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By:

OULIA BOUHOUFANI

Existence and Asymptotic Behavior of Solutions of Certain Hyperbolic Coupled Systems with Variable Exponents

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Pr. Khaled Melkemi,	Univ. Batna 2,	President
Pr. Ilhem Hamchi,	Univ. Batna 2,	Supervisor
Pr. Salim Messaoudi,	Univ. of Sharjah,	Co-supervisor
Pr. Farida Lombarkia,	Univ. Batna 2,	Examiner
Pr. Zouhir Mokhtari,	Univ. of Biskra,	Examiner
MCA. Abdelfeteh Fareh,	Univ. of El Oued,	Examiner

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الملخص

في هذه الاطروحة، نعتبر نوعين من الجمل المزدوجة من معادلات زائديه غير خطية وبأسس متغيرة. فتحت فرضيات مناسبة على المعطيات الابتدائية والأسس المتغيرة، نبرهن عدة نتائج متعلقة بالوجود المحلي، الوجود الشامل، الانفجار في وقت منته وكذلك سلوك الحل. لقد تم الحصول على هذه النتائج بواسطة تقريبات فايدو جالاركين، طريقة المجموعة المستقرة ومقاربة الطاقة وطريقة المضروبات بالإضافة إلى ذلك، نوضح نتائجنا النظرية بواسطة تقديم بعض الاختبارات العددية.

Résumé

Dans cette thèse, on considère deux types de systèmes couplés d'équations hyperboliques non linéaires avec des exposants variables. Sous des hypothèses appropriées sur les données initiales et les exposants variables, on établit plusieurs résultats concernant l'existence locale, l'existence globale, l'explosion en un temps fini ainsi que le comportement de la solution. Ces résultats sont obtenus par les approximations de Faedo-Galerkin, la méthode de l'ensemble stable, l'approche de l'energie et la méthode du multiplicateur. De plus, on illustre nos résultats théoriques en présentant quelques tests numériques.

Abstract

In this thesis, we consider two kind of coupled systems of nonlinear hyperbolic equations with variable-exponents. Under suitable assumptions on the initial data and the variable exponents, we establish several results concerning the local existence, the global existence, the finite-time blow up as well as the decay of the solution. These results are obtained by the Faedo-Galerkin approximations, the stable-set method, the energy approach and the mulitiplier method. In addition, we illustrate our theoretical findings by presenting some numericals tests.

Contents

General Introduction

Literature Review

A considerable effort has been devoted to the study of single wave equation in the case of constant exponents. The following equation, with initial and Dirichlet-boundary conditions,

$$u_{tt} - \Delta u + a\left|u_t\right|^{m-2} u_t = b\left|u\right|^{p-2} u \text{ in } \Omega \times (0,T)$$

has been studied by many researchers. Here Ω is a bounded domain of $\mathbb{R}^n (n \in \mathbb{N}^*)$ with a smooth boundary, $T > 0$ and $m, p \geq 2$. Ball in [5] showed that if $a = 0$, then the source term $b\left|u\right|^{p-2} u$, with $b > 0$, forces the negative-initial-energy solutions to explode in finite time. In [20], Haraux and Zuazua proved that in the absence of the source term, that is $b = 0$, the damping term $a\left|u_t\right|^{m-2} u_t$ assures global existence for arbitrary initial data. In the presence of both terms, the problem was first considered by Levine [25]. He established the blow up for solutions with negative initial energy, when $m = 2$. Georgiev and Todorava [15] considered the case $m > 2$, by introducing a different method and established a blow up result for solution with sufficiently negative initial energy. Messaoudi [30] proved that any solution with negative initial energy only blows up in finite time when $p > m \geq 2$ and then he established a global existence when $m \geq p$.

Concerning the case of equations and systems with constant exponents and weak dissipation, we can cite the works of Mustafa and Messaoudi [40], Benaissa and Mimouni [6], Benaissa and Mokaddem [7], Zennir [52] and Agre and Rammaha [2].

For the class of one wave equation with variable exponent nonlinearity, we mention some recent works. In [3], Antontsev studied the equation:

$$\begin{cases} u_{tt} - div\left(a|\nabla u|^{p-2}\nabla u\right) - \alpha\Delta u_t - bu|u|^{\sigma-2} = f & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u\left(x,0\right) = u_0(x) \text{ and } u_t\left(x,0\right) = u_1(x) & \text{in } \Omega, \end{cases}$$

where $\alpha > 0$ is a constant and a, b, p, σ are given functions. Under specific conditions, he proved the local and global existence of a weak solutions and a blow-up result for certain solutions with non positive initial energy. Guo and Gao [16] took $\sigma\left(x,t\right) = r > 2$ and established a finite time blow up result. In [48], Sun et al. looked into the following equation:

$$u_{tt} - div\left(a\left(x,t\right)\nabla u\right) + c\left(x,t\right)u_t\left|u_t\right|^{q(x,t)-1} = b\left(x,t\right)u\left|u\right|^{p(x,t)-2} \text{ in } \Omega\times(0,T)$$

in a bounded domain, with Dirichlet boundary conditions, and established a blow-up result for certain solutions with positive initial energy. They also gave lower and upper bounds for the blow-up time and provided a numerical example to illustrate their result. After that, Messaoudi and Talahmeh [34] considered the following equation:

$$u_{tt} - div\left(\left|\nabla u\right|^{r(x)-2}\nabla u\right) + au_t\left|u_t\right|^{m(x)-2} = bu\left|u\right|^{p(x)-2} \text{ in } \Omega\times(0,T),$$

where $a, b > 0$ are two constants and m, r, p are given functions. They established a finite-time blow up result for negative initial energy solutions and for certain solutions with positive initial energy. In [36], the same authors studied a decay result for solutions of a nonlinear damped wave equation with variable exponent and presented two numerical applications for their theoretical results. Recently, they gave in [37] an overview of results concerning decay and blow up for nonlinear wave equations involving constant and variable exponents. In [35], Messaoudi et al. studied the equation:

$$u_{tt} - \Delta u + au_t\left|u_t\right|^{m(x)-2} = bu\left|u\right|^{p(x)-2} \text{ in } \Omega\times(0,T).$$

They established the existence and uniqueness of weak local solution, using the Faedo-Galerkin method and proved the finite-time blow up for solutions with negative initial-energy. Very recently, Xiaolei et al. [51] established an asymptotic stability of solutions to quasilinear hyperbolic equations with variable source and damping terms.

Coupled systems of two nonlinear wave equations with constant exponents have been extensively studied. In [18], Hao and Cai obtained several results concerning the local existence, the global existence and the blow up property for positive-initial-energy solutions for the following viscoelastic system

$$\begin{cases} u_{tt} - \Delta u - div\left(g(|\nabla u|^2)\nabla u\right) + \int_0^t h_1(t-s)\triangle u(s)ds & \\ +\left|u_t\right|^{m-1}u_t = f_1(u,v) & \text{in } \Omega\times(0,T), \\ v_{tt} - \Delta v - div\left(g(|\nabla v|^2)\nabla v\right) + \int_0^t h_1(t-s)\triangle v(s)ds & \\ +\left|v_t\right|^{r-1}v_t = f_2(u,v) & \text{in } \Omega\times(0,T), \\ u = v = 0 & \text{on } \partial\Omega\times(0,T), \\ u(0) = u_0 \text{ and } u_t(0) = u_1 & \text{in } \Omega, \\ v(0) = v_0 \text{ and } v_t(0) = v_1 & \text{in } \Omega, \end{cases}$$

where $g, h_1, h_2 \in C^1(\mathbb{R}_+)$ and f_1, f_2 are two given functions. Messaoudi and Said-Houari [33] studied the above system with $g \equiv 0$ and proved a global nonexistence theorem for solutions with positive initial energy. The same system, but in the absense of viscoelastic terms ($h_1 = h_2 = 0$) has been studied by Liang and Gao [26]. They obtained the global nonexistence result for certain solutions with positive initial energy. The last system with $g \equiv 1$ has been investigated in [2], by Agre and Rammaha. They proved local and global existence results of weak solution and established that any weak solution with negative-Initial energy blows up in finite time. This later blow-up result has been improved by Said-Houari [45] for a certain class of initial data with positive initial energy. In [50], Wu considered a system of two viscoelastic wave equations of Kirchhoff type with nonlinear damping and source terms and Drichlet boundary conditions, given by

$$\begin{cases} u_{tt} - M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta u + \int_0^t h_1(t-s)\triangle u(s)ds + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta v + \int_0^t h_2(t-s)\triangle v(s)ds + |v_t|^{r-1} v_t = f_2(u,v). \end{cases}$$

For certain initial data, Wu proved that the decay estimates of the energy depend on the exponents of the damping terms, and also he established the finite time blow up of solutions with non-negative initial energy. After that, Mu and Ma [38] considered the following nonlinear wave equations with Balakrishnan-talor damping,

$$\begin{cases} u_{tt} - (a + b\|\nabla u\|_2^2 + \sigma \int_\Omega \nabla u \nabla u_t dx)\Delta u + \int_0^t h_1(t-s)\triangle u(s)ds + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - (a + b\|\nabla v\|_2^2 + \sigma \int_\Omega \nabla v \nabla v_t dx)\Delta v + \int_0^t h_2(t-s)\triangle v(s)ds + |v_t|^{r-1} v_t = f_2(u,v), \end{cases}$$

and showed that the decay rate of the solution energy is similar to that of the relaxation fuctions and proved that the nonlinear source of polynomial type forces solutions to blow up in finite time. Very recently, Messaoudi and Hassan [32] established a general decay result, for a certain system of viscoelastic wave equations.

Objectives

Our goal, in the first part of this thesis, is to investigate the existence and uniqueness of local weak solution for a coupled system of two hyperbolic equations, using the standard Faedo-Galerkin method and paying more attention to the difficulties caused by the variable exponents. After that, we determine an appropriate relation between the nonlinearities in the damping and source trems, for which there is either finite-time blow up of solutions or global existence. Precisely, we prove that the solution of system (P), treated in Chapter 2, blows up in finite-time if $p^- > \max\{m^+-1, r^+-1\}$ and exists globally in time if $p^- \leq \max\{m^+-1, r^+-1\}$. In the first case, we give some numerical tests illustrating our theoretical findings and in the second one, we obtain a stability result for the solution by using Komornik's inequality.
Our purpose, in the second part of the thesis, is to establish an explicit decay rate of the solution energy to system $(\tilde{P})$, considered in Chapter 3, depending on the variable exponents m, r and the time dependent coefficients α, β, and then give some examples and few numerical tests.
To the best of our knowledge, these problems have not been considred earlier in the literature.

Organization of the thesis

This thesis is divided into three chapters, in addition to the introduction.

- In Chapter 1, we give some preliminaries. We recall, in Section 1.2, the history and definitions of the variable-exponent Lebesgue and Sobolev spaces, the function spaces used throughout this thesis, and then we present a brief overview of some facts and properties of these spaces. Section 1.3 is devoted to some results, which will be used in many places later on.

- In Chapter 2, we study the following coupled system of two hyperbolic equations with initial and boundary conditions and for the unknowns $u(t,x)$ and $v(t,x)$:

$$\begin{cases} u_{tt} - div\left(A\nabla u\right) + \left|u_t\right|^{m(x)-2} u_t = f_1\left(x,u,v\right) & \text{in } \Omega\times(0,T), \\ v_{tt} - div\left(B\nabla v\right) + \left|v_t\right|^{r(x)-2} v_t = f_2\left(x,u,v\right) & \text{in } \Omega\times(0,T), \\ u = v = 0 & \text{on } \partial\Omega\times(0,T), \\ u\left(0\right) = u_0 \text{ and } u_t\left(0\right) = u_1 & \text{in } \Omega, \\ v\left(0\right) = v_0 \text{ and } v_t\left(0\right) = v_1 & \text{in } \Omega, \end{cases} \qquad (P)$$

where $T > 0$, Ω is a bounded domain of $\mathbb{R}^n (n = 1, 2, 3)$ with a smooth boundary $\partial\Omega$ and for all $x \in \overline{\Omega}$ and $(u,v) \in \mathbb{R}^2$,

$$f_1\left(x,u,v\right) = \frac{\partial}{\partial u}F\left(x,u,v\right) \text{ and } f_2\left(x,u,v\right) = \frac{\partial}{\partial v}F\left(x,u,v\right),$$

with

$$F\left(x,u,v\right) = a\left|u+v\right|^{p(x)+1} + 2b\left|uv\right|^{\frac{p(x)+1}{2}},$$

where $a, b > 0$ are two positive constants. p, m and r are given continuous functions on $\overline{\Omega}$.

Our results concerning this problem are summarized as follows:

- In Section 2.2, we push the local existence result of Agre and Rammaha [2], which was established for the case of constant-exponent nonlinearities, to our problem (P). For this purpose, we use the Faedo-Galerkin method and the Banach fixed point theorem, under suitable assumptions on the variable exponents $m(.), r(.)$ and $p(.)$. To the best of our knowledge, this is the first result of this kind and the generalization was not trivial at all.

- Section 2.3 is devoted to the study of the blow-up result of negative-initial-energy solution. Using the energy method, we prove that no solution with negative initial energy of problem (P) can be extended on $[0,\infty)$, if the source terms dominate the damping terms; that is if $p^- > max\left\{m^+ - 1, r^+ - 1\right\}$.

- In Section 2.4, we establish the finite time blow up for certain solutions with positive-initial energy. Under appropriate assumptions on the variable exponents, we prove that the solutions blow up in finite time $T^* > 0$. We also give some numerical applications to illustrate our theoretical results. This work and the local existence result studied in Section 2.2 are the subject of paper by Bouhoufani et al [9].

- Section 2.5 concerns our contribution on global existence and stability of system (P). To the best of our knowledge, there is no result concerning global existence and stability of the hyperbolic coupled system with nonlinearties of variable-exponent type. With specific hypotheses on the parameters of the problem, we prove a global existence theorem, using a Stable-set method. After that, we use the Komornik integral approach to establish that the solution energy has either an exponential decay or a polynomial one depending on the variable exponents. These results have been published in [8], by Bouhoufani and Hamchi.

- Chapter 3 is devoted to the study of a coupled system of nonlinear hyperbolic equations with initial and boundary conditions and variable exponents in the weak dampings:

$$\begin{cases} u_{tt} - \Delta u + \alpha(t)|u_t|^{m(x)-2}u_t + |u|^{p(x)-2}u|v|^{p(x)} = 0 & \text{in } \Omega \times (0,T)\,, \\ v_{tt} - \Delta v + \beta(t)|v_t|^{r(x)-2}v_t + |v|^{p(x)-2}v|u|^{p(x)} = 0 & \text{in } \Omega \times (0,T)\,, \\ u = v = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u\,(0) = u_0 \text{ and } u_t\,(0) = u_1 & \text{in } \Omega, \\ v\,(0) = v_0 \text{ and } v_t\,(0) = v_1 & \text{in } \Omega, \end{cases} \qquad (\tilde{P})$$

where $T > 0$ and Ω is a bounded domain of $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$ with a smooth boundary $\partial\Omega$. $\alpha, \beta : [0, \infty) \longrightarrow (0, \infty)$ are two non-increasing C^1-functions and m, r and p are given continuous functions on $\overline{\Omega}$ satisfying some conditions, to be specified later.

In Section 3.2, we present, without proof, an existence result of global weak solution of problem $(\tilde{P})$. In Section 3.3, under suitable assumptions on the functions α, β and the variable exponents m and r, we establish the decay rate of the solution energy, using the multiplier method. At the end, some numerical examples are given in Subsection 3.3.1 to ulistrate our theoritical results. This study is the subject of a submitted paper by Bouhoufani et al. [10].

Chapter 1

Preliminaries

1.1 Introduction

This chapter deals with Lebesgue and Sobolev spaces with variable exponents, which differ from the classical spaces $L^p(\Omega)$ and $W^p(\Omega)$ in the fact that exponent p is not constant but a function from Ω to $[1, \infty)$. In Section 1.2, we present the history and the definitions of this important class of spaces. After that, we cite some facts and results related to the variable-exponent Lebesgue and Sobolev spaces, needed in the study of our two systems, in Chapters 2 and 3. No proofs are inclouded for these standard results, but references are provided. Other important tools and Lemmas, which are necessary to obtain various estimates, are given in Section 1.3.

1.2 History of Variable Exponents Spaces

Variable Lebesgue spaces appeared for the first time in a paper by W. Orlicz, in 1931. In his paper [43], Orlicz considered the following question: What are the neccessary and sufficient conditions on a real sequence (y_i) under which $\Sigma_i x_i y_i$ converges, for a real sequences (p_i) and (x_i) such that $\Sigma_i x_i^{p_i}$ converges, with $(p_i > 1)$? After this one article, Orlicz concentrated to the theory of the function spaces that now bear his name. In this theory, the space $L^{\varphi}(\Omega)$ is constituted by measurable functions u on Ω such that

$$\varrho(\lambda u) = \int_{\Omega} \varphi(\lambda\, |u(x)|) dx < +\infty,$$

for some $\lambda > 0$, where φ is a real-valued function satisfying certain conditions. Abstracting certain properties of ϱ, a more general class of function spaces called modular spaces were first studied by Nakano [41, 42]. An explicit version of these modular function spacaces was investigated by Polish Mathematicians, like Hudzik and Kamiñska. For more details about the modular function spaces, the interested

reader can see the monograph [39] of Musielak and Orlicz.
The variabele-exponents Lebesgue spaces $L^{p(.)}(\Omega)$ are defined as the Orlicz space $L^{\varphi_{p(.)}}(\Omega)$ where

$$\varphi_{p(.)}(t) = t^{p(.)} \text{ or } \varphi_{p(.)}(t) = \frac{t^{p(.)}}{p(.)},$$

i.e.,

$$L^{p(.)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable in } \Omega : \ \varrho_{p(.)}(\lambda u) = \int_\Omega \varphi_{p(.)}(\lambda |u(x)|)dx < +\infty \right\},$$

for some $\lambda > 0$, equipped with the following Luxembourg-type norm

$$\|u\|_{p(.)} := \inf \left\{ \lambda > 0 : \varrho_{p(.)}(\frac{u}{\lambda}) \leq 1 \right\}.$$

Variabele-exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. Their results originated in a 1961 article by Tsenov [49]. In [46, 47], Sharapudinov introduced the Luxembourg norm and showed that $L^{p(.)}(\Omega)$ is reflexive if the exponent satisfies $1 < essinf\ p \leq esssup\ p < +\infty$. In the mid-80's, Zhikov [53] started a new line of investigation of variable-exponent spaces, by considering variational integrals with non-standard growth conditions. After that, in the early 90's Kovacik and Rakosnk [23] established some basic properties of the Lebesgue and Sobolev spaces in $\mathbb{R}^n$. In the beginning of the new millennium, a great development has been made for the study of variable-exponent spaces. In particulier, a connection was made between the variable-exponent spaces and the variational integrals with non-standard growth and coercivity conditions. It was also observed the relation between these variational problems and the modeling of several phisical phenomena such as electrorheological fluids, image processing,..., etc.

In the following section, we present some results from [24] on the basic properties on $L^{p(.)}(\Omega)$, which we need in the proof of our results. We mention that many results on these properties were proved first by Kovacik and Rakosnik [23] and were later reproved by Fan and Zhao in [13].

1.2.1 Lebesgue Spaces with Variable Exponents

Definition 1.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\mathbb{P}(\Omega, \Sigma, \mu)$ a σ-finite, complete measurable space. Let $\mathbb{P}(\Omega, \mu)$ be the set of all μ-measurable functions $p : \Omega \longrightarrow [1, \infty)$. The function $p \in \mathbb{P}(\Omega, \mu)$ is called a variable exponent on Ω. We define*

$$p^- := essinf_{x\in\Omega}\ p(x) \text{ and } p^+ := esssup_{x\in\Omega}\ p(x).$$

If $p^+ < +\infty$, then p is said to be a bounded variable exponent. If $p \in \mathbb{P}(\Omega, \mu)$, then we define $p' \in \mathbb{P}(\Omega, \mu)$ by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \text{ where } \frac{1}{\infty} := 0.$$

The function p' is called the dual variable exponent of p.

Definition 1.2.2. *We define the Lebesgue space with a variable-exponent p by*

$$L^{p(.)}(\Omega) = \Big\{u : \Omega \longrightarrow \mathbb{R} \ \textit{measurable in}\ \Omega : \ \lim_{\lambda \longrightarrow 0} \varrho_{p(.)}(\lambda u) = 0\Big\}$$

or equivalently

$$L^{p(.)}(\Omega) = \big\{u : \Omega \longrightarrow \mathbb{R} \ \textit{measurable in}\ \Omega : \ \varrho_{p(.)}(\lambda u) < +\infty, \ \textit{for some}\ \lambda > 0\big\},$$

where $\varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx$.
$L^{p(.)}(\Omega)$ *is endowed with the following Luxembourg-type norm*

$$\|u\|_{p(.)} := inf \left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Lemma 1.2.3. *[4] If* $p(.) \equiv p$, *where p is constant. Then*

$$\|u\|_{p(.)} = (\int_\Omega |u(x)|^p\, dx)^{\frac{1}{p}}.$$

Now, we introduce the most important condition on the variable exponent, called the log-Hölder continuity condition, which is necessary to obtain the Poincaré inequality in the variable case.

Definition 1.2.4. *We say that a function* $q : \Omega \longrightarrow \mathbb{R}$ *is log-Hölder continuous on* Ω, *if there exists constant* $\theta > 0$ *such that for all* $0 < \delta < 1$, *we have*

$$|q(x) - q(y)| \leq -\frac{\theta}{\log|x-y|}, \textit{for a.e.}\ x, y \in \Omega, \ \textit{with}\ |x-y| < \delta.$$

We have

Lemma 1.2.5. *[24] Let* Ω *be a domain of* $\mathbb{R}^n$. *If* $p :\longrightarrow \mathbb{R}$ *is a Lipschitz function on* Ω, *then it is log-Hölder continuous on the same set* Ω.

Remark 1.2.6. *The log-Hölder continuity condition on p can be replaced by* $p \in C(\overline{\Omega})$, *if* Ω *is bounded.*

The following results are very important and useful in the sequal.

Theorem 1.2.7. *[24] If* $p \in \mathbb{P}(\Omega, \mu)$ *then* $L^{p(.)}(\Omega, \mu)$ *is a Banach space.*

Lemma 1.2.8. *If* $p : \Omega \longrightarrow [1, \infty)$ *is a measurable function with,* $p^+ < +\infty$ *then,* $C_0^\infty(\Omega)$ *is dense in* $L^{p(.)}(\Omega)$.

In the following lemma, we present the relation between the function $\varrho_{p(.)}(u)$, called the modular function, and the norm $\|u\|_{p(.)}$.

Lemma 1.2.9. *If* $1 < p^- \leq p(x) \leq p^+ < +\infty$ *hold then*

$$min\Big\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\Big\} \leq \varrho_{p(.)}(u) \leq max\Big\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\Big\}, \tag{1.2.1}$$

for any $u \in L^{p(.)}(\Omega)$.

Remark 1.2.10. *If the exponent p is constant, then $p^- = p^+$ and hence $\varrho_{p(.)}(u) = \|u\|_p^p$.*

As in the constant exponent case, we have the following Young's and Hölder's inequalities.

Lemma 1.2.11. ***(Young's Inequality)***
Let $p, q, s \geq 1$ be measurable functions defined on Ω such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e } y \in \Omega.$$

Then, for all $a, b \geq 0$, we have

$$\frac{(ab)^{s(.)}}{s(.)} \leq \frac{(a)^{p(.)}}{p(.)} + \frac{(b)^{q(.)}}{q(.)}.$$

By taking $s = 1$ and $1 < p, q < +\infty$, it follows that, for any $\varepsilon > 0$, we have

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \text{ where } C_\varepsilon = 1/q(\varepsilon p)^{\frac{q}{p}}. \tag{1.2.2}$$

For $p = q = 2$, it comes

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Lemma 1.2.12. ***(Hölder's Inequality)***
Let $p, q, s \geq 1$ be measurable functions defined on Ω satisfying

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e } y \in \Omega.$$

If $f \in L^{p(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$ then $fg \in L^{s(.)}(\Omega)$ and

$$\|fg\|_{s(.)} \leq 2 \|f\|_{P(.)} \|g\|_{q(.)}. \tag{1.2.3}$$

*Case $p = q = 2$ yields the **Cauchy-Schwarz inequality.***

1.2.2 Sobolev Spaces with Variable Exponents

The Sobolev space is a vector space of functions with weak derivatives. One motivation of studying these spaces is that solutions of partial differential equations belong naturaly to Sobolev spaces. In this section, we define the variable exponent Sobolev spaces and cite some important properties and results related to this class of spaces. First, we start by recalling the definition of weak derivatives.

Definition 1.2.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume that $u \in L^1_{loc}(\Omega)$. Let $\alpha := (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ be a multi-index. If there exists $g \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega u \frac{\partial^{\alpha_1 + ... + \alpha_n} \psi}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n} dx = (-1)^{\alpha_1 + ... + \alpha_n} \int_\Omega \psi g dx, \text{ for all } \psi \in C_0^\infty(\Omega),$$

then g is called a weak partial derivative of u of order α.
The function g is denoted by $\partial_\alpha u$ or by $\frac{\partial^{\alpha_1 + ... + \alpha_n} u}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}$.

Definition 1.2.14. *Let $k \in \mathbb{N}$. We define the variable exponent Sobolev space $W^{k,p(.)}(\Omega)$ as follows:*

$$W^{k,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \text{ such that } \partial^{|\alpha|} u \in L^{p(.)}(\Omega) \text{ with } |\alpha| \leq k \right\},$$

where $|\alpha| = \alpha_1 + ... + \alpha_n$, equipped with the following norm

$$\|u\|_{W^{k,p(.)}(\Omega)} := inf \left\{ \lambda > 0 : \ \varrho_{W^{k,p(.)}(\Omega)} \left(\frac{u}{\lambda} \right) \leq 1 \right\} = \sum_{0 \leq |\alpha| \leq k} \|\partial_\alpha u\|_{p(.)},$$

with

$$\varrho_{W^{k,p(.)}(\Omega)}(u) = \sum_{0 \leq |\alpha| \leq k} \varrho_{L^{p(.)}(\Omega)}(\partial_\alpha u).$$

Clearly

$$W^{0,p(.)}(\Omega) = L^{p(.)}(\Omega)$$

and

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

Theorem 1.2.15. *[4] Let $p \in \mathbb{P}(\Omega, \mu)$. The space $W^{k,p(.)}(\Omega)$ is a Banach space, which is seperable if p is bounded and reflexive if $1 < p^- \leq p^+ < +\infty$.*

Definition 1.2.16. *The closure of the set of $W^{k,p(.)}(\Omega)$-functions with compact support in $W^{k,p(.)}(\Omega)$ is the Sobolev space $W_0^{k,p(.)}(\Omega)$ "with zero boundary trace", i.e.,*

$$W_0^{k,p(.)}(\Omega) = \overline{\{u \in W^{k,p(.)}(\Omega) : u = u\chi_K \ for \ a \ compact \ K \subset \Omega\}}$$

Furtheremore, we denote by $H_0^{k,p(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(.)}(\Omega)$ and by $W^{-1,p'(.)}(\Omega)$ the dual space of $W_0^{1,p(.)}(\Omega)$, in the same way as the usual Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$.

Remark 1.2.17.

1. *$H_0^{k,p(.)}(\Omega) \subset W_0^{k,p(.)}(\Omega)$.*
2. *If p is log-Hölder continuous on Ω, then $H_0^{k,p(.)}(\Omega) = W_0^{k,p(.)}(\Omega)$.*
3. *If $p(.) = 2$ and $k = 1$, then we set $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.*

The following theorem plays a fundamental role to establish theorems of existence.

Theorem 1.2.18. *[4] Let $p \in \mathbb{P}(\Omega, \mu)$. The space $W_0^{k,p(.)}(\Omega)$ is a Banach space, which is seperable if p is bounded and reflexive if $1 < p^- \leq p^+ < +\infty$.*

The version of the Poincaré inequality, in the variable exponent case, is presented in the following theorem.

Theorem 1.2.19. *[24]* ***(Poincaré's Inequality)***
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If p satisfies the log-Hölder inequality on Ω, then

$$\|u\|_{p(.)} \leq C \|\nabla u\|_{p(.)}, \ for \ all \ u \in W_0^{1,p(.)}(\Omega),$$

where C is a positive constant deponding on Ω and $p(.)$. In particular, the space $W_0^{1,p(.)}(\Omega)$ has an equivalent norm given by

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.)}\,.$$

We end this section with some essential embedding results.

Lemma 1.2.20. *[4, 24](**Embedding Property**)*
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that $p, q \in C(\overline{\Omega})$ such that

$$1 < p^- \leq p(x) \leq p^+ < +\infty \text{ and } 1 < q^- \leq q(x) \leq q^+ < +\infty, \text{ for all } x \in \overline{\Omega}$$

and $p(x) < q^*(x)$ in $\overline{\Omega}$ with $q^*(x) = \begin{cases} \frac{nq(x)}{n-q(x)}, & \text{if } q^+ < n \\ +\infty, & \text{if } q^+ \geq n. \end{cases}$
Then, the embedding $W^{1,q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

Corollary 1.2.21. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that $p : \overline{\Omega} \longrightarrow (1, \infty)$ is a continuous function such that*

$$2 \leq p^- \leq p(x) \leq p^+ < \frac{2n}{n-2},\ n \geq 3.$$

Then, the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

1.3 Important Lemmas

To establisch the stability result for systems (P) and $(\tilde{P})$, we need the following two Lemmas.

