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# Idempotents et inverses généralisés

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## DEDICATION

To my father and mother.

I dedicate this work to my dear wife who supported me in each step of the way, to my sons Islam, Hadjer

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## NOTATION

Throughout this thesis;

$H$, $K$, $L$ and $F$ are four infinite dimensional complex Hilbert spaces,

$B(H,K)$: the set of all linear bounded operators from $H$ to $K$.

$B(H)$ :the set of all linear bounded operators from $H$ to $H$.

$N(A)$ : the null space of an operator $A \in B(H,K)$.

$R(A)$ : the range space of an operator $A \in B(H,K)$.

$A^*$ : the adjoint of an operator $A \in B(H,K)$.

$P^2 = P$ : a projector.

$P^2 = P = P^*$: an orthogonal projector.

$P_M$ : the orthogonal projector onto the closed subspace $M$ of $H$.

$I$ : the identity operator.

$\oplus$ : a direct sum.

$\oplus^{\perp}$: a direct orthogonal sum.

$\overline{H_1}$ : the closure of $H_1$ in $H$.

$\mathbb{C}^{m,n}$ : the set of all $m \times n$ complex matrices.

$\langle v \rangle$: the subspace spanned by a vector $v$.

$r(A)$ : the rank of a matrix $A \in \mathbb{C}^{m,n}$.

$det(A)$ : the the determinant of a matrix $A \in \mathbb{C}^{n,n}$.

$I_n$ the identity matrix of $\mathbb{C}^{n,n}$.

# Contents

Introduction 1

Chapter 1. Auxiliairy results 5

1.1. On M-P inverse, range of operator, projectors. 5

1.2. On disjoint ranges 7

1.3. Full-rank decomposition 9

Chapter 2. On M-P inverse of a $2 \times 2$ block matrix 14

2.1. On rank additivity condition. 15

2.2. Representations of the M-P inverse of a $2 \times 2$ triangular block matrix under the rank additivity condition. 18

2.3. Representations of the M-P inverse of a $2 \times 2$ triangular block matrix 24

2.4. A generalization of the Banachiewicz -Schur form 26

2.5. Algorithm for computing the M-P inverse of a matrix. 28

2.6. A numerical example 34

Chapter 3. On M-P inverse of a $2 \times 2$ block operator. 38

3.1. Representations for the M-P inverse of a $2 \times 2$ Row block operator with disjoint ranges operators 40

3.2. Representations for the M-P inverse of a $2 \times 2$ triangular block operator with disjoint ranges operators 48

3.3. Representations for the M-P inverse of a $2 \times 2$ block operator with disjoint ranges operators 56

3.4. Representation of the M-P inverse of a $2 \times 2$ row block operator 57

3.5. Representation for the M-P inverse of a $2 \times 2$ block triangular operator 61

3.6. Correction to: Representation of the Moore-Penrose for a class of 2-by-2 block operator valued partial matrices, see ([11],[12]). 62

Chapter 4. On the product of operators with closed range. 75

Chapter 5. On M-P inverse of the sum two operators 80

5.1. Representation of the M-P inverse of the sum two operators 81

5.2. Representations of the M-P inverse of the sum two operators with disjoint ranges. 84

5.3. Extension of the Fill-Fishkind formula. 90

5.4. Proofs of the results of subchapter 3.3 92

References 95

# Introduction

E. H. Moore was the first who gave an explicit definition of a kind of inverse related to an arbitrary matrix, this was in 1920.

In 1955 R. Penrose defined the generalized inverses of matrix, a year later, Rado proved that these two definitions are equivalent, and since then this generalized inverse is called the Moore-Penrose inverse, in 1949 Tseng defined the Moore-Penrose inverse for linear operators in Hilbert spaces, (for more details see [4])

The Moore-Penrose inverse is applied in various area: Bouldin [5] gave a geometric characterization of the condition in terms of the angle between two linear subspaces, Nikaido [37] showed a topological characterization for it. The M-P inverse is also used to solve linear systems, in optimization, in electrical networks see [4]. Also it is used in electrical engineering, see [6], Electronics in [20]. That is why many authors have seen the duty to treat its characterizations such as the sum, the product of matrices, the block operator, the closed product range of the operators closed ranges....etc.

In 1956, Penrose [38] first studied the representation for the generalized inverse of a partitioned complex matrix. In 1960, Greville [22] established a representation for the Moore-Penrose inverse of a partitioned matrix of the form $N = [A_1 : A_2]$, where $A_1$ is a single column. Later in 1964, Cline [10] generalized Greville's result and obtained the

Moore-Penrose inverse of a partitioned matrix of the form $N = [A_1 : A_2]$, where $A_1$ has more than one column.

In 1970, Meyer (see [31] and [32]) explored representations for inner inverses and generalized inverses of $2 \times 2$ block triangular matrices, in 1979 [6] Campbell and Meyer derived simple representations of the Moore-Penrose inverses of $2 \times 2$ triangular block matrices under some conditions. Many of authors established several formulas for various generalized inverses of a $2 \times 2$ block matrix ( also, a $2 \times 2$ block operator) under certain conditions involving Schur complements; (for more details see [4], [6], [8], [11], [39], [41])

The thesis is organized as follows:

**In chapter 1**: We gave the definition of the Moore-Penrose inverse of a linear operator , the definition of proprety of disjoint ranges and some equvalent statements, also the definition of a full-rank decomposition and some related results.

**In chapter 2**: Under rank additivity conditions of the columns (resp, of the rows), we gave new representations of different kinds of a $2 \times 2$ block matrices, we use this representations to obtain the Moore-Penrose inverse of a block triangular matrix, and we give a generalization of the Banachiewicz-Schur form of $M$ with the Schur complement of $A_1$ in $M$. Next, we describe an algorithm to calculate the Moore-Penrose inverse of a matrix $A$ with a numerical example.

**In chapter 3**: We obtain necessary and sufficient conditions for the existence of the Moore-Penrose inverse of block row operator, where at least one of the two operators $A_1^+$ and $A_2^+$ exists and its expressions under the condition $R(A_1) \cap R(A_2) = \{0\}$. If $A_1$ or $A_2$ has a closed range, we will introduce the concept of full-rank decomposition on row block operator. Beyond, we give a new representation of the Moore-Penrose inverse of row operator block, based on full-rank decomposition.

We obtained again the necessary and sufficient conditions for the existence of the Moore-Penrose inverse of triangular block operator and its Moore-Penrose inverse with disjoint ranges operators, and on the other hand we derive a new representation of the Moore-Penrose inverse of triangular block operator. Beyond, we consider a $2 \times 2$ block operator $M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ as sum of two operators $M_1 = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}$, and then, we gave some representaions of the Moore-Penrose inverse of $M$ under the condition $R(M_1^*) \cap R(M_2^*) = \{0\}$.

We show that each representation of the Moore-Penrose inverse under assumptions in [11,Theorem 9; Theorem 10] and in [12,Corollary 13,Corollary 14] does not always represent the Moore-Penrose inverse, our criticism is as follows: firstly we will illustrate with examples that the results of the items of corollaries 13, 14 and theorems 9, 10 are not true, secondly we determine the illogical steps in their proofs and we correct these corollaries and propose some representations of the Moore-Penrose inverse of $M$ with preserving the hypotheses of the corollary 13 and 14 in [12].

**In the chapter 4**:

We give some applications of our results, exactly, we obtain necessary and sufficient conditions for the product of two operators with closed ranges to have a closed range, alos we get some necessary and sufficient conditions for the sum of orthogonal projectors to be Moore-Penrose invertible.

**In the chapter 5**:

From the idea that [17] the closed range operator $A$ admits matrix form with respect to the orthogonal sum of subspaces of $H$ and $K$, we obtain a representation of the Moore-Penrose inverse of the sum of two operators $A$ and $B$ satisfying: $R(A) \perp R(B)$ and $R(A^*) + R(B^*)$ is closed, hence under suitable conditions, we obtian a general representation of the Moore-Penrose inverse of the sum $A + B$, in the closedness conditions for ranges.

We use the notion of the full- rank decomposition of an operator to prove that if $A$ and $B$ have closed ranges and $R(A) \cap R(B) = \{0\}$ and $R(A^*) \cap R(B^*) = \{0\}$ hold, then we have $R(A + B) = R(A) + R(B)$, $R(A^* + B^*) = R(A^*) + R(B^*)$ and the subspaces $R(A + B)$, $R(A) + R(B)$ and $R(A^*) + R(B^*)$ are closed, also that the extension of the Fill-Fishkind formula for $A$ and $B$ with closed ranges is valid keeping the conditions of Fill-Fishkind formula for matrices. On the other hand we get an analogous formula under $R(A) \cap R(B) = \{0\}$ and $R(A^*) \cap R(B^*) = \{0\}$ to Fill-Fishkind formula for $A$ and $B$ having closed ranges and derive certain cases where operator ranges are orthogonal.

CHAPTER 1

# Auxiliairy results

## 1.1. On M-P inverse, range of operator, projectors.

**Definition 1.** *The Moore-Penrose inverse ( for short M-P inverse) of a closed range operator* $A \in B(H,K)$, *is the unique operator* $A^+ \in B(K,H)$ *satisfying the following four Penrose equations*

$(i)\ AA^+A = A, (ii)\ A^+AA^+ = A^+, (iii)\ (AA^+)^* = AA^+, (iv)\ (A^+A)^* = A^+A.$

It is well known that $A^+$ exists for given $A \in B(H,K)$ if and only if ( for short iff ) $R(A)$ is closed. The following lemmas is frequently used

**Lemma 2.** *Let* $A \in B(H,K)$, *then the closedness of any one of the following sets implies the closedness of the ramaining there sets*

$$R(A),\ R(A^*),\ R(AA^*) \text{ and } R(A^*A)$$

*If* $A$ *has a closed range, then*

$$R(A) = R(AA^*) \text{ and } R(A^*) = R(A^*A)$$

*And*

$$A^+ = A^*(AA^*)^+ = (A^*A)^+A^*.$$

**Lemma 3.** [20, *Theorem* 2.2]*: Let* $A \in B(H,K)$ *and* $B \in B(L,K)$*, then*

$$R(A) + R(B) = R((AA^{*}+BB^{*})^{\frac{1}{2}}).$$

**Lemma 4.** *Let* $A \in B(H,K)$*,* $P \in B(K)$ *and* $Q \in B(H)$ *such that* $P$ *and* $Q$ *are projectors then*

1)$PA = A \;\Leftrightarrow\; R(A) \subset R(P)$,

2) $AQ = A \;\Leftrightarrow\; N(Q) \subset N(A)$

3) If $P$ is orthogonal projector and $PA$ has a closed range, then

$$(PA)^{+} = (PA)^{+}P$$

4) When $K = H$, then

$$P = Q \quad \Leftrightarrow \quad R(P) \subset R(Q) \text{ and } N(P) \subset N(Q)$$

**Proof.** 1): On the one hand, it is clear that, $PA = A$ gives us that $R(A) \subset R(P)$, on the other hand, we have $Py = y$; $\forall y \in R(P)$, the hypothesis $R(A) \subset R(P)$ leads to $Py = y$; $\forall y \in R(A)$, which implies that $PAx = Ax$; $\forall x \in H$, Consequently, the item 1) is holds. 2): $\Longleftarrow$); We have $N(Q) \subset N(A) \Longrightarrow \overline{R(A^{*})} \subset R(Q^{*})$, by item 1) we get $Q^{*}A^{*} = A^{*}$ which implies that $AQ = A$, $\Longrightarrow$); $AQ = A \Longrightarrow A(I-Q) = 0 \Longrightarrow R(I-Q) \subset N(A) \Longrightarrow N(Q) \subset N(A)$. 3): Direct verification. 4) : $\Longrightarrow$); It is clear, $\Longleftarrow$); from the item 1) and 2) we obtain: $QP = P$ and $QP = Q$ , which implies that $P = Q$ $\square$

**Lemma 5.** *Let $A$ and $B \in B(H,K)$. If there exists an invertible operator $C \in B(H)$ such that $A = BC$, then $R(A)$ is closed iff $R(B)$ is closed.*

**Proof.** We have $R(A) = R(BC) \subset R(B)$ and $R(B) = R(AC^{-1}) \subset R(A)$, so we deduce that $R(A) = R(B)$. $\square$

**Lemma 6.** *Let $A \in B(H,K)$ with closed range, $B \in B(H)$ and $C \in B(K)$, where $B$ and $C$ are invertible, then:*

1) The operator $B^{-1}A^{+}$ satisfies the equations $(i)$, $(ii)$ and $(iii)$ of M-P inverse for $AB$.

2) The operator $A^{+}C^{-1}$ satisfies the equations $(i)$, $(ii)$ and $(iv)$ of M-P inverse for $CA$.

**Proof.** It is clear. $\square$

**Lemma 7.** *Assume that $A \in B(H,K)$ has a closed range, if there are two operators $X$, $Y \in B(K,H)$; such that $Y$ satisfies the equations $(i)$ and $(iv)$ of the M-P inverse of $A$, and $X$ verifies the equations $(i)$ and $(iii)$ of MP inverse of $A$, then $A^{+} = YAX$.*

**Proof.** Direct verification. $\square$

### 1.2. On disjoint ranges

**Definition 8.** *Let $A \in B(H,K), B \in B(L,K)$, we say that $A$, $B$ are disjoint ranges if $R(A) \cap R(B) = \{0\}$, we denote by $DR$ the set of all these pairs $(A,B)$; i.e.,*

$$DR := \{(A,B) : A \in B(H,K),\ B \in B(L,K) \text{ and } R(A) \cap R(B) = \{0\}\}$$

These two following lemmas give us some necessary and sufficient conditions for two bounded operators to be disjoint ranges.

**Lemma 9.** *Let* $A \in B(H, K)$ *with closed range and* $B \in B(L,K)$, *then the next statements are equivalent:*

1) $(A, B) \in DR$, 3) $N(B) = N(P_{(N(A^*)}B)$,

2) $\overline{R(B^*)} = \overline{R(B^* P_{(N(A^*)})}$, 4) $N(P_{(N(A^*))}B) \subset N(A^*B)$.

**Proof.** We know that $\overline{R(B^*)} = N(B)^{\perp}$ and $\overline{R(B^* P_{(N(A^*)})} = N(P_{(N(A^*)}\ B)^{\perp}$, then 2) $\Leftrightarrow$ 3). Using absurd reasoning to proof both implications of the equivalence; 1) $\Leftrightarrow$ 3): first, $\Rightarrow$); Let $x \in L$ satisfies $P_{(N(A^*)}Bx = 0$ and $Bx \neq 0$, which implies that $AA^+Bx = Bx$ and $Bx \neq 0$, it follows that $Ax' = Bx \neq 0$; where $x' = A^+Bx$, therefore contradiction with the assertion 1). Secondly; ($\Leftarrow$: Let $y \in R\ (A) \cap R\ (B) \neq \{0\}$, there exist $x_1 \neq 0$ and $x_2 \neq 0$ such that $Ax_1 = Bx_2 \neq 0$, form the equation $(i)$ of Penrose, we obtain $AA^+Bx_2 = Bx_2$ and $Bx_2 \neq 0$, then $P_{(N(A^*)}Bx = 0$ and $Bx_2 \neq 0$; hence contradiction. Now, we will see that 1) $\Leftrightarrow$ 4): $\Rightarrow$); If $x \in N\left(P_{N(A^*)}B\right)$, we get $Ax' = Bx$ where $A^+Bx = x'$, as $R\,(A) \cap R\,(B) = \{0\}$, we deduce that $Bx = 0$ then $x \in N(A^+B) = N(A^*B)$. $\Leftarrow$): Let $y \in R\,(A) \cap R\,(B)$, then there exist $x_1 \in H$, $x_2 \in L$ such that $AA^+Ax_1 = Bx_2 = y$ which implies that $AA^+Bx_2 = Bx_2$, or $P_{N(A^*)}Bx_2 = 0$, since $N\left(P_{N(A^*)}B\right) \subset N(A^*B)$, then $A^*Bx_2 = A^*y = 0$, Consequently $y \in R\,(A) \cap N(A^*) = \{0\}$, so $y = 0$. □

We apply the results of the proceding lemma for $A^*$ and $B^*$, we get:

**Lemma 10.** *Let $A \in B(H,K)$ with closed range and $B \in B(L,K)$, then the next statements are equivalent:*

1) $(A^{*}, B^{*}) \in DR$, 3) $N(B^{*}) = N(P_{(N(A)}B^{*})$,

2) $\overline{R(B)} = \overline{R(BP_{(N(A)})}$, 4) $N(P_{(N(A))}B^{*}) \subset N((AB^{*})$.

**Remark 11.** *Through the definition* $(8)$*, if* $C \in B(F,H)$*, then we have*

1) $(A,B) \in DR \ \Leftrightarrow \ (B,A) \in DR.$

2) $(A,B) \in DR \Rightarrow \ (B,AC) \in DR.$

### 1.3. Full-rank decomposition

The full-rank decomposition plays an important role in the theory of the generalized inverses, in particular for determining the expressions of the M-P inverse of an operator; for more information see $[[4],[7]]$. We recall that in $[7]$, Caradus has proved that an operator $A \in B(H,K)$ admits a full-rank decompositon iff there exists an operator $X \in B(K,H)$ that satisfies the equation $(i)$ or iff $A^{+}$ exists.

**Definition 12.** *Let $A \in B(H,K)$ If there exists a Hilbert space $H_A$ and operators $G_A \in B(H, H_A)$; $F_A \in B(H_A, K)$, such that $G_A$ is right invertible, $F_A$ is left invertible and*

$$A = F_A G_A \tag{1-1}$$

*Then we say that* $(1-1)$ *is a full-rank decomposition of* $A$.

**Theorem 13.** *: For any $A \in B(H,K)$, $A$ has a full-rank decomposition iff $A^+$ exists.*

**Proof.** : Effectively, if $F_A G_A$ is a full-rank decomposition of $A$, from the definition previous, it is obvious to verify that $G_A^+ F_A^+$ is the M-P of inverse of $A$, in this case $A^+ = G_A^+ F_A^+$. □

Conversely, From the existence of $A^+$, we have that $R(A)$ is closed and we conclude $R(A)$ is a Hilbert space included in $K$, we define the operators $G_A$ and $F_A$ as follows:

$$G_A \in B(H,\, R(A)),\ \text{such that}\ G_A x = Ax;\ \forall x \in H.$$

And

$$F_A \in B(R(A),\, K),\ \text{such that}\ F_A x = x;\ \forall x \in R(A);\ .$$

It is easy to see that $G_A$ is surjective, and $F_A$ is injective, furthermore $A = F_A G_A$.

We need of the following lemmas:

**Lemma 14.** *If $F_A G_A$ is a full-rank decompositions of $A \in B(H,K)$, then:*

1) $F_A^* F_A$ and $G_A G_A^*$ are invertible.

2) $F_A^+$ is a left inverse of $F_A$, also $G_A^+$ is a right inverse of $G_A$.

3) $R\,(A) = R\,(F_A)$, $N\,(A) = N\,(G_A)$, $R\,(A^*) = R\,(G_A^*)$ and $N\,(A^*) = N(F_A^*)$.

