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Abstract

The aim of this work is to give new necessary and sufficient conditions for the existence of solution and Hermitian solution to some operators equations in Banach and Hilbert spaces and give the general solutions and Hermitian solutions for the first time via the inner inverses of elementary operators and via simple operators. One of the main objective of this thesis is generalization of results of Alegra Dajić and J. J. Koliha in [15] and [16].

Résumé

Le but de ce travail est de donner de nouvelles conditions nécessaires et suffisantes d'existence des solutions et des solutions Hermitiennes de quelques équations opératorielles sur les espaces de Banach et de Hilbert, ensuite donner les formes des solutions et des solutions Hermitiennes pour la première fois via les inveres intérieurs des opérateurs élémentaires et via les opérateurs simple. L'un des principaux objectifs de cette thèse c'est la généralisation des résultats de Alegra Dajić et J. J. Koliha dans [15] and [16].

Key words: Banach spaces, Hilbert spaces, Inner inverse, Elementary operators, Operators equations, Hermitian solutions

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الملخص:

الهدف من هذا العمل هو إعطاء شروط جديدة ضرورية وكافية لوجود حلول وحلول هيرميتية لبعض المعادلات ذات العوامل المعرفة في فضاء بناخ وهيلبرت وإعطاء الحلول العامة والحلول الهيرميتية ولأول مرة عبر المعكوسات الداخلية للعوامل الأولية والعوامل البسيطة.

أحد الأهداف الرئيسية لهذه الرسالة هو تعميم نتائج Alegra Dajic and J. J. Koliha في [15] و [16].

الكلمات الدالة:

فضاء بناخ، فضاء هيلبرت، المعكوس الداخلي، العوامل الأولية، معادلات العوامل، الحلول الهيرميتية.

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Contents

General introduction

Many problems of the theory of linear operators or in other fields of mathematics such as algebra, numerical analysis, optimal control, spectral theory ..., are strongly linked to the notion of invertibility of the elements. This is the reason why some mathematicians have thought of introducing new notions of invertibility "Generalized inverses" which are useful for the solutions to these problems. Among them, we cite I. Fredholm, J. Von Neumann, M. Z Nashed, C. R. Caradus, J. J. Koliha... and many others.

The concept of generalized inverses seems to have been first mentioned in print in 1903 by Ivar Fredholm [22], who formulated a pseudo inverse for a linear integral operator which is not invertible in the ordinary sense.

One year later, in 1904 Hilbert made implicit use of pseudo-inverses when considering the theory of linear ordinary differential equations. In fact, he introduced the notion of the generalized Green's function which was the integral kernel of the pseudo-inverse of the differential operator.

In 1913, W. A. Hurwitz [24] reconsidered the same problem of Fredholm and used the finite dimensionality of the null-space of Fredholm operators to give a simple algebraic construction.

Generalized inverses of differential operators were consequently studied by numerous authors, in particular, Myller (1906), Westfall (1909), Bounitzky in 1909, Elliott (1928), and Reid (1931).

Generalized inverses of differential and integral operators thus antidated the generalized inverses of matrices whose existence was first noted in 1920 by E. H. Moore [42], who defined a unique inverse A^+ called by him the "general reciprocal" for every finite matrix (square or rectangular). E. H. Moore established the existence and uniqueness of A^+ for any A, and gave an explicit form for A^+ in terms of the sub-determinants of A and A^*. His work received practically no attention in the next 30 years, mostly because it used very complicated notation.

In 1951, Bjerhamar [6, 7, 8] recognized the least squares properties of certain generalized inverses and noted the relation between some generalized inverses and solutions to linear systems.

In 1955, Penrose [50] sharpened and extended Bjerhammar's results on linear systems, and showed that Moore's inverse, for a given matrix A, is the unique matrix X satisfying the following four equations

$$\begin{aligned} AXA &= A, \\ XAX &= X, \\ (AX)^* &= AX, \\ (XA)^* &= XA. \end{aligned}$$

The latter discovery has been so important that this unique inverse is now commonly called the Moore Penrose inverse. Since 1955 the theory of generalized inverses and their applications and computational methods have been developing rapidly. Many authors investigated various types of generalized inverses, also generalized inverses which satisfy some of the four Penrose's equations. Generalized inverses which satisfy some, but not all, of the four Penrose equations play an important role in solution of systems of linear equations. The generalized inverse which satisfy the first equation is noted $\{1\}$-inverse and is often called inner generalized inverse. Since we have used this generalized inverses in the general setting of Banach and Hilbert spaces, we can therefore give a general definition. Let E and F are Banach space and $\mathbb{B}(E, F)$ is the algebra of all bounded linear operators and let $A \in \mathbb{B}(E, F)$, then any solution B in $\mathbb{B}(F, E)$ of the equation

$$ABA = A$$

will be called a pseudo-inverse, or $\{1\}$-inverse or inner inverse of A. An operator with a pseudo-inverse will be called regular.

Many results have been obtained on the solvability of equations for matrices and operators on Hilbert and Banach spaces using generalized inverses, see for example [3, 4, 11, 14, 15, 16, 17, 23, 25, 36].

In 1973, Mitra [38] obtained a necessary and sufficient condition for the matrix equations

$$A_1XB_1 = C_1, \;\; A_2XB_2 = C_2$$

to have a common solution.

In 1976, Khatri and Mitra [25] considered matrix equations of various types

$$AX = C; \;\; AX = C, \;\; XB = D \text{ and } AXB = C$$

over the complex field and obtained conditions for Hermitian and non-negative definite solutions and gave explicit solutions based on generalized matrix equations.

In 1979, Phadke and Thakare [51] attempted to describe Hermitian, positive definite and semi-definite solutions for Hilbert space operators of

$$AX = C \text{ and } AXA^* = C,$$

but several of their results are incorrect or have incorrect proofs.

In 1998,J. Groß[23] gives a necessary and sufficient condition for consistency of this matrix equation

$$AXA^* = C$$

and gave a representation of its general Hermitian solution.

In 2004, Q. W. Wang derived the necessary and sufficient for the existence and the expression of the general solution to the same matrix equations over arbitrary regular rings with identity.

In 2007, Alegra Dajić and J.J. Koliha [16] gives conditions for the existence of Hermitian solutions and positive solutions for Hilbert spaces of some operator equations

$$AX = C \text{ and } XB = D,$$

and obtained the formula for the general form of these solutions and corrected the results of Phadke and Thakare [51].

In 2008, Alegra Dajić and J.J. Koliha [15] reviews the precedent equations from a new perspective by studying them in the setting of associative rings with or without involution.

In 2011, N. Li, J. Jiang and W. Wang considered Hermitian solution and skew-Hermitian solutions to a quaternion matrix equation

$$AXA^* + BYB^* = C,$$

In 2015, Alegra Dajić [14] gives conditions for the existence of solutions for some operator equations between Banach spaces and obtains the formula for the general solutions.

The aim of this thesis is to present new necessary and sufficient conditions for the existence of a common solution of the operator equations $M_{A_1,B_1}(X) = C_1$ and $M_{A_2,B_2}(X) = C_2$, using for the first time the generalized inverses of elementary operators, where E, F, G and D are infinite complex Banach spaces and $A_1, A_2 \in \mathbb{B}(F,E)$, $B_1, B_2 \in \mathbb{B}(D,G)$, and M_{A_1,B_1}, M_{A_2,B_2} are the multiplication operators defined on $\mathbb{B}(G,F)$ by $M_{A_1,B_1}(X) = A_1XB_1$ and $M_{A_2,B_2}(X) = A_2XB_2$ and we derive a new and for the first time a representation of the general common solution via the inner inverse of the elementary operator $\Psi = M_{A_1,B_1} + M_{A_2,B_2}$, we apply this result to determine new necessary and sufficient conditions for the existence of a Hermitian solution and a representation of the general Hermitian solution to the operator equation $M_{A,B}(X) = C$, where A, B and C are bounded linear operators on Hilbert spaces. As consequence, we obtain well-known results of Alegra Dajić and J. J. Koliha in [15]. We consider the same system

$$A_1XB_1 = C_1, \quad A_2XB_2 = C_2,$$

where A_1, A_2, B_1, B_2, C_1 and C_2 are linear bounded operators on Hilbert spaces, and we give other necessary and sufficient conditions for the existence of a common solution, we apply the result to determine new necessary and sufficient conditions for the existence of a Hermitian solution and a representation of the general Hermitian solution to the operators equations $AXB = C$, and $AXA^* + BYB^* = C$.

The thesis is organized in four chapters.

Chapter 1 is a reminder on essential notions about inner inverses and classes of bounded linear operators, first we present some basic and important properties of projections, second we introduce basic concepts of inner inverses and we recall its algebraic and topological properties in Banach and Hilbert spaces. We also recall the definitions of some classes of bounded linear operators Hilbert spaces. Finally the convergence and stability concepts of operators and their characterizations are given.

Chapter 2 is divided into 3 sections. In section 2.1, we present the inner inverse for elementary operators in Banach space. In section 2.2, we give new necessary and sufficient conditions for the existence of a common solution of the operator equations

$$M_{A_1,B_1}(X) = C_1 \textit{ and } M_{A_2,B_2}(X) = C_2,$$

using for the first time the inner inverses of the elementary operators and derive a new representation of the general common solution via the inner inverse of the elementary operator $\Psi = M_{A_1,B_1} + M_{A_2,B_2}$ in Banach space. In section 2.3, we apply the previous result to determine new necessary and sufficient condition for the existence of a Hermitian solution and a representation of the general Hermitian solution to the operator equation

$$M_{A,B}(X) = C.$$

in Hilbert space. As consequence, we obtain well known results of Alegra Dajić and J.J. Koliha in [15].

Chapter 3 is divided into 3 sections. In section 3.1, we give new necessary and sufficient conditions for the existence of common solutions to the operator equations

$$A_1XB_1 = C_1, \quad A_2XB_2 = C_2,$$

where A_1, A_2, B_1, B_2, C_1 and C_2 are linear bounded operators defined on Hilbert spaces. In section 3.2, we apply the previous result to determine new necessary and sufficient conditions for the existence of a Hermitian solution and a representation of the general Hermitian solution to the operator equation

$$AXB = C,$$

where A, B and C are linear bounded operators. In section 3.3, we deduce necessary and sufficient condition for the existence of Hermitian solution to the operator equation

$$AXA^* + BYB^* = C,$$

where A, B and C are linear bounded operators.

Chapter 4 This chapter is independent of the two previous ones. We present a study developed by O. L. V. Costa and C. S. Kubrusly [13] "Lyapunov equation for infinite dimensional discrete bilinear systems", published in Systems & Control Letters 17(1991) pp 71-77. The first section of this chapter is about preliminaries, we recall some concepts needed in this chapter. The second section we shall conclude the proof for the equivalence between the assertions

- $r_\sigma(\mathcal{M}) < 1$.
- for every $Y \in \mathbb{G}^+(H)$ there exists a unique solution $X \in \mathbb{G}^+(H)$ for

$$\text{the Lyapunov equation } Y = X - \mathcal{M}(X),$$

where $\mathcal{M}$ is the operator defined on $\mathbb{B}(H)$ (H is a separable complex Hilbert space) by

$$\mathcal{M}(X) = M_{A,A^*}(X) + \sum_{k,l=1}^{\infty} \langle Ce_l, e_k \rangle M_{A_k,A_l^*}(X),$$

and C is a non-negative nuclear operator. This supplies a necessary and sufficient condition for the convergence preserving property between input and state correlation sequences, as required in the mean-square stability problem, for infinite-dimensional discrete bilinear systems.

0.1 Terminology

1. Let E and F be infinite complex Banach spaces
   (a) $\mathbb{B}(E, F)$ be the set of all bounded linear operators from a Banach space E into F.
   (b) $\mathbb{G}(E, F)$ be the set of all invertible operators from $\mathbb{B}(E, F)$.
2. Let H be a complex Hilber space
   (a) $\mathbb{B}^+(H)$ be the set of all self adjoint non-negative operators.
   (b) $\mathbb{G}^+(H)$ be the set of all strictly positive operators.
   (c) $\mathbb{B}_\infty(H)$ be the set of all compact operators.
   (d) $\mathbb{B}_1$ be the set of all nuclear operators.
   (e) $\mathbb{B}_1^+$ be the set of all non-negative nuclear operators.
3. Let $A \in \mathbb{B}(H)$
   (a) A^* the adjoint of A.
   (b) A^- the inner inverse of A
   (c) $\mathcal{R}(A)$ the range of A.
   (d) $\mathcal{N}(A)$ the kernel of A.
   (e) $\sigma(A)$ the spectrum of A.
   (f) $r_\sigma(A)$ the spectral radius of A.
   (g) $\|.\|$ the norm.
   (h) $\langle .; . \rangle$ inner product.
4. Elementary operators.
   Let D and G be two other Banach spaces. Consider $A_1, A_2 \in \mathbb{B}(F, E)$, $B_1, B_2 \in \mathbb{B}(D, G)$.
   (a) The multiplication operator on $\mathbb{B}(G, F)$ induced by A_1, B_1 is
   $$M_{A_1,B_1} : X \to A_1XB_1.$$
   (b) In particular $L_{A_1} = M_{A_1,I}$ and $R_{B_1} = M_{I,B_1}$, where I is the identity operator are the left and the right multiplication operators, respectively.
   (c) The elementary operator Ψ defined on $\mathbb{B}(G, F)$ is the sum of two multiplication operators
   $$\Psi = M_{A_1,B_1} + M_{A_2,B_2}.$$

Chapter 1

Preliminaries

In this introductory chapter, we will introduce some basic concepts and well-known results that facilitate the understanding of this thesis, in particular the projections, as well as some basic notions and theorems of inner inverses of operators in Banach and Hilbert spaces. We also recall the spectrum and the classes of linear bounded operators in Hilbert space that will be used throughout this thesis. Most contents of chapter 1 are taken from [10], [44], [14],[15], [32], [57] and [47].

1.1 Projections

Let E, F be Banach spaces and let $\mathbb{B}(E, F)$ denote the set of all linear bounded operators from E to F.

Proposition 1.1.1. *Each projection P determines a direct sum decompsition of E, namely*

$$E = \mathcal{R}(P) \oplus \mathcal{N}(P). \tag{1.1}$$

Conversely every direct sum decomposition of E determines a projection.

Proof. It is clear that $E = \mathcal{R}(P) + \mathcal{N}(P)$, since each $x \in E$ may be written in the form

$$x = Px + (x - Px).$$

Furthermore, $\forall x \in \mathcal{R}(P)$ are characterized by the fact that $Px = x$.

So, if $x \in \mathcal{R}(P) \cap \mathcal{N}(P)$, then $x = Px = 0$, that is $\mathcal{R}(P) \cap \mathcal{N}(P) = \{0\}$. This proves (1.1). Conversely let $E = M \oplus N$, then $\forall x \in E$ may be written uniquely in the form $x = x_1 + x_2$ with $x_1 \in M$ and $x_2 \in N$. If we define P by $Px = x_1$, then it is clear that P is a linear operator such $\mathcal{R}(P) = M$, $\mathcal{N}(P) = N$, and $P^2 = P$. We call P a projection of E onto M along N. □

Remark 1.1.1. *The operator $(I - P)$ is also a projection of E onto N along M.*

Definition 1.1.1. *A subspace M of a Banach space E is said to have a complemented subspace if there exists a subspace N such that $E = M \oplus N$.*

Theorem 1.1.1. *Let M and N be closed subspaces such that $E = M \oplus N$. Then the projection P of E onto M along N is continuous.*

Proof. Because of the closed graph Theorem it suffices to prove that P is a closed operator. Suppose that

$$x_n \to x \quad \textit{and} \quad Px_n \to y.$$

Then

$$x_n - Px_n \to x - y.$$

Since $Px_n \in M$ and $x_n - Px_n \in N$, it follows that

$$y \in M \quad \textit{and} \quad x - y \in N = \mathcal{N}(P).$$

Then

$$Px - Py = 0 \quad \textit{and} \quad Px = Py = y.$$

Thus P is closed. □

Remark 1.1.2. *The decomposition $E = M \oplus N$ in Theorem 1.1.1 is said to be topological direct sum because M and N are both closed subspaces.*

Lemma 1.1.1.

1. *The range of a continuous projector P on a Banach space E is closed.*

2. *A closed subspace of a Banach space E is complemented if and only if it is the range of some continuous projector in E.*

Proof.

1. Since $\mathcal{R}(P) = \mathcal{N}(I - P)$, thus $\mathcal{R}(P)$ being the nullspace of a continuous linear operator is a closed subspace.

2. Let M be a closed subspace of E. If $M = \mathcal{R}(P)$ for some continuous projector P on E, then $E = \mathcal{R}(P) \oplus \mathcal{N}(P)$ and $\mathcal{N}(P)$ is closed. Thus M is complemented.

   Conversely, if M is complemented, let P be the continuous projector of E onto M, and the result follow.

□

1.2 Inner inverses in Banach spaces

Definition 1.2.1. *Let $A \in \mathbb{B}(E,F)$. An operator $S \in \mathbb{B}(F,E)$ is said to be an inner inverse of A if it satisfies the equation*

$$ASA = A. \tag{1.2}$$

We denote the inner inverse by A^-.

Remark 1.2.1.

1. *$A \in \mathbb{B}(E,F)$ has an inner inverse iff $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are closed and complemented subspaces of E and F respectively.*
2. *If A has an inverse A^{-1} in $\mathbb{B}(E,F)$, then A^{-1} is the only inner inverse of A.*

Definition 1.2.2. *An operator $A \in \mathbb{B}(E,F)$ is called regular if A^- exists.*

Definition 1.2.3. *Let $A \in \mathbb{B}(E,F)$. An operator $S \in \mathbb{B}(F,E)$ is said to be an outer inverse of A if it satisfies the equation*

$$SAS = S. \tag{1.3}$$

Lemma 1.2.1. *If S is an inner inverse of A, then the operator SAS satisfies both equations (1.2) and (1.3).*

Proof. The proof of this assertion is a simple verification. If $ASA = A$, then

$$A(SAS)A = (ASA)SA = ASA = A,$$

and

$$(SAS)A(SAS) = S(ASA)SAS = S(ASA)S = SAS.$$

□

Theorem 1.2.1. *[10] Let $A \in \mathbb{B}(E,F)$. Then*

1. *If $S \in \mathbb{B}(F,E)$ is an inner and outer inverse of A, then AS is a projection of F onto $\mathcal{R}(A)$ along to $\mathcal{N}(S)$ and SA is a projection of E onto $\mathcal{R}(S)$ along to $\mathcal{N}(A)$.*
2. *If $S \in \mathbb{B}(F,E)$ is an inner and outer inverse of A, then $\mathcal{R}(S)$ is a closed complemented subspace of $\mathcal{N}(A)$ and $\mathcal{N}(S)$ is a closed complemented subspace of $\mathcal{R}(A)$.*

3. *Suppose that $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are closed and complemented subspaces of E and F respectively, P is a projection onto $\mathcal{R}(A)$ and M is a complementary subspace to $\mathcal{N}(A)$. Then $S = A_1^{-1}P$ is an inner inverse of A, where A_1 is the restriction of A to M.*

Proof.

