(\mathbb{R}^3)}. \tag{3.33}$$

3.3 Main results

At this stage, we are ready to state the main result of this chapter. To be precise, we will prove the following theorem.

Theorem 3.3.1. *Let $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$ be axisymmetric initial data, then the system (3.1) admits a unique global mild solution. More precisely we have:*

$$(\omega_\theta, r\rho) \in \Big(C^0\big([0,\infty); L^1(\Omega)\big) \cap C^0\big((0,\infty); L^\infty(\Omega)\big)\Big)^2 \tag{3.34}$$

$$\rho \in C^0\big([0,\infty); L^1(\mathbb{R}^3)\big) \cap C^0\big((0,\infty); L^\infty(\mathbb{R}^3)\big) \tag{3.35}$$

Furthermore, for every $p \in [1,\infty)$, there exists some constant $K_p(D_0) > 0$, for which, and for all $t > 0$ the following statements hold

$$\|(\omega_\theta(t), r\rho(t))\|_{L^p(\Omega)\times L^p(\Omega)} \le t^{-(1-\frac{1}{p})}K_p(D_0). \tag{3.36}$$

$$\|\rho(t)\|_{L^p(\mathbb{R}^3)} \le t^{-\frac{3}{2}(1-\frac{1}{p})}K_p(D_0), \tag{3.37}$$

where

$$D_0 \triangleq \|(\omega_0, \rho_0)\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)}.$$

A few comments about the previous Theorem are given by the following remark.

Remark 3.3.2. It is worth pointing out the hypothesis ω_θ is in $L^1(\Omega)$ doesn't implies generally that the associated velocity v is in $L^2(\Omega)$ space. Consequently, the classical energy estimate is not available to derive a uniform bound for the velocity.

The proof is organized in two parts. The first one cares with the local well-posedness topic for (3.1) in the spirit of Gallay and Sverák [36]. We make use of fixed point-method for a system equivalent to (3.1) on product space equipped with an adequate norm with the help of the axisymmetric Biot-Savart law and some norm estimates between the velocity and vorticity. But in our context, we should deal carefully wih the additional terms $\partial_r\rho$ and $\frac{\partial_r\rho}{r}$ which contributes a singularity over the axe (Oz). The remedy is to hide those terms by creating a new function $\widetilde{\rho}$ and exploiting the coupling structure of the system (3.1) for $\kappa = 1$ for introducing a new unknown functions Γ and $\widetilde{\Gamma}$ in the spirit of [53] by setting $\Gamma = \frac{\omega_\theta}{r} - \frac{\rho}{2}$, and $\widetilde{\Gamma} = r\Gamma$. A straightforward computation shows that Γ and $\widetilde{\Gamma}$ solve, respectively

$$\begin{cases} \partial_t\Gamma + v\cdot\nabla\Gamma - (\Delta + \frac{2}{r}\partial_r)\Gamma = 0, \\ \Gamma_{t=0} = \Gamma_0. \end{cases} \tag{3.38}$$

$$\begin{cases} \partial_t\widetilde{\Gamma} + v\cdot\nabla\widetilde{\Gamma} - \frac{v^r}{r}\widetilde{\Gamma} - (\Delta - \frac{1}{r^2})\widetilde{\Gamma} = 0, \\ \widetilde{\Gamma}_{t=0} = r\Gamma_0. \end{cases} \tag{3.39}$$

In fact, in the second part, we shall investigate some a priori estimates for the *coupled functions* in order to derive the global regularity for the system in question. Significant properties of the new unknowns, such as the maximum principle, are gained in this transition and Γ evolves a similar equation and keeps the same boundary conditions than Π in the case of the axisymmetric Navier-Stokes without swirl, see (1.6). As consequence, the function Γ (and eventually $\widetilde{\Gamma}$ after some technical computations) satisfies the boundedness estimate as in (1.7) which will be crucial in the process of deriving the global regularity of our solutions.

3.3.1 Local existence of solutions

We will explore the aforementioned results and some preparatory topics in the previous sections, we shall scrutinize the local well-posedness issue for the system (3.1). For this reason, we rewrite it in view of the divergence-free condition in the following form

$$\begin{cases} \partial_t\omega_\theta + \mathrm{div}_\star(v\omega_\theta) = \Big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big)\omega_\theta - \partial_r\rho & \text{if } (t,r,z)\in\mathbb{R}_+\times\Omega, \\ \partial_t\rho + \mathrm{div}(v\rho) - \Delta\rho = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ (\omega_\theta,\rho)_{|t=0} = (\omega_0\rho_0). \end{cases} \tag{3.40}$$

The direct treatment of the local well-posedness topic for (3.40) in the spirit of [36] for initial data (ω_0,ρ_0) in the critical space $L^1(\Omega)\times L^1(\mathbb{R}^3)$ contributes many technical difficulties. This motivates to add the following new uknown $\widetilde{\rho} \triangleq r\rho$ which solves

$$\partial_t\widetilde{\rho} + \mathrm{div}_\star(v\widetilde{\rho}) = \Big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big)\widetilde{\rho} - 2\partial_r\rho. \tag{3.41}$$

We can be easily seen that $\widetilde{\rho}$ satisfying the same equation as ω_θ with additional source term and their variations are in Ω.

To achieve our topic we will handle with the following equivalent integral formulation

$$\begin{cases} \omega_\theta(t) = \mathbb{S}_1(t)\omega_0 - \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau - \int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau \\ \widetilde{\rho}(t) = \mathbb{S}_1(t)\widetilde{\rho}_0 - \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\widetilde{\rho}(\tau)\big)d\tau - 2\int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau \\ \rho(t) = \mathbb{S}_2(t)\rho_0 - \int_0^t \mathbb{S}_2(t-\tau)\mathrm{div}\big(v(\tau)\rho(\tau)\big)d\tau. \end{cases} \tag{3.42}$$

In order to analysis the above system, we will be working in the following Banach spaces

$$X_T = \Big\{ f \in C^0\big((0,T], L^{4/3}(\Omega)\big) : \|f\|_{X_T} < \infty \Big\},$$
$$Z_T = \Big\{ h \in C^0\big((0,T], L^{4/3}(\mathbb{R}^3)\big) : \|h\|_{Z_T} < \infty \Big\},$$

equipped with the following norms

$$\|f\|_{X_T} = \sup_{0<t\le T} t^{1/4}\|f(t)\|_{L^{4/3}(\Omega)}$$
$$\|h\|_{Z_T} = \sup_{0<t\le T} t^{3/8}\|h(t)\|_{L^{4/3}(\mathbb{R}^3)}.$$

Now, our task is to prove the following result.

Proposition 3.3.3. *Let $(\omega_0,\rho_0) \in L^1(\Omega)\times L^1(\mathbb{R}^3)$, then there exists $T = T(\omega_0,\rho_0)$ such that (3.42) admits a unique local solution satisfying*

$$(\omega_\theta, r\rho, \rho) \in C\big((0,T]; X_T\big) \times C\big((0,T]; X_T\big) \times C\big((0,T]; Z_T\big). \tag{3.43}$$

Proof. We will proceed by the fixed point theorem in the product space $\mathscr{X}_T = X_T \times X_T \times Z_T$ equipped by the norm

$$\|(\omega_\theta, \widetilde{\rho}, \rho)\|_{\mathscr{X}_T} \triangleq \|\omega_\theta\|_{X_T} + \|\widetilde{\rho}\|_{X_T} + \|\rho\|_{Z_T}.$$

For $t\ge 0$, define the free part $(\omega_{\rm lin}(t), \widetilde{\rho}_{\rm lin}(t), \rho_{\rm lin}(t)) = \big(\mathbb{S}_1(t)\omega_0, \mathbb{S}_1(t)(r\rho_0), \mathbb{S}_2(t)\rho_0\big)$, where $\big(\mathbb{S}_1(t), \mathbb{S}_2(t)\big)$ is given in Proposition 3.2.8. In accordance with the **(i)**-Proposition 3.2.11, it is not difficult to check that for $(\omega_0,\rho_0) \in L^1(\Omega)\times L^1(\mathbb{R}^3)$ we have for $T>0$

$$\sup_{0<t\le T} t^{1/4}\|\omega_{\rm lin}(t)\|_{L^{\frac43}(\Omega)} \le C\|\omega_0\|_{L^1(\Omega)}. \tag{3.44}$$

and

$$\sup_{0<t\le T} t^{1/4}\|\widetilde{\rho}_{\rm lin}(t)\|_{L^{\frac43}(\Omega)} \le C\|r\rho_0\|_{L^1(\Omega)} = C\|\rho_0\|_{L^1(\mathbb{R}^3)}. \tag{3.45}$$

On the other hand, the fact that

$$\|\rho_{\rm lin}(t)\|_{L^{\frac43}(\mathbb{R}^3)} = \|r^{\frac34}\rho_{\rm lin}(t)\|_{L^{\frac43}(\Omega)}$$

together with (3.31) stated in Proposition 3.2.12, we further get

$$\sup_{0<t\le T} t^{3/8}\|\rho_{\rm lin}(t)\|_{L^{\frac43}(\mathbb{R}^3)} \le C\|r\rho_0\|_{L^1(\Omega)} = C\|\rho_0\|_{L^1(\mathbb{R}^3)}. \tag{3.46}$$

Combining (4.32), (4.33) and (4.34) to obtain that $(\omega_{\rm lin}, \widetilde{\rho}_{\rm lin}, \rho_{\rm lin}) \in \mathscr{X}_T$.

Next, define the following quantity which will be useful later

$$\Lambda(\omega_0, \rho_0, T) = C\|(\omega_{\rm lin}, \widetilde{\rho}_{\rm lin}, \rho_{\rm lin})\|_{\mathscr{X}_T}. \tag{3.47}$$

We claim that $\Lambda(\omega_0, \rho_0, T) \to 0$ when $T \to 0$. To do this, we employ the fact $(L^{4/3}(\Omega) \cap L^1(\Omega)) \times (L^{4/3}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))$ is a dense space in $L^1(\Omega) \times L^1(\mathbb{R}^3)$. Then for every $\varepsilon > 0$ and every $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$ there exists $(\phi, \psi) \in (L^{4/3}(\Omega) \cap L^1(\Omega)) \times (L^{4/3}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))$ such that

$$\|(\omega_0, \rho_0) - (\phi, \psi)\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)} < \varepsilon.$$

On account of **(i)**-Proposition 3.2.11 we write

$$\begin{aligned}
\|\omega_{\rm lin}(t)\|_{L^{4/3}(\Omega)} &= \|\mathbb{S}_1(t)(\omega_0 - \phi + \phi)\|_{L^{4/3}(\Omega)} \\
&\leq \|\mathbb{S}_1(t)(\omega_0 - \phi)\|_{L^{4/3}(\Omega)} + \|\mathbb{S}_1(t)\phi\|_{L^{4/3}(\Omega)} \\
&\leq \frac{C}{t^{1/4}}\|\omega_0 - \phi\|_{L^1(\Omega)} + C\|\phi\|_{L^{4/3}(\Omega)\cap L^1(\Omega)}.
\end{aligned}$$

Multiply the both sides by $t^{1/4}$ and taking the supremum over $(0, T]$ to get

$$\begin{aligned}
\sup_{0<t\leq T} t^{1/4}\|\omega_{\rm lin}(t)\|_{L^{4/3}(\Omega)} &\leq C\|\omega_0 - \phi\|_{L^1(\Omega)} + CT^{1/4}\|\phi\|_{L^{4/3}(\Omega)\cap L^1(\Omega)} \\
&\leq C\varepsilon + CT^{1/4}\|\phi\|_{L^{4/3}(\Omega)\cap L^1(\Omega)}.
\end{aligned}$$

Thus, by setting

$$C_0(\omega_0, T) = \sup_{0<t\leq T} t^{1/4}\|\omega_{\rm lin}(t)\|_{L^{4/3}(\Omega)} \tag{3.48}$$

and let T (resp. ε) goes to 0, one deduces

$$\lim_{T\to 0} C_0(\omega_0, T) = 0. \tag{3.49}$$

By the same reasoning as above, it holds

$$\sup_{0<t\leq T} t^{1/4}\|\widetilde{\rho}_{\rm lin}(t)\|_{L^{4/3}(\Omega)} \leq C\varepsilon + CT^{1/4}\|\phi\|_{L^{4/3}(\Omega)\cap L^1(\Omega)},$$

with

$$C_1(\widetilde{\rho}_0, T) = \sup_{0<t\leq T} t^{1/4}\|\widetilde{\rho}_{\rm lin}(t)\|_{L^{4/3}(\Omega)}. \tag{3.50}$$

Likewise

$$\lim_{T\to 0} C_1(\widetilde{\rho}_0, T) = 0. \tag{3.51}$$

For $\rho_{\rm lin}$, a new use of Propositions 4.2.13 and 3.2.12 yield

$$\begin{aligned}
\|\rho_{\rm lin}(t)\|_{L^{4/3}(\mathbb{R}^3)} &= \|\mathbb{S}_2(t)(\rho_0 - \psi + \psi)\|_{L^{4/3}(\mathbb{R}^3)} \\
&\leq \|\mathbb{S}_2(t)(\rho_0 - \psi))\|_{L^{4/3}(\mathbb{R}^3)} + \|\mathbb{S}_2(t)\psi\|_{L^{4/3}(\mathbb{R}^3)} \\
&\leq \frac{C}{t^{3/8}}\|\rho_0 - \psi\|_{L^1(\mathbb{R}^3)} + \|\psi\|_{L^{4/3}(\mathbb{R}^3)}
\end{aligned}$$

Now, we multiply the both sides by $t^{3/8}$ and taking the supremum over $(0, T]$ to deduce

$$\begin{aligned}\sup_{0<t\leq T} t^{3/8}\|\rho_{\mathrm{lin}}(t)\|_{L^{4/3}(\mathbb{R}^3)} &\leq C\|\rho-\psi\|_{L^1(\mathbb{R}^3)} + CT^{3/8}\|\psi\|_{L^{4/3}\cap L^1(\mathbb{R}^3)} \\ &\leq C\varepsilon + CT^{3/8}\|\psi\|_{L^{4/3}\cap L^1(\mathbb{R}^3)}.\end{aligned} \tag{3.52}$$

Similarly, by putting

$$C_2(\rho_0, T) = \sup_{0<t\leq T} t^{3/8}\|\rho_{\mathrm{lin}}(t)\|_{L^{4/3}(\mathbb{R}^3)}, \tag{3.53}$$

we shall obtain that

$$\lim_{T\to 0} C_2(\rho_0, T) = 0. \tag{3.54}$$

Collecting (3.49), (3.51) and (3.54), so that by (3.47), we end up with

$$\lim_{T\to 0} \Lambda(\omega_0, \rho_0, T) = 0.$$

Now, we are ready to contract the integral formulation (3.42) in $\mathscr{X}_T$. Doing so, define for $(\omega_\theta, \widetilde{\rho}, \rho) \in \mathscr{X}_T$ the map

$$(0, T] \ni t \mapsto \mathscr{T}(t)(\omega_\theta, \widetilde{\rho}, \rho) \in L^{4/3}(\Omega) \times L^{4/3}(\Omega) \times L^{4/3}(\mathbb{R}^3)$$

by

$$\mathcal{T}(t)(\omega_\theta, \widetilde{\rho}, \rho) = \begin{pmatrix} \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau + \int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau \\ \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\widetilde{\rho}(\tau)\big)d\tau + 2\int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau \\ \int_0^t \mathbb{S}_2(t-\tau)\mathrm{div}\big(v(\tau)\rho(\tau)\big)d\tau \end{pmatrix}. \tag{3.55}$$

We aim at estimating $\mathscr{T}(t)(\omega_\theta, \widetilde{\rho}, \rho)$ in $L^{4/3}(\Omega) \times L^{4/3}(\Omega) \times L^{4/3}(\mathbb{R}^3)$. Due to similarity of the first two lines of (3.55), we will restrict ourselves to analyse the first and the third ones. For $\int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau$, we employ (3.23) in Proposition 3.2.11 and Hölder's inequality with respect to time to obtain

$$\begin{aligned}\|\int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau\|_{L^{\frac{4}{3}}(\Omega)} &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}+1-\frac{3}{4}}}\|v(\tau)\omega_\theta(\tau)\|_{L^1(\Omega)}d\tau \\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}}\|v(\tau)\|_{L^4(\Omega)}\|\omega_\theta(\tau)\|_{L^{\frac{4}{3}}(\Omega)}d\tau.\end{aligned}$$

Thanks to (3.11), it follows that

$$\begin{aligned}\|\int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau\|_{L^{\frac{4}{3}}(\Omega)} &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}}\|\omega_\theta(\tau)\|^2_{L^{\frac{4}{3}}(\Omega)}d\tau \\ &\lesssim \int_0^t \frac{d\tau}{(t-s)^{\frac{3}{4}}\tau^{\frac{1}{2}}}\|\omega_\theta\|^2_{X_T} \\ &\lesssim t^{-\frac{1}{4}}\|\omega_\theta\|^2_{X_T}.\end{aligned}$$

We show next how to estimate $\int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau$ in $L^{\frac{4}{3}}(\Omega)$. In view of Proposition (3.2.12) for $\alpha=0$ and $\beta=\frac{3}{4}$, we get

$$
\begin{aligned}
\|\int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau\|_{L^{\frac{4}{3}}(\Omega)} &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}+\frac{\frac{3}{4}-0}{2}}}\|r^{\frac{3}{4}}\rho\|_{L^{\frac{4}{3}}(\Omega)}d\tau\\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{7}{8}}}\|\rho\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}d\tau\\
&\lesssim \int_0^t \frac{d\tau}{(t-\tau)^{\frac{7}{8}}\tau^{\frac{3}{8}}}\|\rho\|_{Z_T}\\
&\lesssim t^{-\frac{1}{4}}\|\rho\|_{Z_T}.
\end{aligned}
$$

The above estimates combined with (3.47) provide the following inequality

$$
\|\omega_\theta\|_{X_T} \le \Lambda(\omega_0,\rho_0,T) + C\|\omega_\theta\|_{X_T}^2 + C\|\rho\|_{Z_T}. \tag{3.56}
$$

As explained above, the estimate of $\int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\widetilde{\rho}(\tau)\big)d\tau$ can be done along the same lines, so we have

$$
\|\int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\widetilde{\rho}(\tau)\big)d\tau\|_{L^{\frac{4}{3}}(\Omega)} \lesssim t^{-\frac{1}{4}}\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}. \tag{3.57}
$$

we deduce then the following estimate for $\widetilde{\rho}$

$$
\|\widetilde{\rho}\|_{X_T} \le \Lambda(\omega_0,\rho_0,T) + C\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T} + C\|\rho\|_{Z_T}. \tag{3.58}
$$

Let us move to estimate the last line in (3.55). Under the remark $\mathrm{div}(v\rho)=\frac{v^r}{r}\rho+\mathrm{div}_\star(v\rho)$, we write

$$
\begin{aligned}
\int_0^t \|\mathbb{S}_2(t-\tau)\mathrm{div}\big(v(\tau)\rho(\tau)\big)\|_{L^{4/3}(\mathbb{R}^3)}d\tau \;\le\; & \int_0^t \Big\|\mathbb{S}_2(t-\tau)\Big(\frac{v^r(\tau)}{r}\rho(\tau)\Big)\Big\|_{L^{4/3}(\mathbb{R}^3)}d\tau \\
& + \int_0^t \|\mathbb{S}_2(t-\tau)\mathrm{div}_\star(v(\tau)\rho(\tau))\|_{L^{4/3}(\mathbb{R}^3)}d\tau
\end{aligned} \tag{3.59}
$$

So, for the first term, we shall apply (3.31) stated in Proposition 3.2.12 for $\alpha=\frac{3}{4}$ and $\beta=2$ to get

$$
\begin{aligned}
\int_0^t \Big\|\mathbb{S}_2(t-\tau)\Big(\frac{v^r(\tau)}{r}\rho(\tau)\Big)\Big\|_{L^{4/3}(\mathbb{R}^3)}d\tau \;&=\; \int_0^t \Big\|r^{3/4}\mathbb{S}_2(t-\tau)\Big(\frac{v^r(\tau)}{r}\rho(\tau)\Big)\Big\|_{L^{4/3}(\Omega)}d\tau\\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{1-3/4+(2-3/4)/2}}\|v^r(\tau)r\rho(\tau)\|_{L^1(\Omega)}d\tau\\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{7/8}}\|v^r(\tau)\|_{L^4(\Omega)}\|\widetilde{\rho}(\tau)\|_{L^{4/3}(\Omega)}d\tau\\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{7/8}\tau^{1/2}}\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}d\tau\\
&\lesssim t^{-3/8}\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}.
\end{aligned}
$$

Therefore

$$t^{3/8}\int_0^t \Big\|\mathbb{S}_2(t-\tau)\Big(\frac{v^r(\tau)}{r}\rho(\tau)\Big)\Big\|_{L^{4/3}(\mathbb{R}^3)}d\tau \lesssim \|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}.$$

The second term of the r.h.s. in (3.59), will be done by a similar way as above, but we employ (3.32) in Proposition 3.2.12 for $\alpha=\frac{3}{4}$ and $\beta=1$, one may write

$$t^{3/8}\int_0^t \|\mathbb{S}_2(t-\tau)\mathrm{div}_\star(v(\tau)\rho(\tau))\|_{L^{4/3}(\mathbb{R}^3)}d\tau \lesssim \|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}.$$

Gathering the last two estimates and insert them in (3.59), one has

$$t^{3/8}\int_0^t \|\mathbb{S}_2(t-\tau)\mathrm{div}\big(v(\tau)\rho(\tau)\big)\|_{L^{4/3}(\mathbb{R}^3)}d\tau \lesssim \|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T},$$

combined with (3.53), it follows

$$\|\rho\|_{Z_T}\le C_2(\rho_0,T)+\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}. \tag{3.60}$$

Collecting (3.56), (3.58) and (3.60) we finally find the nonlinear system

$$\|\omega_\theta\|_{X_T}\le \Lambda(\omega_0,\rho_0,T)+C\|\omega_\theta\|^2_{X_T}+\|\rho\|_{Z_T}. \tag{3.61}$$

$$\|\widetilde{\rho}\|_{X_T}\le \Lambda(\omega_0,\rho_0,T)+C\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}+\|\rho\|_{Z_T}. \tag{3.62}$$

$$\|\rho\|_{Z_T}\le \Lambda(\omega_0,\rho_0,T)+C\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}. \tag{3.63}$$

In order to better justify the contraction argument, let us denote

$$\mathcal{B}_T(R)\triangleq\{(a,b)\in X_T\times X_T:\ \|(a,b)\|_{X_T\times X_T}<R\}.$$

and we claim, for R,T sufficiently small, $(\omega_\theta,\widetilde{\rho})\in\mathcal{B}_T(R)$.
By substituting (3.63) into (4.37) and (3.62), the contraction argument is satisfied if

$$3\Lambda(\omega_0,\rho_0,T)+\widetilde{C}R^2<R.$$

Since $\Lambda(\omega_0,\rho_0,T)\to 0$ when $T\to 0$, then an usual argument leads to the existence of $T^\star>0$ for which $\|\omega_\theta\|_{X_T}+\|\widetilde{\rho}\|_{X_T}$ remains bounded by R for all $T<T^\star$. Finally by substituting this latest in (3.63) we deduce that $\|\rho\|_{Z_T}$ remains bounded as well for all $T<T^\star$. The local existence and uniqueness follow then from classical fixed-point arguments. The continuity of the solution will be treated after the proof of the next proposition.
This ends the proof of Proposition 3.3.3. □

Remark 3.3.4. In fact in the proof of the local existence above we skip two steps by assuming that $\tilde{\rho}=r\rho$, the rigorous proof should be as the following: In a first step instead of dealing with (3.42), we need first to solve the system

$$\begin{cases}\omega_\theta(t)=\mathbb{S}_1(t)\omega_0-\int_0^t\mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau-\int_0^t\mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau\\[2ex] \widetilde{\rho}(t)=\mathbb{S}_1(t)\widetilde{\rho}_0-\int_0^t\mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\widetilde{\rho}(\tau)\big)d\tau-2\int_0^t\mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau\\[2ex] \rho(t)=\mathbb{S}_2(t)\rho_0-\int_0^t\mathbb{S}_2(t-\tau)\mathrm{div}\big(v(\tau)\frac{\widetilde{\rho}}{r}(\tau)\big)d\tau\end{cases} \tag{3.64}$$

by following the idea developed in the previous proof, and finally we check that $\widetilde{\rho} = r\rho$ by solving a heat-type equation evolving the quantity $\widetilde{\rho} - r\rho$ with 0 initial data.

Remark 3.3.5. In the light of Remark 4.2 from [36], the local lifespan T given by Proposition 3.3.3 above can not be bounded from below, by using only the norm $\|(\omega_0, \rho_0)\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)}$. However, in the case where $(\omega_0, \rho_0) \in \big(L^1(\Omega)\big) \times L^1(\mathbb{R}^3)) \cap \big(L^p(\Omega) \times L^p(\mathbb{R}^3)\big)$, for some $p > 1$, it is easy to provide explicitly a lower bound on T from an upper bound of $\|(\omega_0, \rho_0)\|_{L^p(\Omega)\times L^p(\mathbb{R}^3)}$, by making use of Propositions 3.2.11, 3.2.12, and 4.2.13.

We supply the above local well-posedness result by the following properties of the solution constructed in the previous part. Especially, we will prove.