4) $A^+ A = G_A^+ G_A$ and $AA^+ = FF^+$

**Proof.** 1): $F_A$ is injective means that $F_A^*$ is surjective; ( i.e: $R(F_A^*) = H$), it follows that $F_A^+$ exists and $R(F_A^* F_A) = R(F_A^*)$, therefore $R(F_A^* F_A) = H$, while $F_A^* F_A$ is self-adjoint, so $F_A^* F_A$ is invertible, by the same way we have $G_A G_A^*$ is invertible. 2): Employing item 1) and lemma (2) we get

$$F_A^+ F_A = (F_A^* F_A)^+ F_A^* F_A = (F_A^* F_A)^{-1} F_A^* F_A = I_{H_A}$$

$$G_A G_A^+ = G_A G_A^* (G_A G_A^*)^+ = G_A G_A^* (G_A G_A^*)^{-1} = I_{H_A}$$

Hence, the 2) is holds. The items 3) and 4) are clear. □

We use this below lemma in the proof of theorem (77) to prove the identity $(5 - 10)$

**Lemma 15.** *Let $F_A G_A$, $F_B G_B$ be a full-rank decompositions of $A$ and $B$, resepectively, then we have*

a)

$$\begin{aligned} R(P_{N(B^*)} A) &= R(P_{N(B^*)} F_A), \\ R(P_{N(A^*)} B) &= R(P_{N(A^*)} F_B) \end{aligned}$$

And

$$\begin{aligned} R(P_{N(A)} B^*) &= R(P_{N(A)} G_B^*), \\ R(P_{N(B)} A^*) &= R(P_{N(B)} G_A^*) \end{aligned}$$

b) We suppose that $(A, B) \in DR$ and $P_{N(B^*)}A$ has a closed range, then we have

$$\begin{aligned} (P_{N(B^*)}F_A)^+ &= G_A(P_{N(B^*)}A)^+, \\ (P_{N(A^*)}F_B)^+ &= G_B(P_{N(A^*)}B)^+ \end{aligned}$$

c) We suppose that $(A^*, B^*) \in DR$ and $BP_{N(A)}$ has a closed range, then we have

$$\begin{aligned} (G_B P_{N(A)})^+ &= (BP_{N(A)})^+ F_B \ , \\ (G_A P_{N(B)})^+ &= (AP_{N(B)})^+ F_A \end{aligned}$$

**Proof.** a) The equality $R(P_{N(B^*)}A) = R(P_{N(B^*)}F_A)$ is proved as follows

$R(P_{N(B^*)}F_A) = P_{N(B^*)}R(F_A) = P_{N(B^*)}R(F_A F_A^+) = P_{N(B^*)}R(AA^+) = P_{N(B^*)}R(A) = R(P_{N(B^*)}A)$. Similarly, we can have the other equals.

b) Let $U = P_{N(B^*)}A$ and $V = G_A^+$, we have

$$R((P_{N(B^*)}A)^*(P_{N(B^*)}A)G_A^+) \subset R(A^* P_{N(B^*)}) \subset R(A^*) = R(G_A^*) = R(G_A^+)$$

So, we deduce that

$$(*1) \qquad R(U^*UV) \subset R(V)$$

Now, note that $R(\ G_A^+ G_A^{+^*}(P_{N(B^*)}A)^*) \subset R(G_A^+) = R(G_A^*)$ and by the item 3 of lemma (14) we get $R(\ G_A^+ G_A^{+^*}(P_{N(B^*)}A)^*) \subset R(A^*)$, on the other hande since $(A, B) \in DR$, it follows from the item 2 lemma (9) that $R(\ G_A^+ G_A^{+^*}(P_{N(B^*)}A)^*) \subset R((P_{N(B^*)}A)^*)$

that is

$$(*2) \qquad R(VV^*U^*) \subset R(U^*)$$

According $(*1)$ and $(*2)$ and [17, item (4) of Theorem 2.2; ], then $U$ and $V$ satisfy the reverse order law $(UV)^+ = V^+U^+$, that is $(P_{N(B^*)}AG_A^+)^+ = G_A(P_{N(B^*)}A)^+$, while $P_{N(B^*)}F_A = P_{N(B^*)}AG_A^+$, so the equality $(P_{N(B^*)}F_A)^+ = G_A(P_{N(B^*)}A)^+$ holds. In the same way we get that $(P_{N(A^*)}F_B)^+ = G_B(P_{N(A^*)}B)^+$. Taking the adjoint on both sides of the equalites of item c) and we use the item b) we obtain

$$(P_{N(A)}G_B^*)^+ = F_B^*(P_{N(A)}B^*)^+$$

and

$$(P_{N(B)}G_A^*)^+ = F_A^*(P_{N(B)}A^*)^+$$

We take again the adjoints on both sides of two last equalities, obtaining the item c) $\square$

CHAPTER 2

# On M-P inverse of a $2 \times 2$ block matrix

Let $M$ be a $2 \times\ 2$ block matrix:

$$(2-1) \qquad M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in C^n \oplus C^p \to C^m \oplus C^q$$

In the case, $A_1$ is invertible square matrix , the matrix $S_{A_1} := A_4 - A_3 A_1^{-1} A_2$ is called the Schur complement of $A_1$ in $M$ , where $A_1^{-1}$ is the usual inverse of $A_1$, if we further assume, $M$ is square matrix, then the Schur complement $S_{A_1}$ is invertible, iff $M$ is invertible, in addition, $M^{-1}$ has the form:

$$(2-2) \qquad M^{-1} = \begin{pmatrix} A_1^{-1} + A_1^{-1} A_2 S_{A_1}^{-1} A_3 A_1^{-1} & -A_1^{-1} A_2 S_{A_1}^{-1} \\ -S_{A_1}^{-1} A_3 A_1^{-1} & S_{A_1}^{-1} \end{pmatrix}$$

The expression $(2-2)$ is called the Banachiewicz-Schur form of the matrix $M$. It should be noted that $S_{A_1}$is not always invertible, But his M-P inverse exists always, that is why, Several authors describe generalized inverses of block matrices with Banachiewicz-Schur forms, in $[3, \text{Corollary 2}]$ the M-P inverse of $M$ has the following Banachiewicz–Schur form

$$(2-3) \qquad M^{+}=\begin{pmatrix} A_1^{-1}+A_1^{-1}A_2S_{A_1}^{+}A_3A_1^{-1} & -A_1^{-1}A_2S_{A_1}^{+} \\ -S_{A_1}^{+}A_3A_1^{-1} & S_{A_1}^{+} \end{pmatrix}$$

If and only if

$$(2-4) \qquad R(A_3)\subset R\left(S_{A_1}\right) \text{ and } R(A_2^{*})\subset R\left(S_{A_1}^{*}\right)$$

The Banachiewicz-Schur form of the matrix $M$ has been used in dealing with inverses of block matrices; see [[3],[8],[43] , [46]], for example, in [43] by the matrix rank method, Y. Tian and Y. Takane gave necessary and sufficient conditions for a block matrix to have generalized inverses with Banachiewicz-Schur forms, now, our goal in what follows is to obtain a representation of the M-P inverse of $M$ with the Schur complement of $A_1$ in $M$.

## 2.1. On rank additivity condition.

In this subchapter, we give some assertions equivalent to the rank additivity conditions of the columns, (of rows). We start by this definition:

**Definition 16.** *Let $M$ be given in $(2-1)$, we say that $M$ has the rank additivity condition of the columns if*

$$(2-5) \qquad r(M)=r\begin{pmatrix} A_1 \\ A_3 \end{pmatrix}+r\begin{pmatrix} A_2 \\ A_4 \end{pmatrix}$$

*Also, $M$ has the rank additivity condition of the rows if:*

$$(2-6) \qquad r(M) = r\begin{pmatrix} A_1, & A_2 \end{pmatrix} + r\begin{pmatrix} A_3, & A_4 \end{pmatrix}.$$

*The following lemma contains some other assertions equivalent to the* $(2-5)$.

**Lemma 17.** *Let $M$ be given in $(2-1)$, then the following statements are equivalent:*

1) $M$ has the rank additivity condition of the columns

2) $R(B_1) \cap R(B_2) = \{0\}$. 8) $R(B_1^* B_2) \subset R\left(B_1^* P_{N\left(B_2^*\right)}\right)$.

3) $N(M) = N(B_1) \oplus N(B_2)$. 9) $R(B_2^*) = R\left(B_2^* P_{N\left(B_1^*\right)}\right)$.

4) $R(M^*) = R(B_1^*) \oplus R(B_2^*)$. 10) $N(B_2) = N\left(P_{N\left(B_1^*\right)} B_2\right)$.

5) $N\left(P_{N\left(B_1^*\right)} B_2\right) \subset N(B_1^* B_2)$. 11) $R(B_1^*) = R\left(B_1^* P_{N\left(B_2^*\right)}\right)$.

6) $R(B_2^* B_1) \subset R\left(B_2^* P_{N\left(B_1^*\right)}\right)$. 12) $N(B_1) = N\left(P_{N\left(B_2^*\right)} B_1\right)$.

7) $N\left(P_{N\left(B_2^*\right)} B_1\right) \subset N(B_2^* B_1)$. 13) $r(M) = r(B_1) + r(B_2)$

where: $B_1 = \begin{pmatrix} A_1 \\ A_3 \end{pmatrix}$, $B_2 = \begin{pmatrix} A_2 \\ A_4 \end{pmatrix}$.

**Proof.** ((1) $\Leftrightarrow$ (2)) Follows from the fact that $r([A,B]) = r(A) + r(B) \Longleftrightarrow R(A) \cap R(B) = \{0\}$.((2) $\Leftrightarrow$ (3)), We suppose that $N(M) = N(B_1, B_2) \neq N(B_1) \oplus N(B_2)$, this is equivalent to the existence of $x \notin N(B_1)$ and $x^{'} \notin N(B_2)$, $B_1x = B_2x^{''} \neq 0$ such that $x^{''} = -x^{'}$, it is equivalent to, $R(B_1) \cap R(B_2) \neq \{0\}$. ((3) $\Leftrightarrow$ (4)): let the block matrix $T = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$, it is easy to show that $R(T^*) = R(B_1^*) \oplus R(B_2^*)$, and $N(T) = N(B_1) \oplus N(B_2)$, as $R(M^*)^{\perp} = N(M)$, $R(T^*)^{\perp} = N(T)$ and by operation of orthogonality; $R(M^*) = R(T^*)$ is equivalent to $N(M) = N(T)$. As the orthogonal range of a matrix is equal to the kernel of its adjoint, this gives the equivalence between 5) and 6) ,7) and 8) , 9) and 10) , 11) and 12). Now between ((2) $\Leftrightarrow$ (12)), $\Rightarrow$) : we suppose that $N((I - B_2B_2^+)B_1) \subsetneq N(B_1)$, then there exists $x$ such that $(I - B_2B_2^+)B_1x = 0$ and $B_1x \neq 0$ implie that $B_2B_2^+B_1x = B_1x \neq 0$, then contradiction, $\Leftarrow$): if $R(B_1) \cap R(B_2) \neq \{0\}$, then there exists $0 \neq x \in R(B_1^*)$, $0 \neq x^{'} \in R(B_2^*)$ such that $B_1x = B_2x^{'}$, so that $B_1x = B_2B_2^+B_2x^{'}$ which implies that $(I - B_2B_2^+)B_1x = 0$ and $B_1x \neq 0$, at the end we have contradiction. In the same procedure we find that 2) $\Leftrightarrow$ 10). 3) $\Rightarrow$ 7); it is clear that $N(B_2^+B_1) = N(B_2^*B_1)$,we suppose that there exists $0 \neq x$ where $(I - B_2B_2^+)B_1x = 0$ and $B_2^+B_1x \neq 0$, it also implies that $B_1x + B_2y = 0$ such that $y = -B_2^+B_1x$, then we get a contradiction. In the same procedure we find that 3) $\Rightarrow$ 5). 7) $\Rightarrow$ 2), we suppose there exists $x \in R(B_1^*)$, $x^{'} \in R(B_2^*)$, $x \neq 0$ and $x^{'} \neq 0$, such as $B_1x = B_2x^{'} \neq 0$, whith implies that $B_1x = B_2B_2^+B_2x^{'} \neq 0$ and $B_2^+B_1x$ $\neq 0$, so $B_1x = B_2B_2^+B_1x$; and $B_2^+B_1x \neq 0$, which equivalent to $N((I - B_2B_2^+)B_2) \subseteq N(B_2^*B_1)$, then we get a contradiction. In the same procedure we find that 5) $\Rightarrow$ 2). $\square$

The following lemma contains some other assertions equivalent to the $(2-6)$ :

**Lemma 18.** *Let $M$ be given in $(2-1)$, then the following statements are equivalent:*

1) $M$ has the rank additivity condition of the rows.

2) $R(L_1^*) \cap R(L_2^*) = \{0\}$. 8) $R(L_1 L_2^*) \subset R(L_1 P_{N(L_2)})$.

3) $N(M^*) = N(L_1^*) \oplus N(L_2^*)$. 9) $R(L_2) = R(L_2 P_{N(L_1)})$.

4) $R(M) = R(L_1) \oplus R(L_2)$. 10) $N(L_2^*) = N(P_{N(L_1)} L_2^*)$.

5) $N(P_{N(L_1)} L_2^*) \subset N(L_1 L_2^*)$. 11) $R(L_1) = R(L_1 P_{N(L_2)})$.

6) $R(L_2 L_1^*) \subset R(L_2 P_{N(L_1)})$. 12) $N(L_1^*) = N(P_{N(L_2)} L_1^*)$.

7) $N(P_{N(L_2)} L_1^*) \subset N(L_2 L_1^*)$. 13) $r(M) = r(L_1) + r(L_2)$

Where: $L_1 = \begin{pmatrix} A_1, & A_2 \end{pmatrix}$, $L_2 = \begin{pmatrix} A_3, & A_4 \end{pmatrix}$.

### 2.2. Representations of the M-P inverse of a $2 \times 2$ triangular block matrix under the rank additivity condition.

We obtain in this subchapter, Some representations of the M-P inverse of a $2 \times 2$ block triangular matrix, for the four types of block triangular matrices; under the rank additivity condition.

**Theorem 19.** *Let $M$ be given in $(2-1)$, with $A_3 = 0$, and $Y_1$ be defined by:*

$$(2-7) \qquad Y_1 = \begin{pmatrix} A_1^* G_1^+ & -A_1^* G_1^+ A_2 A_4^+ \\ D_1^* G_1^+ & A_4^+ - D_1^* G_1^+ A_2 A_4^+ \end{pmatrix}$$

*Then: $M^+ = Y_1$ iff one of the statements of lemma $(18)$ holds for $R_1$ and $R_2$. Where:*

$R_1 = \begin{pmatrix} A_1, & A_2 \end{pmatrix}$ and $R_2 = \begin{pmatrix} 0, & A_4 \end{pmatrix}$.
$D_1 = A_2 P_{N(A_4)}$ , $G_1 = A_1 A_1^* + D_1 D_1^*$

**Proof.** For abridge the proof, we can easily check that the matrice $Y_1$ satisfies the equations $(i)$, $(ii)$ and $(iv)$ of the M-P inverse of $M$, only remains to determine some necessary and sufficient conditions for which the projector $MY_1$ is self-adjoint, that is why, we calculate $MY_1$

$$MY_1 = \begin{pmatrix} G_1 G_1^+ & -G_1 G_1^+ A_2 A_4^+ + A_2 A_4^+ \\ 0 & A_4 A_4^+ \end{pmatrix}$$

Clearly, $MY_1$ is self-adjoint iff $G_1 G_1^+ A_2 \ A_4^+ = A_2 A_4^+$, by item 1 of the lemma $(4)$; the last equation equivalents to $R(A_2 A_4^+) \subset R(G_1)$, on the other hand, note that

$$R(A_2 A_4^+) = R(A_2 A_4^*) = R(R_1 R_2^*)$$

And

$$R(G_1) = R\left(\begin{pmatrix} A_1, & D_1 \end{pmatrix}\begin{pmatrix} A_1^* \\ D_1^* \end{pmatrix}\right) = R\begin{pmatrix} A_1, & D_1 \end{pmatrix} = R\left(R_1 P_{N(R_2)}\right)$$

We deduce that, $R(A_2A_4^+) \subset R(G_1)$ iff $R(R_1R_2^*) \subset R(R_1P_{N(R_2)})$, this coincides with the assertion 8) of lemma (18). □

**Theorem 20.** *Let $M$ be given in $(2-1)$, with $A_3 = 0$, and $X_1$ be defined by:*

$$(2-8) \qquad X_1 = \begin{pmatrix} A_1^+ - A_1^+A_2G_2^+D_2^* & -A_1^+A_2G_2^+A_4^* \\ G_2^+D_2^* & G_2^+A_4^* \end{pmatrix}$$

*Then: $M^+ = X_1$ iff one of the statements of the lemma $(17)$ holds for $C_1$ and $C_2$. Where:*

$$C_1 = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} A_2 \\ A_4 \end{pmatrix}.$$

$D_2 = P_{N(A_1^*)}A_2$, $G_2 = D_2^*D_2 + A_4^*A_4$

**Proof.** After calculation, it will be clear that $X_1$ satisfies the equations $(i)$, $(ii)$ and $(iii)$ of M-P inverse of $M$, as $D_2^*A_1 = 0$, then

$$X_1M = \begin{pmatrix} A_1^+A_1 & A_1^+A_2 - A_1^+A_2G_2^+G_2 \\ 0 & G_2^+G_2 \end{pmatrix}$$

Hence, $X_1$ satisfies the equation $(iii)$ of M-P inverse of $M$, iff $A_1^+A_2 = A_1^+A_2G_2^+G_2$, by item 2 of the lemma (4), iff

$$N(G_2) \subset N(A_1^+A_2),$$

That is

$$N(P_{N(C_1^*)}C_2) \subset N(C_1^*C_2)$$

Because:

$$N(G_2) = N\left(\begin{pmatrix} D_2^* & A_4^* \end{pmatrix}\begin{pmatrix} D_2 \\ A_4 \end{pmatrix}\right) = N\left(\begin{pmatrix} D_2 \\ A_4 \end{pmatrix}\right) = N(P_{N(C_1^*)}C_2)$$

And

$$N(A_1^+A_2) = R(A_2^*A_1^{+^*})^{\perp} = (A_2^*R(A_1))^{\perp} = R(A_2^*A_1)^{\perp} = N(A_1^*A_2)$$

Note that $N(P_{N(C_1^*)}C_2) \subset N(C_1^*C_2)$ coincides with the assertion 5) of the lemma (17), which is the desired result. □

In the following, using proofs similar to those of theorems 19 and 20 , we get the following results:

**Corollary 21.** *Let $M$ be given in $(2-1)$, with $A_2 = 0$, and $Y_2$ be defined by:*

$$(2-9) \qquad Y_2 = \begin{pmatrix} A_1^+ - D_3^*G_3^+A_3A_1^+ & D_3^*G_3^+ \\ -A_4^*G_3^+A_3A_1^+ & A_4^*G_3^+ \end{pmatrix}$$

*Then: $M^+ = Y_2$ iff one of the statements of the lemma (18) holds for $R_3$ and $R_4$. Where:*

$D_3 = A_3P_{N(A_1)}$ , $G_3 {=} D_3D_3^* + A_4A_4^*$

$R_3 = \begin{pmatrix} A_1, & 0 \end{pmatrix}$ and $R_4 = \begin{pmatrix} A_3, & A_4 \end{pmatrix}$.

**Corollary 22.** *Let $M$ be given in $(2-1)$, with $A_2 = 0$, and $X_2$ be defined by:*

$$(2-10) \qquad X_2 = \begin{pmatrix} G_4^+ A_1^* & G_4^+ D_4^* \\ -A_4^+ A_3 G_4^+ A_1^* & A_4^+ - A_4^+ A_3 G_4^+ D_4^* \end{pmatrix}$$

*Then: $M^+ = X_2$ iff one of the statements of the lemma $(17)$ holds for $C_3$ and $C_4$. Where:*

$D_4 = P_{N(A_4^*)} A_3$ , $G_4 = A_1^* A_1 + D_4^* D_4$

$C_3 = \begin{pmatrix} A_1 \\ A_3 \end{pmatrix}$, $C_4 = \begin{pmatrix} 0 \\ A_4 \end{pmatrix}$.