Lemma 1.3.1. *[22](**Komornik's Lemma**)*
Consider $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nonincreasing function (and differentiable in the case of Lyaponov hypothesis), $C > 0$ and $\alpha \geq 0$ such that

$$\int_S^\infty E(t)^{1+\alpha} dt \leq CE(S),\ 0 \leq S < \infty.$$

Then, there exists positive constants c and w and $t_0 \geq 0$ such that, for all $t \geq t_0$, we have

$$E(t) \leq \begin{cases} E(0)e^{-\omega t}, & if\ \alpha = 0, \\ Ct^{-1/\alpha}, & if\ \alpha > 0. \end{cases}$$

Lemma 1.3.2. *[29] Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and $\sigma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing C^1-functions, with $\sigma(0) = 0$ and $\sigma(t) \longrightarrow +\infty$ as $t \longrightarrow \infty$. Assume that there exists $q \geq 0$ and $C > 0$ such that*

$$\int_S^\infty \sigma'(t)E(t)^{q+1} dt \leq CE(S),\ 0 \leq S < \infty.$$

Then, there exists positive constants c and w such that, for all $t \geq 0$,

$$E(t) \leq \begin{cases} ce^{-\omega\sigma(t)}, & if\ q = 0, \\ \frac{c}{[1+\sigma(t)]^{1/q}}, & if\ q > 0. \end{cases}$$

The following Lemmas will play an essential role in the proof of several results, in this dissertation.

Lemma 1.3.3. *[4] (**Gronwall's Inequality**)*
Let $f, g : [0, a] \longrightarrow \mathbb{R}$ be continuous and nonnegative. Suppose that there exists a positive constant C such that

$$f(t) \leq C + \int_0^t f(s)g(s)ds, \text{ for all } t \in [0, a].$$

Then,

$$f(t) \leq C \exp G(t),$$

where

$$G(t) = \int_0^t g(s)ds.$$

Lemma 1.3.4. *[26] Let Θ be a positive solution of the ordinary differential inequality*

$$\frac{d\Theta(t)}{dt} \geq C\Theta^{1+\varepsilon}(t), \ t > 0,$$

where $\varepsilon > 0$. If $\Theta(0) > 0$, then the solution ceases to exist for $t \geq \Theta^{-\varepsilon}(0)C^{-1}\varepsilon^{-1}$.

Lemma 1.3.5. *[33] 1- There exist $C_1, C_2 > 0$ such that, for all $x \in \overline{\Omega}$ and $(u, v) \in \mathbb{R}^2$ we have*

$$C_1 \left(|u|^{p(x)+1} + |v|^{p(x)+1} \right) \leq F(x, u, v) \leq C_2 \left(|u|^{p(x)+1} + |v|^{p(x)+1} \right). \tag{1.3.1}$$

2- For all $x \in \Omega$ and $(u, v) \in \mathbb{R}^2$, we have

$$u\ f_1(x, u, v) + v f_2(x, u, v) = (p(x) + 1) F(x, u, v), \tag{1.3.2}$$

where F is given by (2.1.4), $f_1 = \frac{\partial F}{\partial u}$ and $f_2 = \frac{\partial F}{\partial v}$.

Chapter 2

Coupled System of Nonlinear Hyperbolic Equations with Variable-exponents in the Damping and Source terms

2.1 Introduction

In this chapter, we are concerned with the following initial-boundary-value problem:

$$\begin{cases} u_{tt} - div\,(A\nabla u) + |u_t|^{m(x)-2}\, u_t = f_1\,(x,u,v) & \text{in } \Omega \times (0,T)\,, \\ v_{tt} - div\,(B\nabla v) + |v_t|^{r(x)-2}\, v_t = f_2\,(x,u,v) & \text{in } \Omega \times (0,T)\,, \\ u = v = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u\,(0) = u_0 \text{ and } u_t\,(0) = u_1 & \text{in } \Omega, \\ v\,(0) = v_0 \text{ and } v_t\,(0) = v_1 & \text{in } \Omega, \end{cases} \qquad (P)$$

where $T > 0$ and Ω is a bounded domain of $\mathbb{R}^n (n = 1, 2, 3)$ with a smooth boundary $\partial\Omega$.

The study of system (P) is motivated by the description of several models in physical phenomena, such as viscoelastic fluids, filtration processes through a porous media, fluids with temperature dependent viscosity, image processing, or robotics, etc. For more detail, one can see [1, 12].

Before studying the finite-time blow up of solutions with negative initial energy and for certain solutions with positive initial data, in Section 2.2 and 2.3, respectively, and investigating a global existence and stability results of the solutions, in Section 2.4, we state and prove an existence and uniqueness theorem of local weak solutions to problem (P), in the following Section. The proof of this result is based on the Faedo-Galerkin procedure as in [2, 17, 35] for systems with constant exponents.

ASSUMPTIONS:

The Damping terms.

In the system (P), the variable exponents m and r in the two damping terms are given continuous functions on $\overline{\Omega}$ satisfying

$$\begin{array}{ll} 2 \leq m(x), & \text{if } n = 1, 2, \\ 2 \leq m^- \leq m(x) \leq m^+ \leq 6, & \text{if } n = 3, \end{array} \tag{2.1.1}$$

and

$$\begin{array}{ll} 2 \leq r(x), & \text{if } n = 1, 2, \\ 2 \leq r^- \leq r(x) \leq r^+ \leq 6, & \text{if } n = 3, \end{array} \tag{2.1.2}$$

for all $x \in \Omega$, where

$$m^- = \inf_{x \in \Omega} m(x), \; m^+ = \sup_{x \in \Omega} m(x),$$

and

$$r^- = \inf_{x \in \Omega} r(x), \; r^+ = \sup_{x \in \Omega} r(x).$$

The Source terms.

In the right-hand side of the two differential equations of (P), the source terms f_1 and f_2 are given as follows, for all $x \in \overline{\Omega}$ and $(u, v) \in \mathbb{R}^2$:

$$f_1(x, u, v) = \frac{\partial}{\partial u} F(x, u, v) \text{ and } f_2(x, u, v) = \frac{\partial}{\partial v} F(x, u, v), \tag{2.1.3}$$

with

$$F(x, u, v) = a|u + v|^{p(x)+1} + 2b|uv|^{\frac{p(x)+1}{2}}, \tag{2.1.4}$$

where $a, b > 0$ are two positive constants, p is a continuous function on $\overline{\Omega}$ such that

$$\begin{array}{ll} 3 \leq p^- \leq p(x) \leq p^+, & \text{if } n = 1, 2, \\ p(x) = 3, & \text{if } n = 3, \end{array}$$

for all $x \in \Omega$, where

$$p^- = \inf_{x \in \Omega} p(x) \text{ and } p^+ = \sup_{x \in \Omega} p(x).$$

The Matrices A and B.

In the left-hand side of the differential equations of (P), A and B are two symmetric matrices of class $C^1\left(\overline{\Omega} \times [0, \infty)\right)$ such that there exist constants $a_0, b_0 > 0$ for which we have, for all $\xi \in \mathbb{R}^n$,

$$A\xi.\xi \geq a_0 |\xi|^2, \; B\xi.\xi \geq b_0 |\xi|^2 \tag{2.1.5}$$

and

$$A'\xi.\xi \leq 0, \; B'\xi.\xi \leq 0, \tag{2.1.6}$$

where $A' = \frac{\partial A}{\partial t}(., t)$ and $B' = \frac{\partial B}{\partial t}(., t)$.

2.2 Existence and Uniqueness of Local Weak Solution and Decreasingness of the Energy

In the beginning, let us introduce the definition of a weak solution for our system.

Definition 2.2.1. *Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$. A pair of functions (u, v) is said to be a weak solution of (P) on $[0, T)$ if $u, v \in C_\omega\left((0,T), H_0^1(\Omega)\right), u_t, v_t \in C_\omega\left((0,T), L^2(\Omega)\right), u_t \in L^{m(.)}\left(\Omega \times (0,T)\right),\ v_t \in L^{r(.)}\left(\Omega \times (0,T)\right)$ and for all test functions $\Phi, \Psi \in H_0^1(\Omega)$ and all $t \in (0,T)$, we have*

$$\begin{aligned}\int_\Omega u_t \Phi\ dx - \int_\Omega u_1 \Phi\ dx + \int_0^t \int_\Omega |u_t|^{m(x)-2} u_t \Phi\ dx d\tau \\ + \int_0^t \int_\Omega A\nabla u.\nabla\Phi\ dx d\tau - \int_0^t \int_\Omega f_1(x,u,v)\Phi\ dx d\tau = 0\end{aligned}$$

and

$$\begin{aligned}\int_\Omega v_t \Psi dx - \int_\Omega v_1 \Psi\ dx + \int_0^t \int_\Omega |v_t|^{r(x)-2} v_t \Psi\ dx d\tau \\ + \int_0^t \int_\Omega B\nabla v.\nabla\Psi\ dx d\tau - \int_0^t \int_\Omega f_2(x,u,v)\Psi\ dx d\tau = 0.\end{aligned}$$

2.2.1 Existence and Uniqueness of Local Weak Solution

In order to prove an existence theorem of a local weak solution of problem (P), we first consider, as in [27], the following initial-boundary-value problem:

$$\begin{cases} u_{tt} - div\left(A\nabla u\right) + |u_t|^{m(x)-2} u_t = f\left(x,t\right) & \text{in } \Omega \times (0,T), \\ v_{tt} - div\left(B\nabla v\right) + |v_t|^{r(x)-2} v_t = g\left(x,t\right) & \text{in } \Omega \times (0,T), \\ u = v = 0 & \text{on } \partial\Omega \times (0,T), \\ u\left(0\right) = u_0 \text{ and } u_t\left(0\right) = u_1 & \text{in } \Omega, \\ v\left(0\right) = v_0 \text{ and } v_t\left(0\right) = v_1 & \text{in } \Omega, \end{cases} \qquad (Q)$$

where $f, g \in L^2\left(\Omega \times (0,T)\right)$.

Theorem 2.2.2. *Under the above conditions, on m, r, A and B, and for $(u_0, u_1), (v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the problem (Q) has a unique local weak solution (u, v) on $[0, T)$, in the sense of Definition* 2.2.1.

Proof. **UNIQUENESS:**
Assume that (Q) has two weak solutions (u_1, v_1) and (u_2, v_2) on $[0, T)$, in the sense of Definition 2.2.1. Taking $\Phi = u_{1t} - u_{2t}$ and $\Psi = v_{1t} - v_{2t}$, in this definition, we infer that $(u, v) = (u_1 - u_2, v_1 - v_2)$ satisfies the following identities

$$\begin{aligned}\frac{d}{dt}\left[\int_\Omega (u_t^2 + A\nabla u.\nabla u)dx\right] - \int_\Omega A'\nabla u.\nabla u dx \\ + 2\int_\Omega \left(|u_{1t}|^{m(x)-2} u_{1t} - |u_{2t}|^{m(x)-2} u_{2t}\right) u_t dx = 0 \quad (2.2.1)\end{aligned}$$

and

$$\frac{d}{dt}\left[\int_{\Omega}(v_t^2 + B\nabla v.\nabla v)dx\right] - \int_{\Omega} B'\nabla v.\nabla v dx + 2\int_{\Omega}\left(|v_{1t}|^{r(x)-2} v_{1t} - |v_{2t}|^{r(x)-2} v_{2t}\right) v_t dx = 0, \quad (2.2.2)$$

for all $t \in (0,T)$, since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$,

$$\frac{d}{dt}\left(\int_{\Omega} A\nabla u.\nabla u dx\right) = \int_{\Omega} A'\nabla u.\nabla u dx + 2\int_{\Omega} A\nabla u.\nabla u_t dx$$

and

$$\frac{d}{dt}\left(\int_{\Omega} B\nabla v.\nabla v dx\right) = \int_{\Omega} B'\nabla v.\nabla v dx + 2\int_{\Omega} B\nabla v.\nabla v_t dx.$$

By (2.1.6), the equations (2.2.1) and (2.2.2) lead to

$$\frac{d}{dt}\left[\int_{\Omega}(u_t^2 + A\nabla u.\nabla u)dx\right] + 2\int_{\Omega}\left(|u_{1t}|^{m(x)-2} u_{1t} - |u_{2t}|^{m(x)-2} u_{2t}\right)(u_{1t} - u_{2t})dx \le 0 \quad (2.2.3)$$

and

$$\frac{d}{dt}\left[\int_{\Omega}(v_t^2 + B\nabla v.\nabla v)dx\right] + 2\int_{\Omega}\left(|u_{1t}|^{m(x)-2} u_{1t} - |u_{2t}|^{m(x)-2} u_{2t}\right)(u_{1t} - u_{2t})dx \le 0. \quad (2.2.4)$$

Integrating (2.2.3) and (2.2.4) over $(0,t)$, where $t \le T$, we find

$$\int_{\Omega}(u_t^2 + A\nabla u.\nabla u)dx + 2\int_0^t\int_{\Omega}\left(|u_{1t}|^{m(x)-2} u_{1t} - |u_{2t}|^{m(x)-2} u_{2t}\right)(u_{1t} - u_{2t})dxdt \le 0, \quad (2.2.5)$$

and

$$\int_{\Omega}(v_t^2 + B\nabla v.\nabla v)dx + 2\int_0^t\int_{\Omega}\left(|u_{1t}|^{m(x)-2} u_{1t} - |u_{2t}|^{m(x)-2} u_{2t}\right)(u_{1t} - u_{2t})dxdt \le 0. \quad (2.2.6)$$

Since we have, for all $x \in \Omega$ and $Y, Z \in \mathbb{R}$,

$$\left(|Y|^{q(x)-2} Y - |Z|^{q(x)-2} Z\right)(Y - Z) \ge 0,\ q(x) \ge 2 \quad (2.2.7)$$

and from (2.1.5)

$$\int_{\Omega} A\nabla u.\nabla u dx \ge a_0 \|\nabla u\|_2^2 \text{ and } \int_{\Omega} B\nabla v.\nabla v dx \ge b_0 \|\nabla v\|_2^2,$$

then, inequalities (2.2.5) and (2.2.6) give

$$\|u_t\|_2^2 + a_0 \|\nabla u\|_2^2 = 0 \text{ and } \|v_t\|_2^2 + b_0 \|\nabla v\|_2^2 = 0, \text{ respectively.}$$

Therefore, $u_t(x,.) = v_t(x,.) = 0$ on Ω and $\nabla u(.,t) = \nabla v(.,t) = 0$, for all $t \in (0,T)$.

Which implies $u = v = 0$ on $\Omega \times (0,T)$, since $u = v = 0$ on $\partial\Omega \times (0,T)$. This proves the uniqueness.

EXISTENCE:

To prove the existence of a local solution to problem(Q), we procced in several steps:

Step 1. Approximate problem.

Let $\{\omega_j\}_{j=1}^{\infty}$ be an orthonormal basis of $H_0^1(\Omega)$. For all $k \geq 1$, let (u^k, v^k) be a sequence in the finite-dimensional subspace $V_k = span\{\omega_1, \omega_2, ..., \omega_k\}$, defined by

$$u^k(x,t) = \Sigma_{j=1}^k a_j(t)\omega_j(x) \text{ and } v^k(t) = \Sigma_{j=1}^k b_j(t)\omega_j(x), \text{ for all } x \in \Omega \text{ and } t \in (0,T)$$

and satisfying the following approximate problems, denoted by (P_k):

$$\int_\Omega u_{tt}^k(x,t)\omega_j dx + \int_\Omega A\nabla u^k(x,t).\nabla\omega_j dx + \int_\Omega \left|u_t^k(x,t)\right|^{m(x)-2} u_t^k(x,t)\omega_j dx = \int_\Omega f(x,t)\omega_j, \tag{2.2.8}$$

$$\int_\Omega v_{tt}^k(x,t)\omega_j dx + \int_\Omega B\nabla v^k(x,t).\nabla\omega_j dx + \int_\Omega \left|v_t^k(x,t)\right|^{r(x)-2} v_t^k(x,t)\omega_j dx = \int_\Omega g(x,t)\omega_j, \tag{2.2.9}$$

for all $j = 1, 2, ..., k$, with the following initial data

$$\begin{aligned} u^k(0) = u_0^k = \Sigma_{i=1}^k \langle u_0, \omega_i\rangle \omega_i, \ u_t^k(0) = u_1^k = \Sigma_{i=1}^k \langle u_1, \omega_i\rangle \omega_i \\ v^k(0) = v_0^k = \Sigma_{i=1}^k \langle v_0, \omega_i\rangle \omega_i, \ v_t^k(0) = v_1^k = \Sigma_{i=1}^k \langle v_1, \omega_i\rangle \omega_i, \end{aligned} \tag{2.2.10}$$

such that

$$\begin{aligned} u_0^k \longrightarrow u_0 \text{ and } v_0^k \longrightarrow v_0 \ \ in \ H_0^1(\Omega), \\ u_1^k \longrightarrow u_1 \text{ and } v_1^k \longrightarrow v_1 \ \ in \ L^2(\Omega). \end{aligned}$$

This generates a system of k nonlinear ordinary differential equations, which admits a unique local solution (u^k, v^k) in $[0, T_k), T_k < T$, by standard ODE theory.

In the following step, we will show, by a priory estimates, that $T_k = T, \forall k \geq 1$.

Step 2. A priori Estimates.

We multiply (2.2.8) and (2.2.9) by $a_j'(t)$ and $b_j'(t)$, respectively. We sum each result over j, from 1 to k, to obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|u_t^k\|_2^2 + \int_\Omega A\nabla u^k.\nabla u^k dx\right] - \frac{1}{2}\int_\Omega A'\nabla u^k.\nabla u^k dx + \int_\Omega \left|u_t^k(x,t)\right|^{m(x)} dx = \int_\Omega f(x,t)u_t^k(x,t)dx, \tag{2.2.11}$$

$$\frac{1}{2}\frac{d}{dt}\left[\|v_t^k\|_2^2 + \int_\Omega B\nabla v^k.\nabla v^k dx\right] - \frac{1}{2}\int_\Omega B'\nabla v^k.\nabla v^k dx + \int_\Omega \left|v_t^k(x,t)\right|^{r(x)} dx = \int_\Omega g(x,t)v_t^k(x,t)dx. \tag{2.2.12}$$

The integration of (2.2.11) and (2.2.12) over $(0,t)$, with $t \leq T_k$, leads to

$$\|u_t^k\|_2^2 - \|u_1^k\|_2^2 + \int_\Omega A\nabla u^k.\nabla u^k dx - \int_\Omega A(x,0)\nabla u_0^k.\nabla u_0^k dx + 2\int_0^t \int_\Omega \left|u_t^k(x,t)\right|^{m(x)} dxds \tag{2.2.13}$$
$$\leq 2\int_0^t \int_\Omega f(x,t)u_t^k(x,t)dxds.$$

and

$$\|v_t^k\|_2^2 - \|v_1^k\|_2^2 + \int_\Omega B\nabla v^k.\nabla v^k dx - \int_\Omega B(x,0)\nabla v_0^k.\nabla v_0^k dx + 2\int_0^t \int_\Omega \left|v_t^k(x,t)\right|^{r(x)} dxds \tag{2.2.14}$$
$$\leq 2\int_0^t \int_\Omega g(x,t)v_t^k(x,t)dxds,$$

by virtue of (2.1.6). Now, by adding (2.2.13) and (2.2.14), using the assumptions on A, B and Young's inequality (1.2.2), we arrive at

$$\begin{aligned}
&\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + a_0\|\nabla u^k\|_2^2 + b_0\|\nabla v^k\|_2^2 + 2\int_0^{T_k} \int_\Omega \left(\left|u_t^k(x,t)\right|^{m(x)} + \left|v_t^k(x,t)\right|^{r(x)}\right) dxds \\
&\leq \|u_1^k\|_2^2 + \|v_1^k\|_2^2 + \alpha\left\|\nabla u_0^k\right\|_2^2 + \beta\left\|\nabla v_0^k\right\|_2^2 + 2\int_0^T \int_\Omega \left(f(x,t)u_t^k(x,t) + g(x,t)v_t^k(x,t)\right) dxds \\
&\leq \|u_1^k\|_2^2 + \|v_1^k\|_2^2 + \alpha\left\|\nabla u_0^k\right\|_2^2 + \beta\left\|\nabla v_0^k\right\|_2^2 + 2C_\varepsilon\int_0^T \int_\Omega (\mid f(x,t)\mid^2 + \mid g(x,t)\mid^2)dxds \\
&+ 2\varepsilon\int_0^{T_k} \left(\left\|u_t^k\right\|_2^2 + \left\|v_t^k\right\|_2^2\right) ds,
\end{aligned} \tag{2.2.15}$$

where

$$\alpha = \sup_{\Omega\times(0,T)} A(x,t) \text{ and } \beta = \sup_{\Omega\times(0,T)} B(x,t).$$

But

$$u_0^k \longrightarrow u_0 \text{ and } v_0^k \longrightarrow v_0 \ \ in\ H_0^1(\Omega),$$
$$u_1^k \longrightarrow u_1 \text{ and } v_1^k \longrightarrow v_1 \ \ in\ L^2(\Omega)$$

so, inequality (2.2.15) is rewritten as follows

$$\begin{aligned}
&\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + a_0\|\nabla u^k\|_2^2 + b_0\|\nabla v^k\|_2^2 + 2\int_0^{T_k} \int_\Omega \left(\left|u_t^k(x,t)\right|^{m(x)} + \left|v_t^k(x,t)\right|^{r(x)}\right) dxds \\
&\leq M + 2\varepsilon\int_0^{T_k} \left(\left\|u_t^k\right\|_2^2 + \left\|v_t^k\right\|_2^2\right) ds + 2C_\varepsilon\int_0^T \int_\Omega \left(|f(x,t)|^2 + |g(x,t)|^2\right) dxds,\ M > 0.
\end{aligned}$$

Since $f, g \in L^2(\Omega \times (0,T))$, then

$$\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + a_0\|\nabla u^k\|_2^2 + b_0\|\nabla v^k\|_2^2 + 2\int_0^{T_k}\int_\Omega \left(\left|u_t^k(x,t)\right|^{m(x)} + \left|v_t^k(x,t)\right|^{r(x)}\right) dxds \tag{2.2.16}$$

$$\leq C_\varepsilon + 2\varepsilon \int_0^{T_k} (\|u_t^k\|_2^2 + \|v_t^k\|_2^2) ds.$$

This gives, for all $\varepsilon > 0$,

$$\|u_t^k\|_2^2 + \|v_t^k\|_2^2 \leq C_\varepsilon + 2\varepsilon \int_0^{T_k} \left(\|u_t^k\|_2^2 + \|v_t^k\|_2^2\right) ds, \forall t \in [0, T_k].$$

Therefore,

$$\|u_t^k\|_2^2 + \|v_t^k\|_2^2 = \|(u_t^k, v_t^k)\|_2^2 \leq C_\varepsilon, \forall t \in [0, T_k], \forall k \geq 1,$$

by virtue of Gronwall's Lemma 1.3.3. Consequently, estimate (2.2.16) leads to

$$\sup_{(0,T_k)} [\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2] + \int_0^{T_k}\int_\Omega \left(\left|u_t^k(x,t)\right|^{m(x)} + \left|v_t^k(x,t)\right|^{r(x)}\right) dxds \leq C_\varepsilon.$$

Taking $\varepsilon = \frac{1}{2}$ to find

$$\sup_{(0,T_k)} [\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2] + \int_0^{T_k}\int_\Omega \left(\left|u_t^k(x,t)\right|^{m(x)} + \left|v_t^k(x,t)\right|^{r(x)}\right) dxds \leq C,$$

where $C > 0$, for all $T_k < T$ and $k \geq 1$. Therefore, the local solution (u^k, v^k) of system (P_k) can be extended to $(0, T)$ for all $k \geq 1$. Furthermore, we have

$$(u^k)_k, (v^k)_k \text{ are bounded in } L^\infty((0,T), H_0^1(\Omega)),$$

$$(u_t^k)_k \text{ is bounded in } L^\infty((0,T), L^2(\Omega)) \cap L^{m(.)}(\Omega \times (0,T)),$$
$$(v_t^k)_k \text{ is bounded in } L^\infty((0,T), L^2(\Omega)) \cap L^{r(.)}(\Omega \times (0,T)).$$

Consequently, we can extract two subsequences of $(u^k)_k$ and $(v^k)_k$, which we denote by $(u_l)_l$ and $(v_l)_l$, respectively, such that, when $l \to \infty$, we have

$$u^l \to u \text{ and } v^l \to v \text{ weakly * in } L^\infty((0,T), H_0^1(\Omega)),$$

$$u_t^l \to u_t \text{ weakly * in } L^\infty((0,T), L^2(\Omega)) \text{ and weakly in } L^{m(.)}(\Omega \times (0,T)),$$
$$v_t^l \to v_t \text{ weakly * in } L^\infty((0,T), L^2(\Omega)) \text{ and weakly in } L^{r(.)}(\Omega \times (0,T)).$$

Step 3. The Nonlinear terms.
In this step, we show that

$$| u_t^l |^{m(.)-2} u_t^l \to | u_t |^{m(.)-2} u_t \text{ weakly in } L^{\frac{m(.)}{m(.)-1}}(\Omega \times (0,T))$$

and

$$| v_t^l |^{r(.)-2} v_t^l \to | v_t |^{r(.)-2} v_t \text{ weakly in } L^{\frac{r(.)}{r(.)-1}}(\Omega \times (0,T)).$$

By exploiting Hölder's inquality (1.2.3), it results

$$(| u_t^l |^{m(.)-2} u_t^l)_l \text{ is bounded in } L^{\frac{m(.)}{m(.)-1}}(\Omega \times (0,T)),$$

since $(u_t^l)_l$ is bounded in $L^{m(.)}(\Omega \times (0,T))$. It follows that, there exists a subsequence of $(| u_t^l |^{m(.)-2} u_t^l)_l$, still denoted by $(| u_t^l |^{m(.)-2} u_t^l)_l$, for simplicity, such that

$$| u_t^l |^{m(.)-2} u_t^l \to \Phi \text{ weakly in } L^{\frac{m(.)}{m(.)-1}}(\Omega \times (0,T)).$$

In what follows, we prove that $\Phi =| u_t |^{m(.)-2} u_t$.
For this purpose, we set $h(z) =| z |^{m(.)-2} z$ and define the following sequence, for all $l \geq 1$, see [27],

$$S_l = \int_0^T \int_\Omega (h(u_t^l) - h(z))(u_t^l - z), \ \forall z \in L^{m(.)}((0,T), H_0^1(\Omega)).$$

By the inequality (2.2.7), $S_l \geq 0$, for all $l \geq 1$. Replacing u^k by u^l in (2.2.11) and integrating the result over $(0,T)$, we find

$$\begin{aligned} S_l =& \frac{1}{2}\left[\|u_1^l\|_2^2 - \|u_t^l(T)\|_2^2 + \int_\Omega A(x,0)\nabla u_0^l.\nabla u_0^l - \int_\Omega A(x,T)\nabla u^l(T).\nabla u^l(T) \right] \\ & - \int_0^T \int_\Omega h(u_t^l)z - \int_0^T \int_\Omega h(z)(u_t^l - z) + \int_0^T \int_\Omega f u_t^l, \ \forall\, l \geq 1. \end{aligned} \tag{2.2.17}$$

By the definition of $(u_0^l), (u_1^l)$ and since $A \in C^1\left(\overline{\Omega} \times [0,\infty[\right)$ and

$$u^l \to u \text{ weakly * in } L^\infty((0,T), H_0^1(\Omega))$$

we obtain

$$\lim_{l \longrightarrow \infty} \|u_1^l\|_2^2 = \|u_1\|_2^2,$$

$$\liminf_{l \longrightarrow \infty} \int_\Omega A(x,T)\nabla u^l(T).\nabla u^l(T) \geq \int_\Omega A(x,T)\nabla u(T).\nabla u(T)$$

and

$$\lim_{l \longrightarrow \infty} \int_\Omega A(x,0)\nabla u_0^l.\nabla u_0^l dx = \int_\Omega A(x,0)\nabla u_0.\nabla u_0 dx.$$

Also, we have

$$u_t^l \to u_t \text{ weakly * in } L^\infty((0,T), L^2(\Omega)) \text{ and weakly in } L^{m(.)}(\Omega \times (0,T))$$

and

$$h(u_t^l) =| u_t^l |^{m(.)-2} u_t^l \to \Phi \text{ weakly in } L^{\frac{m(.)}{m(.)-1}}(\Omega \times (0,T)).$$