Proposition 3.3.6. *For any $p \in (1, \infty)$, we have*

$$\lim_{t\to 0} t^{(1-\frac{1}{p})}\|\omega_\theta(t)\|_{L^p(\Omega)} = 0,$$

$$\lim_{t\to 0} t^{(1-\frac{1}{p})}\|r\rho(t)\|_{L^p(\Omega)} = 0,$$

$$\lim_{t\to 0} t^{\frac{3}{2}(1-\frac{1}{p})}\|\rho(t)\|_{L^p(\mathbb{R}^3)} = 0.$$

Proof. The proof is based principally on a bootstrap argument similar to that of [36]. For this aim, we will use the notaions

$$N_p(f, T) \triangleq \sup_{0<t\leq T} t^{(1-\frac{1}{p})}\|f\|_{L^p(\Omega)}, \quad J_p(f, T) \triangleq \sup_{0<t\leq T} t^{\frac{3}{2}(1-\frac{1}{p})}\|f\|_{L^p(\mathbb{R}^3)}.$$

$$M_p(f_0, T) \triangleq \sup_{0<t\leq T} t^{(1-\frac{1}{p})}\|\mathbb{S}_1(t)f_0\|_{L^p(\Omega)},$$

$$F_p(f_0, T) \triangleq \sup_{0<t\leq T} t^{\frac{3}{2}(1-\frac{1}{p})}\|\mathbb{S}_2(t)f_0\|_{L^p(\mathbb{R}^3)}.$$

From the properties of the semi-groups $\mathbb{S}_1$ and $\mathbb{S}_2$, we have for all $p \in (1, \infty]$

$$\lim_{T\to 0} M_p(\omega_0, T) = \lim_{T\to 0} M_p(r\rho_0, T) = \lim_{T\to 0} F_p(\rho_0, T) = 0. \tag{3.65}$$

In addition, from the local existence the desired inequalities hold also for $p = \frac{4}{3}$, assuming for a moment that the $L^1(\Omega) \times L^1(\mathbb{R}^3)$-norm of $(\omega_\theta(t), \rho(t))$ is bounded for all t small enough (for $t < T$, with T denotes the local time of existence given by the local existence theorey). Thus, by interpolation the proposition in question holds for all $p \in (1, \frac{4}{3}]$. In order to extend it to the other values of p we consider the Duhamel formula (3.42), and we will argue as in the local existence part, thus we omit some steps to make the presentation simpler. In view of Proposition 3.2.12, we write

$$\|\omega_\theta(t)\|_{L^p(\Omega)} \leq \|\mathbb{S}_1(t)\omega_0\|_{L^p(\Omega)} + C\int_0^{\frac{t}{2}} \frac{\|\omega_\theta\|^2_{L^q(\Omega)}}{(t-\tau)^{\frac{2}{q}-\frac{1}{p}}}d\tau$$
$$+ C\int_{\frac{t}{2}}^t \frac{\|\omega_\theta(\tau)\|_{L^{q_1}(\Omega)}\|\omega_\theta(\tau)\|_{L^{q_2}(\Omega)}}{(t-\tau)^{\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{p}}}d\tau$$
$$+ C\int_0^{\frac{t}{2}} \frac{\|\rho(\tau)\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}}{(t-\tau)^{\frac{1}{2}+\frac{3}{4}-\frac{1}{p}+\frac{3}{8}}}d\tau$$
$$+ C\int_{\frac{t}{2}}^t \frac{\|\rho(\tau)\|_{L^p(\mathbb{R}^3)}}{(t-\tau)^{\frac{1}{2}+\frac{1}{2p}}}d\tau.$$

Under the conditions

$$\frac{1}{2} \leq \frac{2}{q} - \frac{1}{p}, \quad \frac{1}{2} \leq \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{p} < 1, \tag{3.66}$$

we shall obtain

$$N_p(\omega_\theta, T) \leq M_p(\omega_0, T) + C_{p,q}N_q(\omega_\theta, T)^2 + C_{q_1,q_2}N_{q_1}(\omega_\theta, T)N_{q_2}(\omega_\theta, T)$$
$$+ C_pJ_{\frac{4}{3}}(\rho, T) + C_pJ_p(\rho, T). \tag{3.67}$$

We recall that $\widetilde{\rho}$ evolves the same equation as ω_θ, so we have

$$N_p(\widetilde{\rho}, T) \leq M_p(\widetilde{\rho}_0, T) + C_{p,q}N_q(\omega_\theta, T)N_q(\widetilde{\rho}, T) + C_{q_1,q_2}N_{q_1}(\omega_\theta, T)N_{q_2}(\widetilde{\rho}, T)$$
$$+ C_pJ_{\frac{4}{3}}(\rho, T) + C_pJ_p(\rho, T). \tag{3.68}$$

Finally, to claim similar estimate for $J_p(\rho, T)$, first we write

$$\|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq \|\mathbb{S}_2(t)\rho_0\|_{L^p(\mathbb{R}^3)} + C\int_0^{\frac{t}{2}} \frac{\|\omega_\theta\|_{L^{\frac{4}{3}}(\Omega)}\|\widetilde{\rho}\|_{L^{\frac{4}{3}}(\Omega)}}{(t-\tau)^{\frac{1}{2}+1-\frac{1}{p}+\frac{1-\frac{1}{p}}{2}}}d\tau$$
$$+ C\int_{\frac{t}{2}}^t \frac{\|\omega_\theta(\tau)\|_{L^{q_1}(\Omega)}\|\widetilde{\rho}(\tau)\|_{L^{q_2}(\Omega)}}{(t-\tau)^{\frac{1}{2}+\frac{1}{\alpha}-\frac{1}{p}+\frac{1-\frac{1}{p}}{2}}}d\tau,$$

with

$$\frac{1}{\alpha} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{2}.$$

Under the additional condition on p, q_1, q_2

$$\frac{1}{q_1} + \frac{1}{q_2} - \frac{3}{2p} < \frac{1}{2}$$

and for $q = \frac{4}{3}$, we obtain

$$J_p(\rho, T) \leq F_p(\rho_0, T) + C_pN_{\frac{4}{3}}(\omega_\theta, T)N_{\frac{4}{3}}(\widetilde{\rho}, T) + C_{q_1,q_2}N_{q_1}(\omega_\theta, T)N_{q_2}(\widetilde{\rho}, T). \tag{3.69}$$

Plugging (3.69) in (3.67) and (3.68) for $q = \frac{4}{3}$, and by denoting

$$U_p(T) \triangleq N_p(\omega_\theta, T) + N_p(\widetilde{\rho}, T), \quad V_p(T) \triangleq M_p(\omega_0, T) + M_p(\widetilde{\rho}_0, T) + F_p(\rho_0, T),$$

we deduce, that

$$U_p(T) \leq C_{p,q_1,q_2}\big(V_p(T) + U_{\frac{4}{3}}(T)^2 + J_{\frac{4}{3}}(\rho, T) + U_{q_1}(T)U_{q_2}(T)\big).$$

Now, to cover all the range $p \in (\frac{4}{3}, \infty)$, we proceed by the following bootstrap algorithm:
• For $q_1 = q_2 = \frac{4}{3}$ we can check that $U_p(T) \to 0$ as $T \to 0$ for all $1 < p < \frac{3}{2}$.
• Next, by taking $q_1 = q_2$ sufficiently close to $\frac{3}{2}$, we obtain the same result, for all $p < \frac{9}{5}$.
• For $q_1 = q_2 = \frac{8}{5}$, the estimate in question will hold then for all $p < 2$.
• Taking q_1 sufficiently close to 2, the result follows for all $p < \frac{3}{2}q_2$ and for all $q_2 < 2$.
• Finally, we define the sequence p_n by $p_0 = \frac{4}{3}$ and p_n sufficiently close to $\frac{3}{2}p_{n-1}$, by induction, we find that p_n is sufficiently close to $(\frac{3}{2})^n p_0$. Hence, letting n goes to ∞, we can cover all the range $p < \infty$, and thus we obtain

$$U_p(T) \to 0, \quad T \to 0, \quad \text{for all } p \in (1, \infty).$$

Finally, substituting this latest into (3.69), leads to

$$J_p(T) \to 0, \quad T \to 0, \quad \text{for all } p \in (1, \infty). \tag{3.70}$$

This ends the proof of Proposition 3.3.6 provided that we prove the existence of some $C_0 > 0$ for which

$$\|(\omega_\theta(t), \widetilde{\rho}(t), \rho(t))\|_{L^1(\Omega)\times L^1(\Omega)\times L^1(\mathbb{R}^3)} \leq C_0\|(\omega_0, \rho_0)\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)}.$$

From the definition of $\widetilde{\Gamma}$, we have

$$\|\omega_\theta(t)\|_{L^1(\Omega)} \leq \|\widetilde{\Gamma}(t)\|_{L^1(\Omega)} + \|\widetilde{\rho}(t)\|_{L^1(\Omega)}$$

and since $\widetilde{\rho} = r\rho$, our claim is equivalent to

$$\|(\widetilde{\Gamma}(t), \rho(t))\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)} \leq C_0\|(\omega_0, \rho_0)\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)}. \tag{3.71}$$

Let us then prove (3.71), we will restrict ourselves to the estimates of the nonlinear terms since the linear part can be dealt with by applying the properties of the semi-groups proved in the previous section. Thus, according to the equations of $\widetilde{\Gamma}$ and ρ, we need to show that

$$\int_0^t \|\mathbb{S}_1(t-\tau)\mathrm{div}_\star(v\widetilde{\Gamma})(\tau)\|_{L^1(\Omega)}d\tau \lesssim \|(\omega_0, \rho_0)\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)}, \tag{3.72}$$

and

$$\int_0^t \|\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\mathbb{R}^3)}d\tau \lesssim \|(\omega_0, \rho_0)\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)}\,. \tag{3.73}$$

For (3.72), Hölder inequality, axisymmetric Biot-Savart law and the definition of X_T norm, we write

$$\begin{aligned}\int_0^t \|\mathbb{S}_1(t-\tau)\mathrm{div}_\star(v\widetilde{\Gamma})(\tau)\|_{L^1(\Omega)}d\tau &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\|v(\tau)\|_{L^4(\Omega)}\|\widetilde{\Gamma}(\tau)\|_{L^{\frac{4}{3}}(\Omega)}d\tau\\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\|\omega_\theta(\tau)\|_{L^{\frac{4}{3}}(\Omega)}\|\widetilde{\Gamma}(\tau)\|_{L^{\frac{4}{3}}(\Omega)}d\tau\\ &\lesssim \|\omega_\theta\|_{X_T}\|\widetilde{\Gamma}\|_{X_T}\int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}\tau^{\frac{1}{2}}}d\tau\\ &\lesssim \|\omega_\theta\|_{X_T}\|\widetilde{\Gamma}\|_{X_T}.\end{aligned}$$

For (3.73), we have

$$\|\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\mathbb{R}^3)} = \|r\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\Omega)},$$

then in view of Proposition 3.2.12, we infer that

$$\int_0^t \|r\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\Omega)}\,d\tau \lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\,\|vr\rho(\tau)\|_{L^1(\Omega)}\,d\tau,$$

So, the definition of $\widetilde{\rho} = r\rho$ and Hölder inequality yield to

$$\|\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\mathbb{R}^3)} \lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\,\|v(\tau)\|_{L^4(\Omega)}\,\|\widetilde{\rho}(\tau)\|_{L^{\frac{4}{3}}(\Omega)}\,d\tau.$$

The rest of the estimate is then similar to the proof of (3.73) above by replacing $\widetilde{\Gamma}$ by $\widetilde{\rho}$. Finally, (3.72) and (3.73) follow from the local existence theory given by Proposition 3.3.3. $\square$

3.3.2 Continuity of the solution

Our last task of this section is to reach the continuity of the solution stated in (3.34) and (3.35) of the main Theorem 3.3.1. For this aim, we briefly outline the continuity of ω_θ, the rest of quantities can be treated along the same lines. So, we will show that

$$\omega_\theta \in C^0\big((0,T^\star); L^p(\Omega)\big), \quad \forall p \in [1,\infty).$$

To do so, let $0 < t_0 \le t < T^\star$, so we have

$$\begin{aligned}\omega_\theta(t) - \omega_\theta(t_0) = \big(\mathbb{S}_1(t-t_0) - \mathbb{I}\big)\omega_\theta(t_0) &- \int_{t_0}^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau\\ &- \int_{t_0}^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau. \end{aligned} \tag{3.74}$$

The first term (free part) is derived by the same manner as in (3.30), that is to say,

$$\lim_{t\to t_0}\|\big(\mathbb{S}_1(t-t_0) - \mathbb{I}\big)\omega_\theta(t_0,\cdot)\|_{L^p(\Omega)} \to 0. \tag{3.75}$$

Concerning the second term in the r.h.s of (3.74), (3.24) in Proposition 3.2.8 provides

$$\Big\| \int_{t_0}^{t} \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau\Big\|_{L^p(\Omega)} \lesssim \int_{t_0}^{t} \frac{1}{(t-\tau)^{\frac{1}{2}}}\|v(\tau)\|_{L^\infty(\Omega)}\|\omega_\theta(\tau)\|_{L^p(\Omega)}d\tau.$$

By virtue of the following interpolation estimate, see, Proposition 2.3 in [36], we have for some $1<q_1<2<q_2<\infty$

$$\|v(\tau)\|_{L^\infty(\Omega)} \lesssim \|\omega_\theta(\tau)\|^{\sigma}_{L^{q_1}}\|\omega_\theta(\tau)\|^{1-\sigma}_{L^{q_2}}, \quad \text{with} \quad \sigma = \frac{q_1}{2}\frac{q_2-2}{q_2-q_1}\in(0,1),$$

one may conclude that

$$\begin{aligned}
&\Big\| \int_{t_0}^{t} \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau\Big\|_{L^p(\Omega)}\\
&\lesssim \mathrm{essup}_{\tau\in(t_0,T^\star)}\big(\|\omega_\theta(\tau)\|_{L^p(\Omega)}\|\omega_\theta(\tau)\|^{\sigma}_{L^{q_1}}\|\omega_\theta(\tau)\|^{1-\sigma}_{L^{q_2}}\big)\int_{t_0}^{t}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}}\\
&\lesssim \mathrm{essup}_{\tau\in(t_0,T^\star)}\big(\|\omega_\theta(\tau)\|_{L^p(\Omega)}\|\omega_\theta(\tau)\|^{\sigma}_{L^{q_1}}\|\omega_\theta(\tau)\|^{1-\sigma}_{L^{q_2}}\big)(t-t_0)^{\frac{1}{2}},
\end{aligned}$$

which is sufficient to obtain

$$\lim_{t\to t_0}\Big\| \int_{t_0}^{t} \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau\Big\|_{L^p(\Omega)} = 0. \tag{3.76}$$

Let us move to the last term of (3.74) for which we distinguish two cases for p. For $p\in(1,\infty)$, (3.32) stated in Proposition 3.2.12 for $\alpha=0$ and $\beta=\frac{1}{p}$ yield

$$\Big\| \int_{t_0}^{t} \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau\Big\|_{L^p(\Omega)} \lesssim \int_{t_0}^{t}\frac{1}{(t-\tau)^{\frac{1}{2}+\frac{1}{2p}}}\|r^{\frac{1}{p}}\rho(\tau)\|_{L^p(\Omega)}d\tau, \tag{3.77}$$

and the fact that $\rho\in L^\infty\big((0,T^\star);L^p(\mathbb{R}^3)\big)$ ensures that

$$\Big\| \int_{t_0}^{t} \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau\Big\|_{L^p(\Omega)} \lesssim \mathrm{essup}_{\tau\in(t_0,T^\star)}\|\rho(\tau)\|_{L^p(\mathbb{R}^3)}(t-t_0)^{\frac{1}{2}(1-\frac{1}{p})}$$

combined with (3.77), one has

$$\lim_{t\to t_0}\Big\| \int_{t_0}^{t} \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau\Big\|_{L^p(\Omega)} = 0. \tag{3.78}$$

For the case $p=1$, we will work with $\widetilde{\Gamma}$ instead of ω_θ to avoid the source term $\partial_r\rho$. The fact $\|r\rho\|_{L^1(\Omega)}=\|\rho\|_{L^1(\mathbb{R}^3)}$ leads to

$$\|\omega_\theta(t)-\omega_\theta(t_0)\|_{L^1(\Omega)} \le \|\widetilde{\Gamma}(t)-\widetilde{\Gamma}(t_0)\|_{L^1(\Omega)} + \|\rho(t)-\rho(t_0)\|_{L^1(\mathbb{R}^3)},$$

so, the continuity of $\|\omega_\theta(\cdot)\|_{L^p(\Omega)}$ relies then on the continuity of $\|\widetilde{\Gamma}(\cdot)\|_{L^p(\Omega)}$ and $\|\rho(\cdot)\|_{L^1(\mathbb{R}^3)}$.

On the one hand, seen that the equation of $\tilde{\Gamma}$ governs the same equation to that of ω_θ, but without the source term $\partial_r\rho$, hence we follow then the same appraoch as above to prove that

$$\lim_{t\to t_0}\|\widetilde{\Gamma}(t)-\widetilde{\Gamma}(t_0)\|_{L^1(\Omega)}=0. \tag{3.79}$$

On the other hand, ρ solve a transport-diffusion equation, for which the continuity property is well-known to hold, thus we skip the details. Therefore

$$\lim_{t\to t_0}\|\omega_\theta(t)-\omega_\theta(t_0)\|_{L^1(\Omega)}=0.$$

Combining the last estimate with (3.75), (3.76) and (3.78), we achieve the result.

In the case $p=1$ and $t_0=0$, we should just be careful about the term $\partial_r\rho$ which can be completely avoided in the estimates by using the coupled functions Γ and $\tilde{\Gamma}$, the details are left to the reader.

3.4 Global existence

To reach the global existence for the local solution constructed in sections 3.3.1, we will establish some a priori estimates in Lebesgue spaces. For this target,

$$(\omega_\theta,r\rho,\rho)\in C^0\big((0,T];L^p(\Omega)\times L^p(\Omega)\times L^p(\mathbb{R}^3)\big),\quad p\in[1,\infty),\quad T\in(0,T^\star).$$

be a solution of the integral formulation (3.42) and so does (ω_θ,ρ) to the differential equation (3.40) associted to initial data $(\omega_0,\rho_0)\in L^1(\Omega)\times L^1(\mathbb{R}^3)$, where $T^\star$ denotes the maximal time of existence. Our basic idea is to couple the system (3.40) by introducing the new unknown $\Gamma=\Pi-\frac{\rho}{2}$ following [53] with $\Pi=\frac{\omega_\theta}{r}$. Some familiar computations show that Γ obeys

$$\begin{cases}\partial_t\Gamma+v\cdot\nabla\Gamma-(\Delta+\frac{2}{r}\partial_r)\Gamma=0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ \Gamma_{|t=0}=\Gamma_0.\end{cases} \tag{3.80}$$

For our analysis, we need to introduce again the unknown $\widetilde{\Gamma}\triangleq r\Gamma=\omega_\theta-\frac{\widetilde{\rho}}{2}$, which solves

$$\begin{cases}\partial_t\widetilde{\Gamma}+\mathrm{div}_\star(v\widetilde{\Gamma})-(\Delta-\frac{1}{r^2})\widetilde{\Gamma}=0 & \text{if } (t,r,z)\in\mathbb{R}_+\times\Omega,\\ \widetilde{\Gamma}_{|t=0}=\widetilde{\Gamma}_0.\end{cases} \tag{3.81}$$

The role of the new function Γ (resp. $\widetilde{\Gamma}$) for the viscous Boussineq system ($\mathrm{B}_{\mu,\kappa}$) is the same that Π (resp. ω_θ) for the Navier-Stokes equations (NS_μ). For this aim, it is quite natural to treat carefully the properties of Γ and $\widetilde{\Gamma}$.

The starting point of our analysis says that Γ enjoys the strong maximum principle. We will prove the following.

Proposition 3.4.1. *We assume that $\Gamma_0(x_1, x_2, z) > 0$ (or, < 0), then $\Gamma(t, x_1, x_2, z) > 0$ (or, < 0) for any $(x_1, x_2, z) \in \mathbb{R}^3$ and $t > 0$.*

Proof. We follow the formalism recently accomplished in [31]. Up to a regularization of Γ by standard method we can achieve the result as follows: we suppose that $\Gamma_0(x_1, x_2, z) > 0$ (likewise the case $\Gamma_0(x_1, x_2, z) < 0$). Due to the singularity of the term $\frac{2}{r}\partial_r\Gamma$, we can not apply directly the maximum principle. To surmount this hitch, we can be appropriately interpreted the term $\Delta + \frac{2}{r}\partial_r$ as the Laplacian in $\mathbb{R}^5$. Thus we recast (3.80) in $]0,\infty[\times\mathbb{R}^5$ by setting

$$\overline{\Gamma}(t, x_1, x_2, x_3, x_4, z) = \Gamma\Big(t, \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, z\Big)$$

and

$$\begin{aligned}\overline{v}(t, x_1, x_2, x_3, x_4, z) = v^r\Big(t, \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, z\Big)\overline{e}_r \\ + v^z\Big(t, \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, z\Big)\overline{e}_z.\end{aligned}$$

Above,

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, \quad \overline{e}_r = \Big(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}, \frac{x_4}{r}, 0\Big), \quad \overline{e}_z = (0, 0, 0, 0, 1)$$

Thus the equation (3.80) becomes

$$\begin{cases} \partial_t\overline{\Gamma} + \overline{v}\cdot\nabla_5\overline{\Gamma} - \Delta_5\overline{\Gamma} = 0 & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^5, \\ \overline{\Gamma}_{|t=0} = \overline{\Gamma}_0, \end{cases} \tag{3.82}$$

where ∇_5 and Δ_5 designate the gradient and Laplacian operators over $\mathbb{R}^5$ respectively. Consequently, by the strong maximum principle for (3.82), we deduce that

$$\overline{\Gamma} > 0 \quad \text{in} \quad]0,\infty[\times\mathbb{R}^5,$$

which leads to

$$\Gamma > 0 \quad \text{in} \quad]0,\infty[\times\mathbb{R}^3.$$

Thus, the proof is completed. $\square$

The second result cares with the classical L^p-estimate for Γ and showing that $t \mapsto \|\Gamma(t)\|_{L^p(\mathbb{R}^3)}$ is strictly decreasing function for $p \in [1,\infty]$. We will establish the following.

Proposition 3.4.2. *Let v be a smooth divergence-free vector field on $\mathbb{R}^3$ and Γ be smooth solution of (3.80). Then the following assertion holds.*

$$\|\Gamma(t)\|_{L^p(\mathbb{R}^3)} \leq \|\Gamma_0\|_{L^p(\mathbb{R}^3)}, \quad p \in [1,\infty]. \tag{3.83}$$

In particular, for $p \in [1,\infty]$ the map $t \mapsto \|\Gamma(t)\|_{L^p(\mathbb{R}^3)}$ is strictly decreasing.

Proof. Thanks to the Proposition 3.4.1, we can assume that $\Gamma_0 > 0$, thus we have $\Gamma(t) > 0$ for $t \in [0, T]$. We developp an integration by parts and taking into account the $\Gamma-$equation, the fact that div$v = 0$ and the boundary condition over $\partial\Omega$, one has

$$\begin{aligned}
\frac{d}{dt}\|\Gamma(t)\|^p_{L^p(\mathbb{R}^3)} &= p\int_\Omega \partial_t\Gamma(t)\Gamma^{p-1}(t)rdrdz \\
&= -p\int_\Omega v\cdot(\nabla\Gamma)\Gamma^{p-1}rdrdz + p\int_\Omega(\Delta\Gamma)\Gamma^{p-1}rdrdz \\
&\quad + 2p\int_\Omega(\partial_r\Gamma)\Gamma^{p-1}drdz \\
&= -p(p-1)\int_\Omega|\nabla\Gamma|^2\Gamma^{p-2}rdrdz + \int_\Omega\partial_r\Gamma^p drdz \\
&= -p(p-1)\int_\Omega|\nabla\Gamma|^2\Gamma^{p-2}rdrdz + \int_{\mathbb{R}}\Gamma^p(t,0,z)\eta_r dz \\
&= -p(p-1)\int_\Omega|\nabla\Gamma|^2\Gamma^{p-2}rdrdz - \int_{\mathbb{R}}\Gamma^p(t,0,z)dz < 0. \qquad (3.84)
\end{aligned}$$

where $\eta = (\eta_r, \eta_z) = (-1, 0)$ is a outward normal vector over Ω. Thus, integrating in time to obtain the aimed estimate for positive solutions.

Generally, if Γ_0 changes its sign, we procced as follows: we split $\Gamma(t) = \Gamma^+(t) - \Gamma^-(t)$, where $\Gamma^\pm$ solves the following linear equation with the same velocity

$$\begin{cases} \partial_t\Gamma^\pm + v\cdot\nabla\Gamma^\pm - (\Delta + \frac{2}{r}\partial_r)\Gamma^\pm = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ \Gamma^\pm_{|t=0} = \max(\pm\Gamma_0, 0) \geq 0. \end{cases} \qquad (3.85)$$

Arguiging as above to obtain that $\Gamma^\pm$ satisfies (3.83). Thus we have:

$$\begin{aligned}
\|\Gamma(t)\|_{L^p(\mathbb{R}^3)} &\leq \|\Gamma^+(t)\|_{L^p(\mathbb{R}^3)} + \|\Gamma^-(t)\|_{L^p(\mathbb{R}^3)} \\
&\leq \|\Gamma^+_0\|_{L^p(\mathbb{R}^3)} + \|\Gamma^-_0\|_{L^p(\mathbb{R}^3)} = \|\Gamma_0\|_{L^p(\mathbb{R}^3)}.
\end{aligned} \qquad (3.86)$$

If $\Gamma_0 \neq 0$, we distinguish two cases: $\Gamma_0 > 0$ or $\Gamma_0 < 0$. For this two cases the last inequality is strict and consequently (3.83) is also strict. Therefore, $t \mapsto \|\Gamma(t)\|_{L^1(\mathbb{R}^3)}$ is strictly decreasing for $t = 0$, and analogously we deduce that is strictly decreasing over $[0, T]$. □

Now, we state a result which deals with the asymptotic behavior of the coupled function Γ in Lebegue spaces $L^p(\mathbb{R}^3)$. Specifically, we have.

Proposition 3.4.3. *Let $\rho_0, \frac{\omega_0}{r} \in L^1(\mathbb{R}^3)$, then for any smooth solution of (3.80) and $1 \leq p \leq \infty$, we have*

$$\|\Gamma(t)\|_{L^p(\mathbb{R}^3)} \leq \frac{C}{t^{\frac{3}{2}(1-1/p)}}\|\Gamma_0\|_{L^1(\mathbb{R}^3)}, \qquad (3.87)$$

where $\Gamma_0 = \Pi_0 - \frac{\rho_0}{2}$.

Proof. Due to (3.83), the estimate (3.87) is valid for $p = 1$.