**Corollary 23.** *Let $M$ be give in $(2-1)$, with $A_4 = 0$, and $Y_3$ be defined by:*

$$(2-11) \qquad Y_3 = \begin{pmatrix} D_5^* G_5^+ & -D_5^* G_5^+ A_1 A_3^+ + A_3^+ \\ A_2^* G_5^+ & -A_2^* G_5^+ A_1 A_3^+ \end{pmatrix}$$

*Then: $M^+ = Y_3$ iff one of the statements of the lemma $(18)$ holds for $R_5$ et $R_5$. Where:*

$D_5 = A_1 P_{N(A_3)}$ , $G_5 = D_5 D_5^* + A_2 A_2^*$

$R_5 = \begin{pmatrix} A_1, & A_2 \end{pmatrix}$ and $R_6 = \begin{pmatrix} A_3, & 0 \end{pmatrix}$.

**Corollary 24.** *Let $M$ be give in $(2-1)$, with $A_4 = 0$, and $X_3$ be defined by:*

$$(2-12) \qquad X_3 = \begin{pmatrix} G_6^+ D_6^* & G_6^+ A_3^* \\ -A_2^+ A_1 G_6^+ D_6^* + A_2^+ & -A_2^+ A_1 G_6^+ A_3^* \end{pmatrix}$$

*Then:* $M^+ = X_3$ *iff one of the statements of the lemma* (17) *holds for* $C_5$ *and* $C_6$*.*

*Where:*

$$D_6 = P_{N(A_2^*)}A_1 \ , \ G_6 = A_3^*A_3 + D_6^*D_6$$
$$C_5 = \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \text{ and } C_6 = \begin{pmatrix} A_2 \\ 0 \end{pmatrix}.$$

**Corollary 25.** *Let* $M$ *be give in* $(2-1)$*, with* $A_1 = 0$*, and* $Y_4$ *be defined by:*

$$(2-13) \qquad Y_4 = \begin{pmatrix} -A_3^*G_7^+A_4A_2^+ & A_3^*G_7^+ \\ -D_7^*G_7^+A_4A_2^+ + A_2^+ & D_7^*G_7^+ \end{pmatrix}$$

*Then:* $M^+ = Y_4$ *iff one of the statements of the lemma* (18) *holds for* $R_7$ *et* $R_8$*. Where*

$$D_7 = A_4P_{N(A_2)} \ , \ G_7 = A_3A_3^* + D_7D_7^*$$
$$R_7 = \begin{pmatrix} 0, & A_2 \end{pmatrix} \text{ and } R_8 = \begin{pmatrix} A_3 & A_4 \end{pmatrix}.$$

**Corollary 26.** *Let* $M$ *be give in* $(2-1)$*, with* $A_1 = 0$*, and* $X_4$ *be defined by:*

$$(2-14) \qquad X_4 = \begin{pmatrix} -A_3^+A_4G_8^+A_2^* & A_3^+ - A_3^+A_4G_8^+D_8^* \\ G_8^+A_2^* & G_8^+D_8^* \end{pmatrix}$$

Then: $M^+ = X_4$ iff one of the statements of the lemma (17) holds for $C_7$ and $C_8$.

Where:

$$D_8 = P_{N(A_3^*)}A_4 \ , \ G_8 = A_2^*A_2 + D_8^*D_8$$
$$C_7 = \begin{pmatrix} 0 \\ A_3 \end{pmatrix} \text{ and } C_8 = \begin{pmatrix} A_2 \\ A_4 \end{pmatrix}.$$

## 2.3. Representations of the M-P inverse of a $2 \times 2$ triangular block matrix

We obtain in this subchapter, Some representations of the M-P inverse of a $2 \times 2$ block triangular matrix, for the four kinds of triangular block matrices.

**Corollary 27.** *: Let $M$ be given in $(2-1)$, with $A_3 = 0$, then:*

$$(2-15) \qquad M^{+} = \begin{pmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{pmatrix}$$

*Where:*

$N_{11} = A_1^* G_1^+ P_{R(A_1)} + A_1^* G_1^+ F_1 G_2^+ D_2^*$,

$N_{12} = A_1^* G_1^+ F_1 G_2^+ A_4^*$,

$N_{13} = D_1^* G_1^+ P_{R(A_1)} + D_1^* G_1^+ F_1 G_2^+ D_2^* + P_{R(A_4^*)} G_1^+ D_2^*$,

$N_{14} = D_1^* G_1^+ F_1 G_2^+ A_4^* + P_{R(A_4^*)} G_2^+ A_4^*$.

$F_1 = D_1 - P_{R(A_1)} A_2$.

**Proof.** : By $(2-7)$ , $(2-8)$ and the lemma $(7)$, $M^+ = Y_1 M X_1$, by direct computation we get $(2-15)$. □

By the same way, we have:

**Corollary 28.** *: Let $M$ be given in $(2-1)$, with $A_2 = 0$, then:*

$$(2-16) \qquad M^{+} = \begin{pmatrix} N_{21} & N_{22} \\ N_{23} & N_{24} \end{pmatrix}$$

*Where*

$N_{21}= P_{R(A_1^*)}G_4^+A_1^*+D_3^*G_3^+F_2G_4^+A_1^*$

$N_{22}= D_3^*G_3^+F_2G_4^+D_4^*+P_{R(A_1^*)}G_4^+D_4^*+D_3^*G_3^+P_{R(A_4)}$

$N_{23}= A_4^*G_3^+F_2G_4^+A_1^*$

$N_{24}= A_4^*G_3^+F_2G_4^+D_4^*+A_4^*G_3^+P_{R(A_4)}$

$F_2= D_3-P_{R(A_4^*)}A_3.$

**Corollary 29.** *: Let $M$ be given in $(2-1)$, with $A_4=0$, then:*

$$M^+=\begin{pmatrix} N_{31} & N_{32} \\ N_{33} & N_{34} \end{pmatrix} \tag{2-17}$$

*Where*

$N_{31}= D_5^*G_5^+P_{R(A_2)}+D_5^*G_5^+F_3G_6^+D_6^*+P_{R(A_3^*)}G_6^+D_6^*$

$N_{32}= D_5^*G_5^+F_3G_6^+A_3^*+P_{R(A_3^*)}G_6^+A_3^*$

$N_{33}= A_2^*G_5^+F_3G_6^+D_6^*+A_2^*G_5^+P_{R(A_2)}$

$N_{34}= A_2^*G_5^+F_3G_6^+A_3^*$

$F_3= D_5-P_{R(A_2)}A_1.$

**Corollary 30.** *: Let $M$ be given in $(2-1)$, with $A_1=0$, then:*

$$M^+=\begin{pmatrix} N_{41} & N_{42} \\ N_{43} & N_{44} \end{pmatrix} \tag{2-18}$$

*Where*

$N_{41}= A_3^*G_7^+F_4G_8^+A_2^*$

$N_{42}= A_3^*G_7^+F_4G_8^+D_8^*+A_3^*G_7^+P_{R(A_3)}$

$N_{43}= D_7^*G_7^+F_4G_8^+A_2^*+P_{R(A_2^*)}G_8^+A_2^*$

$N_{44}= D_7^*G_7^+P_{R(A_3)}+D_7^*G_7^+F_4G_8^+D_8^*+P_{R(A_2^*)}G_8^+D_8^*$

$F_4= D_7-P_{R(A_3)}A_4$.

## 2.4. A generalization of the Banachiewicz -Schur form

In this subchapter, Let $M$ given in $(2-1)$, with $A_1 \in \mathbb{C}^{n,n}$ is invertible, we give a representation of the M-P inverse of $M$, based on the Schur complement of $A_1$,which we call a generalization of the Banachiewicz-Schur form of $M$.

**Theorem 31.** *: Let $M$ be given in $(2-1)$, such that $A_1 \in \mathbb{C}^{n,n}$ is invertible , then:*

$$M^+= \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix} \qquad (2-19)$$

*Where*

$J_1= A_1^*G_\alpha^+KG_\beta^+A_1^*$.

$J_2= A_1^*G_\alpha^+KG_\beta^+D_\beta^*-A_1^*G_\alpha^+A_2S_{A_1}^+$.

$J_3= D_\alpha^*G_\alpha^+KG_\beta^+A_1^*-S_{A_1}^+A_3G_\beta^+A_1^*$.

$J_4= D_\alpha^*G_\alpha^+KG_\beta^+D_\beta^* -S_{A_1}^+A_3G_\beta^+D_\beta^*-D_\alpha^*G_\alpha^+A_2S_{A_1}^++S_{A_1}^+$

$D_\alpha = A_2P_{N(S_{A_1})}$, $G_\alpha = A_1A_1^*+ D_\alpha D_\alpha^*$, $D_\beta = P_{N(S_{A_1}^*)}A_3$,

$$G_\beta = A_1^* A_1 + D_\beta^* D_\beta, \qquad K = A_1 + A_2 S_{A_1}^+ A_3$$

**Proof.** : As $A$ is invertible, then $M$ admits the following decompositions:

$$(2-20) \qquad M = \begin{pmatrix} I & 0 \\ A_3 A_1^{-1} & I \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ 0 & S_{A_1} \end{pmatrix} := EF$$

And

$$(2-21) \qquad M = \begin{pmatrix} A_1 & 0 \\ A_3 & S_{A_1} \end{pmatrix} \begin{pmatrix} I & A_1^{-1} A_2 \\ 0 & I \end{pmatrix} := GD.$$

It is easy to see that, $D^{-1}G^+$ satisfies the equations $(i)$ and $(iii)$ of the M-P inverse for $M$ and $F^+E^{-1}$ satisfies the equations $(i)$ and $(iv)$ of the M-P inverse for $M$, by corollaries (27) and (28), respectively, we obtain:

$$(2-22) \qquad F^+ = \begin{pmatrix} A_1^* G_\alpha^+ & -A_1^* G_\alpha^+ A_2 S_{A_1}^+ \\ D_\alpha^* G_\alpha^+ & - D_\alpha^* G_\alpha^+ A_2 S_{A_1}^+ + S_{A_1}^+ \end{pmatrix}$$

And

$$(2-23) \qquad G^+ = \begin{pmatrix} G_\beta^+ A_1^* & G_\beta^+ D_\beta^* \\ -S_{A_1}^+ A_3 G_\beta^+ A_1^* & - S_{A_1}^+ A_3 G_\beta^+ D_\beta^* + S_{A_1}^+ \end{pmatrix}$$

After the calculation, we also find that

$$F^+ E^{-1} = \begin{pmatrix} A_1^* G_\alpha^+ A_1 & A_1^* G_\alpha^+ D_\alpha \\ A_1 G_\alpha^+ D_\alpha^* & D_\alpha^* G_\alpha^+ D_\alpha + S_{A_1}^+ S_{A_1} \end{pmatrix}$$

And

$$D^{-1}G^{+}=$$

$$\begin{pmatrix} (I+A_1^{-1}A_2S_{A_1}^{+}A_3A_1)G_\beta^{+}A_1^{*} & G_\beta^{+}D_\beta^{*}+A_1^{-1}A_2S_{A_1}^{+}(A_3G_\beta^{+}D_\beta^{*}-I) \\ -S_{A_1}^{+}A_3G_\beta^{+}A_1^{*} & -S_{A_1}^{+}A_3G_\beta^{+}D_\beta^{*}+S_{A_1}^{+} \end{pmatrix}$$

Then from the lemma (7), we have

$$M^{+}=F^{+}E^{-1}MD^{-1}G^{+}$$

Which is $(2-19)$. □

**Remark 32.** *: By the item 1 of lemma (4) , $R(A_3)\subset R(S_{A_1})$ is equivalent to $P_{R(S_{A_1})}A_3=A_3$, or iff $D_\beta=0$, of the same, $R(A_2^{*})\subset R\left(S_{A_1}^{*}\right)$ is equivalent to $D_\alpha=0$, in this case we can derive from the representation $(2-19)$, the M-P inverse of $M$ with the Banachiewicz-Schur form, which is exactly $(2-3)$, as a special case, if $S_{A_1}$ is invertible, then $(2-19)$ becomes the Banachiewicz-Schur form $(2-2)$, effectively, $(2-19)$ is the generalization of the Banachiewicz-Schur form of $M$.*

### 2.5. Algorithm for computing the M-P inverse of a matrix.

The aim of this subchapter is to introduce an algorithm for calculating the M-P inverse of a matrix $A$, under the condition $rank(A)<\min\{m,n\}$.

In [1] , if $A_1$ is invertible, Aitken is the first to give this factorisation:

$$M = \begin{pmatrix} I & 0 \\ A_3A_1^{-1} & I \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & S_{A_1} \end{pmatrix} \begin{pmatrix} I & A_1^{-1}A_2 \\ 0 & I \end{pmatrix}$$

From here we can find

$$rank(M) = rank(A_1) + rank(S_{A_1})$$

We conclude that

$$rank(M) = rank(A_1) \text{ iff } rank(S_{A_1}) = 0, \text{ or iff } S_{A_1} = 0$$

Which allows us to present this proposition:

**Proposition 33.** *Let $M$ be given in $(2-1)$ and $A_1 \in \mathbb{C}^{n,n}$ is invertible, such that $rank(M) = rank(A_1)$, then:*

(2-24) $$M^+ = \begin{pmatrix} A_1^*T_1^{-1}A_1T_2^{-1}A_1^* & A_1^*T_1^{-1}A_1T_2^{-1}A_3^* \\ A_2^*T_1^{-1}A_1T_2^{-1}A_1^* & A_2^*T_1^{-1}A_1T_2^{-1}A_3^* \end{pmatrix}$$

*Where:* $T_1 = A_1A_1^* + A_2A_2^*$ *and* $T_2 = A_1^*A_1 + A_3^* A_3$

As an application of $(2-24)$ we propose an algorithm for computing the M-P inverse of a matrix.

For all $1 \leq k \leq m$ : $P_{ki}^{(m)}$ is the permutation matrix of row $k$ with row $i$ of order $m$; for all $k \leq i \leq m$, right here $P_{kk}^{(m)}$ is the identity matrix $I_m$

For all $1 \leq k \leq n$ : $Q_{kj}^{(n)}$ is the permutation matrix of column $k$ with column $j$ of order $n$, for all $k \leq j \leq n$, right here $Q_{kk}^{(n)}$ is the identity matrix $I_n$

**Proposition 34.** *: The following three points are satisfied:*

1) $P_{ki}^{(m)}$ and $Q_{kj}^{(n)}$ are unitary matrices.

2) $P_{ki}^{(m)+} = P_{ki}^{(m)}$ and $Q_{kj}^{(n)+} = Q_{kj}^{(n)}$.

3)

$$(P_{ki}^{(m)} A Q_{kj}^{(n)})^{+} = Q_{kj}^{(n)} A^{+} P_{ki}^{(m)}. \quad (2-25)$$

**Proof.** : Clearly that the identity matrix $I_m$ is the permutation between columns $k$ and $i$ of the matrix $P_{ki}^{(m)}$ , other way,

$$P_{ki}^{(m)} P_{ki}^{(m)} = I_m$$

Analogously

$$Q_{kj}^{(n)} Q_{kj}^{(n)} = I_n$$

Now, If $\langle x; y\rangle$ is an inner product on $\mathbb{C}^m$, it is easy to see that $\left\langle P_{ki}^{(m)} x; x\right\rangle$ and $\left\langle x; P_{ki}^{(m)} x\right\rangle$ are equal for every $x \in \mathbb{C}^m$ , then $P_{ki}^{(m)}$ is self-adjoint, consequently 1) and 2) are satisfied. Applying the two previous points to obtain that $(2-25)$ of item 3). $\square$

**Theorem 35.** *: Let $A \in \mathbb{C}^{m,n}$, such that $rank(A) < \min\{m,n\}$, then there is $P_{1i}^{(m)}; P_{2i}^{(m)}; ...; P_{ki}^{(m)}$ and $Q_{1j}^{(n)}; Q_{2j}^{(n)}; ...; Q_{kj}^{(n)}$, that satisfy*

$$(2-26) \qquad P_{ki}^{(m)}...P_{2i}^{(m)}P_{1i}^{(m)}AQ_{1j}^{(n)}Q_{2j}^{(n)}...Q_{kj}^{(n)} = \begin{pmatrix} A_1^{(k)} & A_2^{(k)} \\ A_3^{(k)} & A_4^{(k)} \end{pmatrix}$$

Where, $A_1^{(k)} \in \mathbb{C}^{k;k}$ is invertible and $rank(A) = k$. In this case the M–P inverse of $A$ is:

$$(2-27) \qquad A^{+} = P_{ki}^{(m)}...P_{2i}^{(m)}P_{1i}^{(m)} \begin{pmatrix} A_1^{(k)} & A_2^{(k)} \\ A_3^{(k)} & A_4^{(k)} \end{pmatrix}^{+} Q_{1j}^{(n)}Q_{2j}^{(n)}...Q_{kj}^{(n)}$$

**Proof.** Let $A$ be an $m \times n$ matrix

$$A = \begin{pmatrix} a_{1;1} & a_{1;2} & \cdots & a_{1;n} \\ a_{2;1} & a_{2;2} & \cdots & a_{2;n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m;1} & a_{m;2} & \ldots & a_{m;n} \end{pmatrix}$$

According to the following algorithm, we can find $(2-26)$, □

At the first step (1):

We search a non-zero coefficient $a_{ij} \neq 0$, by default, afterward by permuting the row 1 with the row $i$ and permuting the column 1 with the column $j$, in an other way we premultiply the matrix $A$ by the matrix of permutation $P_{1i}^{(m)}$ and postmultiply by the matrix of permutation $Q_{1j}^{(n)}$, we get:

$$P_{1i}^{(m)} A Q_{1j}^{(n)} := \begin{pmatrix} A_1^{(1)} & A_2^{(1)} \\ A_3^{(1)} & A_4^{(1)} \end{pmatrix}$$

Where:

$$\begin{pmatrix} A_1^{(1)} & A_2^{(1)} \\ A_3^{(1)} & A_4^{(1)} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_{1;1}^{(1)} \end{pmatrix} & \begin{pmatrix} a_{1;2}^{(1)} & \cdots & a_{1;n}^{(1)} \end{pmatrix} \\ \begin{pmatrix} a_{2;1}^{(1)} \\ \vdots \\ a_{m;1}^{(1)} \end{pmatrix} & \begin{pmatrix} a_{2;2}^{(1)} & \cdots & a_{2;n}^{(1)} \\ \vdots & \ddots & \vdots \\ a_{m;2}^{(1)} & \cdots & a_{m;n}^{(1)} \end{pmatrix} \end{pmatrix}$$

At the step (k): we begin again by a similar procedure, we search a coefficient $a_{ij}^{(k-1)}$ for all $k \leq i \leq m$ and $k \leq j \leq n$ by default, where

$$det \begin{pmatrix} a_{1;1}^{(1)} & \cdots & a_{1;j}^{(k-1)} \\ \vdots & \ddots & \vdots \\ a_{i;1}^{(k-1)} & \cdots & a_{i;j}^{(k-1)} \end{pmatrix} \neq 0$$

We put

$$P_{ki}^{(m)} \begin{pmatrix} A_1^{(k-1)} & A_2^{(k-1)} \\ A_3^{(k-1)} & A_4^{(k-1)} \end{pmatrix} Q_{kj}^{(n)} := \begin{pmatrix} A_1^{(k)} & A_2^{(k)} \\ A_3^{(k)} & A_4^{(k)} \end{pmatrix}$$

Where

$$\begin{pmatrix} A_1^{(k)} & A_2^{(k)} \\ A_3^{(k)} & A_4^{(k)} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_{1;1}^{(1)} & \cdots & a_{1;k}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{k;1}^{(k)} & \cdots & a_{k;k}^{(k)} \end{pmatrix} & \begin{pmatrix} a_{1,k+1}^{(k)} & \cdots & a_{1;n}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{k;k+1}^{(k)} & \cdots & a_{k,n}^{(k)} \end{pmatrix} \\ \begin{pmatrix} a_{k+1;1}^{(k)} & \cdots & a_{k+1;k}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{m;1}^{(k)} & \cdots & a_{m;k}^{(k)} \end{pmatrix} & \begin{pmatrix} a_{k+1;k+1}^{(k)} & \cdots & a_{k+1;n}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{m;k+1}^{(k)} & \cdots & a_{m;n}^{(k)} \end{pmatrix} \end{pmatrix}$$

We stop the procedure when, for all $a_{ij}^{(k)}$

$$det \begin{pmatrix} a_{1;1}^{(1)} & \cdots & a_{1;j}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{i;1}^{(k)} & \cdots & a_{i;j}^{(k)} \end{pmatrix} = 0$$

Such that $k+1 \leq i \leq m$ and $k+1 \leq j \leq n$

Since, the matrices of permutations are invertible, and $A_1^{(k)}$ is the largest matrix extracted with $det(A_1^{(k)}) \neq 0$, so

$$rank(A) = rank \begin{pmatrix} A_1^{(k)} & A_2^{(k)} \\ A_3^{(k)} & A_4^{(k)} \end{pmatrix} = rank(A_1^{(k)}) = k$$

Finally, we apply the third points of proposition $(34)$ on $(2-26)$ to find $(2-27)$.