Therefore,

$$\liminf_{l\longrightarrow\infty}\|u_t^l(T)\|_2^2\geq\|u_t(T)\|_2^2,$$

$$\lim_{l\longrightarrow\infty}\int_0^T\int_\Omega fu_t^l=\int_0^T\int_\Omega fu_t,$$

$$\lim_{l\longrightarrow\infty}\int_0^T\int_\Omega h(u_t^l)z=\int_0^T\int_\Omega \Phi z$$

and

$$\lim_{l\longrightarrow\infty}\int_0^T\int_\Omega h(z)(u_t^l-z)=\int_0^T\int_\Omega h(z)(u_t-z),$$

since $f\in L^2(\Omega\times(0,T))$, $z\in L^{m(.)}\left((0,T),H_0^1(\Omega)\right)$ and $H_0^1(\Omega)\subset L^{m(.)}(\Omega)$, by invoking Lemma 1.2.8. Taking $l\to\infty$ in (2.2.17) and substituting the above limits, it yields

$$\begin{aligned}0\leq\limsup_l S_l\leq&\frac{1}{2}\left[\|u_1\|_2^2-\|u_t(T)\|_2^2+\int_\Omega A(x,0)\nabla u_0.\nabla u_0-\int_\Omega A(x,T)\nabla u(T).\nabla u(T)\right]\\&-\int_0^T\int_\Omega\Phi z-\int_0^T\int_\Omega h(z)(u_t-z)+\int_0^T\int_\Omega fu_t,\end{aligned}\tag{2.2.18}$$

since

$$\begin{aligned}\limsup_l\left(-\int_\Omega A(x,T)\nabla u^l(T).\nabla u^l(T)dx\right)&=-\liminf_l\int_\Omega A(x,T)\nabla u^l(T).\nabla u^l(T)dx\\&\leq-\int_\Omega A(x,T)\nabla u(T).\nabla u(T)dx.\end{aligned}$$

On the other hand, if we use u^l instead of u^k in (2.2.8) and integrate the result over $(0,t)$, we find

$$\int_\Omega u_t^l\omega_j-\int_\Omega u_1^l\omega_j+\int_0^t\int_\Omega A\nabla u^l.\nabla\omega_j+\int_0^t\int_\Omega|\,u_t^l\,|^{m(x)-2}\,u_t^l\omega_j=\int_0^t\int_\Omega f\omega_j,\ \forall 1<j<l.$$

This leads to

$$\int_\Omega u_t\omega-\int_\Omega u_1\omega+\int_0^t\int_\Omega A\nabla u.\nabla\omega+\int_0^t\int_\Omega\Phi\omega=\int_0^t\int_\Omega f\omega,\ \forall\omega\in H_0^1(\Omega),$$

since $\cup_{m\geq1}V_m$ is dense in $H_0^1(\Omega)$. Taking the derivative with respect to t, it comes for a.e $t\in[0,T]$, that

$$\int_\Omega u_{tt}\omega+\int_\Omega(A\nabla u.\nabla\omega+\Phi\omega)=\int_\Omega f\omega,\ \forall\omega\in H_0^1(\Omega).\tag{2.2.19}$$

Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, we can set u_t instead of ω in (2.2.19) to get

$$\int_\Omega u_{tt}u_t+\int_\Omega(A\nabla u.\nabla u_t+\Phi u_t)=\int_\Omega fu_t,\tag{2.2.20}$$

By integrating (2.2.20) over $(0,T)$, we arrive at

$$\int_0^T \int_\Omega f u_t = \frac{1}{2}\left[\|u_t(T)\|_2^2 - \|u_1\|_2^2 + \int_\Omega A(x,T)\nabla u(T).\nabla u(T) - \int_\Omega A(x,0)\nabla u_0.\nabla u_0\right] + \int_0^T \int_\Omega \Phi u_t, \tag{2.2.21}$$

Combining (2.2.18) and (2.2.21), we find

$$0 \le \limsup_l S_l \le \int_0^T \int_\Omega \Phi u_t - \int_0^T \int_\Omega \Phi z - \int_0^T \int_\Omega h(z)(u_t - z).$$

So,

$$\int_0^T \int_\Omega [\Phi - h(z)](u_t - z) \ge 0, \ \forall z \in L^{m(.)}((0,T), H_0^1(\Omega)).$$

Under the assumption (2.1.1) and by virtue of Lemma 1.2.8, $H_0^1(\Omega)$ is dense in $L^{m(.)}(\Omega)$. Consequently,

$$\int_0^T \int_\Omega [\Phi - h(z)](u_t - z) \ge 0, \ \forall z \in L^{m(.)}(\Omega \times (0,T)). \tag{2.2.22}$$

Now, let $z = \lambda\omega + u_t, \ \omega \in L^{m(.)}(\Omega \times (0,T))$. Hence, inequality (2.2.22) yields

$$-\lambda \int_0^T \int_\Omega [\Phi - h(\lambda\omega + u_t)]\,\omega \ge 0, \quad \forall \lambda \ne 0.$$

We have two cases:
- If $\lambda > 0$, then

$$\int_0^T \int_\Omega [\Phi - h(\lambda\omega + u_t)]\,\omega \le 0, \quad \forall \omega \in L^{m(.)}(\Omega \times (0,T)).$$

Taking $\lambda \to 0$ and by the continuity of h with respect to λ, it results that

$$\int_0^T \int_\Omega (\Phi - h(u_t))\omega \le 0, \ \forall \omega \in L^{m(.)}(\Omega \times (0,T)). \tag{2.2.23}$$

- If $\lambda < 0$, similarly, we get

$$\int_0^T \int_\Omega (\Phi - h(u_t))\omega \ge 0, \ \forall \omega \in L^{m(.)}(\Omega \times (0,T)). \tag{2.2.24}$$

From (2.2.23) and (2.2.24), we deduce that

$$\int_0^T \int_\Omega (\Phi - h(u_t))\omega = 0, \ \forall \omega \in L^{m(.)}(\Omega \times (0,T)).$$

This gives $\Phi =\mid u_t \mid^{m(x)-2} u_t$. Therefore, inequality (2.2.19) leads to

$$\int_\Omega u_{tt}\omega + \int_\Omega A\nabla u.\nabla\omega + \int_\Omega \mid u_t \mid^{m(x)-2} u_t\omega = \int_\Omega f\omega,\ \forall\omega \in H_0^1(\Omega).$$

Consequently,

$$u_{tt} - div(A\nabla u) + \mid u_t \mid^{m(x)-2} u_t = f \text{ in } D'(\Omega \times (0,T)). \tag{2.2.25}$$

Likewise and since $H_0^1(\Omega)$ is dense in $L^{r(.)}(\Omega)$ (Lemma 1.2.8), we obtain

$$\mid v_t^l \mid^{r(.)-2} v_t^l \to \mid v_t \mid^{r(.)-2} v_t \text{ weakly in } L^{\frac{r(.)}{r(.)-1}}(\Omega \times (0,T))$$

and

$$v_{tt} - div(B\nabla v) + \mid v_t \mid^{r(x)-2} v_t = g \text{ in } D'(\Omega \times (0,T)). \tag{2.2.26}$$

From (2.2.25) and (2.2.26), we conclude that u and v satisfy the two differential equations of system (Q) on $\Omega \times [0,T]$.

Step 4. The Initial Conditions.

- First, we prove that

$$u(x,0) = u_0(x) \textit{ and } v(x,0) = v_0(x).$$

Invoking Lions' Lemma [27] (Lemma 1.2, page 7) and since

$$u^l \rightharpoonup u \text{ weakly * in } L^\infty((0,T), H_0^1(\Omega))$$

and

$$u_t^l \rightharpoonup u_t \text{ weakly * in } L^\infty((0,T), L^2(\Omega)),$$

we deduce that

$$u^l \longrightarrow u \text{ in } C([0,T], L^2(\Omega)).$$

So, for all $x \in \Omega, u^l(x,0)$ makes sense and

$$u^l(x,0) \longrightarrow u(x,0) \text{ in } L^2(\Omega).$$

By the definition of u_0^l, we have

$$u^l(x,0) = u_0^l(x) \longrightarrow u_0(x), \text{ in } H_0^1(\Omega).$$

Therefore, $u(x,0) = u_0(x)$. Similary, we obtain $v(x,0) = v_0(x)$.

- Second, we handle the second intial condition, that is

$$u_t(x,0) = u_1(x) \text{ and } v_t(x,0) = v_1(x).$$

For any $\phi \in C_0^\infty(0,T)$ and $j \le l$, we obtain from (2.2.8) that

$$\begin{aligned}&\int_0^T \int_\Omega u_{tt}^l(x,t)\omega_j(x)\phi(t) + \int_0^T \int_\Omega A\nabla u^l(x,t).\nabla\omega_j(x)\phi(t)\\ &= -\int_0^T \int_\Omega \left|u_t^l(x,t)\right|^{m(x)-2} u_t^l(x,t)\omega_j(x)\phi(t) + \int_0^T \int_\Omega f(x,t)\omega_j(x)\phi(t).\end{aligned} \tag{2.2.27}$$

But

$$\frac{d}{dt}\left(u_t^l\phi(t)\omega_j(x)\right) = \left(u_{tt}^l\phi(t) + u_t^l\phi'(t)\right)\omega_j(x),$$

then

$$\int_\Omega u_{tt}^l\phi(t)\omega_j(x)dx = \int_\Omega \frac{d}{dt}\left(u_t^l\phi(t)\omega_j(x)\right)dx - \int_\Omega u_t^l\phi'(t)\omega_j(x)dx.$$

Therefore,

$$\begin{aligned}&\int_0^T\int_\Omega u_{tt}^l\phi(t)\omega_j(x)dxds\\&= \int_0^T\int_\Omega \frac{d}{dt}\left(u_t^l\phi(t)\omega_j(x)\right)dxds - \int_0^T\int_\Omega u_t^l\phi'(t)\omega_j(x)dxds,\end{aligned}$$

i.e,

$$\begin{aligned}&\int_0^T\int_\Omega u_{tt}^l\phi(t)\omega_j(x)dxds\\&= \int_\Omega \left(u_t^l(x,T)\phi(T) - u_t^l(x,0)\phi(0)\right)\omega_j(x)dx - \int_0^T\int_\Omega u_t^l\phi'(t)\omega_j(x)dxds. \quad (2.2.28)\end{aligned}$$

Since $\phi \in C_0^\infty(0,T)$, then $\phi(T) = \phi(0) = 0$.
Consequently, (2.2.28) leads to

$$\int_0^T\int_\Omega u_{tt}^l\phi(t)\omega_j(x)dxds = -\int_0^T\int_\Omega u_t^l\phi'(t)\omega_j(x)dxds. \quad (2.2.29)$$

By substituting (2.2.29) in (2.2.27), we obtain

$$\begin{aligned}&-\int_0^T\int_\Omega u_t^l(x,t)\omega_j(x)\phi'(t) + \int_0^T\int_\Omega A\nabla u^l(x,t).\nabla\omega_j(x)\phi(t)\\&= -\int_0^T\int_\Omega \left|u_t^l(x,t)\right|^{m(x)-2}u_t^l(x,t)\omega_j(x)\phi(t) + \int_0^T\int_\Omega f(x,t)\omega_j(x)\phi(t),\ \forall l \geq 1.\end{aligned}$$

Taking $l \longmapsto +\infty$ to get

$$\begin{aligned}&-\int_0^T\int_\Omega u_t(x,t)\omega_j(x)\phi'(t) - \int_0^T\int_\Omega div(A\nabla u(x,t))\omega_j(x)\phi(t)\\&= -\int_0^T\int_\Omega |u_t(x,t)|^{m(x)-2}u_t(x,t)\omega_j(x)\phi(t) + \int_0^T\int_\Omega f(x,t)\omega_j(x)\phi(t),\ \forall j \geq 1,\end{aligned}$$

since, $\omega_j \in H_0^1(\Omega) \subset L^{m(.)}(\Omega)$, for all $j \geq 1$.
Thanks to (2.2.29), this gives

$$\begin{aligned}&\int_0^T\int_\Omega u_{tt}(x,t)\omega(x)\phi(t) - \int_0^T\int_\Omega div(A\nabla u(x,t))\omega(x)\phi(t)\\&= -\int_0^T\int_\Omega |u_t(x,t)|^{m(x)-2}u_t(x,t)\omega(x)\phi(t) + \int_0^T\int_\Omega f(x,t)\omega(x)\phi(t),\end{aligned}$$

for all $\omega \in H_0^1(\Omega)$. Consequently,

$$\begin{aligned}&\int_0^T \int_\Omega u_{tt}(x,t)\omega(x)\phi(t)\\&= \int_0^T \int_\Omega \left[div(A\nabla u(x,t)) - |u_t(x,t)|^{m(x)-2} u_t(x,t) + f(x,t)\right] \omega(x)\phi(t).\end{aligned}$$

This means $u_{tt} \in L^{\frac{m(.)}{m(.)-1}}([0,T), H^{-1}(\Omega))$ and u solves the following equation

$$u_{tt} - div(A\nabla u) + |u_t|^{m(x)-2} u_t = f, \text{ in } D'(\Omega \times (0,T)).$$

So, we have

$$u_t \in L^\infty((0,T), L^2(\Omega)) \text{ and } u_{tt} \in L^{\frac{m(.)}{m(.)-1}}([0,T), H^{-1}(\Omega)).$$

By Lions's Lemma [27] (see Lemma 1.2, page 7), we deduce that $u_t \in C([0,T), H^{-1}(\Omega))$. Hence, $u_t(x,0)$ has a meaning, for all $x \in \Omega$, with

$$u_t^l(x,0) \longrightarrow u_t(x,0) \text{ in } H^{-1}(\Omega).$$

In the other hand, by the definition of u_1^l, we have

$$u_t^l(x,0) = u_1^l(x) \longrightarrow u_1(x) \text{ in } L^2(\Omega).$$

Consequently, $u_t(x,0) = u_1(x)$. Similary, we can prove that $v_t(x,0) = v_1(x)$.
Finaly, we deduce that (u,v) is a unique local solution of (Q). □

To prove the existence result for problem (P), we recall the following elementary inequalities:

$$\left||X|^k - |Y|^k\right| \leq C|X-Y|\left(|X|^{k-1} + |Y|^{k-1}\right), \tag{2.2.30}$$

for some constant $C > 0$, all $k \geq 1$ and all $X, Y \in \mathbb{R}$. Also

$$\left||X|^{k'} X - |Y|^{k'} Y\right| \leq C|X-Y|\left(|X|^{k'} + |Y|^{k'}\right), \tag{2.2.31}$$

for some constant $C > 0$, all $k' \geq 0$ and all $X, Y \in \mathbb{R}$.

Theorem 2.2.3. *Suppose that the assumptions of Theorem 2.2.2 are fulfilled and that p, in the coupled terms, satisfies the following conditions on $\overline{\Omega}$*

$$\begin{array}{ll} 3 \leq p^- \leq p(x) \leq p^+ < \infty, & \text{if } n = 1,2, \\ p(x) = 3, & \text{if } n = 3. \end{array} \tag{2.2.32}$$

Then, problem (P) has a unique weak maximal solution (u,v) (in the sense of Defintion 2.2.1) on $[0,T)$, for some $T > 0$. Moreover, the following alternatives hold:

1. $T = +\infty$,
   or
2. $T < +\infty$ *and* $lim_{t\to T}\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) = +\infty$.

Proof. **EXISTENCE:**
Recall that the source terms are defined for all $x \in \Omega$ and $(y,z) \in \mathbb{R}^2$ by

$$f_1(x,y,z) = \frac{\partial}{\partial y} F(x,y,z) \text{ and } f_2(x,y,z) = \frac{\partial}{\partial z} F(x,y,z),$$

where

$$F(x,y,z) = a|y+z|^{p(x)+1} + 2b|yz|^{\frac{p(x)+1}{2}}, \; a,b>0.$$

So,

$$f_1(x,y,z) = (p(x)+1)\left[a|y+z|^{p(x)-1}(y+z) + by|y|^{\frac{p(x)-3}{2}}|z|^{\frac{p(x)+1}{2}}\right]$$

and

$$f_2(x,y,z) = (p(x)+1)\left[a|y+z|^{p(x)-1}(y+z) + bz|z|^{\frac{p(x)-3}{2}}|y|^{\frac{p(x)+1}{2}}\right].$$

Let $y,z \in L^\infty((0,T), H_0^1(\Omega))$. Using Young's inequality (1.2.2) and the Sobolev embeddings (Lemma 1.2.20), then $f_1(y,z)$ and $f_2(y,z)$ are in $L^2(\Omega \times (0,T))$. Indeed, for all $t \in (0,T)$, we have

$$\begin{aligned}
&\int_\Omega |f_1(x,y,z)|^2 dx \le 2\int_\Omega (p(x)+1)^2\left[a^2|y+z|^{2p(x)} + b^2|y|^{p(x)-1}|z|^{p(x)+1}\right]dx \\
&\le 2(p^+ +1)^2\left[a^2\int_\Omega |y+z|^{2p(x)}dx + b^2\int_\Omega |y|^{p(x)-1}|z|^{p(x)+1}dx\right] \\
&\le C_0\left[\int_\Omega |y+z|^{2p^+}dx + \int_\Omega |y+z|^{2p^-}dx + \int_\Omega |y|^{3(p(x)-1)}dx + \int_\Omega |z|^{\frac{3}{2}(p(x)+1)}dx\right],
\end{aligned} \tag{2.2.33}$$

where $C_0 = 2(p^+ +1)^2 max\{a^2, 3b^2\} > 0$. By the embeddings (Lemma 1.2.20) and the fact that $y,z \in L^\infty((0,T), H_0^1(\Omega))$, one can obtain the following results:

- If $n = 1,2$, then

$$1 \le \frac{3}{2}(p^- +1) \le \frac{3}{2}(p^+ +1) \le 2p^+ \le 3(p^+ -1) < \infty,$$

since $3 \le p^- \le p(x) \le p^+ < \infty$. Therefore, estimate (2.2.33) leads to

$$\begin{aligned}
&\int_\Omega |f_1(x,y,z)|^2 dx \\
&\le C_1\left[\|\nabla(y+z)\|_2^{2p^+} + \|\nabla(y+z)\|_2^{2p^-} + \|\nabla y\|_2^{3(p^+-1)} + \|\nabla y\|_2^{3(p^--1)}\right] \\
&+ C_1\left[\|\nabla z\|_2^{\frac{3}{2}(p^++1)} + \|\nabla z\|_2^{\frac{3}{2}(p^-+1)}\right] < +\infty, \; C_1 = C_0C_e.
\end{aligned} \tag{2.2.34}$$

- If $n = 3$, then the Sobolev embeddings used in (2.2.34) take place since

$$1 \le 6 = 2p^- = 2p^+ \le \frac{2n}{n-2} = 6,$$

$$1 \le 6 = \frac{3}{2}(p^- +1) = \frac{3}{2}(p^+ +1) \le \frac{2n}{n-2} = 6,$$

and

$$1 \leq 6 = 3(p^{-} - 1) = 3(p^{+} - 1) \leq \frac{2n}{n-2} = 6.$$

Therefore, estimate (2.2.34) is also saitisfied, when $n = 3$. Consequently, under the assumption (2.2.32), we have, for all $t \in (0, T)$,

$$\int_{\Omega} |f_1(x, y, z)|^2 \, dx < \infty$$

and, similarly,

$$\int_{\Omega} |f_2(x, y, z)|^2 \, dx < \infty.$$

Therefore,

$$f_1(y, z), f_2(y, z) \in L^2(\Omega \times (0, T)).$$

By virtue of Theorem 2.2.2, there exists a unique (u, v), in the sense of Definition 2.2.1 which solves the following problem

$$\begin{cases} u_{tt} - div\,(A\nabla u) + |u_t|^{m(x)-2}\, u_t = f_1(y, z) & \text{in } \Omega \times (0, T)\,, \\ v_{tt} - div\,(B\nabla v) + |v_t|^{r(x)-2}\, v_t = f_2(y, z) & \text{in } \Omega \times (0, T)\,, \\ u = v = 0 & \text{on } \partial\Omega \times (0, T) \\ u\,(0) = u_0 \text{ and } u_t\,(0) = u_1 & \text{in } \Omega, \\ v\,(0) = v_0 \text{ and } v_t\,(0) = v_1 & \text{in } \Omega, \end{cases} \tag{R}$$

since $(u_0, u_1), (v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$.
Now, let $G : W_T \times W_T :\longrightarrow W_T \times W_T$ be a map defined by $G(y, z) = (u, v)$, where

$$W_T = \left\{ w \in L^{\infty}((0, T), H_0^1(\Omega)) / w_t \in L^{\infty}((0, T), L^2(\Omega)) \right\}.$$

W_T is a Banach space with respect to the norm

$$||w||_{W_T}^2 = \sup_{(0,T)} \int_{\Omega} |\nabla w|^2 dx + \sup_{(0,T)} \int_{\Omega} |w_t|^2 dx.$$

In what follows, our task is to prove that G is a contraction mapping from a bounded ball $B(0, d)$ into itself, where

$$B(0, d) = \left\{ (y, z) \in W_T \times W_T / \; \|(y, z)\|_{W_{T_0} \times W_{T_0}} \leq d \right\},$$

for $d > 1$ and $T_0 > 0$ to be fixed later.

$G : B(0, d) \longrightarrow B(0, d)$ **is a map for certain** $d > 0$

Taking $(\Phi, \Psi) = (u_t, v_t)$ in Definition 2.2.1 and integrating each result over $(0, t)$ we get, for all $t \leq T$,

$$\begin{aligned} \frac{1}{2} \|u_t\|_2^2 - \frac{1}{2} \|u_1\|_2^2 + \frac{1}{2} \int_{\Omega} A\nabla u.\nabla u dx - \frac{1}{2} \int_{\Omega} A(x, 0)\nabla u_0.\nabla u_0 dx \\ + \int_0^t \int_{\Omega} |u_t(x, t)|^{m(x)} \leq \int_0^t \int_{\Omega} u_t f_1(y, z) dx ds \end{aligned} \tag{2.2.35}$$

and

$$\frac{1}{2}\|v_t\|_2^2 - \frac{1}{2}\|v_1\|_2^2 + \frac{1}{2}\int_\Omega B\nabla v.\nabla v dx - \frac{1}{2}\int_\Omega B(x,0)\nabla v_0^k.\nabla v_0 dx$$
$$+\int_0^t\int_\Omega |v_t(x,t)|^{r(x)} \le \int_0^t\int_\Omega v_t f_2(y,z)dxds, \tag{2.2.36}$$

by virtue of (2.1.6). Under the assumptions on A and B, inequalities (2.2.35) and (2.2.36) lead to

$$\frac{1}{2}\left(\|u_t\|_2^2 + a_0\|\nabla u\|_2^2\right) \le \frac{1}{2}\left(\|u_1\|_2^2 + \alpha\|\nabla u_0\|_2^2\right) + \int_0^t\int_\Omega u_t f_1(y,z)dxds$$

and

$$\frac{1}{2}\left(\|v_t\|_2^2 + b_0\|\nabla v\|_2^2\right) \le \frac{1}{2}\left(\|v_1\|_2^2 + \beta\|\nabla v_0\|_2^2\right) + \int_0^t\int_\Omega v_t f_2(y,z)dxds,$$

where $\alpha = \sup_{\Omega\times(0,T)} A(x,t)$ and $\beta = \sup_{\Omega\times(0,T)} B(x,t)$. Consequently,

$$\frac{1}{2}(\|u_t\|_2^2 + \|\nabla u\|_2^2) \le \frac{\|u_1\|_2^2 + \alpha\|\nabla u_0\|_2^2}{2C_2} + \frac{1}{C_2}\int_0^t\int_\Omega u_t f_1(y,z)dxds$$

and

$$\frac{1}{2}(\|v_t\|_2^2 + \|\nabla v\|_2^2) \le \frac{\|v_1\|_2^2 + \alpha\|\nabla v_0\|_2^2}{2C_3} + \frac{1}{C_3}\int_0^t\int_\Omega v_t f_2(y,z)dxds,$$

where $C_2 = min\{1, a_0\}$ and $C_3 = min\{1, b_0\}$. Therefore,

$$\frac{1}{2}\|u\|_{W_T}^2 = \frac{1}{2}\sup_{(0,T)}\left(\|u_t\|_2^2 + \|\nabla u\|_2^2\right) \le \lambda_0 + \frac{1}{C_2}\sup_{(0,T)}\int_0^t\int_\Omega u_t f_1(y,z)dxds$$

and

$$\frac{1}{2}\|v\|_{W_T}^2 = \frac{1}{2}\sup_{(0,T)}\left(\|v_t\|_2^2 + \|\nabla v\|_2^2\right) \le \beta_0 + \frac{1}{C_3}\sup_{(0,T)}\int_0^t\int_\Omega v_t f_2(y,z)dxds,$$

where, $\lambda_0 = \frac{\|u_1\|_2^2+\alpha\|\nabla u_0\|_2^2}{2C_2}$ and $\beta_0 = \frac{\|v_1\|_2^2+\beta\|\nabla v_0\|_2^2}{2C_3}$. The addition of the last two inequalities gives

$$\begin{aligned}\frac{1}{2}\|(u,v)\|_{W_T\times W_T}^2 &\le \gamma_0 + C_4\sup_{(0,T)}\int_0^t\left(\int_\Omega u_t f_1(y,z)dx + \int_\Omega v_t f_2(y,z)dx\right)ds\\ &\le \gamma_0 + C_4\sup_{(0,T)}\int_0^t\left(\left|\int_\Omega u_t f_1(y,z)dx\right| + \left|\int_\Omega v_t f_2(y,z)dx\right|\right)ds,\end{aligned} \tag{2.2.37}$$

where $\gamma_0 = \lambda_0+\beta_0$ (depending on the initial data) and $C_4 = \frac{1}{C_2}+\frac{1}{C_3}$ (depending on the two mtrices A, B). Under the assumption (2.2.32) and applying Young's inequality and the Sobolev embeddings, we obtain, for all $t \in (0,T)$,

$$\begin{aligned}
&\left|\int_\Omega u_t f_1(y,z)dx\right| \le (p^+ + 1)\left[a\int_\Omega |u_t|\,|y+z|^{p(x)}\,dx + b\int_\Omega |u_t|\,.\,|y|^{\frac{p(x)-1}{2}}\,|z|^{\frac{p(x)+1}{2}}\,dx\right]\\
&\le (p^+ + 1)\left[\frac{\varepsilon(a+b)}{2}\int_\Omega |u_t|^2\,dx + \frac{2a}{\varepsilon}\int_\Omega |y+z|^{2p(x)}\,dx + \frac{2b}{\varepsilon}\int_\Omega |y|^{p(x)-1}\,|z|^{p(x)+1}\,dx\right]\\
&\le c_1\left[\frac{\varepsilon}{2}\|u_t\|_2^2 + C_\varepsilon\left(\int_\Omega |y+z|^{2p^+} + \int_\Omega |y+z|^{2p^-} + \int_\Omega |y|^{3(p(x)-1)} + \int_\Omega |z|^{\frac{3}{2}(p(x)+1)}\right)\right]\\
&\le c_2\left[\varepsilon\|u_t\|_2^2 + \|\nabla y\|_2^{2p^-} + \|\nabla z\|_2^{2p^-} + \|\nabla y\|_2^{2p^+} + \|\nabla z\|_2^{2p^+}\right]\\
&+ c_2\left[\|\nabla y\|_2^{3(p^- -1)} + \|\nabla y\|_2^{3(p^+ -1)} + \|\nabla z\|_2^{\frac{3}{2}(p^- +1)} + \|\nabla z\|_2^{\frac{3}{2}(p^+ +1)}\right],
\end{aligned} \tag{2.2.38}$$

where ε, c_1, c_2 are positive constants. Likewise, we get

$$\begin{aligned}
\left|\int_\Omega v_t f_2(y,z)dx\right| &\le (p^+ + 1)\left[a\int_\Omega |v_t|\,|y+z|^{p(x)}\,dx + b\int_\Omega |v_t|\,.\,|z|^{\frac{p(x)-1}{2}}\,|y|^{\frac{p(x)+1}{2}}\,dx\right]\\
&\le c_2\left[\varepsilon\|v_t\|_2^2 + \|\nabla y\|_2^{2p^-} + \|\nabla z\|_2^{2p^-} + \|\nabla y\|_2^{2p^+} + \|\nabla z\|_2^{2p^+}\right]\\
&+ c_2\left[\|\nabla z\|_2^{3(p^- -1)} + \|\nabla z\|_2^{3(p^+ -1)} + \|\nabla y\|_2^{\frac{3}{2}(p^- +1)} + \|\nabla y\|_2^{\frac{3}{2}(p^+ +1)}\right].
\end{aligned} \tag{2.2.39}$$

Combining (2.2.38) and (2.2.39), it comes that, for all $t \in (0,T)$,

$$\begin{aligned}
&\int_0^t\left(\left|\int_\Omega u_t f_1(y,z)dx\right| + \left|\int_\Omega v_t f_2(y,z)dx\right|\right)ds \le \varepsilon c_2\int_0^t\left(\|u_t\|_2^2 + \|v_t\|_2^2\right)ds\\
&+ c_2\int_0^t\left(2\|(y,z)\|_{H_0^1\times H_0^1}^{2p^-} + 2\|(y,z)\|_{H_0^1\times H_0^1}^{2p^+} + \|(y,z)\|_{H_0^1\times H_0^1}^{3(p^- -1)} + \|(y,z)\|_{H_0^1\times H_0^1}^{3(p^+ -1)}\right)ds\\
&+ c_2\int_0^t\left(\|(y,z)\|_{H_0^1\times H_0^1}^{\frac{3}{2}(p^- +1)} + \|(y,z)\|_{H_0^1\times H_0^1}^{\frac{3}{2}(p^+ +1)}\right)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sup_{(0,T)}\int_0^t\left(\left|\int_\Omega u_t f_1(y,z)dx\right| + \left|\int_\Omega v_t f_2(y,z)dx\right|\right)ds \le \varepsilon T c_2\|(u,v)\|_{W_T\times W_T}^2\\
&+ 2Tc_2\left(\|(y,z)\|_{W_T\times W_T}^{2p^-} + \|(y,z)\|_{W_T\times W_T}^{2p^+}\right)\\
&+ Tc_2\left(\|(y,z)\|_{W_T\times W_T}^{3(p^- -1)} + \|(y,z)\|_{W_T\times W_T}^{3(p^+ -1)} + \|(y,z)\|_{W_T\times W_T}^{\frac{3}{2}(p^- +1)} + \|(y,z)\|_{W_T\times W_T}^{\frac{3}{2}(p^+ +1)}\right).
\end{aligned} \tag{2.2.40}$$

By substituting (2.2.40) into (2.2.37), we obtain, for some $c_3 > 0$,

$$\begin{aligned}
&\frac{1}{2}\|(u,v)\|_{W_T\times W_T}^2 \le \gamma_0 + \varepsilon T c_3\|(u,v)\|_{W_T\times W_T}^2\\
&+ 2Tc_3\left(\|(y,z)\|_{W_T\times W_T}^{2p^-} + \|(y,z)\|_{W_T\times W_T}^{2p^+}\right)\\
&+ Tc_3\left(\|(y,z)\|_{W_T\times W_T}^{3(p^- -1)} + \|(y,z)\|_{W_T\times W_T}^{3(p^+ -1)} + \|(y,z)\|_{W_T\times W_T}^{\frac{3}{2}(p^- +1)} + \|(y,z)\|_{W_T\times W_T}^{\frac{3}{2}(p^+ +1)}\right).
\end{aligned} \tag{2.2.41}$$

Choosing ε such that $\varepsilon T c_3 = \frac{1}{4}$ and recalling that $\|(y,z)\|_{W_T \times W_T} \leq d$ for some $d > 1$ (large enough), inequality (2.2.41) implies

$$\begin{aligned}
\|(u,v)\|^2_{W_T\times W_T} &\leq 4\gamma_0 + 8Tc_3\left(\|(y,z)\|^{2p^-}_{W_T\times W_T} + \|(y,z)\|^{2p^+}_{W_T\times W_T}\right)\\
&+ 4Tc_3\left(\|(y,z)\|^{\frac{3}{2}(p^-+1)}_{W_T\times W_T} + \|(y,z)\|^{\frac{3}{2}(p^++1)}_{W_T\times W_T}\right)\\
&+ 4Tc_3\left(\|(y,z)\|^{3(p^--1)}_{W_T\times W_T} + \|(y,z)\|^{3(p^+-1)}_{W_T\times W_T}\right)\\
&\leq 4\gamma_0 + Tc_4 d^{3(p^+-1)},\ c_4 > 0.
\end{aligned}$$

Here $d^{3(p^+-1)} = \max\left\{d^{2p^+}, d^{\frac{3}{2}(p^-+1)}\right\}$, since $d > 1$ and $p^+ \geq 3$.
So, if we take d such that $d^2 >> 4\gamma_0$ and $T \leq T_0 = \frac{d^2-4\gamma_0}{c_4 d^{3(p^+-1)}}$, we arrive at

$$4\gamma_0 + Tc_4 d^{3(p^+-1)} \leq d^2.$$

It follows that

$$\|(u,v)\|^2_{W_T\times W_T} \leq d^2.$$

Therefore, $G : B(0,d) \to B(0,d)$.