From the estimate (3.84) we have for $p = 2^n$

$$\begin{aligned} \frac{d}{dt}\|\Gamma(t)\|^p_{L^p(\mathbb{R}^3)} &= p\int_\Omega \partial_t\Gamma(t)\Gamma^{p-1}(t)rdrdz \\ &= -p(p-1)\int_\Omega |\nabla\Gamma|^2\Gamma^{p-2}rdrdz - \int_{-\infty}^{\infty}\Gamma^p(t,0,z)dz \\ &\le -p(p-1)\int_{\mathbb{R}^3}|\nabla\Gamma|^2\Gamma^{p-2}dx \\ &= -p(p-1)\int_\Omega\left|\frac{2}{p}\nabla\Gamma^{\frac{p}{2}}\right|^2 rdrdz = -\frac{4(p-1)}{p}\int_\Omega |\nabla\Gamma^{\frac{p}{2}}|^2 rdrdz. \end{aligned} \tag{3.88}$$

Thanks to the well-known Nash's inequality in general case

$$\int_{\mathbb{R}^N}|f|^2dx \le C\Big(\int_{\mathbb{R}^N}|\nabla f|^2dx\Big)^{1-\gamma}\Big(\int_{\mathbb{R}^N}|f|dx\Big)^{2\gamma}, \quad \gamma = \frac{2}{N+2}. \tag{3.89}$$

one obtains for $N = 3$

$$-\frac{d}{dt}\int_\Omega \Gamma^p(t)rdrdz \ge \frac{4(p-1)}{p}C\Big(\int_\Omega |\Gamma^{\frac{p}{2}}|rdrdz\Big)^{-4/3}\Big(\int_\Omega \Gamma^p rdrdz\Big)^{5/3}.$$

To simplify the presentation, setting $E_p(t) = \|\Gamma(t)\|^p_{\mathbb{R}^3} = \int_\Omega |\Gamma(t)|^p rdrdz$, then the last inequality becomes

$$-\frac{d}{dt}E_p(t) \ge \frac{4(p-1)}{p}CE^{-4/3}_{p/2}(t)E^{5/3}_p(t) \tag{3.90}$$

We prove (3.87) for $p = 2^n$ with nonnegative integers n by induction. Assume that (3.87) is true for $q = 2^k$ with $k \ge 0$, and let $p = 2^{k+1}$. Combined with (3.90)

$$-\frac{d}{dt}E_p(t) \ge \frac{4(p-1)C}{p}\big(C^q_q t^{-\frac{3}{2}(q-1)}\|\Gamma_0\|^q_{L^1(\mathbb{R}^3)}\big)^{-4/3}E^{5/3}_p(t).$$

Thus we have

$$\begin{aligned} \frac{3}{2}\frac{d}{dt}\Big(E_p(t)\Big)^{-2/3} = \frac{-\frac{d}{dt}E_p(t)}{E^{5/3}_p(t)} &\ge \frac{4(p-1)C}{p}C_q^{-4q/3}\|\Gamma_0\|^{-4q/3}_{L^1(\mathbb{R}^3)}t^{2(q-1)} \\ &= \frac{4(p-1)C}{p}C_q^{-2p/3}\|\Gamma_0\|^{-2p/3}_{L^1(\mathbb{R}^3)}t^{(p-2)}. \end{aligned}$$

Hence, integrating in time le last inequality yields

$$E_p^{-2/3}(t) \ge E_p^{-2/3}(0) + \frac{8C}{3p}C_q^{-2p/3}\|\Gamma_0\|^{-2p/3}_{L^1(\mathbb{R}^3)}t^{p-1}.$$

After a few easy computations, we derive the following

$$\|\Gamma(t)\|_{L^p(\mathbb{R}^3)} = E_p^{\frac{1}{p}}(t) \le \Big(\frac{3p}{8C}\Big)^{\frac{3}{2p}}C_q\|\Gamma_0\|_{L^1(\mathbb{R}^3)}t^{-3/2(1-1/p)}.$$

By setting $C_p = \left(\frac{3p}{8C}\right)^{\frac{3}{2p}} C_q$, then (3.87) remains true for $p = 2^{k+1}$. Let us observe that

$$C_p = \left(\frac{3p}{8C}\right)^{\frac{3}{2p}} C_q \;=\; \left(\frac{3}{8C}\right)^{\frac{3}{2^{k+2}}} 2^{\frac{3(k+1)}{2^{k+2}}} C_{2^k}$$
$$\leq \left(\frac{3}{8C}\right)^{\frac{3}{4}\sum_{k\geq 0}\frac{1}{2^k}} 2^{\frac{3}{4}\sum_{k\geq 0}\frac{k+1}{2^k}} C_1 \triangleq C_\infty$$

which means that C_∞ is independtly of p. Letting $p \to \infty$, we deduce that

$$\|\Gamma(t)\|_{L^\infty(\mathbb{R}^3)} \leq C_\infty t^{-3/2} \|\Gamma_0\|_{L^1(\mathbb{R}^3)}. \tag{3.91}$$

For the other values of p, we proceed by complex interpolation to get

$$\|\Gamma(t)\|_{L^p(\mathbb{R}^3)} \leq C \|\Gamma(t)\|^{1/p}_{L^1(\mathbb{R}^3)} \|\Gamma(t)\|^{1-1/p}_{L^\infty(\mathbb{R}^3)},$$

combined with (3.91), so the proof is completed. □

Next, we recall some a priori estimates for $\rho-$equation in Lebesgue spaces. To be precise, we have.

Proposition 3.4.4. *Let $\rho_0 \in L^1(\mathbb{R}^3)$ and $p \in [1, \infty]$, then there exists some non-negative universal constant $C_p > 0$ depending only on p such that for any smooth solution of $\rho-$equation in (3.1), we have*

(i) $\|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq \|\rho_0\|_{L^p(\mathbb{R}^3)}$,

(ii) $\|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq \frac{C_p}{t^{\frac{3}{2}(1-\frac{1}{p})}} \|\rho_0\|_{L^1(\mathbb{R}^3)}$.

Proof. **(i)** Can be done by a routine compuations as shown in Proposition 3.4.2, while **(ii)** can be obtained along the same way as Proposition 3.4.3.

We should mention also that the constant C_p is bounded with respect to p (see the proof of Proposition 3.4.3), and according to the proof of Proposition 3.4.3 C_∞ is given by

$$C_\infty \triangleq \left(\frac{3}{8C}\right) 2^{\frac{3}{4}\sum_{k\geq 0}\frac{k+1}{2^k}} C_1 < \infty \tag{3.92}$$

□

Now, we will prove another type of estimates for the quantities $\widetilde{\Gamma}, \widetilde{\rho}$ and ω_θ. Namely, we establish.

Proposition 3.4.5. *Let $\rho_0, \frac{\omega_0}{r} \in L^1(\mathbb{R}^3)$ and $p \in [1, \infty]$, then there exist a nonnegative constants $\widetilde{C}_p, K_p$, depending only on p and the initial data, such that for any smooth solution of (3.40), (3.41) and (3.81), we have*

(i) $\|\widetilde{\Gamma}(t)\|_{L^p(\Omega)} \leq \frac{\widetilde{C}_p(D_0)}{t^{1-\frac{1}{p}}}$,

(ii) $\|\widetilde{\rho}(t)\|_{L^p(\Omega)} \leq \frac{K_p(D_0)}{t^{1-\frac{1}{p}}}$,

(iii) $\|\omega_\theta(t)\|_{L^p(\Omega)} \lesssim \frac{\widetilde{C}_p(D_0)+K_p(D_0)}{t^{1-\frac{1}{p}}}$,

where

$$D_0 = \|(\omega_0, \rho_0)\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)} \tag{3.93}$$

and

$$\sup_{p\in[1,\infty)} \widetilde{C}_p(s) \triangleq \widetilde{C}_\infty(s) < \infty, \quad \widetilde{C}_p(s) \to 0, \ \textit{as } s \to 0, \quad \forall p \in [1,\infty]. \tag{3.94}$$

Proof. Let us point out that **(iii)** is a consequence of **(i)** and **(ii)**. Thus, we shall focus ourselves to prove **(i)** and **(ii)**.

(i) Due to the similarity of the equation of $\widetilde{\Gamma}$ and the one of ω_θ for the Navier-Stokes (NS_μ) treated in [36], we follow the approach stated in Proposition 5.3 in [36]. The key point consists to employ the following estimate,

$$\Big\|\frac{v^r}{r}\Big\|_{L^\infty(\Omega)} \lesssim \frac{1}{t}\|(\omega_0, r\rho_0)\|_{L^1(\Omega)}. \tag{3.95}$$

Indeed, Proposition 2.6 in [36] gives

$$\Big\|\frac{v^r}{r}\Big\|_{L^\infty(\Omega)} \lesssim \|\omega_\theta\|^{\frac{1}{3}}_{L^1(\Omega)}\Big\|\frac{\omega_\theta}{r}\Big\|^{\frac{2}{3}}_{L^\infty(\Omega)},$$

by using the fact that

$$\Big\|\frac{\omega_\theta}{r}\Big\|_{L^\infty(\Omega)} = \Big\|\frac{\omega_\theta}{r}\Big\|_{L^\infty(\mathbb{R}^3)}$$

combined with $\frac{\omega_\theta}{r} = \frac{\rho}{2} + \Gamma$, together with Propositions 3.4.3 and 3.4.4, lead to (3.95). So, the inequality **(i)** follows then by exploring (3.95) and repeating the outlines of the proof of Proposition 5.3 from [36], the details are left to the reader. We should only mention that the constant $\widetilde{C}_p$ in our proposition is the same as the one from proposition 5.3 in [36], which guaranties (3.94).

(ii) The estimate obviously holds for $p = 1$, whereas in the rest of the proof we shall deal with $p > 3$. The case $p \in (1,3]$ follows by interpolation.

We multiply the $\widetilde{\rho}-$equation by $|\widetilde{\rho}|^{p-1}$, after some integrations by parts we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\widetilde{\rho}(t)\|^p_{L^p(\Omega)} \leq & -4\frac{(p-1)}{p^2}\int_\Omega |\nabla(|\widetilde{\rho}|^{\frac{p}{2}})|^2 drdz \\ & + \Big| -\int_\Omega \text{div}_\star(v\widetilde{\rho})|\widetilde{\rho}|^{p-1}drdz - \int_\Omega \partial_r\rho|\widetilde{\rho}|^{p-1}drdz\Big|. \end{aligned}\tag{3.96}$$

On the one hand, a straightforward computation give

$$-\int_\Omega \mathrm{div}_\star(v\widetilde{\rho})|\widetilde{\rho}|^{p-1}drdz = \left(1-\frac{1}{p}\right)\int_\Omega \frac{v^r}{r}|\widetilde{\rho}|^p drdz,$$

then (3.95) provides

$$-\int_\Omega \mathrm{div}_\star(v\widetilde{\rho})|\widetilde{\rho}|^{p-1}drdz \leq CD_0\left(1-\frac{1}{p}\right)t^{-1}\int_\Omega |\widetilde{\rho}|^p drdz, \tag{3.97}$$

where D_0 is given by (3.93).

On the other hand, the fact $-\partial_r\rho = \frac{-\partial_r\widetilde{\rho}}{r} + \frac{\widetilde{\rho}}{r^2}$ yields

$$-\int_\Omega \partial_r\rho|\widetilde{\rho}|^{p-1}drdz = \left(1-\frac{1}{p}\right)\int_\Omega \frac{|\widetilde{\rho}|^p}{r^2}drdz.$$

Next, let us write

$$\int_\Omega \frac{|\widetilde{\rho}|^p}{r^2}drdz = \mathcal{I}_1 + \mathcal{I}_2, \tag{3.98}$$

with

$$\mathcal{I}_1 \triangleq \int_\Omega \frac{|\widetilde{\rho}|^p}{r^2}\mathbf{1}_{\{r\leq t^{1/2}\}}drdz, \quad \mathcal{I}_2 \triangleq \int_\Omega \frac{|\widetilde{\rho}|^p}{r^2}\mathbf{1}_{\{r>t^{1/2}\}}(r,z)drdz.$$

For $p>3$ we have,

$$\mathcal{I}_1 = \int_\Omega r^{p-3}|\rho|^p\mathbf{1}_{\{r\leq t^{1/2}\}}rdrdz \leq t^{\frac{p-3}{2}}\|\rho\|^p_{L^p(\mathbb{R}^3)}.$$

So, by virtue of Proposition 3.4.4, we infer that

$$\mathcal{I}_1 \leq C_p^p t^{-p} G_0^p, \tag{3.99}$$

where $G_0 \triangleq \|\rho_0\|_{L^1(\mathbb{R}^3)}$, and C_p is the constant given by Proposition 3.4.4.

For the term $\mathcal{I}_2$ an easy computation yields

$$\mathcal{I}_2 \leq t^{-1}\int_\Omega |\widetilde{\rho}|^p drdz. \tag{3.100}$$

Therefore, (3.99) and (3.100) give rise to

$$-\int_\Omega \partial_r\rho|\widetilde{\rho}|^{p-1}drdz \leq \frac{p-1}{p}\left(C_p^p t^{-p}G_0^p + t^{-1}\int_\Omega |\widetilde{\rho}|^p drdz\right). \tag{3.101}$$

Finally, Nash's inequality allows us to write

$$\int_\Omega |\widetilde{\rho}|^p drdz \lesssim \left(\int_\Omega |\nabla(|\widetilde{\rho}|^{\frac{p}{2}})|^2 drdz\right)^{\frac{1}{2}}\left(\int_\Omega |\widetilde{\rho}|^{\frac{p}{2}}drdz\right). \tag{3.102}$$

Since $p > 3$, so we have $1 < \frac{p}{2} < p$, and then by interpolation method, it happens

$$\left(\int_\Omega |\widetilde{\rho}|^{\frac{p}{2}} drdz\right) \lesssim \left(\int_\Omega |\widetilde{\rho}| drdz\right)^{\frac{p}{2(p-1)}} \left(\int_\Omega |\widetilde{\rho}|^p drdz\right)^{\frac{p-2}{2(p-1)}}.$$

Plugging the last inequality in (3.102), it holds

$$\left(\int_\Omega |\widetilde{\rho}|^p drdz\right)^{\frac{p}{2(p-1)}} \lesssim \left(\int_\Omega |\nabla(|\widetilde{\rho}|^{\frac{p}{2}})|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\widetilde{\rho}| drdz\right)^{\frac{p}{2(p-1)}}.$$

Since the inequality we aim to prove holds for $p = 1$, accordingly

$$CG_0^{-\frac{p}{p-1}} \left(\int_\Omega |\widetilde{\rho}|^p drdz\right)^{\frac{p}{p-1}} \leq \left(\int_\Omega |\nabla(|\widetilde{\rho}|^{\frac{p}{2}})|^2 drdz\right). \tag{3.103}$$

Thus, by gathering (3.97), (3.101) and (3.103) and insert them in (3.96), it happens

$$f'(t) \lesssim (p-1)\Big(-\frac{C}{p} G_0^{-\frac{p}{p-1}} \big(f(t)\big)^{\frac{p}{p-1}} + (CD_0 + 1)t^{-1} f(t) + C_p^p t^{-p} G_0^p \Big), \tag{3.104}$$

where, $f(t) \triangleq \int_\Omega |\widetilde{\rho}(t)|^p drdz$.

We recall that one may deduce from Proposition 3.3.6, for all $p \in [1, \infty)$

$$f(t) \leq e_p(D_0)^p t^{-(p-1)}, \quad \forall\, 0 < t < T^\star, \tag{3.105}$$

for some $e_p(D_0) > 0$.

In a first step, we will show that $f(t)$ is finite for all $t > 0$, then we prove that the decay property (3.105) holds as well for all $t > 0$, for a suitable non negative constant $K_p(G_0)$. Indeed, the first step is easy, one should remark that (3.104) implies

$$f'(t) \lesssim (p-1)\Big((CD_0 + 1)t^{-1} f(t) + C_p^p t^{-p} G_0^p \Big).$$

Via, Gronwall inequality on (t_0, t), for some $0 < t_0 < T^\star$, we get for all $t > t_0$

$$f(t) \leq \big(f(t_0) + C_p^p t_0^{1-p}\big) \left(\frac{t}{t_0}\right)^{(p-1)(CD_0+1)}, \tag{3.106}$$

which ensures that $f(t)$ is finite for all $t > 0$.

Now, let us denote

$$\widetilde{T} \triangleq \sup\big\{t > 0 : \ f(t) < K_p^p(D_0) t^{-(p-1)}\big\}, \tag{3.107}$$

where $K_p(D_0)$ will be chosen later, and we will prove that (3.105) holds as well for all $t \in [\widetilde{T}, \widetilde{T} + \varepsilon]$, for some $\varepsilon > 0$, this should be enough to contradict the fact that $\widetilde{T} < \infty$, and we shall conclude then that (3.105) is true for all $t > 0$. If $\widetilde{T}$ is finite then we deduce

$$f(\widetilde{T}) = K_p^p(D_0) \widetilde{T}^{-(p-1)}. \tag{3.108}$$

Now, define g by

$$g(t) \triangleq f(t) - K_p^p(D_0)t^{-(p-1)}.$$

By virtue of (3.104) and (3.108), we find out that

$$g'(\widetilde{T}) \leq \widetilde{T}^{-p}(p-1)\Big(-\frac{C}{p}G_0^{-\frac{p}{p-1}}\big(K_p(D_0)\big)^{\frac{p^2}{p-1}} + K_p^p(D_0) + \Sigma_p(D_0)\Big), \tag{3.109}$$

where $\Sigma_p(D_0) = CD_0 + 1 + C_p^p D_0^p$. Since $\frac{p^2}{p-1} > p$, then if we choose $K_p^p(D_0)$ large enough, in terms of $\Sigma_p(D_0)$ and G_0, we may conclude that

$$g'(\widetilde{T}) < 0,$$

which in particular gives, for $\varepsilon \ll 1$

$$g(\widetilde{T} + \varepsilon) \leq g(\widetilde{T}) = 0.$$

This means that (3.105) holds for $t = \widetilde{T} + \varepsilon$, which contracits the fact that $\tilde{T}$ is finite. The choice of $K_p(D_0)$ can be made as

$$K_p(D_0) = \max\Big\{\Big(C^{-1}pG_0^{\frac{p}{p-1}}\big(CD_0 + 2 + C_p^p D_0^p\big)\Big)^{\frac{1}{p}}, 1\Big\}. \tag{3.110}$$

and we end with, for all $p > 3$

$$\|\widetilde{\rho}\|_{L^p(\Omega)} \leq K_p(D_0)t^{-(1-\frac{1}{p})}. \tag{3.111}$$

By denoting

$$K_\infty(D_0) \triangleq \lim_{p\to\infty} \max\Big\{\Big(C^{-1}pG_0^{\frac{p}{p-1}}\big(CD_0 + 2 + C_\infty^p D_0^p\big)\Big)^{\frac{1}{p}}, 1\Big\} = 1 + C_\infty D_0,$$

from Proposition 3.4.4, C_∞ is finite, hence $K_\infty(D_0)$ is also finite, thus by letting $p \to \infty$ in (3.111), we end up with

$$\|\widetilde{\rho}(t)\|_{L^\infty(\Omega)} \leq K_\infty(D_0)t^{-1}.$$

□

Remark 3.4.6. As pointed out for the Navier-Stokes equations in Remark 5.4 from [36], for the global existence part in our case, we only need to mention that due to Proposition 3.4.5 (resp. Proposition 3.4.4) the $L^p(\Omega)$ norms of $\omega_\theta(t)$ and $r\rho(t)$ (resp. the $L^p(\mathbb{R}^3)$ of $\rho(t)$) can not blow-up in finite time, hence in view of Remark 3.3.5, it turns out that any constructed solution in the previous section is global for positive time, in addition of that, all the assertions (3.34)−(3.37) follow as a consequence of Propositions 3.4.4 and 3.4.5.

Remark 3.4.7. To be more precise about the assertions (3.34) and (3.35) in the case of L^∞, remark that the constants $\tilde{C}_p(D_0)$ and $K_p(D_0)$ given by the last Proposition above do not blow-up as p goes to infinity, which we do not know whether it holds true or not for the constants that appear in the bootstrap argument in the proof of Proposition 3.3.6, this is why we did not say any thing about the L^∞ case in the local existence part. As fact of matter, now while we know that the triplet $\big(\omega_\theta(t_0), r\rho(t_0), \rho(t_0)\big)$ holds to be in $L^\infty(\Omega)\times L^\infty(\Omega)\times L^\infty(\mathbb{R}^3)$, for all $t_0>0$, we can prove that the map

$$t\mapsto \big(\omega_\theta(t), r\rho(t), \rho(t)\big)$$

is continious with value in $L^\infty(\Omega)\times L^\infty(\Omega)\times L^\infty(\mathbb{R}^3)$ just by following exactely the same procedure we showed in the case $p\in(1,\infty)$.

4 Remarks on the global well-posedness of the axisymmetric Boussinesq system with rough initial data

This chapter is the subject of the following manuscript:

Adalet Hanachi, Haroune Houamed and M. Zerguine: *Remarks on the global well-posedness of the axisymmetric Boussinesq system with rough initial data*. Submitted in Journal of Mathematical Physics 2021.

4.1 Introduction

In this chapter, we deal with the global well-posedness of $3D$ axisymmetric viscous Boussinesq system in the context that initial vorticity ω_0 and initial density ρ_0 are both finite Radon measures. The existense and uniqueness will cost us to impose a condition of smallness to the punctual part of the initial data (ω_0, ρ_0). This result is considered as an extension of the previous result whenever the initial data is only Lebesgue integrable.

The success of many researchers in proving the global well posedness for the Navier-Stokes system despite the initial data is singular, has suggests whether this extension remains valid for the Boussinesq system. We were able to answer this question in the affirmative. Before giving this result, it is appropriate to state the most important results in this regard for the Navier Stokes system.

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ \text{div} v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{NS$_\mu$}$$

As we mentioned earlier in the introduction the global regularity of (NS$_\mu$) with a singular initial data has received a considerable attention. Let us mention in particular Cottet in [25], and independently Giga, Miyakawa, and Osada in [41] have established a global result when the initial vorticity ω_0 belongs to $\mathscr{M}(\mathbb{R}^2)$.

This latter result was early enhanced by Gallagher and Gallay in [33], where they established that if $\omega_0 \in \mathscr{M}(\mathbb{R}^2)$, there exists a unique solution $\omega \in C\big((0,\infty); L^1 \cap L^\infty\big)$ so have $\|\omega(t,\cdot)\|_{L^1} \leq \|\omega_0\|_{\mathscr{M}(\mathbb{R}^2)}$ and demonstrated also that such solution is in fact continuously dependent on initial data, deducing that Navier-Stokes equations is globally well-posed in 2D case. For large literature we refer the reader to [39].

For 3D Navier-Stokes equations the classical paradigm à la Leray and à la Kato remains valid. In terms of vorticity $\omega = \nabla \times v$, the situation is very worse because of the additional term $\omega \cdot \nabla v$ in ω's equation. In other words, we have

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega - \mu \Delta \omega = \omega \cdot \nabla v & x \in \mathbb{R}^3, \quad t \in (0,\infty), \\ \omega_{|t=0} = \omega_0. \end{cases} \tag{4.1}$$

The appearance of the term $\omega \cdot \nabla v$ is due to the higher dimension and is often referred to as the vorticity *stretching term*. Note that for 2D case, we have $\omega \cdot \nabla v \equiv 0$, we immediately deduce for $t \geq 0$ that $\|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p}$ for $p \in [1,\infty]$. According to Beale, Kato and Majda criterion [8] this latter boundedness is the main tool to achieve global well-posedness, controlling ω in $L^1_{loc}(\mathbb{R}_+; L^\infty)$. Unhappily, for 3D case, this criterion breaks down because of the *stretching term* which is one of the main sources of difficulties in the well-posedness theory of 3D Navier Stokes. To remedy this situation, we will restrict our selves to axisymmetric velocity vector fields without swirling. More precisely, the velocity v having the form

$$v(t,x) = v^r(t,r,z)\vec{e}_r + v^z(t,r,z)\vec{e}_z. \tag{4.2}$$

Here, (r,θ,z) refers to the cylindrical coordinates in $\mathbb{R}^3$, defined by setting $x = (r\cos\theta, r\sin\theta, z)$ with $0 \leq \theta < 2\pi$ and the triplet $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ indicates the usual frame of unit vectors in the radial, toroidal and vertical directions with the notation

$$\vec{e}_r = \Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big), \quad \vec{e}_\theta = \Big(-\frac{x_2}{r}, \frac{x_1}{r}, 0\Big), \quad \vec{e}_z = (0,0,1).$$

The formula (4.2) allows us to reduce the vorticity to the forme $\omega \triangleq \omega_\theta \vec{e}_\theta$ with $\omega_\theta = \partial_z v^r - \partial_r v^z$. In this case, the stretching term $\omega \cdot \nabla v$ closes to $\frac{v^r}{r}\omega$, in particular the time evolution of ω_θ reads as follows

$$\partial_t \omega_\theta + (v \cdot \nabla)\omega_\theta - \mu \Delta \omega_\theta + \mu \frac{\omega_\theta}{r^2} = \frac{v^r}{r}\omega_\theta, \tag{4.3}$$

with the notation $v \cdot \nabla = v^r \partial_r + v^z \partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$. Moreover, it easy to check that the quantity $\Pi = \frac{\omega_\theta}{r}$ obeys the following transport–diffusion equation

$$\partial_t \Pi + v \cdot \nabla \Pi - \mu\Big(\Delta + \frac{2}{r}\partial_r\Big)\Pi = 0, \quad \Pi_{|t=0} = \Pi_0. \tag{4.4}$$

First, we remark that Π satisfies almost the same equation as ω in the bidimensional situation. Thus, the axisymmetric structure is considered in some sense as a reduction of dimension. Second, an L^p- estimate for (4.4) gives

$$\|\Pi(t)\|_{L^p} \leq \|\Pi_0\|_{L^p}, \quad p \in [1, \infty]. \tag{4.5}$$

This estimate provided a good framework for establish the global well-posedness for (NS_μ) by many researchers, like M. Ukhoviskii and V. Yudovich [80], O. Ladyzhenskaya [61], S. Leonardi, J. Màlek, J. Necăs and M. Pokorný in [64] and H. Abidi [1].

The majority of aforementioned results are accomplished within the framework of finite energy solutions. Further, in terms of the vorticity for rough initial data, the system (NS_μ) has tackled by many authors. It sould be emphisized that in [31], H. Feng and V. Sverák recently settled a global result in time for (NS_μ), in a particular case that the initial vorticity ω_0 is supported on a circle. This result developed lately by Th. Gallay and V. Sverák [36] in the more general case. They also proved that the system (NS_μ) is globally well-posed in time whenever $\omega_0 \in L^1(\Omega)$.

Such result was extended by the same authors to the case ω_0 is only a finite measure with a small atomic part (see [36]). More recently, the same authors proved in [37] that the atomic part of the initial measure can be taken arbitrary large for stemming from circular vortex filaments, i.e $\omega_0 = \gamma\delta_{\bar{r},\bar{z}}$, with $\gamma, \bar{r} > 0$ and $\bar{z} \in \mathbb{R}$. However, the well-posedness in the general case of arbitrary measures is still open.