**Algorithm 36.** *: Given a matrix $A \in \mathbb{C}^{m,n}$, such that $rank(A) < \min\{m, n\}$, to calculate the M-P inverse of $A$, we follow the following steps:*

Step (1): Applying the algorithm that is in the proof of theorem 35 to determine $(2-26)$,

Step (2): Using $(2-24)$ to calculate the M-P inverse of $\begin{pmatrix} A_1^{(k)} & A_2^{(k)} \\ A_3^{(k)} & A_4^{(k)} \end{pmatrix}$,

Step (3): By the items of the proposition 34 we get $(2-27)$ which is the M-P inverse of $A$.

## 2.6. A numerical example

We will give numerical example to illustrate our results, we propose this example:

Let:

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 2 & -1 & 1 & 1 & 0 \end{pmatrix}$$

The first step (1):

Note that $a_{11} = 0$, by permuting the column 1 with the column 2, we obtain

$$P_{11}^{(m)} A Q_{12}^{(n)} = \begin{pmatrix} (1) & \begin{pmatrix} 0 & 2 & 1 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} & \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 \\ 2 & 1 & 1 & 0 \end{pmatrix} \end{pmatrix}$$

The second step (2):

As $det \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \neq 0$, then

$$P_{11}^{(m)} P_{11}^{(m)} A Q_{12}^{(n)} Q_{22}^{(n)} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \end{pmatrix}$$

The third step (3): We have

$$det \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} = 0,$$

By permuting the row 3 with the row 4 , we obtain

$$\begin{aligned} P_{34}^{(m)} P_{22}^{(m)} P_{11}^{(m)} A Q_{12}^{(n)} Q_{22}^{(n)} A Q_{33}^{(n)} &= \begin{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ -1 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \end{pmatrix} \\ : &= \begin{pmatrix} A_1^{(3)} & A_2^{(3)} \\ A_3^{(3)} & A_4^{(3)} \end{pmatrix} \end{aligned}$$

Where

$$det\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ -1 & 2 & 1 \end{pmatrix} \neq 0$$

Now, we apply the proposition (33) for $M = \begin{pmatrix} A_1^{(3)} & A_2^{(3)} \\ A_3^{(3)} & A_4^{(3)} \end{pmatrix}$, first we calculate: $T_1$, $T_1^{-1}$, $T_2$, $T_2^{-1}$, $T_1^{-1}A_1T_2^{-1}$

$$T_1 = \begin{pmatrix} 10 & 8 & 2 \\ 8 & 16 & 7 \\ 2 & 7 & 7 \end{pmatrix} \text{ and } T_1^{-1} = \begin{pmatrix} \frac{7}{38} & \frac{-7}{57} & \frac{4}{57} \\ \frac{-7}{57} & \frac{11}{57} & \frac{-3}{19} \\ \frac{4}{57} & \frac{-3}{19} & \frac{16}{57} \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 8 & 4 \\ 4 & 4 & 10 \end{pmatrix} \text{ and } T_2^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} & \frac{-1}{4} \\ \frac{1}{8} & \frac{3}{16} & \frac{-1}{8} \\ \frac{-1}{4} & \frac{-1}{8} & \frac{1}{4} \end{pmatrix}$$

and

$$T_1^{-1}A_1T_2^{-1} = \begin{pmatrix} \frac{-11}{114} & \frac{-55}{912} & \frac{43}{456} \\ \frac{10}{57} & \frac{31}{456} & \frac{-9}{76} \\ \frac{-7}{114} & \frac{14}{152} & \frac{29}{228} \end{pmatrix}$$

Secondly, we use $(2-24)$:

$$\begin{pmatrix} A_1^{(3)} & A_2^{(3)} \\ A_3^{(3)} & A_4^{(3)} \end{pmatrix}^+ = \begin{pmatrix} \begin{pmatrix} \frac{-1}{228} & \frac{13}{57} & \frac{-7}{19} \\ \frac{-1}{19} & \frac{4}{57} & \frac{14}{57} \\ \frac{3}{19} & \frac{-4}{19} & \frac{5}{19} \end{pmatrix} & \begin{pmatrix} \frac{-1}{228} \\ \frac{-1}{19} \\ \frac{3}{19} \end{pmatrix} \\ \begin{pmatrix} \frac{-13}{228} & \frac{17}{57} & \frac{-7}{57} \\ \frac{7}{57} & \frac{-1}{19} & \frac{-1}{57} \end{pmatrix} & \begin{pmatrix} \frac{-13}{228} \\ \frac{7}{57} \end{pmatrix} \end{pmatrix}$$

Finally by $(2-27)$:

$$A^+ = \begin{pmatrix} \frac{13}{57} & \frac{-1}{228} & \frac{-7}{19} & \frac{-1}{228} \\ \frac{4}{57} & \frac{-1}{19} & \frac{14}{57} & \frac{-1}{19} \\ \frac{-4}{19} & \frac{3}{19} & \frac{5}{19} & \frac{3}{19} \\ \frac{-1}{19} & \frac{7}{57} & \frac{-1}{57} & \frac{7}{57} \\ \frac{17}{57} & \frac{-13}{228} & \frac{-7}{57} & \frac{-13}{228} \end{pmatrix}$$

CHAPTER 3

# On M-P inverse of a $2 \times 2$ block operator.

Recently, the representation and charaterizations of the M-P inverse of block operator on Hilbert space have been considered by many authors, for example, in [11; Lemma 5] under the conditions; $A_4 \in B(L)$ is invertible and $M \in B(H \oplus L, H \oplus L)$, Deng and Du showed that the upper triangular block operator $M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$ is M-P invertible iff $A_1$ has a closed range and in this case they gived a representation of the M-P inverse of $M$. The idea of multiplicative perturbation of an operator of the form $M = XNY$, where $X$ and $Y$ are invertible, allowed the authors Deng, Lui and Wang to give some necessary and sufficient conditions for the existence of $M^+$ and an expression for the multiplicative perturbation of the M-P inverse of a block operator $M \in B(H \oplus L, H \oplus L)$, see[13] .(for more details see ([25], [29], [31], [35], [36], [41])

We use the notation $H \oplus K$ to denote the direct sum of $H$ and $K$, which is also a Hilbert space, endowed with the inner product given by: $\left\langle \binom{h_1}{k_1}, \binom{h_2}{k_2} \right\rangle_{H \oplus K} = \langle h_1,\ h_2 \rangle_H + \langle k_1, k_2 \rangle_K$ , for any $h_i \in H$ and $k_i \in K$, $i = 1; 2$, where $\langle .,\ . \rangle_H$ is an inner product in $H$,

Consider a $2{\times}2$ block operator

$$(3-1) \qquad M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in B(H \oplus L, K \oplus F).$$

**Lemma 37.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$ *be a* $2\times2$ *upper triangular block operator, we assume that* $A_1^+$*,* $A_2^+$ *and* $A_4^+$ *exist, then* $M$ *admits the following decompositions:*

$$(3-2) \qquad M = \begin{bmatrix} A_1 & P_{N(A_1^*)}A_2 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} I & A_1^+A_2 \\ 0 & I \end{bmatrix} := S_1R_1$$

$$(3-3) \qquad M = \begin{bmatrix} I & A_2A_4^+ \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & A_2P_{N(A_4)} \\ 0 & A_4 \end{bmatrix} := R_2S_2$$

$$(3-4) \qquad M = \begin{bmatrix} P_{N(A_2^*)}A_1 & A_2 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_2^+A_1 & I \end{bmatrix} := S_3R_3$$

$$(3-5)$$

$$M = \begin{bmatrix} I & P_{N(A_1^*)}A_2A_4^+ \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & P_{N(A_1^*)}A_2P_{N(A_4)} \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} I & A_1^+A_2 \\ 0 & I \end{bmatrix} := R_4S_4H_4$$

**Proposition 38.** *We assume that $A_1$ and $A_2$ are injective, then* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ *is injective iff* $(A_1, A_2) \in DR$.

**Proof.** We suppose that $R(A_1) \cap R(A_2) \neq \{0\}$, this means that, there is $x \notin N(A_1)$ and $x^{'} \notin N(A_2)$ suth that $A_1 x = A_2(x^{'}) \neq 0$ or $A_1 x + A_2(-x^{'}) \neq 0;$, which is equivalent to the existence $\left(x; -x^{'}\right) \neq (0; 0)$ with $M\binom{x}{x'} = (0; 0)$, that is to say, $M$ is not injective. $\square$

### 3.1. Representations for the M-P inverse of a $2 \times 2$ Row block operator with disjoint ranges operators

First of all, the range of the block operator $M = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ is equal to $R(A_1) + R(A_2) \oplus \{0\}$, because

Let $y \in R(A_1) + R(A_2) \oplus \{0\} \Leftrightarrow \exists\ x_1, x_2$ suth that $\begin{bmatrix} A_1 x_1 + A_2 x_2 \\ 0 \end{bmatrix} = y \Leftrightarrow$ $\exists\ x_1, x_2$ suth that $\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = y \Leftrightarrow y \in R(M)$, then in this case $M^{+}$ exists iff $R(A_1) + R(A_2)$ is closed,

Now, we will present another necessary and sufficient conditions for the existence of the M-P inverse of the row block operator $M$ where at least one of the two operators $A_1^{+}$ and $A_2^{+}$ exists and some representations of the M-P inverse of block row operator with disjoint ranges operators.

**Theorem 39.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ *be a* $2{\times}2$ *row block operator, we assume that* $A_1^+$ *and* $A_2^+$ *exist, then*

$$M^+ = \begin{bmatrix} A_1^+ & 0 \\ A_2^+ & 0 \end{bmatrix} \;\Leftrightarrow\; A_1^+ A_2 = 0 \;\Leftrightarrow\; A_1^* A_2 = 0$$

**Proof.** We have $A_1^+ A_2 = 0 \Leftrightarrow R(A_1) \perp R(A_2)$; because

$$\begin{aligned} A_1^+ A_2 = 0 &\Leftrightarrow \left\langle A_1^\dagger A_2 x,\ y \right\rangle_H = 0, \forall x \in L, \forall y \in H \\ &\Leftrightarrow \left\langle A_2 x,\ A_1^{+^*} y \right\rangle_K = 0, \forall x \in L, \forall y \in K \\ &\Leftrightarrow R\left(A_1^{+^*}\right) \perp R(A_2) \Leftrightarrow R(A_1) \perp R(A_2). \end{aligned}$$

By the same procedure we get that $A_1^* A_2 = 0 \Leftrightarrow R(A_1) \perp R(A_2)$, we deduce that $A_1^+ A_2 = 0 \Leftrightarrow\ A_1^* A_2 = 0$. Now, on the one hand clearly that $A_1^+ A_2 = 0 \Leftrightarrow A_2^+ A_1 = 0$, and hence we obtain that $M^+ = \begin{bmatrix} A_1^+ & 0 \\ A_2^+ & 0 \end{bmatrix}$. On the other hand, if the following equation holds

$$\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^+ & 0 \\ A_2^+ & 0 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$$

Then $A_1 A_1^+ A_2 = 0$, Multiply the left-hand side of the last equation by $A_1^+$ we get $A_1^+ A_2 = 0$. Finally we have proved that

$$M^+ = \begin{bmatrix} A_1^+ & 0 \\ A_2^+ & 0 \end{bmatrix} \Leftrightarrow A_1^+ A_2 = 0. \qquad \square$$

The proof of the following theorem, is based on the fact that the space $H_1 \oplus K_1$ is closed iff $H_1$ and $K_1$ are closed, where $H_1$ and $K_1$ are subspaces of $H$ and $K$, respectively.

**Theorem 40.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ *be a* $2\times 2$ *row block operator, then*

a) If $A_1^+$ exists, then the following statements are equivalent:

1) $M$ has a closed range.

2) $R((P_{N(A_1^*)}A_2))$ is closed.

3) $R(A_1) + R((P_{N(A_1^*)}A_2))$ is closed.

In this case, there exists a linear bounded operator $X$ of the form:

$$X = \begin{bmatrix} A_1^+ - A_1^+ A_2 (P_{N(A_1^*)}A_2)^+ & 0 \\ (P_{N(A_1^*)}A_2)^+ & 0 \end{bmatrix} \tag{3-6}$$

Which satisfies the equations $(i)$, $(ii)$ and $(iii)$ of the M-P inverse of $M$; moreover

$$M^+ = X \quad \Leftrightarrow \quad N(P_{N(A_1^*)}A_2) \subset N(A_1^* A_2) \Leftrightarrow \ (A_1, A_2) \in DR.$$

b) If $A_2^+$ exists, then the following statements are equivalent:

1) $M$ has a closed range.

2) $R(P_{N(A_2^*)}A_1)$ is closed.

3) $R(A_2) + R(P_{N(A_2^*)}A_1)$ is closed.

In this case, there exists a linear bounded operator $Y$ of the form:

$$(3-7) \qquad Y = \begin{bmatrix} (P_{N(A_2^*)}A_1)^+ & 0 \\ A_2^+ - A_2^+ A_1 (P_{N(A_2^*)}A_1)^+ & 0 \end{bmatrix}$$

Which satisfies the equations $(i)$, $(ii)$, $(iii)$ of the M-P inverse of $M$; moroever

$$M^+ = Y \quad \Leftrightarrow \quad N(P_{N(A_2^*)}A_1) \subset N(A_2^* A_1) \;\Leftrightarrow\; (A_1, A_2) \in DR$$

**Proof.** a) By $(3-2)$ of lemma (37) and lemma (5), respectively; where $A_4 = 0$ , we deduce that $M^+$ exists iff $S_1^+$ exists, which also is equivalent by lemma (2) to $R(S_1^* S_1)$ is closed. Now we begin to prove that the item 1) is equivalent to the item 2), as $S_1^* S_1$ has the form:

$$S_1^* S_1 = \begin{bmatrix} A_1^* A_1 & 0 \\ 0 & (P_{N(A_1^*)}A_2)^*(P_{N(A_1^*)}A_2) \end{bmatrix}$$

We have $R(S_1^* S_1) = R(A_1^* A_1) \oplus R((P_{N(A_1^*)}A_2)^*(P_{N(A_1^*)}A_2))$, since $R(A_1^* A_1)$ is closed, it results from the lemma (2) that $S_1^* S_1$ has a closed range, iff $R(P_{N(A_1^*)}A_2)$ is closed. Clearly that $R(S_1) = R(A_1) + (P_{N(A_1^*)}A_2) \oplus \{0\}$, so the items 1) and 3) are equivalent. From $(3-2)$ of lemma (37) and the item 1) of lemma (6), the operator $X = R_1^{-1} S_1^+$ verifies the equations $(i)$, $(ii)$ and $(iii)$ of the M-P inverse for $M$. We need to determine the representation of the M-P inverse of $S$, which through it we calculate $X$, applying the item 3) of the lemma (4), we obtain $(P_{N(A_1^*)}A_2)^+ \, A_1 = 0$, so from the theorem (39) that $S_1^+$ has the form:

$$S_1^+ = \begin{bmatrix} A_1^+ & 0 \\ (P_{N(A_1^*)}A_2)^+ & 0 \end{bmatrix}$$

After calculation of $R_1^{-1}S_1^+$, we get the previous form $(3-6)$ of $X$, consequently of the above that $X = M^+$ iff $X$ satisfies the equation $(iv)$ of the M-P inverse of $M$, (i.e; $(XA)^* = XA$), that is why we need to the formula of $XA$:

$$XA = \begin{bmatrix} (P_{N(A_2^*)}A_1)^+ & 0 \\ A_2^+ - A_2^+A_1(P_{N(A_2^*)}A_1)^+ & 0 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (P_{N(A_2^*)}A_1)^+A_1 & (P_{N(A_2^*)}A_1)^+A_2 \\ A_2^+A_1 - A_2^+A_1(P_{N(A_2^*)}A_1)^+A_1 & A_2^+A_2 - A_2^+A_1(P_{N(A_2^*)}A_1)^+A_2 \end{bmatrix}$$

We know that: $(P_{N(A_2^*)}A_1)^+ = (P_{N(A_2^*)}A_1)^+P_{N(A_2^*)}$, then $XA$ becomes:

$$XA = \begin{bmatrix} (P_{N(A_2^*)}A_1)^+P_{N(A_2^*)}A_1 & 0 \\ A_2^+A_1 - A_2^+A_1(P_{N(A_2^*)}A_1)^+P_{N(A_2^*)}A_1 & A_2^+A_2 \end{bmatrix}$$

So, $(XA)^* = XA$ iff $A_2^+A_1 = A_2^+A_1((P_{N(A_2^*)}A_1)^+P_{N(A_2^*)}A_1)$, or is equivalent by item 2) of lemma (4) to $N(P_{N(A_1^*)}A_2) \subset N(A_1^+A_2) = N(A_1^*A_2)$. We have already illustrated that $(A_1, A_2) \in DR$ is equivalent to $N(P_{N(A_1^*)}A_2) \subset N(A_1^*A_2)$ into the lemma (9). $\square$

To prove that the assertions 1), 2) and 3) of b) are equivalent, it is sufficient to follow the steps of the party a), whereas instead of $(3-2)$ of lemma (37), we apply item $(3-4)$ of lemma (37).

**Remark 41.** *Suppose that $A_1$ and $A_2$ have a closed ranges, then:*

$\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ has a closed range $\Leftrightarrow$ $R(A_1)+R(A_2)$ is closed $\Leftrightarrow$ $R(P_{N(A_2^*)}A_1)$ is closed $\Leftrightarrow$ $R(P_{N(A_1^*)}A_2)$ is closed $\Leftrightarrow$ $R(A_1)+R(P_{N(A_1^*)}A_2)$ is closed $\Leftrightarrow$ $R(A_2)+R(P_{N(A_2^*)}A_1)$ is closed.