$G : B(0,d) \longrightarrow B(0,d)$ **is a contraction**

In what follows, we prove that for T_0 (even smaller), G is a contraction mapping. Let (y_1,z_1) and (y_2,z_2) be in $B(0,d)$ and set $(u_1,v_1) = G(y_1,z_1)$ and $(u_2,v_2) = G(y_2,z_2)$. Then $(u,v) = (u_1-u_2, v_1-v_2)$ is a solution of the following problem, in the sense of Definition 2.2.1,

$$\begin{cases}
u_{tt} - div\,(A\nabla u) + (|u_{1t}|^{m(x)-2}\,u_{1t} - |u_{2t}|^{m(x)-2}\,u_{2t}) & \\
= f_1(y_1,z_1) - f_1(y_2,z_2) & \text{in } \Omega\times(0,T),\\
v_{tt} - div\,(B\nabla v) + (|v_{1t}|^{r(x)-2}\,v_{1t} - |v_{2t}|^{r(x)-2}\,v_{2t}) & \\
= f_2(y_1,z_1) - f_2(y_2,z_2) & \text{in } \Omega\times(0,T),\\
u = v = 0 & \text{on } \partial\Omega\times(0,T),\\
(u(0),v(0)) = (u_t(0),v_t(0)) = (0,0) & \text{in } \Omega.
\end{cases} \qquad (S)$$

Taking $\Phi = u_t$ in the first equation of Definition 2.2.1, we obtain, for all $t \in (0,T)$,

$$\begin{aligned}
&\frac{d}{dt}\left[\|u_t\|_2^2 + \int_\Omega A\nabla u.\nabla u\right] - \int_\Omega A'\nabla u.\nabla u + 2\int_\Omega \left(|u_{1t}|^{m(x)-2}\,u_{1t} - |u_{2t}|^{m(x)-2}\,u_{2t}\right)u_t\\
&= 2\int_\Omega u_t\left(f_1(y_1,z_1) - f_1(y_2,z_2)\right)dx, \qquad (2.2.42)
\end{aligned}$$

since $\frac{d}{dt}\left(\int_\Omega A\nabla u.\nabla u dx\right) = \int_\Omega A'\nabla u.\nabla u dx + 2\int_\Omega A\nabla u.\nabla u_t dx$.
Now, by integrating (2.2.42) over $(0,t)$ and using the initial conditions, we get

$$\begin{aligned}
\|u_t\|_2^2 + \int_\Omega A\nabla u.\nabla u dx - \int_0^t\int_\Omega A'\nabla u.\nabla u dx ds&\\
\leq 2\int_0^t\int_\Omega u_t\left(f_1(y_1,z_1) - f_1(y_2,z_2)\right)dx ds,&
\end{aligned}$$

by virtue of (2.2.7). Under the assumption (2.1.5) and (2.1.6), this gives

$$\|u_t\|_2^2 + a_0 \|\nabla u\|_2^2 \leq 2 \int_0^t \int_\Omega u_t \left(f_1(y_1, z_1) - f_1(y_2, z_2)\right) dxds,$$

for all $t \in (0, T)$. Consequently,

$$\begin{aligned} \|u\|_{W_T}^2 &\leq C \sup_{(0,T)} \int_0^t \int_\Omega u_t \left(f_1(y_1, z_1) - f_1(y_2, z_2)\right) dxds \\ &\leq C \sup_{(0,T)} \int_0^t \int_\Omega |u_t| \left|f_1(y_1, z_1) - f_1(y_2, z_2)\right| dxds, \end{aligned} \tag{2.2.43}$$

where $C = \frac{2}{min\{1,a_0\}}$. By repeating the same computations with $\Psi = v_t$, in the second equation of Definition 2.2.1, we arrive at

$$\begin{aligned} \|v\|_{W_T}^2 &\leq C \sup_{(0,T)} \int_0^t \int_\Omega v_t \left(f_2(y_1, z_1) - f_2(y_2, z_2)\right) dxds \\ &\leq \mathrm{C} \sup_{(0,T)} \int_0^t \int_\Omega |v_t| \left|f_2(y_1, z_1) - f_2(y_2, z_2)\right| dxds, \end{aligned} \tag{2.2.44}$$

for all $t \in (0, T)$, where $\mathrm{C} = \frac{2}{min\{1,b_0\}}$. By exploiting Young's inequality, estimates (2.2.43) and (2.2.44) lead to

$$\|u\|_{W_T}^2 \leq \varepsilon CT \|u\|_{W_T}^2 + C_\varepsilon \sup_{(0,T)} \int_0^t \int_\Omega \left|f_1(y_1, z_1) - f_1(y_2, z_2)\right|^2 dxds$$

and

$$\|v\|_{W_T}^2 \leq \varepsilon CT \|v\|_{W_T}^2 + C_\varepsilon \sup_{(0,T)} \int_0^t \int_\Omega \left|f_2(y_1, z_1) - f_2(y_2, z_2)\right|^2 dxds.$$

By the addition of the last two inequalities and choosing ε small enough, we infer that

$$\begin{aligned} &\|(u, v)\|_{W_T \times W_T}^2 \\ &\leq C_\varepsilon \sup_{(0,T)} \int_0^t \int_\Omega \left[\left|f_1(y_1, z_1) - f_1(y_2, z_2)\right|^2 + \left|f_2(y_1, z_1) - f_2(y_2, z_2)\right|^2\right] dxds. \end{aligned} \tag{2.2.45}$$

Now, for all $t \in (0, T)$, we set $Y = y_1 - y_2,\ Z = z_1 - z_2$ and we estimate

$$\int_\Omega \left|f_1(y_1, z_1) - f_1(y_2, z_2)\right|^2 dx$$

and

$$\int_\Omega \left|f_2(y_1, z_1) - f_2(y_2, z_2)\right|^2 dx.$$

For this purpose, we recall inequalites (2.2.30) and (2.2.31) to obtain the following estimates enjoyed by f_1 and f_2, respectively, as in [2].

$$\begin{aligned}
&|f_1(y_1,z_1)-f_1(y_2,z_2)| \\
&\le C_4\left(|y_1-y_2|+|z_1-z_2|\right)\left(|y_1|^{p(x)-1}+|z_1|^{p(x)-1}+|y_2|^{p(x)-1}+|z_2|^{p(x)-1}\right) \\
&+C_5\,|z_1-z_2|\,.\,|y_1|^{\frac{p(x)-1}{2}}\left(|z_1|^{\frac{p(x)-1}{2}}+|z_2|^{\frac{p(x)-1}{2}}\right) \\
&+C_5\,|y_1-y_2|\,.\,|z_2|^{\frac{p(x)+1}{2}}\left(|y_1|^{\frac{p(x)-3}{2}}+|y_2|^{\frac{p(x)-3}{2}}\right), \qquad (2.2.46)
\end{aligned}$$

and

$$\begin{aligned}
&|f_2(y_1,z_1)-f_2(y_2,z_2)| \\
&\le C_4\left(|y_1-y_2|+|z_1-z_2|\right)\left(|y_1|^{p(x)-1}+|z_1|^{p(x)-1}+|y_2|^{p(x)-1}+|z_2|^{p(x)-1}\right) \\
&+C_5\,|y_1-y_2|\,.\,|z_1|^{\frac{p(x)-1}{2}}\left(|y_1|^{\frac{p(x)-1}{2}}+|y_2|^{\frac{p(x)-1}{2}}\right) \\
&+C_5\,|z_1-z_2|\,.\,|y_2|^{\frac{p(x)+1}{2}}\left(|z_1|^{\frac{p(x)-3}{2}}+|z_2|^{\frac{p(x)-3}{2}}\right), \qquad (2.2.47)
\end{aligned}$$

for some constants $C_4, C_5 > 0$ and for almost all $x \in \Omega$ and all $t \in (0,T)$. So,

$$\int_\Omega |f_1(y_1,z_1)-f_1(y_2,z_2)|^2\,dx \le I_1+I_2+I_3+I_4, \qquad (2.2.48)$$

where

$$I_1 = C_4\int_\Omega |y_1-y_2|^2\left(|y_1|^{2(p(x)-1)}+|z_1|^{2(p(x)-1)}+|y_2|^{2(p(x)-1)}+|z_2|^{2(p(x)-1)}\right)dx,$$

$$I_2 = C_4\int_\Omega |z_1-z_2|^2\left(|y_1|^{2(p(x)-1)}+|z_1|^{2(p(x)-1)}+|y_2|^{2(p(x)-1)}+|z_2|^{2(p(x)-1)}\right)dx,$$

$$I_3 = C_5\int_\Omega |z_1-z_2|^2\,|y_1|^{p(x)-1}\left(|z_1|^{p(x)-1}+|z_2|^{p(x)-1}\right)dx$$

and

$$I_4 = C_5\int_\Omega |y_1-y_2|^2\,|z_2|^{p(x)+1}\left(|y_1|^{p(x)-3}+|y_2|^{p(x)-3}\right)dx.$$

By using Hölder's and Young's inequalities and the Sobolev embeddings, we get the following estimate for a typical term in I_1 and I_2,

$$\begin{aligned}
&\int_\Omega |y_1-y_2|^2\,|y_1|^{2(p(x)-1)}\,dx \le 2\left(\int_\Omega |y_1-y_2|^6 dx\right)^{\frac{1}{3}}\left(\int_\Omega |y_1|^{3(p(x)-1)}\right)^{\frac{2}{3}} \\
&\le C||y_1-y_2||_6^2\left[\left(\int_\Omega |y_1|^{3(p^+-1)}dx\right)^{\frac{2}{3}}+\left(\int_\Omega |y_1|^{3(p^--1)}dx\right)^{\frac{2}{3}}\right] \\
&\le C||\nabla(y_1-y_2)||_2^2\left(||y_1||_{3(p^+-1)}^{2(p^+-1)}+||y_1||_{3(p^--1)}^{2(p^--1)}\right) \\
&\le C||\nabla Y||_2^2\left(||\nabla y_1||_2^{2(p^+-1)}+||\nabla y_1||_2^{2(p^--1)}\right) \\
&\le C||\nabla Y||_2^2\left(||(y_1,z_1)||_{W_T\times W_T}^{2(p^+-1)}+||(y_1,z_1)||_{W_T\times W_T}^{2(p^--1)}\right), \qquad (2.2.49)
\end{aligned}$$

since

- $1 \leq 3(p^- - 1) \leq 3(p^+ - 1) < \infty$, when $n = 1, 2$.
- $1 \leq 3(p^- - 1) = 3(p^+ - 1) = 6 = \frac{2n}{n-2}$, when $n = 3$.

Likewise, we obtain

$$\int_\Omega |y_1 - y_2|^2 |z_2|^{2(p(x)-1)}\, dx \leq C||\nabla Y||_2^2 \left(||(y_2, z_2)||_{W_T \times W_T}^{2(p^+-1)} + ||(y_2, z_2)||_{W_T \times W_T}^{2(p^--1)} \right).$$

Therefore,

$$\begin{aligned} I_1 &\leq 2C||\nabla Y||_2^2 \left(||(y_1, z_1)||_{W_T \times W_T}^{2(p^+-1)} + ||(y_1, z_1)||_{W_T \times W_T}^{2(p^--1)} \right) \\ &\quad + 2C||\nabla Y||_2^2 \left(||(y_2, z_2)||_{W_T \times W_T}^{2(p^+-1)} + ||(y_2, z_2)||_{W_T \times W_T}^{2(p^--1)} \right) \end{aligned}$$

and

$$\begin{aligned} I_2 &\leq 2C||\nabla Z||_2^2 \left(||(y_1, z_1)||_{W_T \times W_T}^{2(p^+-1)} + ||(y_1, z_1)||_{W_T \times W_T}^{2(p^--1)} \right) \\ &\quad + 2C||\nabla Z||_2^2 \left(||(y_2, z_2)||_{W_T \times W_T}^{2(p^+-1)} + ||(y_2, z_2)||_{W_T \times W_T}^{2(p^--1)} \right). \end{aligned}$$

But $(y_1, z_1), (y_2, z_2) \in B(0, d)$, then for all $t \in (0, T)$, we infer

$$I_1 + I_2 \leq C d^{2(p^+-1)} \left(||\nabla Y||_2^2 + ||\nabla Z||_2^2 \right). \tag{2.2.50}$$

Similarly, a typical terms in I_3 can be handled as follows

$$\begin{aligned} &\int_\Omega |z_1 - z_2|^2 |y_1|^{p(x)-1} |z_1|^{p(x)-1}\, dx \leq 2 \left(\int_\Omega |z_1 - z_2|^6 dx \right)^{\frac{1}{3}} \left(\int_\Omega |y_1|^{\frac{3}{2}(p(x)-1)} |z_1|^{\frac{3}{2}(p(x)-1)} \right)^{\frac{2}{3}} \\ &\leq C||z_1 - z_2||_6^2 \left[\left(\int_\Omega |y_1|^{\frac{3}{2}(p(x)-1)} dx \right)^{\frac{2}{3}} + \left(\int_\Omega |z_1|^{\frac{3}{2}(p(x)-1)} dx \right)^{\frac{2}{3}} \right] \\ &\leq C||\nabla(z_1 - z_2)||_2^2 \left(||y_1||_{\frac{3}{2}(p^+-1)}^{(p^+-1)} + ||y_1||_{\frac{3}{2}(p^--1)}^{(p^--1)} + ||z_1||_{\frac{3}{2}(p^+-1)}^{(p^+-1)} + ||z_1||_{\frac{3}{2}(p^--1)}^{(p^--1)} \right) \\ &\leq C||\nabla(z_1 - z_2)||_2^2 \left(||\nabla y_1||_2^{(p^+-1)} + ||\nabla y_1||_2^{(p^--1)} + ||\nabla z_1||_2^{(p^+-1)} + ||\nabla z_1||_2^{(p^--1)} \right) \\ &\leq 2C||\nabla Z||_2^2 \left(||(y_1, z_1)||_{W_T \times W_T}^{(p^+-1)} + ||(y_1, z_1)||_{W_T \times W_T}^{(p^--1)} \right) \end{aligned}$$

and

$$\begin{aligned} \int_\Omega |z_1 - z_2|^2 |y_1|^{p(x)-1} |z_2|^{p(x)-1}\, dx &\leq 2C||\nabla Z||_2^2 \left(||(y_1, z_1)||_{W_T \times W_T}^{(p^+-1)} + ||(y_1, z_1)||_{W_T \times W_T}^{2(p^--1)} \right) \\ &\quad + 2C||\nabla Z||_2^2 \left(||(y_2, z_2)||_{W_T \times W_T}^{(p^+-1)} + ||(y_2, z_2)||_{W_T \times W_T}^{(p^--1)} \right), \end{aligned}$$

since

- $1 \leq \frac{3}{2}(p^- - 1) \leq \frac{3}{2}(p^+ - 1) < \infty$, when $n = 1, 2$.
- $1 \leq \frac{3}{2}(p^- - 1) = \frac{3}{2}(p^+ - 1) = 3 = \frac{2n}{n-2}$, when $n = 3$.

Consequently,

$$I_3 \leq Cd^{p^+-1}||\nabla Z||_2^2, \ \forall t \in (0,T), \tag{2.2.51}$$

since $(y_1,z_1),(y_2,z_2) \in B(0,d)$ and $d>1$.
By using the same arguments, we estimate the terms in I_4, as follows:
Case 1: If $n=1,2$, we have $3 \leq p^- \leq p^+ < \infty$. So,

$$\begin{aligned}
&\int_\Omega |y_1-y_2|^2 |z_2|^{p(x)+1} |y_1|^{p(x)-3}\, dx \\
&\leq 2\left(\int_\Omega |y_1-y_2|^3 dx\right)^{\frac{2}{3}} \left(\int_\Omega |z_2|^{3(p(x)+1)}|y_1|^{3(p(x)-3)}\right)^{\frac{1}{3}} \\
&\leq C||y_1-y_2||_3^2 \left[\left(\int_\Omega |z_2|^{6(p(x)+1)}dx\right)^{\frac{1}{3}} + \left(\int_\Omega |y_1|^{6(p(x)-3)}dx\right)^{\frac{1}{3}}\right] \\
&\leq C||\nabla Y||_2^2 \left(||\nabla z_2||_2^{2(p^++1)} + ||\nabla z_2||_2^{2(p^-+1)} + ||\nabla y_1||_2^{2(p^+-3)} + ||\nabla y_1||_2^{2(p^--3)}\right) \\
&\leq 2C||\nabla Y||_2^2 \left(||(y_2,z_2)||_{W_T\times W_T}^{2(p^++1)} + ||(y_2,z_2)||_{W_T\times W_T}^{2(p^-+1)}\right) \\
&+ 2C||\nabla Y||_2^2 \left(||(y_1,z_1)||_{W_T\times W_T}^{2(p^+-3)} + ||(y_1,z_1)||_{W_T\times W_T}^{2(p^--3)}\right).
\end{aligned}$$

Case 2: If $n=3$, then $p \equiv 3$ on $\overline{\Omega}$. Hence,

$$\begin{aligned}
\int_\Omega |y_1-y_2|^2 |z_2|^{p(x)+1} |y_1|^{p(x)-3}\, dx &= \int_\Omega |y_1-y_2|^2 |z_2|^4\, dx \\
&\leq C\left(\int_\Omega |y_1-y_2|^6\, dx\right)^{\frac{1}{3}} \left(\int_\Omega |z_2|^6\, dx\right)^{\frac{2}{3}} \\
&\leq C||y_1-y_2||_6^2.||z_2||_6^4 \\
&\leq C||\nabla Y||_2^2.||\nabla z_2)||_2^4 \\
&\leq C||\nabla Y||_2^2.||(y_2,z_2)||_{W_T\times W_T}^4.
\end{aligned}$$

So, for all $t \in (0,T)$, we deduce that

$$I_4 \leq C||\nabla Y||_2^2 d^{2(p^++1)}, \tag{2.2.52}$$

since $(y_1,z_1),(y_2,z_2) \in B(0,d)$ and $d>1$.

Finally, by substituting $(2.2.52), (2.2.51)$ and $(2.2.50)$ in $(2.2.48)$, we arrive at

$$\int_\Omega |f_1(y_1,z_1) - f_1(y_2,z_2)|^2\, dx \leq Cd^{2(p^++1)}\left(||\nabla Y||_2^2 + ||\nabla Z||_2^2\right), \tag{2.2.53}$$

for all $t \in (0,T)$. Similarly, we get

$$\int_\Omega |f_2(y_1,z_1) - f_2(y_2,z_2)|^2\, dx \leq Cd^{2(p^++1)}\left(||\nabla Y||_2^2 + ||\nabla Z||_2^2\right). \tag{2.2.54}$$

Now, we replace (2.2.53) and (2.2.54) into (2.2.45) to get

$$\begin{aligned}\|(u,v)\|^2_{W_T\times W_T} &\leq C_\varepsilon d^{2(p^+ +1)} \sup_{(0,T)} \int_0^t \left(\|\nabla Y(s)\|_2^2 + \|\nabla Z(s)\|_2^2\right) ds \\ &\leq C_\varepsilon d^{2(p^+ +1)} T \, \|(Y,Z)\|^2_{W_T\times W_T} \\ &\leq \gamma T_0 \, \|(Y,Z)\|^2_{W_T\times W_T} ,\end{aligned}$$

where $\gamma = C_\varepsilon d^{2(p^+ +1)}$.
Therefore, if we take T_0 small enough, we get for $0 < k < 1$

$$\|(u,v)\|^2_{W_T\times W_T} \leq k \, \|(Y,Z)\|^2_{W_T\times W_T} .$$

Thus,

$$\|G(y_1,z_1) - G(y_2,z_2)\|^2_{W_T\times W_T} \leq k \, \|(y_1,z_1) - (y_2,z_2)\|^2_{W_T\times W_T} .$$

This proves that $G : B(0,d) \longrightarrow B(0,d)$ is a contraction. Then, the Banach-fixed-point theorem guarantees the existence of a unique $(u,v) \in B(0,d)$, such that $G(u,v) = (u,v)$, which is a local weak solution of (P).
UNIQUENESS:
Suppose that (P) has two weak solutions (u_1,v_1) and (u_2,v_2) on $[0,T)$, in the sense of Definition 2.2.1. Then, $(u,v) = (u_1 - u_2, v_1 - v_2)$ satisfies, for all $t \in (0,T)$,

$$\begin{aligned}&\frac{d}{dt}\left[\|u_t\|_2^2 + \int_\Omega A\nabla u.\nabla u\right] - \int_\Omega A'\nabla u.\nabla u + 2\int_\Omega \left(|u_{1t}|^{m(x)-2} u_{1t} - |u_{2t}|^{m(x)-2} u_{2t}\right) u_t \\ &= 2\int_\Omega u_t \left(f_1(u_1,v_1) - f_1(u_2,v_2)\right) dx\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dt}\left[\|v_t\|_2^2 + \int_\Omega B\nabla v.\nabla v\right] - \int_\Omega B'\nabla v.\nabla v + 2\int_\Omega \left(|v_{1t}|^{r(x)-2} v_{1t} - |v_{2t}|^{r(x)-2} v_{2t}\right) v_t \\ &= 2\int_\Omega v_t \left(f_2(u_1,v_1) - f_2(u_2,v_2)\right) dx,\end{aligned}$$

by the density of $H_0^1(\Omega)$ in $L^2(\Omega)$ and in fact that

$$\frac{d}{dt}\left(\int_\Omega A\nabla u.\nabla u dx\right) = \int_\Omega A'\nabla u.\nabla u dx + 2\int_\Omega A\nabla u.\nabla u_t dx.$$

We integrate each result over $(0,t)$, with $t \leq T$. The addition of the two results yields (as in (2.2.43) and (2.2.44)),

$$\begin{aligned}\|(u_t,v_t)\|_2^2 + \|(\nabla u,\nabla v)\|_2^2 &\leq C\int_0^t \int_\Omega |u_t| \, |f_1(u_1,v_1) - f_1(u_2,v_2)| \, dxdt \\ &+ C\int_0^t \int_\Omega |v_t| \, |f_2(u_1,v_1) - f_2(u_2,v_2)| \, dxdt.\end{aligned}$$

Under the assumption (2.2.32) and applying similar arguments as in above, we arrive at

$$y(t) = \|(u_t, v_t)\|_2^2 + \|(\nabla u, \nabla v)\|_2^2 \leq C_\varepsilon \int_0^t \left(\|(u_t(s), v_t(s))\|_2^2 + \|(\nabla u(s), \nabla v(s))\|_2^2\right) ds,$$

for all $t \in (0, T)$. Gronwall's lemma leads to

$$\|(u_t, v_t)\|_2^2 + \|(\nabla u, \nabla v)\|_2^2 = 0, \text{ for all } t \in (0, T).$$

Thanks to the boundary conditions, we get $u = v = 0$ on $\Omega \times (0, T)$. This prove the uniqueness of the solution of (P).
- For the proof of the alternative statement, we use the idea in [14]. □

Remark 2.2.4. *Theorem* 2.2.3 *is a generalization of the local existence of Agre and Rammaha [2], which dealt with constant exponents only, to the situation of variable exponents.*

2.2.2 Decreasingness of the Energy

We define the energy functional associated to system (P) for all $t \in [0, T)$ by

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \frac{1}{2}\int_\Omega (A\nabla u.\nabla u + B\nabla v.\nabla v)\, dx - \int_\Omega F(x, u, v)\, dx, \tag{2.2.55}$$

Lemma 2.2.5. *The energy functional E is a decreasing function.*

Proof. By multiplying the first differential equation in (P) by u_t, the second one by v_t, integrating the two equations over Ω, adding the two results and using the boundary condition in (P), we get

$$\begin{aligned} E'(t) &= -\int_\Omega |u_t|^{m(x)}\, dx - \int_\Omega |v_t|^{r(x)}\, dx + \frac{1}{2}\int_\Omega \left(A'\nabla u.\nabla u + B'\nabla v.\nabla v\right) dx \\ &\leq 0, \end{aligned} \tag{2.2.56}$$

by virtue of (2.1.6). □

2.3 Blow up of Negative Initial Energy Solution

The purpose of this Section is to show that any solution (u, v) of problem (P) blows up in finite time, i.e, there exists $T^* > 0$, such that

$$lim_{t\to T^*}\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) = +\infty.$$

under the following conditions

$$E(0) < 0 \text{ and } max\,\{m^+ - 1, r^+ - 1\} < p^-,$$

in addition to the assumptions of Theorem 2.2.3. First, we state and prove some preliminary results.