As regards to 3D Boussinesq system ($\mathrm{B}_{\mu,\kappa}$), the well-posedness subject has been considerably explored. Danchin and Paicu revisited in [27] the solutions à la Leray and à la Fujita-Kato for ($\mathrm{B}_{\mu,\kappa}$) in the case $\kappa = 0$ and demonstrated that solutions are global and unique in time under smallness condition. In the same way, Abidi-Hmidi-Keraani [3] asserted that ($\mathrm{B}_{\mu,\kappa}$) admits a unique global solution in axisymmetric setting whenever $v_0 \in H^1(\mathbb{R}^3)$, $\Pi_0 \in L^2(\mathbb{R}^3), \rho_0 \in L^2 \cap L^\infty$ with $(\mathrm{supp}\ \rho_0) \cap (Oz) = \emptyset$ and $P_z(\mathrm{supp}\ \rho_0)$ is a compact set in $\mathbb{R}^3$ to prohibit the violent singularity $\frac{\partial_r \rho}{r}$, with P_z being the orthogonal projector over (Oz). By the same process, this problem has been considered by Hmidi-Rousset in [53] for $\kappa \geqslant 0$. First, they declined the assumption on the support of the density. Second, they took advantage of the coupling phenomena between the vorticity and the density by introducing a new unknown $\Gamma = \Pi - \frac{\rho}{2}$ which satisfies the equation

$$\partial_t \Gamma + v \cdot \nabla \Gamma - \Big(\Delta + \frac{2}{r}\partial_r\Big)\Gamma = 0, \quad \Gamma_{|t=0} = \Gamma_0.$$

We can easily notice that Γ plays the same role as Π for the Navier-Stokes system (NS_μ).

The main interest of Γ is to derive by a simple way a priori estimates for Π. More recently, H. Houamed and M. Zerguine conducted a new result in the sense that exploited the axisymmetric structure on the velocity and the crititical regularity à la Fujita-Kato to assert that ($\mathrm{B}_{\mu,\kappa}$), for $\kappa = 0$, possesses a unique global solution as long as and $(v_0,\rho_0)\in H^{\frac12}\cap\dot B^0_{3,1}\times L^2\cap\dot B^0_{3,1}$. Finally, in the case $\kappa=\mu>0$, we succeed lately in [44] in performing a new result of global well-posedeness for $(\mathrm{B}_{\mu,\kappa})$ in the setting of (ω_0,ρ_0) is axisymmetric and belonging to the critical Lebesgue spaces $L^1(\Omega)\times L^1(\mathbb{R}^3)$ whose idea inspired from [36] concerning Navier-Stokes equations (NS_μ).

4.1.1 Aims

The current chapter occupies with a topic of the global well-posedness for the system $(\mathrm{B}_{\mu,\kappa})$ given under vorticity-density formulation[1]

$$\begin{cases}\partial_t\omega_\theta+v\cdot\nabla\omega_\theta-\frac{v^r}{r}\omega_\theta-\big(\Delta-\frac{1}{r^2}\big)\omega_\theta=-\partial_r\rho,\\ \partial_t\rho+v\cdot\nabla\rho-\Delta\rho=0\\ (\omega_\theta,\rho)_{|t=0}=(\omega_0,\rho_0).\end{cases}\tag{4.6}$$

We aim to extend our result latterly established in [44] to the larger class of initial data of measure type, i.e, for $(\omega_0,\rho_0)\in\mathscr{M}(\Omega)\times\mathscr{M}(\mathbb{R}^3)$, where, $\mathscr{M}(\mathbb{X})$ denotes the set of finite Radon measures on $\mathbb{X}\in\{\mathbb{R}^3,\Omega\}$ being such that

$$\|\mu\|_{\mathscr{M}(\mathbb{X})}\triangleq\sup_{\{\varphi\in C_0(\mathbb{X}),\|\varphi\|_{L^\infty(\mathbb{X})}\le1\}}|\langle\mu,\varphi\rangle|<\infty,\tag{4.7}$$

with $\langle\cdot,\cdot\rangle$ symbolizes the pairing between $\mathscr{M}(\mathbb{X})$ and $C_0(\mathbb{X})$ which is defined by

$$\langle\mu,\varphi\rangle\triangleq\int_{\mathbb{X}}\varphi(x)d\mu(x).$$

Due to (4.7) the space $\mathscr{M}(\Omega)$ is the topological dual of $C_0(\Omega)$, so the Banach-Alaoglu theorem, insures that the unit ball in $\mathscr{M}(\Omega)$ is a sequentially compact set for the weak topology in the following sense:

$$\lim_{n\to\infty}\langle\mu_n,\varphi\rangle=\langle\mu,\varphi\rangle.\tag{4.8}$$

Each $\mu\in\mathscr{M}(\Omega)$ can be decomposed in unique way as

$$\mu=\mu_{ac}+\mu_{sc}+\mu_{pp},\quad \mu_{ac}\perp\mu_{sc}\perp\mu_{pp}\tag{4.9}$$

and

$$\|\mu\|_{\mathscr{M}(\Omega)}=\|\mu_{ac}\|_{\mathscr{M}(\Omega)}+\|\mu_{sc}\|_{\mathscr{M}(\Omega)}+\|\mu_{pp}\|_{\mathscr{M}(\Omega)}.$$

where, in the sequel we denote by:
- μ_{ac} is a measure which is absolutely continuous with respect to Lebesgue measure,

[1] For simplicity, we take $\kappa=\mu=1$. However, we should mention that our arguments hold also for $\kappa=\mu>0$.

that is $\dfrac{d\mu_{ac}}{dx} = f$ for some $f \in L^1(\Omega)$,
• μ_{sc} is a singular continuous measure which has no atom but is supported on a set of zero Lebesgue measure.
• μ_{pp} is is punctual part (an atomic measure), $\mu_{pp} = \sum_{n\geq 1} \lambda_n \delta_{a_n}$, $(\lambda_n) \subset \mathbb{R}$, $(a_n) \subset \Omega$, with δ_{a_n} stands to be the Dirac measure supported at $a_n \in \Omega$.

4.1.2 Statement of the main results

This subsection addresses to state the main results of this chapter and thrash out the headlines of their proofs.

The main contribution of this chapter is dedicated to extending the result of Theorem 3.3.1 to more general case of initial data, that is to say the initial data belonging to the class of finite measure over $\Omega \times \mathbb{R}^3$. Specifically, we shall prove the following theorem.

Theorem 4.1.1. *There exists a non negative constant $\varepsilon > 0$ such that the following hold. Let $(\omega_0, \rho_0) \in \mathscr{M}(\Omega) \times \mathscr{M}(\mathbb{R}^3)$ with ρ_0 being axisymmetric in the sense of Definition 4.2.3 and*

$$\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)} + \|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)} \leq \varepsilon, \tag{4.10}$$

then, the Boussinesq system (4.6) admits a unique global mild axisymmetric solution (ω_θ, ρ) such that

$$(\omega_\theta, \rho) \in C^0\big((0,\infty); L^1(\Omega) \cap L^\infty(\Omega)\big) \times C^0\big((0,\infty); L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\big),$$
$$r\rho \in C^0\big((0,\infty); L^1(\Omega) \cap L^\infty(\Omega)\big).$$

Furthermore, for every $p \in (1, \infty]$, we have

$$\limsup_{t\to 0} t^{\frac{3}{2}(1-\frac{1}{p})} \|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq C\varepsilon, \quad \limsup_{t\to 0} t^{1-\frac{1}{p}} \|(\omega_\theta(t), r\rho(t))\|_{L^p(\Omega)\times L^p(\Omega)} \leq C\varepsilon$$

and

$$\limsup_{t\to 0} \|(\omega_\theta(t), \rho(t))\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)} < \infty.$$

Moreover, we have that $(\omega_\theta(t), \rho(t)) \rightharpoonup (\omega_0, \rho_0)$ as $t \to 0$.

Few remarks are in order.

Remark 4.1.2. Observe that Theorem 4.1.1 covers a class of initial data which is considerably larger than the one treated by Theorem 3.3.1. However, the smallness condition for the atomic parts is crucial in our arguments to guarantee the existence and the uniqueness. Nevertheless, we should point out that it is probably possible to construct global solutions for arbitrary large initial data (see [31] for a precise result of that in the case of the vortex rings). Such result is based on smoothing out the initial data, hence, the uniqueness is to be dealt with separately to the existence part, which, on the other hand, stands to be open in general for the time being even for the case the Navier-Stokes equations.

Remark 4.1.3. The serious drawback that arises in Theorem 4.1.1 is how to give a rigorous and suitable sense to the initial data associated with the quantity $r\rho$ if the initial density ρ_0 is a finite measure. More precisely, it is important to make a choice that does not perturb the weak continuity of the solution near $t = 0$. To get around this, we should understand and give some general notions on the axisymmetric measures in $\mathbb{R}^3$ (see, the next section). In broad terms, one should notice that, with Theorem 3.3.1 on the hand, the most challenging part in the proof of Theorem 4.1.1 is the understanding of the solution near $t = 0$. Indeed, after some $t_0 > 0$, the solution becomes more regular and, hence, the arguments of Theorem 3.3.1 would apply to garantee estimates alike (3.36) and (3.37), globally in time. In other words, to be more precise on the main novel part in Theorem 4.1.1, we refer to Theorem 4.3.7 that we shall prove at the end of this chapter.

Remark 4.1.4. Remark that the case $p = \infty$ is missing in the statement of Theorem 3.3.1. However, as we shall see later on, the estimates of Theorem 3.3.1 hold also in this case. To justify our claim, we will outline the idea to get the L^∞-estimate in the proof of Proposition 4.3.3 below.

Structure of the chapter. We discuss concisely the steps of the proof of Theorem 4.1.1 and the structure of the chapter. The local well-posedness will be done via the classical fixed point formalism in an adequate functional spaces as in the proof of Theorem 4.3.1. But before doing so, as noticed in the proof of Theorem 3.3.1, since the quantity $r\rho$ will play a significant role in our analysis, we have to give a suitable sense to the limit of $r\rho(t)$, as t tends to 0, in the case of initial measure-type density ρ_0. For a better understanding of this limit, we state in Section 4.2 a succinct about the measure theory, in particular the push forward of a measure by a measurable function in the general case. This can be also considered as a preamble to introduce the concept of axisymmetric measures. In the second part of Section 4.2, we shall recall some nice properties of the semi–groups associated to the system in study. Thereafter, Section 4.3 contains three parts: In the first one, we prove a result which can be seen as an intermediate case between Theorem 3.3.1 and Theorem 4.1.1 where we assume that $(\omega_0, \rho_0) \in \mathscr{M}(\Omega) \times L^1(\mathbb{R}^3)$. The details of the proof we provide in this part should help to simplify the presentation of the proof of Theorem 4.1.1. Then, in the second part, we shall prove a general version of the local well-posedness (see Theorem 4.3.7) which implies the local results of Theorem 4.1.1. Finally, in the last part of Section 4.3, we outline the idea that allows to globalize the local results we prove in the first two parts.

4.2 The framework preliminaries

In this preparatory section, we gather the basic ingredients freely explored during this work. We begin with a self-contained abstract on some notions from measures theory. Then, we recall some estimates of the heat semi-group in Lebesgue spaces.

4.2.1 Results on measure theory

We embark on the measure tool, where we state the notion of the push-forward of measure, some properties, and we give a new concept about the axisymmetric measure. To illustrate this notion, we introduce two examples. Overall, we claim the action of the axisymmetric measure on the heat semigroup. We end this section with some properties concerning the restriction of any axisymmetric measure on Ω to define the quantity $r\rho$.

Definition 4.2.1. Let (X_1, Σ_1) and (X_2, Σ_2) be two measurable spaces and μ be a positive measure on (X_1, Σ_1). Let F be a measurable mapping from X_1 into X_2. The *push-forward* measure of μ by F, denoted $F_\star\mu$, is defined as

$$\begin{aligned} F_\star\mu : &\Sigma_2 \to [0,\infty] \\ &B \longmapsto F_\star\mu(B) \triangleq \mu(F^{-1}(B)). \end{aligned}$$

The main feature of the above definition is the fact that it is useful in the following generalization formula of change of variable (see Sections 3.6–3.7 in [10])

Theorem 4.2.2. *A measurable function g on X_2 is integrable with respect to the push-forward measure $F_\star\mu$ if and only if the composition $g \circ F$ is integrable with respect to the measure μ. As well, we have*

$$\int_{X_2} g d(F_\star\mu) = \int_{X_1} g \circ F d\mu. \tag{4.11}$$

The typical example that concerns the axisymmetric structure given as follows. For $\alpha \in [0, 2\pi)$, define $\mathfrak{R}_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ by $x \mapsto \mathfrak{R}_\alpha x$, with

$$\mathfrak{R}_\alpha \triangleq \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x^t = (x_1, x_2, x_3). \tag{4.12}$$

An elementary calculus shows that $\mathfrak{R}_\alpha$ is an orthogonal 3×3 matrix, with $\mathfrak{R}_\alpha^{-1} = \mathfrak{R}_{-\alpha}$. In addition, by exploring Definition 4.2.1, for $\mu \in \mathscr{M}(\mathbb{R}^3)$, we can define its push-forward measure $\mathfrak{R}_\alpha\mu$ as an element of $\mathscr{M}(\mathbb{R}^3)$. Moreover, Theorem 4.2.2 provides us the following identity

$$\langle \mathfrak{R}_\alpha\mu, \varphi \rangle = \langle \mu, \varphi \circ \mathfrak{R}_{-\alpha} \rangle, \quad \forall \varphi \in C^0(\mathbb{R}^3).$$

Based on that, we state the following definition.

Definition 4.2.3 (Axisymmetric measure). A Radon measure $\mu \in \mathscr{M}(\mathbb{R}^3)$ is said to be axisymmetric if and only if it is stable by the push-forward mapping $\mathfrak{R}_\alpha$, for all $\alpha \in [0, 2\pi)$. i.e, if

$$\mathfrak{R}_\alpha\mu = \mu, \quad \forall \alpha \in [0, 2\pi). \tag{4.13}$$

Remark 4.2.4. The above definition says that the measure μ is axisymmetric if and only if it is stable by the push-forward mapping $\mathfrak{R}_\alpha$, for all $\alpha \in [0, 2\pi)$. We can, moreover, check that this definition is equivalent to

$$\mu(B) = \frac{1}{2\pi} \int_0^{2\pi} \mathfrak{R}_\alpha \mu(B) d\alpha, \quad \forall B \in \mathscr{B}(\mathbb{R}^3).$$

Or, also equivalently

$$d\mu = \frac{1}{2\pi} \int_0^{2\pi} d(\mathfrak{R}_\alpha \mu) d\alpha.$$

To illustrate the above definition we state here two typical examples of an axisymmetric measure.

- *Absolutely continuous axisymmetric measures* : If $\mu \in \mathscr{M}_{ac}(\mathbb{R}^3)$, then there exists an integrable function $f_\mu \in L^1(\mathbb{R}^3)$ such that

$$\mu(B) = \int_B f_\mu(x) dx, \quad \forall B \subset \mathbb{R}^3.$$

  In this case, we can check that μ is axisymmetric in the sense of Definition 4.2.3 if and only if f_μ is an axisymmetric function in the classical sense.

- *Atomic axisymmetric measures* : Let $a \in \mathbb{R}^3$ and $\mu = \delta_a$, one can check that μ is axisymmetric in the sense of Definition 4.2.3 if and only if $\{a\}$ is stable by rotation around the (oz) axis. In other words, δ_a is axisymmetric if and only if $a \in (oz)$. More generally, if $A \subset \mathbb{R}^3$, then $\mu = \delta_A$ is axisymmetric if and only if, there exists $(r, z) \in \mathbb{R}^+ \times \mathbb{R}$ such that

$$A = \bigcup_{\theta \in [0,2\pi]} \big\{(r\cos\theta, r\sin\theta, z)\big\}.$$

As a consequence of the above properties, we have the following elementary result.

Proposition 4.2.5. *Let $\mu \in \mathscr{M}(\mathbb{R}^3)$ be an axisymmetric measure, then the function $x \mapsto e^{t\Delta}\mu(x)$ is axisymmetric.*

Proof. We need to show that, for all $\alpha \in [0, 2\pi)$, there holds

$$e^{t\Delta}\mu(x) = e^{t\Delta}\mu(\mathfrak{R}_\alpha x).$$

To do so, we write

$$\begin{aligned} e^{t\Delta}\mu(\mathfrak{R}_\alpha(x)) &= \frac{1}{(4\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{|\mathfrak{R}_\alpha x - y|^2}{4t}} d\mu(y) \\ &= \frac{1}{(4\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-|\mathfrak{R}_\alpha|^2 \frac{|x - \mathfrak{R}_{-\alpha} y|^2}{4t}} d\mu(y). \end{aligned}$$

Now, exploring the fact that $|\mathfrak{R}_\alpha|^2 = 1$ and by taking $g_{t,x}(y) = \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|x-y|^2}{4t}}$, then Theorem 4.2.2 implies

$$
\begin{aligned}
e^{t\Delta}\mu(\mathfrak{R}_\alpha(x)) &= \int_{\mathbb{R}^3} g_{t,x} \circ \mathfrak{R}_{-\alpha}(y) d\mu(y) \\
&= \int_{\mathbb{R}^3} g_{t,x}(y) d(\mathfrak{R}_\alpha \mu(y))
\end{aligned}
$$

Now, because μ is an axisymmetric measure, we infer that for all $\alpha \in [0, 2\pi)$

$$
\int_{\mathbb{R}^3} g_{t,x}(y) d(\mathfrak{R}_\alpha \mu(y)) = \int_{\mathbb{R}^3} g_{t,x}(y) d\mu(y).
$$

This completes the proof. □

Remark 4.2.6. For any PDE in general, and for the Boussinesq system in particular, if we are looking for special solutions, then, we need to impose some kind of suitable compatibility condition for the initial data that fits well with the desired structure of the solutions. Definition 4.2.3 plays exactly that role here. In other words, it provides the requirement of the structure of the initial data that allows to study the existence of axisymmetric solutions. Proposition 4.2.5 is then a typical example of the propagation of this special geometric structure of the initial data for all times $t \geqslant 0$.

Remark 4.2.7. Let us also mention that a similar result to Proposition 4.2.5 would apply to more general equations such as

$$
f = e^{t\Delta}\mu + B(f, f), \tag{4.14}
$$

if B is a bi-linear operator, preserving the axisymmetric structure[2], and if a contraction argument is applicable to the system (4.14) in some time-space Banach space X_T. Indeed, the proof of such result relies on proving that the sequence

$$
\begin{cases}
f_n = e^{t\Delta}\mu + B(f_{n-1}, f_{n-1}), & n \geq 1, \\
f_0 = e^{t\Delta}\mu.
\end{cases}
$$

converges to some axisymmetric limit f in the space X_T. This can be done easily under the aforementioned assumptions on B and on the space X_T.

Lemma 4.2.8 (**Action on test functions**). *Let μ be an axisymmetric measure, then, for any $\varphi \in C^0(\mathbb{R}^3)$, we have that*

$$
\langle \mu, \varphi \rangle = \langle \mu, \varphi_{\mathrm{axi}} \rangle ,
$$

where, φ_{axi} is the axisymmetric part of φ given by

$$
\varphi_{\mathrm{axi}} \triangleq \frac{1}{2\pi} \int_0^{2\pi} \varphi \circ \mathfrak{R}_\alpha d\alpha. \tag{4.15}
$$

[2] In the case of the Boussinesq system, B takes the form $B(\rho, \rho) = \int_0^t e^{(t-s)\Delta}(v \cdot \nabla\rho)(s)ds$, and v is related to ρ through the Navier-Stokes equations.

Proof. The definition of axisymmetric measure and the identity 4.13 allow us to write

$$\langle \mu, \varphi\rangle = \langle \mathfrak{R}_{-\alpha}\mu, \varphi\rangle, \quad \forall \alpha \in [0, 2\pi).$$

Then, straightforward computation yield

$$\begin{aligned}\langle \mu, \varphi\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle \mathfrak{R}_{-\alpha}\mu, \varphi\rangle d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle \mu, \varphi\circ\mathfrak{R}_{\alpha}\rangle d\alpha \\ &= \langle \mu, \frac{1}{2\pi}\int_0^{2\pi} \varphi\circ\mathfrak{R}_{\alpha} d\alpha\rangle \\ &= \langle \mu, \varphi_{\text{axi}}\rangle.\end{aligned}$$

The lemma is then proved. □

Remark 4.2.9. We point out again that the function φ_{axi} is indeed axisymmetric due to the following elementary computation, valid for all $\theta \in [0, 2\pi)$

$$\begin{aligned}\varphi_{\text{axi}}\circ\mathfrak{R}_{\theta} &= \frac{1}{2\pi}\int_0^{2\pi} \varphi\circ\mathfrak{R}_{\alpha+\theta} d\alpha \\ &= \frac{1}{2\pi}\int_{\theta}^{2\pi+\theta} \varphi\circ\mathfrak{R}_{\gamma} d\gamma \\ &= \frac{1}{2\pi}\int_0^{2\pi} \varphi\circ\mathfrak{R}_{\gamma} d\gamma \\ &= \varphi_{\text{axi}}.\end{aligned}$$

Lemma 4.2.8 says then that, when we deal with axisymmetric measures, we can restrict our test functions to be axisymmetric ones.

Now, let us define the function $\mathfrak{F}$ as the mapping from $\Omega\times[0, 2\pi)$ into $\mathbb{R}^3$, with $\Omega = (0, \infty)\times\mathbb{R}$, given by

$$\mathfrak{F}(r, z, \theta) \triangleq \mathfrak{R}_{\theta}\cdot(r, 0, z)^t = (r\cos\theta, r\sin\theta, z). \tag{4.16}$$

The following proposition will serve latter to prove our main Theorem 4.1.1.

Proposition 4.2.10. *Let μ be an axisymmetric measure in $\mathscr{M}(\mathbb{R}^3)$. Then, the mapping $\widetilde{\mu}$ defined on $C^0(\Omega)$ as*

$$\begin{cases} \langle \widetilde{\mu}, \psi\rangle \triangleq \displaystyle\int_{\mathbb{R}^3} \phi_{\psi} d\mu, \qquad \forall \psi \in C^0(\Omega) \\ \phi_{\psi}(x, y, z) \triangleq \psi(\sqrt{x^2+y^2}, z), \end{cases} \tag{4.17}$$

belongs to $\mathscr{M}(\Omega)$ and satisfies, for any function $\varphi \in C^0(\mathbb{R}^3)$ and for all $\theta \in [0, 2\pi)$

$$\int_{\Omega} \varphi_{\text{axi}}\circ\mathfrak{F}(r, z, \theta) d\tilde{\mu}(r, z) = \int_{\Omega} \varphi_{\text{axi}}\circ\mathfrak{F}(r, z, 0) d\tilde{\mu}(r, z) = \int_{\mathbb{R}^3} \varphi d\mu, \tag{4.18}$$

where, φ_{axi} is given by (4.15). Moreover, we have that

$$\|\widetilde{\mu}\|_{\mathscr{M}(\Omega)} = \|\mu\|_{\mathscr{M}(\mathbb{R}^3)}. \tag{4.19}$$

Furthermore, the following holds. If $\mu = \mu_{ac}+\mu_{pp}+\mu_{sc}$ is the Lebesgue decomposition of μ with μ_{ac}, μ_{pp} and μ_{sc} denote, respectively, the absolute continuous part of μ, its atomic and its singular continuous part, then, $\widetilde{\mu} = \widetilde{\mu}_{ac} + \widetilde{\mu}_{pp} + \widetilde{\mu}_{sc}$ with

$$\|\widetilde{\mu}_{ac}\|_{\mathscr{M}(\Omega)} = \|\mu_{ac}\|_{\mathscr{M}(\mathbb{R}^3)},$$

$$\|\widetilde{\mu}_{pp}\|_{\mathscr{M}(\Omega)} = \|\mu_{pp}\|_{\mathscr{M}(\mathbb{R}^3)},$$

$$\|\widetilde{\mu}_{sc}\|_{\mathscr{M}(\Omega)} = \|\mu_{sc}\|_{\mathscr{M}(\mathbb{R}^3)}.$$

Remark 4.2.11. Remark that, in view of Lemma 4.2.8, the equality (4.18) yields, for any axisymmetric function $\varphi \in C^0(\mathbb{R}^3)$ and for all $\theta \in [0, 2\pi)$

$$\int_\Omega \varphi \circ \mathfrak{F}(r,z,\theta) d\tilde{\mu}(r,z) = \int_\Omega \varphi \circ \mathfrak{F}(r,z,0) d\tilde{\mu}(r,z) = \int_{\mathbb{R}^3} \varphi d\mu, \tag{4.20}$$

Proof. From the definition of $\widetilde{\mu}$, we can easily check that it belongs to $\mathscr{M}(\Omega)$. We shall then focus on the proof of (4.18). Remark first that the fact that φ_{axi} is axisymmetric insures

$$\varphi_{\text{axi}} \circ \mathfrak{F}(r,z,\theta) = \varphi_{\text{axi}} \circ \mathfrak{F}(r,z,0), \quad \forall (r,z,\theta) \in \Omega \times [0,2\pi).$$

which is a direct consequence of the fact that

$$\varphi_{\text{axi}}(x,y,z) = \varphi_{\text{axi}}(\sqrt{x^2+y^2}, 0, z) = \varphi_{\text{axi}} \circ \mathfrak{F}(\sqrt{x^2+y^2}, z, 0), \quad \forall (x,y,z) \in \mathbb{R}^3. \tag{4.21}$$

The first equality on the l.h.s of (4.18) follows then. For the second equality, we only have to use the definition of $\widetilde{\mu}$ together with (4.21) to infer that

$$\int_\Omega \varphi_{\text{axi}} \circ \mathfrak{F}(r,z,0) d\tilde{\mu}(r,z) = \int_{\mathbb{R}^3} \varphi_{\text{axi}} d\mu.$$

Consequently, the equality on the r.h.s on (4.18) follows by applying Lemma 4.2.8.