**Theorem 42.** *Let $M = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ be a $2\times 2$ row block operator suth that $R(A_2)$, $R(A_1)$ and $R(M)$ are closed, then the following statements are equivalent:*

a) $(A_1, A_2) \in DR$,

b) $M^+$ has the form

$$M^+ = \begin{bmatrix} (P_{N(A_2^*)}A_1)^+ & 0 \\ (P_{N(A_1^*)}A_2)^+ & 0 \end{bmatrix} := Z \qquad (3-8)$$

**Proof.** a)⇒b): Let $Z = \begin{bmatrix} (P_{N(A_2^*)}A_1)^+ & 0 \\ (P_{N(A_1^*)}A_2)^+ & 0 \end{bmatrix}$, Now we will see that $Z$ satisfies the equations of the M-P inverse of $M$, firstly, applying the item 3 of lemma (4) we get that $Z$ satisfies the equation $(iv)$:

$$(*) \qquad ZM = \begin{bmatrix} (P_{N(A_2^*)}A_1)^+ P_{N(A_2^*)}A_1 & 0 \\ 0 & (P_{N(A_1^*)}A_2)^+ P_{N(A_2^*)}A_2 \end{bmatrix}$$

Remark that $(A_1, A_2) \in DR$ is equivalent, by lemma (9) to at each one of these equalities

$$N((P_{N(A_2^*)}A_1)^+ P_{N(A_2^*)}A_1) = N(A_1),\ N((P_{N(A_1^*)}A_2)^+ P_{N(A_1^*)}A_2) = N(A_2)$$

Hence, the multiplication of the equality $(*)$ on the left by $M$ and using the item 2 of lemma (4), we find that $Z$ satisfies the equation $(i)$, and the multiplication of the equality $(*)$ on the right by $Z$ we find that $Z$ satisfies the equation $(ii)$. It remains see that $Z$ satisfies the equation $(iii)$, it results from the equations $(i)$ and $(ii)$ that $MZ$ which has the matrix form below is a projection

$$MZ = \begin{bmatrix} A_1(P_{N(A_2^*)}A_1)^+ + A_2(P_{N(A_1^*)}A_2)^+ & 0 \\ 0 & 0 \end{bmatrix}$$

and we have

$$R(MZ) = R(M),\ N(MZ) = N(Z) = N(A_1(P_{N(A_2^*)}A_1)^+ + A_2(P_{N(A_1^*)}A_2)^+) \oplus F$$

We consider the orthogonal projection

$$Q = \begin{bmatrix} P_{R(A_1)+R(A_2),N(A_1^*)\cap N(A_2^*)} & 0 \\ 0 & 0 \end{bmatrix} \in B(K \oplus F, K \oplus F)$$

From where

$$R(Q) = R(M) \text{ and } N(Q) = N(A_1^*) \cap N(A_2^*) \oplus F,$$

We will see that $MZ = Q$. These inclusions are easy to check

$$N(A_1^*) \cap N(A_2^*) \subset N(A_1^* P_{N(A_2^*)}) \subset N(A_1(P_{N(A_2^*)}A_1)^+)$$

And

$$N(A_1^*) \cap N(A_2^*) \subset N(A_2^* P_{N(A_1^*)}) \subset N(A_2(P_{N(A_1^*)}A_2)^+),$$

Hence

$$N(A_1^*) \cap N(A_2^*) \subset N(A_1(P_{N(A_2^*)}A_1)^\dagger + A_2(P_{N(A_1^*)}A_2)^+)$$

Which implies that

$$N(A_1^*) \cap N(A_2^*) \oplus F \subset N(MZ)$$

Consequently $N(Q) \subset N(MZ)$ and $R(Q) = R(M)$, it follows from the item 4 of lemma (4) that $MZ = Q$.

b)⇒a): $M^+$ has the form $(3-8)$ then the equality $MM^+M = M$ is satisfied and it is equivalent to

$$\begin{bmatrix} A_1(P_{N(A_2^*)}A_1)^+A_1 & A_2(P_{N(A_1^*)}A_2)^+A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$$

Which implies that $A_1(P_{N(A_2^*)}A_1)^+A_1 = A_1$, next using the items 3 and 2 of lemma (4), we obtain $N(P_{N(A_2^*)}A_1) = N(A_1)$ that is the item 3 of lemma (9), or equivalently $(A_1, A_2) \in DR$. □

### 3.2. Representations for the M-P inverse of a $2 \times 2$ triangular block operator with disjoint ranges operators

We obtain the necessary and sufficient conditions for the existence of the M-P inverse of triangular block operator and its M-P inverse with disjoint ranges operators.

In reality, there are four positions of the triangular block operator, we will only study the case where $M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$, because the remaining cases can be converted into upper block triangular operator, for example:

Consider the operator $M = \begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix}$, we get $N = VMU = \begin{bmatrix} A_4 & A_3 \\ 0 & A_1 \end{bmatrix}$.

where: $U = \begin{bmatrix} 0 & I_H \\ I_L & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & I_K \\ I_F & 0 \end{bmatrix}$

**Theorem 43.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$ *be a* $2\times 2$ *upper block triangular operator,if* $A_1^+$ *exists, then* $M$ *has a closed range exists iff* $R(A_4^*) + R(A_2^* P_{N(A_1^*)})$ *is closed, in this case there exists an operator* $X$ *of the form:*

$$(3-9) \qquad X = \begin{bmatrix} A_1^+ - A_1^+A_2G^+D^* & -A_1^+A_2G^+A_4^* \\ G^+D^* & G^+A_4^* \end{bmatrix}$$

Which satisfies the three equations $(i)$, $(ii)$ and $(iii)$ of the M-P inverse of $M$; where $D = P_{N(A_1^*)}A_2$ and $G = D^*D + A_4^*A_4$, moroever, then the following statements are equivalent:

1) $M^+ = X$,

2) $N(G) \subset N(A_1^+A_2)$,

3) $\left( \begin{pmatrix} A_1 \\ 0 \end{pmatrix}, \begin{pmatrix} A_2 \\ A_4 \end{pmatrix} \right) \in DR.$

**Proof.** The $(3-2)$ of lemma $(37)$ and lemma $(5)$, implies that $M^+$exists if and only if $S_1^+$ exists, iff $S_1^*S_1$ has a closed range, while $S_1^*S_1 = \begin{bmatrix} A_1^*A_1 & 0 \\ 0 & G \end{bmatrix}$, it indicates that $M^+$ exists is equivalent to $R(G)$ is closed, notice that the operator $G$ is positive semi-definite, then $R(G)$ is closed iff $R(G^{\frac{1}{2}})$ is closed, and the lemma $(3)$ gives us that $R(G^{\frac{1}{2}}) = R(A_4^*)+R(A_2^*P_{N(A_1^*)})$, as a consequence $M^+$ exists iff $R(A_4^*)+R(A_2^*P_{N(A_1^*)})$ is closed. Returning to the decomposition $M = S_1R_1$, we assume that $S_1$ has a closed range, we know that $S_1^+ = (S_1^*S_1)^+S_1^*$ then by the item 1) of lemma $(3)$ the operator $X = R_1^{-1}(S_1^*S_1)^+S_1^*$ verifies the three equations $(i)$, $(ii)$ and $(iii)$ of the M-P inverse of $M$, at the end, from calculation, we obtain $(3-9)$. Now we will prove that the three items are equivalent, 1)$\Leftrightarrow$ 2): Clearly that

$$(XM)^* = XM \Leftrightarrow M^+ = X,$$

We have

$$XM = \begin{bmatrix} A_1^+A_1 & A_1^+A_2 - A_1^+A_2G^+G \\ 0 & G^+G \end{bmatrix}$$

Then, $(XM)^* = XM \Leftrightarrow A_1^+A_2 = A_1^+A_2G^+G$, by item 2) of lemma (4), it follows that $(XM)^* = XM \Leftrightarrow N(G) \subset N(A_1^+A_2)$. Consider the block operator $W = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}$, where $B_1 = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}$ $B_2 = \begin{bmatrix} A_2 \\ A_4 \end{bmatrix}$, then by the lemma (9), we get 2)$\Leftrightarrow$3). □

**Theorem 44.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$ *be a* $2\times2$ *upper triangular block operator,if* $A_4^+$ *exists, then* $M$ *has a closed range iff* $R(A_1) + R(A_2P_{N(A_4)})$ *is closed, in this case there exists an operator* $Y$ *of the form*

$$(3-10) \qquad Y = \begin{bmatrix} A_1^*E^+ & -A_1^*E^+A_2A_4^+ \\ R^*E^+ & A_4^+ - R^*E^+A_2A_4^+ \end{bmatrix}$$

*which satisfies the three conditions* $(i)$, $(ii)$ *and* $(iv)$ *of the M-P inverse of* $M$, *where* $R = A_2P_{N(A_4)}$ *and* $E = A_1\ A_1^* + RR^*$, *moroever, then the following statements are equivalent:*

1) $M^+ = Y$,

2) $R(A_2A_4^+) \subset R(E)$

3) $(\begin{pmatrix} 0 & A_4 \end{pmatrix}^*, \begin{pmatrix} A_1 & A_2 \end{pmatrix}^*) \in DR$.

**Proof.** From $(3-3)$ of the lemma (37), lemma (5) and lemma (2), respectively, $M^+$ exists iff $S_2S_2^*$ has a closed range, note that $S_2S_2^* = \begin{bmatrix} E & 0 \\ 0 & A_4^*A_4 \end{bmatrix}$, afterwards, way similarly to the proof of the theorem (43) we have $M^+$ exists iff $R(A_1) + R(A_2P_{N(A_4)})$ is closed, again in the decomposition $M = R_2S_2$, we assume $S_2$ to have closed range, then $S_2^+ = S_2^*(S_2S_2^*)^+$, and by the item 2) of lemma (6), we have $Y = S_2^*(S_2S_2^*)^+R_2^{-1}$ satisfies the three equations $(i)$, $(ii)$ and $(iv)$ of M-P inverse of $M$, wich is $(3-10)$. Similarly to the proof of the 1) ⇔2) in the theorem (43), we can prove that 1)⇔2) of this theorem. If we replace $A$ and $B$ by $\begin{pmatrix} 0 & A_4 \end{pmatrix}$ and $\begin{pmatrix} A_1 & A_2 \end{pmatrix}$, respectively, in lemma 10 , the item 3) is equivalent to $N(E) \subset N(A_4^{+^*}A_2^*)$, next since $E$ is self-adjoint with closed range and $R(A_2A_4^+) \subset \overline{R(A_2A_4^+)}$, so

$$N(E) \subset N(A_4^{+^*}A_2^*) \Leftrightarrow R(A_2A_4^+) \subset R(E)$$

Consequently, 2)⇔3). □

**Proposition 45.** *a) In the theorem* (43)*, If* $(A_1, A_2) \in DR$*, then* $M^+ = X$.

b) In the theorem (44), If $(A_2^*, A_4^*) \in DR$, then $M^+ = Y$.

**Proof.** a) We put $C_1 = \begin{pmatrix} A_1 \\ 0 \end{pmatrix}$ and $C_2 = \begin{pmatrix} A_2 \\ A_4 \end{pmatrix}$. Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in R(C_1) \cap R(C_2)$, then there exist $x$, $x'$ such that $y_1 = A_1x = A_2x'$ and $y_2 = 0 = A_2x'$, now under the assumption of the item a), we get that $y_1 = 0$ and $y = 0$, so $(C_1, C_2) \in DR$, it follows by theorem (40) that $M^+ = X$. Similar to the proof of a), we can prove the item b). $\square$

**Remark 46.**

1) In the theorem (43):

*a*) We assume $A_1$ is surjective; (i.e., $A_1 A_1^+ = I$), then the M-P inverse of $M$ exists if and only if $A_4^+$ exists; in addition

$$M^+ = X \Leftrightarrow N(A_4) \subset N(A_1^+ A_2).$$

*b*) if $A_4$ is injective; (i.e, $A_4^+ A_4 = I$), then the M-P inverse of $M$ exists, because $R(A_4^*) = L$, in this case the positive operator $G$ is invertible, and we have $M^+ = X$

3) In the lemma (44):

*a*) if $A_4$ is injective, then $M^+$ exists iff $A_1^+$ exists, in addition

$$M^+ = Y \Leftrightarrow R(A_2 A_4^+) \subset R(A_1).$$

*b*) if $A_1$ is surjective, then the M-P inverse of $M$ exists, in addition $M^+ = Y$, because $E$ is invertible.

**Theorem 47.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$ *be a* $2\times 2$ *upper triangular block operator, we assume that* $A_1^+$ *and* $A_4^+$ *exist, then* $M^+$*has a close range iff* $R(P_{N(A_1^*)} A_2 P_{N(A_4)})$ *is closed, in this case, the following statements are equivalent:*

1) The M-P inverse of $M$ has the form:

$$(3-11) \qquad M^+ = \begin{bmatrix} A_1^+ - A_1^+A_2T^+ & A_1^+A_2T^+A_2A_4^+ - A_1^+A_2A_4^+ \\ T^\dagger & A_4^+ - T^+A_2A_4^+ \end{bmatrix}$$

2) $R(P_{N(A_1^*)}A_2A_4^+) \subset R(T)$ and $N(T) \subset N(A_1^+A_2P_{N(A_4)})$.

3) $(A_1, A_2P_{N(A_4)}) \in DR$ and $(A_4^*, A_2^*P_{N(A_1^*)}) \in DR$.

Where $T = P_{N(A_1^*)}A_2P_{N(A_4)}$.

**Proof.** Applying theorem (43), $M^+$ exists iff $R(A_2^*P_{N(A_1^*)}) + R(A_4^*)$ is closed, or equivalently to $\begin{bmatrix} A_2^*P_{N(A_1^*)} & A_4^* \\ 0 & 0 \end{bmatrix}^+$ exists, and by the party b) of theorem (40), if and only if $(P_{N(A_4)}A_2^*P_{N(A_1^*)})^+$ exists, using the adjoint of operator, $M^+$ exists iff $(P_{N(A_1^*)}A_2P_{N(A_4)})^+$ exists. □

1) ⇔2):

From the item $(3-5)$ of lemma (37), $M$ is equal to $R_4S_4H_4$, clearly that $R_4$ and $H_4$ are invertible and

$$R_4^{-1} = \begin{bmatrix} I & -P_{N(A_1^*)}A_2A_4^+ \\ 0 & I \end{bmatrix}, \quad H_4^{-1} = \begin{bmatrix} I & -A_1^+A_2 \\ 0 & I \end{bmatrix}$$

It is simple to see that $H_4^{-1}S_4^+R_4^{-1}$ satisfies the conditions $(i)$, $(ii)$ of the M-P inverse of $M$, thus

$$R(A_1) \perp R(T) \text{ and } R(A_4^*) \perp R(T^*)$$

Or equivalently

$$A_1^+T = 0 \text{ and } TA_4^+ = 0$$

Then we can check that $S_4^+$ has the form

$$S_4^+ = \begin{bmatrix} A_1^+ & 0 \\ T^+ & A_4^+ \end{bmatrix}$$

And then we have:

$$(3-12) \qquad H_4^{-1}S_4^+R_4^{-1} = \begin{bmatrix} A_1^+ - A_1^+A_2T^+ & A_1^+A_2T^+A_2A_4^+ - A_1^+A_2A_4^+ \\ T^+ & A_4^+ - T^+A_2A_4^+ \end{bmatrix}$$

Note that both sides to the right of the $(3-11)$ and $(3-12)$ are equal, beyond, we wil show that $R(P_{N(A_1^*)}A_2A_4^+) \subset R(T)$ and $N(T) \subset N(A_1^+A_2P_{N(A_4)})$ are necessary and sufficient conditions for which $M^+ = H_4^{-1}S_4^+R_4^{-1}$.

We have

$$MH_4^{-1}S_4^+R_4^{-1} = \begin{bmatrix} A_1A_1^+ & TT^+P_{N(A_1^*)}A_2A_4^+ - P_{N(A_1^*)}A_2A_4^+ \\ 0 & A_4A_4^+ \end{bmatrix}$$

So, $MH_4^{-1}S_4^+R_4^{-1}$ is self-adjoint is equivalent to $TT^+P_{N(A_1^*)}A_2A_4^+ = P_{N(A_1^*)}A_2A_4^+$, and by item 1) of lemma (4),

$$MH_4^{-1}S_4^+R \Leftrightarrow \; R(P_{N(A_1^*)}A_2A_4^+) \subset R(T)$$

Also, we have

$$H_4^{-1}S_4^{+}R_4^{-1}M = \begin{bmatrix} A_1^{+}A_1 & A_1^{+}A_2P_{N(A_4)} - A_1^{+}A_2P_{N(A_4)}T^{+}T \\ 0 & A_4^{+}A_4 \end{bmatrix}$$

Remark that $H_4^{-1}S_4^{+}R_4^{-1}M$ is self-adjoint iff $A_1^{+}A_2P_{N(A_4)} = A_1^{+}A_2P_{N(A_4)}T^{+}T$, then by the item 2) of lemma (4),

$$H_4^{-1}S_4^{+}R_4^{-1}M \Leftrightarrow N(T) \subset N(A_1^{+}A_2P_{N(A_4)}).$$

2) $\Leftrightarrow$3): We have

$$R(P_{N(A_1^*)}A_2A_4^{+}) \subset R(T) \ \Leftrightarrow N(T^*) \subset N(A_4^{+^*}A_2^*P_{N(A_1^*)}),$$

It follows from lemma (10) that

$$N(T^*) \subset N(A_4^{+^*}A_2^*P_{N(A_1^*)}) \Leftrightarrow \ (A_4^*, A_2^*P_{N(A_1^*)}) \in DR.$$

Also by lemma (9)

$$N(T) \subset N(A_1^{+}A_2P_{N(A_4)}) \Leftrightarrow (A_1, A_2P_{N(A_4)}) \in DR.$$

**Corollary 48.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$ *be a* $2\times 2$ *upper triangular block operator, such that* $A_1^{+}$, $A_4^{+}$ *and* $M^{+}$ *exist. If* $(A_1, A_2) \in DR$ *and* $(A_2^*, A_4^*) \in DR$*, then* $M^{+}$ *has the representation* $(3-11)$.

**Corollary 49.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$ *be a* $2\times 2$ *upper triangular block operator, such that* $A_1^+$, $A_2^+$ *and* $A_4^+$ *are exist, then:*

$$M^+ = \begin{bmatrix} A_1^+ & 0 \\ A_2^+ & A_4^+ \end{bmatrix} \Leftrightarrow A_1^*A_2 = 0 \text{ and } A_2A_4^* = 0.$$

## 3.3. Representations for the M-P inverse of a $2 \times 2$ block operator with disjoint ranges operators

We obtain representations of the M-P inverse of a $2 \times 2$ block operator under condition

$$(\begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix}, \begin{pmatrix} A_3^* \\ A_4^* \end{pmatrix}) \in DR$$

We get the proofs of this result in the end of chapter 5.

**Theorem 50.** *Let* $M$ *be defined as in* $(3-1)$ *with closed range such that* $R(A_1) + R(A_2)$ *and* $R(A_3) + R(A_4)$ *are closed, if* $(\begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix}, \begin{pmatrix} A_3^* \\ A_4^* \end{pmatrix}) \in DR$, *then*

$$(3-13) \qquad M^+ = \begin{bmatrix} A_1^*S_1^+ - W_1^+\Upsilon_1^+ZS_1^+ & W_1^+\Upsilon_1^+ \\ A_2^*S_1^+ - W_2\Upsilon_1^+ZS_1^+ & W_2\Upsilon_1^+ \end{bmatrix}$$

*Where* $S_1 = A_1A_1^* + A_2A_2^*$, $Z = A_3A_1^* + A_4A_2^*$, $W_1 = A_3 - ZS_1^+A_1$, $W_2 = A_4 - ZS_1^+A_2$ $\Upsilon_1 = W_1W_1^* + W_2W_2^*$.