2.3.1 Preliminary Results

As in Komornik [22], we introduce

$$\Omega_+ = \{x \in \Omega \ / \ |u(x,t)| \geq 1\} \text{ and } \Omega_- = \{x \in \Omega \ / \ |u(x,t)| < 1\},$$

and we define H by

$$H(t) = -E(t), \text{ for all } t \in [0, \ T). \tag{2.3.1}$$

From the definition of E and its decreasingness (2.2.56), it follows that

$$H(0) \leq H(t) \leq \int_\Omega F(x,u,v)\,dx, \text{ for all } t \in [0, \ T).$$

By (1.3.1) and since $E(0) < 0$, then there exists $C_2 > 0$ such that

$$0 < H(0) \leq H(t) \leq C_2(\rho(u) + \rho(v)), \text{ for all } t \in [0, \ T), \tag{2.3.2}$$

where

$$\rho(u) = \int_\Omega |u|^{p(x)+1}\,dx \text{ and } \rho(v) = \int_\Omega |v|^{p(x)+1}\,dx.$$

Lemma 2.3.1. *There exists $C_3 > 0$ such that*

$$\|u\|_{p^-+1}^{p^-+1} + \|v\|_{p^-+1}^{p^-+1} \leq C_3(\rho(u) + \rho(v)). \tag{2.3.3}$$

Proof. Since $p^- \leq p(.) \leq p^+$, one has

$$\begin{aligned}
\rho(u) &= \int_{\Omega_+} |u|^{p(x)+1}\,dx + \int_{\Omega_-} |u|^{p(x)+1}\,dx \\
&\geq \int_{\Omega_+} |u|^{p^-+1}\,dx + \int_{\Omega_-} |u|^{p^++1}\,dx \\
&\geq \int_{\Omega_+} |u|^{p^-+1}\,dx + c_1\left(\int_{\Omega_-} |u|^{p^-+1}\,dx\right)^{\frac{p^++1}{p^-+1}}, \ c_1 > 0,
\end{aligned}$$

which implies

$$\rho(u) \geq \int_{\Omega_+} |u|^{p^-+1}\,dx \text{ and } \left(\frac{\rho(u)}{c_1}\right)^{\frac{p^-+1}{p^++1}} \geq \int_{\Omega_-} |u|^{p^-+1}\,dx.$$

By addition, it results

$$\begin{aligned}
\|u\|_{p^-+1}^{p^-+1} &\leq \rho(u) + c_2 (\rho(u))^{\frac{p^-+1}{p^++1}}, \ c_2 > 0 \\
&\leq \rho(u) + \rho(v) + c_2 (\rho(u) + \rho(v))^{\frac{p^-+1}{p^++1}} \\
&= (\rho(u) + \rho(v)) \left[1 + c_2 (\rho(u) + \rho(v))^{\frac{p^- - p^+}{p^++1}} \right].
\end{aligned}$$

But, from (2.3.2), we have

$$\rho(u) + \rho(v) \geq H(0) / C_2.$$

Therefore,

$$\|u\|_{p^-+1}^{p^-+1} \leq (\rho(u) + \rho(v)) \left[1 + c_2 (H(0)/C_2)^{\frac{p^- - p^+}{p^++1}} \right].$$

So,

$$\|u\|_{p^-+1}^{p^-+1} \leq c_3 (\rho(u) + \rho(v)), c_3 > 0.$$

Similarly, we find

$$\|v\|_{p^-+1}^{p^-+1} \leq c_3 (\rho(u) + \rho(v)).$$

Thus, (2.3.3) is verified. □

Corollary 2.3.2. *There exist two constants $C_4, C_5 > 0$ such that*

$$\int_\Omega |u|^{m(x)} dx \leq C_4 \left[(\rho(u) + \rho(v))^{\frac{m^+}{p^-+1}} + (\rho(u) + \rho(v))^{\frac{m^-}{p^-+1}} \right], \tag{2.3.4}$$

and

$$\int_\Omega |v|^{r(x)} dx \leq C_5 \left[(\rho(u) + \rho(v))^{\frac{r^+}{p^-+1}} + (\rho(u) + \rho(v))^{\frac{r^-}{p^-+1}} \right]. \tag{2.3.5}$$

Proof. Since $p^- \geq \max\{m^+, r^+\}$, it follows that

$$\begin{aligned}
\int_\Omega |u|^{m(x)} dx &\leq \int_{\Omega_+} |u|^{m^+} dx + \int_{\Omega_-} |u|^{m^-} dx \\
&\leq c_1 \left(\int_{\Omega_+} |u|^{p^-+1} dx \right)^{\frac{m^+}{p^-+1}} + c_1 \left(\int_{\Omega_-} |u|^{p^-+1} dx \right)^{\frac{m^-}{p^-+1}} \\
&\leq c_1 \left(\|u\|_{p^-+1}^{m^+} + \|u\|_{p^-+1}^{m^-} \right), \ c_1 > 0.
\end{aligned}$$

By recalling Lemma 2.3.1, we arrive at

$$\int_\Omega |u|^{m(x)} dx \leq C_4 \left[(\rho(u) + \rho(v))^{\frac{m^+}{p^-+1}} + (\rho(u) + \rho(v))^{\frac{m^-}{p^-+1}} \right], \ C_4 > 0.$$

Likewise, we obtain

$$\int_\Omega |v|^{r(x)}\,dx \le C_5\left[(\rho(u)+\rho(v))^{\frac{r^+}{p^-+1}}+(\rho(u)+\rho(v))^{\frac{r^-}{p^-+1}}\right],\ C_5>0.$$

□

2.3.2 Blow up Result

In this subsection, we state and prove our main result.

Theorem 2.3.3. *Suppose that the above assumptions hold. Then, any solution of the system (P) blows up in finite time.*

Proof. For small $\varepsilon>0$ to be fixed later, we define the following auxiliary functional

$$G(t)=H^{1-\sigma}(t)+\varepsilon\int_\Omega (uu_t+vv_t)\,dx,\ \text{for all } t\in[0,T),$$

where

$$0<\sigma\le \min\left\{\frac{p^--m^++1}{(p^-+1)(m^+-1)},\frac{p^--r^++1}{(p^-+1)(r^+-1)},\frac{p^--1}{2(p^-+1)}\right\}. \tag{2.3.6}$$

Our purpose is to show that G satisfies a differential inequality which leads to a blow up in finite time. It will be carried out in the following four steps.
Step 1: By using the two differential equations in (P) and Green's formula, we obtain for all $t\in(0,T)$,

$$\begin{aligned}G'(t)&=(1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\&+\varepsilon\int_\Omega (uf_1(x,u,v)+vf_2(x,u,v))\,dx-\varepsilon\int_\Omega(A\nabla u.\nabla u+B\nabla v.\nabla v)\,dx\\&-\varepsilon\int_\Omega\left(|u_t|^{m(x)-2}u_tu+|v_t|^{r(x)-2}v_tv\right)dx.\end{aligned} \tag{2.3.7}$$

Invoking Lemma 1.3.5, we get

$$\begin{aligned}\int_\Omega (uf_1(x,u,v)+vf_2(x,u,v))\,dx&=\int_\Omega (p(x)+1)F(x,u,v)\,dx\\&\ge (p^-+1)\int_\Omega F(x,u,v)\,dx.\end{aligned} \tag{2.3.8}$$

The definitions of E and H lead to

$$\int_\Omega(A\nabla u.\nabla u+B\nabla v.\nabla v)\,dx=2\int_\Omega F(x,u,v)\,dx-(\|u_t\|_2^2+\|v_t\|_2^2)-2H(t). \tag{2.3.9}$$

By inserting (2.3.9) and (2.3.8) in (2.3.7), it results

$$\begin{aligned} G'(t) \geq & (1-\sigma) H^{-\sigma}(t) H'(t) + 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + 2\varepsilon H(t) \\ & + \varepsilon \left(p^- - 1\right) \int_\Omega F(x,u,v)\, dx \\ & - \varepsilon \int_\Omega \left(|u| |u_t|^{m(x)-1} + |v| |v_t|^{r(x)-1}\right) dx, \end{aligned} \tag{2.3.10}$$

which gives

$$\begin{aligned} G'(t) \geq & (1-\sigma) H^{-\sigma}(t) H'(t) + 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \varepsilon c_1 \int_\Omega F(x,u,v)\, dx \\ & + 2\varepsilon H(t) - \varepsilon \int_\Omega \left(|u| |u_t|^{m(x)-1} + |v| |v_t|^{r(x)-1}\right) dx, \end{aligned} \tag{2.3.11}$$

where $c_1 = p^- - 1 > 0$, since $p^- > 1$.
Step 2: In this step, we estimate the last two terms in the right hand-side of (2.3.11), which we note by

$$I_1 := \int_\Omega |u| |u_t|^{m(x)-1}\, dx \text{ and } I_2 := \int_\Omega |v| |v_t|^{r(x)-1}\, dx.$$

Applying the following Young inequality

$$XY \leq \frac{\delta^\lambda}{\lambda} X^\lambda + \frac{\delta^{-\beta}}{\beta} Y^\beta, \text{ for all } X,\ Y \geq 0,\ \ \delta > 0 \text{ and } \frac{1}{\lambda} + \frac{1}{\beta} = 1,$$

with

$$X = |u|,\ Y = |u_t|^{m(x)-1},\ \lambda = m(x),\ \beta = \frac{m(x)}{m(x)-1} \text{ and } \delta > 0,$$

we obtain

$$I_1 \leq \int_\Omega \frac{\delta^{m(x)}}{m(x)} |u|^{m(x)}\, dx \ + \int_\Omega \frac{m(x)-1}{m(x)} \delta^{-m(x)/(m(x)-1)} |u_t|^{m(x)}\, dx. \tag{2.3.12}$$

By taking

$$\delta = \left[K H^{-\sigma}(t)\right]^{\frac{1-m(x)}{m(x)}},$$

where K is a large constant to be chosen later, we arrive at

$$I_1 \leq \frac{K^{1-m^-}}{m^-} \int_\Omega [H(t)]^{\sigma(m(x)-1)} |u|^{m(x)}\, dx + \frac{m^+ - 1}{m^-} K H^{-\sigma}(t) \int_\Omega |u_t|^{m(x)}\, dx. \tag{2.3.13}$$

Using (2.2.56), we have

$$\begin{aligned} H'(t) &= \int_\Omega |u_t|^{m(x)}\, dx + \int_\Omega |v_t|^{r(x)}\, dx - \frac{1}{2} \int_\Omega \left(A' \nabla u . \nabla u + B' \nabla v . \nabla v\right) dx \\ &\geq \int_\Omega |u_t|^{m(x)}\, dx, \end{aligned} \tag{2.3.14}$$

by virtue of (2.1.6). On the other hand, we have $H(t) \geq H(0) > 0$.

Therefore,

$$\begin{aligned}\int_\Omega [H(t)]^{\sigma(m(x)-1)} |u|^{m(x)}\,dx &= \int_\Omega \left[\frac{H(t)}{H(0)}\right]^{\sigma(m(x)-1)} [H(0)]^{\sigma(m(x)-1)} |u|^{m(x)}\,dx \\ &\leq c_2 [H(t)]^{\sigma\left(m^+-1\right)} \int_\Omega [H(0)]^{\sigma(m(x)-1)} |u|^{m(x)}\,dx,\ c_2>0,\end{aligned}$$

since $m(x) \leq m^+$. But for all $x \in \Omega$, we have

$$[H(0)]^{\sigma(m(x)-1)} \leq c_3,\ c_3>0.$$

So,

$$\int_\Omega [H(t)]^{\sigma(m(x)-1)} |u|^{m(x)}\,dx \leq c_4 [H(t)]^{\sigma\left(m^+-1\right)} \int_\Omega |u|^{m(x)}\,dx,\ c_4>0. \tag{2.3.15}$$

Replace (2.3.15) and (2.3.14) in (2.3.13) to find

$$I_1 \leq c_4 \frac{K^{1-m^-}}{m^-} [H(t)]^{\sigma\left(m^+-1\right)} \int_\Omega |u|^{m(x)}\,dx + \frac{m^+-1}{m^-} K H^{-\sigma}(t) H'(t). \tag{2.3.16}$$

Likewise, we can prove that

$$I_2 \leq c_5 \frac{K^{1-r^-}}{r^-} [H(t)]^{\sigma\left(r^+-1\right)} \int_\Omega |v|^{r(x)}\,dx + \frac{r^+-1}{r^-} K H^{-\sigma}(t) H'(t),\ c_5>0. \tag{2.3.17}$$

Also, from (2.3.2), we obtain

$$[H(t)]^{\sigma\left(m^+-1\right)} \leq c_6 (\rho(u)+\rho(v))^{\sigma\left(m^+-1\right)},\ c_6>0.$$

Combining with (2.3.4), we get

$$\begin{aligned}[H(t)]^{\sigma\left(m^+-1\right)} \int_\Omega |u|^{m(x)}\,dx &\leq c_7 (\rho(u)+\rho(v))^{\sigma\left(m^+-1\right)+\frac{m^+}{p^-+1}} \\ &\quad + c_7 (\rho(u)+\rho(v))^{\sigma\left(m^+-1\right)+\frac{m^-}{p^-+1}},\ c_7>0.\end{aligned} \tag{2.3.18}$$

Now, under the condition (2.3.6) and using the following algebraic inequality

$$z^\tau \leq z+1 \leq \left(1+\frac{1}{a}\right)(z+a),\ \text{for all } z \geq 0,\ 0<\tau\leq 1 \text{ and } a>0, \tag{2.3.19}$$

with

$$z = \rho(u)+\rho(v),\ a = H(0),\ \tau = \sigma\left(m^+-1\right) + \frac{m^+}{p^-+1}$$

and then with $\tau = \sigma (m^+ - 1) + \frac{m^-}{p^-+1}$, respectively, we obtain

$$(\rho(u) + \rho(v))^{\sigma\left(m^+-1\right)+\frac{m^+}{p^-+1}} \leq \left[1 + \frac{1}{H(0)}\right] (\rho(u) + \rho(v) + H(0)) \\ \leq \gamma (\rho(u) + \rho(v) + H(t)) \quad (2.3.20)$$

and

$$(\rho(u) + \rho(v))^{\sigma\left(m^+-1\right)+\frac{m^-}{p^-+1}} \leq \gamma (\rho(u) + \rho(v) + H(t)), \quad (2.3.21)$$

where $\gamma = 1 + \frac{1}{H(0)}$. By replacing (2.3.21) and (2.3.20) into (2.3.18), it follows

$$[H(t)]^{\sigma\left(m^+-1\right)} \int_\Omega |u|^{m(x)}\, dx \leq c_8 (\rho(u) + \rho(v) + H(t)), \ c_8 > 0. \quad (2.3.22)$$

Similar computations lead to

$$[H(t)]^{\sigma\left(r^+-1\right)} \int_\Omega |v|^{r(x)}\, dx \leq c_9 (\rho(u) + \rho(v) + H(t)), \ c_9 > 0. \quad (2.3.23)$$

Incerting (2.3.22) into (2.3.16), we get

$$I_1 \leq c_{10} \frac{K^{1-m^-}}{m^-} (\rho(u) + \rho(v) + H(t)) + \frac{m^+ - 1}{m^-} K H^{-\sigma}(t) H'(t), \ c_{10} > 0. \quad (2.3.24)$$

and (2.3.23) into (2.3.17) to obtain

$$I_2 \leq c_{11} \frac{K^{1-r^-}}{r^-} (\rho(u) + \rho(v) + H(t)) + \frac{r^+ - 1}{r^-} K H^{-\sigma}(t) H'(t), \ c_{11} > 0. \quad (2.3.25)$$

Step 3: Now, we estimate G'.
By substituting (2.3.25) and (2.3.24) into (2.3.11), it yields

$$G'(t) \geq (1 - \sigma - \varepsilon M) H^{-\sigma}(t) H'(t) + 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + 2\varepsilon H(t) \\ + \varepsilon c_{12} (\rho(u) + \rho(v)) - \varepsilon c_{10} \frac{K^{1-m^-}}{m^-} [\rho(u) + \rho(v) + H(t)] \\ - \varepsilon c_{11} \frac{K^{1-r^-}}{r^-} [\rho(u) + \rho(v) + H(t)], \ c_{12} > 0. \quad (2.3.26)$$

where $M = K\left(\frac{m^+-1}{m^-} + \frac{r^+-1}{r^-}\right)$. Thus,

$$G'(t) \geq (1 - \sigma - \varepsilon M) H^{-\sigma}(t) H'(t) + 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ + \varepsilon \left(2 - \frac{K^{1-m^-}}{m^-} c_{10} - \frac{K^{1-r^-}}{r^-} c_{11}\right) H(t) \\ + \varepsilon \left(c_{12} - \frac{K^{1-m^-}}{m^-} c_{10} - \frac{K^{1-r^-}}{r^-} c_{11}\right) (\rho(u) + \rho(v)). \quad (2.3.27)$$

For large value of K, we can find $c_{13} > 0$ such that

$$G'(t) \geq (1 - \sigma - \varepsilon M) H^{-\sigma}(t) H'(t) + \varepsilon c_{13} \left(\|u_t\|_2^2 + \|v_t\|_2^2 + H(t) + \rho(u) + \rho(v) \right). \tag{2.3.28}$$

Once K is fixed (hence M), we pick ε small enough so that

$$1 - \sigma - \varepsilon M \geq 0 \text{ and } G(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega (u_0 u_1 + v_0 v_1)\, dx > 0.$$

On the other hand, from Lemma 2.2.5, we have $H'(t) \geq 0$. Hence, there exists $\Upsilon > 0$ such that

$$G'(t) \geq \varepsilon \Upsilon \left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \rho(u) + \rho(v) \right). \tag{2.3.29}$$

Therefore,

$$G(t) \geq G(0) > 0, \text{ for all } t \in [0, T).$$

Step 4: The completion of the proof.
By the definition of G, we have

$$\begin{aligned} G^{1/(1-\sigma)}(t) &\leq \left(H^{1-\sigma}(t) + \varepsilon \int_\Omega |u u_t + v v_t|\, dx \right)^{1/(1-\sigma)} \\ &\leq 2^{\sigma/(1-\sigma)} \left(H(t) + \left(\varepsilon \int_\Omega (|u u_t| + |v v_t|)\, dx \right)^{1/(1-\sigma)} \right) \\ &\leq c_{14} \left(H(t) + \left(\int_\Omega (|u|\,|u_t| + |v|\,|v_t|)\, dx \right)^{1/(1-\sigma)} \right), \ c_{14} > 0, \end{aligned} \tag{2.3.30}$$

since,

$$(X + Y)^\delta \leq 2^{\delta - 1} \left(X^\delta + Y^\delta \right), \text{ for all } X, Y \geq 0 \text{ and } \delta > 1. \tag{2.3.31}$$

Also, we have

$$\begin{aligned} \left(\int_\Omega (|u|\,|u_t| + |v|\,|v_t|)\, dx \right)^{1/(1-\sigma)} &\leq 2^{\sigma/(1-\sigma)} \left(\int_\Omega |u|\,|u_t|\, dx \right)^{1/(1-\sigma)} \\ &\quad + 2^{\sigma/(1-\sigma)} \left(\int_\Omega |v|\,|v_t|\, dx \right)^{1/(1-\sigma)}. \end{aligned} \tag{2.3.32}$$

Since $p^- > 1$, Hölder's and Young's inequalities give

$$\begin{aligned} \left(\int_\Omega |u|\,|u_t|\, dx \right)^{1/(1-\sigma)} &\leq \|u\|_2^{1/(1-\sigma)} \|u_t\|_2^{1/(1-\sigma)} \\ &\leq c_{15} \|u\|_{p^-+1}^{1/(1-\sigma)} \|u_t\|_2^{1/(1-\sigma)}, \ c_{15} > 0 \\ &\leq c_{16} \left(\|u\|_{p^-+1}^{\mu/(1-\sigma)} + \|u_t\|_2^{\beta/(1-\sigma)} \right), \ c_{16} > 0, \end{aligned} \tag{2.3.33}$$

where $\frac{1}{\mu} + \frac{1}{\beta} = 1$. If we take $\beta = 2(1 - \sigma)$, we get $\mu / (1 - \sigma) = 2 / (1 - 2\sigma)$.
Then,

$$\left(\int_\Omega |u|\,|u_t|\, dx \right)^{1/(1-\sigma)} \leq c_{16} \left(\|u\|_{p^-+1}^{2/(1-2\sigma)} + \|u_t\|_2^2 \right). \tag{2.3.34}$$

From Lemma 2.3.1, estimate (2.3.34) leads to

$$\left(\int_{\Omega} |u| \, |u_t| \, dx\right)^{1/(1-\sigma)} \leq c_{17} \left((\rho(u) + \rho(v))^{\tau} + \|u_t\|_2^2\right),$$

where $c_{17} > 0$ and $\tau = 2/(p^- + 1)(1 - 2\sigma)$. Again, by using (2.3.6) and (2.3.19), we obtain

$$\left(\int_{\Omega} |u| \, |u_t| \, dx\right)^{1/(1-\sigma)} \leq c_{18} \left(\rho(u) + \rho(v) + H(t) + \|u_t\|_2^2\right), \ c_{18} > 0, \quad (2.3.35)$$

since $\tau \leq 1$.
Similarly,

$$\left(\int_{\Omega} |v| \, |v_t| \, dx\right)^{1/(1-\sigma)} \leq c_{18} \left(\rho(v) + \rho(v) + H(t) + \|v_t\|_2^2\right). \quad (2.3.36)$$

By substituting (2.3.36) and (2.3.35) into (2.3.32), it results

$$\left(\int_{\Omega} (|u| \, |u_t| + |v| \, |v_t|) \, dx\right)^{1/(1-\sigma)} \leq c_{19} \left(\rho(u) + \rho(v) + \|u_t\|_2^2 + \|v_t\|_2^2 + H(t)\right), c_{19} > 0.$$

Hence, inequality (2.3.30) becomes

$$G^{1/(1-\sigma)}(t) \leq c_{20} \left(\rho(u) + \rho(v) + H(t) + \|u_t\|_2^2 + \|v_t\|_2^2\right), \ c_{20} > 0. \quad (2.3.37)$$

By combining (2.3.37) and (2.3.29), we obtain

$$G'(t) \geq C G^{1/(1-\sigma)}(t), \text{ for all } t \in [0, T), C > 0.$$

Therefore, by virtue of Lemma 1.3.4, the solution (u, v) of system (P) blows up in a finite time. □

2.4 Blow up of Solution with Positive Initial Data

In this section, we discuss the blow up of certain solutions of problem (P), with positive initial energy, paying more attention to the difficulties caused by the variable exponents $m(.), r(.)$ and $p(.)$. This section consists of three subsections. In the first one, we give some preliminary results. Subsection 2 is devoted to the statement and the proof of the blow-up theorem. In the last subsection, we present two numerical examples to illustrate our theoretical findings.

ASSUMPTIONS:

As in Section 2.3, we suppose that the conditions of Theorem 2.2.3 are fulfilled and that the source term, in each differential equation of system (P), dominates the damping; namely, we require that

$$\max\left\{m^{+}-1,r^{+}-1\right\}<p^{-}. \tag{2.4.1}$$

To obtain the finite time blow up result for system (P), we set

$$\alpha_1=\left(k\left(p^{-}+1\right)\right)^{\frac{1}{1-p^{-}}},\ E_1=\left(\frac{1}{2}-\frac{1}{p^{-}+1}\right)\alpha_1^2, \tag{2.4.2}$$

where

$$k=\left(a2^{\frac{p^{-}+1}{2}}+2b\right)\left(\frac{B_*^2}{c_0}\right)^{\frac{p^{-}+1}{2}},\ c_0=\min\{a_0,b_0\}>0$$

and B_* is the best constant of the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^{p(.)+1}(\Omega)$, and we derive the following results.

2.4.1 Preliminary Results

Lemma 2.4.1. *Assume that*

$$0\leq E(0)<E_1 \tag{2.4.3}$$

and

$$\alpha_1<\left(\int_\Omega\left(A\nabla u_0.\nabla u_0+B\nabla v_0.\nabla v_0\right)dx\right)^{1/2}\leq\left(\frac{c_0}{2B_*^2}\right)^{1/2}.$$

Then, there exists $\alpha_2>\alpha_1$ such that

$$\alpha_2\leq\left(\int_\Omega\left(A\nabla u.\nabla u+B\nabla v.\nabla v\right)dx\right)^{1/2},\ \text{for all } t\in[0,T). \tag{2.4.4}$$

Proof. From the definition of the energy, it results that

$$E(t)\geq\frac{1}{2}\int_\Omega\left(A\nabla u.\nabla u+B\nabla v.\nabla v\right)dx-\int_\Omega F(x,u,v)\,dx.$$

If we set

$$\alpha=\left(\int_\Omega\left(A\nabla u.\nabla u+B\nabla v.\nabla v\right)dx\right)^{1/2} \tag{2.4.5}$$

then

$$E(t)\geq\frac{1}{2}\alpha^2-\int_\Omega F(x,u,v)\,dx. \tag{2.4.6}$$

From (2.1.5), we have

$$\|\nabla u\|_2^2+\|\nabla v\|_2^2\leq\frac{\int_\Omega\left(A\nabla u.\nabla u+B\nabla v.\nabla v\right)dx}{c_0}.$$

So

$$\|\nabla u\|_2^2+\|\nabla v\|_2^2\leq\frac{\alpha^2}{c_0}. \tag{2.4.7}$$

On the other hand, from the definition of F, we have

$$\int_\Omega F(x,u,v) = a\int_\Omega |u+v|^{p(x)+1}\,dx + 2b\int_\Omega |uv|^{\frac{p(x)+1}{2}}\,dx.$$

Invoking Lemma 1.2.9, we obtain

$$\begin{aligned}\int_\Omega F(x,u,v) \le a\max\left\{\|u+v\|_{p(.)+1}^{p^-+1}, \|u+v\|_{p(.)+1}^{p^++1}\right\} \\ + 2b\max\left\{\|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^-+1}{2}}, \|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^++1}{2}}\right\}.\end{aligned} \tag{2.4.8}$$

The embedding Theorem 1.2.20 and (2.3.31) lead to

$$\begin{aligned}\|u+v\|_{p(.)+1} &\le B_*\|\nabla(u+v)\|_2 \\ &\le B_*\left[(\|\nabla u\|_2 + \|\nabla v\|_2)^2\right]^{1/2} \\ &\le B_*\left[2\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\right]^{1/2}.\end{aligned}$$

Using (2.4.7), this gives

$$\|u+v\|_{p(.)+1} \le \left(\frac{2B_*^2\alpha^2}{c_0}\right)^{1/2}.$$

Consequently,

$$\|u+v\|_{p(.)+1}^{p^-+1} \le \left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}} \text{ and } \|u+v\|_{p(.)+1}^{p^++1} \le \left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}}.$$

Thus

$$\max\left\{\|u+v\|_{p(.)+1}^{p^-+1}, \|u+v\|_{p(.)+1}^{p^++1}\right\} \le \max\left\{\left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}}, \left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}}\right\}. \tag{2.4.9}$$

Similarly, Hölder's (1.2.3) and Young's inequalities and the embedding theorem give

$$\begin{aligned}\|uv\|_{\frac{p(.)+1}{2}} &\le 2\|u\|_{p(.)+1}\|v\|_{p(.)+1} \\ &\le \|u\|_{p(.)+1}^2 + \|v\|_{p(.)+1}^2 \\ &\le B_*^2\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right).\end{aligned}$$

Again, by (2.4.7) we arrive at

$$\|uv\|_{\frac{p(.)+1}{2}} \le \frac{B_*^2\alpha^2}{c_0}.$$

Then,

$$\|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^-+1}{2}} \leq \left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}} \text{ and } \|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^++1}{2}} \leq \left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}}.$$

So,

$$\max\left\{\|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^-+1}{2}}, \|uv\|_{\frac{p(.)+1}{2}}^{\frac{p^++1}{2}}\right\} \leq \max\left\{\left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}}, \left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}}\right\}. \quad (2.4.10)$$

Replacing (2.4.9) and (2.4.10) in (2.4.8), it comes

$$\begin{aligned}\int_\Omega F(x,u,v) \leq a \max&\left\{\left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}}, \left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}}\right\} \\ &+ 2b \max\left\{\left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}}, \left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}}\right\}. \quad (2.4.11)\end{aligned}$$

Now, by inserting (2.4.11) into (2.4.6), we obtain

$$E(t) \geq h(\alpha), \text{ for all } \alpha \geq 0, \quad (2.4.12)$$

where

$$\begin{aligned}h(\alpha) := \frac{1}{2}\alpha^2 - a \max&\left\{\left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}}, \left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}}\right\} \\ - 2b \max&\left\{\left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}}, \left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}}\right\}.\end{aligned}$$

For α in $\left[0, \left(\frac{c_0}{2B_*^2}\right)^{1/2}\right]$, one can easily check that

$$\frac{B_*^2\alpha^2}{c_0} \leq \frac{2B_*^2\alpha^2}{c_0} \leq 1.$$

Consequently, we have

$$\left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}} \geq \left(\frac{2B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}} \text{ and } \left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^-+1}{2}} \geq \left(\frac{B_*^2\alpha^2}{c_0}\right)^{\frac{p^++1}{2}}.$$

Thus, inequality (2.4.12) leads to

$$E(t) \geq \frac{1}{2}\alpha^2 - \left(a2^{\frac{p^-+1}{2}} + 2b\right)\left(\frac{B_*^2}{c_0}\right)^{\frac{p^-+1}{2}} \alpha^{p^-+1}.$$

That is

$$E(t) \geq g(\alpha), \text{ for all } \alpha \in \left[0, \left(\frac{c_0}{2B_*^2}\right)^{1/2}\right], \tag{2.4.13}$$

where

$$g(\alpha) = \frac{1}{2}\alpha^2 - k\alpha^{p^-+1}.$$

It is easy to verify that g is strictly increasing on $[0, \alpha_1)$ and strictly decreasing on $[\alpha_1, +\infty)$. Therefore, since

$$E(0) < E_1 \text{ and } E_1 = g(\alpha_1),$$

we can find $\alpha_2 > \alpha_1$ such that $g(\alpha_2) = E(0)$. But,

$$\alpha_0 = \left(\int_\Omega (A\nabla u_0.\nabla u_0 + B\nabla v_0.\nabla v_0)\, dx\right)^{1/2} \in \left[\alpha_1, \left(\frac{c_0}{2B_*^2}\right)^{1/2}\right],$$

then by (2.4.13), we get

$$g(\alpha_2) = E(0) \geq g(\alpha_0).$$

This implies that $\alpha_0 \geq \alpha_2$. Consequently $\alpha_2 \in (\alpha_1, \left(\frac{c_0}{2B_*^2}\right)^{1/2}]$.
To establish (2.4.4), we suppose on the contrary that

$$\left(\int_\Omega (A\nabla u(.,t^*).\nabla u(.,t^*) + B\nabla v(.,t^*).\nabla v(.,t^*))\, dx\right)^{1/2} < \alpha_2,$$

for some $t^* \in [0, T)$. By the continuity of $\left(\int_\Omega A\nabla u.\nabla u + B\nabla v.\nabla v dx\right)^{1/2}$ and since $\alpha_2 > \alpha_1$, we can choose t^* such that

$$\left[\int_\Omega (A\nabla u(.,t^*).\nabla u(.,t^*) + B\nabla v(.,t^*).\nabla v(.,t^*))\, dx\right]^{1/2} > \alpha_1.$$

The decreasingness of g on $\left[\alpha_1, \left(\frac{c_0}{2B_*^2}\right)^{1/2}\right]$ and the inequality (2.4.13) imply that

$$\begin{aligned} E(t^*) &\geq g\left(\left[\int_\Omega (A\nabla u(.,t^*).\nabla u(.,t^*) + B\nabla v(.,t^*).\nabla v(.,t^*))\, dx\right]^{1/2}\right) \\ &> g(\alpha_2) = E(0). \end{aligned}$$

This is impossible since $E(t) \leq E(0)$ for all $t \in [0.T)$. Thus, inequality (2.4.4) is established. □

Now, we set

$$H(t) = E_1 - E(t), \text{ for all } t \in [0,\ T) \tag{2.4.14}$$

and present the following lemma.