Now, concerning the size of $\widetilde{\mu}$, we only outline the proof of (4.19), meanwhile, the proof of the estimates for the decomposed parts is straightforward (see also the two examples below). From the definition of $\widetilde{\mu}$, it is easy to check that

$$\|\widetilde{\mu}\|_{\mathscr{M}(\Omega)} \leqslant \|\mu\|_{\mathscr{M}(\mathbb{R}^3)}.$$

On the other hand, (4.18) provides the converse inequality

$$\|\mu\|_{\mathscr{M}(\mathbb{R}^3)} \leqslant \|\widetilde{\mu}\|_{\mathscr{M}(\Omega)}.$$

This ends the proof of Proposition 4.2.10 $\square$

For the sake of clarity, we provide the following two typical examples:

- *An example in* $\mathscr{M}_{ac}(\mathbb{R}^3)$: If μ is an axisymmetric measure in $\mathscr{M}_{ac}(\mathbb{R}^3)$, then, its associated density f_μ is an axisymmetric function in $L^1(\mathbb{R}^3)$. In this case, $\widetilde{\mu}$ is the measure in $\mathscr{M}(\Omega)$ associated to the $L^1(\Omega)$-density function $f_{\widetilde{\mu}}$ give by
$$f_{\widetilde{\mu}}(r,z) = 2\pi r\, f_\mu(r,0,z).$$

- *An example in* $\mathscr{M}_{pp}(\mathbb{R}^3)$*:* We saw that $\mu = \delta_A$ is an axisymmetric measure if and only if A is invariant by rotation around the axis (oz) (i.e, A is a circle with axis (oz)). In this case, we have
$$\begin{aligned} \langle \widetilde{\mu}, \psi\rangle &= \langle \delta_A, \phi_\psi\rangle \\ &= \sum_{(a_1,a_2,a_3)\in A} \psi(\sqrt{a_1^2+a_2^2}, a_3) \\ &= \left\langle \delta_{\widetilde{A}}, \psi\right\rangle, \end{aligned}$$
where, $\widetilde{A} = \Big\{(\sqrt{a_1^2+a_2^2}, a_3) : (a_1,a_2,a_3)\in A)\Big\}$. In particular, if $A = (0,0,a)$ then $\widetilde{\mu} = \delta_{(0,a)}$ and more generally, for any $r \geqslant 0$, if $A_r = \cup_{\theta\in[0,2\pi)} \{(r\cos\theta, r\sin\theta, a)\}$, then $\widetilde{\mu} = \delta_{(r,a)}$.

4.2.2 Semi-group estimates

In this subsection, we recall some technical results concerning the semi-groups appearing in the study of the Boussinesq system in question. For more details about these results, we refer the reader to [44, 34, 36].

In the sequel, we shall be using the following notations: For $i \in \{1,2\}$, we denote by $(\mathbb{S}_i(t))_{t\geq 0}$ the semi-groups defined as the propagators associated with the following two linear equations respectively
$$\begin{cases} \partial_t f - \left(\Delta - \frac{1}{r^2}\right) f = 0, \\ f_{|t=0} = f_0. \end{cases} \tag{4.22}$$
$$\begin{cases} \partial_t f - \Delta f = 0, \\ f_{|t=0} = f_0. \end{cases} \tag{4.23}$$

The following proposition states some $L^p - L^q$ estimates in the case of initial data in $\mathcal{M}(\Omega)$. The proof can be found in [36].

Proposition 4.2.12. *For any* $\mu \in \mathscr{M}(\Omega)$*, we have*
$$\sup_{t>0}\, t^{1-\frac{1}{q}} \|\mathbb{S}_1(t)\mu\|_{L^q(\Omega)} \leq C\|\mu\|_{\mathscr{M}(\Omega)}, \quad q\in[1,\infty] \tag{4.24}$$

and

$$L_q(\mu) \triangleq \limsup_{t\uparrow 0}\ t^{1-\frac{1}{q}}\|\mathbb{S}_1(t)\mu\|_{L^q(\Omega)} \le C\|\mu_{pp}\|_{\mathscr{M}(\Omega)}, \quad q\in(1,\infty], \tag{4.25}$$

where, μ_{pp} *is the atomic part of* μ.

Finally, in the spirit of the previous proposition, we state a quite similar estimates for the semi–group $\mathbb{S}_2(t)$ in the space $\mathbb{R}^3$ instead of Ω. The proof is similar to the proof of Proposition 4.2.12

Proposition 4.2.13. *Let* $1\le p\le q\le\infty$. *Assume that* $f\in L^p(\mathbb{R}^3)$, *then*

$$\|\mathbb{S}_2(t)f\|_{L^q(\mathbb{R}^3)} \le \frac{C}{t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}}\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.26}$$

Moreover, if $f=\mu\in\mathscr{M}(\mathbb{R}^3)$, *then the above estimate holds by taking* $p=1$ *and by replacing* $\|f\|_{L^1(\mathbb{R}^3)}$ *by* $\|f\|_{\mathscr{M}(\mathbb{R}^3)}$. *In addition of that, the following assertion holds*

$$\widetilde{L}_q(\mu) \triangleq \limsup_{t\to 0}\ t^{\frac{3}{2}(1-\frac{1}{q})}\|\mathbb{S}_2(t)\mu\|_{L^q(\mathbb{R}^3)} \le C\|\mu_{pp}\|_{\mathscr{M}(\mathbb{R}^3)}, \quad \forall q\neq 1, \tag{4.27}$$

where, μ_{pp} *is the atomic part of* μ.

4.3 Proof of the main results

4.3.1 Median case: Only the initial vorticity is a finite measure

Before stating the proof of the main result, we embark this section by a particular result concerning the global well-posedness topic for (4.6) in the case where the initial density is Lebesgue-integrable and the initial vorticity is a finite measure. The arguments of the proof for this result will be considered as the platform to proving the Theorem 4.1.1.

Theorem 4.3.1. *There exist non negative constants* ε *and* C *such that the following hold. Let* $(\omega_0,\rho_0)\in\mathscr{M}(\Omega)\times L^1(\mathbb{R}^3)$ *with* ρ_0 *axisymmetric and* $\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)}\le\varepsilon$, *then, the Boussinesq system* (4.6) *admits a unique global axisymmetric mild solution* (ω_θ,ρ) *satisfying*

$$(\omega_\theta,\rho)\in C^0\big((0,\infty);L^1(\Omega)\cap L^\infty(\Omega)\big)\times C^0\big([0,\infty);L^1(\mathbb{R}^3)\big)\cap C^0\big((0,\infty);L^\infty(\mathbb{R}^3)\big),$$

$$r\rho\in C^0\big([0,\infty);L^1(\Omega)\big)\cap C^0\big(L^\infty(0,\infty);L^\infty(\Omega)\big).$$

Furthermore, for every $p\in(1,\infty]$, *we have*

$$\limsup_{t\to 0}\ t^{\frac{3}{2}(1-\frac{1}{p})}\|\rho(t)\|_{L^p(\mathbb{R}^3)} = \limsup_{t\to 0}\ t^{1-\frac{1}{p}}\|r\rho(t)\|_{L^p(\Omega)} = 0$$

and

$$\limsup_{t\to 0} t^{1-\frac{1}{p}}\|\omega_\theta(t)\|_{L^p(\Omega)} \leq C\varepsilon.$$

Moreover, we have

$$\limsup_{t\to 0} \|\omega_\theta(t)\|_{L^1(\Omega)} < \infty, \quad \lim_{t\to 0}\|\rho(t)-\rho_0\|_{L^1(\mathbb{R}^3)} = 0$$

and $\omega_\theta(t) \rightharpoonup \omega_0$ *as* $t \to 0$.

The proof of Theorem 4.3.1 will be done in four steps. We begin with the proof of the local well-posedness for the integral equations (4.30) below (Proposition 4.3.2), where, we cover the limits stated in Theorem 4.3.1 for $p = \frac{4}{3}$. Then, we provide a self contained proof of the remaining cases of p by a bootstrap argument (Proposition 4.3.3). Thereafter, we establish the contunuity in time and the convergence to the initial data in Proposition 4.3.5. Finally, the globalization the local solution we construct in step one is postponed to the end of the next section.

All in all, along the proof of the three incoming propositions, we will only highlight the big lines of the proof since the idea is the pretty much similar to the section 3.3. Nevertheless, we shall provide the details of the crucial new technical issues.

First, note that the Boussinesq system (4.6) can be written in the following form

$$\begin{cases} \partial_t\omega_\theta + \mathrm{div}_\star(v\omega_\theta) - \Big(\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r - \frac{1}{r^2}\Big)\omega_\theta = -\partial_r\rho \\ \partial_t\rho + \mathrm{div}(v\rho) - \Delta\rho = 0 \\ (\omega_\theta,\rho)_{|t=0} = (\omega_0,\rho_0). \end{cases} \tag{4.28}$$

Hence, according to the result of the section 3.3 the direct treatment of the local well-posedness topic for (4.28) in the spirit of [36] for initial data (ω_0,ρ_0) in the critical space requires the introduction of a new quantity $\widetilde{\rho} \triangleq r\rho$. The outcome system of this new unknown is given by the following parabolic equation

$$\partial_t\widetilde{\rho} + \mathrm{div}_\star(v\widetilde{\rho}) - \Big(\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r - \frac{1}{r^2}\Big)\widetilde{\rho} = -2\partial_r\rho. \tag{4.29}$$

Thus, we shall consider the following system

$$\begin{cases} \omega_\theta(t) = \mathbb{S}_1(t)\omega_0 - \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau - \int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau \\[2ex] \widetilde{\rho}(t) = \mathbb{S}_1(t)\widetilde{\rho}_0 - \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\widetilde{\rho}(\tau)\big)d\tau - 2\int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau \\[2ex] \rho(t) = \mathbb{S}_2(t)\rho_0 - \int_0^t \mathbb{S}_2(t-\tau)\mathrm{div}\big(v(\tau)\rho(\tau)\big)d\tau. \end{cases} \tag{4.30}$$

where $\widetilde{\rho}_0 = r\rho_0$. In order to study the above system, we use the Banach spaces.

$$X_T = \Big\{ f \in C^0\big((0,T], L^{4/3}(\Omega)\big) : \|f\|_{X_T} < \infty\Big\},$$

$$Z_T = \Big\{ h \in C^0\big((0,T], L^{4/3}(\mathbb{R}^3)\big) : \|h\|_{Z_T} < \infty \Big\},$$

equipped with the following norms

$$\|f\|_{X_T} = \sup_{0<t\leq T} t^{1/4}\|f(t)\|_{L^{4/3}(\Omega)}, \ \|h\|_{Z_T} = \sup_{0<t\leq T} t^{3/8}\|h(t)\|_{L^{4/3}(\mathbb{R}^3)}.$$

The local wellposedness of (4.30) is then given by the following proposition

Proposition 4.3.2. *There exist non negative constants ε and C such that the following hold. Let $(\omega_0, \rho_0) \in \mathscr{M}(\Omega) \times L^1(\mathbb{R}^3)$ with ρ_0 axisymmetric and $\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)} \leq \varepsilon$, then, (4.30) admits a unique local solution $(\omega_\theta, \widetilde{\rho}, \rho)$, defined for all positive $t \leqslant T = T(\omega_0, \rho_0)$ such that*

$$(\omega_\theta, \widetilde{\rho}, \rho) \in C\big((0,T]; X_T\big) \times C\big((0,T]; X_T\big) \times C\big((0,T]; Z_T\big). \tag{4.31}$$

Moreover, if the size of initial data is small enough, the local time of existence T can be taken arbitrarily large.

Proof. We closely follow the demonstration of Proposition 3.3.3 with minor modifications due to the particularity of initial data. In view of Proposition 3.2.11, Proposition 3.2.12 and Proposition 4.2.12, we have for $T > 0$

$$\sup_{0<t\leq T} t^{1/4}\|\mathbb{S}_1(t)\omega_0\|_{L^{\frac{4}{3}}(\Omega)} \leq C\|\omega_0\|_{\mathscr{M}(\Omega)}. \tag{4.32}$$

and

$$\sup_{0<t\leq T} t^{1/4}\|\mathbb{S}_1(t)\widetilde{\rho}_0\|_{L^{\frac{4}{3}}(\Omega)} \leq C\|r\rho_0\|_{L^1(\Omega)} = C\|\rho_0\|_{L^1(\mathbb{R}^3)}. \tag{4.33}$$

On the other hand, the fact that

$$\|\mathbb{S}_2(t)\rho_0\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} = \|r^{\frac{3}{4}}\mathbb{S}_2(t)\rho_0\|_{L^{\frac{4}{3}}(\Omega)}$$

together with the first estimate stated in Proposition 3.2.12, we further get

$$\sup_{0<t\leq T} t^{3/8}\|\mathbb{S}_2(t)\rho_0\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq C\|r\rho_0\|_{L^1(\Omega)} = C\|\rho_0\|_{L^1(\mathbb{R}^3)}. \tag{4.34}$$

Combining (4.32), (4.33) and (4.34) to obtain $\big(\omega_{\text{lin}}, \widetilde{\rho}_{\text{lin}}, \rho_{\text{lin}}\big) \in \mathscr{X}_T$ with

$$(\omega_{\text{lin}}(t), \widetilde{\rho}_{\text{lin}}(t), \rho_{\text{lin}}(t)) = (\mathbb{S}_1(t)\omega_0, \mathbb{S}_1(t)\widetilde{\rho}_0, \mathbb{S}_2(t)\rho_0)$$

and

$$\mathscr{X}_T \triangleq X_T \times X_T \times Z_T$$

Now, from (4.32), (4.33) and (4.34), we have

$$\Lambda(\omega_0, \widetilde{\rho}_0, \rho_0, T) \triangleq C\|(\omega_{\text{lin}}, \widetilde{\rho}_{\text{lin}}, \rho_{\text{lin}})\|_{\mathscr{X}_T} \leq C_0\big(\|\omega_0\|_{\mathscr{M}(\Omega)} + \|\rho_0\|_{L^1(\mathbb{R}^3)}\big). \tag{4.35}$$

Moreover, according to (4.25) and (4.27), we have[3]

$$\limsup_{T\to 0} \Lambda(\omega_0, \widetilde{\rho}_0, \rho_0, T) \le \varepsilon C \tag{4.36}$$

The estimates of the bilinear terms can be done as in the proof of Proposition 3.3.3. Hence, we get the nonlinear system

$$\begin{cases} \|\omega_\theta\|_{X_T} \le \|\omega_{\text{lin}}\|_{X_T} + C\|\omega_\theta\|^2_{X_T} + C\|\rho\|_{Z_T} \\ \\ \|\widetilde{\rho}\|_{X_T} \le \|\widetilde{\rho}_{\text{lin}}\|_{X_T} + C\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T} + C\|\rho\|_{Z_T} \\ \\ \|\rho\|_{Z_T} \le \|\rho_{\text{lin}}\|_{Z_T} + C\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}. \end{cases} \tag{4.37}$$

for some universal constant $C > 0$. By substituting $\|\rho\|_{Z_T}$ in the two first equations of (4.37), we readily get

$$\|\omega_\theta\|_{X_T} + \|\widetilde{\rho}\|_{X_T} \le \Lambda(\omega_0, \widetilde{\rho}_0, \rho_0, T) + \widetilde{C}\big(\|\omega_\theta\|_{X_T} + \|\widetilde{\rho}\|_{X_T}\big)^2. \tag{4.38}$$

To complete the contraction argument, let us fix $R > 0$ such that $2\widetilde{C}R < 1$ and define the ball

$$\mathcal{B}_T(R) \triangleq \{(a,b) \in X_T \times X_T : \|(a,b)\|_{X_T\times X_T} < R\},$$

for $(\omega_\theta, \widetilde{\rho}) \in \mathcal{B}_T(R)$ the contraction argument is satisfied if $\Lambda(\omega_0, \widetilde{\rho}_0, \rho_0, T) \le R/2$. The last requirement can be realized in either case

— $C_0\big(\|\omega_0\|_{\mathscr{M}(\Omega)} + \|\rho_0\|_{L^1(\mathbb{R}^3)}\big) \le R/2$ for any $T > 0$, or

— $C\varepsilon \le R/2$ for $T > 0$ is small enough, depending on $\omega_{0,pp}$ and ρ_0 (this is possible because $\Lambda(\omega_0, \widetilde{\rho}_0, \rho_0, T) \to C\varepsilon$ when $T \to 0$).

In other words, we can prove the global well-posedness if the initial data is sufficiently small, or the local well-posedness if only the atomic part $\omega_{0,pp}$ is small. The rigorous construction of the solution can be done by the standard fixed point schema. This concludes the proof of Proposition 4.3.2. □

Remark that the local solution constructed above becomes instantly integrable after time $t > 0$. Hence, all the a priori estimates proved in first Theorem 3.3.1 remains valid for all $t > 0$. However, for the sake of completeness, we provide in the following proposition the precise statement of more properties of our solution.

Proposition 4.3.3. *Let $(\omega_\theta, \widetilde{\rho}, \rho)$ be the solution of (4.30) obtained by Proposition 4.3.2 associated to initial data $(\omega_0, \widetilde{\rho}_0, \rho_0) \in \mathscr{M}(\Omega) \times L^1(\Omega) \times L^1(\mathbb{R}^3)$. Then for any $p \in (1, \infty]$, we have*

$$\lim_{t\to 0} t^{(1-\frac{1}{p})}\|\omega_\theta(t)\|_{L^p(\Omega)} \le C\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)}, \tag{4.39}$$

[3] Remark that since $\rho_0 \in L^1(\mathbb{R}^3)$, then $\rho_{0,pp} = 0$.

$$\lim_{t\to 0} t^{(1-\frac{1}{p})}\|\widetilde{\rho}(t)\|_{L^p(\Omega)} = 0, \tag{4.40}$$

$$\lim_{t\to 0} t^{\frac{3}{2}(1-\frac{1}{p})}\|\rho(t)\|_{L^p(\mathbb{R}^3)} = 0. \tag{4.41}$$

For $p = 1$, the above quantities are bounded as $t \to 0$.

Remark 4.3.4. As aforementioned, Proposition 4.3.3 can be proved along the same lines as the proof of Proposition 3.3.6. However, we should mention that the case $p = \infty$ is missing in Proposition 3.3.6, therefore, we provide below a complimentary proof that treats this case as well.

Proof. Let us first recall the following notation from Proposition 3.3.6

$$N_p(f,T) \triangleq \sup_{0<t\le T} t^{(1-\frac{1}{p})}\|f\|_{L^p(\Omega)}, \quad J_p(f,T) \triangleq \sup_{0<t\le T} t^{\frac{3}{2}(1-\frac{1}{p})}\|f\|_{L^p(\mathbb{R}^3)}.$$

$$M_p(f_0,T) \triangleq \sup_{0<t\le T} t^{(1-\frac{1}{p})}\|\mathbb{S}_1(t)f_0\|_{L^p(\Omega)}, \quad F_p(f_0,T) \triangleq \sup_{0<t\le T} t^{\frac{3}{2}(1-\frac{1}{p})}\|\mathbb{S}_2(t)f_0\|_{L^p(\mathbb{R}^3)}.$$

According to Proposition 3.2.11, Proposition 3.2.12 and Proposition 4.2.12 we find for all $p \in (1,\infty]$

$$\lim_{t\to 0} M_p(\omega_0,T) \le C\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)} \tag{4.42}$$

and

$$\lim_{t\to 0} M_p(\widetilde{\rho}_0,T) = \lim_{t\to 0} F_p(\rho_0,T) = 0. \tag{4.43}$$

Thanks to the Proposition 4.3.2, the result in the case $p = \frac{4}{3}$ is already proved. By interpolation we find the result for $p \in (1, \frac{4}{3}]$ as long as the $L^1(\Omega) \times L^1(\mathbb{R}^3)-$ norm of $(\omega_\theta(t), \rho(t))$ remains bounded in a neighborhood of $t = 0$. Let us suppose for a moment that this is true and we get back to prove this claim later.

Doing so, it remains to prove the result for $p > \frac{4}{3}$. For this purpose, we can proceed by a bootstrap argument as in the proof of Proposition 4.3.3.

In view of Proposition 3.2.5, Proposition 3.2.11 and Proposition 3.2.12, we have

$$\begin{aligned}\|\omega_\theta(t)\|_{L^p(\Omega)} \le \|\mathbb{S}_1(t)\omega_0\|_{L^p(\Omega)} &+ C\int_0^{\frac{t}{2}} \frac{\|\omega_\theta\|^2_{L^q(\Omega)}}{(t-\tau)^{\frac{2}{q}-\frac{1}{p}}}d\tau + C\int_{\frac{t}{2}}^t \frac{\|\omega_\theta(\tau)\|_{L^{q_1}(\Omega)}\|\omega_\theta(\tau)\|_{L^{q_2}(\Omega)}}{(t-\tau)^{\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{p}}}d\tau \\ &+ C\int_0^{\frac{t}{2}} \frac{\|\rho(\tau)\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}}{(t-\tau)^{\frac{13}{8}-\frac{1}{p}}}d\tau + C\int_{\frac{t}{2}}^t \frac{\|\rho(\tau)\|_{L^p(\mathbb{R}^3)}}{(t-\tau)^{\frac{1}{2}+\frac{1}{2p}}}d\tau.\end{aligned}$$

Under the conditions

$$\frac{1}{2} \le \frac{2}{q} - \frac{1}{p}, \quad \frac{1}{2} \le \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{p} < 1, \tag{4.44}$$

Thus, straightforward computation yields

$$N_p(\omega_\theta,T)\leq M_p(\omega_0,T)+C_{p,q}N_q(\omega_\theta,T)^2+C_{q_1,q_2}N_{q_1}(\omega_\theta,T)N_{q_2}(\omega_\theta,T)+C_pJ_{\frac43}(\rho,T)+C_pJ_p(\rho,T). \tag{4.45}$$

Since $\widetilde{\rho}$ satisfies a similar equation to that of ω_θ, we infer that

$$\begin{aligned} N_p(\widetilde{\rho},T)\leq M_p(\widetilde{\rho}_0,T) &+ C_{p,q}N_q(\omega_\theta,T)N_q(\widetilde{\rho},T)\\ &+ C_{q_1,q_2}N_{q_1}(\omega_\theta,T)N_{q_2}(\widetilde{\rho},T)+C_pJ_{\frac43}(\rho,T)+C_pJ_p(\rho,T). \end{aligned} \tag{4.46}$$

Finally, similar arguments yield[4]

$$J_p(\rho,T)\leq F_p(\rho_0,T)+C_pN_{\frac43}(\omega_\theta,T)N_{\frac43}(\widetilde{\rho},T)+C_{q_1,q_2}N_{q_1}(\omega_\theta,T)N_{q_2}(\widetilde{\rho},T), \tag{4.47}$$

for any q_1,q_2 such that

$$\frac{1}{q_1}+\frac{1}{q_2}-\frac{3}{2p}<\frac12. \tag{4.48}$$

Now, by plugging (4.47) in (4.45) and (4.46), we find for $q=\frac43$

$$\begin{aligned} N_p(\omega_\theta,T) \leq\ & C_{p,q_1,q_2}\big(M_p(\omega_0,T)+F_p(\rho_0,T)+N_{\frac43}(\omega_\theta,T)^2+N_{\frac43}(\omega_\theta,T)N_{\frac43}(\widetilde{\rho},T)\\ &+J_{\frac43}(\rho,T)+N_{q_1}(\omega_\theta,T)N_{q_2}(\omega_\theta,T)+N_{q_1}(\omega_\theta,T)N_{q_2}(\widetilde{\rho},T)\big), \end{aligned}$$

and

$$\begin{aligned} N_p(\widetilde{\rho},T)\leq C_{p,q_1,q_2}\big(M_p(\widetilde{\rho}_0,T) &+ F_p(\rho_0,T)+N_{\frac43}(\omega_\theta,T)N_{\frac43}(\widetilde{\rho},T)\\ &+ J_{\frac43}(\rho,T)+N_{q_1}(\omega_\theta,T)N_{q_2}(\widetilde{\rho},T)\big). \end{aligned}$$

Now, to cover all the range $p\in(\frac43,\infty)$, we can proceed by a bootstrap argument as in Proposition 3.3.6. The only difference we should point out here is the fact that $\lim\limits_{T\to0}M_p(\omega_0,T)$ is not necessary zero, but, it satisfies

$$\lim_{T\to0}M_p(\omega_0,T)\leqslant C\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)},\quad \forall p\in(1,\infty].$$

Thus, we obtain

$$\lim_{T\to0}N_p(\omega_\theta,T)\leqslant C\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)},\quad \text{and}\quad \lim_{T\to0}N_p(\widetilde{\rho},T)=0,\quad \text{for all } p\in(1,\infty).$$

Finally, substituting this latest into (4.47), leads to

$$\lim_{T\to0}J_p(\rho,T)=0,\quad \text{for all } p\in(1,\infty).$$

In order to treat the case $p=\infty$, we need to avoid some technical issues arising from the restriction (4.48). To this end, we chose first $q_1=\frac32$, $q_2=4$ and $p=\infty$ in

[4] We refer also to inequality (3.69) for more details.

