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THÈSE

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Par

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Intitulée

Convergence d'échelle en homogénéisation des fonctionnelles intégrales non convexes

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الملخص:

في هذه الأطروحة، قمنا بدراسة معمقة حول مفاهيم المجموعات و الدوال E- محدّبة مع مناقشة خصائصها الأساسية، ثم تم إدخال التحليل Γ-التقارب و التقارب السلمي الخاص بالتكاملات الدالية

E-محدبة
في إطار المساهمة، قمنا بتصحيح بعض الأعمال الخاصة بالمتراجحات التزايدية العامة و نظمها و كذلك من أجل التحسين الخطي، تم تقديم فئة جديدة من دوال النواة ذات الحاجز المزدوج، المختلفة عن تلك الموجودة.

Abstract:

In this thesis, an in-depth study is started on the concepts of E-convex sets and functions with a discussion of its basic properties, then we introduced the limit analysis by Γ- convergence and the convergence of scale of E-

convex functional integrals.

As part of the contribution, we corrected some works on general variational inequalities and their systems and also, we presented for linear optimization, a new class of kernel functions with double barrier term, different from the existing ones.

Résumé :

Dans cette thèse, une étude approfondie est débutée sur les concepts d’ensembles et de fonctions E-convexes avec une discussion sur ses propriétés de base, ensuite, on a introduit l’analyse limite par Γ-convergence et la convergence d’échelle des intégrales fonctionnelles E-convexes.

Dans le cadre de la contribution, nous avons corrigé quelques travaux sur les inégalités variationnelles générales et leurs systèmes et également, on a présenté pour l’optimisation linéaire, une nouvelle classe de fonctions noyaux à double terme barrière, différentes de celles existantes.

Contents

Introduction

Convex Analysis, Variational Inequalities and linear programming are importance fields of mathematics which plays a vital role in optimization. The main ingredients of convex analysis is related to convex sets and convex functions which are fundamentally employed in convex optimization problems. However, many sets and functions are nonconvex, which restrains the development and application of mathematical programming especially theories of optimization. Therefore, the research on convexity and generalized convexity is one of the most important aspects in mathematical programming.

During the past years, exactlly In 1999, E.A.Youness [41] generalised the concepts of convex sets and convex functions and presented the definitions of E-convex set and E-convex function, and discussed basic properties and conclusions of E-convex set and E-convex function, and then Muhammad Aslam Noor [27] used this concept on general variational inequalities theories which was introduced by Stampacchia in 1964,it has appeared as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in physics, biology, social sciences, and economics...

For linear programming. also known as Operations research Linearity indicates that there are no variables raised to higher powers, exponential or logarithmic. This class of problems involves minimizing (or maximizing) a linear objective function whose variables are real numbers constrained to satisfy a system of linear equality and inequality. The fields of application of these problems are very numerous both in the nature of the problems addressed (planning and control of production, distribution in networks) and in the industrial sectors: manufacturing industry, energy (oil, gas, electricity), transport, telecommunications, industry, finance. The most well-known methods of solving linear programming problems in real numbers are simplex method and interior point methods. This methods, turned out to solve practical problems efficiently. Interest in linear programming develops rapidly, and by 1951 its use spread to industry. Today it is almost impossible to find an laboratory that is not using linear programming in some form.

This Thesis is structured as follows:

In chapter 1 the consepts of E-convex set, E-convex function and semi-E-convex function are introduced and their properties are given. We also correcred the results obtained in Youness [41] concerning the characterization of an E-convex function f in terms of its E-epigraph and gave a weak condition for a lower semicontinuous function on $\mathbb{R}^n$ to be an E-convex function fo a linear map$E$. The limit analysis by Γ-convergence techniques and scale convergence of E-convex integral functionals is studied in chapter 2.

In chapter 3, a critical view about Muhammad Aslam noor's paper [27] are given and we discussed them in [8]. Also we present three studies on system of general variational inequalities:

1. System of Nonlinear General Variational inequalities Involving g-relaxed cocoercive mappings.

2. Corrections to a Paper "on Projection Algorithms for Solving System of General Variational Inequalities."

3. On General convergence analysis for two-step projection methods and applications to variational problems.

Finally in chapter 4 a new class of kernel functions which differs from the existing kernel functions in which it has a double barrier term is introduced for linear optimization.

Chapter 1

About E-convexity

1.1 On E-Convex and semi E-convex Functions for a Linear Map E

Duca and Lupsa [12] show that the results obtained in Youness [41] concerning the characterization of an E-convex function f in terms of its E-epigraph are incorrect. In this chapter we introduce the correct form of this Theorem which will be used in our study (see [7])

1.1.1 Preliminaries

Let M be a nonempty subset of $\mathbb{R}^n$ and let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a map. We recall:

Définition 1.1.1. *[41] A set $M \subseteq \mathbb{R}^n$ is said to be E-convex in $\mathbb{R}^n$ if*

$$tE(x) + (1-t)E(y) \in M,$$

for each $x, y \in M$ and all $t \in [0,1]$.

Remark 1.1.2. *If M is an E-convex set, than it may not be a classical convex set. For example, $M = \left[-1, \frac{-1}{2}\right] \cup [0,1]$ and $E(x) = x^2, \forall x \in \mathbb{R}$. Clearly, this is an E-convex set but not a classical convex set.*

Définition 1.1.3. *[41] A function $f : M \to \mathbb{R}$ is said to be E-convex on M if M is E-convex and*

$$f\left(tE(x) + (1-t)E(y)\right) \leq tf\left(E\left(x\right)\right) + (1-t)f\left(E\left(y\right)\right),$$

for each $x, y \in M$ and all $t \in [0,1]$.

Définition 1.1.4. *[38] A function $f : M \to \mathbb{R}$ is said to be semi-E-convex on M if M is E-convex and*

$$f\left(tE(x) + (1-t)E(y)\right) \leq tf\left(x\right) + (1-t)f\left(y\right),$$

for each $x, y \in M$ and all $t \in [0,1]$.

Remark 1.1.5. *[38] A semi-E-convex function on an E-convex set is not necessary an E-convex function.*

Proposition 1.1.6. *[38] Suppose the function $f:\mathbb{R}^n \to \mathbb{R}$ is E-convex on an E-convex set $M \subseteq \mathbb{R}^n$. Then f is semi-E-convex on set M if and only if $f(E(x)) \leq f(x)$ for each $x \in M$.*

Définition 1.1.7. *[38] We define a map $E \times I$ as follows:*

$$\begin{array}{rcl} E \times I : \mathbb{R}^n \times \mathbb{R} & \to & \mathbb{R}^n \times \mathbb{R} \\ (x,t) & \to & (E \times I)(x,t) = (E(x),t)\,. \end{array}$$

Définition 1.1.8. *[17] A function $f:\mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous if and only if for every real number α, the set $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is closed.*

The following Proposition characterize lower semi-continuous functions which shall be used in the sequal.

Proposition 1.1.9. *[17] A function $f:\mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous if and only if its epigraph epi $(f) = \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \alpha\}$ is closed.*

Définition 1.1.10. *Let $(x,s),(y,t) \in \mathbb{R}^{n+1}$, with $x,y \in \mathbb{R}^n$ and $s,t \in \mathbb{R}$. The line segment $[(x,s),(y,t)]$ (with endpoints (x,s) and (y,t)) is the segment*

$$\{\alpha(x,s) + (1-\alpha)(y,t) : 0 \leq \alpha \leq 1\}\,.$$

If $(x,s) \neq (y,t)$, the interior $](x,s),(y,t)[$ of $[(x,s),(y,t)]$ is the segment

$$\{\alpha(x,s) + (1-\alpha)(y,t) : 0 < \alpha < 1\}\,.$$

In a similar way, we can define $[(x,s),(y,t))$ and $((x,s),(y,t)]$.

1.1.2 Main results

The Theorem (3.1) of Youness [41] concerning the characterization of an E-convex function f in terms of its E-epigraph is modified to the following Theorem.

Theorem 1.1.11. *Let $E:\mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map, then the function $f:\mathbb{R}^n \to \mathbb{R}$ is E-convex on $\mathbb{R}^n$ if and only if its E-epigraph is $E \times I$-convex on $\mathbb{R}^n \times \mathbb{R}$. Where $E - e(f) = \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} : f(E(x)) \leq \alpha\}$.*

Proof. Suppose that f is E- convex, let $(x_1,\alpha_1),(x_2,\alpha_2) \in E - e(f)$. We have

$$\begin{aligned} f(E(\alpha E(x_1) + (1-\alpha)E(x_2))) &= f(\alpha E(x_1) + (1-\alpha)E(x_2)), \\ &\leq \alpha f(E(x_1)) + (1-\alpha) f(E(x_2)), \\ &\leq \alpha\alpha_1 + (1-\alpha)\alpha_2. \end{aligned}$$

Thus $\alpha\,(E\times I)\,(x_1,\alpha_1)+(1-\alpha)\,(E\times I)\,(x_2,\alpha_2)\in E-e\,(f)\,.$
For the converse part suppose that $E-e\,(f)$ is $E\times I$-convex.
We see that $(x_1,f\,(E\,(x_1)))\,,(x_2,f\,(E\,(x_2)))\in E-e\,(f)$ then $\alpha\,(E\times I)\,(x_1,f\,(E\,(x_1)))+(1-\alpha)\,(E\times I)\,(x_2,f\,(E\,(x_2)))\in E-e\,(f)$. Therfore

$$f\,(\alpha E(x_1)+(1-\alpha)E(x_2))\le \alpha f\,(E\,(x_1))+(1-\alpha)f\,(E\,(x_2))\,,$$

This completes the proof □

We are now in a position to state the main results of this chapter.

Theorem 1.1.12. *Let $E:\mathbb{R}^n\to\mathbb{R}^n$ be a linear and idempotent map, $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi-continuous. Suppose that there exists an $\alpha\in\,]0,1[$ such that for all $x,y\in\mathbb{R}^n$, $s,t\in\mathbb{R}$ such that $f\,(E(x))<s,\ f\,(E(y))<t$,*

$$f\,(\alpha E(x)+(1-\alpha)E(y))<\alpha s+(1-\alpha)t,$$

then f is E-convex.