1. Suppose that A has an inner and outer inverse, so that there exists an operator S which satisfies equation (1.2) and (1.3).

   Then, since
   $$(AS)^2 = ASAS = AS,$$
   and
   $$(SA)^2 = SASA = SA,$$
   it is clear that AS and SA are projection.

   Clearly, $\mathcal{R}(AS) \subseteq \mathcal{R}(A)$. Conversely, for each $y \in \mathcal{R}(A)$ there exists $x \in E$ such that $y = Ax$, we can write
   $$Ax = ASAx,$$
   so that
   $$\mathcal{R}(A) \subseteq \mathcal{R}(AS).$$
   In a similar way, we have $\mathcal{N}(S) \subseteq \mathcal{N}(AS)$ and if $ASx = 0$, then from equation (1.3) we know that
   $$Sx = SASx = 0,$$
   so that
   $$\mathcal{N}(AS) \subseteq \mathcal{N}(S).$$
   This means that, AS is a projection onto $\mathcal{R}(A)$ along to $\mathcal{N}(S)$.

   On the other hand, clearly $\mathcal{R}(SA) \subseteq \mathcal{R}(S)$. Conversely, for each $y \in \mathcal{R}(S)$ such that $y = Sx$ we can write
   $$Sx = SASx,$$
   so that
   $$\mathcal{R}(S) \subseteq \mathcal{R}(SA).$$
   In a similar way, we have $\mathcal{N}(A) \subseteq \mathcal{N}(SA)$ and if $SAx = 0$, then from equation (1.2) we know that
   $$Ax = ASAx = 0,$$
   so that
   $$\mathcal{N}(SA) \subseteq \mathcal{N}(A).$$
   This means that SA is a projection onto $\mathcal{R}(S)$ along to $\mathcal{N}(A)$.

2. According to property 1, there exist projections $SA \in \mathbb{B}(E)$ and $AS \in \mathbb{B}(F)$ such that
$$\mathcal{N}(SA) = \mathcal{N}(A), \qquad \mathcal{R}(AS) = \mathcal{R}(A),$$
On other hand, There exist closed subspaces $\mathcal{R}(S)$ and $\mathcal{N}(S)$ such that
$$E = \mathcal{N}(A) \oplus \mathcal{R}(S), \qquad F = \mathcal{N}(S) \oplus \mathcal{R}(A),$$
Thus, $\mathcal{R}(S)$ is a closed complemented subspace of $\mathcal{N}(A)$ and $\mathcal{N}(S)$ is a closed complemented subspace of $\mathcal{R}(A)$.

3. Suppose that $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented subspaces in E and F respectively
$$E = \mathcal{N}(A) \oplus M, \qquad F = N \oplus \mathcal{R}(A), \tag{1.4}$$
let $A_1 = A/M$, and let P be the continuous projection of F onto $\mathcal{R}(A)$ along N. Since A_1 is a bijective map of M onto $\mathcal{R}(A)$, which are both closed subspaces, it follows from the inverse mapping theorem that A_1^{-1} is continuous, hence $A_1^{-1}P$ is also continuous. Clearly $A(A_1^{-1}P)A = A$. This proves that $S = A_1^{-1}P$ is a bounded inner inverse for A.

□

Remark 1.2.2. *Note that if S is an inner inverse and not an outer inverse, then the arguments in the first and second part of Theorem 1.2.1, can be applied to the operator SAS to show that the conclusions are true in general.*

Theorem 1.2.2. *[57] Let $A \in \mathbb{B}(E, F)$. Then the following conditions are equivalent:*

1. *There exist projections $P \in \mathbb{B}(E)$ and $Q \in \mathbb{B}(F)$ such that*
$$\mathcal{R}(P) = \mathcal{N}(A), \qquad \mathcal{R}(Q) = \mathcal{R}(A), \tag{1.5}$$

2. *There exist closed subspaces M and N such that*
$$E = \mathcal{N}(A) \oplus M, \qquad F = N \oplus \mathcal{R}(A), \tag{1.6}$$

3. *A has an inner inverse.*

Proof.
(1) ⇒ (2) Clear from Lemma 1.1.1.

(2) $\Rightarrow$ (3) Suppose that M and N are closed subspaces such that

$$E = \mathcal{N}(A) \oplus M, \qquad F = N \oplus \mathcal{R}(A),$$

from the third part of Theorem 1.2.1, $S = A_1^{-1}P$ is an inner inverse of A. Thus (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) Suppose that S is an inner inverse of A. Then from Theorem 1.2.1, SA is a projection and $\mathcal{N}(A) = \mathcal{N}(SA)$, also AS is a projection and $\mathcal{R}(A) = \mathcal{R}(AS)$. So (3) $\Rightarrow$ (1), with $P = I - SA$ and $Q = AS$. $\square$

Examples

1. All projections are regular, in fact, if $P^2 = P$ then P is its own inner inverse.
2. A compact operator K with infinite dimensional range does not admit a bounded inner inverse since $\mathcal{R}(K)$ is not closed.
3. Let A be a linear operator of finite rank ($\dim \mathcal{R}(A) < \infty$). Then A has a bounded inner inverse. Because $\mathcal{R}(A)$ being finite dimensional, is complemented in F. Clearly A is bounded, $\mathcal{N}(A)$ is closed and the quotient space $E/\mathcal{N}(A)$ is finite dimensional, from which it follows that $\mathcal{N}(A)$ has a finite dimensional complement; hence $\mathcal{N}(A)$ is topologically complemented.
4. Let $A : E \to F$ be a Fredholm operator. Then A has a bounded inner inverse. This follows from Theorem 1.2.2, since $\mathcal{N}(A)$ is complemented in E, and $\mathcal{R}(A)$ is closed and has finite codimension, so $\mathcal{R}(A)$ is complemented in F. In particular , any operator of the form $A = I - K$, where K is compact has a bounded inner inverse.

Remark 1.2.3. *We have seen in Theorem 1.2.2 how to characterize the set of inner inverses in term of the formula $S = A_1^{-1}P$. However, in many situations, this is not so useful since we may not be able to describe all the projections onto $\mathcal{R}(A)$ and $\mathcal{N}(A)$. We are able to describe the set of inner inverses in another way; we first need a simple lemma.*

Lemma 1.2.2. *[10] Let $A, B \in \mathbb{B}(E, F)$ are regular operators and $C \in \mathbb{B}(E, F)$, then the operator equation*

$$AXB = C, \tag{1.7}$$

has a solution if and only if

$$AA^{-}CB^{-}B = C. \tag{1.8}$$

In which case, the general solution is

$$X = A^{-}CB^{-} + U - A^{-}AUBB^{-}, \tag{1.9}$$

where $U \in \mathbb{B}(F, E)$ is an arbitrary operator.

Proof. Suppose that X is a solution of equation (1.7), then

$$C = AXB = AA^{-}AXBB^{-}B = AA^{-}CB^{-}B,$$

so the condition (1.8) is necessary. If $AA^{-}CB^{-}B = C$ holds, then $X_0 = A^{-}CB^{-}$ is a solution of the operator equation (1.7).

Now suppose that the condition (1.8) holds. Then clearly every operator of the form (1.9) is a solution of (1.7).

Conversely, if X is a solution of (1.7) then $X - A^{-}CB^{-}$ is a solution of the equation $AXB = 0$.

Let $X_0 = X - A^{-}CB^{-}$, so that $X_0 = X_0 - A^{-}AX_0BB^{-}$, then

$$X - A^{-}CB^{-} = X_0 - A^{-}AX_0BB^{-},$$

from which (1.9) follows. □

Corollary 1.2.1. *Let $A \in \mathbb{B}(E, F)$ is regular operator and $C \in \mathbb{B}(E, F)$, the operator equation*

$$AX = C,$$

has a solution if and only if

$$AA^{-}C = C.$$

In which case, the general solution is

$$X = A^{-}C + (I_E - A^{-}A)U,$$

where $U \in \mathbb{B}(E)$ is an arbitrary operator.

Corollary 1.2.2. *Let $B \in \mathbb{B}(E, F)$ is regular operator and $D \in \mathbb{B}(E, F)$, then the operator equation*

$$XB = D,$$

has a solution if and only if

$$DB^{-}B = D.$$

In which case, the general solution is

$$X = DB^{-} + U(I_F - BB^{-}),$$

where $U \in \mathbb{B}(F)$ is an arbitrary operator.

Theorem 1.2.3. *If $A \in \mathbb{B}(E, F)$ has an inner inverse $A^{-} \in \mathbb{B}(F, E)$ satisfying equations*

$$AA^{-}A = A \quad \text{and} \quad A^{-}AA^{-} = A^{-},$$

then the set of inner inverses of A consists of all operators of the form

$$A^{-} + U - A^{-}AUAA^{-},$$

where $U \in \mathbb{B}(F, E)$ is an arbitrary operator.

Proof. We know that $AA^-A = A$. From Lemma 1.2.2, we conclude that the other inner inverses are given by

$$A^-AA^- + U - A^-AUAA^- = A^- + U - A^-AUAA^-,$$

where $U \in \mathbb{B}(F, E)$ is an arbitrary operator □

Remark 1.2.4.

- *In [1], Arens considers the problem of finding all left inverses i.e. all solutions of the equation*
$$BA = I.$$
He shows that if B_0 is any left inverse, then the family of left inverses is given by
$$B = B_0 + V(I - AB),$$
where V is an arbitrary operator.

Corollary 1.2.3. *Suppose that M is a closed complemented subspace of Banach space E and that P_0 is a projection of E onto M. Then the family of projections of E onto M is given by*
$$P = P_0 + P_0U(I - P_0),$$
where $U \in \mathbb{B}(E)$ is an arbitrary operator.

Proof. Clearly P_0 is its own inner inverse. Hence, from the Theorem 1.2.3 all other inner inverses of P_0 can be written

$$\begin{aligned} B &= P_0 + U - P_0^2UP_0^2, \\ &= P_0 + U - P_0UP_0. \end{aligned}$$

But from such an inner inverse, we can obtain a projection onto M:

$$\begin{aligned} P &= P_0B, \\ &= P_0 + P_0U - P_0UP_0, \\ &= P_0 + P_0U(I - P_0). \end{aligned}$$

Since all projections can be obtained by this way, the result follows. □

1.3 Inner inverses in the Hilbert spaces case

In Hilbert space, it is well-known that every closed subspace is complemented. Therefore an operator in Hilbert space is regular if and only if it has closed range. Let H, K be Hilbert spaces and let $\mathbb{B}(H,K)$ denote the set of all linear bounded operators from H to K.

Lemma 1.3.1. *Let $A \in \mathbb{B}(H,K)$ be regular operator, M and N are complemented subspaces of $\mathcal{R}(A)$ and $\mathcal{N}(A)$ respectively. Then there exist an inner inverse $A^- \in \mathbb{B}(K,H)$ of A, such that*

1. $(I - A^-A)$ *is a projection onto* $\mathcal{N}(A)$.
2. $\mathcal{R}(AA^-) = \mathcal{R}(A)$.
3. $\mathcal{N}(AA^-) = M$.
4. $\mathcal{R}(I - A^-A) = \mathcal{N}(A^-A) = \mathcal{N}(A)$.
5. $\mathcal{N}(I - A^-A) = \mathcal{R}(A^-A) = N$.

Theorem 1.3.1. *[44] Let $A \in \mathbb{B}(H,K)$. The following statements are equivalent*

1. *A has a bounded inner inverse.*
2. *A^* has a bounded inner inverse.*
3. *$\mathcal{R}(A)$ is closed.*
4. *$\mathcal{R}(A^*)$ is closed.*
5. *T has a bounded right inverse on $\mathcal{R}(A)$.*
6. *The restriction of A to $\mathcal{N}(A)^\perp$ has a bounded inverse.*

Before giving the inner inverses of matrix of operators we recall that if H and K are Hilbert spaces, then the Cartesian product space $H \times K$ is itself a Hilbert space and $H \times K$ will be denoted by $H \oplus K$.

Lemma 1.3.2. *[47] Suppose that E is another Hilbert space. Let $A \in \mathbb{B}(H,K)$, $B \in \mathbb{B}(H,E)$ are regular operators, then $\begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{B}(H, K \oplus E)$ is regular if only and if $\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$ is regular.*

Proof. Suppose that $\begin{pmatrix} A^- & B^- \end{pmatrix}$ is an inner inverse of $\begin{pmatrix} A \\ B \end{pmatrix}$, then

$$\begin{pmatrix} A \\ B \end{pmatrix}\begin{pmatrix} A^- & B^- \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix},$$

this implies that the equalities $A(A^-A + B^-B) = A$ and $B(A^-A + B^-B) = B$ holds. So,

$$\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} A^- & B^- \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix},$$

thus $\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$ is regular.

Conversely, if $\begin{pmatrix} A^- & B^- \\ 0 & 0 \end{pmatrix}$ is an inner inverse of $\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$, then the equalities $A(A^-A + B^-B) = A$ and $B(A^-A + B^-B) = B$ holds. So

$$\begin{pmatrix} A \\ B \end{pmatrix}\begin{pmatrix} A^- & B^- \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix},$$

thus, $\begin{pmatrix} A \\ B \end{pmatrix}$ is regular. □

Lemma 1.3.3. [47] *Let $A \in \mathbb{B}(H, K)$ and $B \in \mathbb{B}(E, K)$ be regular operators. Then*

$\begin{pmatrix} A & B \end{pmatrix} \in \mathbb{B}(H \oplus E, K)$ is regular if and only if $S = (I_K - AA^-)B$ is regular.

In this case

$$\begin{pmatrix} A & B \end{pmatrix}^- = \begin{pmatrix} A^- - A^-BS^-(I_K - AA^-) \\ S^-(I_K - AA^-) \end{pmatrix}. \tag{1.10}$$

Lemma 1.3.4. [16] *Let $A \in \mathbb{B}(H, K)$ and $B \in \mathbb{B}(H, E)$ be regular operators. Then the regularity of any one of the following operators implies the regularity of the remaining three operators:*

$$D = B(I_H - A^-A),\ M = A(I_H - B^-B),\ \begin{pmatrix} A \\ B \end{pmatrix} \text{ and } \begin{pmatrix} B \\ A \end{pmatrix}.$$

In this case an inner inverse of $\begin{pmatrix} A \\ B \end{pmatrix}$ is given by

$$\begin{pmatrix} A \\ B \end{pmatrix}^- = \begin{pmatrix} (I_H - B^-B)M^- & B^- - (I_H - B^-B)M^-AB^- \end{pmatrix}. \tag{1.11}$$

1.4 The spectrum of bounded linear operators

Let $A \in \mathbb{B}(E)$. For $\lambda \in \mathbb{C}$, if the operator $\lambda I - A$ has an inverse, which is linear, we denote it $R(\lambda, A)$ the inverse operator, that is

$$R(\lambda, A) = (\lambda I - A)^{-1}, \tag{1.12}$$

and call it the resolvent operator of A at λ. The name resolvent is appropriate, since $R(\lambda, A)$ helps to solve the equation

$$(\lambda I - A)x = y.$$

Thus

$$x = R(\lambda, A)y,$$

provided $R(\lambda, A)$ exists. More important, the investigation of properties $R(\lambda, A)$ will be basic for an understanding of the operator A itself. Naturally, many properties of $(\lambda I - A)$ (or simply $(\lambda - A)$) and $R(\lambda, A)$ depend on λ, and spectral theory is concerned with those properties. For instance, we shall interested in the set of all λ in the complex plane such that $R(\lambda, A)$ exists and bounded. For our investigation of $R(\lambda, A)$, we shall need some basic concepts in the spectral theory which are given as follows

Definition 1.4.1. *The resolvent set of A is*

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda - A) \text{ has an inverse in } \mathbb{B}(E)\}, \tag{1.13}$$

its complement

$$\sigma(A) = \mathbb{C} \setminus \rho(A), \tag{1.14}$$

is called the spectrum of A. The number

$$r_\sigma(A) = sup\{|\lambda|; \lambda \in \sigma(A)\}, \tag{1.15}$$

is called the spectral radius of A.

for further reference [32, 57] we collect some important facts about spectrum, resolvent operator and spectral radius in the following theorem.

Theorem 1.4.1. *Let $A \in \mathbb{B}(E)$.*

1. *The resolvent identities*

$$\forall \lambda, \mu \in \rho(A), \;\; R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A),$$

moreover $R(\lambda, A)$ and $R(\mu, A)$ commute for $\lambda,\ \mu \in \rho(A)$.

2. *if* $\lambda \in \rho(A)$ *and* $|\lambda - \mu| \leq \|R(\lambda, A)\|^{-1}$, *then* $\mu \in \rho(A)$. *Thus* $\rho(A)$ *is open set in* $\mathbb{C}$.

3. *The resolvent is an analytic map. Moreover*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n (R(\lambda_0, A))^{n+1},$$

for all $\lambda_0 \in \rho(A)$ *such that* $|\lambda - \lambda_0| < \|R(\lambda_0, A\|^{-1}$.

4. $\sigma(A)$ *is closed in* $\mathbb{C}$.

Theorem 1.4.2. *Let* $A \in \mathbb{B}(E)$. *Then*

1. $\sigma(A)$ *is nonempty compact set in* $\mathbb{C}$.

2. *The spectral radius is given by the Gelfand formula*

$$r_\sigma(A) \;=\; \lim_{n \longrightarrow +\infty} \|A^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|A^n\|^{\frac{1}{n}}.$$

3. *We have*

$$r_\sigma(A) \leq \|A\|,$$

equality holdes, for example, if E *is a Hilbert space and* A *is normal, it means commutes with its adjoint.*

4. *The Neumann series*

$$R(\lambda, A) = \frac{1}{\lambda}(I - \frac{1}{\lambda}A)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n$$

converges in $\mathbb{B}(E)$ *for each* $\lambda \in \mathbb{C}$ *with* $|\lambda| > r_\sigma(A)$.

5. *for every* λ *with* $|\lambda| > \|A\|$ *we have* $\lambda \in \rho(A)$ *and*

$$\|R(\lambda, A)\| \leq \frac{1}{|\lambda| - \|A\|}. \tag{1.16}$$

Note that the spectrum of bounded operator is never empty nor equal to $\mathbb{C}$.

example 1.4.1. *Let* $E = \ell^1(\mathbb{Z})$ *be the space of all summable complex sequences*

$$x = (x_n)_n = (..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...),$$

indexed by the integers, with the usual norm. For any $\epsilon \in \mathbb{R}$, *let* $A_\epsilon \in \mathbb{B}(E)$ *be defined by* $A_\epsilon(x) = y$, *where* $x = (x_n)_n$ *and* $y = (y_n)_n$ *are related by*

$$y = \begin{cases} x_{k-1} & \text{if } k \neq 0 \\ \epsilon x_{-1} & \text{if } k = 0 \end{cases}$$

Then, we have

$$\sigma(A_0) = \overline{\mathbb{D}},$$

where $\mathbb{D}$ *denotes the open complex unit disc. On the other hand,*

$$\sigma(A_\epsilon) = \mathbb{S} = \partial\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}(\epsilon \neq 0).$$

So, here the spectrum collapses when ϵ *changes from zero to a nonzero value.*

The spectrum $\sigma(A)$ is partitioned into three disjoint sets as follows:

- **The point spectrum** $\sigma_p(A)$ **of** A, is the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not injective. $\lambda \in \sigma_p(A)$ is called an eigenvalue of A and for this λ there exists a non zero vector x such that $Ax = \lambda x$ called an eignvector corresponding to λ.