Lemma 2.4.2. *We have*

$$0 < H(0) \leq H(t) \leq \int_\Omega F(x,u,v)\,dx, \text{ for all } t \in [0,\ T). \tag{2.4.15}$$

and

$$\int_\Omega F(x,u,v)\,dx \geq k\alpha_2^{p^-+1}. \tag{2.4.16}$$

Proof. From Lemma 2.2.5 and inequality (2.4.3), we have

$$0 < E_1 - E(0) = H(0) \leq H(t) \tag{2.4.17}$$

and by (2.4.6), we infer

$$H(t) \leq E_1 - \frac{1}{2}\alpha^2 + \int_\Omega F(x,u,v)\,dx.$$

Since $E_1 = g(\alpha_1)$ and $\alpha \geq \alpha_2 > \alpha_1$, then

$$\begin{aligned} H(t) &\leq \left(g(\alpha_1) - \frac{1}{2}\alpha_1^2\right) + \int_\Omega F(x,u,v)\,dx \\ &\leq -k\alpha_1^{p^-+1} + \int_\Omega F(x,u,v)\,dx \leq \int_\Omega F(x,u,v)\,dx. \end{aligned}$$

This prove the first inequality.
To prove the second estimate, we use (2.4.6) and the decreasingness of g to get

$$E(0) \geq\ E(t) \geq \frac{1}{2}\alpha^2 - \int_\Omega F(x,u,v)\,dx.$$

Consequently,

$$\int_\Omega F(x,u,v)\,dx \geq \frac{1}{2}\alpha^2 - E(0).$$

But $E(0) = g(\alpha_2)$ and $\alpha \geq \alpha_2$, so

$$\int_\Omega F(x,u,v)\,dx > \frac{1}{2}\alpha_2^2 - g(\alpha_2) = k\alpha_2^{p^-+1}.$$

□

Remark 2.4.3. *We note that Lemma* 2.3.1 *and Corollary* 2.3.2 *remain valid in this case.*

Now, we state and prove our main result.

2.4.2 Blow up Result

Theorem 2.4.4. *Assume that the assumptions of Lemma* 2.4.1 *hold. Then, any solution of system* (P) *blows up in finite time.*

Proof. We recall that the auxiliary functional is given by

$$G(t) = H^{1-\sigma}(t) + \varepsilon \int_{\Omega} (uu_t + vv_t)\, dx, \text{ for all } t \in [0,T),$$

where $\varepsilon > 0$ is a constant to be fixed later and

$$0 < \sigma \le \min\left\{ \frac{p^- - m^+ + 1}{(p^- + 1)(m^+ - 1)}, \frac{p^- - r^+ + 1}{(p^- + 1)(r^+ - 1)}, \frac{p^- - 1}{2(p^- + 1)} \right\}. \tag{2.4.18}$$

Our goal is to show that G satisfies the conditions in Lemma 1.3.4.
For all $t \in [0,T)$, we have

$$\begin{aligned} G'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) \\ &+ \varepsilon \int_{\Omega} (uf_1(x,u,v) + vf_2(x,u,v))\, dx - \varepsilon \int_{\Omega} (A\nabla u.\nabla u + B\nabla v.\nabla v)\, dx \\ &- \varepsilon \int_{\Omega} \left(|u_t|^{m(x)-2} u_t u + |v_t|^{r(x)-2} v_t v \right) dx. \end{aligned} \tag{2.4.19}$$

By using inequality (1.3.2), it comes

$$\begin{aligned} \int_{\Omega} (uf_1(x,u,v) + vf_2(x,u,v))\, dx &= \int_{\Omega} (p(x)+1) F(x,u,v)\, dx \\ &\ge \left(p^- + 1\right) \int_{\Omega} F(x,u,v)\, dx. \end{aligned} \tag{2.4.20}$$

By the definitions of H and E, we obtain

$$\int_{\Omega} (A\nabla u.\nabla u + B\nabla v.\nabla v)\, dx = 2\int_{\Omega} F(x,u,v)\, dx - \|u_t\|_2^2 - \|v_t\|_2^2 + 2E_1 - 2H(t). \tag{2.4.21}$$

If we insert (2.4.21) and (2.4.20) in (2.4.19), it results

$$\begin{aligned} G'(t) &\ge (1-\sigma) H^{-\sigma}(t) H'(t) + 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) + 2\varepsilon H(t) \\ &- 2\varepsilon E_1 + \varepsilon \left(p^- - 1\right) \int_{\Omega} F(x,u,v)\, dx \\ &- \varepsilon \int_{\Omega} \left(|u|\, |u_t|^{m(x)-1} + |v|\, |v_t|^{r(x)-1} \right) dx. \end{aligned} \tag{2.4.22}$$

Using (2.4.16), we get

$$E_1 \le \left(k\alpha_2^{p^-+1} \right)^{-1} E_1 \int_{\Omega} F(x,u,v)\, dx.$$

Hence, (2.4.22) becomes

$$\begin{aligned} G'(t) &\ge (1-\sigma) H^{-\sigma}(t) H'(t) + 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) + \varepsilon c_1 \int_{\Omega} F(x,u,v)\, dx \\ &+ 2\varepsilon H(t) - \varepsilon \int_{\Omega} \left(|u|\, |u_t|^{m(x)-1} + |v|\, |v_t|^{r(x)-1} \right) dx, \end{aligned}$$

where $c_1 = p^- - 1 - 2\left(k\alpha_2^{p^-+1}\right)^{-1} E_1 > 0$, since $\alpha_2 > \alpha_1$.

By repeating the Steps (2,3 and 4), of the proof of Theorem 2.3.3, we arrive at

$$G'(t) \geq CG^{1/(1-\sigma)}(t), \text{ for all } t \in [0,T),$$

where $C > 0$. By invoking Lemma 1.3.4, we deduce that the solution (u, v) of system (P) blows up in a finite time $T^* > 0$. □

2.4.3 Numerical Tests

In this subsection, some numerical experiments have been performed to illustrate the theoretical results in Theorem 2.4.4, where we have used the Free Fem++ software and Matlab. We solve the system (P) under specific initial data and Dirichlet boundary conditions. We exploit a numerical scheme based on the finite element method in space and the Newmark method in time [54, 55].
We consider problem (P) in two space-dimensions and take the functions m, r and p fulfilling the assumptions (2.1.1), (2.1.2) and (2.2.32).
Precisely, we have

$$m(x,y) = 2 + \frac{1}{1+x^2},$$

$$r(x,y) = 2 + \frac{1}{1+y^2}$$

and

$$p(x,y) = 3 + \frac{2}{1+x^2+y^2}$$

and the source terms are given by (2.1.3) and (2.1.4) with $a = b = 1$. Whereas, the matrices A and B are given as follows

$$A = \left(1+e^{-t}\right)\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$B = \left(1+\frac{1}{1+t}\right)\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}.$$

Test 1: For the first test, we consider a circular domain

$$\Omega_1 = \{(x, y)/x^2 + y^2 < 1\}$$

with a triangulation discretization (see the mesh-grid in Figure 2.1) which consists of 281 triangles and 162 degrees of freedoms [44] and use the following initial conditions:

$$u_0(x, y) = 2(1 - x^2 - y^2),$$

$$v_0(x, y) = 3(1 - x^2 - y^2)$$

and

$$u_1 = v_1 = 0.$$

We run our code with a time step $\Delta t = 10^{-3}$, which is small enough to catch the blow-up behavior.

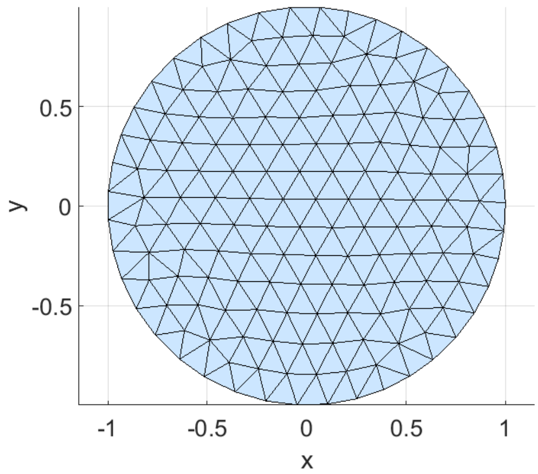
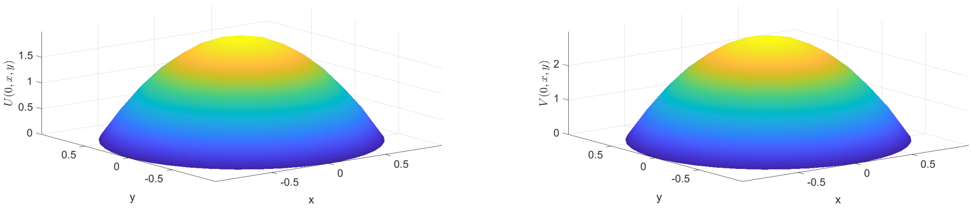
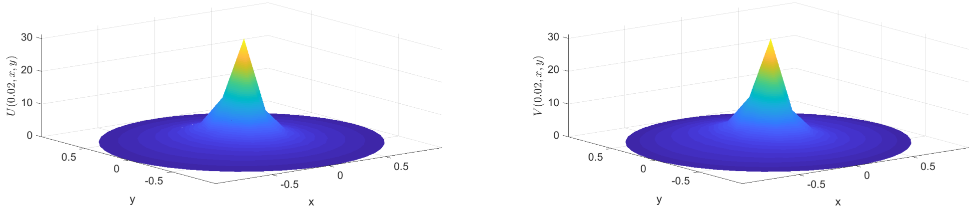


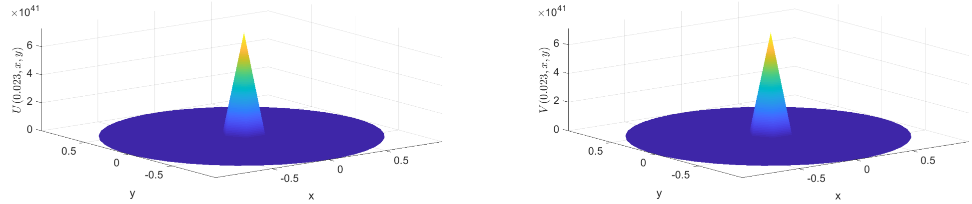
FIGURE 2.1: Uniform mesh grid of Ω_1.



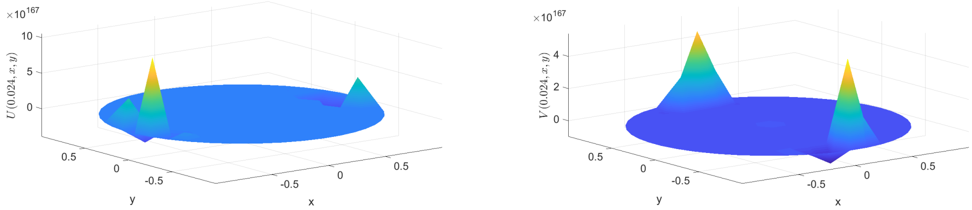
(A) $t = 0$



(B) $t = 0.02$



(C) $t = 0.023$



(D) $t = 0.024$

FIGURE 2.2: The numerical results of Test 1 at different times.

Figure 2.2 shows the approximate numerical results of the solution (u, v) at different time iterations $t = 0$, $t = 0.02$, $t = 0.023$ and $t = 0.024$, where the left column shows the approximate values of u and the right column shows the approximate values of v. Notice that the blow-up is occurring at instant $t = 0.024$.

Figure 2.3 presents the numerical values of the functional H defined by (2.4.14) during the time iterations. It shows the blow-up of the energy of the system (P).

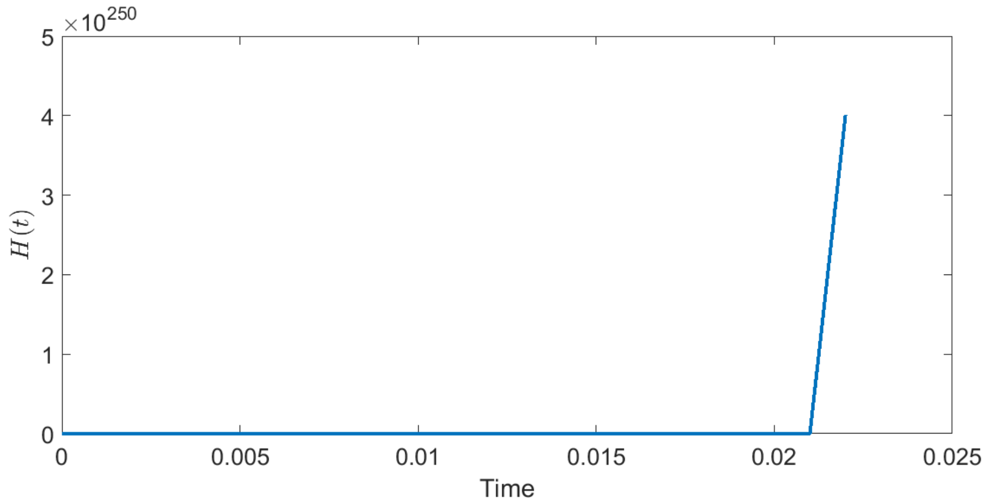


FIGURE 2.3: Test 1: The blow-up of H in finite time.

Test 2: For the second test, we consider an elliptical domain

$$\Omega_2 = \left\{ (x,y) / \frac{x^2}{4} + y^2 < 1 \right\}$$

with a triangulation discretization (see the mesh-grid in Figure 2.4) which consists of 311 triangles and 180 degrees of freedoms [44] and take the following initial conditions:

$$u_0(x,y) = 2(1 - \frac{x^2}{4} - y^2),\ \ v_0(x,y) = 3(1 - \frac{x^2}{4} - y^2) \text{ and } u_1 = v_1 = 0.$$

We run our code with a time step $\Delta t = 5 \cdot 10^{-4}$, which is small enough to catch the blow-up behavior.

In Figure 2.5, we show the approximate numerical results of the solution (u,v) at different time iterations $t = 0$, $t = 0.02$, $t = 0.0205$ and $t = 0.021$, where the left column shows the approximate values of u and the right column shows the approximate values of v. Notice that the blow-up takes place at instant $t = 0.021$.

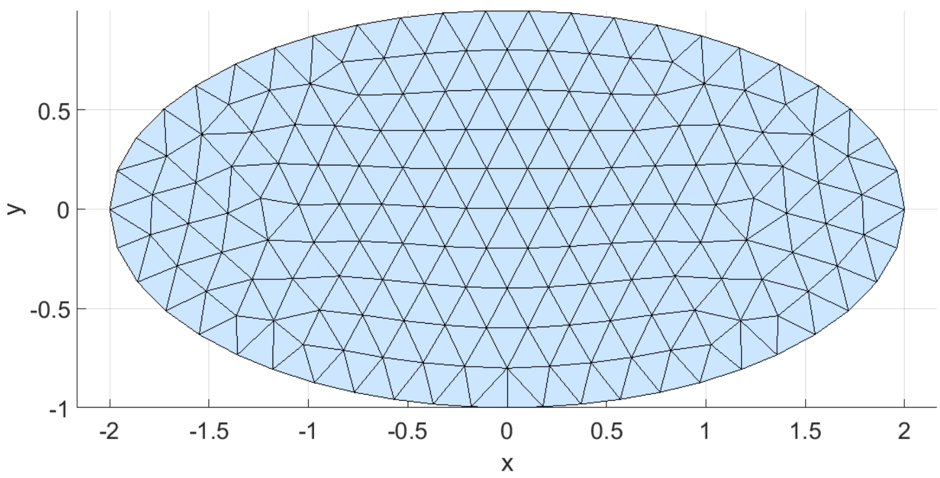
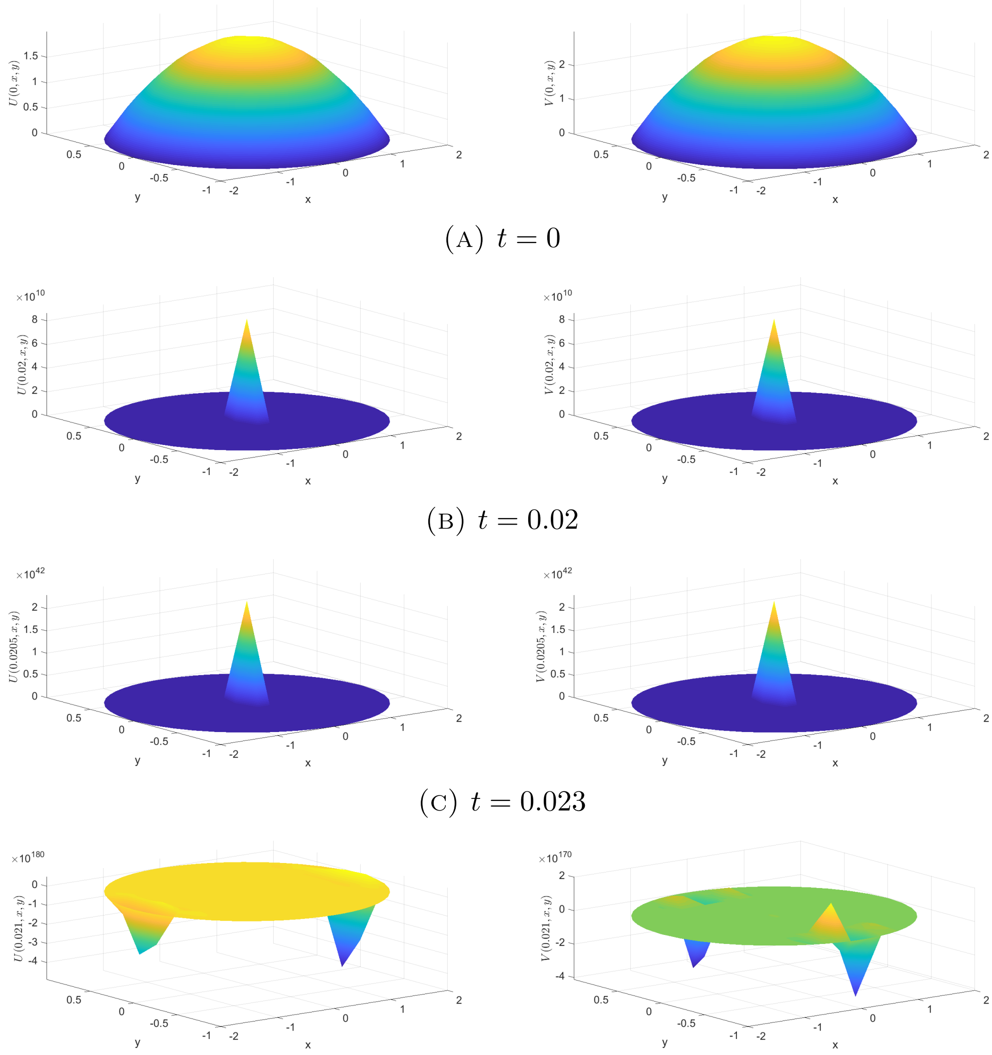


FIGURE 2.4: Uniform mesh grid of Ω_2.



(A) $t = 0$

(B) $t = 0.02$

(C) $t = 0.023$

(D) $t = 0.024$

FIGURE 2.5: The numerical results of Test 2 at different times.

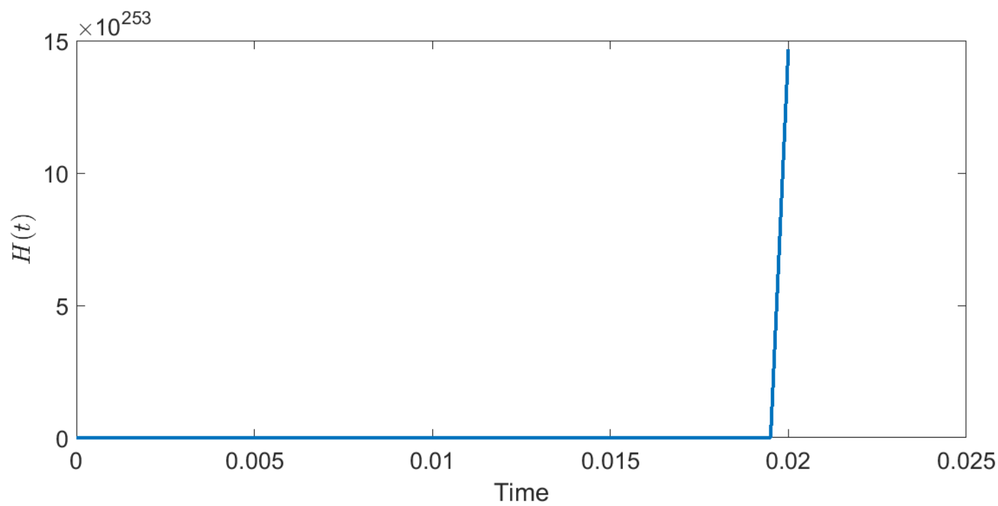


FIGURE 2.6: Test 2: The blow-up of H in finite time.

For Test 2, the numerical values of the functional H are presented in Figure 2.6. Observe the blow-up of the function H from $t = 0.02$.

2.5 Global Existence and Decay Rates of Solution

Our goal in this section is to establish the global existence and the decay rate of solution of system (P) , under appropriate conditions on the initial data. In subsection 1, we state and prove a global existence theorem, for initial data in certain stable set. Second, we prove that the decay estimates of the energy solution are exponential or polynomial deponding on the exponents $m(.)$ and $r(.)$.
For this purpose, we introduce the two functionals defined for all $t \in (0,T)$ by

$$I(t) = \int_\Omega (A\nabla u.\nabla u + B\nabla v.\nabla v)\, dx - \left(p^+ + 1\right) \int_\Omega F(x,u,v)\, dx \tag{2.5.1}$$

and

$$J(t) = \frac{1}{2}\int_\Omega (A\nabla u.\nabla u + B\nabla v.\nabla v)\, dx - \int_\Omega F(x,u,v)\, dx.$$

Clearly, we have

$$E(t) = J(t) + \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right),$$

where E is the energy functional associated to system (P) (see in (2.2.55)).

2.5.1 Global Existence result

Lemma 2.5.1. *Suppose that $I(0) > 0$ and*

$$\beta = \frac{C_2(p^+ + 1)}{c_0} max \left\{ B_*^{p^- + 1} \left(\frac{2(p^+ + 1)}{p^+ - 1} E(0)\right)^{\frac{p^- - 1}{2}}, B_*^{p^+ + 1} \left(\frac{2(p^+ + 1)}{p^+ - 1} E(0)\right)^{\frac{p^+ - 1}{2}} \right\} < 1.$$

Then

$$I(t) > 0, \text{ for all } t \in (0,T). \tag{2.5.2}$$

Proof. By continuity of I, there exists T_m in $]0,T)$ such that

$$I(t) \geq 0,\ \forall t \in (0,T_m]. \tag{2.5.3}$$

In what follows, we will prove that this inequality is strict.
For all $t \in (0,T)$, we have

$$\begin{aligned} J(t) &= \frac{1}{2}\int_\Omega (A\nabla u.\nabla u + B\nabla v.\nabla v)\, dx - \frac{1}{p^+ + 1}\left[\int_\Omega (A\nabla u.\nabla u + B\nabla v.\nabla v)\, dx - I(t)\right] \\ &= \frac{p^+ - 1}{2(p^+ + 1)}\int_\Omega (A\nabla u.\nabla u + B\nabla v.\nabla v)\, dx + \frac{1}{p^+ + 1} I(t). \end{aligned}$$

So, using (2.1.5)

$$J(t) \geq \frac{c_0 (p^+ - 1)}{2(p^+ + 1)} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{1}{p^+ + 1} I(t). \tag{2.5.4}$$

By (2.5.3), we obtain

$$J(t) \geq \frac{c_0 (p^+ - 1)}{2(p^+ + 1)} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right), \ \forall t \in (0, T_m].$$

Thus

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq \frac{2(p^+ + 1)}{c_0 (p^+ - 1)} J(t).$$

The definition of E leads to

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq \frac{2(p^+ + 1)}{c_0 (p^+ - 1)} E(t)$$

By the decreasingness of E, we find

$$\|\nabla u\|_2^2 \leq \frac{2(p^+ + 1)}{c_0 (p^+ - 1)} E(0). \tag{2.5.5}$$

From Lemma 1.2.9, we have

$$\int_\Omega |u|^{p(x)+1} dx \leq max \ \{\|u\|_{p(.)+1}^{p^- + 1}, \|u\|_{p(.)+1}^{p^+ + 1}\}.$$

By (1.2.1) and the embedding, we get

$$\begin{aligned} \int_\Omega |u|^{p(x)+1} dx &\leq max\{B_*^{p^- + 1} \|\nabla u\|_2^{p^- + 1}, B_*^{p^+ + 1} \|\nabla u\|_2^{p^+ + 1}\} \\ &\leq max \ \{B_*^{p^- + 1} \|\nabla u\|_2^{p^- - 1}, B_*^{p^+ + 1} \|\nabla u\|_2^{p^+ - 1}\} \|\nabla u\|_2^2, \end{aligned}$$

Recalling (2.5.5), we obtain

$$\begin{aligned} &\int_\Omega |u|^{p(x)+1} dx \\ &\leq max \left\{ B_*^{p^- + 1} \left(\frac{2(p^+ + 1)}{c_0 (p^+ - 1)} E(0)\right)^{\frac{p^- - 1}{2}}, B_*^{p^+ + 1} \left(\frac{2(p^+ + 1)}{c_0 (p^+ - 1)} E(0)\right)^{\frac{p^+ - 1}{2}} \right\} \|\nabla u\|_2^2, \end{aligned}$$

for all $t \in (0, T_m]$. Then,

$$\int_\Omega |u|^{p(x)+1} dx \leq \frac{c_0 \beta}{C_2 (p^+ + 1)} \|\nabla u\|_2^2,$$

and similarly,

$$\int_\Omega |v|^{p(x)+1} dx \leq \frac{c_0 \beta}{C_2 (p^+ + 1)} \|\nabla v\|_2^2.$$

By addition, it comes

$$\int_{\Omega}\left(|u|^{p(x)+1}+|v|^{p(x)+1}\right)dx\leq\frac{c_0\beta}{C_2\left(p^{+}+1\right)}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right). \tag{2.5.6}$$

On the other hand, by (1.3.1), we have

$$\int_{\Omega}F\left(x,u,v\right)dx\leq C_2\int_{\Omega}\left(|u|^{p(x)+1}+|v|^{p(x)+1}\right)dx.$$

By (1.3.1), it results

$$\int_{\Omega}F\left(x,u,v\right)dx\leq\frac{c_0\beta}{p^{+}+1}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right). \tag{2.5.7}$$

Since $\beta<1$, then

$$\int_{\Omega}F\left(x,u,v\right)dx<\frac{c_0}{p^{+}+1}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right),\forall t\in\left(0,T_m\right].$$

By inserting this inequality in (2.5.1) and using the assumption (2.1.5), it yields

$$I\left(t\right)>\int_{\Omega}\left(A\nabla u.\nabla u+B\nabla v.\nabla v\right)dx-c_0\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\geq0,$$

for all $t\in\left[0,T_m\right]$. Consequently,

$$I\left(T_m\right)>0.$$

From the decreasingness of E and the condition on β, we find

$$\frac{C_2\left(p^{+}+1\right)}{c_0}\beta_m\leq\beta<1,$$

where

$$\beta_m=max\left\{B_*^{p^{-}+1}\left(\frac{2\left(p^{+}+1\right)}{c_0\left(p^{+}-1\right)}E\left(T_m\right)\right)^{\frac{p^{-}-1}{2}},B_*^{p^{+}+1}\left(\frac{2\left(p^{+}+1\right)}{c_0\left(p^{+}-1\right)}E\left(T_m\right)\right)^{\frac{p^{+}-1}{2}}\right\}.$$

By repeating this procedure, we can extend T_m to T. This proves (2.5.2). □

Theorem 2.5.2. *Under the assumptions of Lemma* 2.5.1*, the local solution* (u,v) *of system* (P) *exists globally.*

Proof. By the definition of E and using (2.5.2) and (2.5.4), it results

$$\begin{aligned}E\left(t\right)&=J\left(t\right)+\frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\&\geq\frac{c_0\left(p^{+}-1\right)}{2\left(p^{+}+1\right)}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right),\forall t\in\left(0,T\right).\end{aligned}$$

Therefore, there exists a constant $C > 0$ such that

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2 \leq CE(t). \tag{2.5.8}$$

Using the decreasingness of E, we obtain

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2 \leq CE(0), \forall t \in (0,T).$$

Consequently, by the alternative statement, (u,v) exists globally. □

2.5.2 Decay Rates of Solution

To study the decay of the solution energy of (P), we give the following Lemma.