Proof. By Theorem (1.1.11), it is sufficient to show that $E-e\,(f)$ is $E\times I$-convex as a subset of $\mathbb{R}^n\times\mathbb{R}$. By contradiction, suppose that there exist $(x_1,\alpha_1),(x_2,\alpha_2)\in E-e\,(f)$ (with $x_1,x_2\in\mathbb{R}^n$ and $\alpha_1,\alpha_2\in\mathbb{R}$) and $\alpha_0\in\,]0,1[$ such that,
$(\alpha_0E\,(x_1)+(1-\alpha_0)\,E\,(x_2)\,,\alpha_0\alpha_1+(1-\alpha_0)\,\alpha_2)\notin E-e\,(f)\,.$
Let $x_0=\alpha_0E\,(x_1)+(1-\alpha_0)\,E\,(x_2)$ and $\lambda_0=\alpha_0\alpha_1+(1-\alpha_0)\,\alpha_2$, then $(x_0,\lambda_0)\notin E-e\,(f)$.
Using the fact that E is an idempotent map, we see that,
$(E\,(x_1)\,,\alpha_1),(E\,(x_2)\,,\alpha_2)\in E-e\,(f)\,.$ Let

$$A=E-e\,(f)\cap[(E\,(x_1)\,,\alpha_1),(x_0,\lambda_0)]$$

and

$$B=E-e\,(f)\cap[(x_0,\lambda_0)\,,(E\,(x_2)\,,\alpha_2)]\,.$$

Since f is lower semi continuous, E is continuous (linear on $\mathbb{R}^n$), by Proposition (1.1.9) *epi* $(f\circ E)=E-e\,(f)$ is a closed subset of $\mathbb{R}^n\times\mathbb{R}$. Consequently, A and B are bounded and closed subsets of $\mathbb{R}^n\times\mathbb{R}$. Also we have $(x_0,\lambda_0)\notin A$ and $(x_0,\lambda_0)\notin B$. Thus there exist $Z_A=(x_3,\alpha_3)\in A$ and $Z_B=(x_4,\alpha_4)\in B$ such that,

$$\min_{Z\in A}\|Z-(x_0,\lambda_0)\|=\|Z_A-(x_0,\lambda_0)\|$$

and

$$\min_{Z\in B}\|Z-(x_0,\lambda_0)\|=\|Z_B-(x_0,\lambda_0)\|\,.$$

Hence, we have

$$]Z_A,Z_B[\,\cap E-e\,(f)=\emptyset. \tag{1.1}$$

On the other hand, since $Z_A\in E-e\,(f)$ and $Z_B\in E-e\,(f)$, we get
$f\,(E(x_3))<\alpha_3+\varepsilon,\ f\,(E(x_4))<\alpha_4+\varepsilon$ for each $\varepsilon>0$.

Since $\alpha(\alpha_3+\varepsilon)+(1-\alpha)(\alpha_4+\varepsilon)=\alpha\alpha_3+(1-\alpha)\alpha_4+\varepsilon$. By the hypothesis of the theorem, we obtian

$$f(\alpha E(x_3)+(1-\alpha)E(x_4))<\alpha\alpha_3+(1-\alpha)\alpha_4+\varepsilon.$$

Since ε is an arbitrary positive real number, it follows that

$$f(E(\alpha x_3+(1-\alpha)x_4))\le\alpha\alpha_3+(1-\alpha)\alpha_4. \tag{1.2}$$

Using (1.2) we get

$$(\alpha x_3+(1-\alpha)x_4,\alpha\alpha_3+(1-\alpha))\,\alpha_4)\in E-e(f).$$

Therfore

$$\alpha Z_A+(1-\alpha)Z_B\in E-e(f)$$

which contradicts (1.1). Thus, we conclude that $E-e(f)$ is $E\times I$-convex. □

Theorem 1.1.13. *Let $E:\mathbb{R}^n\to\mathbb{R}^n$ be a linear and idempotent map, $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi-continuous. Then f is E-convex if and only if there exists an $\alpha\in\,]0,1[$ such that for all $x,y\in\mathbb{R}^n$*

$$f(\alpha E(x)+(1-\alpha)E(y))\le\alpha f(E(x))+(1-\alpha)f(E(y)).$$

Proof. Follows from Theorem (1.1.12), with $s=f(E(x))+\varepsilon$ and $t=f(E(y))+\varepsilon$ for each $\varepsilon>0$, then taking $\varepsilon\to 0$. □

Theorem 1.1.14. *Let $E:\mathbb{R}^n\to\mathbb{R}^n$ be a linear and idempotent map, $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi-continuous. Then f is E-convex if and only if for all $x,y\in\mathbb{R}^n$, there exists an $\alpha\in\,]0,1[$ (α depends on x,y) such that*

$$f(\alpha E(x)+(1-\alpha)E(y))\le\alpha f(x)+(1-\alpha)f(y). \tag{1.3}$$

Proof. Let $f:\mathbb{R}^n\to\mathbb{R}$ E-convex. From Definition (1.1.3), it follows that, for all $x,y\in\mathbb{R}^n$, there exists an $\alpha\in\,]0,1[$ such that (1.3) holds. For the converse part, By Theorem (1.1.11), it is sufficient to show that $E-e(f)$ is $E\times I$-convex as a subset of $\mathbb{R}^n\times\mathbb{R}$. By contradiction, suppose that there exist $(x_1,\alpha_1),(x_2,\alpha_2)\in E-e(f)$ (with $x_1,x_2\in\mathbb{R}^n$ and $\alpha_1,\alpha_2\in\mathbb{R}$) and $\alpha_0\in\,]0,1[$ such that:
$(\alpha_0E(x_1)+(1-\alpha_0)E(x_2),\alpha_0\alpha_1+(1-\alpha_0)\alpha_2)\notin E-e(f)$.
Let $x_0=\alpha_0E(x_1)+(1-\alpha_0)E(x_2)$ and $\lambda_0=\alpha_0\alpha_1+(1-\alpha_0)\alpha_2$, then $(x_0,\lambda_0)\notin E-e(f)$.
We follow the proof of Theorem (1.1.12). Having defined A, B, $Z_A=(x_3,\alpha_3)$, $Z_B=(x_4,\alpha_4)$, we find that:

$$]Z_A,Z_B[\cap epi(f)=\emptyset. \tag{1.4}$$

On the other hand, by the hypothesis of the theorem , for $x=x_3$ and $y=x_4$, there exists an $\alpha\in\,]0,1[$ such that

$$f(\alpha E(x_3)+(1-\alpha)E(x_4))\le\alpha f(E(x_3))+(1-\alpha)f(E(x_4)). \tag{1.5}$$

Using (1.5) we get

$$f\left(E\left(\alpha x_3+(1-\alpha)x_4\right)\right)\leq \alpha\alpha_3+(1-\alpha)\alpha_4.$$

So,

$$\alpha Z_A+(1-\alpha)Z_B\in E-e\left(f\right)$$

which contradicts (1.4). Thus, we conclude that $E-e\left(f\right)$ is $E\times I$-convex. □

According to Theorems (1.1.13) and (1.1.14) with $E=Id_{\mathbb{R}^n}$, we have the following results about convex functions.

Theorem 1.1.15. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi continuous. Then f is convex if and only if there exists an $\alpha\in\left]0,1\right[$ such that, for all $x,y\in\mathbb{R}^n$,*

$$f\left(\alpha x+(1-\alpha)y\right)\leq \alpha f\left(x\right)+(1-\alpha)f\left(y\right).$$

Corollary 1.1.1. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi continuous. Then f is convex if and only if for all $x,y\in\mathbb{R}^n$,*

$$f\left(\frac{1}{2}\left(x+y\right)\right)\leq \frac{1}{2}\left[f\left(x\right)+f\left(y\right)\right].$$

Theorem 1.1.16. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi continuous. Then f is convex if and only if for all $x,y\in\mathbb{R}^n$, there exists an $\alpha\in\left]0,1\right[$ (α depends on x,y) such that*

$$f\left(\alpha x+(1-\alpha)y\right)\leq \alpha f\left(x\right)+(1-\alpha)f\left(y\right).$$

1.1.3 Conclusion

Remark 1.1.17. *In definition (1.1.3), if we take a linear map E we get the Definition of convexity of $(f\circ E)$ on M (in the case when M is convex).*

According to Theorems (1.1.15), (1.1.16) and corollary (1.1.1) with remarek (1.1.17) we have the following results

Theorem 1.1.18. *Let $E:\mathbb{R}^n\to\mathbb{R}^n$ be a linear map, $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi-continuous. Then f is E-convex if and only if there exists an $\alpha\in\left]0,1\right[$ such that for all $x,y\in\mathbb{R}^n$*

$$f\left(\alpha E(x)+(1-\alpha)E(y)\right)\leq \alpha f\left(E(x)\right)+(1-\alpha)f\left(E(y)\right).$$

Corollary 1.1.2. *Let $E:\mathbb{R}^n\to\mathbb{R}^n$ be a linear map, $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi-continuous. Then f is E-convex if and only if for all $x,y\in\mathbb{R}^n$,*

$$f\left(\frac{1}{2}\left(E(x)+E(y)\right)\right)\leq \frac{1}{2}\left[f\left(E(x)\right)+f\left(E(y)\right)\right].$$

Theorem 1.1.19. *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then f is E-convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in]0,1[$ (α depends on x, y) such that*

$$f\left(\alpha E(x) + (1-\alpha)E(y)\right) \leq \alpha f\left(E(x)\right) + (1-\alpha) f\left(E(y)\right).$$

Using Proposition (1.1.6) we 'll find results about semi-E-convex functions :

Theorem 1.1.20. *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous and $f\left(E\left(x\right)\right) \leq f\left(x\right)$ for all $x \in \mathbb{R}^n$. Then f is semi-E-convex if and only if there exists an $\alpha \in]0,1[$ such that for all $x, y \in \mathbb{R}^n$*

$$f\left(\alpha E(x) + (1-\alpha)E(y)\right) \leq \alpha f\left(E(x)\right) + (1-\alpha) f\left(E(y)\right).$$

Corollary 1.1.3. *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous and $f\left(E\left(x\right)\right) \leq f\left(x\right)$ for all $x \in \mathbb{R}^n$. Then f is semi-E-convex if and only if for all $x, y \in \mathbb{R}^n$,*

$$f\left(\frac{1}{2}\left(E(x) + E(y)\right)\right) \leq \frac{1}{2}\left[f\left(E(x)\right) + f\left(E(y)\right)\right].$$

Theorem 1.1.21. *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous and $f\left(E\left(x\right)\right) \leq f\left(x\right)$ for all $x \in \mathbb{R}^n$. Then f is semi-E-convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in]0,1[$ (α depends on x, y) such that*

$$f\left(\alpha E(x) + (1-\alpha)E(y)\right) \leq \alpha f\left(E(x)\right) + (1-\alpha) f\left(E(y)\right).$$

1.2 A Critical View and corrections of F.Mirzapour's Paper

Recently, F.Mirzapour [16] published a paper in HJMS entitled by "On semi-E-convex and quassi semi-E-convex functions", volume 41 (6) (2012), 841-845. But, this results are incorrect. In this notes some results are given to correct the main results in [16]

1.2.1 Commentaries

1 Throughout his proofs of theorems, the condition $f\left(E\left(x\right)\right) \leq f\left(x\right)$ for all $x \in \mathbb{X}$ was not applied any where.