- **The continuous spectrum** $\sigma_c(A)$ **of** A, is the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is injective but its range is not closed.

- **The residual spectrum** $\sigma_r(A)$ **of** A, is the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is injective but its range is not dense in E.

Remark 1.4.1.

1. *If* E *is finite dimension, then* $\sigma_c(A) = \sigma_r(A) = \phi$.

2. *If* E *is a Hilbert space and* A *is a self-adjoint operator then* $\sigma_r(A) = \phi$.

3. *If* $\lambda \in \sigma_c(A)$, *then* λ *is not eigenvalue of* A *or of* A^*.

1.5 The classes of bounded linear operators

In this section we will investigate some classes of bounded linear operators A on a complex Hilbert spaces.

Definition 1.5.1. *Let* $A \in \mathbb{B}(H, K)$, *then the unique operator* $A^* \in \mathbb{B}(K, H)$ *such that*

$$\forall x \in H,\ \forall y \in K : \langle Ax, y\rangle = \langle x, A^*y\rangle,$$

is called the adjoint of A.

Definition 1.5.2. *An operator $A \in \mathbb{B}(H)$ is said to be*

1. *Hermitian (or self-adjoint): if $A^* = A$.*
2. *strict positive: if and only if $\exists \gamma > 0$, such that $\langle Ax, x\rangle \geq \gamma \|x\|^2$, $\forall x \in H$.*
3. *Unitary: if $A^*A = I = AA^*$, where I is the identity operator on H.*
4. *Isometry: if $A^*A = I$*
5. *Projection: if $A^2 = A$.*
6. *Contraction: if $\|A\| \leq 1$.*
7. *Strict contraction: if $\|A\| < 1$.*

Proposition 1.5.1. *If $A \in \mathbb{B}(H)$ is Hermitian then*

$$\langle Ax, x\rangle = 0, \ \forall x \in H \ if \ and \ only \ if \ A = 0.$$

1.5.1 Non-negative operators

Definition 1.5.3. *An operator $A \in \mathbb{B}(H)$ is said to be non-negative if and only if*

$$\langle Ax, x\rangle \geq 0, \ \forall x \in H.$$

Remark 1.5.1. *If A is a non-negative operator, then A is Hermitian.*

Theorem 1.5.1. *Every non-negative $A \in \mathbb{B}(H)$ has a unique operator $T \in \mathbb{B}(H)$ such that $T^2 = A$, then $T = A^{\frac{1}{2}}$ is called square root operator of A.*

Theorem 1.5.2. *Let $A \in \mathbb{B}(H)$, then A^*A is a non-negative operator in $\mathbb{B}(H)$ and the unique square root of A^*A is defined by*

$$|A|^2 = (A^*A),$$

That is

$$|A| = (A^*A)^{1/2},$$

and $|A|$ is called the absolute value of A.

1.5.2 Compact operators

Throughout this section E, F and G will denote normed linear spaces and $\mathbb{B}_\infty(E, F)$ the set of all compact operators of a normad space E into F. If $E = F$, we write $\mathbb{B}_\infty(E) = \mathbb{B}_\infty(E, E)$ the set of all compact operators on a normed space E.

Definition 1.5.4. *[57] A linear operator $A : E \longrightarrow F$ is compact if, for each bounded subset Ω of E, $A(\Omega)$ is relatively compact in F. Since F is a metric space, A is compact if and only if for each bounded sequence $\{x_n\}$ in E, the sequence $\{Ax_n\}$ contains a subsequence converging to some limit in F.*

Theorem 1.5.3. *[57] The set $\mathbb{B}_\infty(E, F)$ is a subspace $\mathbb{B}(E, F)$. if F is complete, this subspace is closed.*

Theorem 1.5.4. *[57] Suppose that $A \in \mathbb{B}(E, F)$ and $B \in \mathbb{B}(F, G)$. If one the operator A or B is compact, then BA is compact.*

Remark 1.5.2.

1. *If E is infinite dimensional, then the identity operator I on E, is not compact.*
2. *If A is a compact operator whose domain E is infinite dimensional, then A cannot have a bounded inverse. Since, if we suppose that A is invertible, then $A^{-1}A = I$, on E must be compact, which would imply $dim E < \infty$.*

Theorem 1.5.5. *[33] If $A \in \mathbb{B}_\infty(E, F)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then $\mathcal{R}(\lambda I - A)$ is closed.*

Theorem 1.5.6. *[33] If $A \in \mathbb{B}_\infty(E, F)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mathcal{N}(\lambda I - A) = 0$, then*

$$\mathcal{R}(\lambda I - A) = H.$$

1.5.3 Nuclear operators

Recall that when E is a Banach space, the dual space $E' = \mathbb{B}(E, \mathbb{C})$, consists of the bounded linear functionals x' on E, it is a Banach space with the norm

$$\|x\|_{E'} = inf\{|x'(x)| : x \in E, \|x\| = 1\}.$$

Definition 1.5.5. *Let E and F Banach space, An operator $A \in \mathbb{B}(E, F)$ is nuclear if there exist sequences $(a'_j) \subset E'$, $(b_j) \subset F$, and (λ_j) is a set of complex numbers obeying $\sum_{j=1}^{\infty} |\lambda_j| < \infty$, with $\|a'_j\| \leq 1$ and $\|b_j\| \leq 1$ for all j, such that*

$$Ax = \sum_{j=1}^{\infty} \lambda_j a'_j(x) b_j, \tag{1.17}$$

for all $x \in E$.

Proposition 1.5.2. *every nuclear operator is compact*

Proof. Note that the series in (1.17) is absolutely convergent since

$$\|\lambda_j a_j'(x) b_j\| \leq |\lambda_j| \|x\|, \tag{1.18}$$

and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. We shall show that a nuclear operator is compact.

Suppose that A is given by (1.17), and define A_n, $n = 1, 2, 3, ...,$ by

$$A_n x = \sum_{j=1}^{n} \lambda_j a_j'(x) b_j,$$

Clearly, $A_n \in \mathbb{B}(E, F)$ and $dim \mathcal{R}(A_n) \leq n$. Furthermore, (1.18) implies that

$$\|Ax - A_n x\| \leq (\sum_{j=n+1}^{\infty} |\lambda_j|) \|x\|,$$

which shows that $\|A - A_n\| \to 0$ as $n \to \infty$. since each A_n is compact, so is A by Theorem 1.5.3. □

In the following we suppose that H and K are two separable Hilbert spaces

Definition 1.5.6. *[52] An operator $A \in \mathbb{B}(H, K)$ is said to be a nuclear operator if there exists a sequence $(a_j)_{j \in \mathbb{N}}$ in K and a sequence $(b_j)_{j \in \mathbb{N}}$ in H such that*

$$Ax = \sum_{j=1}^{\infty} a_j \langle b_j, x \rangle_H, \ for \ all \ x \in H, \tag{1.19}$$

and

$$\sum_{j=1}^{\infty} \|a_j\| \cdot \|b_j\|_H < \infty. \tag{1.20}$$

The space of all nuclear operators from H to K is denoted by $\mathbb{B}_1(H, K)$.

Remark 1.5.3. *If $A \in \mathbb{B}_1(H)$ is non-negative, then A is called trace class.*

Proposition 1.5.3. *[52] The space $\mathbb{B}_1(H, K)$ endowed with the norm*

$$\|A\|_{\mathbb{B}_1(H,K)} = \inf\{ \sum_{j \in \mathbb{N}} \|a_j\| \cdot \|b_j\|_H \ such \ that \ Ax = \sum_{j=1}^{\infty} a_j \langle b_j, x \rangle_H, \ x \in H \}, \tag{1.21}$$

is a Banach space.

Definition 1.5.7. *[52] Let $A \in \mathbb{B}(H)$ and let $\{e_k,\ k \in \mathbb{N}\}$, be an orthonormal basis of H. Then we define*

$$trA = \sum_{k\in\mathbb{N}} \langle Ae_k, e_k\rangle_H, \tag{1.22}$$

if the series is convergent.

This definition could depend on the choice of the orthonormal basis. However, note the following result concerning nuclear operators.

Proposition 1.5.4. *[52] If $A \in \mathbb{B}_1(H)$, then trA is well-defined independently of the choice of the orthonormal basis $\{e_k, k \in \mathbb{N}\}$. Moreover we have that*

$$|trA| \leq \|A\|_{\mathbb{B}_1(H)}, \tag{1.23}$$

Proof. Let $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ be sequences in H such that

$$Ax = \sum_{j\in\mathbb{N}} a_j \langle b_j, x\rangle_H,$$

for all $x \in H$ and $\sum_{j\in\mathbb{N}} \|a_j\|_H \cdot \|b_j\|_H < \infty$.

Then we get for any orthonormal basis $\{e_k,\ k \in \mathbb{N}\}$ of H that

$$\langle Ae_k, e_k\rangle_H = \sum_{j\in\mathbb{N}} \langle e_k, a_j\rangle_H \cdot \langle e_k, b_j\rangle_H,$$

and therefore

$$\begin{aligned}
\sum_{j\in\mathbb{N}} |\langle Ae_k, e_k\rangle_H| &\leq \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} |\langle e_k, a_j\rangle_H \cdot \langle e_k, b_j\rangle_H|, \\
&\leq \sum_{j\in\mathbb{N}} (\sum_{k\in\mathbb{N}} |\langle e_k, a_j\rangle_H|^2)^{\frac{1}{2}} \cdot (\sum_{k\in\mathbb{N}} |\langle e_k, b_j\rangle_H|^2)^{\frac{1}{2}}, \\
&= \sum_{j\in\mathbb{N}} \|a_j\|_H \cdot \|b_j\|_H < \infty.
\end{aligned}$$

This implies that we can exchange the summation to get that

$$\sum_{k\in\mathbb{N}} \langle Ae_k, e_k\rangle_H = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \langle e_k, a_j\rangle_H \cdot \langle e_k, b_j\rangle_H = \sum_{j\in\mathbb{N}} \langle a_j, b_j\rangle_H,$$

and the assertion follows. □

1.5.4 Correlation operators

Definition 1.5.8. *[29] for any $h \in H$, let $h \circ h \in \mathbb{B}_1^+(H)$ be defined as*

$$(h \circ h)x = \langle x; h\rangle h \quad for\,all \quad x \in H. \tag{1.24}$$

Where $\mathbb{B}_1^+(H) = \{T \in \mathbb{B}_\infty^+(H) : tr(T) < \infty\}$, The class of all non-negative nuclear operators.

1.6 Convergence and stability of operators

Definition 1.6.1. *[32] Let $\{A_n\}_{n\geq 0}$ be a sequence of operators on $\mathbb{B}(H, K)$.*

1. *A sequence $\{A_n\}_{n\geq 0}$ is uniformly convergent if and only if there exists an operator $A \in \mathbb{B}(H, K)$ such that $\lim\limits_{n\longrightarrow\infty} \|(A_n - A)\| = 0$, denoted by $A_n \xrightarrow{u} A$.*
2. *A sequence $\{A_n\}_{n\geq 0}$ is strongly convergent if and only if there exists an operator $A \in \mathbb{B}(H, K)$ such that $\lim\limits_{n\longrightarrow\infty} \|(A_n x - Ax)\| = 0$, $\forall x \in H$, denoted by $A_n \xrightarrow{s} A$.*
3. *A sequence $\{A_n\}_{n\geq 0}$ is weakly convergent if and only if there exists an operator $A \in \mathbb{B}(H, K)$ such that $\forall x \in H, \forall y \in K :\ \lim\limits_{n\longrightarrow\infty} \langle A_n x, x\rangle = \langle Ax, y\rangle$, $\forall x \in H$, denoted by $A_n \xrightarrow{w} A$.*

Definition 1.6.2. *[32] Let $A \in \mathbb{B}(H, K)$,*

1. *A is uniformly stable if the power sequence $\{A^n\}_{n\geq 0}$ converges uniformly to the null operator (i.e. $\|A^n\| \longrightarrow 0$, as $n \longrightarrow \infty$).*
2. *A is strongly stable if the power sequence $\{A^n\}_{n\geq 0}$ converges strongly to the null operator (i.e. $\|A^n x\| \longrightarrow 0$, as $n \longrightarrow \infty$, $\forall x \in H$).*
3. *A is weakly stable if the power sequence $\{A^n\}_{n\geq 0}$ converges weakly to the null operator (i.e. $\langle A^n x; y\rangle \longrightarrow 0$, as $n \longrightarrow \infty$, $\forall x, y \in H$).*

Remark 1.6.1. *Let $\{A^n\}_{n\geq 0}$ the power sequence in $\mathbb{B}(H, K)$, then*

$$r_\sigma(A) < 1 \;\Leftrightarrow\; A^n \xrightarrow{u} 0 \implies A^n \xrightarrow{s} 0 \implies A^n \xrightarrow{w} 0.$$

Proposition 1.6.1. *[32] Two Hilbert spaces H and K, are topologically isomorphic if and only if they are unitarily equivalent. Therefore*

1. *$\mathbb{G}(H, K) \neq \emptyset$ if and only if $\{U \in \mathbb{G}(H, K) : U^{-1} = U^*\} \neq \emptyset$.*

2. *For each* $A \in \mathbb{B}(H)$ *and* $W \in \mathbb{G}(H,K)$, $\|WAW^{-1}\| = \left\| |W|\, A\, |W|^{-1} \right\|$.

3. $A \in \mathbb{B}(H)$ *is similar to a contraction (resp. strict contraction) if and only if there exists* $X \in \mathbb{G}^{+}(H)$ *such that* $\|XAX^{-1}\| \leq 1$ *(resp.* $\|XAX^{-1}\| < 1$*).*

4. *For any* $A \in \mathbb{B}(H)$, $X \in \mathbb{G}^{+}(H)$ *and* $\alpha \in \,]0,\infty[$

$$\left\|XAX^{-1}\right\| \leq \alpha \text{ if and only if } (\alpha^2 X^2 - A^* X^2 A) \in \mathbb{B}^{+}(H).$$

5.

$$\left\|XAX^{-1}\right\| < \alpha \text{ if and only if } (\alpha^2 X^2 - A^* X^2 A) \in \mathbb{G}^{+}(H).$$

Proposition 1.6.2. *[32] Let* $A \in \mathbb{B}(H,K)$*, The following assertions are equivalent:*

1. $A^n \xrightarrow{u} 0$.

2. $r_\sigma(A) < 1$.

3. $\|A^n\| \leq \beta\alpha^n,$ *for every* $n \geq 0,$ *for some* $\beta \geq 1$ *and* $\alpha \in (0,1)$.

4. $\sum_{n=0}^{\infty} \|A^n\|^p < \infty, \ \forall p > 0$.

5. $\sum_{n=0}^{\infty} \|A^n x\|^p < \infty, \ \forall x \in H, \ \forall p > 0$.

Theorem 1.6.1. *[31] Let* $A \in \mathbb{B}(H)$*. The following assertions are equivalent:*

1. $r_\sigma(A) < 1$.

2. $\mathcal{M} = M_{A^*,A}$ *is similar to a strict contraction.*

3. *There exists* $X \in \mathbb{G}^{+}(H)$ *such that*

$$X - M_{A^*,A}(X) \in \mathbb{G}^{+}(H).$$

4. *For every* $Y \in \mathbb{G}^{+}(H)$ *there exists* $X \in \mathbb{G}^{+}(H)$ *such that*

$$Y = X - M_{A^*,A}(X).$$

for any $Y \in \mathbb{G}^{+}(H)$, X *is given by*

$$X = \sum_{k=0}^{\infty} A^{*k} Y A^k \tag{1.25}$$

Proof. $4 \Rightarrow 3$
For every $Y \in \mathbb{G}^+(H)$, there exists $X \in \mathbb{G}^+(H)$ such that
$Y = X - M_{A^*,A}(X)$, then $X - M_{A^*,A}(X) \in \mathbb{G}^+(H)$.

$3 \Rightarrow 2$
Suppose that there exists $X \in \mathbb{G}^+(H)$ such that
$X - M_{A^*,A}(X) \in \mathbb{G}^+(H)$. According to Proposition 1.6.1, we have

$$\left\|XAX^{-1}\right\| < \alpha \Leftrightarrow \alpha^2 X^2 - A^* X^2 A \in \mathbb{G}^+(H).$$

Let $\alpha = 1$ and $X = W^2 \in \mathbb{G}^+(H)$, then

$$X - M_{A^*,A}(X) \in \mathbb{G}^+(H),$$

this implies that

$$\left\|X^{1/2} A (X^{-1})^{1/2}\right\| < 1.$$

Thus A is similar to a strict contraction.

$2 \Rightarrow 1$
Suppose that A is similar to a strict contraction, then there exists $W \in \mathbb{G}(H)$ such that $\|WAW^{-1}\| < 1$. Since

$$r_\sigma(A) = r_\sigma\left(WAW^{-1}\right) \leq \left\|WAW^{-1}\right\|.$$

Thus

$$r_\sigma(A) < 1.$$

$1 \Rightarrow 4$
Suppose that $r_\sigma(A) < 1$, so that $\sum\limits_{k=0}^{\infty} \left\|A^k\right\|^2 < \infty$.

Let $Y = R^2 \in \mathbb{G}^+(H)$, then $\forall n \geq 0$, we have

$$\left\|R^{-1}\right\|^{-2} \|x\|^2 \leq \|Rx\|^2 \leq \sum_{k=0}^{n} \left\|RA^k x\right\|^2 \leq \|R\|^2 (\sum_{k=0}^{n} \left\|A^k\right\|^2) \|x\|^2.$$

We define a map

$$\begin{array}{rcll} W: & H & \rightarrow & \ell_2(H) \\ & x & \mapsto & Wx = (Rx, RAx, ..), \end{array}$$

W is linear and bounded with

$$\|Wx\|^2 = \sum_{k=0}^{\infty} \left\|RA^k x\right\|^2, \forall x \in H.$$

So that W has a bounded inverse in $\mathcal{R}(W)$, then $W \in \mathbb{G}(H)$. Thus

$$|W|^2 \in \mathbb{G}^+(H).$$

Note that

$$\begin{aligned}
\left\langle \left(R^2 - |W|^2 + A^* |W|^2 A\right) x, x \right\rangle &= \left\langle \left(R^2 - |W|^2\right) x, x \right\rangle + \left\langle A^* |W|^2 Ax, x \right\rangle, \\
&= \left\langle \left(R^2 - |W|^2\right) x, x \right\rangle + \|WAx\|^2, \\
&= \|Rx\|^2 - \|Wx\|^2 + \|WAx\|^2, \\
&= \|Rx\|^2 - \sum_{k=0}^{\infty} \left\|RA^k x\right\|^2 + \sum_{k=0}^{\infty} \left\|RA^{k+1} x\right\|^2, \\
&= \|Rx\|^2 - \sum_{k=0}^{\infty} \left\|RA^k x\right\|^2 + \sum_{k'=1}^{\infty} \left\|RA^{k'} x\right\|^2, \\
&= \|Rx\|^2 - \|Rx\|^2 = 0, \ \forall x \in H
\end{aligned}$$

this implies that

$$R^2 - |W|^2 + A^* |W|^2 A = 0.$$

So that

$$R^2 = |W|^2 - A^* |W|^2 A,$$

let $X = |W|^2$ and $Y = R^2$, we get (4).