**Corollary 51.** *Let $M$ be defined as in $(3-1)$ with closed range such that $R(A_1)+R(A_2)$ and $R(A_3)+R(A_4)$ are closed, if* $R\begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix} \perp R\begin{pmatrix} A_3^* \\ A_4^* \end{pmatrix}$, *then*

$$(3-14) \qquad M^+ = \begin{bmatrix} A_1^*S_1^+ & A_3^*S_2^+ \\ A_2^*S_1^+ & A_4^*S_2^+ \end{bmatrix}$$

*Where* $S_1 = A_1A_1^* + A_2A_2^*$, $S_2 = A_3A_3^* + A_4A_4^*$

**Theorem 52.** *Let $M$ be defined as in $(3-1)$ with closed range such that $R(A_1)+R(A_2)$ and $R(A_3)+R(A_4)$ are closed, if* $(\begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix}, \begin{pmatrix} A_3^* \\ A_4^* \end{pmatrix}) \in DR$, *then*

$$(2-15) \qquad M^+ = \begin{bmatrix} W_3\Upsilon_2^+S_1S_1^+ & W_1\Upsilon_1^+S_2S_2^+ \\ W_4\Upsilon_2^+S_1S_1^+ & W_2\Upsilon_1^+S_2S_2^+ \end{bmatrix}$$

*Where* $S_1 = A_1A_1^* + A_2A_2^*$, $S_2 = A_3A_3^* + A_4A_4^*$, $Z = A_3A_1^* + A_4A_2^*$, $W_3 = A_1 - Z^*S_2^+A_3$, $W_4 = A_2 - Z^*S_2^+A_4$, $\Upsilon_1 = W_1W_1^* + W_2W_2^*$ *and* $\Upsilon_2 = W_3W_3^* + W_4W_4^*$.

### 3.4. Representation of the M-P inverse of a $2 \times 2$ row block operator

We give a representation of M-P inverse of row operator block, based on full-rank decomposition.

**Theorem 53.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ *be a $2\times2$ block operator, if $F_{A_1}G_{A_1}$ and $F_{D_1}G_{D_1}$ are a full-rank decompositions of $A_1$ and $D_1 = P_{N(A_1^*)}A_2$, respectively, then $M$*

*has a full-rank decomposition as follows:*

$$(3-16) \qquad M = \begin{bmatrix} F_{A_1} & F_{D_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_{A_1} & F_{A_1}^{+}A_2 \\ 0 & G_{D_1} \end{bmatrix} := F_M G_M$$

*In addition, the M-P inverse of $M$ has the form:*

$$(3-17) \qquad M^{+} = \begin{bmatrix} G_{A_1}^{*} T_1^{-1} F_{A_1}^{+}(I - A_2 D_1^{+}) & 0 \\ L_1^{*} T_1^{-1} F_{A_1}^{+}(I - A_2 D_1^{+}) - D_1^{+} & 0 \end{bmatrix}$$

Where:

$D_1 = P_{N(A_1^*)} A_2$

$L_1 = F_{A_1}^{+} A_2 (I - D_1^{+} D_1)$

$T_1 = G_{D_1} G_{D_1}^{*} + L_1 L_1^{*}$

**Proof.** The decomposition $(3-16)$ is obtained by this way:

$$M = \begin{bmatrix} A_1 & D_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & A_1^{+}A_2 \\ 0 & I \end{bmatrix}$$

$$M = \begin{bmatrix} F_{A_1} & F_{D_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_{A_1} & 0 \\ 0 & G_{D_1} \end{bmatrix} \begin{bmatrix} I & A_1^{+}A_2 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} F_{A_1} & F_{D_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_{A_1} & A_1^{+}A_2 \\ 0 & G_{D_1} \end{bmatrix} := F_M G_M \qquad \square$$

Now, we will illustrate through the following three points that $F_M G_M$ is a full-rank decomposition of $M$

**Point 1**: From the assumptions and the definition (12), there are two complex Hilbert spaces $H_{A_1}$ and $H_{D_1}$, Consequently, $K_M := H_{A_1} \oplus H_{D_1}$ is also complex Hilbert space, thus, observe that:

$$F_M \in B(H_M, K \oplus F),$$

And

$$G_M \in B(H \oplus L, H_M).$$

**Point 2**:

Since $R(D_1) \subset R\left(P_{N(A_1^*)}\right) = N(A_1^*)$, then $R(A_1) \perp R(D_1)$, it automatically implies that $R(A_1) \cap R(D_1) = \{0\}$, in this case, the proposition (38) proves that the block operator $F_M$ is injective.

**Point 3**:

Applying of the lemma (5) we get

$$R(G_M) = R\left(\begin{bmatrix} G_{A_1} & 0 \\ 0 & G_{D_1} \end{bmatrix}\begin{bmatrix} I & A_1^+ A_2 \\ 0 & I \end{bmatrix}\right) = R\left(\begin{bmatrix} G_{A_1} & 0 \\ 0 & G_{D_1} \end{bmatrix}\right)$$

That is

$$R(G_M) = R(G_{A_1}) \oplus R(G_{D_1}) = K \oplus F$$

Then $G_M$ is surjective.

Now, as $R(A_1) \perp R(D_1)$, then by the Lemma (14), $R(F_{A_1}) \perp R(F_{D_1})$, it follows from theorem (39) that:

$$F_M^+ = \begin{bmatrix} F_{A_1}^+ & 0 \\ F_{D_1}^+ & 0 \end{bmatrix}$$

Since $G_{D_1}$ is surjective then by the item $(3-10)$ of theorem $(44)$ :

$$G_M^+ = \begin{bmatrix} G_{A_1} & F_{A_1}^+ A_2 \\ 0 & G_{D_1} \end{bmatrix}^+ = \begin{bmatrix} G_{A_1}^* T_1^{-1} & -G_{A_1}^* T_1^{-1} F_{A_1}^+ A_2 G_{D_1}^+ \\ L_1^* T_1^{-1} & G_{D_1}^+ - R^* T_1^{-1} F_{A_1}^+ A_2 G_{D_1}^+ \end{bmatrix}$$

Finally, $M^+ = G_M^+ F_M^+$ is the formula $(3-17)$.

**Theorem 54.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ *be a* $2\times 2$ *block operator, if* $F_{A_2} G_{A_2}$ *and* $F_{D_2} G_{D_2}$ *are full-rank decompositions of* $A_2$ *and* $D_2 = P_{N(A_2^*)} A_1$*, respectively, then the decomposition :*

$$(3-18) \qquad M = \begin{bmatrix} F_{D_2} & F_{A_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_{D_2} & 0 \\ F_{A_2}^{\dagger} A_1 & G_{A_2} \end{bmatrix} := F_M G_M$$

Is a full-rank decomposition of $M$. Therefore, the block operator $M^{\dagger}$ has the form:

$$(3-19) \qquad M^+ = \begin{bmatrix} L_2^* T_2^{-1} F_{A_2}^+ (I - A_1 D_2^+) - D_2^+ & 0 \\ G_{A_2}^* T_2^{-1} F_{A_2}^+ (I - A_1 D_2^+) & 0 \end{bmatrix}$$

Where

$L_2 = F_{A_2}^+ A_1 (I - D_2^+ D_2)$

$T_2 = G_{D_2} G^*_{D_2} + L_2 L^*_2$.

### 3.5. Representation for the M-P inverse of a $2 \times 2$ block triangular operator

We derive from our results ( theorems (43), (44) ) a representation of the M-P inverse of triangular block operator.

**Theorem 55.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$ *be a* $2\times 2$ *upper triangular block operator, if* $A_1^+$, $A_4^+$ *exist, then the following statements are equivalent:*

1) $M$ has a close range.

2) $R(A_4^*) + R(A_2^* P_{N(A_1^*)})$ is closed.

3) $R(A_1) + R(A_2 P_{N(A_4)})$ is closed.

And the M-P inverse of $M$ has the follows representation :

$$M^+ = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} \tag{3-20}$$

Where:

$N_1 = A_1^* E^+ P_{R(A_1)} + A_1^* E^+ (D - A_2 P_{R(A_4^*)}) G^+ D^*$.

$N_2 = -A_1^* E^+ (D - A_2 P_{R(A_4^*)}) G^+ A_4^*$

$N_3 = R^* E^+ P_{R(A_1)} + (R^* E^+ (R - A_1 A_1^+ A_2) + P_{R(A_4^*)}) G^+ D^*$.

$N_4 = (R^* E^+ (R - A_1 A_1^+ A_2) + P_{R(A_4^*)}) G^+ D^*$

**Proof.** From theorems (43) and (44), we obtain that, the items 1), 2), 3) are equivalent, now the representation M-P the inverse of $M$ which is in $(3-20)$, it follows from $(3-9)$, $(3-10)$, and lemma (7). □

### 3.6. Correction to: Representation of the Moore-Penrose for a class of 2-by-2 block operator valued partial matrices, see ([11],[12]).

In this subchapter will prove that the results [11; Theorem 9, Theorem 10] and [12,Corollary 13, Corollary14] are not true. Our objective is to discover and see that the representations of the M-P inverse in each item of the [12;Corollary13, Corollary14], [11; Theorem 9, Theorem 10] are not true. That is why, we give two examples, the first is a counter-example and the second illustrates the illogical step in the proofs of these result. Next, we will propose their corrections.

The result below are copies of the reference [11] and [12] without changing the notations:

Now we consider [11,Theorem 9, Theorem 10] :

**Theorem 56.** [11, *Theorem* 9] *Let* $M$ *be defined as Eqn .(6),* $R(A), R(D)$ *be closed*

*such that* $AC^* = 0$ *and* $D^*C = 0$*.*

(1) If $R(A) \cap R(B) = \{0\}$, then $M$ is MP invertible if and only if $R(C)$ and $R(B_0)$ are closed and

$$M^{+}=\begin{bmatrix} A^{+} & C^{+} \\ B_0^{+}+(D^{+}D+B_0^{+}B_0-B_0^{+}B & (D^{+}D+B_0^{+}B_0- \\ )\triangle(B-B_0)^{*}(I-B_0B_0^{+}) & B_0^{+}B)\triangle D^{*} \end{bmatrix}$$

Where $\triangle=(D^{*}D+(B-B_0)^{*}(I-B_0B_0^{+})(B-B_0))^{+}$ and $B_0=(I-AA^{+})B(I-D^{+}D)$

(2) If $R(D^{*})\cap R(B^{*})=\{0\}$, then $M$ is MP invertible if and only if $R(C)$ and $R(B_0)$ are closed and

$$M^{+}=\begin{bmatrix} A^{*}\triangle(AA^{+}+B_0B_0^{+}-BB_0^{+}) & C^{+} \\ B_0^{+}+(I-B_0^{+}B_0)(B-B_0)^{*}\triangle(AA^{+}+ & \\ B_0B_0^{+}-BB_0^{+}) & D^{+} \end{bmatrix}$$

Where $\triangle=(AA^{*}+(B-B_0)(I-B_0^{+}B_0)(B-B_0)^{*})^{+}$ and $B_0=(I-AA^{+})B(I-D^{+}D)$

(3) If $R(A)\cap R(B)=\{0\}$ and $R(D^{*})\cap R(B^{*})=\{0\}$, then $M$ is MP invertible if and only if $R(C)$ and $R(B)$ are closed and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{+}=\begin{bmatrix} A^{+} & C^{+} \\ B^{+} & D^{+} \end{bmatrix}$$

**Proof.** (1) since $R(A)\cap R(B)=\{0\}$, $R(A)$ and $R(D)$ are closed, $S'$ has the form

$$(7) \qquad S' = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_1 & B_2 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} N(A) \\ R(A^*) \\ R(D^*) \\ N(D) \end{bmatrix} \rightarrow \begin{bmatrix} N(A^*) \\ R(A) \\ R(D) \\ N(D^*) \end{bmatrix}$$

.

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. □

**Theorem 57.** [11, *Theorem* 10] *Let $M$ be defined as Eqn.(6), $R(B)$, $R(C)$ be closed such that $BD^* = 0$ and $C^*D = 0$.*

(1) If $R(A) \cap R(B) = \{0\}$, then $M$ is MP invertible if and only if $R(A_0)$ and $R(D)$ are closed and

$$M^+ = \begin{bmatrix} A_0^+ + (C^+C + A_0^+A_0 - A_0^+A\ )\triangle_0 & (C^+C + A_0^+A_0 - \\ (A - A_0)^*(I - A_0A_0^+) & A_0^+A)\triangle_0C^* \\ B^+ & D^+ \end{bmatrix}$$

Where $\triangle_0 = (C^*C + (A - A_0)^*(I - A_0A_0^+)(A - A_0))^+$ and $A_0 = (I - BB^+)A(I - C^+C)$

(2) If $R(A^*) \cap R(C^*) = \{0\}$, then $M$ is MP invertible if and only if $R(A_0)$ and $R(D)$ are closed and

$$M^+ = \begin{bmatrix} A_0^+ + (I - A_0^+ A_0)(A - A_0)^* \triangle_0 (BB^+ + A_0 A_0^+ - AA_0^+) & C^+ \\ B^* \triangle_0 (BB^+ + A_0 A_0^+ - AA_0^+) & D^+ \end{bmatrix}$$

Where $\triangle_0 = (BB^* + (A - A_0)(I - A_0^+ A_0)(A - A_0)^*)^+$ and $A_0 = (I - BB^+)A(I - C^+C)$

3) If $R(A) \cap R(B) = \{0\}$ and $R(A^*) \cap R(C^*) = \{0\}$, then $M$ is MP invertible if and only if $R(A)$ and $R(D)$ are closed and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^+ = \begin{bmatrix} A^+ & C^+ \\ B^+ & D^+ \end{bmatrix}$$

**Proof.** (1) since $R(A) \cap R(B) = \{0\}$, $R(B)$ and $R(C)$ are closed, $S_0$ has the form

$$(8) \qquad S_0 = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} N(C) \\ R(C^*) \\ R(B^*) \\ N(B) \end{bmatrix} \rightarrow \begin{bmatrix} N(B^*) \\ R(B) \\ R(C) \\ N(C^*) \end{bmatrix}$$

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. $\square$

And similarly we consider [12,Corollary13, Corollary14] :

**Corollary 58.** [12, *Corollary* 13] *Suppose that the 2-by-2 upper triangular matrix*

$\Gamma$*is given as in Theorem* 11

1) If $R(A) \cap R(C) = \{0\}$, then

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^{+} = \begin{bmatrix} A^{+} & 0 \\ C_0^{+}+(B^{+}B + C_0^{+}C_0 - C_0^{+}C \\ )\triangle(C - C_0)^{*}(I - C_0C_0^{+}) & (B^{+}B + C_0^{+}C_0 - C_0^{+}C \\ & )\triangle B^{*} \end{bmatrix}$$

Where $\triangle = (B^{*}B + (C - C_0)^{*}(I - C_0C_0^{+})(C - C_0))^{+}$ and $C_0 = (I - AA^{+})C(I - B^{+}B)$

2) If $R(C^{*}) \cap R(B^{*}) = \{0\}$, then

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^{+} = \begin{bmatrix} A^{*}\triangle(AA^{+} + C_0C_0^{+} - CC_0^{+}) & 0 \\ C_0^{+}+(I - C_0^{+}C_0)(C - C_0)^{*}\triangle(AA^{+}+C_0C_0^{+} & \\ -CC_0^{+}) & B^{+} \end{bmatrix}$$

$$C_0^{+}+(I - C_0^{+}C_0)(C - C_0)^{*}\triangle(AA^{+}+ \\ C_0C_0^{+} - CC_0^{+})$$

Where $\triangle = (A^*A + (C - C_0)(I - C_0^+ C_0)(C - C_0)^*)^+$ and $C_0 = (I - AA^+)C(I - B^+B)$

3) If $R(A) \cap R(C) = \{0\}$ and $R(B^*) \cap R(C^*) = \{0\}$, then

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^+ = \begin{bmatrix} A^+ & 0 \\ C^+ & B^+ \end{bmatrix}$$

**Proof.** (1) since $R(A) \cap R(C) = \{0\}$, $R(A)$ and $R(B)$ are closed, $\Gamma$ has the form

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & C_1 & C_2 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} N(A) \\ R(A^*) \\ R(B^*) \\ N(B) \end{bmatrix} \rightarrow \begin{bmatrix} N(A^*) \\ R(A) \\ R(B) \\ N(B^*) \end{bmatrix} \tag{12}$$

.

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. □

**Corollary 59.** [12, *Corollary* 14] *Let $A \in B(H)$, $B \in B(K), C \in B(K,H)$ and $R(A)$ be closed. Then the 2-by-2 block operator valued $\Gamma$ is MP invertible if and only if*

*$R(C^*(I-AA^+)+R(B^*)$ is closed , moreover, and*

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^{\{1\}} = \begin{bmatrix} A^+ - A^+C(B^*B + C^*(I- & -A^+C(B^*B + C^*(I- \\ AA^+)C)^+C^*(I-AA^+) & AA^+)C)^+B^* \\ (B^*B + C^*(I-AA^+)C)^+C^*( & (B^*B + C^*(I- \\ I-AA^+) & AA^+)C)^+B^* \end{bmatrix}$$

*Moreover, if $R(A) \cap R(C) = \{0\}$, then*

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^{+} = \begin{bmatrix} A^+ & 0 \\ (B^*B + C^*C)^+C^* & (B^*B + C^*C)^+B^* \end{bmatrix}$$

**Proof.** .............

"If $R(A) \cap R(C) = \{0\}$, then $C_1 = 0$ in equation (13)"

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We give an example concerning corollaries 13 and 14 of reference [12]

**Example 60.** *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, *it is clear that $R(A) \cap R(C) = \{0\}$ and $R(C^*) \cap R(B^*) = \{0\}$ hold, then:*

a) According to item (1) of corollary 13 in [12], the representation of the M-P inverse of $\Gamma$ has the form

$$(3-21) \qquad \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \end{bmatrix} := \Omega_1$$

b) According to item (3) of corollary 13 in [12], the representation of the M-P inverse of $\Gamma$ has the form

$$(3-22) \qquad \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} := \Omega_2$$

c) According to corollary 14 in [12], the representation of the M-P inverse of $\Gamma$ has the form

$$(3-23) \qquad \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \end{bmatrix} := \Omega_3$$

When, we multiply the representations $\Omega_1$ and $\Omega_2$ on the left by $\Gamma$, we find:

$$(3-24) \qquad \Gamma\Omega_1 = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$(3-25) \qquad \Gamma\Omega_2 = \begin{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

When we multiply the representation $\Omega_3$ on the right by $\Gamma$, we find:

$$(3-26) \qquad \Omega_3\Gamma = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

We note that the representations $\Omega_1$ and $\Omega_2$ dont satisfy the equation $(iii)$ of the M-P inverse of $\Gamma$ (ie, $\Gamma\Omega_1$ and $\Gamma\Omega_2$ are not self-adjoint), while $\Omega_3$ does not satisfy the equations $(iv)$ of the inverse of M-P of $M$, (ie, $\Omega_3\Gamma$ is not self-adjoint), it results that $\Omega_1 \neq \Gamma^+$, $\Omega_2 \neq \Gamma^+$ and $\Omega_3 \neq \Gamma^+$, so the items of corollary 13 in [12] are not true, also the representation of the inverse of M-P of $\Gamma$ under the condition $R(A) \cap R(C) = \{0\}$ in corollary 14 in [12] is not true, this allows us to ask the following question " where are the illogical steps in the proofs of corollaries 13 and 14 in [12]? " in the following we answer it:

Suppose that $A,\ B \in B(L, F)$ have closed ranges, always $C$ has the following matrix decomposition with respect to orthogonal sums $L = R(B^*) \oplus^{\perp} N(B)$ and $F = N(A^*) \oplus^{\perp} R(A)$:

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} R(B^*) \\ N(B) \end{bmatrix} \to \begin{bmatrix} N(A^*) \\ R(A) \end{bmatrix} \qquad (3-27)$$

In [12] , Deng and Du. considered, under the condition $R(A) \cap R(C) = \{0\}$, that $C$ from $R(B^*) \oplus^{\perp} N(B)$ into $N(A^*) \oplus^{\perp} R(A)$ has the form $C = \begin{bmatrix} C_1 & C_2 \\ 0 & 0 \end{bmatrix}$ Where $C_3 = 0$ and $C_4 = 0$ , also

$\Gamma = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ has the form

$$\Gamma = \begin{bmatrix} 0 & 0 & C_1 & C_2 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} N(A) \\ R(A^*) \\ R(B^*) \\ N(B) \end{bmatrix} \to \begin{bmatrix} N(A^*) \\ R(A) \\ R(B) \\ N(B^*) \end{bmatrix}$$

which is denoted by (12) in [12, proof of corollary 13]

The following counter-example illustrates that even if $R(A) \cap R(C) = \{0\}$ then $C_3$ and $C_4$ are not always nulle,

**Example 61.** *Let* $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$, *and* $B = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, *We have* $R(A) \cap R(C) = \{0\}$, $R(B^*) = \left\langle \binom{1}{0} \right\rangle$, $N(B) = \left\langle \binom{0}{1} \right\rangle$, $N(A^*) = \left\langle \binom{0}{1} \right\rangle$ *and* $R(A) = \left\langle \binom{1}{0} \right\rangle$. *Now note that* $C\binom{1}{0} = \binom{2}{1} = 1\binom{0}{1} + 2\binom{1}{0}$ *and* $C\binom{0}{1} = \binom{2}{1} = 1\binom{0}{1} + 2\binom{1}{0}$, *then* $C$ *has the form*

$$(3-28) \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} : \begin{bmatrix} \left\langle \binom{1}{0} \right\rangle \\ \left\langle \binom{0}{1} \right\rangle \end{bmatrix} \to \begin{bmatrix} \left\langle \binom{0}{1} \right\rangle \\ \left\langle \binom{1}{0} \right\rangle \end{bmatrix}$$

*By identification between* $(3-27)$ *and* $(3-28)$, *we get* $C_1 = C_2 = 1$ *and* $C_3 = C_4 = 2$.