Lemma 2.5.3. *Suppose that the assumptions of Lemma* 2.5.1 *are fulfilled. Then, there exists a positive constant C_3, such that the global solution (u,v) of (P) satisfies*

$$\int_\Omega |u(t)|^{m(x)}\, dx + \int_\Omega |v(t)|^{r(x)}\, dx \leq C_3 E(t) \text{ for all } t \geq 0. \tag{2.5.9}$$

Proof. Let $\tilde{\mathrm{B}}$ be the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega)$, then we have

$$\begin{aligned}\int_\Omega |u(t)|^{m(x)}\, dx &\leq max\left\{\tilde{B}^{m^-}\|\nabla u\|_2^{m^-}, \tilde{B}^{m^+}\|\nabla u\|_2^{m^+}\right\}\\ &\leq max\left\{\tilde{B}^{m^-}\|\nabla u\|_2^{m^--2}, \tilde{B}^{m^+}\|\nabla u\|_2^{m^+-2}\right\}\|\nabla u\|_2^2.\end{aligned}$$

By (2.5.8), this gives

$$\int_\Omega |u(t)|^{m(x)}\, dx \leq c_1 E(t),$$

for all $t \geq 0$, where c_1 is a positive constant. Similarly, we get

$$\int_\Omega |v(t)|^{r(x)}\, dx \leq c_2 E(t),\ c_2 > 0.$$

By addition of the last two inequalities, we obtain the desired result. □

Theorem 2.5.4. *Under the assumptions of Lemma* 2.5.1*, there exists two constants $c, w > 0$ satisfying the following decay estimates:*

$$E(t) \leq \begin{cases} \frac{c}{(1+t)^{2/(\lambda^+-2)}}, & \forall t \geq 0,\ if\ \lambda^+ > 2,\\ ce^{-\omega t}, & \forall t \geq 0,\ if\ \lambda^+ = 2,\end{cases} \tag{2.5.10}$$

where $\lambda^+ = max\ \{m^+, r^+\}$.

Proof. Let $T > S > 0$ and $q \geq 0$ to be specified later. Multiplying the first equation of (P) by uE^q and the second one by vE^q and integrating each result over $\Omega \times (S,T)$, we obtain

$$\int_S^T \int_\Omega E^q(t)\left[u(t)\,u_{tt}(t) - u(t)\,div\,(A\nabla u(t)) + u(t)\,|u_t(t)|^{m(x)-2}\,u_t(t)\right] dxdt$$
$$= \int_S^T \int_\Omega E^q(t)\,u(t)\,f_1(x,u,v)\,dxdt$$

and

$$\int_S^T \int_\Omega E^q(t)\left[v(t)\,v_{tt}(t) - v(t)\,div\,(A\nabla v(t)) + v(t)\,|v_t(t)|^{r(x)-2}\,v_t(t)\right] dxdt$$
$$= \int_S^T \int_\Omega E^q(t)\,v(t)\,f_2(x,u,v)\,dxdt.$$

So, we get

$$\int_S^T \int_\Omega E^q(t)\left[(u(t)\,u_t(t))_t - |u_t(t)|^2 + A\nabla u(t)\,.\nabla u(t) + u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}\right] dxdt$$
$$= \int_S^T \int_\Omega E^q(t)\,u(t)\,f_1(x,u,v)\,dxdt, \tag{2.5.11}$$

and

$$\int_S^T \int_\Omega E^q(t)\left[(v(t)\,v_t(t))_t - |v_t(t)|^2 + B\nabla v(t)\,.\nabla v(t) + v(t)\,v_t(t)\,|v_t(t)|^{r(x)-2}\right] dxdt$$
$$= \int_S^T \int_\Omega E^q(t)\,v(t)\,f_2(x,u,v)\,dxdt. \tag{2.5.12}$$

Now, we add and subtract the following two terms

$$\int_S^T \int_\Omega E^q(t)\left[\beta A\nabla u(t)\,.\nabla u(t) + (1+\beta)\,|u_t(t)|^2\right] dxdt$$

and

$$\int_S^T \int_\Omega E^q(t)\left[\beta B\nabla v(t)\,.\nabla v(t) + (1+\beta)\,|v_t(t)|^2\right] dxdt,$$

to (2.5.11) and (2.5.12), repectively, exploit (2.1.5) and (2.5.7) and results to obtain

$$(1-\beta)\int_S^T E^q(t)\int_\Omega \left(A\nabla u(t)\,.\nabla u(t) + B\nabla v(t)\,.\nabla v(t) + |u_t(t)|^2 + |v_t(t)|^2\right) dxdt$$
$$+ \int_S^T E^q(t)\int_\Omega \left[(u(t)\,u_t(t) + v(t)\,v_t(t))_t - (2-\beta)\left(|u_t(t)|^2 + |v_t(t)|^2\right)\right] dxdt$$
$$+ \int_S^T E^q(t)\int_\Omega \left(u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2} + v(t)\,v_t(t)\,|v_t(t)|^{r(x)-2}\right) dxdt \tag{2.5.13}$$
$$= -\int_S^T E^q(t)\int_\Omega \left[\beta\,(A\nabla u(t)\,.\nabla u(t) + B\nabla v(t)\,.\nabla v(t)) - (p(x)+1)\,F(x,u,v)\right] dxdt \le 0.$$

Since

$$\begin{aligned}
&\frac{d}{dt}\left(E^{q}(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx\right)\\
&=qE^{q-1}(t)\,E'(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx\\
&+E^{q}(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)_t dx,
\end{aligned}$$

then

$$\begin{aligned}
E^{q}(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)_t dx &= \frac{d}{dt}\left(E^{q}(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx\right)\\
&-qE^{q-1}(t)\,E'(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx.
\end{aligned}$$

Replacing this term in (2.5.13), we find

$$\begin{aligned}
2(1-\beta)\int_S^T E^{q+1}(t)\,dt &\le q\int_S^T E^{q-1}(t)\,E'(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dxdt\\
&-\int_S^T \frac{d}{dt}\left(E^{q}(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx\right)dt\\
&-\int_S^T E^{q}(t)\int_{\Omega}\left(u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}+v(t)\,v_t(t)\,|v_t(t)|^{r(x)-2}\right)dxdt\\
&+(2-\beta)\int_S^T E^{q}(t)\int_{\Omega}\left(|u_t(t)|^2+|v_t(t)|^2\right)dxdt. \qquad (2.5.14)
\end{aligned}$$

In what follows, we estimate the terms in the right-hand side of (2.5.14).
First, by exploiting Young's and Poincaré's inequality, we obtain

$$\begin{aligned}
&q\int_S^T E^{q-1}(t)\,E'(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dxdt\\
&\le \frac{q}{2}\int_S^T E^{q-1}(t)\left(-E'(t)\right)(\|u(t)\|_2^2+\|u_t(t)\|_2^2+\|v(t)\|_2^2+\|v_t(t)\|_2^2)dt\\
&\le C\int_S^T E^{q-1}(t)\left(-E'(t)\right)(\|\nabla u(t)\|_2^2+\|\nabla v(t)\|_2^2+\|u_t(t)\|_2^2+\|v_t(t)\|_2^2)dt,
\end{aligned}$$

where C is a positive generic constant. By (2.5.8), this yields

$$\begin{aligned}
q\int_S^T E^{q-1}(t)\,E'(t)\int_{\Omega}\left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dxdt &\le C\int_S^T E^{q}(t)\left(-E'(t)\right)dt\\
&\le CE^{q+1}(S)-CE^{q+1}(T)\\
&\le CE^{q}(0)\,E(S)\le CE(S)\,. \qquad (2.5.15)
\end{aligned}$$

For the second term, we have

$$
\begin{aligned}
&-\int_S^T \frac{d}{dt}\left(E^q(t)\int_\Omega \left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx\right)dt \\
&\qquad = E^q(S)\left(\int_\Omega \left(u(x,S)\,u_t(x,S)+v(x,S)\,v_t(x,S)\right)dx\right) \\
&\qquad - E^q(T)\left(\int_\Omega \left(u(x,T)\,u_t(x,T)+v(x,T)\,v_t(x,T)\right)dx\right) \qquad (2.5.16)
\end{aligned}
$$

Again, by (2.5.8), we get

$$
\left|\int_\Omega u(x,S)\,u_t(x,S)\,dx\right| \le \frac{C}{2}\int_\Omega |\nabla u(S)|^2\,dx+\frac{1}{2}\int_\Omega |u_t(S)|^2\,dx \le CE(S),
$$

$$
\left|\int_\Omega v(x,S)\,v_t(x,S)\,dx\right| \le \frac{C}{2}\int_\Omega |\nabla v(S)|^2\,dx+\frac{1}{2}\int_\Omega |v_t(S)|^2\,dx \le CE(S)
$$

and similarly

$$
\left|\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \le \frac{C}{2}\int_\Omega |\nabla u(T)|^2\,dx+\frac{1}{2}\int_\Omega |u_t(T)|^2\,dx \le CE(S),
$$

$$
\left|\int_\Omega v(x,T)\,v_t(x,T)\,dx\right| \le \frac{C}{2}\int_\Omega |\nabla v(T)|^2\,dx+\frac{1}{2}\int_\Omega |v_t(T)|^2\,dx \le CE(S).
$$

Therefore, (2.5.16) leads to

$$
\begin{aligned}
-\int_S^T \frac{d}{dt}\left(E^q(t)\int_\Omega \left(u(t)\,u_t(t)+v(t)\,v_t(t)\right)dx\right)dt &\le CE^{q+1}(S) \\
&\le CE^q(0)\,E(S)\le CE(S).
\end{aligned}
\qquad (2.5.17)
$$

Concerning the third term, we have

$$
\begin{aligned}
&-\int_S^T E^q(t)\int_\Omega \left(u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}+v(t)\,v_t(t)\,|v_t(t)|^{r(x)-2}\right)dxdt \\
&\quad \le \int_S^T E^q(t)\int_\Omega |u(t)|\,|u_t(t)|^{m(x)-1}\,dx+\int_S^T E^q(t)\int_\Omega |v(t)|\,|v_t(t)|^{r(x)-1}\,dx \\
&\quad = J_1+J_2.
\end{aligned}
$$

Using the following Young's inequality:

$$
XY\le \frac{\varepsilon}{\delta}X^\delta+\frac{1}{\delta'\varepsilon^{\delta'/\delta}}Y^{\delta'},\ \text{for all } X,\ Y\ge 0,\ \varepsilon>0 \text{ and } \frac{1}{\delta}+\frac{1}{\delta'}=1,
$$

with

$$
X=|u(t)|\,,\ Y=|u_t(t)|^{m(x)-1}\,,\ \delta=m(x)\,,\ \delta'=\frac{m(x)}{m(x)-1} \text{ and } \varepsilon\in\,]0,1[\,,
$$

we find

$$J_1 \le \int_S^T E^q(t)\left[\varepsilon \int_\Omega \frac{1}{m(x)} |u(t)|^{m(x)}\,dx + \int_\Omega \frac{m(x)-1}{m(x)\,\varepsilon^{1/(m(x)-1)}} |u_t(t)|^{m(x)}\,dx\right] dt.$$

Since for all $x \in \Omega$, we have $m(x) \ge 2$ then $\varepsilon^{1/(m(x)-1)} \ge \varepsilon$. Hence,

$$J_1 \le \int_S^T E^q(t)\left[\frac{\varepsilon}{2}\int_\Omega |u(t)|^{m(x)}\,dx + \frac{1}{\varepsilon}\int_\Omega |u_t(t)|^{m(x)}\,dx\right] dt.$$

Likewise,

$$J_2 \le \int_S^T E^q(t)\left[\frac{\varepsilon}{2}\int_\Omega |v(t)|^{r(x)}\,dx + \frac{1}{\varepsilon}\int_\Omega |v_t(t)|^{r(x)}\,dx\right] dt.$$

By addition and use of Lemma 2.2.5 and (2.5.9), we arrive at

$$\begin{aligned} J_1 + J_2 &\le \frac{\varepsilon}{2}\int_S^T E^q(t)\int_\Omega \left(|u(t)|^{m(x)} + |v(t)|^{r(x)}\right) dxdt + \frac{1}{\varepsilon}\int_S^T E^q(t)\left(-E'(t)\right) dt \\ &\le \varepsilon C \int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S). \end{aligned}$$

So,

$$\begin{aligned} -\int_S^T E^q(t)\int_\Omega \Big(u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2} + v(t)\,v_t(t)\,|v_t(t)|^{r(x)-2}\Big)\,dxdt& \\ \le \varepsilon C \int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S).& \end{aligned} \tag{2.5.18}$$

Finally, we estimate the last term in (2.5.14) as follows:
We have

$$\int_S^T E^q(t)\int_\Omega \left(|u_t(t)|^2 + |v_t(t)|^2\right) dxdt = J_3 + J_4,$$

where

$$J_3 = \int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt,\ \ J_4 = \int_S^T E^q(t)\int_\Omega |v_t(t)|^2\,dxdt.$$

We set, similarly to [22]

$$\Omega_+ = \{x \in \Omega \ / \ |u_t(x,t)| \ge 1\},\ \ \Omega_- = \{x \in \Omega \ / \ |u_t(x,t)| < 1\}.$$

Therefore,

$$\begin{aligned} J_3 &= \int_S^T E^q(t)\left[\int_{\Omega_-}|u_t(t)|^2\,dx + \int_{\Omega_+}|u_t(t)|^2\,dx\right]dt \\ &\leq C\int_S^T E^q(t)\left[\left(\int_{\Omega_-}|u_t(t)|^{\lambda^+}\,dx\right)^{2/\lambda^+} + \left(\int_{\Omega_+}|u_t(t)|^{\lambda^-}\,dx\right)^{2/\lambda^-}\right]dt \\ &\leq C\int_S^T E^q(t)\left[\left(\int_{\Omega_-}|u_t(t)|^{m(x)}\,dx\right)^{2/\lambda^+} + \left(\int_{\Omega_+}|u_t(t)|^{m(x)}\,dx\right)^{2/\lambda^-}\right]dt, \end{aligned}$$

where

$$\lambda^- = \ min\left\{m^-, r^-\right\},\ \lambda^+ = \ max\left\{m^+, r^+\right\}.$$

This leads to

$$\begin{aligned} J_3 &\leq C\int_S^T E^q(t)\left[\left(\int_{\Omega}|u_t(t)|^{m(x)}\,dx\right)^{2/\lambda^+} + \left(\int_{\Omega}|u_t(t)|^{m(x)}\,dx\right)^{2/\lambda^-}\right]dt \\ &\leq C\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^+}dt + C\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^-}dt. \end{aligned}$$

Similarly, we obtain

$$J_4 \leq C\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^+}dt + C\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^-}dt.$$

By addition, this yields

$$J_3 + J_4 \leq C\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^+}dt + C\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^-}dt. \tag{2.5.19}$$

We claim that

$$J_3 + J_4 \leq \varepsilon C\int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S).$$

Indeed, we distinguish two cases:
Case 1: if $\lambda^+ = 2$, then $\lambda^- = 2$. Hence,

$$J_3 + J_4 \leq C\int_S^T E^q(t)\left(-E'(t)\right)dt \leq CE(S) \leq \varepsilon C\int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S).$$

Case 2: if $\lambda^+ > 2$, we use Young's inequality with

$$\delta = (q+1)/q \text{ and } \delta' = q+1$$

to obtain

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^+}dt \leq \varepsilon C\int_S^T E^{q+1}(t)\,dt + C_\varepsilon\int_S^T \left(-E'(t)\right)^{2(q+1)/\lambda^+}dt.$$

By taking $q = \lambda^+/2 - 1$, it comes

$$\begin{aligned}\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^+} dt &\leq \varepsilon C \int_S^T E^{q+1}(t)\,dt + C_\varepsilon \int_S^T \left(-E'(t)\right) dt. \\ &\leq \varepsilon C \int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S)\,. \end{aligned} \tag{2.5.20}$$

For the second term in the right-hand side of (2.5.19), we have the following two cases:

- if $\lambda^- = 2$, then

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^-} dt = \int_S^T E^q(t)\left(-E'(t)\right) dt \leq CE(S)\,.$$

  Adding this inquality to (2.5.20), we find

$$J_3 + J_4 \leq \varepsilon C \int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S)\,.$$

- if $\lambda^- > 2$, we use Young's inequality with

$$\delta = \lambda^-/\left(\lambda^- - 2\right) \text{ and } \delta' = \lambda^-/2$$

  to obtain

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^-} dt \leq \varepsilon C \int_S^T E(t)^{q\lambda^-/\left(\lambda^- - 2\right)} dt + C_\varepsilon E(S)\,.$$

  Since, $q\lambda^-/(\lambda^- - 2) = q + 1 + (\lambda^+ - \lambda^-)/(\lambda^- - 2)$, then

$$\begin{aligned}\int_S^T E^q(t)\left(-E'(t)\right)^{2/\lambda^-} dt &\leq \varepsilon C\left(E(S)\right)^{\left(\lambda^+ - \lambda^-\right)/\left(\lambda^- - 2\right)} \int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S) \\ &\leq \varepsilon C \int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S)\,.\end{aligned}$$

  Therefore, for $\lambda^- \geq 2$, we have

$$J_3 + J_4 \leq \varepsilon C \int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S)\,. \tag{2.5.21}$$

  Consequently,

$$\begin{aligned}\int_S^T E^q(t)\int_\Omega \left(\left|u_t(t)\right|^2 + \left|v_t(t)\right|^2\right) dxdt &= J_3 + J_4 \\ &\leq \varepsilon C \int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S)\,.\end{aligned} \tag{2.5.22}$$

Finally, by inserting (2.5.22), (2.5.18), (2.5.17) and (2.5.15) in (2.5.14), we obtain

$$2(1-\beta)\int_S^T E^{q+1}(t)\,dt \leq \varepsilon C \int_S^T E^{q+1}(t)\,dt + C_\varepsilon E(S).$$

Since, $q = \lambda^+/2 - 1$, it follows

$$2(1-\beta)\int_S^T E^{\lambda^+/2}(t)\,dt \leq \varepsilon C \int_S^T E^{\lambda^+/2}(t)\,dt + C_\varepsilon E(S).$$

Chosing ε small enough, we get

$$\int_S^T E^{\lambda^+/2}(t)\,dt \leq CE(S).$$

When $T \longrightarrow \infty$, it yields

$$\int_S^\infty E^{\lambda^+/2}(t)\,dt \leq CE(S).$$

By applying Komornik's integral inequality given in Lemma 1.3.1, we obtain the decay estimates (2.5.10). □

Chapter 3

Coupled System of Nonlinear Hyperbolic Equations with Variable-exponents in the Damping and Attractive terms

3.1 Introduction

This chapter deals with the following initial-boundary-value problem:

$$\begin{cases} u_{tt} - \Delta u + \alpha(t)|u_t|^{m(x)-2}u_t + |u|^{p(x)-2}u|v|^{p(x)} = 0 & \text{in } \Omega \times (0,T)\,, \\ v_{tt} - \Delta v + \beta(t)|v_t|^{r(x)-2}v_t + |v|^{p(x)-2}v|u|^{p(x)} = 0 & \text{in } \Omega \times (0,T)\,, \\ u = v = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u\,(0) = u_0,\ u_t\,(0) = u_1 & \text{in } \Omega, \\ v\,(0) = v_0,\ v_t\,(0) = v_1 & \text{in } \Omega, \end{cases} \qquad (\tilde{P})$$

where $T > 0$ and Ω is a bounded domain of $\mathbb{R}^n\,(n \in \mathbb{N}^*)$ with a smooth boundary $\partial\Omega$. $\alpha, \beta : [0, \infty) \longrightarrow (0, \infty)$ are two non-increasing C^1-functions and m, r and p are given continuous functions on $\overline{\Omega}$ satisfying some conditions to be specified later.

This class of coupled systems of nonlinear wave equations with variable exponents occur in the mathematical modeling of various physical phenomnas such as flows of electro-rheological fluids or fluids with temperature dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing, thermorheological fluids, or robotics, etc. Our system $(\tilde{P})$ can be regarded as a model for interaction between two fields describing the motion of two nonlinear "smart" materials. For more details, the interested reader can see [1, 12].
Under suitable assumptions on the functions α, β and the variable exponents m, r and p, we establish the decay of the solution energy, by using the multiplier method. Then, we give some numerical examples.

ASSUMPTIONS:

For all $x \in \Omega$, we assume that

$$\begin{aligned} &2 \leq m(x), && \text{if } n = 1,2, \\ &2 \leq m^- \leq m(x) \leq m^+ \leq \frac{2n}{n-2}, && \text{if } n \geq 3, \end{aligned} \tag{H.1}$$

$$\begin{aligned} &2 \leq r(x), && \text{if } n = 1,2, \\ &2 \leq r^- \leq r(x) \leq r^+ \leq \frac{2n}{n-2}, && \text{if } n \geq 3, \end{aligned}$$

and

$$\begin{aligned} &2 \leq p(x), && \text{if } n = 1,2, \\ &2 \leq p^- \leq p(x) \leq p^+ \leq \frac{n}{n-2}, && \text{if } n \geq 3. \end{aligned} \tag{H.2}$$

3.2 Existence of Global Weak Solution

Definition 3.2.1. *Let $(u_0, u_1), (v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. A pair of functions (u, v) is said to be a weak solution of $(\tilde{P})$ on $[0, T)$ if $u, v \in C_\omega((0,T), H_0^1(\Omega)), u_t, v_t \in C_\omega((0,T), L^2(\Omega)), u_t \in L^{m(.)}(\Omega \times (0,T)),\ v_t \in L^{r(.)}(\Omega \times (0,T))$ and for all test functions $\Phi, \Psi \in H_0^1(\Omega)$ and all $t \in (0, T)$, we have*

$$\begin{aligned} &\int_\Omega u_t \Phi\ dx - \int_\Omega u_1 \Phi\ dx + \int_0^t \int_\Omega \alpha(\tau)|u_t|^{m(x)-2} u_t \Phi\ dx d\tau \\ &\quad + \int_0^t \int_\Omega \nabla u \nabla \Phi\ dx d\tau + \int_0^t \int_\Omega |u_t|^{p(x)-2} u |v|^{p(x)} \Phi\ dx d\tau = 0 \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega v_t \Psi dx - \int_\Omega v_1 \Psi\ dx + \int_0^t \int_\Omega \beta(\tau)|v_t|^{r(x)-2} v_t \Psi\ dx d\tau \\ &\quad + \int_0^t \int_\Omega \nabla v \nabla \Psi\ dx d\tau + \int_0^t \int_\Omega |v_t|^{p(x)-2} v |u|^{p(x)} \Psi\ dx d\tau = 0. \end{aligned}$$

Proposition 3.2.2. *Assume that the above assumptions hold. Then, for any initial data $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$, there exists a unique global weak solution (u, v) of $(\tilde{P})$ (in the sense of Definition 3.2.1) defined in $[0, T)$ for all $T > 0$. Moreover, we have the energy inequality*

$$E(t) + \int_s^t \alpha(\tau) \int_\Omega |u_t|^{m(x)} dx d\tau + \int_s^t \beta(\tau) \int_\Omega |v_t|^{r(x)} dx d\tau \leq E(s), \tag{3.2.1}$$

for $0 \leq s \leq t \leq T$, where

$$E(t) =: \frac{1}{2}\left[\|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2\right] + \int_\Omega \frac{|uv|^{p(x)}}{p(x)} dx.$$

Remark 3.2.3. *The proof of this proposition can be established by the Faedo-Galerkin approximation and combining arguments from [2, 17, 35] for coupled systems with constant exponents.*

3.3 Decay Rates of Solutions

In this subsection, we state and prove our main result.

Theorem 3.3.1. *Suppose that the assumptions (H.1) and (H.2) hold. Assume, further, that* $\int_0^\infty \alpha(s)ds = \infty$ *and* $\int_0^\infty \beta(s)ds = \infty$*. Then, there exists two constants* $c, \omega > 0$ *such that the solution of* $(\tilde{P})$ *satisfies, for all* $t \geq 0$,

$$E(t) \leq \begin{cases} ce^{-\omega \int_0^t \gamma(s)ds}, & if\ \lambda^+ = 2, \\ \frac{c}{\left(1+\int_0^t \gamma(s)ds\right)^{2/(\lambda^+-2)}}, & if\ \lambda^+ > 2, \end{cases}$$

where

$$\lambda^+ = max\ \{m^+, r^+\}\ and\ \gamma = min\{\alpha, \beta\}.$$

Proof. Let $T > S > 0$ and $q \geq 0$ to be specified later. Multiplying the first differential equation of $(\tilde{P})$ by $\gamma E^q u$, the second one by $\gamma E^q v$ and integrating each result over $\Omega \times (S, T)$, we obtain

$$\int_S^T \gamma(t)E^q(t) \int_\Omega \left[uu_{tt} - u\Delta u + \alpha(t)|u_t|^{m(x)-2}u_t u\right] dxdt$$
$$= - \int_S^T \gamma(t)E^q(t) \int_\Omega |uv|^{p(x)} dxdt$$

and

$$\int_S^T \gamma(t)E^q(t) \int_\Omega \left[vv_{tt} - v\Delta v + \beta(t)|v_t|^{r(x)-2}v_t v\right] dxdt$$
$$= - \int_S^T \gamma(t)E^q(t) \int_\Omega |uv|^{p(x)} dxdt.$$

These can be rewritten as:

$$\int_S^T \gamma(t)E^q(t) \int_\Omega \left[(uu_t)_t - u_t^2 + |\nabla u|^2 + \alpha(t)|u_t|^{m(x)-2}u_t u\right] dxdt$$
$$= - \int_S^T \gamma(t)E^q(t) \int_\Omega |uv|^{p(x)} dxdt \tag{3.3.1}$$

and

$$\int_S^T \gamma(t)E^q(t) \int_\Omega \left[(vv_t)_t - v_t^2 + |\nabla v|^2 + \beta(t)|v_t|^{r(x)-2}v_t v\right] dxdt$$
$$= - \int_S^T \gamma(t)E^q(t) \int_\Omega |uv|^{p(x)} dxdt. \tag{3.3.2}$$

We add and subtract the following two terms

$$\left(-\int_S^T \gamma(t)E^q(t)\int_\Omega u_t^2 dxdt\right) \text{ and } \left(-\int_S^T \gamma(t)E^q(t)\int_\Omega v_t^2 dxdt\right)$$

to (3.3.1) and (3.3.2), respectively. The addition of the two results yields

$$\begin{aligned}
&\int_S^T \gamma E^q \int_\Omega (u_t^2 + v_t^2 + |\nabla u|^2 + |\nabla v|^2) dxdt \\
&= -\int_S^T \gamma E^q \int_\Omega (uu_t + vv_t)_t dxdt + 2\int_S^T \gamma E^q \int_\Omega (u_t^2 + v_t^2) dxdt \\
&- \int_S^T \gamma E^q \int_\Omega \left(\alpha |u_t|^{m(x)-2} u_t u + \beta |v_t|^{r(x)-2} v_t v\right) dxdt \\
&- 2\int_S^T \gamma E^q \int_\Omega |uv|^{p(x)} dxdt.
\end{aligned} \tag{3.3.3}$$

Recalling the expression of E, equality (3.3.3) becomes

$$\begin{aligned}
2\int_S^T \gamma E^{q+1} dt = &-\int_S^T \gamma E^q \int_\Omega (uu_t + vv_t)_t dxdt + 2\int_S^T \gamma E^q \int_\Omega (u_t^2 + v_t^2) dxdt \\
&- \int_S^T \gamma E^q \int_\Omega \alpha \left(|u_t|^{m(x)-2} u_t u + \beta |v_t|^{r(x)-2} v_t v\right) dxdt \\
&+ \int_S^T \gamma E^q \int_\Omega \left(\frac{2}{p(x)} - 2\right) |uv|^{p(x)} dxdt.
\end{aligned}$$

Then,

$$\begin{aligned}
2\int_S^T \gamma E^{q+1} dt \le &-\int_S^T \gamma E^q \int_\Omega (uu_t + vv_t)_t dxdt + 2\int_S^T \gamma E^q \int_\Omega (u_t^2 + v_t^2) dxdt \\
&- \int_S^T \gamma E^q \int_\Omega \left(\alpha |u_t|^{m(x)-2} u_t u + \beta |v_t|^{r(x)-2} v_t v\right) dxdt,
\end{aligned} \tag{3.3.4}$$

since $p(x) > 1$, for all $x \in \Omega$.
On the other hand, we have for $a.e.$ $t \in [S, T]$

$$\frac{d}{dt}\left(\gamma E^q \int_\Omega (uu_t + vv_t)\, dx\right) = (\gamma E^q)' \int_\Omega (uu_t + vv_t)\, dx + \gamma E^q \int_\Omega (uu_t + vv_t)_t\, dx,$$

which gives,

$$\gamma E^q \int_\Omega (uu_t + vv_t)_t\, dx = \frac{d}{dt}\left(\gamma E^q \int_\Omega (uu_t + vv_t)\, dx\right) - (\gamma E^q)' \int_\Omega (uu_t + vv_t)\, dx. \tag{3.3.5}$$

Substituting (3.3.5) in (3.3.4), we arrive at

$$2\int_S^T \gamma E^{q+1} dt \le I_1 + I_2 + I_3 + I_4. \tag{3.3.6}$$

In what follows, we estimate I_i, for $i = 1, ..., 4$.