Suppose that $Q^2 \in \mathbb{G}^+(H)$ is another solution of the equation

$$R^2 = |W|^2 - A^* |W|^2 A,$$

then

$$\begin{aligned}
\left\|RA^k x\right\|^2 &= \left\langle RA^k x, RA^k x \right\rangle, \\
&= \left\langle A^{*^k} R^2 A^k x, x \right\rangle, \\
&= \left\langle A^{*^k} \left(Q^2 - A^* Q^2 A\right) A^k x, x \right\rangle, \\
&= \left\langle A^{*^k} Q^2 A^k x, x \right\rangle - \left\langle A^{*^{k+1}} Q^2 A^{k+1} x, x \right\rangle, \\
&= \left\|QA^k x\right\|^2 - \left\|QA^{k+1} x\right\|^2,
\end{aligned}$$

hence

$$\begin{aligned}
\sum_{k=0}^{n} \left\|RA^k x\right\|^2 &= \sum_{k=0}^{n} (\left\|QA^k x\right\|^2 - \left\|QA^{k+1} x\right\|^2), \\
&= \|Qx\|^2 - \left\|QA^{n+1} x\right\|^2,
\end{aligned}$$

then $\forall x \in H$, $\forall k \geq 0$, we have

$$\begin{aligned} \|Wx\|^2 &= \lim_{n\to\infty}\sum_{k=0}^{n} \left\|RA^k x\right\|^2, \\ &= \lim_{n\to\infty}(\|Qx\|^2 - \left\|QA^{n+1}x\right\|^2), \\ &= \|Qx\|^2, \end{aligned}$$

because $r_\sigma(A) < 1$, thus

$$\begin{aligned} \left\langle \left(|W|^2 - Q^2\right) x, x\right\rangle &= \|Wx\|^2 - \|Qx\|^2 \\ &= 0,\ \forall x \in H, \end{aligned}$$

therefore

$$Q^2 = |W|^2.$$

So that X is the unique solution of

$$Y = X - M_{A^*,A}(X), \quad \forall Y \in \mathbb{G}^+(H).$$

Finally, let $Y = X - M_{A^*,A}(X)$, then

$$\begin{aligned} \sum_{k=0}^{n} A^{*k} Y A^k &= \sum_{k=0}^{n} A^{*k} X A^k - \sum_{k=0}^{n} A^{*k+1} X A^{k+1}, \\ &= X - A^{*^{n+1}} X A^{n+1} \in \mathbb{B}^+(H),\ \forall n \geq 0, \end{aligned}$$

where X is the solution of the equation $Y = X - M_{A^*,A}(X)$, let

$$X_n = \sum_{k=0}^{n} A^{*k} Y A^k,$$

then

$$\begin{aligned} \|X_n - X\| &= \left\|A^{*^{n+1}} X A^{n+1}\right\|, \\ &\leq \|X\| \left\|A^{n+1}\right\|^2, \forall n \geq 0, \end{aligned}$$

Hence

$$\lim_{n\to\infty} \|X_n - X\| \leq \lim_{n\to\infty} \|X\| \left\|A^{n+1}\right\|^2.$$

Since $r_\sigma(A) < 1$ (i.e. $\|A^n\| \to 0$), so that

$$\lim_{n\to\infty} \|X_n - X\| = 0,$$

therefore

$$X = \sum_{k=0}^{\infty} A^{*k} Y A^k.$$

□

Chapter 2

Operator equations and elementary operators

2.1 Introduction

This chapter is the subject of an article [37] written in collaboration with Pr Lombarkia published in Linear and Multilinear Algebra.
Let E, F, G and D be infinite complex Banach spaces and $\mathbb{B}(F,E)$ the Banach space of all bounded linear operators from F into E.

Consider $A_1, A_2 \in \mathbb{B}(F,E)$, $B_1, B_2 \in \mathbb{B}(D,G)$. Let

$$M_{A_1,B_1} : X \to A_1 X B_1$$

be the multiplication operator on $\mathbb{B}(G,F)$ induced by A_1, B_1. In particular

$$L_{A_1} = M_{A_1,I} \text{ and } R_{B_1} = M_{I,B_1},$$

where I is the identity operator are the left and the right multiplication operators respectively. The elementary operator Ψ defined on $\mathbb{B}(G,F)$ is the sum of two multiplication operators

$$\Psi = M_{A_1,B_1} + M_{A_2,B_2}.$$

In this chapter, we give necessary and sufficient conditions for the existence of a common solution of the operator equations $M_{A_1,B_1}(X) = C_1$ and $M_{A_2,B_2}(X) = C_2$, and we derive a new representation of the general common solution via the inner inverse of the elementary operator $\Psi = M_{A_1,B_1} + M_{A_2,B_2}$, we apply this result to determine new necessary and sufficient conditions for the existence of a Hermitian solution and a representation of the general Hermitian solution to the operator equation $M_{A,B}(X) = C$, where A, B and C are bounded linear operators on Hilbert spaces. As consequence, we obtain well known results of Alegra Dajić and J. J. Koliha in [15].

2.2 Inner inverses of elementary operators

Lemma 2.2.1. *[10] Let $A_1, A_2 \in \mathbb{B}(F, E)$, $B_1, B_2 \in \mathbb{B}(D, G)$, such that $\Psi = M_{A_1,B_1} + M_{A_2,B_2}$ is regular. Then the operator equation*

$$\Psi(X) = C,$$

has a solution if and only if

$$\Psi\Psi^-(C) = C.$$

A representation of the general solution is

$$X = \Psi^-(C) + (I_{\mathbb{B}(G,F)} - \Psi^-\Psi)(U),$$

where $U \in \mathbb{B}(G, F)$ is an arbitrary operator.

It can be easily seen that in the case, where $A_1 = A \in \mathbb{B}(F, E)$, $B_1 = I$ and $A_2 = B_2 = 0$, Lemma 2.2.1 reduces to the well-known result, here related as

Corollary 2.2.1. *Let $A \in \mathbb{B}(F, E)$, such that A is regular. Then the operator equation*

$$L_A(X) = C,$$

has a solution $X \in \mathbb{B}(E)$ if and only if

$$L_{AA^-}(C) = C.$$

A representation of the general solution is

$$X = L_{A^-}(C) + (I - L_{A^-A})(U),$$

where $U \in \mathbb{B}(E)$ is an arbitrary operator.

Proof. The operator equation $L_A(X) = C$ has a solution if and only if

$$L_A(L_A)^-(C) = C.$$

Since

$$L_{A^-A} = L_{A^-}L_A,$$

it follows that if A^- is the inner inverse of A, then $(L_A)^- = L_{A^-}$ is the inner inverse of L_A.

Consequently, from Lemma 2.2.1, we get the result. □

It can be easily seen that in the case, where $A_1 = A_2 = A \in \mathbb{B}(F, E)$, $B_1 = B_2 = B \in \mathbb{B}(D, G)$, Lemma 2.2.1 reduces to the well-known result, here related as

Corollary 2.2.2. *Let $A \in \mathbb{B}(F,E)$ and $B \in \mathbb{B}(D,G)$ be regular. Then the operator equation*

$$M_{A,B}(X) = C,$$

has a solution if and only if

$$M_{AA^-,B^-B}(C) = C.$$

A representation of the general solution is

$$X = M_{A^-,B^-}(C) + (I - M_{A^-A,BB^-})(U),$$

where $U \in \mathbb{B}(G,F)$ is an arbitrary operator.

Proof. Since

$$M_{A,B}M_{A^-,B^-} = M_{AA^-,B^-B},$$

it follows that, if A^- and B^- are the inner inverses of A and B, then $(M_{A,B})^- = M_{A^-,B^-}$ is the inner inverse of $M_{A,B}$.

Consequently from Lemma 2.2.1, we get the result. □

2.3 Common solution to the pair of equations $M_{A_1,B_1}(X) = C_1$ and $M_{A_2,B_2}(X) = C_2$

In the following section, we give necessary and sufficient conditions for the existence of a common solutions of the operators equations

$$M_{A_1,B_1}(X) = C_1 \quad and \quad M_{A_2,B_2}(X) = C_2,$$

we suppose that all the spaces are complex Banach spaces.

Theorem 2.3.1. *Let $A_1 \in \mathbb{B}(F,E)$, $A_2 \in \mathbb{B}(F,N)$,$B_1 \in \mathbb{B}(D,G)$ and $B_2 \in \mathbb{B}(M,G)$, such that A_1, B_1 and $\Psi = M_{A_2,B_2} + M_{-A_2A_1^-A_1,B_1B_1^-B_2}$ are regular, then the pair of equations*

$$M_{A_1,B_1}(X) = C_1, \tag{2.1}$$

$$M_{A_2,B_2}(X) = C_2, \tag{2.2}$$

has a common solution if and only if

$$M_{A_1A_1^-,B_1^-B_1}(C_1) = C_1,$$

and

$$\Psi\Psi^-(C_2 - A_2A_1^-C_1B_1^-B_2) = C_2 - A_2A_1^-C_1B_1^-B_2.$$

If a common solution to (2.1) and (2.2) exists, a representation of the general common solution is given by

$$\begin{aligned} X &= M_{A_1^-,B_1^-}(C_1) + \Psi^-(C_2 - A_2A_1^-C_1B_1^-B_2) + (I - \Psi^-\Psi)(V) \\ &- M_{A_1^-A_1,B_1B_1^-}[\Psi^-(C_2 - A_2A_1^-C_1B_1^-B_2) + (I - \Psi^-\Psi)(V)], \end{aligned} \tag{2.3}$$

where $V \in \mathbb{B}(G, F)$ is an arbitrary operator.

Proof. Suppose that the pair of equations (2.1) and (2.2) has a common solution. Since equation (2.1) has solution, then from Corollary 2.2.2 we have that

$$M_{A_1A_1^-,B_1^-B_1}(C_1) = C_1,$$

and the representation of the general solution to equation (2.1) is

$$X_1 = M_{A_1^-,B_1^-}(C_1) + (I_{\mathbb{B}(D,E)} - M_{A_1^-A_1,B_1B_1^-})(U), \tag{2.4}$$

where $U \in \mathbb{B}(D, E)$ is an arbitrary operator.

Because equations (2.1) and (2.2) have a common solution, there exists an operator $U_0 \in \mathbb{B}(D, E)$, such that X_1 satisfies

$$M_{A_2,B_2}(X_1) = C_2.$$

Thus

$$\begin{aligned} M_{A_2,B_2}(X_1) &= M_{A_2A_1^-,B_1^-B_2}(C_1) + M_{A_2,B_2}(U_0) + M_{-A_2A_1^-A_1,B_1B_1^-B_2}(U_0) \\ &= C_2, \end{aligned}$$

which implies that

$$\Psi(U_0) = C_2 - A_2A_1^-C_1B_1^-B_2, \tag{2.5}$$

has a solution, applying Lemma 2.2.1, the equation (2.5) has a solution if and only if

$$\Psi\Psi^-(C_2 - A_2A_1^-C_1B_1^-B_2) = C_2 - A_2A_1^-C_1B_1^-B_2. \tag{2.6}$$

Now assume that

$$M_{A_1A_1^-,B_1^-B_1}(C_1) = C_1,$$

and

$$\Psi\Psi^-(C_2 - A_2A_1^-C_1B_1^-B_2) = C_2 - A_2A_1^-C_1B_1^-B_2.$$

We can show that X as defined in (2.3) is a common solution to (2.1) and (2.2). Let X_0 an arbitrary common solution to (2.1) and (2.2). Then we have

$$M_{A_2,B_2}(X_0) + M_{-A_2A_1^-A_1,B_1B_1^-B_2}(X_0) = C_2 - A_2A_1^-C_1B_1^-B_2,$$

or

$$\Psi(X_0) = C_2 - A_2A_1^-C_1B_1^-B_2.$$

Letting $V = X_0$ in (2.3), we have that

$$\begin{aligned} X &= M_{A_1^-,B_1^-}(C_1) + \psi^-(C_2 - A_2A_1^-C_1B_1^-B_2) + (I - \Psi^-\Psi)(X_0) \\ &- M_{A_1^-A_1,B_1B_1^-}[\Psi^-(C_2 - A_2A_1^-C_1B_1^-B_2) + (I - \Psi^-\Psi)(X_0)], \\ &= M_{A_1^-,B_1^-}(C_1) + X_0 + \Psi^-(C_2 - A_2A_1^-C_1B_1^-B_2) - \Psi^-\Psi(X_0) \\ &- M_{A_1^-A_1,B_1B_1^-}[\Psi^-(C_2 - A_2A_1^-C_1B_1^-B_2) - \Psi^-\Psi)(X_0)] + M_{A_1^-A_1,B_1B_1^-}(X_0), \\ &= M_{A_1^-,B_1^-}(C_1) - M_{A_1^-A_1,B_1B_1^-}(X_0) + X_0, \\ &= X_0. \end{aligned}$$

Hence X as defined in (2.3) is a representation of the general common solution to (2.1) and (2.2). □

Corollary 2.3.1. *Let $A \in \mathbb{B}(F,E)$, $B \in \mathbb{B}(M,G)$, $C_1 \in \mathbb{B}(G,E)$ and $C_2 \in \mathbb{B}(M,F)$ such that A and $\Psi = R_B + M_{-A^-A,B} = M_{(I-A^-A),B}$ are regular, Then the pair of operator equations*

$$L_AX = C_1, \tag{2.7}$$

$$R_BX = C_2, \tag{2.8}$$

has a common solution if and only if

$$L_AL_{A^-}C_1 = C_1,$$

and

$$\Psi\Psi^-(C_2 - A^-C_1B) = C_2 - A^-C_1B.$$

If a common solution to (2.7) and (2.8) exists, a representation of the general common solution is given by

$$\begin{aligned} X &= L_{A^-}(C_1) + \Psi^-(C_2 - A^-C_1B) + (I - \Psi^-\Psi)(V) \\ &- L_{A^-A}[\Psi^-(C_2 - A^-C_1B) + (I - \Psi^-\Psi)(V)], \end{aligned}$$

where $V \in \mathbb{B}(G,F)$ is an arbitrary operator.

In the following corollary, we can see that Corollary 2.3.1 reduces to the well-known result of Dajić and Koliha [15, Theorem 4.5]

Corollary 2.3.2. *([15, Theorem 4.5]) Let $A, C_1 \in \mathbb{B}(F,E)$, $B, C_2 \in \mathbb{B}(G,E)$, and let A and B be regular. Then the equations*

$$AX = C_1 \text{ and } XB = C_2, \tag{2.9}$$

have a common solution $X \in \mathbb{B}(E)$ *if and only if*

$$\mathcal{R}(C_1) \subset \mathcal{R}(A),\ \mathcal{N}(B) \subset \mathcal{N}(C_2) \text{ and } AC_2 = C_1 B.$$

The general common solution is of the form

$$X = A^- C_1 + C_2 B^- - A^- A C_2 B^- + (I - A^- A) U (I - BB^-), \tag{2.10}$$

where $U \in \mathbb{B}(E)$ *is an arbitrary operator.*

Proof. If A and B are regular, then the multiplication operator $\Psi = M_{(I-A^-A),B}$ is regular and $M_{(I-A^-A),B^-}$ is the inner inverse of $M_{(I-A^-A),B}$.
Applying Corollary 2.3.1, we get that the equations

$$L_A(X) = C_1,\ R_B(X) = C_2$$

have a common solution $X \in \mathbb{B}(E)$ if and only if

$$L_{AA^-}(C_1) = C_1 \quad \text{and} \quad \Psi\Psi^-(C_2 - A^- C_1 B) = C_2 - A^- C_1 B.$$

The equation $L_{AA^-}(C_1) = C_1$, implies that

$$AA^- C_1 = C_1,$$

which is equivalent to

$$\mathcal{R}(C_1) \subset \mathcal{R}(A).$$

The condition

$$\Psi\Psi^-(C_2 - A^- C_1 B) = C_2 - A^- C_1 B$$

implies that

$$M_{(I-A^-A),B^-B}(C_2 - A^- C_1 B) = C_2 - A^- C_1 B,$$

consequently

$$(I - A^- A) C_2 B^- B + A^- A A^- C_1 B = C_2, \tag{2.11}$$

applying A to the equation (2.11) we get

$$AA^- C_1 B = AC_2,$$

since

$$AA^- C_1 = C_1.$$

Hence

$$AC_2 = C_1 B,$$

using in equation (2.11), the equality

$$AC_2 = C_1 B,$$

we get

$$C_2 B^- B = C_2,$$

which is equivalent to

$$\mathcal{N}(B) \subset \mathcal{N}(C_2).$$

From Corollary 2.3.1, the general solution is of the form

$$\begin{aligned} X &= L_{A^-}(C_1) + \Psi^-(C_2 - A^- C_1 B) + (I - \Psi^- \Psi)(U) \\ &- L_{A^- A}[\Psi^-(C_2 - A^- C_1 B) + (I - \Psi^- \Psi)(U)], \end{aligned}$$

where $U \in \mathbb{B}(E)$ is an arbitrary operator, it follows that

$$\begin{aligned} X &= A^- C_1 + M_{(I-A^-A),B^-}(C_2 - A^- C_1 B) + (I - M_{(I-A^-A),B^-B})(U) \\ &- A^- A(M_{(I-A^-A),B^-B})(U), \end{aligned}$$

using the equality $AC_2 = C_1 B$, we get

$$X = A^- C_1 + C_2 B^- - A^- A C_2 B^- + (I - A^- A) U (I - BB^-),\ U \in \mathbb{B}(E).$$

□

2.4 Hermitian solution to the operator equation $M_{A,B}(X) = C$

In this section, we suppose that $F = H$ and $E = K$ are complex Hilbert spaces and we determine conditions for the existence of a Hermitian solution to the operator equation $M_{A,B}(X) = C$.