We derive from the previous example these remarks:

**Remark 62.** *1) The illogical step in the proof of corollary* 13 *in* [12] *is that the matrix representation which is noted by* (12), *also in the proof of corollary* 14 *of* [12], *the phrase:*

*"If* $R(A) \cap R(C) = \{0\}$, *then* $C_1 = 0$ *in equation* (13)*"*

**Remark 63.** *The illogical steps in the proofs of theorem* 9 *and* 10 *in* [11] *are due to the matrix representations which are noted by* (7) *and* (8).

Noted that the condition $R(A) \cap R(C) = \{0\}$ does not necessarily imply this inclusion $R(C) \subset N(A^*)$, But it follows from lemma (4) that $R(C) \subset N(A^*) \Leftrightarrow P_{N(A^*)}C = C$ While $P_{N(A^*)}C = C \Leftrightarrow A^+AC = 0$ Similarly to proof of theorem 13 $A^+AC = 0 \Leftrightarrow R(A) \perp R(C)$, So

$$R(C) \subset N(A^*) \Leftrightarrow R(A) \perp R(C)$$

**Remark 64.** *from the last equivalence, if one replaces the interssection between the ranges in the hypotheses of the items of* [11;*Theorem* 9, *Theorem* 10] *and* [12,*Corollary* 13, *Corollary*14] *by the orthogonality the results and their proofs remain true, to illustrate this remark, we propose for example:*

The corollaries 13 in [12] may be refomulated as follows:

**Corollary 65.** [12, *Corollary* 13] *Suppose that the 2-by-2 upper triangular matrix* $\Gamma = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ *is given as in Theorem* 11

1) If $R(A) \perp R(C)$, then

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^+ = \begin{bmatrix} A^+ & 0 \\ C_0^+ + (B^+B + C_0^+C_0 - C_0^+C)\triangle(C - C_0)^*(I - C_0C_0^+) & (B^+B + C_0^+C_0 - C_0^+C)\triangle B^* \end{bmatrix}$$

Where $\triangle = (B^*B + (C - C_0)^*(I - C_0C_0^+)(C - C_0))^+$ and $C_0 = (I - AA^+)C(I - B^+B)$

2) If $R(C^*) \perp R(B^*)$, then

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^+ = \begin{bmatrix} A^*\triangle(AA^+ + C_0C_0^+ - CC_0^+) & 0 \\ C_0^+ + (I - C_0^+C_0)(C - C_0)^*\triangle(AA^+ + C_0C_0^+ - CC_0^+) & B^+ \end{bmatrix}$$

Where $\triangle = (A^*A + (C - C_0)(I - C_0^+C_0)(C - C_0)^*)^+$ and $C_0 = (I - AA^+)C(I - B^+B)$

3) If $R(A) \perp R(C)$ and $R(C^*) \perp R(B^*)$, then

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^+ = \begin{bmatrix} A^+ & 0 \\ C^+ & B^+ \end{bmatrix}$$

**Remark 66.** *We propose instead of* [11;*Theorem* 9*, Theorem* 10] *and* [12,*Corollary* 13*, Corollary*14], *we can use our results in proposition* (45) *and the corollaries* (48) *and* (49) *in this thesis .*

CHAPTER 4

# On the product of operators with closed range.

Let $A \in B(H,K)$ and $B \in B(L,H)$,with closed ranges, the following problem: "when the product of two operators with closed ranges has closed range", has been studied for the first time in 1973 by Bouldin [5], in his work based on the notion of the angle between two closed subspaces $M$ and $N$, to demonstrate that: $AB$ has closed range iff the angle of Dixmier between $R(B)$ and $N(A) \cap [N(A) \cap R(B)]^{\perp}$ is positive. From the angles of Friedrichs and Dixmier in [14, Theorem 22], Deutsch proved that the product $AB$ has closed range iff $c(N(A)\ ,\ R(B)\ ) < 1$, iff $N(A) + R(B)$ is closed or equivalently $N(B^{*}) + R(A^{*})$ is closed. Another author, Izumino used the lower bound $\gamma(A)$ of $A$ ($\gamma(A)$, defined by $\gamma(A) = \inf\{\|Ax\| : x \in (\ker A)^{\perp}, \|x\| = 1\}$) in [28, Corollary 2.5] , to prove the equivalence between: (i) $AB$ has closed range and; (ii) $P_{N(A)} + P_{R(B)}$ has closed range, (iii) $N(A) + R(B)$ is closed.

We apply our main results, to give some necessary and sufficient conditions equivalent for the product of two operators with closed ranges to have closed range.

**Proposition 67.** *Let $A \in B(K,L)$ and $B \in B(H,K)$, assume that $A^{+}$ and $B^{+}$exist, Then the following statements are equivalent:*

1) $(AB)^{+}$exists,

2) $\left(P_{R(A^*)}P_{R(B)}\right)^+$ exists,

3) $\begin{bmatrix} P_{N(A)} & P_{R(B)} \\ 0 & 0 \end{bmatrix}^+$ exists,

4) $N(A)+R(B)$ is closed,

5) $\begin{bmatrix} P_{N(B^*)} & P_{R(A^*)} \\ 0 & 0 \end{bmatrix}^+$ exists,

6) $N(B^*)+R(A^*)$ is closed,

7) $(P_{N(B^*)}\ P_{N(A)})^+$ exists,

8) $\left(P_{N(A)}+P_{R(B)}\right)^+$ exists,

9) $\left(P_{N(B^*)}+P_{R(A^*)}\right)^+$ exsits,

10) $\begin{bmatrix} B & I \\ 0 & A \end{bmatrix}^+$ exists,

11) $B^*(I-(A^*A+I)^{-1})B$ has a closed range,

12) $A(I-(BB^*+I)^{-1})A^*$ has a closed range

**Proof.** It is clear that 3) $\Leftrightarrow$ 4), 5) $\Leftrightarrow$ 6). Note that

$$R(AB) = AR(B) = AR(P_{R(B)}) = R(AP_{R(B)})$$

Hence, □

$R(AB)$ is closed iff $R(AP_{R(B)})$ is closed, and by the lemma (2), $R(AP_{R(B)})$ is closed means that $R(P_{R(B)}A^*)$ is closed, as $R(P_{R(B)}A^*)\ = R(P_{R(B)}\ P_{R(A^*)})$, we deduce that 1) $\Leftrightarrow$ 2) are equivalent.

Using the party a) of the theorem (40), we get that 2) $\Leftrightarrow$ 3)$\Leftrightarrow$ 5).

By the party b) of the theorem (40), we obtain the equivalence 3) ⇔7).

Applying of the lemma (2), we have the equivalences 3) ⇔ 8) and 5) ⇔ 9).

Using the theorem (44) we have that 4) ⇔ 10).

We put $M = \begin{bmatrix} B & I \\ 0 & A \end{bmatrix}$, It follows from the lemma (2) that the statement 10 holds iff $MM^*$ has a closed range, since we have:

$$MM^* = \begin{bmatrix} BB^*+I & A^* \\ A & AA^* \end{bmatrix} =$$

$$\begin{bmatrix} I & 0 \\ A(BB^*+I)^{-1} & I \end{bmatrix} \begin{bmatrix} BB^*+I & 0 \\ 0 & A(I-(BB^*+I)^{-1})A^* \end{bmatrix} \begin{bmatrix} I & (BB^*+I)^{-1}A^* \\ 0 & I \end{bmatrix}$$

Remark that the left and right matrices in the previous equation are invertible, and $BB^*+I$ is invertible, so $M^+$ exists iff the statement 12 holds; i.e., 10) ⇔12).

Similarly, we can obtain that 10) ⇔11).

**Corollary 68.** *Let $P$ and $Q$ be orthogonal projections in $B(H)$, then the following statements are equivalent:*

1) $(PQ)^+$ exists,

2) $R(I-P)+R(Q)$ is closed,

3) $R(P)+R(I-Q)$ is closed,

4) $((I-P)(I-Q))^+$ exists,

5) $(I-P+Q)^+$ exists,

6) $(I+P-Q)^+$exists.

7) $\begin{bmatrix} P & I-Q \\ 0 & 0 \end{bmatrix}^+$ exists,

8) $\begin{bmatrix} I-P & Q \\ 0 & 0 \end{bmatrix}^+$ exists,

9) $P - P(Q+I)^{-1}P$ has a closed range,

10) $Q - (P+I)^{-1}Q$ has a closed range.

**Proposition 69.** *Let* $M = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ *a be* $2 \times 2$ *block operator , we assume that* $A_1^+$ *and* $A_2^+$ *exist, then the following statements are equivalent:*

1) $M$ has a closed range.

2) $R(A_1) + R(A_2)$ is closed.

3) $\begin{bmatrix} P_{R(A_1)} & P_{R(A_2)} \\ 0 & 0 \end{bmatrix}^+$ exists.

4) $\left(P_{N(A_1^*)} P_{R(A_2)}\right)^+$ exists.

5) $\begin{bmatrix} P_{N(A_1^*)} & P_{N(A_2^*)} \\ 0 & 0 \end{bmatrix}^+$ exists.

6) $\left(P_{R(A_1)} P_{N(A_2^*)}\right)^+$ exists.

7) $N(A_1^*) + N(A_2^*)$ is closed.

8) $\left(P_{R(A_1)} + P_{R(A_2)}\right)^+$ exists.

9) $\left(P_{N(A_1^*)} + P_{N(A_2^*)}\right)^+$ exists.

**Proof.** In the beginning, clearly that 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3), also 5) $\Leftrightarrow$ 7).

From the party a) of the theorem (40) we get 5) $\Leftrightarrow$ 6).

Using the party b) of theorem (40) and lemma (3) we obtain 3) $\Leftrightarrow$ 6), 4) $\Leftrightarrow$ 5).

By the lemma (2) we have 3) $\Leftrightarrow$ 8) and 5) $\Leftrightarrow$ 7) . $\square$

**Corollary 70.** *Let $P$ and $Q$ be orthogonal projectors in $B(H)$, then the following statements are equivalent:*

1) $(P+Q)^+$ exists.

2) $R(P)+R(Q)$ is closed.

3) $\begin{bmatrix} P & Q \\ 0 & 0 \end{bmatrix}^+$ exists.

4)$((I-P)Q)^+$ exists.

5) $(P(I-Q))^+$ exists.

6) $\begin{bmatrix} I-P & Q \\ 0 & 0 \end{bmatrix}^+$ exists.

7) $(I-P-Q+PQ)^+$ exists.

8) $N(I-P)+N(I-Q)$ is closed.

9) $(2I-P-Q)^+$ exists.

CHAPTER 5

# On M-P inverse of the sum two operators

For the special case where, $A$ and $B$ are matrices with both $AB^* = 0$ and $A^*B = 0$, M. R. Hestenes [26] has shown that $(A+B)^+ = A^+ + B^+$, four years later, Cline [9] has developed some representations for the M-P inverse of the sum $A + B$, where $A$ and $B$ satisfying only the single condition $AB^* = 0$. This result is derived as a particular case of a representation for the M-P inverse of the sum of two matrices, without the previous conditions, by C. G Hung and T. L. Markham in [27].

In [19]; Fill and Fishkind exhibit a neat relationship between the M-P inverse of a sum of two square matrices $A$ and $B$ and the M-P inverse of the individual terms, this is the Fill–Fishkind formula: $(A + B)^+ = (I - S)A^+(I - T) + SB^+T$, Provided that $R(A) \cap R(B) = \{0\}$ and $R(A^*) \cap R(B^*) = \{0\}$, Where: $S = (P_{N(B)^\perp} P_{N(A)})^+$ and $T = (P_{N(A^*)} P_{N(B^*)^\perp})^+$, Recently, in the setting of Hilbert spaces, for $A$ , $B \in B(H, K)$, Arias, Corach and Maestripieri in [2, Theorem 5.2] extend the Fill- fishkind formula to $A$ and $B$ with closed ranges, satisfying the assumptions: $R(A) \cap R(B) = \{0\}$ and $R(A^*) \cap R(B^*) = \{0\}$, $R(A + B) = R(A) + R(B)$ and $R(A^* + B^*) = R(A^*) + R(B^*)$, a year after, Djikić in [16,Theorem 2.4] obtained the Fill-Fishkind formula for $A$ and $B$ with closed ranges satisfying these weak assumptions: $A$ and $B$ coincide on $R(A^*) \cap$

$R(B^{*})$, $R(A) \cap R(B) = \{0\}$ and $R(A+B)$ is closed, or these $A$ and $B$ coincide on $R(A^{*}) \cap R(B^{*})$, $R(A) \cap R(B) = \{0\}$, $R(A+B) = R(A) + R(B)$ and $R(A^{*}+B^{*}) = R(A^{*}) + R(B^{*})$.

### 5.1. Representation of the M-P inverse of the sum two operators

In this subchapter, we use the orthogonal sums of subspaces, for obtain a representation of the M-P inverse of sum two operators, in the closedness conditions for ranges.

We assume that the operator $A$ has a closed range, the operator $A$ has the following matrix form with respect to the orthogonal sums $K = R(A) \oplus^{\perp} N(A^{*})$ and $H = R(A^{*}) \oplus^{\perp} N(A)$ :

$$(5-1) \qquad A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} R(A^{*}) \\ N(A) \end{pmatrix} \rightarrow \begin{pmatrix} R(A) \\ N(A^{*}) \end{pmatrix}$$

Where $A_{11}$ is invertible. Moreover,

$$(5-2) \qquad A^{+} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} R(A) \\ N(A^{*}) \end{pmatrix} \rightarrow \begin{pmatrix} R(A^{*}) \\ N(A) \end{pmatrix}$$

To obtain the identity $(5-3)$, using the matrix forms of $A$ and $B$ with respect to the orthogonal sums above of $K$ and $H$, to transform the sum $A+B$ into a $2 \times 2$ block operator block, which is the $(5-4)$, hence by the theorem $(44)$ we get $(5-5)$ which is equivalent by identification to $(5-3)$.

**Theorem 71.** *If $R(A) \perp R(B)$; then $(A+B)^+$ exists iff $\Omega_A^+$ exists, and $(A+B)^+$ can be expressed as:*

$$(A+B)^+ = \Omega_A^+ + (I - \Omega_A^\dagger B) J_A^+ (\Delta_A^* + A^*) \tag{5-3}$$

*Where:*

$\Omega_A = BP_{N(A)}$,

$\Delta_A = (I - \Omega_A^+ \Omega_A) B$,

$J_A = A^* A + \Delta_A^* \Delta_A$

**Proof.** Under the assumption $R(A) \perp R(B)$, then $B$ has the matrix form:

$$B = \begin{bmatrix} 0 & 0 \\ B_{13} & B_{14} \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}$$

By the addition between $A$ and $B$ we have the matrix form of $A + B$

$$A + B = \begin{bmatrix} A_{11} & 0 \\ B_{13} & B_{14} \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \tag{5-4}$$

Hence,

$$\Omega_A = BP_{N(A)} = \begin{bmatrix} 0 & 0 \\ 0 & B_{14} \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}$$

$$\Delta_A = (I - \Omega_A \Omega_A^+) B = \begin{bmatrix} 0 & 0 \\ P_{N(B_{14}^*)} B_{13} & 0 \end{bmatrix}$$

And

$$J_A = A^*A + \Delta_A^*\Delta_A =$$

$$\begin{bmatrix} A_{11}^*A_{11}+(P_{N(B_{14}^*)}B_{13})^*(P_{N(B_{14}^*)}B_{13}) & 0 \\ 0 & 0 \end{bmatrix} := \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$$

It is clear that $\Omega_A^+$ exists iff $B_{14}^+$ exists, on the other hand as $A_{11}$ is invertible, we have

$$(*3) \qquad A + B = \begin{bmatrix} I & 0 \\ B_{13}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & B_{14} \end{bmatrix}$$

it follows from the lemma $(*3)$ that $(A+B)^+$ exists iff $B_{14}^+$ exists, then it is automatically $(A+B)^+$ exists iff $\Omega_A^+$ exists. We will find the expression $(5-3)$, applying the theorem (44), we get

$$(5-5) \qquad (A+B)^+ = \begin{bmatrix} \Sigma^+A_{11}^* & \Sigma^+(P_{N(B_{14}^*)}B_{13})^* \\ -B_{14}^+B_{13}\Sigma^+A_{11}^* & B_{14}^+ - B_{14}^+B_{13}\Sigma^+(P_{N(B_{14}^*)}B_{13})^* \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & B_{14}^+ \end{bmatrix} + \begin{bmatrix} \Sigma^+A_{11}^* & 0 \\ -B_{14}^+B_{13}\Sigma^+A_{11}^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Sigma^+(P_{N(B_{14}^*)}B_{13})^* \\ 0 & -B_{14}^+B_{13}\Sigma^+(P_{N(B_{14}^*)}B_{13})^* \end{bmatrix}$$

By identification

$$(A+B)^+ = \Omega_A^+ + (I - \Omega_A^+B)J_A^+A^* + (I - \Omega_A^+B)J_A^+\Delta_A^*$$

$$= \Omega_A^+ + (I - \Omega_A^+B)J_A^+(\Delta_A^* + A^*) \qquad \square$$

In the general case, $R(A) \perp R(B)$ is not alwyas verified, that is why, we use the notion of orthogonal projection to determine two bounded linears operators $\bar{A}, \bar{B} \in$

$B(H, K)$, satisfy

$$A+B=\bar{A}+\bar{B} \qquad \text{with} \qquad R(\bar{A}) \perp R(\bar{B})$$

For example, note that:

$A+B=P_{N(A^*)}B+A(I-A^+B)$, we consider $\bar{A}=P_{N(A^*)}B$ and $\bar{B}=A(I-A^+B)$, it is easy to see that $\bar{A}^*\bar{B}=0$, which is equivalent to $R(\bar{A}) \perp R(\bar{B})$, now if $R(\bar{A})$ is closed, thus as consequence of the theorem (71), we get a representation of M-P inverse of the sum $A+B$, see the following theorem.