2 F. Mizapour found $(\widetilde{x}, s)$ and $(\widetilde{y}, t)$ such that:

$$\min_{a \in A} \|a - (x_0, \lambda_0)\| = \|(\widetilde{x}, s) - (x_0, \lambda_0)\|,$$

and

$$\min_{a \in B} \|a - (x_0, \lambda_0)\| = \|(\widetilde{y}, t) - (x_0, \lambda_0)\|.$$

Where A and B are bounded and closed subsets of $\mathbb{X} \times \mathbb{R}$

This existence is true just in the finite dimention case, for example $\mathbb{X} = \mathbb{R}^n$.

3 (The last line on page 842) and (line 25 on page 843). He passed from this inequality

$$f\left(\alpha E(\widetilde{x})+(1-\alpha)E(\widetilde{y})\right)\leq \alpha s+(1-\alpha)t$$

to

$$\alpha\left(\widetilde{x},s\right)+(1-\alpha)\left(\widetilde{y},t\right)\in epi\left(f\right)$$

and he created a contradiction with (2,1) on theorem 2.1 and a contradiction with (2,3) on theorem 2.2. This passage is not true, because the inequality

$$f\left(\alpha E(\widetilde{x})+(1-\alpha)E(\widetilde{y})\right)\leq \alpha s+(1-\alpha)t$$

implies

$$\left(\alpha E\left(\widetilde{x}\right)+(1-\alpha)E\left(\widetilde{y}\right),\alpha s+(1-\alpha)t\right)\in epi\left(f\right).$$

Hence

$$\alpha\left(E\left(\widetilde{x}\right),s\right)+(1-\alpha)\left(E\left(\widetilde{y}\right),t\right)\in epi\left(f\right),$$

then there is no contradiction with (2,1) and (2,3).

4 (Line 33 on page 844). He passed from this inequality

$$f\left(\alpha E(\widetilde{x})+(1-\alpha)E(\widetilde{y})\right)\leq \lambda^{*}$$

to

$$\alpha E\left(\widetilde{x}\right)+(1-\alpha)E\left(\widetilde{y}\right)\in F_{\lambda^{*}}$$

and he created a contradiction with (6). This passage is true, but there is no contradiction with (6).

Comments on his Example

He found that

1 $f\left(E\left(x\right)\right)\nleq f\left(x\right)$

2 For each $x,y\geq 0$, there exists $\lambda\in\,]0,1[$ such that

$$f\left(\alpha E(x)+(1-\alpha)E(y)\right)\leq \alpha f\left(x\right)+(1-\alpha)f\left(y\right).$$

The two results is not true because :

1 Let $x\in[0,+\infty[$,we have:

a if $x=0$ then $E\left(0\right)=0$, $f\left(E\left(0\right)\right)=0$, $f\left(0\right)=0$.

b If $x \notin \mathbb{Q}$ then $E(x) = 0,\ f(E(x)) = 0,\ 0 < f(x)$.

c If $x = \dfrac{m}{n} \in \mathbb{Q}$, $(m,n) = 1$, then $E(x) = \dfrac{1}{n}$, $f(E(x)) = \begin{cases} 1 & if \quad n = 1 \\ \frac{2}{n} & if \quad n > 1 \end{cases}$, and

$$f(x) = \begin{cases} 1 & if \quad n = m \\ m^2 & if \quad n = 1, m > 1 \\ \frac{2m}{n} & if \quad m < n \end{cases}.$$

We conclude that for all $x \in [0, +\infty[\ :\ f(E(x)) \leq f(x)$.

2 Here is a counterexemple, if we take $x = \frac{1}{4}$, $y = 1$ we 'll find :

$f(\lambda E(x) + (1-\lambda)E(y)) = \dfrac{\lambda}{2} + 2(1-\lambda)$,

$\lambda f(x) + (1-\lambda) f(y) = \dfrac{\lambda}{2} + (1-\lambda)$

and $f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(x) + (1-\lambda) f(y) \Rightarrow 2 \leq 1$

wich contradicts $2 > 1$.

1.2.2 Results for semi-E-convex functions

Lemma 1.2.1. *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map. Consider $(\overline{x}, u) \in [(E(x), s), (E(y), t)]$. Then*

$$E(\overline{x}) = \overline{x}.$$

Proof. Let $(\overline{x}, u) \in [(E(x), s), (E(y), t)]$, then there exist $\alpha \in [0,1]$, such that $(\overline{x}, u) = \alpha(E(x), s) + (1-\alpha)(E(y), t)$. Using the fact that E is linear and idempotent map, we have

$$\begin{aligned} (E \times I)(\overline{x}, u) &= (E(\alpha E(x) + (1-\alpha)E(y)), \alpha s + (1-\alpha)t) \\ &= (\alpha E(x) + (1-\alpha)E(y), \alpha s + (1-\alpha)t) \\ &= (\overline{x}, u). \end{aligned}$$

On the other hand $(E \times I)(\overline{x}, u) = (E(\overline{x}), u)$, therfore $E(\overline{x}) = \overline{x}$. □

We shall make use the following three sets:

$$H_{Sci} = \{f : \mathbb{R}^n \to \mathbb{R},\ f \text{ is lower semi continuous}\}, \tag{1.6}$$
$$H_{L,I} = \{E : \mathbb{R}^n \to \mathbb{R}^n,\ E \text{ is linear and idempotent}\} \tag{1.7}$$

and for each $E \in H_{L,I}$ we define H_E as follows:

$$H_E = \{f \in H_{Sci},\ f(E(x)) \leq f(x) \text{ for all } x \in \mathbb{R}^n\} \tag{1.8}$$

Theorem 1.2.2. *Let $E \in H_{L,I}$, and $f \in H_E$. Suppose that there exists an $\alpha \in]0,1[$ such that for all $x, y \in \mathbb{R}^n$, $s, t \in \mathbb{R}$ such that $f(x) < s$, $f(y) < t$,*

$$f(\alpha E(x) + (1-\alpha)E(y)) < \alpha s + (1-\alpha)t.$$

Then f is semi-E-convex.

Proof. By Theorem (1.1.11), it is sufficient to show that $epi(f)$ is $E \times I$-convex as a subset of $\mathbb{R}^n \times \mathbb{R}$. By contradiction, suppose that there exist $(x_1, \alpha_1), (x_2, \alpha_2) \in epi(f)$ (with $x_1, x_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$) and $\alpha_0 \in]0,1[$ such that,
$(\alpha_0 E(x_1) + (1-\alpha_0) E(x_2), \alpha_0\alpha_1 + (1-\alpha_0)\alpha_2) \notin epi(f)$.
Let $x_0 = \alpha_0 E(x_1) + (1-\alpha_0) E(x_2)$ and $\lambda_0 = \alpha_0\alpha_1 + (1-\alpha_0)\alpha_2$, then $(x_0, \lambda_0) \notin epi(f)$.
Using the fact that $f \in H_E$, we see that $(E(x_1), \alpha_1), (E(x_2), \alpha_2) \in epi(f)$. Let

$$A = epi(f) \cap [(E(x_1), \alpha_1), (x_0, \lambda_0)]$$

and

$$B = epi(f) \cap [(x_0, \lambda_0), (E(x_2), \alpha_2)].$$

Since $f \in H_{Sci}$, by Proposition (1.1.9), $epi(f)$ is a closed subset of $\mathbb{R}^n \times \mathbb{R}$. Consequently, A and B are bounded and closed subsets of $\mathbb{R}^n \times \mathbb{R}$.
Also we have $(x_0, \lambda_0) \notin A$ and $(x_0, \lambda_0) \notin B$. Thus there exist $Z_A = (x_3, \alpha_3) \in A$ and $Z_B = (x_4, \alpha_4) \in B$ such that,

$$\min_{Z \in A} \|Z - (x_0, \lambda_0)\| = \|Z_A - (x_0, \lambda_0)\|$$

and

$$\min_{Z \in B} \|Z - (x_0, \lambda_0)\| = \|Z_B - (x_0, \lambda_0)\|.$$

Hence, we have

$$]Z_A, Z_B[\cap epi(f) = \emptyset. \tag{1.9}$$

On the other hand, since $Z_A \in epi(f)$ and $Z_B \in epi(f)$, we get
$f(x_3) < \alpha_3 + \varepsilon$, $f(x_4) < \alpha_4 + \varepsilon$ for each $\varepsilon > 0$.
Since $\alpha(\alpha_3 + \varepsilon) + (1-\alpha)(\alpha_4 + \varepsilon) = \alpha\alpha_3 + (1-\alpha)\alpha_4 + \varepsilon$. By the hypothesis of the theorem, we obtian

$$f(\alpha E(x_3) + (1-\alpha)E(x_4)) < \alpha\alpha_3 + (1-\alpha)\alpha_4 + \varepsilon.$$

Since ε is an arbitrary positive real number, it follows that

$$f(\alpha E(x_3) + (1-\alpha)E(x_4)) \le \alpha\alpha_3 + (1-\alpha)\alpha_4. \tag{1.10}$$