- $I_1 = -\left[\gamma E^q \int_\Omega (uu_t + vv_t)\, dx\right]_S^T$.
  Using Young's and Poincaré inequalities and the definition of E, we obtain

$$\left|\int_\Omega (uu_t + vv_t)\, dx\right| \le \frac{c_e}{2}\left[\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u_t\|_2^2 + \|v_t\|_2^2\right] \le CE(t), \quad (3.3.7)$$

where c_e is the Poincaré constant. Therefore,

$$\begin{aligned} I_1 =& \gamma(S)E^q(S)\int_\Omega (u(x,S)\, u_t(x,S) + v(x,S)\, v_t(x,S))\, dx \\ &- \gamma(T)E^q(T)\int_\Omega (u(x,T)u_t(x,T) + v(x,T)v_t(x,T))\, dx \\ &\le C\left[\gamma(S)E^{q+1}(S) + \gamma(T)E^{q+1}(T)\right] \le C\gamma(S)E^{q+1}(S) \le CE(S), \quad (3.3.8)\end{aligned}$$

by the non-increasingness property of the two functions γ and E.

- $I_2 = \int_S^T (\gamma' E^q + q\gamma E^{q-1}E') \int_\Omega (uu_t + vv_t)\, dxdt$.
  Again by (3.3.7), we get

$$\begin{aligned} I_2 &\le C\int_S^T \left(\gamma' E^q + q\gamma E^{q-1}E'\right) E(t)dt \\ &\le C|\int_S^T \gamma' E^{q+1}dt| + C|\int_S^T q\gamma E^q E' dt| \\ &\le CE^{q+1}(S)|\int_S^T \gamma' dt| + Cq\gamma(S)|\int_S^T E^q E' dt| \\ &\le CE^{q+1}(S)\left[\gamma(S) - \gamma(T)\right] + CE(S) \le CE(S). \quad (3.3.9)\end{aligned}$$

- $I_3 = 2\int_S^T \gamma E^q \int_\Omega (u_t^2 + v_t^2) dxdt$.
  We have

$$\begin{aligned} I_3 &= 2\int_S^T \gamma E^q \int_\Omega |u_t|^2 dxdt + 2\int_S^T \gamma E^q \int_\Omega |v_t|^2 dxdt \\ &= J_1 + J_2.\end{aligned}$$

Therefore, by Hölder's inequality and the definition of λ^+, we obtain

$$\begin{aligned} J_1 &= 2\int_S^T \gamma E^q \left[\int_{\Omega_-} |u_t|^2 dx + \int_{\Omega_+} |u_t|^2 dx\right] dt \\ &\le C\int_S^T \gamma E^q \left(\int_{\Omega_-} |u_t|^{\lambda^+} dx\right)^{2/\lambda^+} dt + C\int_S^T \gamma E^q \int_{\Omega_+} |u_t|^{m(x)} dxdt \\ &\le C\int_S^T \gamma E^q \left(\int_{\Omega_-} |u_t|^{m(x)} dx\right)^{2/\lambda^+} dt + C\int_S^T E^q \left(\gamma \int_{\Omega_+} |u_t|^{m(x)} dx\right) dt,\end{aligned}$$

such that Ω_+ and Ω_- are introduced in Subsection 2.3.1. This yields

$$\begin{aligned}
J_1 &\leq C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})} E^q \left(\gamma \int_\Omega |u_t|^{m(x)}dx\right)^{2/\lambda^+} + C\int_S^T E^q \left(\gamma \int_\Omega |u_t|^{m(x)}dx\right) dt \\
&\leq C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})} E^q \left(\alpha \int_\Omega |u_t|^{m(x)}dx\right)^{2/\lambda^+} + C\int_S^T E^q \left(\alpha \int_\Omega |u_t|^{m(x)}dx\right) dt \\
&\leq C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})} E^q \left(-E'\right)^{2/\lambda^+} dt + C\int_S^T E^q(-E')dt \\
&\leq C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})} E^q \left(-E'\right)^{2/\lambda^+} dt + CE(S),
\end{aligned}$$

using (3.2.1) and the definition of γ. Similarly, we find

$$J_2 \leq C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})} E^q \left(-E'\right)^{2/\lambda^+} dt + CE(S).$$

By addtion of J_1 and J_2, it results

$$I_3 \leq C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})} E^q \left(-E'\right)^{2/\lambda^+} dt + CE(S).$$

Two cases are possible:
Case 1: if $\lambda^+ = 2$ then,

$$\begin{aligned}
I_3 &\leq C\int_S^T E^q \left(-E^{'}\right) dt + CE(S) \\
&\leq C\left[E^{q+1}(S) - E^{q+1}(T)\right] + CE(S) \leq CE(S).
\end{aligned}$$

Case 2: if $\lambda^+ > 2$, we exploit Young's inequality, with

$$\delta = q+1 \text{ and } \delta' = (q+1)/q,$$

to get, for any $\varepsilon > 0$,

$$I_3 \leq \varepsilon C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})(\frac{q+1}{q})} E^{q+1} dt + C_\varepsilon \int_S^T \left(-E'\right)^{\frac{2(q+1)}{\lambda^+}} dt + CE(S).$$

If we take $\varepsilon = \frac{1}{2C}$ and $q = \frac{\lambda^+}{2} - 1$, then

$$\begin{aligned}
I_3 &\leq \frac{1}{2}\int_S^T \gamma E^{q+1} dt + C_\varepsilon \int_S^T \left(-E'\right) dt + CE(S) \\
&\leq \frac{1}{2}\int_S^T \gamma E^{q+1} dt + CE(S).
\end{aligned}$$

Therefore, for $\lambda^+ \geq 2$

$$I_3 \leq \frac{1}{2}\int_S^T \gamma E^{q+1} dt + CE(S). \tag{3.3.10}$$

- $I_4 = -\int_S^T \gamma E^q \int_\Omega \left(\alpha|u_t|^{m(x)-2}u_t u + \beta|v_t|^{r(x)-2}v_t v\right) dxdt.$
  Since α and β are bounded functions on $\mathbb{R}_+$, then

$$\begin{aligned}I_4 &\leq C\int_S^T \gamma E^q \int_\Omega |u||u_t|^{m(x)-1}dxdt + C\int_S^T \gamma E^q \int_\Omega |v||v_t|^{r(x)-1}dxdt\\ &= J_3 + J_4.\end{aligned}$$

Now, applying Young's inequality with

$$\delta(x) = \frac{m(x)}{m(x)-1} \text{ and } \delta'(x) = m(x),$$

we obtain, for all $\varepsilon > 0$

$$J_3 \leq \int_S^T \gamma E^q \left[\varepsilon\int_\Omega |u|^{m(x)}dx + \int_\Omega C_\varepsilon(x)|u_t|^{m(x)}dx\right] dt,$$

where

$$C_\varepsilon(x) = \frac{[m(x)-1]^{m(x)-1}}{[m(x)]^{m(x)}\,\varepsilon^{m(x)-1}}.$$

Likewise,

$$J_4 \leq \int_S^T \gamma E^q \left[\varepsilon\int_\Omega |v|^{r(x)}dx + \int_\Omega C'_\varepsilon(x)|v_t|^{r(x)}dx\right] dt,$$

where

$$C'_\varepsilon(x) = \frac{[r(x)-1]^{r(x)-1}}{[r(x)]^{r(x)}\,\varepsilon^{r(x)-1}}.$$

By addition, we get

$$I_4 \leq \int_S^T \gamma E^q \int_\Omega \left(\varepsilon|u|^{m(x)} + \varepsilon|v|^{r(x)} + C_\varepsilon(x)|u_t|^{m(x)} + C'_\varepsilon(x)|v_t|^{r(x)}\right) dxdt. \tag{3.3.11}$$

Now, we have the following estimate, using (H.1),

$$\begin{aligned}J_5 &= \varepsilon\int_S^T \gamma E^q \int_\Omega (|u|^{m(x))} + |v|^{r(x)})dxdt\\ &\leq \varepsilon C\int_S^T \gamma E^q \int_\Omega (|u|^{m_-} + |u|^{m_+} + |v|^{r_-} + |v|^{r_+})\,dxdt\\ &\leq \varepsilon C\int_S^T \gamma E^q\,(\|\nabla u\|_2^{m_-} + \|\nabla u\|_2^{m_+} + \|\nabla v\|_2^{r_-} + \|\nabla v\|_2^{r_+})\,dt\\ &\leq \varepsilon C\int_S^T \gamma E^{q+1}\left(E^{\frac{m_-}{2}-1} + E^{\frac{m_+}{2}-1} + E^{\frac{r_-}{2}-1} + E^{\frac{r_+}{2}-1}\right) dt\\ &\leq \varepsilon C\left(E(0)^{\frac{m_-}{2}-1} + E(0)^{\frac{m_+}{2}-1} + E(0)^{\frac{r_-}{2}-1} + E(0)^{\frac{r_+}{2}-1}\right)\int_S^T \gamma E^{q+1}dt,\end{aligned}$$

since, $m^-, r^- \geq 2$. By taking

$$\varepsilon = \frac{\left(E(0)^{\frac{m_-}{2}-1} + E(0)^{\frac{m_+}{2}-1} + E(0)^{\frac{r_-}{2}-1} + E(0)^{\frac{r_+}{2}-1}\right)^{-1}}{2C},$$

it results

$$J_5 \leq \frac{1}{2}\int_S^T \gamma E^{q+1} dt.$$

Moreover, $C_\varepsilon(x), C'_\varepsilon(x)$ will be bounded since $m(x)$ and $r(x)$ are bounded. Consequently, inequality (3.3.11) becomes

$$\begin{aligned}
I_4 &\leq \frac{1}{2}\int_S^T \gamma E^{q+1} dt + C\int_S^T \gamma E^q \left(|u_t|^{m(x)} + |v_t|^{r(x)}\right) dxdt \\
&\leq \frac{1}{2}\int_S^T \gamma E^{q+1} dt + C\int_S^T E^q \left(\alpha|u_t|^{m(x)} + \beta|v_t|^{r(x)}\right) dxdt \\
&\leq \frac{1}{2}\int_S^T \gamma E^{q+1} dt + C\int_S^T E^q(-E'(t))dt \\
&\leq \frac{1}{2}\int_S^T \gamma E^{q+1} dt + CE(S). \qquad (3.3.12)
\end{aligned}$$

Finally, by inserting (3.3.12), (3.3.10), (3.3.9) and (3.3.8) in (3.3.6), it results

$$\int_S^T \gamma E^{q+1} dt \leq CE(S).$$

Taking $T \longrightarrow \infty$, we get

$$\int_S^\infty \gamma E^{q+1}(t) dt \leq CE(S).$$

Invoking Lemma 1.3.2 with $\sigma(t) = \int_0^t \gamma(s)ds$, we obtain the desired result.

□

In the special case, α and β are constants, we have the following corollary

Corollary 3.3.2. *Assume that assumptions (H.1) and (H.2) hold. Then, there exist two constants $c, \omega > 0$ such that the solution of $(\tilde{P})$ satisfies, for all $t \geq 0$,*

$$E(t) \leq \begin{cases} ce^{-\omega t}, & if\ \lambda^+ = 2, \\ \frac{c}{(1+t)^{2/(\lambda^+-2)}}, & if\ \lambda^+ > 2. \end{cases}$$

SOME EXAMPLES

We end this section with some examples illustrating our stability result.

- If $\alpha(t) = \beta(t) = \frac{1}{1+t}$, then the estimate in Theorem 3.3.1 gives

$$E(t) \leq \begin{cases} \frac{c}{(1+t)^\omega}, & if\ \lambda^+ = 2, \\ \frac{c}{(1+\ln(1+t))^{2/(\lambda^+-2)}}, & if\ \lambda^+ > 2. \end{cases}$$

- If $\alpha(t) = \frac{1}{(1+t)^a}, \beta(t) = \frac{1}{(1+t)^b}$, for $0 \leq b < a < 1$, then the estimate in Theorem 3.3.1 gives

$$E(t) \leq \begin{cases} ce^{\frac{-\omega}{1-a}\left[(1+t)^{(1-a)}-1\right]}, & if\ \lambda^+ = 2, \\ c/\left(1 + \frac{1}{1-a}\left[(1+t)^{(1-a)} - 1\right]\right)^{2/\left(\lambda^+-2\right)}, & if\ \lambda^+ > 2. \end{cases}$$

- If $\alpha(t) = 1/(2+t)\ln(2+t), \beta(t) = 1/(2+t)^2(\ln(2+t))^2$, the estimate in Theorem 3.3.1 gives

$$E(t) \leq \begin{cases} c\left(\frac{\ln 2}{\ln(2+t)}\right)^{\omega}, & if\ \lambda^+ = 2, \\ c/\left[1 + \ln\left(\frac{\ln(2+t)}{\ln 2}\right)\right]^{2/\left(\lambda^+-2\right)}, & if\ \lambda^+ > 2. \end{cases}$$

3.3.1 Numericals Tests

In this subsection, we illustrate numerically the theoretical results of the present work. We solve the system $(\tilde{P})$ under the corresponding initial and boundary conditions. The nonlinear system $(\tilde{P})$ is discritized using the classical second order finite difference method in time and space. In addition, we implement the stable and conservative scheme of Lax-Wendroff. For more details and similar techniques, we refer to [21]. Here we give five performed tests, for $\Omega =]0,1[$ and $[0,T] = [0,20]$:

- **TEST 1**: Based on the result of Theorem 3.3.1 and the result explained in the example 1, we obtain the polynomial decay of the energy (E):

$$E_1(t) \leq E_f^1(t) = \frac{c}{(1+t)^w},$$

for two positive constants c and w. For this test, we use the functions:

$$m(x) = r(x) = 2;\ p(x) = 2 - \frac{1}{1+x};\ \forall x \in \Omega,$$
$$\alpha(t) = \beta(t) = \frac{1}{1+t};\ \forall t > 0.$$

- **TEST 2**: In Test 2, we examine the second result explained in the example 1, we obtain a logarithmic-polynomial decay of the energy (E):

$$E_2(t) \leq E_f^2(t) = \frac{c}{(1+\ln(1+t))^2},$$

for a positive constants c. For this test, we use the functions:

$$m(x) = 2 \quad \text{and } r(x) = 2 + \frac{1}{1+x};\ p(x) = 2 - \frac{1}{1+x};\ \forall x \in \Omega,$$

$$\alpha(t) = \beta(t) = \frac{1}{1+t};\ \forall t > 0.$$

- **TEST 3**: In Test 3, we examine the result explained in the example 2, we obtain a sub-exponential-type decay of the energy (E):

$$E_3(t) \leq E_f^3(t) = c_1 e^{-c_2\sqrt{t}},$$

for two positive constants c_1 and c_2. For this test, we use the functions:

$$\begin{aligned} m(x) = r(x) = 2;\ p(x) = \ 1 + \frac{1}{1+x};\ \forall x \in \Omega, \\ \alpha(t) = \beta(t) = \frac{1}{\sqrt{1+t}};\ \forall t > 0. \end{aligned}$$

- **TEST 4**: In Test 4, we examine the second result explained in our example 2, we obtain a polynomial-type decay of the energy (E):

$$E_4(t) \leq E_f^4(t) = \frac{c}{(1+t)^w},$$

for two positive constants c and w. For this test, we use the functions:

$$\begin{aligned} m(x) = r(x) = 2 + \frac{1}{1+x};\ p(x) = \ 1 + \frac{1}{1+x};\ \forall x \in \Omega, \\ \alpha(t) = \beta(t) = \frac{1}{\sqrt{1+t}};\ \forall t > 0. \end{aligned}$$

- **TEST 5**: In Test 5, we examine a result explained in the example 3, we obtain a logarithmic-polynomial decay of the energy (E):

$$E_5(t) \leq E_f^5(t) = \frac{c}{(\ln(2+t))^w},$$

for two positive constants c and w. For this test, we use the functions:

$$\begin{aligned} m(x) = r(x) = \ p(x) = \ 2;\ \forall x \in \Omega, \\ \alpha(t) = \frac{1}{(2+t)\ln(1+t)} \text{ and } \beta(t) = \frac{1}{((2+t)\ln(1+t))^2};\ \forall t > 0. \end{aligned}$$

It should be stressed that the numerical stability of the implemented method is ensured by taking in consideration the Courant-Friedrichs-Lewy (CFL) inequality $\Delta t << 0.5\Delta x$, where Δt represents the numerical time step and Δx the numerical spatial step. The spatial interval $\Omega =]0,1[$ is subdivided into 200 subintervals and the temporal interval $[0,T] = [0,20]$ is deduced from the stability condition above.

Using the Free Fem++ software in addition to Matlab, we run our code for 10000 time steps $\Delta t = 2 \cdot 10^{-3}$ under the following initial conditions:

$$\begin{aligned} u(x,0) = \sin(\pi x) \quad \text{and} \quad v(x,0) = -\sin(\pi x) \quad \text{in}\,]0,1[\,, \\ u_t(x,0) = 1 \quad \text{and} \quad v_t(x,0) = 1 \quad \text{in}\,]0,1[\,. \end{aligned}$$

Our computational simulations show in Figures 3.1–3.5(left) all decay types. We restrict ourselves to plot three cross-section cuts for the numerical solution (u,v) at $x = 0.25$, $x = 0.5$ and at $x = 0.75$. For all components of the solutions, the decay behavior is clearly demonstrated for all tests. Moreover, the dotted curves in Figures 3.1–3.5(right) represent the corresponding upper bound of the energy function $E_f^i(t)$ for $i = \ldots, 5$.

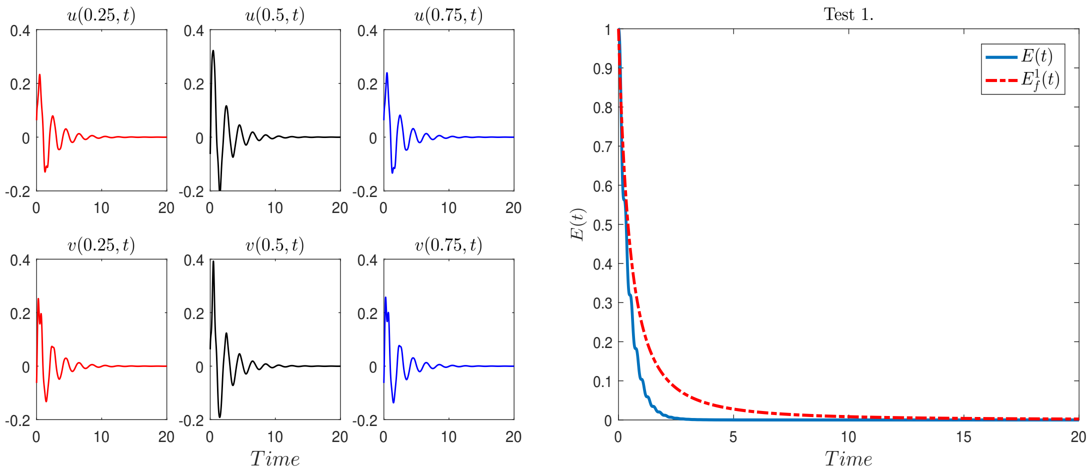


FIGURE 3.1: TEST 1: Damping cross section waves, energy decay and the upper bound function $E_f^2(t) = \dfrac{1}{(1+t)^2}$.

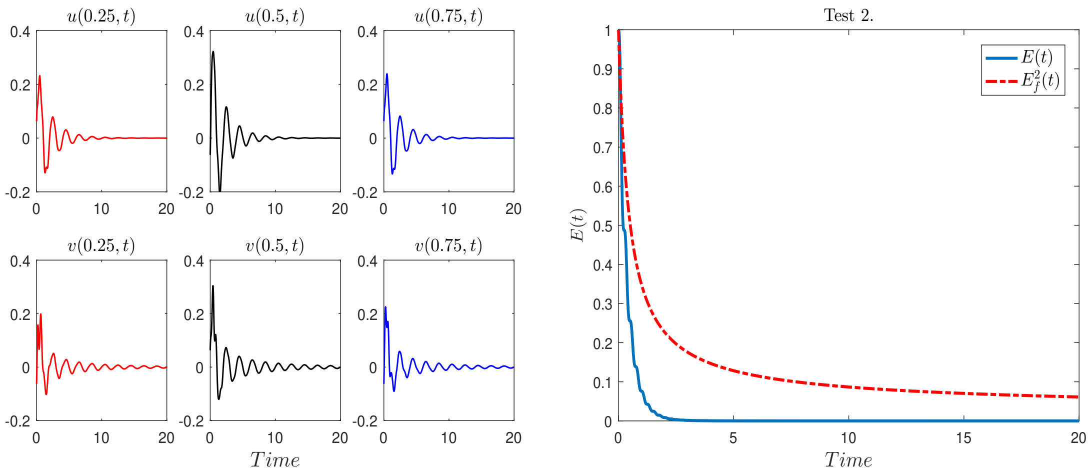


FIGURE 3.2: TEST 2: Damping cross section waves, energy decay and the upper bound function $E_f^2(t) = \dfrac{1}{(1+\ln(1+t))^2}$.

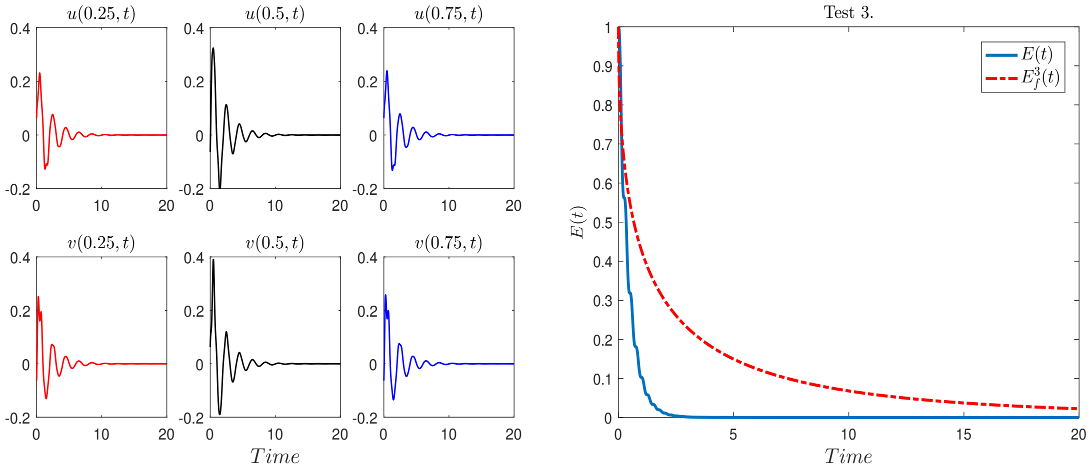


FIGURE 3.3: TEST 3: Damping cross section waves, energy decay and the upper bound function $E_f^3(t) = e^{-0.85\sqrt{t}}$.

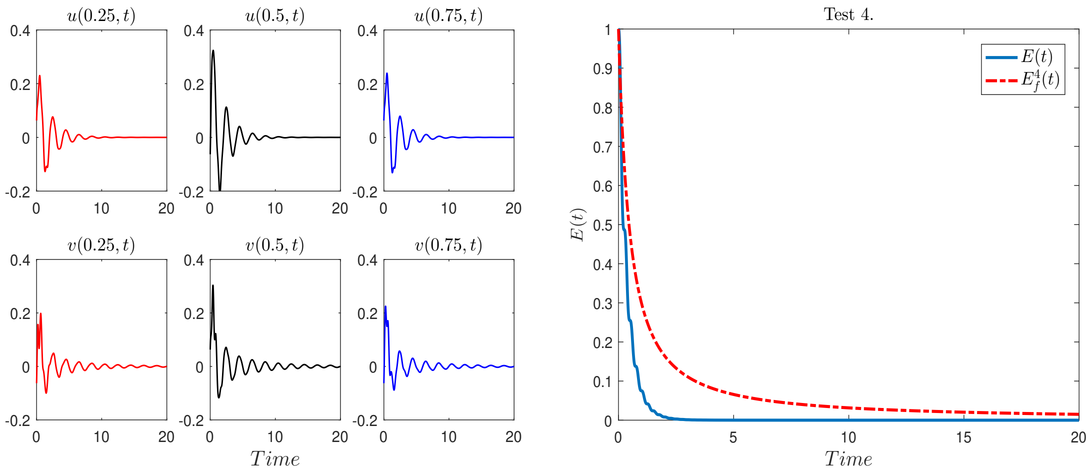


FIGURE 3.4: TEST 4: Damping cross section waves, energy decay and the upper bound function $E_f^4(t) = \dfrac{1}{(-1+2\sqrt{1+t})^2}$.

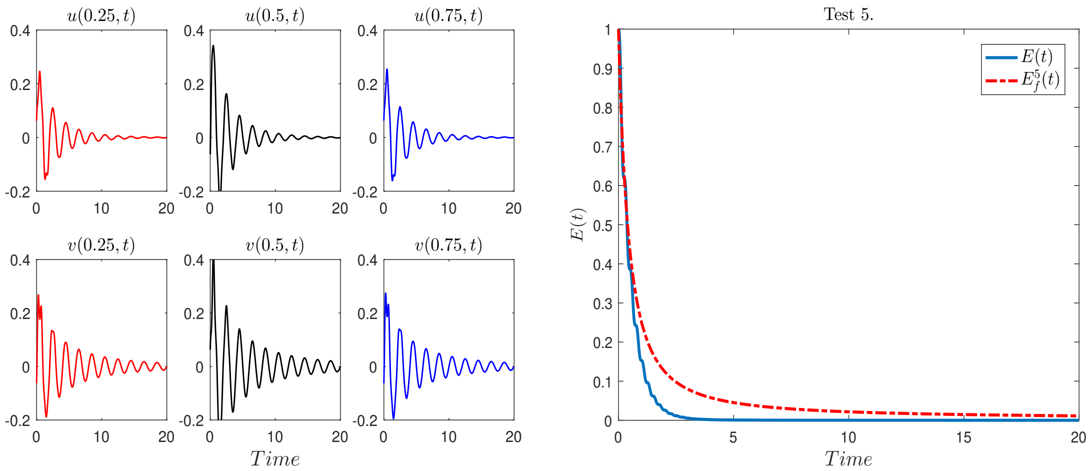


FIGURE 3.5: TEST 5: Damping cross section waves, energy decay and the upper bound function $E_f^5(t) = \dfrac{(\ln(2))^3}{(\ln(2+t))^3}$.

Conclusion and Open Problems

CONCLUSION:

In this work, we studied two coupled systems of nonlinear hyperbolic equations where the coupling and damping terms are of non-standard forms. Under appropriate assumptions, on the initial data and the exponents of nonlinearity, we obtained several results concerning existence, decay and blow up of solutions. For the first system, we proved an existence and uniqueness theorem of local weak solution, a finite time of blow up result, global existence and decay rates. We also gave some numerical tests to illustrate our theoretical results. Whereas, concerning the second system, which is a coupled system of two weakly damped hyperbolic equations, we established decay results in terms of the damping exponents and the coefficients. For the latter problem, we gave some examples and presented few numerical tests to illustrate our theoretical findings. All numerical tests came in agreement with the theoretical results and the particular examples.
Our results generalize and improve many previous results in the literature.

OPEN PROBLEMS:

The following open questions can be made regarding the material presented in this thesis.

1. Study of the existence of coupled systems with exponent variables for the case $n > 3$.
2. Study of certain problems with Biarmonic operateur (Existence, blow up and stability).
3. Study of some equations and systems in more general spaces (Besov, Orlicz, ...,etc).

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