(4.45) and (4.46). Remark that this choice of (q_1, q_2, p) is admissible by the relation (4.44), hence, we obtain

$$N_\infty(\omega_\theta, T) \leq f_1(T) + CJ_\infty(\rho, T). \tag{4.49}$$

$$N_\infty(\widetilde{\rho}, T) \leq f_2(T) + CJ_\infty(\rho, T), \tag{4.50}$$

with

$$f_1(T) = M_\infty(\omega_0, T) + CN_{\frac{4}{3}}(\omega_\theta, T)^2 + CN_{\frac{3}{2}}(\omega_\theta, T)N_4(\omega_\theta, T) + CJ_{\frac{4}{3}}(\rho, T) \underset{T\to 0}{\leqslant} C\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)},$$

$$f_2(T) = M_\infty(\widetilde{\rho}_0, T) + CN_{\frac{4}{3}}(\omega_\theta, T)N_{\frac{4}{3}}(\widetilde{\rho}, T) + CN_{\frac{3}{2}}(\omega_\theta, T)N_4(\widetilde{\rho}, T) + CJ_{\frac{4}{3}}(\rho, T) \underset{T\to 0}{\longrightarrow} 0\,.$$

Now, we need to deal with $J_\infty(\rho, T)$. By exploring again the properties of the heat semi-group as in the case $p < \infty$, we infer that

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\mathbb{S}_2(t)\rho_0\|_{L^\infty(\mathbb{R}^3)} + C\int_0^{\frac{t}{2}} \frac{\|\omega_\theta\|_{L^{\frac{4}{3}}(\Omega)}\|\widetilde{\rho}\|_{L^{\frac{4}{3}}(\Omega)}}{(t-\tau)^{\frac{1}{2}+\frac{3}{2}}}d\tau + C\int_{\frac{t}{2}}^t \frac{\|v(\tau)\|_{L^\infty(\mathbb{R}^3)}\|\rho(\tau)\|_{L^q(\mathbb{R}^3)}}{(t-\tau)^{\frac{1}{2}+\frac{3}{2q}}}d\tau. \tag{4.51}$$

To assert that the last term on the r.h.s above is finite, we need to chose q such that $\frac{1}{2} + \frac{3}{2q} < 1$. A possible choice is then $q = 6$. On the other hand, remark that due to the Biot-Savart law, we have, for some $1 < m < 2 < \ell < \infty$

$$\|v(\tau)\|_{L^\infty(\Omega)} \lesssim \|\omega_\theta(\tau)\|^{\alpha}_{L^m(\Omega)}\|\omega_\theta(\tau)\|^{1-\alpha}_{L^\ell(\Omega)}, \qquad \text{for } \alpha = \frac{m}{2}\frac{\ell-2}{\ell-m} \in (0,1).$$

Moreover, since $m, \ell < \infty$, the previous estimates of ω_θ together with a straightforward computation yield

$$\|v(\tau)\|_{L^\infty(\Omega)} \lesssim \tau^{-\frac{1}{2}}. \tag{4.52}$$

Finally, we infer that

$$t^{\frac{3}{2}}\|\rho(t)\|_{L^\infty(\mathbb{R}^3)} \leq t^{\frac{3}{2}}\|\mathbb{S}_2(t)\rho_0\|_{L^\infty(\mathbb{R}^3)} + CN_{\frac{4}{3}}(\omega_\theta, T)N_{\frac{4}{3}}(\widetilde{\rho}, T) + J_6(\rho, T)\sup_{\tau\in(0,T)}\Big(\tau^{\frac{1}{2}}\|v(\tau)\|_{L^\infty(\Omega)}\Big).$$

It is easy then to conclude that

$$\lim_{T\to 0} J_\infty(\rho, T) = 0,$$

and, therefore, we get from (4.49) and (4.50)

$$\lim_{T\to 0} N_\infty(\omega_\theta, T) \leqslant C\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)}, \quad \text{and} \quad \lim_{T\to 0} N_\infty(\widetilde{\rho}, T) = 0.$$

This ends the proof of Proposition 4.3.3 provided that we prove the following claim

$$\|(\omega_\theta(t), \widetilde{\rho}(t), \rho(t))\|_{L^1(\Omega)\times L^1(\Omega)\times L^1(\mathbb{R}^3)} \lesssim \|(\omega_0, \rho_0)\|_{\mathscr{M}(\Omega)\times L^1(\mathbb{R}^3)}\,.$$

From the definition of $\widetilde{\Gamma}$ and the fact $\widetilde{\rho} = r\rho$, the above claim is equivalent to

$$\|(\widetilde{\Gamma}(t), \rho(t))\|_{L^1(\Omega)\times L^1(\mathbb{R}^3)} \lesssim \|(\omega_0, \rho_0)\|_{\mathscr{M}(\Omega)\times L^1(\mathbb{R}^3)}\,. \tag{4.53}$$

Let us then prove that (4.53). We will restrict ourselves to the estimates of the nonlinear terms since the linear parts can be treated by applying the properties of semi-groups recalled in the previous section. So, according to the equations of $\widetilde{\Gamma}$ and ρ, we must show that

$$\int_0^t \left\|\mathbb{S}_1(t-\tau)\mathrm{div}_\star(v\widetilde{\Gamma})(\tau)\right\|_{L^1(\Omega)} d\tau \lesssim \|(\omega_0,\rho_0)\|_{\mathscr{M}(\Omega)\times L^1(\mathbb{R}^3)} \tag{4.54}$$

and

$$\int_0^t \|\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\mathbb{R}^3)} d\tau \lesssim \|(\omega_0,\rho_0)\|_{\mathscr{M}(\Omega)\times L^1(\mathbb{R}^3)}\,. \tag{4.55}$$

For (4.54), Hölder's inequality, Biot Savart law and the definition of the space X_T lead to

$$\begin{aligned}\int_0^t \left\|\mathbb{S}_1(t-\tau)\mathrm{div}_\star(v\widetilde{\Gamma})(\tau)\right\|_{L^1(\Omega)} d\tau &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\|v(\tau)\|_{L^4(\Omega)}\left\|\widetilde{\Gamma}(\tau)\right\|_{L^{\frac{4}{3}}(\Omega)} d\tau\\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\|\omega_\theta(\tau)\|_{L^{\frac{4}{3}}(\Omega)}\|\widetilde{\Gamma}(\tau)\|_{L^{\frac{4}{3}}(\Omega)}d\tau\\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}\tau^{\frac{1}{2}}}d\tau\,\|\omega_\theta\|_{X_T}\|\widetilde{\Gamma}\|_{X_T}\\ &\lesssim \|\omega_\theta\|_{X_T}\|\widetilde{\Gamma}\|_{X_T}.\end{aligned}$$

To treat (4.55), we use the fact that

$$\|\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\mathbb{R}^3)} = \|r\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\Omega)}\,,$$

then we use first Proposition 3.2.12 to infer that

$$\int_0^t \|r\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\Omega)} d\tau \lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\|vr\rho(\tau)\|_{L^1(\Omega)} d\tau.$$

Therefore, the identity $\widetilde{\rho} = r\rho$, Hölder's inequality and the Biot–Savart law yield

$$\begin{aligned}\|\mathbb{S}_2(t-\tau)\mathrm{div}(v\rho)(\tau)\|_{L^1(\mathbb{R}^3)} &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\|v(\tau)\|_{L^4(\Omega)}\|\widetilde{\rho}(\tau)\|_{L^{\frac{4}{3}}(\Omega)} d\tau\\ &\lesssim \|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}.\end{aligned}$$

Hence, the estimates in X_T lead to (4.54) and (4.55), thereafter, (4.53) follows. □

To complete the proof of the Theorem 4.3.1, it remains to outline the proof of the continuity of the solution and the convergence to the initial data. Precisely, we shall prove the following

Proposition 4.3.5. *Let (ω_0,ρ_0) be the initial data to system (4.6) that satisfies the assumptions of Theorem 4.3.1. Let $(\omega_\theta, r\rho,\rho)$ be the local solution given by the fixed point argument such that*

$$(\omega_\theta, r\rho) \in \Big(L^\infty\big((0,T);L^1(\Omega)\cap L^\infty(\Omega)\big)\Big)\times\Big(L^\infty\big([0,T);L^1(\Omega)\big)\cap L^\infty\big((0,T);L^\infty(\Omega)\big)\Big),$$

$$\rho \in L^\infty\big([0,T);L^1(\mathbb{R}^3)\big) \cap L^\infty\big((0,T);L^\infty(\mathbb{R}^3)\big).$$

Then

$$(\omega_\theta, r\rho) \in \Big(C^0\big((0,T);L^1(\Omega)\cap L^\infty(\Omega)\big)\Big) \times \Big(C^0\big([0,T);L^1(\Omega)\big)\cap C^0\big((0,T);L^\infty(\Omega)\big)\Big), \tag{4.56}$$

$$\rho \in C^0\big([0,T);L^1(\mathbb{R}^3)\big) \cap C^0\big((0,T);L^\infty(\mathbb{R}^3)\big). \tag{4.57}$$

Moreover, we have the following convergence to the initial

$$\omega_\theta(t) \rightharpoonup \omega_0, \qquad \text{as } t \to 0 \tag{4.58}$$

and

$$\lim_{t\to 0} \|\rho(t) - \rho_0\|_{L^1(\mathbb{R}^3)} = 0. \tag{4.59}$$

Proof. Assertions (4.56) and (4.57) concern the continuity of the solution away from 0, this can be done by the same way as in previous Subsection 3.3.2, since the solution satisfies, for all $t_0 \in (0,T]$

$$\big(\omega_\theta(t_0), \widetilde{\rho}(t_0), \rho(t_0)\big) \in L^p(\Omega)\times L^p(\Omega)\times L^p(\mathbb{R}^3), \quad \forall p \in [1,\infty].$$

Let us now investigate the convergence to the initial data (4.58) and (4.59). It should be noted that the major difficulty in this part is the weak convergence of the vorticity towards the initial datum. We begin with the proof of the limit (4.59) which does not differ a lot from the proof given in Subsection 3.3.2.

Indeed, (4.59) is an easy consequence of

$$\limsup_{t\to 0} \|\rho(t) - \mathbb{S}_2(t)\rho_0\|_{L^1(\mathbb{R}^3)} = 0 \tag{4.60}$$

Hence, we should focus on the proof of (4.60), for $t > 0$. By using Proposition 3.2.12 for $\alpha = \beta = 1$, we get

$$\begin{aligned}
\|\rho(t) - \mathbb{S}_2(t)\rho_0\|_{L^1(\mathbb{R}^3)} &\le \int_0^t \|r\mathbb{S}_2(t)\mathrm{div}(v(\tau)\rho(\tau))\|_{L^1(\Omega)}d\tau \\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac12}}\|v(\tau)r\rho(\tau))\|_{L^1(\Omega)}d\tau \\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac12}}\|v(\tau)\|_{L^4(\Omega)}\|r\rho(\tau))\|_{L^{\frac43}(\Omega)}d\tau.
\end{aligned}$$

Then, Biot–Savart's law yields,

$$\begin{aligned}
\|\rho(t) - \mathbb{S}_2(t)\rho_0\|_{L^1(\mathbb{R}^3)} &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac12}}\|\omega_\theta(\tau)\|_{L^{\frac43}(\Omega)}\|\widetilde{\rho}(\tau))\|_{L^{\frac43}(\Omega)}d\tau \\
&\lesssim \|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_t}\int_0^t \frac{1}{(t-\tau)^{\frac12}\tau^{\frac12}} \\
&\le C\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_t}.
\end{aligned}$$

Thus (4.60) follows from the fact that $\lim_{t\to 0} \|\widetilde{\rho}\|_{X_t} = 0$.
We turn now to prove (4.58) and we follow the idea in [36]. We begin by proving the following claim

$$\limsup_{t\to 0} \|\omega_\theta(t) - \mathbb{S}_1(t)\omega_0\|_{L^1(\Omega)} = 0 \tag{4.61}$$

As mentioned earlier, the linear term of $\partial_r\rho$ is an obstacle in the L^1-estimate of ω_θ. To avoid the estimation of this term, we use the coupling $\widetilde{\Gamma} = \omega_\theta - \frac{\widetilde{\rho}}{2}$. First, note that (4.59) gives

$$\limsup_{t\to 0} \|\widetilde{\rho}(t) - \mathbb{S}_1(t)\widetilde{\rho}_0\|_{L^1(\Omega)} = 0. \tag{4.62}$$

Thus (4.61) is equivalent to the following assertion

$$\limsup_{t\to 0} \|\widetilde{\Gamma}(t) - \mathbb{S}_1(t)\widetilde{\Gamma}_0\|_{L^1(\Omega)} = 0, \tag{4.63}$$

where, $\widetilde{\Gamma}_0 = \omega_0 - \frac{\widetilde{\rho}_0}{2} \in \mathscr{M}(\Omega)$. Let us define the functional $\mathcal{F}$ by

$$(\mathcal{F}g)(t) \triangleq \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star(v(\tau)g(\tau))d\tau.$$

We emphasize that the following estimate holds true, for any $g \in X_T$

$$\|\mathcal{F}g(t)\|_{L^1(\Omega)} + \|\mathcal{F}g\|_{X_t} \leqslant \widetilde{C}\|\omega_\theta\|_{X_t}\|g\|_{X_t}, \quad \forall t \leqslant T. \tag{4.64}$$

The proof of (4.64) can be done by using the estimates of subsection 4.2.2. On the other hand, from the equation of $\widetilde{\Gamma}$, we have

$$\widetilde{\Gamma} - \widetilde{\Gamma}_{\mathrm{lin}} = \big(\mathcal{F}(\widetilde{\Gamma}_{\mathrm{lin}}) - \mathcal{F}(\widetilde{\Gamma})\big) - \mathcal{F}(\widetilde{\Gamma}_{\mathrm{lin}}) \tag{4.65}$$

where $\widetilde{\Gamma}_{\mathrm{lin}}(\cdot) = \mathbb{S}_1(\cdot)\widetilde{\Gamma}_0$. Let R be the radius of the ball in which we applied the fixed-point argument to construct the local solution[5]. Hence, we have

$$\|\widetilde{\rho}_{\mathrm{lin}}\|_{X_T} + \|\omega_{\mathrm{lin}}\|_{X_T} + \|\widetilde{\rho}\|_{X_T} + \|\omega_\theta\|_{X_T} \leq 2R$$

Let us also define the two quantities δ and $\ell_p(\omega_0)$ by

$$\delta \triangleq \limsup_{T\to 0} \|\widetilde{\Gamma} - \widetilde{\Gamma}_{\mathrm{lin}}\|_{X_T}$$

and

$$\ell_p(\omega_0) \triangleq \limsup_{t\to 0} t^{1-\frac{1}{p}}\|\mathcal{F}(\omega_{\mathrm{lin}})(t)\|_{L^p(\Omega)}, \quad p \in [1,\infty].$$

Note first that, we have

$$\limsup_{T\to 0} \|\widetilde{\rho} - \widetilde{\rho}_{\mathrm{lin}}\|_{X_T} = \limsup_{T\to 0} \|\mathcal{F}(\rho_{\mathrm{lin}})\|_{X_T} = 0.$$

[5] That is to say, R is such that $2\widetilde{C}R < 1$, where $\widetilde{C} > 0$ is defined by (4.38)

Similarly, by a bootstrap argument (Proposition 4.3.3), we can prove that

$$\limsup_{t\to 0} t^{1-\frac{1}{p}}\|\mathcal{F}(\rho_{\mathrm{lin}})\|_{L^p(\Omega)} = 0, \quad \forall p \in [1,\infty].$$

Thus, by definition of $\widetilde{\Gamma}_{\mathrm{lin}}$ and by linearity of the functional $\mathcal{F}$, we deduce that

$$\limsup_{t\to 0} t^{1-\frac{1}{p}}\|\mathcal{F}(\widetilde{\Gamma}_{\mathrm{lin}})\|_{L^p(\Omega)} = \ell_p(\omega_0), \quad p \in [1,\infty]. \tag{4.66}$$

We resume now from (4.65). Note that (4.64) together with (4.66) yield

$$\begin{aligned}\delta &\le \widetilde{C}\limsup_{t\to 0}\Big(\|\omega_\theta\|_{X_t}\|\widetilde{\Gamma}-\widetilde{\Gamma}_{\mathrm{lin}}\|_{X_t}\Big) + \ell_{\frac{4}{3}}(\omega_0)\\ &\le 2\widetilde{C}R\delta + \ell_{\frac{4}{3}}(\omega_0).\end{aligned}$$

We end up with $\delta = 0$ because, $\ell_p(\omega_0) = 0$, for all $p \in [1,\infty]$ [6], and $2\widetilde{C}R < 1$.

We are now in position to prove (4.63). Again, using (4.65) and (4.64) infer that

$$\limsup_{t\to 0}\|\widetilde{\Gamma}(t)-\widetilde{\Gamma}_{\mathrm{lin}}(t)\|_{L^1(\Omega)} \le C\|\omega_\theta\|_{X_T}\limsup_{t\to 0}\|\widetilde{\Gamma}-\widetilde{\Gamma}_{\mathrm{lin}}\|_{X_t} + \ell_1(\omega_0) = 0,$$

where, we have used the fact that $\delta = \ell_1(\omega_0) = 0$. Consequently, we obtain, in view of (4.62)

$$\limsup_{t\to 0}\|\omega_\theta(t)-\omega_{\mathrm{lin}}(t)\|_{L^1(\Omega)} = 0.$$

This ends the hard part of the proof. To prove the weak limit towards to initial vorticity, we only need to use the fact that[7]

$$\omega_{\mathrm{lin}}(t,r,z) = \frac{1}{4\pi t}\int_\Omega \frac{\tilde{r}^{1/2}}{r^{1/2}}\mathcal{N}_1\Big(\frac{t}{r\tilde{r}}\Big)e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}}\,d\omega_0(\tilde{r},\tilde{z}).$$

From the above formula, we can check then that $\omega_{\mathrm{lin}} \rightharpoonup \omega_0$ and finally (4.58) follows. The proof of the proposition is then achieved. □

Remark 4.3.6. We should point out that, the propositions we proved in this subsection do not say anything about the global well-posedness, it is all about the local theory. However, one can prove that the local solution we construct in Proposition 4.3.2 can be in fact extended to be global in time. We postpone the details of that to the the last subsection of this chapter.

[6] See the last part in [36, Section 4] and [34, Section 2.3.4] for a detailed proof of the fact $\ell_p(\omega_0) = 0$.
[7] See Section 3.2.2 .

4.3.2 Proof of Theorem 4.1.1: All initial data are finite measures

In this subsection, we shall outline the proof of Theorem 4.1.1. More precisely, we will focus on the local well-posedness matter then we give the details for the proof of the global estimates at the end of this section.

As we pointed out before, the most challenging part is how to give a rigorous and suitable sense to the initial data of the quantity $r\rho$ if the initial density ρ_0 is only a finite measure. Also, it is important to make a choice that does not perturb the continuity of the solution near $t = 0$[8]. In the case where ρ_0 is an axisymmetric function in $L^1(\mathbb{R}^3)$, we saw in the proof of Theorem 4.3.1 that $\widetilde{\rho}_0 = r\rho_0 \in L^1(\Omega)$ with

$$\|\widetilde{\rho}_0\|_{L^1(\Omega)} = \frac{1}{2\pi}\|\rho_0\|_{L^1(\mathbb{R}^3)}.$$

Hence, the general case where ρ_0 is axisymmetric measure should fulfill this properties as well. More precisely, if ρ_0 is a finite axisymmetric measure in $\mathscr{M}(\mathbb{R}^3)$, then, we should look for a measure $\widetilde{\rho}_0$ in $\mathscr{M}(\Omega)$ that satisfies

$$\|\widetilde{\rho}_0\|_{\mathscr{M}(\Omega)} = \frac{1}{2\pi}\|\rho\|_{\mathscr{M}(\mathbb{R}^3)}. \tag{4.67}$$

The best candidate is then inspired by Proposition 4.2.10. More precisely, we shall define $\widetilde{\rho}_0$ as

$$\begin{cases} \langle \widetilde{\rho}_0, \psi\rangle \triangleq \frac{1}{2\pi}\displaystyle\int_{\mathbb{R}^3} \phi_\psi d\rho_0, \qquad \forall \psi \in C^0(\Omega) \\ \\ \phi_\psi(x,y,z) \triangleq \psi(\sqrt{x^2+y^2}, z), \end{cases} \tag{4.68}$$

The factor $\frac{1}{2\pi}$ is added for a compatibility reason[9] and all the results of Proposition 4.2.10 and the remark thereafter hold modulo that factor.

In the sequel, we denote by $C^\infty_{c,axi}(\mathbb{R}^3)$ the space of axisymmetric functions φ belonging to $C^\infty_c(\mathbb{R}^3)$ and satisfying the following boundary conditions

$$\varphi|_{r=0} = \partial_r\varphi|_{r=0} = 0.$$

For such test function φ, we also adopt the identification $\varphi \circ \mathfrak{F} \approx \psi$, where $\mathfrak{F}$ is defined by (4.16). Moreover, for simplicity we write

$$\langle f, \psi\rangle_\Omega$$

instead of

$$\langle f, \varphi\circ\mathfrak{F}\rangle_\Omega\,,$$

[8] Or at least the weak continuity near $t = 0$ as we will see later on.
[9] see identity (4.90) which is why we should define $\widetilde{\rho}_0$ by (4.68).

for any distribution f on Ω. Let us consider μ to be any measure in $\mathcal{M}(\Omega)$ and we set the goal of this part to the understanding of the following integral system

$$\begin{cases} \omega_\theta(t) = \mathbb{S}_1(t)\omega_0 - \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\omega_\theta(\tau)\big)d\tau - \int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau, \\ \widetilde{\rho}(t) = \mathbb{S}_1(t)\mu - \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star\big(v(\tau)\widetilde{\rho}(\tau)\big)d\tau - 2\int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau, \\ \rho(t) = \mathbb{S}_2(t)\rho_0 - \int_0^t \mathbb{S}_2(t-\tau)\mathrm{div}\big(v(\tau)\frac{\widetilde{\rho}}{r}(\tau)\big)d\tau. \end{cases} \tag{4.69}$$

Above, for $t>0$ and $(r,z) = (\sqrt{x^2+y^2}, z)$, we consider the identification of axisymmetric functions

$$\omega_\theta = \omega_\theta(t,x,y,z) = \omega_\theta(t,r,z), \quad \rho = \rho(t,x,y,z) = \rho(t,r,z)$$

and $\widetilde{\rho}$ is, for now, an unknown function of the form $\widetilde{\rho} = \widetilde{\rho}(t,r,z)$. Remark that the system (4.69) is equivalent to ($\mathrm{B}_{\mu,\kappa}$) if $(\omega_\theta, \widetilde{\rho}, \rho)$ is regular enough and if $\widetilde{\rho} = r\rho$ and $\mu = r\rho_0$, at least for integrable initial density.

The following theorem, which is the main result of this section, is a general version of the local results in Theorem 4.1.1.

Theorem 4.3.7. *Let (ω_0, ρ_0, μ) be in $\mathscr{M}(\Omega)\times\mathscr{M}(\mathbb{R}^3)\times\mathscr{M}(\Omega)$, such that ρ_0 is axisymmetric in the sense of Definition 4.2.3. Then, the following hold*

(i) ***Local well-posedness of*** (4.69)**.** *There exists a non negative constant ε such that, if*
$$\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)} + \|\mu_{pp}\|_{\mathscr{M}(\Omega)} + \|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)} \le \varepsilon, \tag{4.70}$$
then, there exists $T = T(\omega_0,\rho_0,\mu) > 0$ for which (4.69) *has a unique solution, defined on $[0,T]$, and satisfying, for all $p\in[1,\infty]$*
$$\sup_{t\in(0,T]} \Big\{ t^{1-\frac{1}{p}} \|(\omega(t),\widetilde{\rho}(t))\|_{L^p(\Omega)\times L^p(\Omega)} + t^{\frac{3}{2}(1-\frac{1}{p})}\|\rho(t)\|_{L^p(\mathbb{R}^3)} \Big\} \lesssim \|(\omega_0,\mu)\|_{\mathscr{M}(\Omega)\times\mathscr{M}(\Omega)} + \|\rho_0\|_{\mathscr{M}(\mathbb{R}^3)}. \tag{4.71}$$

(ii) ***Weak convergence to the initial data.*** *For all $\varphi\in C^\infty_{c,Axi}(\mathbb{R}^3)$, we have*[10]
$$\lim_{t\to0}\langle\omega_\theta(t)|\psi\rangle_\Omega = \langle\omega_0|\psi\rangle_\Omega \tag{4.72}$$
$$\lim_{t\to0}\langle\widetilde{\rho}(t)|\psi\rangle_\Omega = \langle\mu|\psi\rangle_\Omega\,, \tag{4.73}$$
$$\lim_{t\to0}\langle\rho(t)|\varphi\rangle_{\mathbb{R}^3} = \langle\rho_0|\varphi\rangle_{\mathbb{R}^3}\,. \tag{4.74}$$

(iii) ***Local well-posedness of the Boussinesq system*** ($\mathrm{B}_{\mu,\kappa}$)**.** *Moreover, if $\mu = \widetilde{\rho}_0$ is given by* (4.68)*, then the condition on the size of the initial data* (4.70) *can be replaced by*
$$\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)} + \|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)} \le \widetilde{\varepsilon}, \tag{4.75}$$

[10] We recall that we are using the identification $\psi\approx\varphi\circ\mathfrak{F}$.

for some $\widetilde{\varepsilon} > 0$. *Also, we have*

$$\lim_{t\to 0}\langle\widetilde{\rho}(t)|\psi\rangle_\Omega = \lim_{t\to 0}\langle r\rho(t)|\psi\rangle_\Omega = \langle\widetilde{\rho}_0|\psi\rangle_\Omega\,, \quad \forall\psi\in C_c^\infty(\Omega),$$

$$\widetilde{\rho}(t) = r\rho(t), \quad \forall t>0$$

and (ω_θ,ρ) *is actually the unique solution of the Boussinesq system* $(\mathrm{B}_{\mu,\kappa})$ *on* $[0,T]$.

Proof. We prove the results of the above theorem in the order given in its statement

• **Proof of (i): Local well-posedness of system** (4.69)**.**

We have to prove the existence of some $T>0$, and a unique solution $(\omega_\theta,\widetilde{\rho},\rho)\in X_T\times X_T\times Z_T$ to (4.69). This can be done by a fixed point argument, more precisely, by following exactly the same idea explored in the proof of Theorem 4.3.1. To do so, the free part $(\mathbb{S}_1(t)\omega_0,\mathbb{S}_1(t)\mu,\mathbb{S}_2(t)\rho_0)$ has to be small enough in $X_T\times X_T\times Z_T$, as T is close to zero and the nonlinear parts have to be estimated by using the properties of the semi-groups stated in the subsection 4.2.2. Indeed, by employing the results of the subsection 4.2.2 we can get the same estimates obtained in the proof of Proposition 4.3.2

$$\begin{cases} \|\omega_\theta\|_{X_T}\le\|\mathbb{S}_1(\cdot)\omega_0\|_{X_T}+C\|\omega_\theta\|_{X_T}^2+C\|\rho\|_{Z_T}\\[2mm] \|\widetilde{\rho}\|_{X_T}\le\|\mathbb{S}_1(\cdot)\mu\|_{X_T}+C\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}+C\|\rho\|_{Z_T}\\[2mm] \|\rho\|_{Z_T}\le\|\mathbb{S}_2(.)\rho_0\|_{Z_T}+C\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}, \end{cases} \tag{4.76}$$

for some universal constant $C>0$. The system (4.76) yields then to the following estimate, up to a suitable modification in C

$$A_T \triangleq \|\omega_\theta\|_{X_T}+\|\widetilde{\rho}\|_{X_T}\le A_{0,T}+CA_T^2, \tag{4.77}$$

where, $A_{0,T}$ is given by

$$A_{0,T}\overset{def}{=}\|\mathbb{S}_1(\cdot)\omega_0\|_{X_T}+\|\mathbb{S}_1(\cdot)\mu\|_{X_T}+C\|\mathbb{S}_2(.)\rho_0\|_{Z_T}. \tag{4.78}$$

The local well-posedness follows then by usual arguments if $\lim\limits_{T\to 0}A_{0,T}$ is small enough.

Now, in order to measure the size of $A_{0,T}$ for small T, we use Proposition 4.2.12 and Proposition 4.2.13 to get

$$\lim_{T\to 0}A_{0,T}\le\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)}+\|\mu_{pp}\|_{\mathscr{M}(\Omega)}+C\,\|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)}\,, \tag{4.79}$$

which gives, for some $\widetilde{C}>0$

$$\lim_{T\to 0}A_{0,T}\le\widetilde{C}\big(\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)}+\|\mu_{pp}\|_{\mathscr{M}(\Omega)}+\|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)}\big). \tag{4.80}$$

Thus, if the r.h.s. of the last inequality above is small enough, then the fixed point argument guarantees the local well-posedness of (4.69). That is to say, there exist $\varepsilon>0$, such that if

$$\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)}+\|\mu_{pp}\|_{\mathscr{M}(\Omega)}+\|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)}\le\varepsilon, \tag{4.81}$$

then there exists $T > 0$ for which (4.69) has a unique solution $(\omega_\theta, \widetilde{\rho}, \rho)$ in $X_T \times X_T \times Z_T$.
Remark that the fixed point argument gives in particular the estimate (4.71) for $p = \frac{4}{3}$. The proof of estimate (4.71) for all $p \in [1, \infty]$ can be done by a Bootstrap argument. The details of that are exactly the same as in the proof of Proposition 4.3.3. Assertion (i) is then proved.