Since $Z_A \in A \subset [(E(x_1), \alpha_1), (E(x_2), \alpha_2)]$ and
$Z_B \in B \subset [(E(x_1), \alpha_1), (E(x_2), \alpha_2)]$. By Lemma (1.2.1) we have $E(x_3) = x_3$ and $E(x_4) = x_4$. Using (1.10) we get

$$(\alpha x_3 + (1-\alpha)x_4, \alpha\alpha_3 + (1-\alpha))\alpha_4) \in epi(f).$$

Therfore

$$\alpha Z_A + (1-\alpha)Z_B \in epi(f)$$

which contradicts (1.9). Thus, we conclude that $epi(f)$ is $E \times I$-convex. □

Theorem 1.2.3. *Let $E:\mathbb{R}^n\to\mathbb{R}^n$ be a linear and idempotent map, $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi-continuous and $f(E(x))\le f(x)$ for all $x\in\mathbb{R}^n$. Then f is semi-E-convex if and only if there exists an $\alpha\in\,]0,1[$ such that for all $x,y\in\mathbb{R}^n$*

$$f(\alpha E(x)+(1-\alpha)E(y))\le \alpha f(x)+(1-\alpha)f(y).$$

Proof. Follows from Theorem (1.2.2) with $s=f(x)+\varepsilon$ and $t=f(y)+\varepsilon$ for each $\varepsilon>0$, then taking $\varepsilon\to 0$. □

Corollary 1.2.1. *Let $E:\mathbb{R}^n\to\mathbb{R}^n$ be a linear and idempotent map, $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi-continuous and $f(E(x))\le f(x)$ for all $x\in\mathbb{R}^n$. Then f is semi-E-convex if and only if for all $x,y\in\mathbb{R}^n$,*

$$f\left(\frac{1}{2}(E(x)+E(y))\right)\le\frac{1}{2}[f(x)+f(y)].$$

Theorem 1.2.4. *Let $E:\mathbb{R}^n\to\mathbb{R}^n$ be a linear and idempotent map, $f:\mathbb{R}^n\to\mathbb{R}$ be lower semi-continuous and $f(E(x))\le f(x)$ for all $x\in\mathbb{R}^n$. Then f is semi-E-convex if and only if for all $x,y\in\mathbb{R}^n$, there exists an $\alpha\in\,]0,1[$ (α depends on x,y) such that*

$$f(\alpha E(x)+(1-\alpha)E(y))\le \alpha f(x)+(1-\alpha)f(y). \tag{1.11}$$

Proof. Let $f:\mathbb{R}^n\to\mathbb{R}$ be semi-E-convex. From Definition (1.1.4), it follows that, for all $x,y\in\mathbb{R}^n$, there exists an $\alpha\in\,]0,1[$ such that (1.11) holds. For the converse part, By Theorem (1.1.11), it is sufficient to show that *epi* (f) is $E\times I$-convex as a subset of $\mathbb{R}^n\times\mathbb{R}$. By contradiction, suppose that there exist $(x_1,\alpha_1),(x_2,\alpha_2)\in epi\,(f)$ (with $x_1,x_2\in\mathbb{R}^n$ and $\alpha_1,\alpha_2\in\mathbb{R}$) and $\alpha_0\in\,]0,1[$ such that:
$(\alpha_0E(x_1)+(1-\alpha_0)E(x_2),\alpha_0\alpha_1+(1-\alpha_0)\alpha_2)\notin epi\,(f)$.
Let $x_0=\alpha_0E(x_1)+(1-\alpha_0)E(x_2)$ and $\lambda_0=\alpha_0\alpha_1+(1-\alpha_0)\alpha_2$, then $(x_0,\lambda_0)\notin epi\,(f)$.
We follow the proof of Theorem (1.2.2). Having defined A, B, $Z_A=(x_3,\alpha_3)$, $Z_B=(x_4,\alpha_4)$, we find that:

$$]Z_A,Z_B[\,\cap\, epi\,(f)=\emptyset. \tag{1.12}$$

On the other hand, by the hypothesis of the theorem , for $x=x_3$ and $y=x_4$, there exists an $\alpha\in\,]0,1[$ such that

$$f(\alpha E(x_3)+(1-\alpha)E(x_4))\le \alpha f(x_3)+(1-\alpha)f(x_4). \tag{1.13}$$

Since $Z_A\in[(E(x_1),\alpha_1),(E(x_2),\alpha_2)]$, $Z_B\in[(E(x_1),\alpha_1),(E(x_2),\alpha_2)]$. By Lemma (1.2.1) we have $E(x_3)=x_3$ and $E(x_4)=x_4$. Using (1.13) we get

$$f(\alpha x_3+(1-\alpha)x_4)\le\alpha\alpha_3+(1-\alpha)\alpha_4.$$

So,

$$\alpha Z_A+(1-\alpha)Z_B\in epi\,(f)$$

which contradicts (1.12). Thus, we conclude that *epi* (f) is $E\times I$-convex. □

If we take $E = Id_{\mathbb{R}^n}$, we get $E \in H_{L,I}$, and $H_E = H_{Sci}$. Then we find results about convex functions.

Corollary 1.2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then f is convex if and only if there exists an $\alpha \in]0,1[$ such that, for all $x, y \in \mathbb{R}^n$,*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y).$$

Corollary 1.2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then f is convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in]0,1[$ (α depends on x, y) such that*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y).$$

Corollary 1.2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then f is convex if and only if for all $x, y \in \mathbb{R}^n$,*

$$f\left(\frac{1}{2}(x+y)\right) \leq \frac{1}{2}[f(x) + f(y)].$$

1.2.3 Results for quassi-semi-E-convex functions

Définition 1.2.5. *[38] A function $f : M \to \mathbb{R}$ is said to be quassi-semi-E-convex on M if M is E-convex and*

$$f(tE(x) + (1-t)E(y)) \leq \max\{f(x), f(y)\},$$

for each $x, y \in M$ and all $t \in [0,1]$.

Proposition 1.2.6. *[38] The function $f : \mathbb{R}^n \to \mathbb{R}$ is quassi-semi-E-convex on $\mathbb{R}^n$ if and only if the level set $K_\alpha = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is E-convex on $\mathbb{R}^n$ for each $\alpha \in \mathbb{R}$.*

Lemma 1.2.7. *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and idempotent map. Consider $\overline{x} \in [(E(x), (E(y)]$. Then*

$$E(\overline{x}) = \overline{x}.$$

Proof. Let $\overline{x} \in [(E(x), (E(y)]$, then there exist $\alpha \in [0,1]$, such that $\overline{x} = \alpha E(x) + (1-\alpha) E(y)$. Using the fact that E is linear and idempotent map, we have

$$\begin{aligned} E(\overline{x}) &= E(\alpha(E(x) + (1-\alpha)(E(y)) \\ &= \alpha E(x) + (1-\alpha) E(y) \\ &= \overline{x}. \end{aligned}$$

This completes the proof. □

Theorem 1.2.8. *Let $E \in H_{L,I}$, and $f \in H_E$. Then f is quassi-semi-E-convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in]0,1[$ (α depends on x, y) such that*

$$f(\alpha E(x) + (1-\alpha) E(y)) \leq \max\{f(x), f(y)\}.$$

Proof. By Proposition (1.2.6), it is sufficient to show that $K_\lambda = \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ is E -convex on $\mathbb{R}^n$ for each $\lambda \in \mathbb{R}$. By contradiction, suppose that there exists a real number λ_0 such that the set $\{x \in \mathbb{R}^n : f(x) \leq \lambda_0\}$ is not E-convex. Thus, there exist $\alpha_0 \in]0,1[$ and $x, y \in K_{\lambda_0}$, such that $\alpha_0 E(x) + (1-\alpha_0)E(y) \notin K_{\lambda_0}$.
Let $x_0 = \alpha_0 E(x) + (1-\alpha_0)E(y)$, then $x_0 \notin K_{\lambda_0}$. Using the fact that $f \in H_E$, we see that $E(x), E(y) \in K_{\lambda_0}$. Let

$$A = K_{\lambda_0} \cap [(E(x), x_0]$$

and

$$B = K_{\lambda_0} \cap [(E(y), x_0]$$

Since $f \in H_{Sci}$, by Proposition (1.1.9), K_{λ_0} is a closed subset of $\mathbb{R}^n$. Consequently, A and B are bounded and closed subsets of $\mathbb{R}^n$.
Also we have $x_0 \notin A$, and $x_0 \notin B$. Thus, there exist $Z_A \in A$ and $Z_B \in B$ such that,

$$\min_{Z \in A} \|Z - x_0\| = \|Z_A - x_0\|$$

and

$$\min_{Z \in B} \|Z - x_0\| = \|Z_B - x_0\|.$$

Hence, we have

$$]Z_A, Z_B[\cap K_{\lambda_0} = \emptyset \tag{1.14}$$

On the other hand, by the hypothesis of the theorem, for $Z_A, Z_B \in \mathbb{R}^n$ there exists an $\alpha \in]0,1[$ such that,

$$f(\alpha E(Z_A) + (1-\alpha)E(Z_B)) \leq \max\{f(Z_A), f(Z_B)\}. \tag{1.15}$$

Since $Z_A \in K_{\lambda_0}$, $Z_B \in K_{\lambda_0}$. We have,

$$f(Z_A) \leq \lambda_0 \text{ and } f(Z_B) \leq \lambda_0. \tag{1.16}$$

Combining (1.15) and (1.16), we obtain,

$$f(\alpha E(Z_A) + (1-\alpha)E(Z_B)) \leq \lambda_0 \tag{1.17}$$

By Lemma (1.2.7) we have $E(Z_A) = Z_A$ and $E(Z_B) = Z_B$. Using (1.17) we get

$$f(\alpha Z_A + (1-\alpha)Z_B) \leq \lambda_0.$$

So,

$$\alpha Z_A + (1-\alpha)Z_B \in K_{\lambda_0},$$

which contradicts (1.14). Thus, we conclude that $K_\lambda = \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ is E-convex on $\mathbb{R}^n$ for each $\lambda \in \mathbb{R}$. □

Corollary 1.2.5. *Let $E \in H_{L,I}$, and $f \in H_E$. Then f is quassi-semi-E-convex if and only if there exists an $\alpha \in]0,1[$ such that for all $x, y \in \mathbb{R}^n$*

$$f(\alpha E(x) + (1-\alpha)E(y)) \leq \max\{f(x), f(y)\}.$$

If we take $E = Id_{\mathbb{R}^n}$, we have that $E \in H_{L,I}$, and $H_E = H_{Sci}$. Then we find results about quassi convex functions.