Lemma 2.4.1. *Let $A \in \mathbb{B}(H, K)$, $B \in \mathbb{B}(K, H)$ and $\Psi = M_{B^*,A^*} + M_{-B^*A^-A,BB^-A^*}$, such that A, B and Ψ are regular. Then the operator equation*

$$M_{A,B}(X) = C, \tag{2.12}$$

has a Hermitian solution if and only if the pair of equations

$$M_{A,B}(X) = C \text{ and } M_{B^*,A^*}(X) = C^*, \tag{2.13}$$

has a common solution, a representation of the general Hermitian solution to (2.12) is of the form

$$X_H = \frac{X + X^*}{2},$$

where X is the representation of the general common solution to equations (2.13).

Proof. From Theorem 2.3.1, the pair of operator equations (2.13) has a common solution if and only if

$$M_{AA^-,B^-B}(C) = C,$$

and

$$\Psi\Psi^-(C^* - B^*A^-CB^-A^*) = C^* - B^*A^-CB^-A^*,$$

a representation of the general common solution to (2.13) is

$$\begin{aligned} X &= M_{A^-,B^-}(C) + \Psi^-(C^* - B^*A^-CB^-A^*) + (I - \Psi^-\Psi)(V) \\ &- M_{A^-A,BB^-}[\Psi^-(C^* - B^*A^-CB^-A^*) + (I - \Psi^-\Psi)(V)]. \end{aligned}$$

Clearly, X_H is a Hermitian solution to (2.12). □

Corollary 2.4.1. *Let $A \in \mathbb{B}(H,K)$ and $\Psi = R_{A^*} + M_{-A^-A,A^*} = M_{(I-A^-A),A^*}$ be regular operators. Then the operator equation*

$$L_A(X) = C, \tag{2.14}$$

has a Hermitian solution if and only if the pair of equations

$$L_A(X) = C \text{ and } R_{A^*}(X) = C^*, \tag{2.15}$$

has a common solution, a representation of the general Hermitian solution to (2.14) is of the form

$$X_H = \frac{X + X^*}{2},$$

where X is the representation of the general common solution to equations (2.15).

Proof. The result follows from Lemma 2.4.1 and Corollary 2.3.1. □

In the following corollary, we can see that Corollary 2.4.1 reduces to the well-known result of Dajic and Koliha [15, Theorem 4.6]

Corollary 2.4.2. *([15, Theorem 4.6]) Let $A, C \in \mathbb{B}(H,K)$ be regular operator. then the operator equation*

$$AX = C,$$

has a Hermitian solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A) \quad \text{and} \quad AC^* \in \mathbb{B}(K) \text{ is Hermitian.}$$

The general Hermitian solution is of the form

$$X_H = A^-C + (I - A^-A)(A^-C)^* + (I - A^-A)U(I - A^-A)^*,\ U^* = U \in \mathbb{B}(H).$$

Proof. If A is regular, then the left multiplication operator $\Psi = L_A$ is regular and L_{A^-} is the inner inverse of L_A,

applying Corollary 2.4.2 and Corollary 2.3.1, we get that the equation

$$L_A(X) = C$$

has a Hermitian solution $X_H \in \mathbb{B}(H)$ if and only if

$$L_{AA^-}(C) = C,$$

and

$$\Psi\Psi^-(C^* - A^-CA^*) = C^* - A^-CA^*.$$

The equation $L_{AA^-}(C) = C$ implies that

$$AA^-C = C,$$

which is equivalent to

$$\mathcal{R}(C) \subset \mathcal{R}(A),$$

the condition

$$\Psi\Psi^-(C^* - A^-CA^*) = C^* - A^-CA^*,$$

implies that

$$M_{(I-A^-A),(A^*)^-A^*}(C^* - A^-CA^*) = C^* - A^-CA^*,$$

consequently

$$C^*(A^-)^*A^* - A^-AC^*(A^-)^*A^* - A^-C(AA^-A)^* + A^-AA^-C(AA^-A)^* = C^* - A^-CA^*,$$

hence

$$A^-AC^* - A^-CA^* = 0, \tag{2.16}$$

applying A to the equation (2.16), we get

$$AC^* = CA^*.$$

From Corollary 2.4.1, the Hermitian solution is of the form

$$X_H = \frac{X + X^*}{2},$$

where X is the representation of the general common solution to equation (2.15). Then

$$\begin{aligned} X &= L_{A^-}(C) + \Psi^-(C^* - A^-CA^*) + (I - \Psi^-\Psi)(U) \\ &- L_{A^-A}[\Psi^-(C^* - A^-CA^*) + (I - \Psi^-\Psi)(U)], \end{aligned}$$

where $U \in \mathbb{B}(K)$ is an arbitrary operator, it follows that

$$\begin{aligned} X &= A^{-}C + M_{(I-A^{-}A),(A^{*})^{-}}(C^{*} - A^{-}CA^{*}) + (I - M_{(I-A^{-}A),A^{*}(A^{*})^{-}})(U) \\ &- A^{-}A(I - M_{(I-A^{-}A),A^{*}(A^{*})^{-}})(U), \end{aligned}$$

using the equality $AC^{*} = CA^{*}$, we have

$$X = A^{-}C + (I - A^{-}A)(A^{-}C)^{*} + (I - A^{-}A)U(I - A^{-}A)^{*}.$$

Consequently, we get

$$X_H = A^{-}C + (I - A^{-}A)(A^{-}C)^{*} + (I - A^{-}A)U(I - A^{-}A)^{*},\ where\ U^{*} = U \in \mathbb{B}(H).$$

□

Chapter 3

Hermitian solutions to some operators equations

3.1 Introduction

This chapter is the subject of an article [9] written in collaboration with Pr F. Lombarkia accepted in Facta universitatis (Niš). Ser. Math. Inform.
We consider the same system us in the second chapter

$$A_1XB_1 = C_1, \quad A_2XB_2 = C_2,$$

where A_1, A_2, B_1, B_2, C_1 and C_2 are linear bounded operators on Hilbert spaces, and we give other necessary and sufficient conditions for the existence of a common solution, we apply the result to determine new necessary and sufficient conditions for the existence of a Hermitian solution and a representation of the general Hermitian solution to the operators equations $AXB = C$, and $AXA^* + BYB^* = C$. As consequence, we obtain well-known results of [16]

3.2 Common solution to the pair of equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$

Let A_1, A_2, B_1, B_2, C_1 and C_2 are linear bounded operators defined on Hilbert spaces H, K, E, L, N and G. Before enouncing our main results, we need the following lemmas

Lemma 3.2.1. *[14] Suppose that $A_1 \in \mathbb{B}(H,K)$, $A_2 \in \mathbb{B}(H,E)$, $B_1 \in \mathbb{B}(L,G)$, $B_2 \in \mathbb{B}(N,G)$, $S_1 = A_2(I_H - A_1^-A_1)$ and $M_1 = (I_G - B_1B_1^-)B_2$ are regular operators. Then*

$$T_1 = (I_E - S_1S_1^-)A_2A_1^- \text{ and } D_1 = B_1^-B_2(I_N - M_1^-M_1),$$

are regular with inner inverses $T_1^- = A_1 A_2^-$ *and* $D_1^- = B_2^- B_1$.

Lemma 3.2.2. *[15] Let* $A, B \in \mathbb{B}(H, K)$ *are regular operators and* $C, D \in \mathbb{B}(H, K)$, *then the pair of operators equations*

$$AX = C \text{ and } XB = D, \tag{3.1}$$

have a common solution is

$$AA^- C = C,\ DB^- B = D, \text{ and } AD = CB, \tag{3.2}$$

or equivalently

$$\mathcal{R}(C) \subset \mathcal{R}(A),\ \mathcal{R}(D^*) \subset \mathcal{R}(B^*),\ \text{and } AD = CB.$$

A representation of the general solution is

$$X = A^- C + DB^- - A^- ADB^- + (I_E - A^- A)V(I_F - BB^-), \tag{3.3}$$

where $V \in \mathbb{B}(K)$ *is an arbitrary operator.*

In the following theorem, we give necessary and sufficient conditions for the existence of a common solution of the operator equations

$$A_1 X B_1 = C_1, \quad A_2 X B_2 = C_2.$$

Theorem 3.2.1. *Suppose that* $A_1 \in \mathbb{B}(H, K)$, $A_2 \in \mathbb{B}(H, E)$, $B_1 \in \mathbb{B}(L, G)$, $B_2 \in \mathbb{B}(N, G)$, $M_1 = (I_G - B_1 B_1^-)B_2$ *and* $S_1 = A_2(I_H - A_1^- A_1)$ *are regular operators and* $C_1 \in \mathbb{B}(L, K)$, $C_2 \in \mathbb{B}(N, E)$, *then the following statement are equivalent*

1. *The pair of equations*

$$A_1 X B_1 = C_1, \quad A_2 X B_2 = C_2, \tag{3.4}$$

   have a common solution X.

2. *There exists two operators* $U \in \mathbb{B}(N, K)$ *and* $V \in \mathbb{B}(L, E)$, *such that the operator equation* $AXB = C$ *is solvable, where*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix}.$$

3. *For* $i = 1, 2$, $\mathcal{R}(C_i) \subset \mathcal{R}(A_i)$, $\mathcal{R}(C_i^*) \subset \mathcal{R}(B_i^*)$ *and*

$$T_1C_1D_1 = T_2C_2D_2,$$

where $T_1 = (I_E - S_1S_1^-)A_2A_1^-$, $T_2 = (I_E - S_1S_1^-)$, $D_1 = B_1^-B_2(I_N - M_1^-M_1)$ *and* $D_2 = (I_N - M_1^-M_1)$.

Proof.
(1) $\Leftrightarrow$ (2) We have

$$\begin{aligned} AXB = C \quad &\Leftrightarrow \quad \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X \begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix}. \\ &\Leftrightarrow \quad \begin{cases} A_1XB_1 = C_1, \\ A_1XB_2 = U, \\ A_2XB_1 = V, \\ A_2XB_2 = C_2. \end{cases} \end{aligned}$$

Since the pair of equations

$$A_1XB_1 = C_1 \text{ and } A_2XB_2 = C_2,$$

has a common solution. Then, by applying Lemma 1.2.2, the operator equations

$$A_1XB_2 = U \text{ and } A_2XB_1 = V,$$

have a solution if and only if

$$U = A_1A_1^-UB_2^-B_2 \text{ and } V = A_2A_2^-VB_1^-B_1,$$

respectively. This is equivalent, there exists two operators $U \in \mathbb{B}(N,K)$ and $V \in \mathbb{B}(L,E)$, such that the operator equation $AXB = C$ is solvable. Thus (1) $\Leftrightarrow$ (2)

(2) $\Rightarrow$ (3) According to Lemma 1.2.2, the operator equation $AXB = C$ has a solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A) \quad and \quad \mathcal{R}(C^*) \subset \mathcal{R}(B^*),$$

then, we deduce that

$$for \quad i = 1,2, \quad \mathcal{R}(C_i) \subset \mathcal{R}(A_i) \quad and \quad \mathcal{R}(C_i^*) \subset \mathcal{R}(B_i^*). \tag{3.5}$$

On the other hand, we have

$$\begin{aligned} T_1C_1D_1 &= (I_E - S_1S_1^-)A_2A_1^-C_1B_1^-B_2(I_N - M_1^-M_1), \\ &= (I_E - S_1S_1^-)A_2A_1^-A_1X_0B_1B_1^-B_2(I_N - M_1^-M_1), \end{aligned} \tag{3.6}$$

where X_0 is the common solution of the pair of equations (3.4).

Let

$$S_1 = A_2(I_H - A_1^- A_1) \ and \ M_1 = (I_G - B_1 B_1^-)B_2.$$

This implies that

$$A_2 A_1^- A_1 = A_2 - S_1 \ \ and \ \ B_1 B_1^- B_2 =_2 - M_1. \tag{3.7}$$

We insert (3.7) in (3.6), to obtain

$$\begin{aligned} T_1 C_1 D_1 &= (I_E - S_1 S_1^-)(A_2 - S_1)X_0(B_2 - M_1)(I_N - M_1^- M_1), \\ &= (I_E - S_1 S_1^-)(A_2 - S_1)X_0(B_2(I_N - M_1^- M_1), \\ &= (I_E - S_1 S_1^-)A_2 X_0 B_2(I_N - M_1^- M_1). \end{aligned}$$

Therefore

$$T_1 C_1 D_1 = T_2 C_2 D_2. \tag{3.8}$$

From (3.5) and (3.8), we deduce that (2) $\Rightarrow$ (3). Conversely, since

$$T_1 C_1 D_1 = T_2 C_2 D_2,$$

then

$$\mathcal{R}(T_2 C_2) \subset \mathcal{R}(T_1) \ and \ \mathcal{R}(D_1^* C_1^*) \subset \mathcal{R}(D_2^*).$$

By applying Lemma 3.2.2, there exist $U \in \mathbb{B}(N, K)$ which is the common solution to the pair of equations

$$\begin{cases} T_1 U = T_2 C_2, \\ U D_2 = C_1 D_1, \end{cases} \tag{3.9}$$

given by

$$U = T_1^- T_2 C_2 + C_1 D_1 D_2^- - T_1^- T_1 C_1 D_1 D_2^- + (I_K - T_1^- T_1)Z(I_N - D_2 D_2^-),$$

where $Z \in \mathbb{B}(N, K)$ is an arbitrary operator. On the other hand, we have

$$U = A_1 A_1^- U B_2^- B_2,$$

then

$$\begin{aligned} U &= A_1 A_1^- T_1^- T_2 C_2 B_2^- B_2 + A_1 A_1^- C_1 D_1 D_2^- B_2^- B_2 - A_1 A_1^- T_1^- T_1 C_1 D_1 D_2^- B_2^- B_2 \\ &+ A_1 A_1^- (I - T_1^- T_1) Z (I - D_2 D_2^-) B_2^- B_2. \end{aligned}$$

After simplification we obtain

$$U = C_1 D_1 + T_1^- (I_E - S_1 S_1^-) C_2 M_1^- M_1 + (A_1 A_1^- - T_1^- T_1) Z M_1^- M_1 \tag{3.10}$$

where $Z \in \mathbb{B}(N, K)$ is an arbitrary operator.

In the same way, since

$$T_1C_1D_1 = T_2C_2D_2,$$

then

$$\mathcal{R}(T_1C_1) \subset \mathcal{R}(T_2) \text{ and } \mathcal{R}(D_2^*C_2^*) \subset \mathcal{R}(D_1^*).$$

it follows from Lemma 3.2.2 that there exist $V \in \mathbb{B}(L, E)$ which is the common solution to the pair of equations

$$\begin{cases} T_2V = T_1C_1, \\ VD_1 = C_2D_2, \end{cases} \tag{3.11}$$

given by

$$V = T_2^-T_1C_1 + C_2D_2D_1^- - T_2^-T_2C_2D_2D_1^- + (I - T_2^-T_2)Z'(I - D_1D_1^-),$$

where $Z' \in \mathbb{B}(L, E)$ is an arbitrary operator. On the other hand, we have

$$V = A_2A_2^-VB_1^-B_1.$$

After simplification we obtain

$$V = T_1C_1 + S_1S_1^-C_2(I_N - M_1^-M_1)D_1^- + S_1S_1^-Z'(B_1^-B_1 - D_1D_1^-), \tag{3.12}$$

where $Z' \in \mathbb{B}(L, E)$ is an arbitrary operator.

Thus, there exists $U \in \mathbb{B}(N, K)$ and $V \in \mathbb{B}(L, E)$ solutions of the pair of equations (3.9), (3.11) and as for $i = 1, 2$, we have $A_iA_i^-C_i = C_i$ and $C_iB_i^-B_i = C_i$, we obtain

$$\begin{aligned} AA^-CB^-B &= \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix}, \\ &= C. \end{aligned}$$

So that, the operator equation $AXB = C$ is solvable and (3) $\Rightarrow$ (2). □

Theorem 3.2.2. *Suppose that $A_1 \in \mathbb{B}(H, K)$, $A_2 \in \mathbb{B}(H, E)$, $B_1 \in \mathbb{B}(L, G)$, $B_2 \in \mathbb{B}(N, G)$, $M_1 = (I_G - B_1B_1^-)B_2$ and $S_1 = A_2(I_H - A_1^-A_1)$ are regular operators and $C_1 \in \mathbb{B}(L, K)$, $C_2 \in \mathbb{B}(N, E)$, when any one of the conditions (2), (3) of Theorem 3.2.1 holds, a general common solution to the pair of equations (3.4) is given by*

$$\begin{aligned} X = {} & (A_1^-C_1 + (I_H - A_1^-A_1)S_1^-(V - A_2A_1^-C_1))B_1^-(I_G - B_2M_1^-(I_G - B_1B_1^-)) \\ & + (A_1^-U + (I_H - A_1^-A_1)S_1^-(C_2 - A_2A_1^-U))M_1^-(I_G - B_1B_1^-) + F \\ & - (A_1^-A_1 + (I_H - A_1^-A_1)S_1^-S_1)F(B_1B_1^- + M_1M_1^-(I_G - B_1B_1^-)), \end{aligned} \tag{3.13}$$

where $F \in \mathbb{B}(G,H)$ *is an arbitrary operator and* U, V *are given by*

$$\begin{aligned} U &= C_1B_1^-B_2(I_N - M_1^-M_1) + A_1A_2^-(I_E - S_1S_1^-)C_2M_1^-M_1 + A_1A_1^-ZM_1^-M_1 \\ &- A_1A_2^-(I_E - S_1S_1^-)A_2A_1^-ZM_1^-M_1, \end{aligned}$$

and

$$\begin{aligned} V &= (I_E - S_1S_1^-)A_2A_1^-C_1 + S_1S_1^-C_2(I_N - M_1^-M_1)B_2^-B_1 + S_1S_1^-Z'B_1^-B_1 \\ &- S_1S_1^-Z'B_1^-B_2(I_N - M_1^-M_1)B_2^-B_1, \end{aligned}$$

where $Z \in \mathbb{B}(N,K)$, $Z' \in \mathbb{B}(L,E)$ *are arbitrary operators.*

Proof. From Theorem 3.2.1, we get that the pair of equations (3.4) has a common solution equivalently the two conditions (2) and (3) holds.
On the other hand, since the pair of equations (3.4) is equivalent to

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X \begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix}. \tag{3.14}$$

According to Lemmas 1.3.3 and 1.3.4, we have
$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in \mathbb{B}(H, K \oplus E)$ *and* $\begin{pmatrix} B_1 & B_2 \end{pmatrix} \in \mathbb{B}(L \oplus N, G)$ are regular with inner inverses

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^- = \begin{pmatrix} (I_E - A_2^-A_2)S_1^- & A_2^- - (I_E - A_2^-A_2)S_1^-A_1A_2^- \end{pmatrix}, \tag{3.15}$$

$$\text{and} \qquad \begin{pmatrix} B_1 & B_2 \end{pmatrix}^- = \begin{pmatrix} B_1^- - B_1^-B_2M_1^-(I_G - B_1B_1^-) \\ M_1^-(I_G - B_1B_1^-) \end{pmatrix}, \tag{3.16}$$

respectively. Using Lemma 1.2.2, we deduce that the general solution of (3.14) is given by

$$X = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^- \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \end{pmatrix}^- + F - \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^- \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} F \begin{pmatrix} B_1 & B_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \end{pmatrix}^-. \tag{3.17}$$

By substituting (3.15) and (3.16) in (3.17), we get the solution X as defined in (3.13) such that U, V are given in (3.10) and (3.12) respectively and $F \in \mathbb{B}(G,H)$ is an arbitrary operator □

3.3 Hermitian solution to the operator equation $AXB = C$

Based on Theorem 3.2.1 and Theorem 3.2.2, in this section we give necessary and sufficient conditions for the existence of Hermitian solutions to the operator equation

$$AXB = C,$$

and we obtain the general Hermitian solution to this operator equation. Before enouncing our main results we have the following lemma

Lemma 3.3.1. *Let $A \in \mathbb{B}(H,K)$ and $B \in \mathbb{B}(K,H)$, such that A, B, $S_1 = B^*(I_H - A^-A)$ and $M_1 = (I_H - BB^-)A^*$ are regular, then the operator equation*

$$AXB = C, \tag{3.18}$$

has a Hermitian solution if and only if the pair of operator equations

$$AXB = C \quad and \quad B^*XA^* = C^*, \tag{3.19}$$

has a common solution, a representation of the general Hermitian solution to (3.18) is of the form

$$X_H = \frac{X + X^*}{2},$$

where X is the representation of the general common solution to equations (3.19).