• Proof of (ii): Weak convergence to the initial data.

Let us introduce the following linear operators

$$\mathcal{F}_1(f)(t) = \int_0^t \mathbb{S}_1(t-\tau)\mathrm{div}_\star(vf)(\tau)d\tau,$$

$$\mathcal{F}_2(g)(t) = \int_0^t \mathbb{S}_2(t-\tau)\mathrm{div}(vg)(\tau)d\tau$$

and

$$\mathcal{G}(\rho)(t) = \int_0^t \mathbb{S}_1(t-\tau)\partial_r\rho(\tau)d\tau,$$

where, v is the velocity associated with the unique solution $(\omega_\theta, \widetilde{\rho}, \rho)$ constructed in the previous step. Hence, our integral system (4.69) can be rewritten as

$$\begin{cases} \omega_\theta(t) = \mathbb{S}_1(t)\omega_0 - \mathcal{F}_1(\omega_\theta)(t) - \mathcal{G}(\rho)(t) \\ \widetilde{\rho}(t) = \mathbb{S}_1(t)\mu - \mathcal{F}_1(\widetilde{\rho})(t) - 2\mathcal{G}(\rho)(t) \\ \rho(t) = \mathbb{S}_2(t)\rho_0 - \mathcal{F}_2(\frac{\widetilde{\rho}}{r})(t), \end{cases} \tag{4.82}$$

First, we point out that, for every $\varphi \in C_c^\infty(\mathbb{R}^3)$ and $\psi \in C_c^\infty(\Omega)$, we have

$$\lim_{t\to 0}\int_\Omega \mathbb{S}_1(t)\omega_0\psi drdz = \int_\Omega \psi(r,z)d(\omega_0(r,z)),$$

$$\lim_{t\to 0}\int_\Omega \mathbb{S}_1(t)\mu\psi drdz = \int_\Omega \psi(r,z)d(\mu(r,z)),$$

$$\lim_{t\to 0}\int_{\mathbb{R}^3} \mathbb{S}_2(t)\rho_0\varphi dx = \int_{\mathbb{R}^3} \varphi(x)d(\rho_0(x)).$$

Thus, in order to prove the convergence to the initial data in (4.82), we need to show that the terms containing the operators $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ tend weakly to zero as t goes to 0. Let us begin by proving that

$$\lim_{t\to 0}\int_{\mathbb{R}^3} \mathcal{F}_2(\frac{\widetilde{\rho}}{r})(t)\varphi(x)dx = 0. \tag{4.83}$$

Remark that the operators div and $\mathbb{S}_2(t)$ commute, whereas an integration by parts followed by the Proposition 3.2.12 and Biot–Savart law yield, in view of the notation

$\widetilde{\nabla\varphi} = (\nabla\varphi)\circ\mathfrak{F}$

$$\Big|\int_{\mathbb{R}^3}\mathcal{F}_2(\frac{\widetilde{\rho}}{r})\varphi(x)dx\Big| \lesssim \int_0^t\int_\Omega \big|r\mathbb{S}_2(t-\tau)(v\frac{\widetilde{\rho}}{r})\cdot\widetilde{\nabla\varphi}(r,z)\big|drdzd\tau$$
$$\lesssim \int_0^t \|\omega_\theta(\tau)\|_{L^{\frac{4}{3}}(\Omega)}\|\widetilde{\rho}(\tau)\|_{L^{\frac{4}{3}}(\Omega)}d\tau\|\widetilde{\nabla\varphi}\|_{L^\infty(\Omega)}$$
$$\lesssim \int_0^t \frac{d\tau}{\tau^{\frac{1}{2}}}\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}\|\widetilde{\nabla\varphi}\|_{L^\infty(\Omega)}$$
$$\lesssim t^{\frac{1}{2}}\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}\|\widetilde{\nabla\varphi}\|_{L^\infty(\Omega)},$$

This is enough to guarantee (4.83).

For the rest of the limits, we will restrict our selves to the ones appearing in the equation of $\widetilde{\rho}$ due to the similarity of the equations of $\widetilde{\rho}$ and ω_θ. Let us point out first that the operators ∂_r and $\mathbb{S}_1(t)$ do not commute. To overcome this issue, let us rewrite the equation of $\widetilde{\rho}$ in terms of $\mathbb{S}_2$. To do so, owing to the fact that

$$\mathrm{div}_\star(v\widetilde{\rho}) = \mathrm{div}(v\widetilde{\rho}) - \frac{v^r}{r}\widetilde{\rho},$$

then the equation of $\widetilde{\rho}$, given by

$$\partial_t\widetilde{\rho} - \Delta\widetilde{\rho} + \mathrm{div}_\star(v\widetilde{\rho}) + \frac{\widetilde{\rho}}{r^2} = -2\partial_r\rho,$$

can be written in the integral form as

$$\widetilde{\rho}(t) = \mathbb{S}_2(t)\mu - \int_0^t \mathbb{S}_2(t-\tau)\mathrm{div}(v\widetilde{\rho})d\tau + \int_0^t \mathbb{S}_2(t-\tau)\big(\frac{v^r}{r}\widetilde{\rho}\big)d\tau - \int_0^t \mathbb{S}_2(t-\tau)\frac{\widetilde{\rho}}{r^2}d\tau$$
$$-2\int_0^t \mathbb{S}_2(t-\tau)\partial_r\rho d\tau. \tag{4.84}$$

Except of the first term on the r.h.s above, all the rest of terms should go to 0 (in distributional sense) in order to reach our claim. Indeed, by using the fact that the operator div commutes with $\mathbb{S}_2(t)$, Proposition 3.2.12 and the Biot-Savart law, we obtain

$$\Big|\int_0^t\int_\Omega \mathbb{S}_2(t-\tau)\mathrm{div}(v\widetilde{\rho})\psi(r,z)drdzd\tau\Big| \lesssim \int_0^t\int_\Omega \big|\mathbb{S}_2(t-\tau)(v\widetilde{\rho})\cdot\nabla\psi(r,z)\big|drdzd\tau$$
$$\lesssim \int_0^t \|\omega_\theta(\tau)\|_{L^{\frac{4}{3}}(\Omega)}\|\widetilde{\rho}(\tau)\|_{L^{\frac{4}{3}}(\Omega)}d\tau\|\nabla\psi\|_{L^\infty(\Omega)}.$$

We continue as in the proof of (4.83) to obtain

$$\Big|\int_0^t\int_\Omega \mathbb{S}_2(t-\tau)\mathrm{div}(v\widetilde{\rho})\psi(r,z)drdzd\tau\Big| \lesssim t^{\frac{1}{2}}\|\omega_\theta\|_{X_T}\|\widetilde{\rho}\|_{X_T}\|\nabla\psi\|_{L^\infty(\Omega)},$$

which tends to 0 as t goes to 0. For the 3^{rd} term on the r.h.s of (4.84), we proceed by using again Proposition 3.2.12, and the Biot–Savart law to infer that

$$\Big|\int_0^t\int_\Omega \mathbb{S}_2(t-\tau)\big(\frac{v^r}{r}\widetilde{\rho}\big)\psi(r,z)drdzd\tau\Big| = \Big|\int_0^t\int_\Omega r\mathbb{S}_2(t-\tau)\big(\frac{v^r}{r}\widetilde{\rho}\big)\frac{\psi(r,z)}{r}drdzd\tau\Big|$$
$$\lesssim \int_0^t \|\omega_\theta(\tau)\|_{L^{\frac{4}{3}}(\Omega)}\|\widetilde{\rho}(\tau)\|_{L^{\frac{4}{3}}(\Omega)}d\tau\|\frac{\psi}{r}\|_{L^\infty(\Omega)}.$$

Again, we continue as in the proof of (4.83) to get

$$\Big| \int_0^t \int_\Omega \mathbb{S}_2(t-\tau)\big(\frac{v^r}{r}\widetilde{\rho}\big)\psi(r,z)drdzd\tau \Big| \lesssim t^{\frac{1}{2}} \|\omega_\theta\|_{X_T} \|\widetilde{\rho}\|_{X_T} \|\frac{\psi}{r}\|_{L^\infty(\Omega)},$$

which tends to 0 as t goes to 0. For the 4^{th} term on the r.h.s of (4.84), we use again Proposition 3.2.12 and the Biot–Savart law to find that

$$\begin{aligned}
\Big| \int_0^t \int_\Omega \mathbb{S}_2(t-\tau)\frac{\widetilde{\rho}}{r^2}\psi(r,z)drdzd\tau \Big| &= \Big| \int_0^t \int_\Omega r^2\mathbb{S}_2(t-\tau)\frac{\widetilde{\rho}}{r^2}\frac{\psi(r,z)}{r^2}drdzd\tau \Big| \\
&\lesssim \int_0^t \|\widetilde{\rho}(\tau)\|_{L^{\frac{4}{3}}(\Omega)} d\tau \|\frac{\psi}{r^2}\|_{L^4(\Omega)} \\
&\lesssim \int_0^t \frac{d\tau}{\tau^{\frac{1}{4}}} \|\widetilde{\rho}\|_{X_T} \|\frac{\psi}{r^2}\|_{L^4(\Omega)} \\
&\lesssim t^{\frac{3}{4}} \|\widetilde{\rho}\|_{X_T} \|\frac{\psi}{r^2}\|_{L^4(\Omega)},
\end{aligned}$$

which tends to 0 as t goes to 0. Finally, for the last term in (4.84), implementing again Proposition 3.2.12 and Biot–Savart law yield

$$\begin{aligned}
\Big| \int_0^t \int_\Omega \mathbb{S}_2(t-\tau)\partial_r\rho\psi(r,z)drdzd\tau \Big| &= \Big| \int_0^t \int_\Omega r^{\frac{3}{4}}\mathbb{S}_2(t-\tau)\partial_r\rho\frac{\psi(r,z)}{r^{\frac{3}{4}}}drdzd\tau \Big| \\
&\lesssim \int_0^t \frac{1}{\tau^{\frac{1}{2}}} \|r^{\frac{3}{4}}\rho(\tau)\|_{L^{\frac{4}{3}}(\Omega)} d\tau \|\frac{\psi}{r^{\frac{3}{4}}}\|_{L^4(\Omega)} \\
&\lesssim \int_0^t \frac{1}{\tau^{\frac{1}{2}}} \|\rho(\tau)\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} d\tau \|\frac{\psi}{r^{\frac{3}{4}}}\|_{L^4(\Omega)} \\
&\lesssim \int_0^t \frac{d\tau}{\tau^{\frac{7}{8}}} \|\rho\|_{Z_T} \|\frac{\psi}{r^{\frac{3}{4}}}\|_{L^4(\Omega)} \\
&\lesssim t^{\frac{1}{8}} \|\rho\|_{Z_T} \|\frac{\psi}{r^{\frac{3}{4}}}\|_{L^4(\Omega)},
\end{aligned}$$

which tends to 0 as t goes to 0. All in all, we deduce that $\widetilde{\rho}(t)$ tends to μ (in distributional sense) as t goes to 0. Similar arguments can be used to prove that $\omega_\theta(t)$ tends to ω_0 when t goes to 0.

Remark 4.3.8. We should point out that the computations of this step hold true whenever the test function ψ is in $C^1(\Omega)$ such that

$$\|\nabla\psi\|_{L^\infty(\Omega)} + \|\frac{\psi}{r}\|_{L^\infty(\Omega)} + \|\frac{\psi}{r}\|_{L^4(\Omega)} + \|\frac{\psi}{r^2}\|_{L^4(\Omega)} < \infty. \tag{4.85}$$

Such a condition is automatically satisfied if we take $\psi = \varphi \circ \mathfrak{F}$, for any $\varphi \in C^\infty_{c,Axi}(\mathbb{R}^3)$. Indeed, the boundary conditions on the test function belonging to $C^\infty_{c,Axi}(\mathbb{R}^3)$, together with Taylor expansion near $r = 0$ would clearly imply (4.85).

• **Proof of (iii): Local well-posedness of the Boussinesq system ($\mu = \widetilde{\rho}_0$).** Now, we assume that μ and ρ_0 are connected by the formula (4.68). That is, we set

$\mu = \widetilde{\rho}_0$, where

$$\begin{cases} \langle \widetilde{\rho}_0, \psi \rangle \triangleq \frac{1}{2\pi} \displaystyle\int_{\mathbb{R}^3} \phi_\psi d\rho_0, \quad \forall \psi \in C^0(\Omega), \\ \\ \phi_\psi(x,y,z) \triangleq \psi(\sqrt{x^2+y^2}, z). \end{cases} \tag{4.86}$$

Proposition 4.2.10 and the remark thereafter yield, for all $\varphi \in C^\infty_{c,Axi}(\mathbb{R}^3)$

$$\frac{1}{2\pi} \langle \rho_0 | \varphi \rangle_{\mathbb{R}^3} = \langle \widetilde{\rho}_0 | \varphi \circ \mathfrak{F} \rangle_\Omega = \langle \widetilde{\rho}_0 | \psi \rangle_\Omega\,, \tag{4.87}$$

and

$$\|\widetilde{\rho}_{0,pp}\|_{\mathscr{M}(\Omega)} = \frac{1}{2\pi} \|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)}\,.$$

This estimate on the size of $\|\widetilde{\rho}_{0,pp}\|_{\mathscr{M}(\Omega)}$ can be used then in (4.80) to obtain, for some $\widetilde{C}_0 > 0$

$$\lim_{T\to 0} A_{0,T} \le \widetilde{C}_0\big(\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)} + \|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)} \big).$$

Hence, it is obvious that, up to a modification in ε given by (4.81), then (4.81) can be replaced by

$$\|\omega_{0,pp}\|_{\mathscr{M}(\Omega)} + \|\rho_{0,pp}\|_{\mathscr{M}(\mathbb{R}^3)} \le \widetilde{\varepsilon}. \tag{4.88}$$

The local well-posedness of (4.69) is then guaranteed as long as (4.88) is satisfied. This ends the proof of the first part of (iii). Now, since $\rho(t)$ is axisymmetric, belonging to $L^1(\mathbb{R}^3)$ for all $t > 0$, then $r\rho(t)$ belongs to $L^1(\Omega)$ and a change of variables gives

$$\frac{1}{2\pi} \langle \rho(t) | \varphi \rangle_{\mathbb{R}^3} = \langle r\rho(t) | \varphi \circ \mathfrak{F} \rangle_\Omega\,. \tag{4.89}$$

Hence, the weak limits (as t tends to 0) proved in the previous step, together with (4.87) and (4.89) yield

$$\lim_{t\to 0} \langle r\rho(t) | \varphi \circ \mathfrak{F} \rangle_\Omega = \frac{1}{2\pi} \lim_{t\to 0} \langle \rho(t) | \varphi \rangle_{\mathbb{R}^3} = \frac{1}{2\pi} \langle \rho_0 | \varphi \rangle_{\mathbb{R}^3} = \langle \widetilde{\rho}_0 | \varphi \circ \mathfrak{F} \rangle_\Omega\,. \tag{4.90}$$

Consequently, for all $\psi \in C_c^\infty(\Omega)$, we obtain

$$\lim_{t\to 0} \langle r\rho(t) | \psi \rangle_\Omega = \langle \widetilde{\rho}_0 | \psi \rangle_\Omega = \lim_{t\to 0} \langle \widetilde{\rho}(t) | \psi \rangle_\Omega\,. \tag{4.91}$$

Moreover, remark that for all $t > 0$, the quantity $\sigma(t) \triangleq r\rho(t)$ satisfies the equation

$$\partial_t \sigma - \left(\Delta - \frac{1}{r^2} \right) \sigma + \mathrm{div}_\star(v\widetilde{\rho}) = -2\partial_r \rho.$$

This, together with (4.91) yield to the following system for σ

$$\begin{cases} \partial_t \sigma - \left(\Delta - \frac{1}{r^2}\right) \sigma + \mathrm{div}_\star(v\widetilde{\rho}) = -2\partial_r \rho, \quad (t,r,z) \in \mathbb{R}_*^+ \times \Omega, \\ \sigma_{|t=0} = \widetilde{\rho}_0, \end{cases} \tag{4.92}$$

where the initial condition is to be understood in the weak sense given by (4.91). We recall, on the other hand, that $\widetilde{\rho}$ satisfies the system

$$\begin{cases} \partial_t \widetilde{\rho} - \left(\Delta - \frac{1}{r^2}\right) \widetilde{\rho} + \mathrm{div}_\star(v\widetilde{\rho}) = -2\partial_r \rho, & (t,r,z) \in \mathbb{R}_*^+ \times \Omega, \\ \widetilde{\rho}_{|t=0} = \widetilde{\rho}_0 \end{cases} \tag{4.93}$$

One finds then that the quantity $\widetilde{\rho} - r\rho$ satisfies a heat equation with zero inputs. It is easy then to deduce that $r\rho(t) = \widetilde{\rho}(t)$, for all $t > 0$.
We emphasize that this characterization of $\widetilde{\rho}$ would imply that (ω_θ, ρ) solves the Boussinesq system. Moreover, all the estimates proved for $\widetilde{\rho}$ hold for for the quantity $r\rho$. Theorem 4.3.7 is then proved. □

4.3.3 Global well-posedness

The results proved in Theorem 4.3.7 provide information only on the local well-posedness whenever the initial data (ω_0, ρ_0) is suitable and lies in $\mathscr{M}(\Omega) \times \mathscr{M}(\mathbb{R}^3)$. However, one can in fact extend the local solution to be defined for all $t > 0$. Indeed, from the proof of Theorem 4.3.7, we deduce that there exists $t_0 \in (0,T)$ such $(\omega_\theta(t_0), \rho(t_0)) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$. Hence, Theorem 3.3.1 insures the existence of a unique solution of the Boussinesq system with initial data $(\omega_\theta(t_0), \rho(t_0))$, denoted for now by $(\bar{\omega}, \bar{\rho})$. This solution is defined on $[t_0, \infty)$ and satisfies in particular, for all $p \in [1, \infty]$

$$\sup_{t \geqslant t_0} (t - t_0)^{1-\frac{1}{p}} \|(\bar{\omega}(t), r\bar{\rho}(t))\|_{L^p(\Omega)\times L^p(\Omega)} + \sup_{t \geqslant t_0} (t - t_0)^{\frac{3}{2}(1-\frac{1}{p})} \|\bar{\rho}(t)\|_{L^p(\mathbb{R}^3)} < \infty.$$

On the other hand, the local solution (ω_θ, ρ) constructed in Theorem 4.3.7 satisfies, for all $p \in [1, \infty]$

$$\sup_{t \in [t_0, T]} (t - t_0)^{1-\frac{1}{p}} \|(\omega_\theta(t), r\rho(t))\|_{L^p(\Omega)\times L^p(\Omega)} + \sup_{t \in [t_0, T]} (t - t_0)^{\frac{3}{2}(1-\frac{1}{p})} \|\rho(t)\|_{L^p(\mathbb{R}^3)} < \infty.$$

To conclude, we only need to use the above estimates and repeat the arguments leading to the uniqueness in the proof of Theorem 4.3.7 to end up with $(\omega_\theta, \rho) \equiv (\bar{\omega}, \bar{\rho})$ on $[t_0, T]$. This proves that the local solution is uniquely extendable to a global one.

5 Appendix

5.1 Derivation of the Navier-Stokes-Boussinesq equations

Our goal in this section is to develop the equations that control the motion of a stratified fluid in a rotating environment. These equations are then somewhat simplified by taking advantage of the so-called Boussinesq approximation. In 1903, Boussinesq stated for the first time the conditions under which the famous " Approximation " applied:
Il fallait encore observer que, dans la plupart des mouvements provoqués par la chaleur sur nos fluides pesants, les volumes ou les densités se conservent à trés peu prés, quoique la variation correspondante du poids de l'unité de volume soit justement la cause des phénomènes qu'il s'agit d'analyser. De là résulte la possibilité de négliger les variations de la densité là oû elles ne sont pas multipliées par la gravité g, tout en conservant dans les calculs leur produit par celle-ci. [12]
Boussinesq's equations are given by a system that combines the velocity field and the density.

Without appreciable loss of precision, we can note that the Boussinesq approximation is based on the following remark: *In most geophysical systems, the fluid density varies, slightly around a mean value.* [77].

For example for air which is compressible or seawater (where the salinity is negligible), it can be assumed in most cases that the density of the fluid ρ, does not deviate much from one mean reference value, ρ_0, so we can write.

$$\rho = \rho_0 + \rho'(x, y, z, t), \qquad |\rho'| << \rho_0 \tag{5.1}$$

where the variance is ρ' due to the stratification present in the fluid or from the fact that its movement is small compared to the reference value ρ_0.

5.1.1 Continuity equation

Thanks to the fundamental principle in fluid mechanics is that :*the mass be conserved* which we can formulate mathematically as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_1) + \frac{\partial}{\partial y}(\rho v_2) + \frac{\partial}{\partial z}(\rho v_3) = 0 \tag{5.2}$$

where (v_1, v_2, v_3) are the three components of velocity, due to (5.1), we can write

$$\rho_0(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}) \quad + \quad \rho'(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z})$$
$$+(\frac{\partial \rho'}{\partial t} + v_1\frac{\partial \rho'}{\partial x} + v_2\frac{\partial \rho'}{\partial y} + v_3\frac{\partial \rho'}{\partial z}) = 0 \tag{5.3}$$

Geophysical flows indicate that the relative variations of density in time and space are not larger than - and usually much less than - the relative variations of the velocity field. Consequently the terms of the third group in (5.3) are of the same order as - if not much less than - those of the second. While the terms of this second group are always much lower than those of the first because $|\rho'| << \rho_0$. Therefore, only this first group of terms needs to be kept, and we write

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0 \tag{5.4}$$

Remark 5.1.1. From a physical point of view, the conservation of mass has become the conservation of volume.

5.1.2 Derivation of velocity equations

Momentum equations

Newton's second law says " *mass times acceleration equals the sum of forces.* " Recall that in the frame where we neglect the rotation, the absolute acceleration has the following three components $(\frac{dv_1}{dt}, \frac{dv_2}{dt}, \frac{dv_3}{dt})$. In fluid is better stated per unit volume with density replacing mass (due to $\rho = \frac{m}{V}$), so the Newton's second law gives

$$\begin{cases} x : \rho\dfrac{dv_1}{dt} = -\dfrac{\partial p}{\partial x} + \dfrac{\partial \tau^{xx}}{\partial x} + \dfrac{\partial \tau^{xy}}{\partial y} + \dfrac{\partial \tau^{xz}}{\partial z} \\[2ex] y : \rho\dfrac{dv_2}{dt} = -\dfrac{\partial p}{\partial y} + \dfrac{\partial \tau^{xy}}{\partial x} + \dfrac{\partial \tau^{yy}}{\partial y} + \dfrac{\partial \tau^{yz}}{\partial z} \\[2ex] z : \rho\dfrac{dv_3}{dt} = -\dfrac{\partial p}{\partial z} - \rho g + \dfrac{\partial \tau^{xz}}{\partial x} + \dfrac{\partial \tau^{yz}}{\partial y} + \dfrac{\partial \tau^{zz}}{\partial z} \end{cases} \tag{5.5}$$

where $-\nabla p$ is the pressure force and the viscous force is due to the derivatives of the stress tensor where its components is given by

$$\begin{cases} \tau^{xx} = \mu(\frac{\partial v_1}{\partial x} + \frac{\partial v_1}{\partial x}); \tau^{xy} = \mu(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}); \tau^{xz} = (\frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x}) \\ \tau^{yy} = \mu(\frac{\partial v_2}{\partial y} + \frac{\partial v_2}{\partial y}); \tau^{yz} = \mu(\frac{\partial v_2}{\partial z} + \frac{\partial v_3}{\partial y}); \\ \tau^{zz} = \mu(\frac{\partial v_3}{\partial z} + \frac{\partial v_3}{\partial z}); \end{cases} \tag{5.6}$$

About the extraction of these components you can see [46]. μ is called the coefficient of dynamic viscosity.

Remark 5.1.2. Let us mention that the acceleration in a fluid is not counted as the rate of change in velocity at a fixed location but as the change in velocity of a fluid particle as it moves along with the flow, the time derivatives in the acceleration components $\frac{dv_1}{dt}$, $\frac{dv_2}{dt}$ and $\frac{dv_3}{dt}$, consist of both the local time rate of change and the so-called advective terms:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z}$$

Remark 5.1.3. Note that the Navier-Stokes equations result of (5.5) after series of algebraic manipulations. The significant role in the simplification is due to the incompressibility condition ($\mathrm{div} v = 0$).

The treatment of the two first equations of the previous system (5.5) is the same due to the appearance of ρ only in the left side. Like $|\rho'| << \rho_0$ and with kinematic viscosity $\nu = \frac{\mu}{\rho_0}$, we find

$$\begin{cases} \frac{dv_1}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\Delta v_1 \\ \frac{dv_2}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\Delta v_2. \end{cases} \tag{5.7}$$

We turn now to addressing the third equation $z-$momentum equation, where ρ is appearing in both sides of the equation. With respect to the term on the left, we deal with it in the same way as before, i.e. we neglect ρ' compared to ρ_0. While for the right-hand side, that is not available, because there is a ρ multiplied by g, which indicates the weight of the fluid. That weight causes the pressure to increase with depth (or the pressure decreases with height, depending on whether one thinks of the ocean or Atmosphere). With the ρ_0 part of the density goes a hydrostatic pressure p_0, which is a function of z only:

$$p = p_0(z) + p'(x, y, z, t) \quad \text{with} \quad p_0(z) = P_0 - \rho_0 g z$$

so that $\dfrac{dp_0}{dz} = -\rho_0 g$, and the equation of $z-$momentum becomes

$$\frac{dv_3}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} + \nu \Delta v_3 \tag{5.8}$$

At this point, we cannot simplify further because the term $\rho' g$ is responsible for the buoyancy forces which are considered a crucial ingredient in geophysical fluid dynamics.

Remark 5.1.4. Note that the hydrostatic pressure $p_0(z)$ can be subtracted from p in the reduced momentum equations (5.7), because it has no derivatives with respect to x and y, and is dynamically inactive.

Remark 5.1.5. The equations (5.7) and (5.8) can be seen as three equations providing the three velocity components v_1, v_2 and v_3. An equation for ρ is given by the conservation of mass.