Corollary 1.2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then f is quassi convex if and only if for all $x, y \in \mathbb{R}^n$, there exists an $\alpha \in]0, 1[$ (α depends on x, y) such that*

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}.$$

Corollary 1.2.7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous. Then f is quassi convex if and only if there exists an $\alpha \in]0, 1[$ such that for all $x, y \in \mathbb{R}^n$*

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}.$$

Chapter 2

Homogenization of E-convex integral functionals with scale convergence

2.1 Introduction

The introduction of the scale convergence and its application to the study of the $\Gamma-$convergence of non-periodically oscillating functionals is due to Mascarenhas and Toader[1]. More precisely, they studied the $\Gamma-$convergence of the sequence of functionals:

$$J_n(v) = \int_\Omega f(x, \alpha_n(x), E\,(v))dx,$$

where, Ω is an open bounded subset of $\mathbb{R}^N$, Π is a metrizable compact space, $\alpha_n : \Omega \to \Pi$ is a sequence of measurable functions, $v \in L^2(\Omega)$ and $E : \xi \to E\,(\xi)$ is a linear function from $\mathbb{R}$ into $\mathbb{R}$, i.e $(E\,(\xi) = \xi)$.
$f : (x, \lambda, \xi) \to f\,(x, \lambda, \xi)$ is a function from $\Omega \times \Pi \times \mathbb{R}$ into $\mathbb{R}^+$, wich satisfies the following conditions:

- f is (ξ)-convex;
- there exists $C > 0$, such that for every $\xi \in \mathbb{R}$

$$\frac{1}{C}\left(|\xi|^2\right) \leq f(x, \lambda, \xi) \leq C\left(1 + |\xi|^2\right),\ a.e\ (x, \lambda) \in \times\Omega \times \Pi \tag{2.1}$$

- f, $\dfrac{\partial f}{\partial \xi}$ are $(x)-$measurable, $(\lambda, \xi)-$continuous

By using, the theory of $\Gamma-$convergence, the fundamental theorem for Young measures and the scale convergence, Mascarenhas and Toader proved that the following integral functional:

$$J^{\text{hom}}(v) = \inf\left\{\int_\Omega \int_\Pi f(x, \lambda, w(x, \lambda))d\mu,\ w \in \mathcal{H}, v\,(x) = \int_\Pi w(x, \lambda)d\mu_x\right\}$$

is the $\Gamma-$limit of the sequence (J_n), for every $v \in L^2(\Omega)$, with $\mathcal{H}=L^2_\mu(\Omega \times \Pi)$, μ is the Young measure assaciated to (α_n) and μ_x is the disintegration of μ with respect to Lesbegue measure.

In this study, we deal with the same sequence of the nonperiodic integral functionals:

$$F_n(v) = \int_\Omega f(x, \alpha_n(x), E\,(v(x)))dx,$$

where,

- $E : \mathbb{R} \to \mathbb{R}$ is a k-Lipschitz function and $g\,(0) = 0$.
- f is $E-(\xi)$-convex;

and we will prove that

$$F^{\text{hom}}(v) = \inf \left\{ \int_\Omega \int_\Pi f(x, \lambda, E\,(w(x, \lambda)))d\mu,\ w \in \mathcal{H}, v\,(x) = \int_\Pi w(x, \lambda)d\mu_x \right\}$$

is the $\Gamma-$limit of the sequence (F_n)
The chapter is organized as follows:
Some useful results concerning Young measures are given, the so-called scale convergence and related results are introduced, as well as a proof of the main result.

2.2 Young measures and the measurable functions

Let O be an open bounded subset of $\mathbb{R}^N$ and let S be a metrizable space. We denote by: dx the Lebesgue measure on $\mathbb{R}^N$; $\mathcal{F}\,(O)$ the family of all Lebesgue measurable subsets of O and $\mathcal{B}\,(S)$ the Borel $\sigma-$field of S; $\mathcal{M}^+\,(O \times S)$ the set of the positive Radon measures.

Définition 2.2.1. *1. A Young measure on $O \times S$, is an any $\mu \in \mathcal{M}^+\,(O \times S)$, whose projection on O is dx, i.e. $\mu\,(A \times S) = dx\,(A)$, for all $A \in \mathcal{F}\,(O)$.*

2. The function Ψ: $O \times S \to \mathbb{R}$ is said to be:

   (a) An integrand if Ψ is $F(O) \otimes B(S)$-measurable

   (b) A normal integrand if Ψ is an integrand and $\Psi(x, .)$ is lower semicontinuous for dx-a.e. x in Ω

   (c) A Carathéodory integrand if Ψ is an integrand and $\Psi(x, .)$ is continuous for dx-a.e. x in Ω.

3. A sequence $(\Psi_n)_n$ in $L^1\left(\Omega, \mathbb{R}^N\right)$ is uniformly integrable if:

(a) $(\Psi_n)_n$ *is bounded in* $L^1\left(\Omega, \mathbb{R}^N\right)$

(b) for every $A \in \Omega$ $m_N(A) \to 0 \Rightarrow \sup\limits_n \int\limits_A \|(\Psi_n(x))_n\| \, dx \to 0$

We denote by $\mathcal{Y}(O \times S)$, the set of all Young measures on $O \times S$, and we say that, the sequence μ_n narrow converges to μ in $\mathcal{Y}(O \times S)$ and we write $\mu_n \overset{nar}{\to} \mu$, if for each Ψ in $Cth^b(O \times S)$ (the set of the Caratheodory bounded integrands), we have $\langle \Psi, \mu_n \rangle \to \langle \Psi, \mu \rangle$.

Theorem 2.2.2. *[disintegration [24]]*
Let $\mu \in \mathcal{Y}(O \times S)$, *then for a.e* $-x$ *in* O,*there exists a probability measures* μ_x *from* S, *such that for all* $\Psi : O \times S \to \mathbb{R}^+$, $\mu-$*measurable*

$$\int_{O \times S} \Psi(x, \lambda) d\mu = \int_O \int_S \Psi(x, \lambda) d\mu_x dx.$$

Then we write $\mu = \mu_x \otimes dx$

Let $\alpha : O \to S$, be a measurable function and $G : x \to (x, \alpha(x))$ from O into $O \times S$, the graph mapp of α. Denotting by $\mu_\alpha = dx \circ G^{-1}$, the image measure of dx on O by G. Then $\mu_\alpha \in \mathcal{Y}(O \times S)$, and for every $A \in \mathcal{F}(O)$ and every $B \in \mathcal{B}(S)$ $\mu_\alpha(A \times B) := dx(A \cap \alpha^{-1}(B))$. So, for each $\mu_\alpha-$measurable function $\Psi : O \times S \to \mathbb{R}^+$, we have

$$\int_{O \times S} \Psi(x, \lambda) \, d\mu_\alpha = \int_O \Psi(x, \alpha(x)) \, dx.$$

By using Therem 2.2.2, we obtain $\mu_\alpha = \delta_\alpha \otimes dx$, here δ_α denotes the Dirac measure of α. μ_α is said the Young measure associated to α. For a measurable sequence $\alpha_n : O \to S$, we say that

1. μ in $\mathcal{Y}(O \times S)$ is generated by α_n, if the sequence of the Young measures associated to α_n, narrow converges to μ. Or equivalently for all ϕ in $Cth^b(O \times S)$

$$\int_O \phi(x, \alpha_n(x)) \, dx \to \int_{O \times S} \phi(x, \lambda) \, d\mu.$$

2. The sequence μ_n in $\mathcal{Y}(O \times S)$ is tight if, for every $\eta > 0$, there exists a compact space $K_\eta \subset S$, such that $\sup\limits_n \mu_n \{O \times (S \setminus K_\eta)\} < \eta$, or $\sup\limits_n dx \{x \in O;\ \alpha_n(x) \in (S \setminus K_\eta)\} < \eta$, if μ_n is associated to α_n.

Note that, if $S = \mathbb{R}^N$, and α_n is a bounded sequence in $L^1\left(O; \mathbb{R}^N\right)$, then the sequence of theire associated Young measure is tight. If now, S is a compact space, then the sequence μ_n of the Young measure associated to α_n is tight.

Theorem 2.2.3 (see [24]). *Eevery tight sequence* μ_n *in* $Y(O \times S)$, *admits a subsequence* μ_{n_k} *witch narrow converges in* $Y(O \times S)$.

Proposition 2.2.4. *If the sequence (μ_n) is relatively compact in $\mathcal{Y}(O \times S_1)$ and if the sequence (ν_n) is relatively compact in $\mathcal{Y}(O \times S_2)$. Then, the sequence (μ_n, ν_n) is relatively compact in $\mathcal{Y}(O \times S_1 \times S_2)$.*

Theorem 2.2.5. *[Fundamental theorem for Young measures [24]]*
Let $\alpha_n : O \to S$ be a sequence of measurable functions, such that the sequence of their associated Young measures narrow converges to μ, then

(a) If $\psi : O \times S \to \mathbb{R}$ is a normal integrand such that, the sequence of the negative parts $\left\{\psi(x, \alpha_n(x))^-\right\}$ is uniformly integrable in O, then

$$\int_{O\times S} \psi(x, \lambda)\, d\mu \leq \liminf_n \int_O \psi(x, \alpha_n(x))\, dx.$$

(b) If $\psi : O\times S \to \mathbb{R}$ is a Carathéodory integrand such that, the sequence $\{\psi(x, \alpha_n(x))\}$ is uniformly integrable in O, then

$$\int_{O\times S} \psi(x, \lambda)\, d\mu = \lim_n \int_O \psi(x, \alpha_n(x))\, dx.$$

2.3 The scale convergence

In order to treat the periodic homogenization, the notion of two scale convergence is developed in [18] and [19]. As remarked by [25] the two scale limit represents in fact the barycenter of a Young measure. More recently, [23] introduced the scale convergence, which generalizes the multiscale convergence introduced by [18] and [19] This new concept, seems to be a powerful tool to study by $\Gamma-$convergence, the nonperiodic case.