Proof. From Theorem 3.2.1 the pair of operator equations (3.19) has a common solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A) \ \ and \ \ \mathcal{R}(C^*) \subset \mathcal{R}(B^*),$$

and

$$(I_K - S_1S_1^-)B^*A^-CB^-A^*(I_K - M_1^-M_1) = (I_K - S_1S_1^-)C^*(I_K - M_1^-M_1).$$

A representation of the general common solution to equations (3.19) is given by (3.13) in Theorem 3.2.2, where $A_1 = A$, $B_1 = B$, $C_1 = C$, $A_2 = B^*$, $B_2 = A^*$ and $C_2 = C^*$. Clearly X_H is a Hermitian solution to (3.18). □

From the above proof and Theorem 3.2.2, we obtain the following corollary.

Corollary 3.3.1. *Let $A \in \mathbb{B}(H,K)$, $B \in \mathbb{B}(K,H)$, $M_1 = (I_H - BB^-)A^*$ and $S_1 = B^*(I_H - A^-A)$ are regular operators and $C \in \mathbb{B}(K)$, then the operator equation*

$$AXB = C,$$

has a Hermitian solution if and only if

1. *$\mathcal{R}(C) \subset \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subset \mathcal{R}(B^*)$,*

2. *$(I_K - S_1S_1^-)B^*A^-CB^-A^*(I_K - M_1^-M_1) = (I_K - S_1S_1^-)C^*(I_K - M_1^-M_1)$.*

In this case, a representation of the general Hermitian solution is of the form

$$X_H = \frac{X + X^*}{2},$$

where

$$\begin{aligned} X &= (A^- C + (I_H - A^- A)S_1^-(V - B^* A^- C))B^-(I_H - A^* M_1^-(I_H - BB^-)) \\ &\quad + (A^- U + (I_H - A^- A)S_1^-(C^* - B^* A^- U))M_1^-(I_H - BB^-) + F \\ &\quad - (A^- A + (I_H - A^- A)S_1^- S_1)F(BB^- + M_1 M_1^-(I_H - BB^-), \end{aligned} \tag{3.20}$$

where $F \in \mathbb{B}(H)$ is an arbitrary operator and U, V are given by

$$\begin{aligned} U &= CB^- A^*(I_K - M_1^- M_1) + A(B^*)^-(I_K - S_1 S_1^-)C^* M_1^- M_1 + AA^- Z M_1^- M_1 \\ &- A(B^*)^-(I_K - S_1 S_1^-)B^* A^- Z M_1^- M_1, \end{aligned}$$

and

$$\begin{aligned} V &= (I_K - S_1 S_1^-)B^* A^- C + S_1 S_1^- C^*(I_K - M_1^- M_1)(A^*)^- B + S_1 S_1^- Z' B^- B \\ &- S_1 S_1^- Z' B^- A^*(I_K - M_1^- M_1)(A^*)^- B, \end{aligned}$$

where $Z, Z' \in \mathbb{B}(K)$ are arbitrary operators.

Corollary 3.3.2. *Let $A \in \mathbb{B}(H, K)$, $C \in \mathbb{B}(K)$ such that A is regular and $C^* = C$. Then the operator equation*

$$AXA^* = C, \tag{3.21}$$

has a Hermitian solution $X \in \mathbb{B}(H)$ if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A).$$

A representation of the general Hermitian solution is

$$X = A^- C(A^-)^* + Z - A^- AZ(A^- A)^*, \tag{3.22}$$

where $Z \in \mathbb{B}(H)$ is an arbitrary Hermitian operator.

Proof. We put $B = A^*$ in Corollary 3.3.1 we get the result. □

example 3.3.1. *Let the operator equation $AXA^* = C$, such that $A : \ell^2 \to \mathbb{C}$ is defined by*

$$A(x_1, x_2, ...) = \sum_{k=1}^{\infty} \frac{x_k}{2^{k-1}},$$

and $C : \mathbb{C} \to \mathbb{C}$ is defined by

$$C(\lambda) = \frac{\lambda}{24}.$$

It is easy to check that $A \in \mathbb{B}(\ell^2, \mathbb{C})$. *We have* $A^* : \mathbb{C} \to l^2$ *is defined by*

$$A^*(\lambda) = \lambda(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...),$$

for all $\lambda \in \mathbb{C}$. *The inner inverse* $A^- : \mathbb{C} \to l^2$ *of* A *is defined by*

$$A^-(\lambda) = \frac{3\lambda}{4}(1, \frac{1}{2}, \frac{1}{4}, ...).$$

We have

$$\begin{aligned} AA^-C(\lambda) &= AA^-(\frac{\lambda}{24}), \\ &= \frac{\lambda}{32}\sum_{k=0}^{\infty}(\frac{1}{4})^k, \\ &= \frac{\lambda}{32} \times \frac{4}{3}, \\ &= C(\lambda). \end{aligned}$$

Hence, from Corollary 3.3.2 we deduce that the operator equation $AXA^* = C$ *has a Hermitian solution given by*

$$X_H = AC(A^-)^* + Z + A^-AZ(A^-A)^*.$$

Where Z *is an arbitrary Hermitian operator.*

As a consequence of Corollary 3.3.1 we obtain the well-known Theorem of Alegra Dajić and J.J. Koliha [16, Theorem 3.1].

Corollary 3.3.3. *[16, Theorem 3.1] Let* $A, C \in \mathbb{B}(H, K)$ *such that* A *is a regular operator. Then the operator equation*

$$AX = C,$$

has a Hermitian solution $X \in \mathbb{B}(H)$ *if and only if*

$$AA^-C = C \text{ and } AC^* \text{ is Hermitian.}$$

The general Hermitian solution is of the form

$$X = A^-C + (I_H - A^-A)(A^-C)^* + (I_H - A^-A)Z'(I_H - A^-A)^*,$$

where $Z' \in \mathbb{B}(H)$ *is an arbitrary Hermitian operator.*

Proof. By applying Corollary 3.3.1, the operator equation $AX = C$ has a Hermitian solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A),$$

which is equivalent to

$$AA^{-}C = C,$$

and

$$(I_H - I_H + A^{-}A)A^{-}CA^{*} = (I_H - I_H + A^{-}A)C^{*},$$

this implies that

$$CA^{*} = AC^{*}.$$

Hence, AC^{*} is Hermitian. In this case,

$$\begin{aligned} X &= [A^{-}C + (I_H - A^{-}A)(A^{-}C + (I_H - A^{-}A)C^{*}(A^{*})^{-} \\ &+ (I_H - A^{-}A)Z'(I_H - A^{-}A)^{*} - A^{-}C)], \\ &= A^{-}C + (I_H - A^{-}A)(A^{-}C)^{*} + (I_H - A^{-}A)Z'(I_H - A^{-}A)^{*}. \end{aligned}$$

It follows that,

$$\begin{aligned} X_H &= \frac{X + X^{*}}{2}, \\ &= A^{-}C + (I_H - A^{-}A)(A^{-}C)^{*} + (I_H - A^{-}A)Z'(I_H - A^{-}A)^{*}. \end{aligned}$$

□

3.4 Hermitian solutions to the operator equation $AXA^{*} + BYB^{*} = C$

In this section, we determine conditions for the existence of a Hermitian solution to the operator equation $AXA^{*} + BYB^{*} = C$.

Theorem 3.4.1. *Let $A, B \in \mathbb{B}(H, K)$ and $A_1 = (I_K - AA^{-})B$, $C_1 = (I_K - AA^{-})C$ and $S_2 = B(I_H - A_1^{-}A_1)$ be all regular and $C \in \mathbb{B}(K)$ is Hermitian. Then the operator equation*

$$AXA^{*} + BYB^{*} = C, \tag{3.23}$$

has a Hermitian solution if and only if

1. $A_1A_1^{-}(I_K - AA^{-})C(B^{*})^{-}B^{*} = (I_K - AA^{-})C$,
2. $(I_K - S_2S_2^{-})[C - BA_1^{-}(I_K - AA^{-})C(B^{*})^{-}B^{*}](I_K - (A^{-})^{*}A^{*}) = 0$.

In this case, a representation of the general Hermitian solution is of the form

$$(X_H, Y_H) = (\frac{X+X^*}{2}, \frac{Y+Y^*}{2}),$$

where X *and* Y *are given by*

$$\begin{cases} X = A^-(C - BYB^*)(A^*)^- + F - A^-AF(A^-A)^*, \\ Y = A_1^-(I_K - AA^-)C(B^*)^- + (I_H - A_1^-A_1)S_2^-[V - BA_1^-(I_K - AA^-)C](B^*)^- \\ \quad +U - [A_1^-A_1 + (I_H - A_1^-A_1)S_2^-S_2]UB^*(B^*)^-, \end{cases}$$

and

$$V = (I_K - S_2S_2^-)BA_1^-(I_K - AA^-)C + S_2S_2^-C(I_K - (A^-)^*A^*)(A_1^*)^-B^* \\ + S_2S_2^-Z(B^*)^-(I_H - A_1^*(A_1^-)^*)B^*,$$

with $F \in \mathbb{B}(H)$*,* $U \in \mathbb{B}(H)$ *and* $Z \in \mathbb{B}(K)$ *are arbitrary Hermitian operators.*

Proof. The operator equation (3.23) is equivalent to

$$AXA^* = C - BYB^*. \tag{3.24}$$

Applying Corollary 3.3.2, the operator equation (3.24) has a Hermitian solution if and only if

$$\begin{aligned} \mathcal{R}(C - BYB^*) \subset \mathcal{R}(A) &\Leftrightarrow AA^-(C - BYB^*) = (C - BYB^*), \\ &\Leftrightarrow (I - AA^-)(C - BYB^*) = 0. \end{aligned} \tag{3.25}$$

Then, (3.25) is equivalent to the operator equation

$$A_1YB^* = C_1, \tag{3.26}$$

with $A_1 = (I_K - AA^-)B$, $C_1 = (I_K - AA^-)C$. From Corollary 3.3.1, the operator equation (3.26) has a Hermitian solution if and only if

$$\begin{aligned} \mathcal{R}(C_1) \subset \mathcal{R}(A_1) &\Leftrightarrow A_1A_1^-C_1 = C_1, \\ &\Leftrightarrow A_1A_1^-(I_K - AA^-)C = (I_K - AA^-)C, \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \mathcal{R}(C_1^*) \subset \mathcal{R}(B) &\Leftrightarrow C_1(B^*)^-B^* = C_1, \\ &\Leftrightarrow (I_K - AA^-)C(B^*)^-B^* = (I_K - AA^-)C. \end{aligned} \tag{3.28}$$

From (3.27) and (3.28), we get

$$A_1A_1^-(I_K - AA^-)C(B^*)^-B^* = (I_K - AA^-)C.$$

On the other hand, we have

$$(I_K - S_2 S_2^-) B A_1^- (I_K - AA^-) C (B^*)^- A_1^* = (I_K - S_2 S_{2-}) C (I_K - (A^-)^* A^*),$$

this implies that

$$(I_K - S_2 S_2^-)[C - B A_1^- (I_K - AA^-) C (B^*)^- B^*](I_K - (A^-)^* A^*) = 0.$$

A representation of the general Hermitian solution to the operator equation (3.26) is of the form

$$Y_H = \frac{Y + Y^*}{2},$$

where Y is given by (3.20) in Corollary (3.3.1) such that $A = A_1$, $B = B^*$ and $C = C_1$.

$$\begin{aligned} Y &= A_1^- (I_K - AA^-) C (B^*)^- + (I_H - A_1^- A_1) S_2^- [V - B A_1^- (I_K - AA^-) C](B^*)^- \\ &+ U - [A_1^- A_1 + (I_H - A_1^- A_1) S_2^- S_2] U B^* (B^*)^-, \end{aligned}$$

and

$$\begin{aligned} V = (I_K - S_2 S_2^-) B A_1^- (I_K - AA^-) C + S_2 S_2^- C (I_K - (A^-)^* A^*)(A_1^*)^- B^* \\ + S_2 S_2^- Z (B^*)^- (I_H - A_1^* (A_1^-)^*) B^*, \end{aligned}$$

with $U \in \mathbb{B}(H)$ and $Z \in \mathbb{B}(K)$ are arbitrary Hermitian operators. We return to the operator equation

$$AXA^* = C - BYB^*,$$

in order to fined the Hermitian solution X. By Corollary 3.3.2, the operator equation (3.24) has a Hermitian solution if and only if

$$\mathcal{R}(C - BYB^*) \subset \mathcal{R}(A).$$

So the operator equation (3.24) has a Hermitian solution X is given by

$$X_H = X = A^- (C - BYB^*)(A^*)^- + F - A^- A F (A^- A)^*,$$

with $F \in \mathbb{B}(H)$ is an arbitrary Hermitian operator. □

Chapter 4

Lyapunov equation for infinite dimensional discrete bilinear systems

4.1 Introduction

Control theorists in recent years developed extensively the mathematical theory of so-called bilinear systems.

Bilinear systems are an important subclass of nonlinear dynamical systems, with numerous applications in engineer, biology, ecology, physical process, and economics. The main reason for this is that offers considerable intrinsic theoretical interests since they form a transitive class between the linear and the general nonlinear problems.

Since the beginning of the 1970's, they have attracted the attention of many researchers for example Mohler (1973), Bruni, Dipillo, and Koch (1974) Espana and Landau (1978), Brockett (1979), and Mohler and Kolodziej (1980)..., Some of them focused on continuous-time systems, and others on discrete-time. A simple example is as follows.

example 4.1.1. ***Automobiles***(*Mohler, 1987*)

The frictional force between an automobile brake shoe and drum is nearly proportional to the product of the orthogonal force u_1 between the surfaces and their relative velocity. Thought actually involving by the mechanical brake is commonly approximated by

$$f_b = c_b u_1 \frac{dx}{dt}.$$

Then, by a summation of engine force u_2 with inertial, braking, and other frictional *forces, it is seen from Newton's second law that the state of the vehicle is given by*

$$\begin{aligned} F &= m\frac{d^2x}{dt^2} = -kc_f\frac{dx}{dt} - kf_b + u_2 \\ &= -kc_f\frac{dx}{dt} - kc_b u_1\frac{dx}{dt} + u_2 \\ \frac{d^2x}{dt^2} &= \frac{-kc_f}{m}\frac{dx}{dt} - \frac{kc_b u_1}{m}\frac{dx}{dt} + \frac{u_2}{m}. \end{aligned}$$

Let $x_1 = x,\ x_2 = \frac{dx_1}{dt}$, *then we have the state equation is as follows:*

$$\frac{dX}{dt} = AX + u_1 BX + Cu_2, \tag{4.1}$$

where X *is composed of* x_1*, position, and* x_2*, velocity;* $C = [0, 1/m]^T$;

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -kc_f/m \end{pmatrix};$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & -kc_b/m \end{pmatrix};$$

here k *is a proportionality constant,* c_f *is a vehicle frictional constant,* c_b *is a brake constant, and* m *is vehicle mass. Here (4.1) is a bilinear systems.*

Since most models in the real world are nonlinear, recently more and more attention to the stability of nonlinear systems. Specifically, the stability of discrete bilinear systems operating in a stochastic environment, where sufficient conditions for mean-square stability were established. Many results have been obtained of this stability, in 1985, Kubrusly and Costa [35] gives the necessary and sufficient conditions for mean-square stability of finite-dimensional discrete bilinear systems driven by random sequences. In 1986, Kubrusly [30] got mean square stability conditions for discrete bilinear systems only independence and wide sense stationarity are required for the second-order disturbance sequences involved. In 1989, X. Yang et al [66] drove sufficient conditions ensuring the stability and asymptotic stability of discrete bilinear systems with output feedback. Also, in [67], X. Yang et al gives mean square stability conditions for stochastic models without the assumption of stationarity for the random noise. After, in 1991 Costa and C. S. Kubrusly present a study was motivated by the earlier works on finite-dimensional stochastic bilinear systems in [30] and on infinite-dimensional deterministic linear systems. Suppose in this chapter that H is a separable nontrivial complex Hilbert space.

Consider two operators sequences $\{R_i \in \mathbb{B}(H); i \geq 0\}$, and $\{X_i \in \mathbb{B}(H); i \geq 0\}$ recursively defined by a linear autonomous difference equation

$$X_{i+1} = \mathcal{M}(X_i) + R_{i+1}, \qquad X_0 = R_0, \tag{4.2}$$

where

$$\mathcal{M} : \mathbb{B}(H) \longrightarrow \mathbb{B}(H).$$

Recall that $\{X_i \in \mathbb{B}(H); i \geq 0\}$ converges in $\mathbb{B}(H)$ whenever $\{R_i \in \mathbb{B}(H); i \geq 0\}$ converges in $\mathbb{B}(H)$ if and only if

$$r_\sigma(\mathcal{M}) < 1. \tag{4.3}$$

Let us consider first a well-known particular case $\mathcal{M} = \mathcal{M}_A$, where $\mathcal{M}_A \in \mathbb{B}[\mathbb{B}(H)]$ is the multiplication operator defined by

$$\mathcal{M}_A(X) = M_{A,A^*}(X) = AXA^*, \qquad \forall X \in \mathbb{B}(H),$$

for some $A \in \mathbb{B}(H)$. Since $\|\mathcal{M}_A^i\| = \|M_{A,A^*}^i\| = \|A^i\|^2$ for every $i \geq 0$, then

$$r_\sigma(\mathcal{M}_A) = r_\sigma(A)^2,$$

according to the Gelfand formula for the spectral radius. Hence,

$$r_\sigma(\mathcal{M}_A) < 1 \quad \text{if and only if} \quad r_\sigma(A) < 1.$$

However, in [31], C.Kubrusly proved that $r_\sigma(A) < 1$ if and only if

$$\text{for every } Y \in \mathbb{G}^+(H) \text{ there exists a unique solution } X \in \mathbb{G}^+(H)$$

$$\text{for the Lyapunov equation } Y = X - \mathcal{M}_A(X). \tag{4.4}$$

The purpose of this section is to show that the equivalence between (4.3) and (4.4) still holds for a more general case where, instead of setting $\mathcal{M} = \mathcal{M}_A$, we take $\mathcal{M} = \mathcal{M}_A + \mathcal{J}$ with $\mathcal{J}$ is an operator in $\mathbb{B}[\mathbb{B}(H)]$ defined as follows:

$$\mathcal{J}(X) = \sum_{k,l=1}^{\infty} \langle Ce_l; e_k \rangle A_k X A_l^* \qquad \forall X \in \mathbb{B}(H),$$

where

- $\{A_k \in \mathbb{B}(H); k \geq 0\}$ is an arbitrary bounded sequence of operators,
- $C \in \mathbb{B}_1^+(H)$ is any nonnegative nuclear operator,
- $\{e_k; k \geq 1\}$ is a suitable orthonormal basis for H ensuring convergence for the above infinite series.