To better understand this topic, we refer to Beckers and Roisin [77] , Geoffrey[38], Turner [81] and the references therein.

5.2 About axisymmetric Biot Savart Law

Recalling that in the cylindrical coordinates and in the class of axisymmetric vector fields without swirl the velocity is given by $v = (v^r, 0, v^z)$ with v^r and v^z are independtly of $\theta-$variable, ω_θ its vorticity defined from Ω into $\mathbb{R}^3$ by

$$\omega = \mathrm{curl} v = curl \begin{pmatrix} v^r \\ 0 \\ v^z \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_z v^r - \partial_r v^z \\ 0 \end{pmatrix} \tag{5.9}$$

Remark 5.2.1. For any vector field $v = v_\theta e_\theta$, we have

$$\mathrm{curl} v = \mathrm{curl} \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -\partial_z v_\theta \\ 0 \\ \frac{\partial_r (r v_\theta)}{r} \end{pmatrix}. \tag{5.10}$$

From (5.9) and (5.10), we notice that the application $v \mapsto \mathrm{curl} v$ takes any axisymmetric fields without swirl to axisymmetric fields of the form $\omega_\theta e_\theta$ which is " pure swil" and vice versa.

The divergence-free condition $\mathrm{div} v = 0$ turns out to be

$$\mathrm{div} v = \frac{1}{r}\partial_r (r v^r) + \partial_z v^z = 0.$$

therefore

$$\partial_r (r v^r) + \partial_z (r v^z) = 0. \tag{5.11}$$

the equation (5.11) can be written as

$$\mathrm{curl}\begin{pmatrix} rv^z \\ 0 \\ -rv^r \end{pmatrix} = 0. \tag{5.12}$$

Remark 5.2.2. If the flow is irrotational (its rotation is zero at any point), in mathematical terms, the velocity vector is then the gradient of the potential

Thanks to the remark 5.2.2 and under the homogeneous boundary conditions $v^r = \partial_r v^z = 0$, we can build a scalar function $\Omega \ni (r,z) \mapsto \psi(r,z) \in \mathbb{R}$ which called *axisymmetric stream function* and satisfying

$$\nabla\psi = \begin{pmatrix} \partial_r\psi \\ \partial_z\psi \end{pmatrix} = \begin{pmatrix} rv^z \\ -rv^r \end{pmatrix} \tag{5.13}$$

Hence (5.13) can be written as

$$v^r = -\frac{1}{r}\partial_z\psi, \quad v^z = \frac{1}{r}\partial_r\psi. \tag{5.14}$$

and

$$\begin{pmatrix} v^r \\ 0 \\ v^z \end{pmatrix} = \begin{pmatrix} -\frac{1}{r}\partial_z\psi \\ 0 \\ \frac{1}{r}\partial_r\psi \end{pmatrix} = \mathrm{curl}\begin{pmatrix} 0 \\ \frac{\psi}{r} \\ 0 \end{pmatrix} \tag{5.15}$$

Consequently, one obtains that ψ evolves the following linear elliptic inhomogeneous equation

$$-\frac{1}{r}\partial_r^2\psi + \frac{1}{r^2}\partial_r\psi - \frac{1}{r}\partial_z^2\psi = \omega_\theta,$$

with the boundary conditions $\psi(0,z) = \partial_r\psi(0,z) = 0$. By setting $\mathcal{L} = -\frac{1}{r}\partial_r^2 + \frac{1}{r^2}\partial_r - \frac{1}{r}\partial_z^2$, one finds the following boundary value problem

$$\begin{cases} \mathcal{L}\psi(r,z) = \omega_\theta(r,z) & \text{if} \quad (r,z) \in \Omega \\ \psi(r,z) = \partial_r\psi(r,z) = 0 & \text{if} \quad (r,z) \in \Omega, \end{cases} \tag{5.16}$$

where $\partial\Omega = \{(r,z) \in \mathbb{R}^2 : r = 0\}$. To solve the elliptic problem (5.16), it is efficient to write the solution v in terms of vector potential A i.e $v = \mathrm{curl}A$. In accordance with (5.15), we can write A as

$$A = \begin{pmatrix} 0 \\ A_\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\psi}{r} \\ 0 \end{pmatrix} \tag{5.17}$$

We aim at estimating A, to do this it suffices to evaluate A_θ at points x with cylindrical coordinates $(r,0,z)$. Recall that in cartesian coordinates A is given as

$$A(x) = \int_{\mathbb{R}^3} G(x-y)\omega(y)dy \tag{5.18}$$

with $G(x) = \frac{1}{4\pi}\frac{1}{|x|}$.

Remark 5.2.3. The last result is due to curlcurl $= -\Delta + \nabla$div.

Let (r', θ', z') be the cylindrical coordinates of y. A straightforward computation yields

$$A(x) = \int_{-\infty}^{+\infty}\int_0^{+\infty}\int_0^{2\pi} \frac{\omega_\theta}{4\pi\sqrt{r^2 - 2rr'\cos\theta' + r'^2 + (z-z')^2}} \begin{pmatrix} -\sin\theta' \\ \cos\theta' \\ 0 \end{pmatrix} r' d\theta' dr' dz' \tag{5.19}$$

By setting

$$A(r, r', z, z') = \int_0^{2\pi} \frac{r'}{4\pi\sqrt{r^2 - 2rr'\cos\theta' + r'^2 + (z-z')^2}} \begin{pmatrix} -\sin\theta' \\ \cos\theta' \\ 0 \end{pmatrix} d\theta' \tag{5.20}$$

Thus in view of (5.20), and the fact

$$\int_0^{2\pi} \frac{r'\sin\theta' d\theta'}{\sqrt{r^2 - 2rr'\cos\theta' + r'^2 + (z-z')^2}} = 0$$

(5.20) takes the form

$$A(r, r', z, z') = \begin{pmatrix} 0 \\ A_\theta(r, r', z, z') \\ 0 \end{pmatrix} \tag{5.21}$$

with

$$A_\theta(r, r', z, z') = \int_0^{2\pi} \frac{r'\cos\theta' d\theta'}{\sqrt{r^2 - 2rr'\cos\theta' + r'^2 + (z-z')^2}} \tag{5.22}$$

Plug (5.22) in (5.19), we get

$$A(x) = A_\theta(r, z) e_\theta(x)$$

where

$$A_\theta(r, z) = \int_{-\infty}^{+\infty}\int_0^{+\infty} A_\theta(r, r', z, z')\omega_\theta(r', z') dr' dz' \tag{5.23}$$

Combining (5.23) and (5.17) to obtain

$$\psi(r, z) = \int_{-\infty}^{+\infty}\int_0^{+\infty} rA_\theta(r, r', z, z')\omega_\theta(r', z') dr' dz' \tag{5.24}$$

Combining the last formula (5.24) with the problem (5.16), it results that the formula (5.24) inverts the operator $\mathcal{L}$ and therefore the function $rA_\theta(r, r', z, z')$ is considered as Green's function of the operator $\mathcal{L}$. After a suitable change of variables, we find

$$\psi(r, z) \triangleq \mathcal{L}^{-1}\omega_\theta(r, z) = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\sqrt{\tilde{r}r}}{2\pi} F\left(\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{\tilde{r}r}\right)\omega_\theta(\tilde{r}, \tilde{z}) d\tilde{r} d\tilde{z}, \tag{5.25}$$

where the function $F :]0, \infty[\to \mathbb{R}$ is expressed as follows:

$$F(s) = \int_0^\pi \frac{\cos\alpha d\alpha}{(2(1-\cos\alpha) + s)^{1/2}}. \tag{5.26}$$

Consequently, the boundary value problem (5.16) admits a unique solution.

Bibliography

[1] H. Abidi: Résultats de régularité de solutions axisymétriques pour le système de Navier-Stokes. Bull. Sc. Math. **132**, no 7, 592-624 (2008).

[2] H. Abidi and T. Hmidi: On the global well-posedness for Boussinesq System. J. Diff. Equa. 233 (1), 199–220 (2007).

[3] H. Abidi, T. Hmidi and S. Keraani: On the global regularity of axisymmetric Navier-Stokes-Boussinesq system. Discrete Contin. Dyn. Sys. 29 (3), 737–756 (2011).

[4] F. Andreu-Vaillo, V. Caselles and J. M. Mazon: Parabolic Quasilinear Equation Minimizing Linear Growth Functionls. Springer Basel AG 2004.

[5] Ben-Artzi: Global solutions of two-dimensional Navier-Stokes and Euler equations. Arch. Anal. Rational Mech. 128, 329-358 Springer-Verlag (1994).

[6] K. B. Athreya and S. N. Lahiri: Measure Theory and Probability Theory. Springer Science + Business Media, LLC 2006.

[7] H. Bahouri, J. -Y. Chemin and R. Danchin: Fourier analysis and nonlinear partial differential equations. Springer-Verlag Berlin Heidelberg 2011.

[8] J. T. Beale, T. Kato and A. Majda: Remarks on the breakdown of smooth solutions for the $3D-$Euler equations. Commun. Math. Phys. 94, 61–66 (1984).

[9] J.-M. Bony: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. de l'Ecole Norm. Sup. (14), 209–246 (1981).

[10] Bogachev V I: Measure Theory. Volume I, Springer-Verlag Berlin Heidelberg 2007.

[11] Bogachev V II : Measure Theory. Volume II, Springer-Verlag Berlin Heidelberg 2007.

[12] P. A. Bois: Joseph Boussinesq (1842-1929) Universitaire lillois et Académi-

cien. Laboratoire de Mécanique de Lille, UMR 8107 UFR de Mathématiques, Bt. M3, U.S.T.L., F-59655 Villeneuve d'Ascq.

[13] Y. Brenier: Optimal transport, convection, magnetic relaxation and generalized Boussinesq equations. J. Nonlinear Sci. 19, 547–570 (2009).

[14] H. Brezis: Remarks on the preceding paper by M. Ben-Artzi.Global solutions of two-dimensional Navier-Stokes and Euler equations. Arch. Rational Mech. Anal. 128 (1994).

[15] J. R. Cannon and E. Dibenedetto: The Initial Value Problem for the Boussinesq Equations with Data in L^p. Lecture Notes in Math. 771, Berlin-Heidelberg-New York: Springer, 129–144 (1980).

[16] M. Cannone, M. and F. Planchon: Self-similar solutions for Navier-Stokes equations in $\mathbb{R}^3$, Commun. Partial Differential Equations 21, 179-193 (1996).

[17] D. Chae, S.-K. Kim and H.-S. Nam: Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations. Nagoya Math. J. 155, 55-80 (1999).

[18] D. Chae and H.-S. Nam: Local existence and blow-up criterion for the Boussinesq equations. Proc. Roy. Soc. Edinburgh Sect. A 127, 935-946 (1997).

[19] D. Chae: Global regularity for the $2D-$Boussinesq equations with partial viscous terms. Advances in Math. 203 (2), 497–513 (2006).

[20] J. -Y. Chemin: A remark on the inviscid limit for two-dimensional incompressible fluis. Communications in partial differential equations 21, no 11-12, 1771-1779 (1996).

[21] J. -Y. Chemin: Perfect incompressible Fluids. Oxford University Press 1998.

[22] J. -Y. Chemin and I. Gallagher: On the global well-posedness of the 3-D Navier-Stokes equations with large initial data. Annales Scientifiques de lÉcole Normale Supèrieure de Paris, 39 , 679-698 (2006).

[23] J. -Y. Chemin and I. Gallagher: Well-posedness and stability results for Navier-Stokes equations in $\mathbb{R}^3$. Annales de lÍnstitut H. Poincaré, Analyse non Linéaire, 26 , no. 2, 599â624 (2009).

[24] D. Córdoba, C. Fefferman and R. De La LLave: On squirt singularities in hydrodynamics. SIAM, J. Math. Anal. 36 204-213 (2004).

[25] G. -H. Cottet: Equations de Navier-Stokes dans le plan avec tourbillon initial mesure. C. R. Acad. Sci. Paris Sér. I Math. 303, 105-108 (1986).

[26] R. Danchin and M. Paicu: Global well-posedness issues for the inviscid Boussinesq system with Yudovichs type data. Comm. Math. Phys. 290, 114 (2009).

[27] R. Danchin and M. Paicu: Le théorème de Leary et le théorème de Fujita-Kato pour le système de Boussinesq partiellement visqueux. Bulletin de la S. M. F. 136, 261-309 (2008).

[28] J. R. Cannon and E. Dibenedetto: The Initial Value Problem for the Boussinesq Equations with Data in L^p. Lecture Notes in Math. 771, Berlin-Heidelberg-New York: Springer, pp. 129-144 (1980).

[29] P. Dreyfuss and H. Houamed: Uniqueness result for Navier-Stokes-Boussinesq equations with horizontal dissipation. Preprint. arXiv:1904.00437.

[30] K- J. Engel and R.Nagel: One-Parameter Semigroups for linear Evolution Equation. Springer-Verlag, New York 2000.

[31] H. Feng and V. Šverák: On the Cauchy problem for axi-symmetric vortex rings. Arch. Rational Mech. Anal. 215, 89-123 (2015).

[32] H. Fujita and T. Kato: On the Navier-Stokes initial value problem. I. Archive for rational mechanics and analysis, **16** , 269–315 (1964).

[33] I. Gallagher and Th. Gallay: Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity. Math. Ann. **332**, 287-327 (2005).

[34] Th. Gallay: Stability and interaction of vorticies in two dimensional viscous flows. Disc. Cont. Dyn. Systems Ser. S **5**, 1091-1131 (2012).

[35] Th. Gallay and C. E. Wayne: Global stabilityof vortex solutions of the two-dimensional Navier-Stokes equation. Comm. Math. Phys. **255**, 97-129 (2005).

[36] Th. Gallay and V. Šverák: Remarks on the Cauchy problem for the axisymmetric Navier-Stokes equations. Confluentes Mathematici **7**, 67-92 (2015).

[37] Th. Gallay and V. Šverák: Uniqueness of axisymmetric viscous flows originating from circular vortex filament. Annales de l'ENS 52, 1025-1071 (2019).

[38] K. V. Geoffrey: Atmospheric and oceanic fluid dynamics. Cambridge University Press 2017.

[39] P. Germain: Équations de Navier-Stokes dans $\mathbb{R}^2$: existence et comportement asymptotique de solutions d'énergie infinie. Bull. Sci. math. 130, 123-151 (2006).

[40] Y. Giga and T. Miyakawa: Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces. Comm. Partial Diff. Equations **14**, 577-618 (1989).

[41] Y. Giga, T. Miyakawa and H. Osada: Two-dimensional Navier-Stokes flow with measures as initial vorticity. Arch. Rational Mech. Anal. 104, 223-250 (1988).

[42] L. Grafakos: Modern Fourier analysis. Graduate Texts in Mathematics, second edition. Springer 2009.

[43] B. Guo: Spectral method for solving two-dimensional Newton-Boussineq equation. Acta Math. Appl. Sinica 5, 27–50 (1989).

[44] A. Hanachi, H. Houamed and M. Zerguine: On the global well-posedness of axisymmetric Boussinesq system in critical Lebesgue spaces. Discrete and Continuous Dynamical Systems - A, 40 (11) : 6473-6506 (2020).

[45] Hewitt, Edwin, Strongerg, Kari: Real and abstract analysis. A modern treatment of the theory of functions of real variable, Graduate texts in mathematics. Vol 25, Berlin, Heidelberg, New York. Springer-Verlag (1965).

[46] H. Houamed: Mathematical analysis of some three-dimensional fluid and magnetohydrodynamic models. HAL open science, 2020.

[47] H. Houamed and M. Zerguine: On the global solvability of the axisymmetric Boussinesq system with critical regularity. Nonlinear Anal, 200, 26pp (2020).

[48] H. Koch and D. Tataru: Well-posedness for the Navier-Stokes equations. Adv. Math. 157, no. 1, p. 22â35 (2001).

[49] T. Hmidi and S. Keraani: On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity. Adv. Differential Equations 12 (4), 461–480 (2007).

[50] T. Hmidi and S. Keraani: On the global well-posedness of the Boussinesq system with zero viscosity. Indiana Univ. Math. J. 58 (4), 1591–1618 (2009).

[51] T. Hmidi, S. Keraani and F. Rousset: Global well-posedness for an Euler-Boussinesq system with critical dissipation. Comm. Partial Differential Equations 36 (3), 420–445 (2011).

[52] T. Hmidi, S. Keraani and F. Rousset: Global well-posedness for a Navier-Stokes-Boussinesq system with critical dissipation. J. Differential Equations 249, 2147–2174 (2010).

[53] T. Hmidi and F. Rousset: Global well-posedness for the Navier-Stokes-Boussinesq system with axisymmetric data. Ana. Inst. H. Poincaré Anal. Non linéaire, 27 (5), 1227–1246 (2010).

[54] T. Hmidi and F. Rousset: Global well-posedness for the Euler-Boussinesq system with axisymmetric data. Jornal of Functional Analysis 260, 745–776 (2011).

[55] T. Hmidi and M. Zerguine: On the global well-posedness of the Euler-Boussinesq system with fractional dissipation. Physica D. 239, 1387–1401 (2010).

[56] T. Hmidi and M. Zerguine: Inviscid limit axisymmetric Navier-Stokes system. Differential and Integral Equations, 22 no 11-12, 1223-1246 (2009).

[57] T. Hmidi and M. Zerguine: Vortex patch for stratified Euler equations. Commun. Math. Sci.Vol. 12, No. 8, 1541–1563 (2014).

[58] T. Y. Hou and C. Li: Global well-Posedness of the viscous Boussinesq equations. Discrete and Continuous Dynamical Systems. Series A, 12 (1), 1–12 (2005).

[59] T. Kato: The Navier-stokes equation for an incompressible fluid in $\mathbb{R}^2$ with a measure as the initial vorticity. Differential Integral Equations **7**, 949-966 (1994).

[60] T. Kato and G. Ponce: Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p_s(\mathbb{R}^2)$. Rev. Mat. Iberoamericana 2, no. 1-2, 73-88 (1986).

[61] O. Ladyzhenskaya: Unique solvability in the large of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry. Zap. Nauchn. Semin. Leningr. Otd. Math. Inst. Steklova **7** (in Russian), 155–177 (1968).

[62] A. Larios, E. Lunasin and E. S. Titi: Global well-posedness for the $2d-$Boussinesq system with anisotropic viscosity without heat diffusion. J. Differential Equations 255, no. 9, 2636-2654 (2013).

[63] P. -G. Lemarié-Rieusset: Recent Developments in the Navier Stokes Problem. Chapman and Hall/CRC Research Notes in Mathematics, 431, 2002.

[64] S. Leonardi, J. Màlek, J. Necas, M. Pokornỳ: On axially symmetric flows in $\mathbb{R}^3$, Z. Anal. Anwendungen 18, 639â649 (1999).

[65] G. Leoni: A first course in Sobolev spaces. Graduate Studies in Mathematics. Volume 105. AMS 2009.

[66] J. Leray: Essai sur le mouvement dún liquide visqueux emplissant léspace , Acta Math. 63 p. 193â248 (1933).

[67] X. Liu, M. Wang and Z. Zhang: Local well-posedness and blow-up criterion of the Boussinesq equations in critical Besov spaces. Journal of Mathematical Fluid Mechanics, 12, 280-292 (2010).

[68] A. J. Majda and A. L. Bertozzi: Vorticity and incompressible flow. Cambridge University press 2002.

[69] Y. Meyer: Wavelets, Paraproducts and Navier-Stokes equations. In Current developments in mathematics, 1996 (Cambridge, MA), Int. Press, Boston, MA, p. 105â212 1997.

[70] C. Miao and L. Xue: On the global well-posedness of a class of Boussinesq-Navier-Stokes systems. arXiv:0910.0311v1.

[71] C. Miao and X.Zheng: Global well-posedness for axisymmetric Boussinesq system with hor- izontal viscosity. J. Math. Pures Appl. (9), 101, 842-872 (2014).

[72] C. Miao and X. Zheng: On the global well-posedness for the Boussinesq system with horizontal dissipation. Comm. Math. Phys. 321, 33-67 (2013).

[73] A. Pazy: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag. New York. 1983.

[74] J. Pedlosky: Geophysical fluid dynamics. NewYork: Springer-Verlag 1987.

[75] Renzo L. Ricca: An Introduction to the Geometry and Topology of Fluid Flows. Springer-Science +Business Media, B.V 2000.

[76] J. C. Robinson, J. L. Rodrigo and W. Sadowski: The three-dimensional Navier-Stokes equations. Cambridge university press 2016.

[77] B. -C. Roisin and J. M. Beckers: Introduction to Geophysical Fluid Dynamics Physical and Numerical Aspects. Academic Press 2009.

[78] T. Shirota and T. Yanagisawa: Note on global existence for axially symmetric solutions of the Euler system. Proc. Japan Acad. Ser. A Math. Sci. **70**, no. 10, 299–304 (1994).

[79] V. Šveràk: Selected topics in fluid mechanics. Lectures notes of an introduc-

tory graduate course taught in 2011/2012, available at the following URL: http://www.math.umn.edu/sverak/course-notes2011.pdf.

[80] M. Ukhovskii and V. Yudovich: Axially symmetric flows of ideal and viscous fluids filling the whole space. Journal of applied mathematics and mechanics, **32**, 52-69 (1968).

[81] J. S. Turner: Buoyancy effects in fluids. Cambridge University Press, 1973.

[82] Weinan. E and C. Shu: Small-scale structures in Boussinesq convection. Phys. Fluids 6, 49-58 (1994).

[83] M. Weissler: The Navier-Stokes initial value problem in L^p. Arch. Rational Mech. Anal. 74, 219-230 (1980).

[84] M. Zerguine: The regular vortex patch for stratified Euler equations with critical fractional dissipation. J. Evol. Equ. 15, 667-698 (2015).

ملخص:

أطروحتنا هذه من بين العديد التي تناولت أحد المواضيع المهمة لميكانيكا الموائع، حيث تركزت دراستنا حول نظام بوسينسك والذي يعتبر تعميما لنظام نافييه ستوكس أين تكون الكثافة فيه ثابتة. وهذا الأخير يعتبر من بين المسائل الاستثنائية التي خصصت لها جائزة مليون دولار. من خلال هذه الأطروحة قمنا بدراسة وجود ووحدانية الحل لنظام بوسينسك المتماثل محوريا. إن من بين الدوافع لهذه الدراسة وجود تشابه بين حالة التماثل المحوري والتي يكون فيها شعاع السرعة في الإحداثيات الأسطوانية مستقلا عن الزاوية وبين الحالة ثنائية الأبعاد حيث تلعب دراسة الدوامة دورا كبيرا في إثبات الوجود والوحدانية. اعتمدنا في دراستنا هذه على نتائج تخص نظام نافييه ستوكس المتماثل محوريا وبما أن نظام بوسينسك يشمل مجهولا إضافيا وهو الكثافة، حيث يكون هذا الأخير معرفا على مجموعة تختلف عن مجموعة تعريف الدوامة، لذلك أنشأنا مجهولا جديدا يقوم بالموازنة بين المجهولين السابقين ويلعب دورا فعالا في إثبات الوجود المحلي لحل ذو طاقة لا نهائية وفي التقديرات الأولية لأجل شمولية الحل. كما وسعت هذه الدراسة لتشمل حالة المعطيات الأولية من فضاء القياسات المنتهية، حيث واجهنا تحديا كبيرا وهو كيفية إعطاء معنى مدقق ومناسب للمعطيات الأولية. لأجل ذلك قمنا بإنشاء بعض المفاهيم كالقياس المتناظر محوريا ومن خلال بعض القضايا التي أثبتناها استطعنا إثبات أن نظام بوسينسك المتماثل محوريا في هذه الحالة موضوع جيدا.

الكلمات المفتاحية: نظام بوسينسك المتماثل محوريا، الفضاءات الحرجة، القياسات المتماثلة محوريا، التموضع الجيد الشامل.

Abstract:

This thesis is one of many studies that deal with one of the important subjects of fluid mechanics, where our study is focused on the Boussinesq system, which is a generalization of the Navier-Stokes system. It is one of the premium problems, which is specified for its resolution a price of one million dollars. In this thesis, we have studied the existence and uniqueness of the solution of the axisymmetric Boussinesq system. This study is motivated by the existence of a similarity between the axisymmetric case, in which the velocity in cylindrical coordinates is independent of the angle, and the two-dimensional case, where the study of the vorticity plays a major role in proving the existence and uniqueness. In this study, we have relied on the results of the axisymmetric Navier-Stokes system. Since the Boussinesq system has an additional unknown, which is the density, where is defined on a different set from the defining set of the vorticity, we have defined a new unknown that balances the previous ones. The new unknown plays an important role in proving the local existence of an infinite energy solution and in the a priori estimates. This study has also been extended to include the case of initial data in the space of finite measures. The most challenging part is how to give a rigorous and suitable sense to the initial data of the new unknown. For this, we created concepts such as the axisymmetric measure and through some results; we were able to prove that the Boussinesq system is well posed in this case.

keywords: *Axisymmetric Boussinesq system*, *Critical Spaces, Axisymmetric Measure, Global Well-Posedness.*

Résumé :

Cette thèse fait partie des nombreuses études qui traitent l'un des sujets importants de la mécanique des fluides, où notre étude s'accentue sur le système de Boussinesq, qui est une généralisation du système de Navier-Stokes. C'est l'un des rares problèmes dont il lui a été attribué le prix d’un million de dollars. Dans cette thèse, nous avons étudié l'existence et l'unicité de la solution du système de Boussinesq axisymétrique. Cette étude est motivée par l'existence d'une similitude entre le cas axisymétrique, dont lequel la vitesse en coordonnées cylindriques est indépendante de l'angle, et du cas bidimensionnel, où l'étude du tourbillon joue un rôle majeur pour prouver l'existence et l’unicité. Dans cette étude, nous nous sommes appuyés sur les résultats du système de Navier-Stokes axisymétrique, et puisque le système de Boussinesq a un inconnu supplémentaire, qui est la densité, où est définie sur un ensemble différent de celui du tourbillon, nous avons défini un nouveau inconnu qui équilibre les inconnus précédents. Le nouveau inconnu joue un rôle important dans la preuve de l'existence locale d'une solution d'énergie infinie et dans les estimations à priori. Cette étude a également été étendue au cas des données initiales dans l'espace des mesures finies. Le plus difficile est de donner un sens rigoureux et adapté aux données initiales du nouvel inconnu. Pour cela, nous avons créé des concepts tels que la mesure axisymétrique et à travers quelques résultats nous avons pu prouver que le système Boussinesq est bien posé dans ce cas.

Mots clés : *Système de Boussinesq Axisymétrique, Espaces Critique, Mesure Axisymétrique, Bien-Posé globale.*