Définition 2.3.1. *The sequence (v_n) in $L^2(\Omega), \alpha_n$-converges to $v \in L^2_\mu(\Omega \times \Pi)$ if for all $\phi \in L^2(\Omega; C(\Pi))$;*

$$\int_\Omega v_n(x)\phi(x, \alpha_n(x))dx \to \int_{\Omega\times\Pi} v(x, \lambda)\phi(x, \lambda)d\mu,$$

We will say that v is the $\alpha_n -$ limt of the sequence v_n.

The definition 2.3.1 is justified in view of the following compactness theorem

Theorem 2.3.2. *[23] From each bounded sequence (v_n) in $L^2(\Omega)$ there exist a subsequence (v_{n_k}), witch α_{n_k}-converges to $w \in L^2_\mu(\Omega \times \Pi)$).*
In particular, for a.e. x in Ω

$$v_{n_k} \rightharpoonup \int_\Pi w(x, \lambda)d\mu_x \text{ weakly in } L^2(\Omega).$$

Proposition 2.3.3. *[23] Let w be in $L^2(\Omega; C(\Pi))$ and $w_n(x) = w(x, \alpha_n(x))$. Then, for all Carathéodory integrand $\phi : \Omega \times \Pi \times \mathbb{R} \to \mathbb{R}$ such that, there exist a positive constant C and $p \in L^1(\Omega)$ satisfying, for all $(\lambda, \xi) \in \Pi \times \mathbb{R}$*

$$|\phi(x,\lambda,\xi)| \leq C(p(x) + |\xi|^2), a.e\ x \in \Omega.$$

We have

$$\int_\Omega \phi(x, \alpha_n(x), w_n(x))dx \to \int_{\Omega\times\Pi} \phi(x,\lambda, w(x,\lambda))d\mu$$

In particular (w_n), α_n*-converges to* w.

Proposition 2.3.4. *[25]*
$L^2(\Omega; C(\Pi))$ *is dense in* $\mathcal{H} = L^2_\mu(\Omega \times \Pi)$.

2.4 The main result

Let (X,τ) be a Banach space, and let $\{F_n,\ F,\ n \in \mathbb{N}\}$ be a family of functions mapping X into $\mathbb{R} \cup \{+\infty\}$. Let us recall the following notion of $\Gamma-$convergence,. For an overview about $\Gamma-$convergence, we refer the reader to [7] and [8].

Définition 2.4.1. *[17] We say that the sequence* $(F_n)_{n\in\mathbb{N}}$ $\Gamma-$*convergences to* F *at* x *in* X *iff the two following sentences hold:*

1. *for every sequence* $(x_n)_{n\in\mathbb{N}}$*, converging to* x *in* X*,* $F(x) \leq \liminf_{n\to+\infty} F_n(x_n)$.
2. *for every* $x \in X$*, there exists a sequence* (x_n) *of* X*, converging to* x *such that,* $F(x) = \liminf_{n\to+\infty} F_n(x_n)$.

We are now in a position to state the main result of this section.
Let $\{F_n; F^{\text{hom}}\}$ be the familly of the integrales functionals defined in the introduction. Then we have:

Theorem 2.4.2. F_n *Γ−converges weakly to* F^{hom} *in* $L^2(\Omega)$.

Proof. We prove the assertions (1) and (2), in definition (2.4.1), of the $\Gamma-$convergence.

Poof of (**1**)*:* Let v, v_n be a sequence in $L^2(\Omega)$, such that $v_n \rightharpoonup v$ weakly
Let (v_{n_k}) be the subsequence of (v_n) extracted in theorem (2.3.2) still denoted (v_n), which α_{n_k}-converges to $w \in \mathcal{H}$, with $v(x) = \int_\Pi w(x,\lambda)d\mu_x$, $a.e\ x \in \Omega$
From proposition (2.3.4), there exists a sequence (w^k) in $L^2(\Omega; C(\Pi))$ such that

$$\left\|w^k - w\right\|_{\mathcal{H}} < \frac{1}{k}.$$

Define $w^k_n(x) := w^k(x, \alpha_n(x))$, using the fact that f is $E - \xi$-convex, we have

$$F_n(v_n) \geq F_n\left(w^k_n\right) + \int_\Omega \frac{\partial f}{\partial \xi}\left(x, \alpha_n, E\left(w^k_n\right)\right)\left(E(v_n) - E\left(w^k_n\right)\right)dx.$$

Therfore

$$F_n(v_n) \geq F_n\left(w_n^k\right) + \int_\Omega \frac{\partial f}{\partial \xi}\left(x, \alpha_n, E\left(w_n^k\right)\right) E\left(v_n\right) dx - \int_\Omega \frac{\partial f}{\partial \xi}\left(x, \alpha_n, E\left(w_n^k\right)\right) E\left(w_n^k\right) dx$$

By (2.1), and the fact that f is $E-\xi$-convex, we have :

$$\left|\frac{\partial f}{\partial \xi}\left(x, \lambda, E(\xi)\right)\right| \leq C\left|1 + |E(\xi)|\right|$$

and since $E\left(0\right) = 0$ we obtain

$$\left|\frac{\partial f}{\partial \xi}\left(x, \lambda, E\left(\xi\right)\right)\right| \leq C\left|1 + |\xi|\right|$$

So $\frac{\partial f}{\partial \xi}\left(x, \alpha_n, g\left(w_n^k\right)\right)$ belong to $L^2(\Omega; C(\Pi))$,

Taking respectivly in proposition (2.3.3) $\phi(x, \lambda, \xi) = E\left(\xi\right) \frac{\partial f}{\partial \xi}(x, \lambda, E\left(\xi\right))$, and $\phi(x, \lambda, \xi) = f\ \left(x, \lambda, E\left(\xi\right)\right)$ we have respectively when $n \to +\infty$

$$\int_\Omega \frac{\partial f}{\partial \xi}\left(x, \alpha_n, E\left(w_n^k\right)\right) E\left(w_n^k\right) dx \to \int_{\Omega\times\Pi} \frac{\partial f}{\partial \xi}\left(x, \lambda, E\left(w^k\left(x, \lambda\right)\right)\right) E\left(w^k\left(x, \lambda\right)\right) d\mu$$

and

$$F_n\left(w_n^k\right) \to \int_{\Omega\times\Pi} f(x, \lambda, E\left(w^k\left(x, \lambda\right)\right))d\mu$$

Since $\frac{\partial f}{\partial \xi}\left(x, \alpha_n, E\left(w_n^k\right)\right) E\left(v_n\right)$ belong to $L^1(\Omega; C(\Pi))$. Therfore one can apply the statement (b) of theorem (2.2.5). Hence when $n \to +\infty$

$$\int_\Omega \frac{\partial f}{\partial \xi}\left(x, \alpha_n, E\left(w_n^k\right)\right) E\left(v_n\right) dx \to \int_{\Omega\times\Pi} \frac{\partial f}{\partial \xi}\left(x, \lambda, E\left(w^k\left(x, \lambda\right)\right)\right) E\left(w\left(x, \lambda\right)\right) d\mu$$

For fixed k, we obtain:

$$\begin{aligned} \underset{n\to+\infty}{liminf} F_n\left(v_n\right) \ &\geq \ \int_{\Omega\times\Pi} f(x, \lambda, E\left(w^k\left(x, \lambda\right)\right))d\mu + \\ &+ \int_{\Omega\times\Pi} \frac{\partial f}{\partial \xi}\left(x, \lambda, E\left(w^k\left(x, \lambda\right)\right)\right)\left(E\left(w\left(x, \lambda\right)\right) - E\left(w^k\left(x, \lambda\right)\right)\right) d\mu \end{aligned}$$

On the other hand, since w^k is close to w in the norm of $\mathcal{H}$
one obtians:

$$\underset{n\to+\infty}{liminf} F_n\left(v_n\right) \geq \int_{\Omega\times\Pi} f(x, \lambda, E\left(w^k\left(x, \lambda\right)\right))d\mu + O\left(\left\|w^k - w\right\|_{\mathcal{H}}\right)$$

By (2.1), and the fact that f is $E-\xi$-convex, we have :

$$|f(x,\lambda,E(\xi_1))-f(x,\lambda,E(\xi_2))|\leq C\,|1+|E(\xi_1)|+|E(\xi_2)||\,|E(\xi_1)-E(\xi_2)|$$

and since $E(0)=0$ we obtain

$$\left|f\left(x,\lambda,E\left(w^k\right)\right)-f\left(x,\lambda,E\left((w)\right)\right)\right|\leq C\left|1+\left|w^k\right|+|w|\right|\left|w^k-w\right|\leq\frac{1}{k}$$

Then

$$\underset{k\to+\infty}{lim}\int_{\Omega\times\Pi}f(x,\lambda,E\left(w^k\left(x,\lambda\right)\right))d\mu=\int_{\Omega\times\Pi}f(x,\lambda,E\left(w\left(x,\lambda\right)\right))d\mu$$

Finally,

$$\liminf_{n\to+\infty}F_n(v_n)\geq\int_{\Omega\times\Pi}f(x,\lambda,E\left(w(x,\lambda)\right))d\mu\geq F^{\mathrm{hom}}(v).$$

Proof of **(ii)** *Let v be an element of $L^2(\Omega)$. We prove that, there exists a sequence $(\overline{v}_n)$ in $L^2(\Omega)$ such that*

$$\begin{cases}\overline{v}_n\rightharpoonup v\text{ in }L^2(\Omega)\text{ weakly;}\\ \lim_n F_n(\overline{v}_n)=F^{\mathrm{hom}}(v).\end{cases}$$

Let $\left(w^k\right)$ be a minimizing sequence of the following minimizing problem

$$\inf\left\{\int_\Omega\int_\Pi f\left(x,\lambda,g\left(w(x,\lambda)\right)\right)d\mu;w\in\mathcal{H},\ v\left(x\right)=\int_\Pi w(x,\lambda)d\mu_x\right\}.$$

By the proposition (2.3.4), there exists $\left(\overline{w}^k\right)$ in $L^2(\Omega;C(\Pi))$ such that $\left\|\overline{w}^k-w^k\right\|_{\mathcal{H}}<\frac{1}{k}$. Then, for all $\varphi\in L^2(\Omega)$

$$\left|\int_{\Omega\times\Pi}\left[\overline{w}^k(x,\lambda)-w^k(x,\lambda)\right]\varphi(x)d\mu\right|\leq\frac{1}{k}\left\|\varphi\right\|_{L^2(\Omega)},$$