This supplies a necessary and sufficient condition for the convergence preserving property between input and state correlation sequences, as required in the mean-square stability problem, for infinite-dimensional discrete bilinear systems.

4.2 Preliminaries

Consider a bounded sequence of operators $\{A_k \in \mathbb{B}(H); k \geq 1\}$, and let $\{e_k; k \geq 1\}$ be an orthonormal basis for H. Given $C \in \mathbb{B}_1^+(H)$, set $\mathcal{J}_n \in \mathbb{B}[\mathbb{B}(H)]$ as follows. For each integer $n \geq 1$

$$\mathcal{J}_n(X) = \sum_{k,l=1}^{n} \langle Ce_l; e_k \rangle M_{A_k, A_l^*}(X) \qquad \forall X \in \mathbb{B}(H)$$

Assumption

$$\mathcal{J}_n \to \mathcal{J} \in \mathbb{B}[\mathbb{B}(H)] \quad as \quad n \to \infty \quad in \quad \mathbb{B}[\mathbb{B}(H)].$$

Under the above assumption, write

$$\mathcal{J}(X) = \sum_{k,l=1}^{\infty} \langle Ce_l; e_k \rangle M_{A_k, A_l^*}(X), \qquad \forall X \in \mathbb{B}(H),$$

and, given $A \in \mathbb{B}(H)$, let $\mathcal{M} \in \mathbb{B}[\mathbb{B}(H)]$ be defined as

$$\mathcal{M}(X) = M_{A,A^*}(X) + \mathcal{J}(X), \qquad \forall X \in \mathbb{B}(H).$$

Remark 4.2.1. *Note that the very definition of $\{\mathcal{J}_n \in \mathbb{B}[\mathbb{B}(H)], n \geq 1\}$ in terms of bounded sequence $\{A_k \in \mathbb{B}(H); k \geq 0\}$ and a non-negative nuclear operator C is not enough to ensure its convergence as proved in an example given in [13]*

For any $W \in \mathbb{G}^+(H)$ let $\mathcal{W} \in \mathbb{G}[\mathbb{B}(H)]$ is the multiplication operator defined by

$$\mathcal{W}(X) = M_{W,W}(X) = WXW, \qquad \forall X \in \mathbb{B}(H),$$

so that $\mathcal{W}^{-1} \in \mathbb{G}[\mathbb{B}(H)]$ is such that

$$\mathcal{W}^{-1}(X) = M_{W^{-1},W^{-1}}(X) = W^{-1}XW^{-1}, \quad \forall X \in \mathbb{B}(H).$$

Proposition 4.2.1. *For every $W \in \mathbb{G}^+(H)$*

$$(P1) \qquad \mathcal{W}^{-1}\mathcal{M}\mathcal{W}^{-1}(\mathbb{B}^+(H)) \subseteq \mathbb{B}^+(H)$$

$$(P2) \quad \|\mathcal{W}^{-1}\mathcal{M}\mathcal{W}^{-1}\| = \|\mathcal{W}^{-1}\mathcal{M}\mathcal{W}^{-1}(I)\|$$

Proof. Consider the direct sum $\mathcal{H} = \mathbb{C} \oplus H$, which is a Hilbert space with inner product given by

$$\langle \mathbf{x}; \mathbf{y} \rangle = \xi\bar{v} + \langle x; y \rangle$$

for all $\mathbf{x} = \xi \oplus x$ and $\mathbf{y} = v \oplus y$ in $\mathcal{H}$, where $\xi, v \in \mathbb{C}$, and $x, y \in H$.
Given the orthonormal basis $\{e_k; k \geq 1\}$ for H set, for each $k \geq 0$

$$f_k = \begin{cases} 1 \oplus 0 & k = 0, \\ 0 \oplus e_k & k \geq 1; \end{cases}$$

so that $\{f_k; k \geq 0\}$ is an orthonormal basis for $\mathcal{H}$.
Let $\mathbf{C} = (1 \oplus C) \in \mathbb{B}_1^+(\mathcal{H})$ be the direct sum of the identity on $\mathbb{C}$ with $C \in \mathbb{B}_1^+(H)$, so that

$$\langle \mathbf{C}\mathbf{x}; \mathbf{y} \rangle = \langle \xi \oplus Cx; v \oplus y \rangle = \xi \bar{v} + \langle Cx; y \rangle,$$

for all $\mathbf{x} = \xi \oplus x \in \mathcal{H}$ and $\mathbf{y} = v \oplus y \in \mathcal{H}$. In particular

$$\langle \mathbf{C} f_l; f_k \rangle = \begin{cases} 1 & if k = l = 0 \\ \langle C e_l; e_k \rangle & if \quad k, l \geq 1, \\ 0 & otherwise. \end{cases}$$

Thus, for each $n \geq 1$, set $\mathcal{M}_n \in \mathbb{B}[\mathbb{B}(\mathcal{H})]$ as follows

$$\begin{aligned} \mathcal{M}_n(X) &= M_{A,A^*}(X) + \mathcal{J}_n(X) \\ &= AXA^* + \mathcal{J}_n(X) \\ &= \sum_{k,l=0}^{n} \langle \mathbf{C} f_l; f_k \rangle A_k X A_k^*, \end{aligned}$$

where $A_0 = A \in \mathbb{B}(\mathcal{H})$. So that

$$\mathcal{M}_n \to \mathcal{M} \in \mathbb{B}[\mathbb{B}(\mathcal{H})] \quad as \quad n \to \infty \text{ in } \mathbb{B}[\mathbb{B}(\mathcal{H})],$$

where

$$\begin{aligned} \mathcal{M}(X) &= AXA^* + \mathcal{J}(X) \\ &= \sum_{k,l=0}^{\infty} \langle \mathbf{C} f_l; f_k \rangle A_k X A_k^*, \qquad \forall X \in \mathbb{B}(\mathcal{H}). \end{aligned}$$

According to the convergence assumption on $\{\mathcal{J}_n \in \mathbb{B}[\mathbb{B}(\mathcal{H})]; n \geq 1\}$.

Now, since $\mathbf{C} \in \mathbb{B}_1^+(\mathcal{H})$ the spectral theorem says that

$$\mathbf{C}\mathbf{x} = \sum_{j=0}^{\infty} \gamma_j \langle x; f'_j \rangle f'_j, \qquad \forall x \in H,$$

for some orthonormal basis $\{f'_j, j \geq 0\}$ for $\mathcal{H}$, where $\gamma_j \geq 0$ for every $j \geq 0$.

Then, by continuity of the inner product, for every $k, l \geq 0$

$$\langle \mathbf{C} f_l; f_k \rangle = \sum_{j=0}^{\infty} \gamma_j \langle f_l; f'_j \rangle \langle f'_j; f_k \rangle.$$

Hence, for each $n \geq 1$, for all $x,\ y \in \mathcal{H}$

$$\begin{aligned}
\langle \mathcal{M}_n(X)x; y \rangle &= \sum_{k,l=0}^{n} \sum_{j=0}^{\infty} \gamma_j \langle f_l; f'_j \rangle \langle f'_j; f_k \rangle \langle X A_l^* x; A_k^* y \rangle, \\
&= \sum_{j=0}^{\infty} \gamma_j \sum_{k,l=0}^{n} \langle X \langle f_l; f'_j \rangle A_l^* x; \langle f_k; f'_j \rangle A_k^* y \rangle, \\
&= \sum_{j=0}^{\infty} \gamma_j \langle X \sum_{l=0}^{n} \langle f_l; f'_j \rangle A_l^* x; \sum_{k=0}^{n} \langle f_k; f'_j A_k^* y \rangle.
\end{aligned}$$

Since addition is continuous. Thus, if $X \in \mathbb{B}^+(\mathcal{H})$, then , for every $n \geq 1$

$$\langle \mathcal{M}_n(X)x; x \rangle = \sum_{j=0}^{\infty} \gamma_j \| X^{1/2} \sum_{k=0}^{n} \langle f_l; f'_j \rangle A_k^* x \|^2, \ \ \forall x \in \mathcal{H},$$

which implies that $\mathcal{M}_n(X) \in \mathbb{B}^+(\mathcal{H})$. Therefore,

$$\mathcal{M}(\mathbb{B}^+(\mathcal{H})) \subseteq \mathbb{B}^+(\mathcal{H}), \tag{4.5}$$

because $\mathbb{B}^+(\mathcal{H})$ is closed in $\mathbb{B}(\mathcal{H})$.

Now, by using Schwars inequality twice, and recalling that

$$\| \mathcal{M}_n(I)^{1/2} \| = \| \mathcal{M}_n(I) \|^{1/2}$$

since $\mathcal{M}_n(I) \in \mathbb{B}^+(\mathcal{H})$ for every $n \geq 1$, we get

$$\begin{aligned}
|\langle \mathcal{M}_n(X)x; y \rangle| &\leq \|X\| \sum_{j=0}^{\infty} \gamma_j \| \sum_{l=0}^{n} \langle f_l; f'_j \rangle A_l^* x \| \| \sum_{k=0}^{n} \langle f_k; f'_j \rangle A_k^* y \|, \\
&\leq \|X\| (\sum_{j=0}^{\infty} \gamma_j \| \sum_{l=0}^{n} \langle f_l; f'_j \rangle A_l^* x \|^2)^{1/2} (\sum_{j=0}^{\infty} \gamma_j \| \sum_{k=0}^{n} \langle f_k; f'_j \rangle A_k^* y \|^2)^{1/2}, \\
&= \|X\| \langle \mathcal{M}_n(I)x; x \rangle^{1/2} \langle \mathcal{M}_n(I)y; y \rangle^{1/2}, \\
&= \|X\| \| \mathcal{M}_n(I)^{1/2} x \| \| \mathcal{M}_n(I)^{1/2} y \|, \\
&\leq \|X\| \| \mathcal{M}_n(I) \| \|x\| \|y\|.
\end{aligned}$$

for all $x,\ y \in \mathcal{H}$ and every $n \geq 1$, so that

$$\|\mathcal{M}_n(X)\| = \sup_{\|x\|=\|y\|=1} |\langle \mathcal{M}_n(X)x; y\rangle| \leq \|X\|\|\mathcal{M}_n(I)\|$$

for all $X \in \mathbb{B}(H)$ and every $n \geq 1$. Hence,

$$\|\mathcal{M}_n\| = \|\mathcal{M}_n(I)\|,$$

because

$$\|\mathcal{M}_n(I)\| \leq \|\mathcal{M}_n\| = \sup_{\|X\|=1} \|\mathcal{M}_n(X)\| \leq \|\mathcal{M}_n(I)\|,$$

for every $n \geq 1$. Therefore, since $\mathcal{M}_n \to \mathcal{M}$ as $n \to \infty$ in $\mathbb{B}[\mathbb{B}(H)]$,

$$\|\mathcal{M}\| = \|\mathcal{M}(I)\|. \tag{4.6}$$

Finally, take an arbitrary $W \in \mathbb{G}^+(\mathcal{H})$ and set $\tilde{A}_k = W^{-1}A_kW \in \mathbb{B}(\mathcal{H})$ for every $k \geq 0$. Thus,

$$\begin{aligned} \mathcal{W}^{-1}\mathcal{M}\mathcal{W}(X) &= W^{-1}\mathcal{M}(WXW)W^{-1} \\ &= \sum_{k,l=0}^{\infty} \langle Cf_l; f_k\rangle \tilde{A}_k X \tilde{A}_k^*, \ \ \forall X \in \mathbb{B}(\mathcal{H}). \end{aligned}$$

Hence, (4.5) implies $(P1)$, and (4.6) implies $(P2)$. □

Lemma 4.2.1. *For any $W \in \mathbb{G}^+(H)$ and any $\alpha \in (0, \infty)$.*

1. $\|\mathcal{W}^{-1}\mathcal{M}\mathcal{W}\| \leq \alpha \Leftrightarrow \alpha W^2 - \mathcal{M}(W^2) \in \mathbb{B}^+(H)$,

2. $\|\mathcal{W}^{-1}\mathcal{M}\mathcal{W}\| < \alpha \Leftrightarrow \alpha W^2 - \mathcal{M}(W^2) \in \mathbb{G}^+(H)$.

Proof. Take an arbitrary $W \in \mathbb{G}^+(H)$ and set $\tilde{\mathcal{M}} = \mathcal{W}^{-1}\mathcal{M}\mathcal{W} \in \mathbb{B}[\mathbb{B}(H)]$, so that

$$\tilde{\mathcal{M}}(I) = W^{-1}\mathcal{M}(W^2)W^{-1} \in [\mathbb{B}(H)].$$

Thus, for any $\alpha \in (0; \infty)$

$$\begin{aligned} \langle(\alpha W^2 - \mathcal{M}(W^2))x; x\rangle &= \langle(\alpha W^2 - WW^{-1}\tilde{\mathcal{M}}(I)W^{-1}W)x; x\rangle, \\ &= \langle(\alpha W - W^{-1}\tilde{\mathcal{M}}(I)W^{-1}W)x; Wx\rangle, \\ &= \langle(\alpha WW^{-1}W - W^{-1}\tilde{\mathcal{M}}(I)W^{-1}W)x; Wx\rangle, \\ &= \langle(\alpha WW^{-1} - W^{-1}\tilde{\mathcal{M}}(I)W^{-1})Wx; Wx\rangle, \\ &= \langle(\alpha I - \tilde{\mathcal{M}}(I))Wx; Wx\rangle, \quad \forall x \in H. \end{aligned}$$

Hence

$$\alpha W^2 - \mathcal{M}(W^2) \in \mathbb{B}^+(H)(\in \mathbb{G}^+(H)),$$

if and only if

$$\alpha I - \tilde{\mathcal{M}}(I) \in \mathbb{B}^+(H)(\in \mathbb{G}^+(H)),$$

which in turn is equivalent to

$$\|\tilde{\mathcal{M}}(I)^{1/2}\| \leq \alpha^{1/2}(< \alpha^{1/2}),$$

since $\tilde{\mathcal{M}}(I) \in \mathbb{B}^+(H)$ by proposition 4.2.1 (P1).
However, by Proposition 4.2.1 (P2)

$$\|\tilde{\mathcal{M}}\| = \|\tilde{\mathcal{M}}(I)\| = \|\tilde{\mathcal{M}}(I)^{1/2}\|^2.$$

Thus, the above inequality is equivalent to

$$\|\tilde{\mathcal{M}}\| \leq \alpha(< \alpha).$$

□

Proposition 4.2.2. *[32] Let $A \in \mathbb{B}(H, K)$, The following assertions are equivalent:*

1. *There exists $A^{-1} \in \mathbb{B}(\mathcal{R}(A), K)$.*
2. *$\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = \overline{\mathcal{R}(A)}$.*
3. *There exists a real constant $\alpha > 0$ such that $\|Ax\| \geq \alpha\|x\|$, for all $x \in H$ (i.e. A is bounded below).*

4.3 Stability for discrete bilinear systems

In the following section we shall conclude the announced proof for the equivalence between assertions (4.3) and (4.4) with $\mathcal{M} = \mathcal{M}_A$ replaced by $\mathcal{M} = \mathcal{M}_A + \mathcal{J}$.

Theorem 4.3.1. *The following assertions are equivalent:*

1. *$r_\sigma(\mathcal{M}) < 1$.*
2. *$\mathcal{M}$ is similar to a strict contraction.*
3. *There exists $X \in \mathbb{G}^+(H)$ such that*

$$X - \mathcal{M}(X) \in \mathbb{G}^+(H).$$

4. For every $Y \in \mathbb{G}^+(H)$ there exists $X \in \mathbb{G}^+(H)$ such that

$$Y = X - \mathcal{M}(X).$$

Moreover, if the above holds, then the solution $X \in \mathbb{G}^+(H)$ of the Lyapunov equation $Y = X - \mathcal{M}(X)$, for any $Y \in \mathbb{G}^+(H)$, is unique and given by

$$X = \sum_{j=0}^{\infty} \mathcal{M}^i(Y) = (I - \mathcal{M})^{-1}(Y), \tag{4.7}$$

where I standing for the identity in $\mathbb{B}[\mathbb{B}(H)]$.

Proof. $4 \Rightarrow 3$, is trivially verified.

$3 \Rightarrow 2$ Suppose that there exists $X \in \mathbb{G}^+(H)$ such that $X - \mathcal{M}(X) \in \mathbb{G}^+(H)$. According to Lemma 4.2.1, we have

$$\|\mathcal{W}^{-1}\mathcal{M}\mathcal{W}\| < \alpha \Leftrightarrow \alpha W^2 - \mathcal{M}(W^2) \in \mathbb{G}^+(H).$$

Let $\alpha = 1$ and $X = W^2$, then

$$X - \mathcal{M}(X) \in \mathbb{G}^+(H),$$

this implies that

$$\|\mathcal{W}^{-1}\mathcal{M}\mathcal{W}\| < 1.$$

Thus, $\mathcal{M}$ is similar to a strict contraction.

$2 \Rightarrow 1$ Suppose that $\mathcal{M}$ is similar to a strict contraction, then there exist $W \in \mathbb{G}(H)$ such that $\|\mathcal{W}^{-1}\mathcal{M}\mathcal{W}\| < 1$. Since

$$r_\sigma(\mathcal{W}^{-1}\mathcal{M}\mathcal{W}) \leq \|\mathcal{W}^{-1}\mathcal{M}\mathcal{W}\|,$$

and

$$r_\sigma(\mathcal{W}^{-1}\mathcal{M}\mathcal{W}) = r_\sigma(\mathcal{M}).$$

Thus

$$r_\sigma(\mathcal{M}) < 1.$$

$1 \Rightarrow 4$ Suppose that $r_\sigma(\mathcal{M}) < 1$, so that $\sum_{j=0}^{\infty}\|\mathcal{M}^j\| < \infty$.