**Theorem 72.** *We assume that $\bar{A}$ and $\Omega_{\bar{A}}$ have closed ranges, then:*

$$(5-6) \qquad (A+B)^+=\Omega_{\bar{A}}^+ + (I-\Omega_{\bar{A}}^+\bar{B})\ J_{\bar{A}}^+(\Delta_{\bar{A}}^*+\bar{A}^*)$$

Where:

$\Omega_{\bar{A}}=\bar{B}P_{N(\bar{A})}$,

$\Delta_{\bar{A}}=(I-\Omega_{\bar{A}}\Omega_{\bar{A}}^+)\bar{B}$,

$J_{\bar{A}}=\bar{A}^*\bar{A}+\Delta_{\bar{A}}^*\Delta_{\bar{A}}$.

### 5.2. Representations of the M-P inverse of the sum two operators with disjoint ranges.

In this subchapter, we assume that $A$ and $B$ have a closed ranges, by the full-rank decomposition of operators we give some representations of the M-P inverse of sum two operators with disjoint ranges. In what follows we need the following definition,

**Definition 73.** *We say that $A, B$ have the range additivity property if $R(A+B) = R(A) + R(B)$. We denote by $R$ the set of all these pairs $(A, B)$, i.e.,*

$$R := \{(A, B) : A, B \in L(H, K) \text{ and } R(A+B) = R(A) + R(B)\}$$

**Theorem 74.** *We have*

1) If $(A, B) \in DR$, then $(A^*, B^*) \in R$, and $R(A+B)$ is closed iff $R(A^*) + R(B^*)$ is closed.

2) If $(A^*, B^*) \in DR$, then $(A, B) \in R$, and $R(A+B)$ is closed iff $R(A) + R(B)$ is closed.

3) If $(A, B) \in DR$ and $(A^*, B^*) \in DR$, then

$$(A, B) \in R, \ (A^*, B^*) \in R,$$

In addition, $R(A+B)$, $R(A) + R(B)$ and $R(A^*) + R(B^*)$ are closed.

**Proof.** Let $F_A G_A$ and $F_B G_B$ are full-rank decomposition of $A$ and $B$ with $H_A = R(A)$ and $H_B = R(B)$, we consider the operator

$$M_0 = \begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix} \in B(H \oplus L, K \oplus F)$$

We have

$$(5-7) \qquad M_0 = \begin{bmatrix} F_A & F_B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_A & 0 \\ G_B & 0 \end{bmatrix} := A_0 B_0$$

Where

$$B_0 : H \oplus L \to R(A) \oplus R(B), \;\; A_0 : R(A) \oplus R(B) \to K \oplus F$$

1): Since $(A, B) \in DR$, it follows from the proposition (38) that $A_0$ is injective, which equivalent to $A_0^*$ is surjective; i.e. $R(A_0^*) = R(A) \oplus R(B)$, so $A_0$ has a closed range, now remark that

$$\begin{aligned} R(M_0^*) &= R(B_0^* A_0^*) = B_0^* R(A_0^*) = B_0^* R(A_0^* A_0^{*^+}) = B_0^* R((A_0^+ A_0) \\ &= B_0^* R(I) = R(B_0^*) \end{aligned}$$

And by the item 3 of lemma (14) that

$$R(B_0^*) = R(G_A^*) + R(G_B^*) \oplus \{0\} = R(A^*) + R(B^*) \oplus \{0\}$$

Hence,

$$R(M_0^*) = R(A^*) + R(B^*) \oplus \{0\}$$

As $R(M_0^*) = R(A^* + B^*) \oplus \{0\}$ , So

$$R(A^* + B^*) \oplus \{0\} = R(A^*) + R(B^*) \oplus \{0\}$$

Which implies that

$$R(A^* + B^*) = R(A^*) + R(B^*)$$

From the last equality we deduce that $R(A+B)$ is closed iff $R(A^*) + R(B^*)$ is closed. 2): To prove the item 2 taking the adjoint on both side of $(5-7)$ and applying the item 1. 3): we already showed in items 1 and 2 that the equalities below are satisfied

$$R(A+B) = R(A) + R(B),\ R(A^* + B^*) = R(A^*) + R(B^*)$$

Note that $B_0$ is surjective because by the Proposition (38), $B_0^*$ is injective, on the other hand we showed that $A_0$ is injective, it follows from the of lemma (14) that $A_0B_0$ is full-rank decomposition of $A+B$, which means that $A+B$ has a closed range, of course it results from the two last equalities that $R(A) + R(B)$ and $R(A^*) + R(B^*)$ are closed. □

**Corollary 75.** *If* $(A^*, B^*) \in DR$ *and* $R(A) \perp R(B)$*, we have:*

$$(A+B)^+ = (BP_{N(A)})^+ + (I - (BP_{N(A)})^+ B)A^+ \tag{5-8}$$

**Proof.** From the item 3 of theorem (74), $(A+B)^+$ exists and $R(A)+R(B)$ is closed, which implies that $(BP_{N(A)})^+$ exists by proposition (67), it follows from the items 2) and 3) of lemma (10) that $R(\Omega_A\Omega_A^+) = R(BB^+)$ and $N(\Omega_A\Omega_A^+) = N(BB^+)$, so the item 4 of lemma (4) we get that $\Omega_A\Omega_A^+ = BB^+$ consequently, $\Delta_A = (I - \Omega_A\Omega_A^+)B = 0$, so the substitution of $\Delta_A$ by the nul operator in $(5-3)$, we obtain $(5-8)$. □

Similarly, we can prove this corollary:

**Corollary 76.** *If* $(A, B) \in DR$ *and* $R(A^*) \perp R(B^*)$, *we have:*

$$(5-9) \qquad (A+B)^+ = (P_{N(A^*)}B)^+ + (I - (P_{N(A^*)}B)^+B)A^+$$

**Theorem 77.** *If* $(A, B) \in DR$ *and* $(A^*, B^*) \in DR$, *then*

$$(5-10) \qquad (A+B)^+ = (BP_{N(A)})^+B(P_{N(A^*)}B)^+ + (AP_{N(B)})^+A(P_{N(B^*)}A)^+$$

**Proof.** The subspaces $R(A+B), R(A)+R(B)$ and $R(A^*)+R(B^*)$ are closed by the theorem (74), it follows that the M-P inverses that appear in the identity $(5-10)$ exist. □

Let $M_0$ be as in $(5-7)$, it results from the proposition (38) that $\begin{bmatrix} F_A & F_B \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} G_A^* & G_B^* \\ 0 & 0 \end{bmatrix}$ are injective, so $\begin{bmatrix} G_A & 0 \\ G_B & 0 \end{bmatrix}$ is surjective, then $A_0B_0$ is a full-rank decomposition of $M_0$,

in this case we have

$$M_0^+ = \begin{bmatrix} G_A & 0 \\ G_B & 0 \end{bmatrix}^+ \begin{bmatrix} F_A & F_B \\ 0 & 0 \end{bmatrix}^+ := B_0^+ A_0^+$$

Now from the item a) of lemma (15) and theorem (73), $(P_{N(B^*)}F_A)^+$, $(P_{N(A^*)}F_B)^+$, $(G_BP_{N(A)})^+$ and $(G_AP_{N(B)})^+$ exist, hence from $B_0^+ = \left(B_0^{*+}\right)^*$ and using the theorem (42) we get $M$

$$M_0^+ = \begin{bmatrix} (G_A P_{N(G_B)})^+ & (G_B P_{N(G_A)})^+ \\ & 0 \end{bmatrix} \begin{bmatrix} (P_{N(F_B^*)} F_A)^+ & 0 \\ (P_{N(F_A^*)} F_B)^+ & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (G_A P_{N(B)})^+ (P_{N(B^*)} F_A)^+ + (G_B P_{N(A)})^+ (P_{N(A^*)} F_B)^+ & 0 \\ 0 & 0 \end{bmatrix}$$

Using the equality of item b) and c) of lemma (15), we get

$$M_0^+ = \begin{bmatrix} (AP_{N(B)})^+ A (P_{N(B^*)} A)^+ + (BP_{N(A)})^+ B (P_{N(A^*)} B)^+ & 0 \\ 0 & 0 \end{bmatrix}$$

And as

$$M_0^+ = \begin{bmatrix} (A+B)^+ & 0 \\ 0 & 0 \end{bmatrix}$$

The by identification

$$(A+B)^+ = (AP_{N(B)})^+ A (P_{N(B^*)} A)^+ + (BP_{N(A)})^+ B (P_{N(A^*)} B)^+ \ .$$

**Corollary 78.** *In the previous theorem, if* $R(A^*) \perp R(B^*)$ *we obtain the identity* $(5-11)$, *also if* $R(A) \perp R(B)$ *we obtain the identity* $(5-12)$,

$$(5-11) \qquad (A+B)^+ = B^+ B (P_{N(A^*)} B)^+ + A^+ A (P_{N(B^*)} A)^+$$

*And*

$$(5-12) \qquad (A+B)^{+}= (BP_{N(A)})^{+}BB^{+}+(AP_{N(B)})^{+}AA^{+}$$

**Proof.** We have □

$R(A^{*}) \perp R(B^{*}) \Longleftrightarrow BA^{*}=0 \Longleftrightarrow B^{+}BA^{*}=0 \Longleftrightarrow A^{*}-B^{+}BA^{*}=A^{*} \Longleftrightarrow AP_{N(B)}=A$, so $(AP_{N(B)})^{+}=A^{+}$, also $(BP_{N(A)})^{+}=B^{+}$ and we replace $(AP_{N(B)})^{+}$ and $(BP_{N(A)})^{+}$ by $A^{+}$ and $B^{+}$ in $(5-10)$ we obtain $(5-11)$. By the same way we can prove $(5-12)$.

### 5.3. Extension of the Fill-Fishkind formula.

In section 5 of the article [2], Arias, Corach and Maestripieri. extended the formula of Fill-Fishkind to the infinite Hilbert space case, by adding two other conditions to the property of the additivity of ranges.

From the theorem below, we see that the Fill-Fishkind formula remains valid in infinite dimensional Hilbert spaces under the same conditions of the case of matrices.

**Theorem 79.** *If* $(A,B)\in DR$ *and* $(A^{*},B^{*})\in DR$*, then*

$$(5-13) \qquad (A+B)^{+}= (I-S)A^{+}(I-T)+SB^{+}T$$

*Where:* $S=(P_{N(B)^{\perp}}P_{N(A)})^{+}$ *and* $T=(P_{N(A^{*})}P_{N(B^{*})^{\perp}})^{+}$.

**Proof.** From the item 3) of the theorem (74), $(A+B)^{+}$ exists and $R(A^{*})+R(B^{*})$ is closed (resp., $R(A)+R(B)$ is closed) which implies by the proposition (67) that

$S$ exists (resp., $T$ exists). As $B$ has a closed range, it results that $P_{N(B)^\perp} = B^+B$ and $P_{N(B^*)^\perp} = BB^+$, it follows from the lemma (4) that $BS = B(P_{N(B)^\perp}P_{N(A)})^+ = B(P_{N(B)^\perp}P_{N(A)})(P_{N(B)^\perp}P_{N(A)})^+ = BS^+S$, on the other hand, since $R(A) \cap R(B) = \{0\}$, then by the item 3 of lemma (10) that $N(S^+S) = N(S) = N(P_{N(B)^\perp}) = N(B^+B) = N(B)$, using the item 2 of lemma (4) we obtain $BS^+S = B$, we deduce that $BS = B$, by the same we get $TB = B$, also by the lemma (4) we obtain $AS = 0$ and $TA = 0$. Now we will check that $(I - S)A^+(I - T) + SB^+T)$ satisfies the equations of M-P inverse of $A + B$

The equations $(iii)$:

$(A + B)((I - S)A^+(I - T) + SB^+T) =$

$(A + B)(A^+ - SA^+ - A^+T + SA^+T + SB^+T) =$ (or $AS = 0$ and $BS = B$)

$AA^+ - AA^+T + BA^+ - BA^+ - BA^+T + BA^+T + BB^+T =$

$AA^+ - AA^+T + BB^+T =$ ...by the item 3 of lemma (4)

$AA^+ - AA^+T + T = AA^+ + (I - AA^+)T =$ ...by the item 3 of lemma (4)

$AA^+ + (I - AA^+)P_{N(B^*)^\perp}T = AA^+ + T^+ T$

The equations $(iv)$:

$((I - S)A^+(I - T) + SB^+T)(A + B) =$

$(A^+ - SA^+ - A^+T + SA^+T + SB^+T)(A + B) =$ (or $TA = 0$ and $TB = B$)

$= A^+A - SA^+A + A^+B - SA^+B - A^+B + SA^+B + SB^+B =$

$A^+A - SA^+A + SB^+B =$ ...by the item 3 of lemma (4)

$A^+A - SA^+A + S = A^+A + (I - A^+A)S =$ ...by the item 3 of lemma (4)

$A^+A + (I - A^+A)P_{N(A)}S = A^+A + SS^+$

The equations $(i)$:

$(A+B)((I-S)A^{+}(I-T)+SB^{+}T)(A+B) = \ldots$by $(iii)$

$(AA^{+}+T^{+}T)(A+B) = \ldots$by $TA=0, TB=B$ and $P_{N(B^*)^{\perp}} = BB^{+}$

$A+(AA^{+}B+T^{+}B) = A+(AA^{+}B+(P_{N(A^*)}BB^{+})B) = A+B$

The equations $(ii)$:

$((I-S)A^{+}(I-T)+SB^{+}T)(A+B)((I-S)A^{+}(I-T)+SB^{+}T) = \ldots$by $(iv)$

$(A^{+}A+S^{+}S)(A^{+}-SA^{+}-A^{+}T+SA^{+}T+SB^{+}T) =$ (or $AS=0$ and $BS=B$)

$A^{+}-A^{+}T+SS^{+}A^{+}-SA^{+}-SS^{+}A^{+}T+SA^{+}T+SB^{+}T = \ldots$ by $S^{+}A^{+}=0$

$A^{+}-A^{+}T-SA^{+}+SA^{+}T+SB^{+}T = ((I-S)A^{+}(I-T)+SB^{+}T)$. □

## 5.4. Proofs of the results of subchapter 3.3

It suffices to demonstrate the theorem (50) below:

**Theorem 80.** *Theorem* (50)*: Let $M$ be defined as in* $(3-1)$ *with closed range such that $R(A_1)+R(A_2)$ and $R(A_3)+R(A_4)$ are closed, if* $\left(\begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix}, \begin{pmatrix} A_3^* \\ A_4^* \end{pmatrix}\right) \in DR$, *then*

$$M^{+} = \begin{bmatrix} A_1^*S_1^{+} - W_1^{+}\Upsilon_1^{+}ZS_1^{+} & W_1^{+}\Upsilon_1^{+} \\ A_2^*S_1^{+} - W_2\Upsilon_1^{+}ZS_1^{+} & W_2\Upsilon_1^{+} \end{bmatrix}$$

*Where* $S_1 = A_1A_1^*+A_2A_2^*$, $Z = A_3A_1^*+A_4A_2^*$, $W_1 = A_3-ZS_1^{+}A_1$, $W_2 = A_4-ZS_1^{+}A_2$ $\Upsilon_1 = W_1W_1^*+W_2W_2^*$

**Proof.** We have

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix} =: M_1 + M_2$$

Clearly that the assumptions of corollary (75) are satisfied for $M_1$ and $M_2$,we deduce from $(5-8)$ that

$$M^+ = (M_2 P_{N(M_1)})^+ + (I - (M_2 P_{N(M_1)})^+ M_2) M_1^+$$

Next we know that, $M_1^+ = M_1^*(M_1 M_1^*)^+$, then we get

$$M_1^+ = \begin{bmatrix} A_1^* S_1^+ & 0 \\ A_2^* S_1^+ & 0 \end{bmatrix} \text{ and } P_{N(M_1)} = \begin{bmatrix} I - A_1^* S_1^+ A_1 & -A_1^* S_1^+ A_2 \\ -A_2^* S_1^+ A_1 & I - A_2^* S_1^+ A_2 \end{bmatrix}$$

$$M_2 P_{N(M_1)} = \begin{bmatrix} 0 & 0 \\ A_3 - Z S_1^+ A_1 & A_4 - Z S_1^+ A_2 \end{bmatrix} := \begin{bmatrix} 0 & 0 \\ W_1 & W_2 \end{bmatrix}$$

Applying $(M_2 P_{N(M_1)})^+ = (M_2 P_{N(M_1)})^*((M_2 P_{N(M_1)})(M_2 P_{N(M_1)})^*)^+$ we obtain

$$(M_2 P_{N(M_1)})^+ = \begin{bmatrix} 0 & W_1 \Upsilon_1^+ \\ 0 & W_2 \Upsilon_1^+ \end{bmatrix}$$

On the other hand

$$(I - (M_2 P_{N(M_1)})^+ M_2) M_1^+ = \begin{bmatrix} A_1^* S_1^+ - W_1 \Upsilon_1^+ Z S_1^+ & 0 \\ A_2^* S_1^+ - W_2 \Upsilon_1^+ Z S_1^+ & 0 \end{bmatrix}$$

Finally

$$M^+ = (M_2 P_{N(M_1)})^+ + (I - (M_2 P_{N(M_1)})^+ M_2) M_1^+ =$$

$$\begin{bmatrix} 0 & W_1 \Upsilon_1^+ \\ 0 & W_2 \Upsilon_1^+ \end{bmatrix} + \begin{bmatrix} A_1^* S_1^+ - W_1 \Upsilon_1^+ Z S_1^+ & 0 \\ A_2^* S_1^+ - W_2 \Upsilon_1^+ Z S_1^+ & 0 \end{bmatrix} = \begin{bmatrix} A_1^* S_1^+ - W_1 \Upsilon_1^+ Z S_1^+ & W_1 \Upsilon_1^+ \\ A_2^* S_1^+ - W_2 \Upsilon_1^+ Z S_1^+ & W_2 \Upsilon_1^+ \end{bmatrix}$$

□

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FACULTÉ DES MATHÉMATIQUES ET D´INFORMATIQUE

DÉPARTEMENT DES MATHÉMATIQUS

**THÈSE**

Pour obtenir le titre de Docteur en Sciences

Spécialité :

MATHEMATIQUES

Présenté par

**Kara Abdessalam**

Intitulée:

# Idempotents et inverses généralisés

Soutenue le 05/10/2021, devant un jury :

| | | |
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| **Prf. Ameur Seddik** | Université de batna 2, | Président |
| **Prf. Guedjiba Said** | Université de batna 2 | Directeur de thèse |
| **Prf. Mennouni Abdelaziz** | Université de batna 2, | Examinateur |
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