Therefore

$$\begin{aligned}\lim_{k\to+\infty}\int_{\Omega\times\Pi}\overline{w}^k\varphi d\mu &= \lim_{k\to+\infty}\int_{\Omega\times\Pi}w^k\varphi d\mu=\\ \lim_{k\to+\infty}\int_\Omega\int_\Pi w^k\varphi d\mu_x(\lambda)dx &= \int_\Omega\varphi(x)v(x)dx.\end{aligned}$$

In other hand, we have

$$\begin{aligned}\left|\int_{\Omega\times\Pi}f\left(x,\lambda,E\left(\overline{w}^k\right)\right)-f\left(x,\lambda,E\left(w^k\right)\right)d\mu\right| &\leq \int_{\Omega\times\Pi}C\left(1+\left|\overline{w}^k\right|+\left|w^k\right|\right)\left|\overline{w}^k-w^k\right|d\mu\\ &\leq \frac{C}{k};\end{aligned}$$

Then

$$\lim_{k\to+\infty}\int_{\Omega\times\Pi} f\left(x,\lambda,E\left(\overline{w}^k\right)\right)d\mu=\lim_{k\to+\infty}\int_{\Omega\times\Pi} f\left(x,\lambda,E\left(w^k\right)\right)d\mu=F^{\mathrm{hom}}(v).$$

Setting $\overline{w}_n^k(x):=\overline{w}^k(x,\alpha_n(x))$, from the proposition (2.3.3), we have, when $n\to+\infty$

$$\int_\Omega f\left(x,\alpha_n\left(x\right),E\left(\overline{w}_n^k\left(x\right)\right)\right)dx\to\int_{\Omega\times\Pi} f\left(x,\lambda,E\left(\overline{w}^k\right)\right)d\mu$$

and

$$\int_\Omega \overline{w}_n^k(x)\varphi(x)dx\to\int_{\Omega\times\Pi}\overline{w}^k(x,\lambda)\varphi(x)d\mu$$

For $k=k\left(n\right)$ such that $k\left(n\right)\geq n$ we obtain, when :$n\to+\infty$

$$\int_\Omega f\left(x,\alpha_n\left(x\right),E\left(\overline{w}_n^{k(n)}\left(x\right)\right)\right)dx\to F^{\mathrm{hom}}(v).$$

$$\begin{aligned}\int_\Omega \overline{w}_n^{k(n)}(x)\varphi(x)dx &\to \int_\Omega\int_\Pi w\left(x\right)\varphi(x)d\mu_x\left(\lambda\right)dx\\ &= \int_\Omega v\left(x\right)\varphi(x)dx\end{aligned}$$

for all $\varphi\in L^2(\Omega)$.
Finally setting: $\overline{v}_n:=\overline{w}_n^{k(n)}$ if $n=k\left(n\right)$, $\overline{v}_n:=v$ if $n\neq k\left(n\right)$,
The sequence $\overline{v}_n$ satisfies

$$\begin{gathered}\overline{v}_n\rightharpoonup v\text{ in }L^2(\Omega)\text{ weakly}\\ \lim_{n\to+\infty}F_n(\overline{v}_n)=F^{\mathrm{hom}}(v).\end{gathered}$$

□

Chapter 3

On general and system of General variational inequalities

3.1 Corrections to a Paper "on Projection Algorithms for Solving System of General Variational Inequalities

This section is to illustrate that the main result of the paper [Muhammad Aslam Noor, Khalida Inayat Noor, Projection algorithms for solving system of general variational inequalities, Nonlinear Analysis. 70 (2009) 2700-2706] is incorrect. We also present and study the new iterative method (3.1.6) to correct the main result of [29].

Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let K be a closed convex set in H.
For given nonlinear operators T_1, T_2, g, h. We consider the problem of finding $(x^*, y^*) \in K \times K$ such that :

$$\begin{cases} \langle \rho T_1 (y^*, x^*) + x^* - g (y^*) , g (v) - x^* \rangle \geq 0, \forall v \in H : g (v) \in K, \rho > 0 \\ \langle \eta T_2 (x^*, y^*) + y^* - h (x^*) , h (v) - y^* \rangle \geq 0, \forall v \in H : h (v) \in K, \eta > 0 \end{cases}$$

which is called the system of general variational inequalities (SGHVID).
For this purpose, we recall the following well knowns concept.

Définition 3.1.1. *A mapping $T : H \to H$ is called λ-Lipschitz continous if there exist constant $\lambda > 0$, such that :*

$$\forall x, y \in H : \|T (x) - T (y)\| \leq \lambda \|x - y\|$$

Définition 3.1.2. *A mapping $T : H \to H$ is called rlaxed (α, β)-cocoercive if there exist constants $\alpha > 0$, $\beta > 0$ such that :*

$$\forall x, y \in H : \langle T(x) - T(y), x - y \rangle \geq -\alpha \|T(x) - T(y)\|^2 + \beta \|x - y\|^2$$

Proposition 3.1.3. *Let K be a closed convex set in H, for given an element $z \in H$, $x \in K$ satisfies the inequality*

$$\langle x - z, y - x \rangle \geq 0, \forall y \in K$$

if and only if

$$x = P_K(z)$$

where P_K is a projection of H into K.

It is known that P_K is a nonexpansive mapping, i.e

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|, \forall x, y \in H.$$

Using Proposition 3.1.3, we can easily show that, finding the solution $(x^*, y^*) \in K \times K$ of (SGHVID) is equivalent to finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} x^* = (1 - \alpha_n) x^* + \alpha_n P_K [g(y^*) - \rho T_1(y^*, x^*)] \\ y^* = (1 - \alpha_n) y^* + \alpha_n P_K [h(x^*) - \eta T_2(x^*, y^*)] \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.
In the following section, We show that the proof of M. A. Noor and K. I. Noor in [29] is incorrect.

About [29]

M. A. Noor and K. I. Noor used the following iterative algorithm for solving the problem (SGHVID)

Algorithm 3.1.4 (Algorithm 3.1 in [29])**.** *For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_K [g(y_n) - \rho T_1(y_n, x_n)] \\ y_{n+1} = P_K [h(x_{n+1}) - \eta T_2(x_{n+1}, y_n)] \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

Theorem 3.1.5 (Theorem 4.1 in [29])**.** *Let (x^*, y^*) be the solution of SGHVID. Suppose that $T_1 : H \times H \to H$ is relaxed (γ_1, r_1) −cocoercive and μ_1 Lipschitzian in the first variable, and $T_2 : H \times H \to H$ is relaxed (γ_2, r_2) −cocoercive and μ_2 Lipschitzian in the first variable. Let g be relaxed (γ_3, r_3) −cocoercive and μ_3 Lipschitz and let h be relaxed*

(γ_4, r_4) −cocoercive and μ_4 Lipschitz continous. If

$$\left|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \frac{\sqrt{\left(r_1 - \gamma_1\mu_1^2\right)^2 - \mu_1^2 k_1 \left(2 - k_1\right)}}{\mu_1^2}, r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{k_1\left(2 - k_1\right)}, k_1 < 1,$$

$$\left|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \frac{\sqrt{\left(r_2 - \gamma_2\mu_2^2\right)^2 - \mu_2^2 k_2 \left(2 - k_2\right)}}{\mu_2^2}, r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{k_2\left(2 - k_2\right)}, k_2 < 1,$$

where

$$k_1 = \sqrt{1 - 2\left(r_3 - \gamma_3\mu_3^2\right) + \mu_3^2}$$

$$k_2 = \sqrt{1 - 2\left(r_4 - \gamma_4\mu_4^2\right) + \mu_4^2},$$

and $\alpha_n \in [0, 1]$, $\sum_{n=0}^{\infty} \alpha_k = \infty$, then for arbitrarily chosen initial points $x_0, y_0 \in K$, x_n and y_n obtained from Algorithm (3.2.2) converge strongly to x^* and y^* respectively.

Next we will prove that theirs proof is incorrect. Let us consider the following text quoted from the proof of (Theorem 4.1 in [29]).

Proof. To prove the result, we need first to evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$.

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1 - \alpha_n) x_n + \alpha_n P_K [g(y_n) - \rho T_1(y_n, x_n)] - (1 - \alpha_n) x^* - \alpha_n P_K [g(y^*) - \rho T_1(y^*, x^*) \\
&\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \|P_K [g(y_n) - \rho T_1(y_n, x_n)] - P_K [g(y^*) - \rho T_1(y^*, x^*)]\| \\
&\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \|[g(y_n) - \rho T_1(y_n, x_n)] - [g(y^*) - \rho T_1(y^*, x^*)]\| \\
&= (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \|y_n - y^* - \rho [T_1(y_n, x_n) - T_1(y^*, x^*)]\| \\
&\quad + \alpha_n \|y_n - y^* - [g(y_n) - g(y^*)]\|
\end{aligned}$$

From the relaxed (γ_1, r_1) −cocoercive and μ_1 Lipschitzian definition for the first variable on T_1, we have:

$$\begin{aligned}
\|y_n - y^* - \rho [T_1(y_n, x_n) - T_1(y^*, x^*)]\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n, x_n) - T_1(y^*, x^*), y_n - y^*\rangle \\
&\quad + \rho^2 \|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\
&\leq \|y_n - y^*\|^2 + 2\rho\gamma_1 \|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\
&\quad - 2\rho r_1 \|y_n - y^*\|^2 + \rho^2 \|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 . \\
&= \left[1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2\right] \|y_n - y^*\| .^2
\end{aligned}$$

Commentaries

1. There are a clear mistakes in the above formulation so it is not true because:
    - We can't apply the relaxed (γ_1, r_1) −cocoercive definition for the first variable on T_1, (the second variable of T_1 in $\langle T_1(y_n, x_n) - T_1(y^*, x^*), y_n - y^*\rangle$ is not equal).

- Also with $\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2$, we can't apply the Lipschitz continuity definition for the first variable on T_1.

2. The same error used when, they evaluated $\|x_{n+1} - x^* - \eta[T_2(x_{n+1}, y_n) - T_2(x^*, y^*)]\|^2$.

Main result

Now we suggest and analyze the following iterative method for solving the problem SGHVID.