According to Proposition 4.2.1 (P1) with $W = I$ we get by induction on j that $\mathcal{M}^j(Y) \in \mathbb{B}^+(H)$ for every $j \geq 0$ and $Y \in \mathbb{G}^+(H)$. Then, for every $n \geq 0$, $\forall x \in H$

$$\begin{aligned} \|(Y^{1/2})^{-1}\|^2\|x\|^2 &\leq \|Y^{1/2}x\|^2, \\ &\leq \sum_{j=0}^{n}\|\mathcal{M}^j(Y)^{1/2}x\|^2, \\ &\leq (\sum_{j=0}^{\infty}\|\mathcal{M}^j\|)\|Y\|\|x\|^2, \end{aligned} \tag{4.8}$$

Hence we may define a map

$$\begin{array}{rcl}\Phi : H & \longrightarrow & l_2(H) \\ x & \longmapsto & \Phi x = \oplus_{j=0}^{\infty}\mathcal{M}^j(Y)^{1/2}x\end{array}$$

Φ is linear and bounded with

$$\|\Phi x\|^2_{l_2(H)} = \sum_{j=0}^{\infty}\|\mathcal{M}^j(Y)^{1/2}x\|^2,\ \forall x \in H. \tag{4.9}$$

From (4.8) and (4.9), the operator Φ is bounded below and from the Proposition 4.2.2 we have, Φ has a bounded inverse in $\mathcal{R}(\Phi)$. Thus

$$\Phi^*\Phi \in \mathbb{G}^+(H).$$

By the continuity of the inner product we get, for every $x \in H$,

$$\begin{aligned}\langle \mathcal{M}(\Phi^*\Phi)x; x\rangle &= \langle \sum_{k,l=0}^{\infty}\langle Ce_l; e_k\rangle A_k\Phi^*\Phi A_l^* x; x\rangle, \\ &= \sum_{k,l=0}^{\infty}\langle Ce_l; e_k\rangle\langle \Phi A_l^* x; \Phi A_k^* x\rangle_{l_2(H)}.\end{aligned}$$

However, for each $k, l \geq 0$ and every $x \in H$,

$$\begin{aligned}\langle \Phi A_l^* x; \Phi A_k^* x\rangle_{l_2(H)} &= \sum_{j=0}^{\infty}\langle \mathcal{M}^j(Y)^{\frac{1}{2}}A_l^* x; \mathcal{M}^j(Y)^{\frac{1}{2}}A_k^* x\rangle, \\ &= \sum_{j=0}^{\infty}\langle A_k\mathcal{M}^j(Y)A_l^* x; x\rangle, \\ &= \langle A_k(\sum_{j=0}^{\infty}\mathcal{M}^j(Y))A_l^* x; x\rangle,\end{aligned}$$

since $\{\sum_{j=0}^{n}\mathcal{M}^j;\ n \geq 0\}$ converges in $\mathbb{B}[\mathbb{B}(H)]$ whenever $r_\sigma(\mathcal{M}) < 1$. Therefore, for all $x \in H$,

$$\begin{aligned}
\langle \mathcal{M}(\Phi^*\Phi)x; x\rangle &= \langle \sum_{k,l=0}^{\infty} \langle Ce_l; e_k\rangle A_k (\sum_{j=0}^{\infty} \mathcal{M}^j(Y)) A_l^* x; x\rangle, \\
&= \langle \mathcal{M}(\sum_{j=0}^{\infty} \mathcal{M}^j(Y))x; x\rangle, \\
&= \sum_{j=0}^{\infty} \langle \mathcal{M}^{j+1}(Y)x; x\rangle, \\
&= \sum_{j=1}^{\infty} \|\mathcal{M}^j(Y)^{1/2}x\|^2, \\
&= \|\Phi x\|_{l_2(H)}^2 - \|Y^{1/2}x\|^2,
\end{aligned}$$

so that, for all $x \in H$,

$$\begin{aligned}
\langle (Y - \Phi^*\Phi + \mathcal{M}(\Phi^*\Phi))x; x\rangle &= \|Y^{1/2}x\|^2 - \|\Phi x\|_{l_2(H)}^2 + \langle \mathcal{M}(\Phi^*\Phi)x; x\rangle, \\
&= 0,
\end{aligned}$$

Hence

$$Y = \Phi^*\Phi - \mathcal{M}(\Phi^*\Phi).$$

Thus, $1 \Rightarrow 4$ with $X = \Phi^*\Phi \in \mathbb{G}^+(H)$. Moreover, such an operator is unique. Indeed, if $Y = X - \mathcal{M}(X) \in \mathbb{G}^+(H)$ for some $X \in \mathbb{G}^+(H)$, then for all $x \in H$ and every $j \geq 0$

$$\begin{aligned}
\|\mathcal{M}^j(Y)^{1/2}x\|^2 &= \langle \mathcal{M}^j(Y)x; x\rangle, \\
&= \|\mathcal{M}^j(X)^{1/2}x\|^2 - \|\mathcal{M}^{j+1}(X)^{1/2}x\|^2.
\end{aligned}$$

Hence, for every $x \in H$,

$$\begin{aligned}
\|\Phi x\|_{l_2(H)}^2 &= \lim_{n \longrightarrow \infty} \sum_{j=0}^{n} \|\mathcal{M}^j(Y)^{1/2}x\|^2, \\
&= \lim_{n \longrightarrow \infty} (\|X^{1/2}x\|^2 - \|\mathcal{M}^{n+1}(X)^{1/2}x\|^2), \\
&= \|X^{1/2}x\|^2,
\end{aligned}$$

since

$$\begin{aligned}
\|\mathcal{M}^n(X)^{1/2}x\|^2 &\leq \|\mathcal{M}^n\|\|X\|\|x\|^2 \\
&\longrightarrow 0 \quad as \;\; n \longrightarrow \infty,
\end{aligned}$$

because $r_\sigma(\mathcal{M}) < 1$. Therefore

$$\begin{aligned} \langle(\Phi^*\Phi - X)x; x\rangle &= \|\Phi x\|^2_{l_2(H)} - \|X^{1/2}x\|^2, \\ &= 0, \ \forall x \in H. \end{aligned}$$

So that $X = \Phi^*\Phi$, which proves uniqueness. Finally, if $X \in \mathbb{G}^+(H)$ is the solution of $Y = X - \mathcal{M}(X) \in \mathbb{G}^+(H)$, then for each $n \geq 0$

$$\begin{aligned} \sum_{j=0}^{n} \mathcal{M}^j(Y) &= \sum_{j=0}^{n} (\mathcal{M}^j(X) - \mathcal{M}^{j+1}(X)), \\ &= X - \mathcal{M}^{n+1}(X) \in \mathbb{B}^+(H). \end{aligned}$$

Thus

$$\begin{aligned} \|\sum_{j=0}^{n} \mathcal{M}^j(Y) - X\| &= \|\mathcal{M}^{n+1}(X)\|, \\ &\leq \|\mathcal{M}^{n+1}\|\|X\| \longrightarrow 0 \ as \ n \longrightarrow \infty, \end{aligned}$$

because $r_\sigma(\mathcal{M}) < 1$. Hence

$$\sum_{j=0}^{n} \mathcal{M}^j(Y) \longrightarrow X \ as\, n \longrightarrow \infty \quad \text{in} \ \mathbb{B}(H).$$

However, since $r_\sigma(\mathcal{M}) < 1$ implies that

$$(I - \mathcal{M}) \in \mathbb{G}[\mathbb{B}(H)],$$

and that $\{\sum_{j=0}^{n} \mathcal{M}^j \in \mathbb{B}[\mathbb{B}(H)]; n \geq 0\}$ converges in $\mathbb{B}[\mathbb{B}(H)]$ to

$$(I - \mathcal{M})^{-1} \in \mathbb{G}[\mathbb{B}(H)].$$

□

Conclusion

In this work we presented new necessary and sufficient conditions for the existence of solutions and Hermitian solutions to some operators equations in Banach and Hilbert spaces and we got the general solutions and Hermitian solutions via the inner inverses of elementary operators and via simple operators. Our results generalize and improve many previous results in the literature.

Bibliography

[1] R. Arens, Dense inverse limit rings, Michigan Math. J., 5 (1958), 169-182.

[2] Z. Bakhtiari and S. M. Vaezpour, Positive solutions to the system of operator equations $T_iX = U_i$ and $T_iXV_i = U_i$, Journal of Math. Anal., 5 (2016), 102-117.

[3] J. K. Baksalary and R. Kala, The matrix equation $AX - YD = C$, Linear Algebra and Appl, 25 (1979), 41-43.

[4] J. K. Baksalary and R. Kala, The matrix equation $AXB + CYD = E$, Linear Algebra and Appl, 30 (1980), 141-147.

[5] A. Ben-Israel, T.N.E. Greville, Generalized Inverses, Theory and Applications, second ed, Springer, 2003.

[6] A. Bjerhamar, A Generalized Matrix Algebra, Trans. Roy. Inst. Tech. Stockholm. 124 (1958), 32

[7] A. Bjerhammar, Application of calculus of matrices to method of least squares with special reference to geodetic calculations, Trans. Roy. Inst. Tech. Stockholm (1951), no. 49, 86 pp. (2 plates).

[8] A. Bjerhammar, Rectangular reciprocal matrices, with special reference to geodetic calculations, Bull. Géodésique (1951), 188-220.

[9] A. Boussaid and F. Lombarkia, Hermitian solutions to the equation $AXA^* + BYB^* = C$, for Hilbert space operators, Accepted in Facta universitatis (Niš). Ser. Math. Inform.

[10] S. R. Caradus, Generalised Inverse and operator Theory, Queen's Paper in Pure and Appl Math, vol. 50, Quenn's Univ., Kingston, ON, 1978.

[11] X.W. Chang, J. Wang, The symmetric solutions of the matrix equations $AX + YA = C$, $AXA^T + BYB^T = C$ and $(A^TXA, B^TXB) = (C, D)$, Linear Algebra Appl. 179 (1993), 171–189.

[12] S. L. Compbell, CD, Meyer, Generalized inverses of linear transformations, Pitman Publishing limited, 1979, Reprinting, SIMA, 2010.

[13] O. L. V. Costa and C. S. Kubrusly, Lyapunov equation for infinite dimensional discrete bilinear systems, Systems & Control Letters, 17 (1991), 71-77.

[14] A. Dajić, Common solution of linear equations in ring, with application, Electonic Journal of Linear Algebra, Volume 30, 2015.

[15] A. Dajić, J.J. Koliha, Equations $ax = c$ and $xb = d$ in rings and rings with involution with applications to Hilbert space operators, Linear Algebra and its appl, 429 (2008), 1779-1809.

[16] A. Dajić and J.J. Koliha, Positive solution to the equation $AX = C$ and $XB = D$ for hilbert space operators, J. Math. Anal. Appl. 333 (2007), 567-576.

[17] Y. Deng, X. Hu, On solutions of matrix equation $AXA^T + BYB^T = C$, J. Comput. Math. 23 (2005), 17–26.

[18] D.S. Djordjević, V. Rakočević, Lectures on generalized inverses, Faculty of Sciences and Mathematics, University of Nis, (2008).

[19] R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, Proc. Amer. Math. Soc., 17(1966), 413-415.

[20] F. O. Farid, M. S. Moslehian, Qing-Wen Wang, Zhong-Cheng Wu, On the Hermitian solutions to a system of adjointable operator equations, Linear Algebra and its Appl., 437 (2012), 1854–1891.

[21] P. A. Fillmore and J. P. Williams, On operator ranges, Adv. Math, 7 (1971), 244-281.

[22] I. Fredholm, Sur une classe d' quations fonctionelles. Acta Math. 27(1903), 365-390.

[23] J. Gro, A note on the general Hermitian solution to $AXA^* = B$, Bull. Malaysian Math. Soc. (Second Series), 21 (1998), 57-62.

[24] W. A. Hurwitz, Or the pseudo-resolvent to the kernel uf an integral equation. Trans. Amer. Math. Soc. 13(1912), 405-418.

[25] C. G. Khatri and S. K. Mitra, Hermitian and nonnegative definite solution of linear matrix equations, SIAMJ. Appl. Math., 31 (1976), 579-585.

[26] P. Kirrinnis, Fast algorithms for the Sylvester equation $AX - XB = C$. Theor Comput Sci. 259 (2001), 623–638.

[27] J.J. Koliha, V. Rakočević, Invertibility of the diference of idempotents, Linear Multilinear Algebra, 51 (2003), 97-110.

[28] J.J. Koliha, V. Rakočević, I. Starskraba, The diference and sum of projectors, Linear Algebra Appl., 388 (2004), 279-288.

[29] C. S. Kubrusly, Mean square stability dor discrete bounded linear systems in Hilbert space, SIAM J. Control and Optimisation, Vol. 23, No. 1, 1985.

[30] C. S. Kubrusly, On discrete stochastic bilinear systems stability, J. Math. Anal. Appl. 113(1986), 36-58.

[31] C. S. Kubrusly, A Note On The Lyapunov Equation For Discrete Linear Systems In Hilbert Space , App. Math. Lett. Vol. 2, (1989), 349-352.

[32] C. S. Kubrusly, An Introduction to Models and Decompositions in Operator Theory, Birkhuser, Boston, 1997.

[33] C. S. Kubrusly, Spectral Theory of Operators on Hilbert Spaces, Birkhuser, London, 2012.

[34] C. S. Kubrusly and O. L. V. Costa, Mean-square stability for discrete bilinear systems in Hilbert space , Systems & Control Letters, 19(1992), 205-211.

[35] C. S. Kubrusly and O. L. V. Costa, Mean Square Stability Conditions for Discrete Stochastic Bilinear Systems, IEEE Transactions on Automatic Control, 30(1985), 1082–1087.

[36] N. Li, J. Jiang and W. Wang, Hermitian solution to a quaternion matrix equation, Appl. Mechanics and materials, 50-51 (2001), 391-395.

[37] Lombarkia F, Boussaid A (2019) Operator equations and inner inverses of elementary operators, Linear and Multilinear Algebra, DOI: 10.1080/03081087.2019.1652239.

[38] S. K. Mitra, Common solution to a pair of linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$, Proc. Cambridge Philos. Soc., 74 (1973), 213-216.

[39] S. K. Mitra, A pair of simultaneous linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ and a matrix programming problem, Linear Algebra Appl, 131 (1990), 107-123.

[40] M. Mohammadzadeh Karizaki, M. Hassani and S. S. Dragomir, Explicit solution to modular operator equation $TXS^* - SX^*T^* = A$, Krag. Journal of math., 2 (2016), 280-289.

[41] M. Mohammadzadeh Karizaki and D. S. Djordjevi, Solution to some solvable modular operator equation, Func. Analysis, Appr. and Comp., 1 (2016), 7-13.

[42] E. H. Moore, General Analysis Part I: Memoirs Amer. Philos. Soc. 1(1935), especially pp. 147-209.

[43] Z. Mousavi, R. Eskandari, M. S. Moslehian and F. Mirzapour, Operator equation $AX + YB = C$ and $AXA^* + BYB^* = C$ in Hilbert $\mathbb{C}^*$- modules, Linear Alg. App., 517 (2017), 85-98.

[44] M. Z. Nashed, Generalized inverses, Theory and Application, Academic Press, NY, (1976).

[45] A. Navarra, P. L. Odell and D. M. Young, A representation of the general common solution to the matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ with applications, Comput. Math. Appl, 41 (2001), 929-935.

[46] A. B. Ozguler. The equation $AXB{+}CYD{=}E$ over a principal ideal domain. SIAM J. Matrix Anal. Appl, 12-3 (1991), 581-591.

[47] P. Patrício and R. Puystjens, About the von Neumann regularity of triangular block matrices, Linear Algebra and its Appl, 332/334 (2001), 485–502.

[48] YX Peng, XY Hu, L. Zhang, An iteration method for the symmetric solutions and the optimal approximation solution of the matrix equation AXB = C. Appl Math Comput, 3(160) (2005), 763–777.

[49] ZY Peng, XY Hu, The reflexive and anti-reflexive solutions of the matrix equation AX = B, Linear Algebra Appl, 375 (2003), 147–155.

[50] R. Penrose, A generalized inverse for matrices. Mathematic Proceedings of the Cambridge Phil. Soc. 51(1955), 406-413.

[51] S. V. Phadke and N. K. Thakare, Generalized inverses and operator equations, Linear Algebra and its Appl, 23 (1979), 191-199.

[52] C. Prévôt and M. Röckner, A Concise Course on Stochastic Partial Differential Equations, Springer, 2007.

[53] W. T. Reid, Generalized inverses of differential and integral operators, pp. 1-25 in Theory and Applications of Generalized Inverses of Matrices: Symposium Proceedings, Texas Technological College, Lubbcck , Texas, 1968.

[54] P. Sam Johnson, Multiplication operators with closed range in operator algebra, J. Analysis Num. Theory 1, No. 1, 1-5 (2013).

[55] N. Shinozaki and M. Sibuya, Consistency of a pair of matrix equations with an application, Keio engineering reports, 27 (1974), 141-146.

[56] DC. Sorensen, AC. Antoulas, Sylvester equation and approximate balanced reduction. Linear Algebra Appl. 351/352 (2002), 671–700.

[57] A. E. Taylor and D. C. Lay, Introduction to functional analysis, New York, 1980.

[58] J. W. Van Der Woude, On the existence of a common solution X to the matrix equations $A_iXB_j = C_{ij}$, $(i, j) \in \Gamma$. Linear Algebra Appl, 375 (2003), 135-145.

[59] J. Von Neumann, On regular rings. Proc. Natl. Acad. Sci USA 1936, 22 (12), 707-713.

[60] Q. W. Wang, A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity, Linear Algebra Appl., 384 (2004), 43-54.

[61] Q. W. Wang and Z. H. He, Some matrix equations with applications, Linear Multilinear Algebra, Volume 60 (2012), 1327-1353.

[62] G. Wang, Y. Wei, S. Qiao, Generalized inverses: theory and computations, Science Press, (2003).

[63] Q. Xu, Common hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$, Linear Algebra and its Appl, 429 (2008), 1-11.

[64] Q. Xu, L. Sheng, Y. Gu, The solutions to some operator equations, Linear Algebra and its Appl, 429 (2008), 1997-2024.

[65] G. Xu, M.Wei, D. Zheng, On solutions of matrix equation $AXB+CYD = F$, Linear Algebra Appl. 279 (1998), 93–109.

[66] X. Yang, Lung-Kee Chen, AND R. M. Burton, Stability of discrete bilinear systems with output feedback, Internat. J. Conrrol50 (1989), 2085-2092.

[67] X. Yang, R. R. Mohler, AND L. K. Chen, Random parameter discrete bilinear system stability, in "Proceedings of the 28th IEEE Conference on Decision and Control," Vol. 2, pp. 1196-1200, 1989.