Algorithm 3.1.6. *For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T_1(y_n, x_n)] \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_K[h(x_n) - \eta T_2(x_n, y_n)] \end{cases}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Special cases

1/ For $T_1 = T_2 = T$ in Algorithm (3.1.6), we arrive at

Algorithm 3.1.7. *For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T(y_n, x_n)] \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_K[h(x_n) - \eta T(x_n, y_n)] \end{cases}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Which is the aproximate solvability of the following system:

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - g(y^*), g(v) - x^* \rangle \geq 0, \forall v \in H : g(v) \in K, \rho > 0 \\ \langle \eta T(x^*, y^*) + y^* - h(x^*), h(v) - y^* \rangle \geq 0, \forall v \in H : h(v) \in K, \eta > 0 \end{cases}$$

2/ For $g = h$ in Algorithm (3.1.6), we get

Algorithm 3.1.8. *For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T_1(y_n, x_n)] \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_K[g(x_n) - \eta T_2(x_n, y_n)] \end{cases}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Which is the aproximate solvability of the following system:

$$\begin{cases} \langle \rho T_1(y^*, x^*) + x^* - g(y^*), g(v) - x^* \rangle \geq 0, \forall v \in H : g(v) \in K, \rho > 0 \\ \langle \eta T_2(x^*, y^*) + y^* - h(x^*), g(v) - y^* \rangle \geq 0, \forall v \in H : g(v) \in K, \eta > 0 \end{cases}$$

3/ For $T_1 = T_2 = T$, and $g = h$ in Algorithm (3.1.6), we have the following Algorithm

Algorithm 3.1.9. *For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T(y_n, x_n)] \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_K[g(x_n) - \eta T(x_n, y_n)] \end{cases}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Which is the aproximate solvability of the following system:

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - g(y^*), g(v) - x^* \rangle \geq 0, \forall v \in H : g(v) \in K, \rho > 0 \\ \langle \eta T(x^*, y^*) + y^* - h(x^*), g(v) - y^* \rangle \geq 0, \forall v \in H : g(v) \in K, \eta > 0 \end{cases}$$

Now we present the convergence criteria of Algorithm (3.1.6) under some suitable conditions and this is the main result of this part.

Theorem 3.1.10. *Let (x^*, y^*) be the solution of SGHVID. Suppose that $T_1 : H \times H \to H$ is relaxed $(\gamma_1, r_1)-$cocoercive and μ_1-Lipschitzian in the first variable and let T_1 be λ_1-Lipschitz continous in the second variable. Let $T_2 : H \times H \to H$ is relaxed $(\gamma_2, r_2)-$cocoercive and μ_2 Lipschitzian in the first variable and let T_2 be λ_2-Lipschitz continous in the second variable. Let g be relaxed $(\gamma_3, r_3)-$cocoercive and μ_3 Lipschitz and let h be relaxed $(\gamma_4, r_4)-$cocoercive and μ_4-Lipschitz continous. If*

$$\begin{cases} k_1 < \frac{1}{2}, r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{\frac{3}{4} - k_1^2 + k_1}, \\ \left|\rho - \frac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \frac{\sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2\left[\frac{3}{4} - k_1^2 + k_1\right]}}{\mu_1^2}, \rho < \frac{1}{2\lambda_1} \end{cases} \tag{3.1}$$

$$\begin{cases} k_2 < \frac{1}{2}, r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{\frac{3}{4} - k_2^2 + k_2}, \\ \left|\eta - \frac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \frac{\sqrt{(r_2 - \gamma_2\mu_2^2)^2 - \mu_2^2\left[\frac{3}{4} - k_2^2 + k_2\right]}}{\mu_2^2}, \eta < \frac{1}{2\lambda_2} \end{cases} \tag{3.2}$$

where

$$k_1 = \sqrt{1 - 2(r_3 - \gamma_3\mu_3^2) + \mu_3^2}$$
$$k_2 = \sqrt{1 - 2(r_4 - \gamma_4\mu_4^2) + \mu_4^2},$$

and $\alpha_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_k = \infty$, then for arbitrarily chosen initial points $x_0, y_0 \in K$, x_n and y_n obtained from Algorithm (3.1.6) converge strongly to x^* and y^* respectively.

Proof. To prove the result, we need first to evaluate $\|x_{n+1}-x^*\|$ for all $n \geq 0$.

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|(1-\alpha_n)(x_n-x^*)+\alpha_n[P_K[g(y_n)-\rho T_1(y_n,x_n)]-P_K[g(y^*)-\rho T_1(y^*,x^*)]]\| \\
&\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|P_K[g(y_n)-\rho T_1(y_n,x_n)]-P_K[g(y^*)-\rho T_1(y^*,x^*)]\| \\
&\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|[g(y_n)-\rho T_1(y_n,x_n)]-[g(y^*)-\rho T_1(y^*,x^*)]\| \\
&\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|y_n-y^*-\rho[T_1(y_n,x_n)-T_1(y^*,x^*)]\| \\
&\quad +\alpha_n\|y_n-y^*-[g(y_n)-g(y^*)]\| \\
&\leq \alpha_n\|y_n-y^*-\rho[T_1(y_n,x_n)-T_1(y^*,x_n)+T_1(y^*,x_n)-T_1(y^*,x^*)]\| \\
&\quad +\alpha_n\|y_n-y^*-[g(y_n)-g(y^*)]\|+(1-\alpha_n)\|x_n-x^*\| \\
&\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|y_n-y^*-\rho[T_1(y_n,x_n)-T_1(y^*,x_n)]\| \\
&\quad +\rho\alpha_n\|T_1(y^*,x_n)-T_1(y^*,x^*)\|+\alpha_n\|y_n-y^*-[g(y_n)-g(y^*)]\|
\end{aligned}$$

From the relaxed .(γ_1, r_1) −cocoercive for the first variable on T_1, we have

$$\begin{aligned}
\|y_n-y^*-\rho[T_1(y_n,x_n)-T_1(y^*,x_n)]\|^2 &= \|y_n-y^*\|^2-2\rho\langle T_1(y_n,x_n)-T_1(y^*,x_n), y_n-y^*\rangle \\
&\quad +\rho^2\|T_1(y_n,x_n)-T_1(x_n,y^*)\|^2 \\
&\leq -2\rho\left[-\gamma_1\|T_1(y_n,x_n)-T_1(y^*,x_n)\|^2+r_1\|y_n-y^*\|^2\right] \\
&\quad +\|y_n-y^*\|^2+\rho^2\|T_1(y_n,x_n)-T_1(y^*,x_n)\|^2 \\
&\leq 2\rho\gamma_1\|T_1(y_n,x_n)-T_1(y^*,x_n)\|^2-2\rho r_1\|y_n-y^*\|^2 \\
&\quad +\|y_n-y^*\|^2+\rho^2\|T_1(y_n,x_n)-T_1(y^*,x_n)\|^2
\end{aligned}$$

From the μ_1-Lipschitzian definition for the first variable on T_1, we have:

$$\|y_n-y^*-\rho[T_1(y_n,x_n)-T_1(y^*,x_n)]\|^2 \leq \left[1+2\rho\gamma_1\mu_1^2-2\rho r_1+\rho^2\mu_1^2\right]\|y_n-y^*\|^2$$

In a similar way, using the (γ_3, r_3) −cocoercivity and μ_3-Lipschitz continuity of the operator g; we have:

$$\|y_n-y^*-[g(y_n)-g(y^*)]\| \leq k_1\|y_n-y^*\|$$

From the λ_1-Lipschitzian definition for the second variable on T_1, we have:

$$\|T_1(y^*,x_n)-T_1(y^*,x^*)\| \leq \lambda_1\|x_n-x^*\|$$

As a result, we have:

$$\|x_{n+1}-x^*\| \leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\theta_1\|y_n-y^*\|+\alpha_n\rho\lambda_1\|x_n-x^*\| \tag{3.3}$$

where,

$$\theta_1 = k_1+\left[1+2\rho\gamma_1\mu_1^2-2\rho r_1+\rho^2\mu_1^2\right]^{\frac{1}{2}}$$

Similary we have:

$$\|y_{n+1}-y^*\| \leq (1-\alpha_n)\|y_n-y^*\|+\alpha_n\theta_2\|x_n-x^*\|+\alpha_n\eta\lambda_2\|y_n-y^*\| \tag{3.4}$$

where,

$$\theta_2 = k_2 + \left[1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2\right]^{\frac{1}{2}}.$$

It is clear from the conditions (3.1) and (3.2) that,

$$\theta_1 + \eta\lambda_2 < 1 \quad \text{and} \quad \theta_2 + \rho\lambda_1 < 1.$$

Then from (3.3) and (3.4),

$$\begin{aligned}\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| + \alpha_n\rho\lambda_1\|x_n - x^*\| \\ &+ (1-\alpha_n)\|y_n - y^*\| + \alpha_n\theta_2\|x_n - x^*\| + \alpha_n\eta\lambda_2\|y_n - y^*\| \\ &\leq (1-\alpha_n)[\|x_n - x^*\| + \|y_n - y^*\|] + \sigma\alpha_n[\|x_n - x^*\| + \|y_n - y^*\|]\end{aligned}$$

where,

$$\sigma = \max(\theta_1 + \eta\lambda_2, \theta_2 + \rho\lambda_1) < 1.$$

Set

$$z_n = \|x_n - x^*\| + \|y_n - y^*\|.$$

So,

$$z_{n+1} \leq (1 - (1-\sigma)\alpha_n) z_n$$

wich implies that:

$$z_{n+1} \leq \prod_{k=0}^{k=n}(1 - (1-\sigma)\alpha_k) z_0$$

Since $0 < \sigma < 1$ and $\sum_{k=0}^{\infty}\alpha_k = \infty$, it implies in light of [35] that $\lim_{n\to+\infty}\prod_{k=0}^{k=n}((1-(1-\sigma)\alpha_k)) = 0$ therfore $x_n \to x^*$ and $y_n \to y^*$. □

Corollary 3.1.1. *We can replace the conditions (3.1) and (3.2) by (3.5) and (3.6) where,* $0 < p_1 < 1$ *and* $0 < p_2 < 1$.

$$\begin{cases} k_1 < p_1, r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{-k_1^2 + 2p_1k_1 + 1 - p_1^2}, \\ \left|\rho - \dfrac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \dfrac{\sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2[-k_1^2 + 2p_1k_1 + 1 - p_1^2]}}{\mu_1^2} \\ \rho < \dfrac{1-p_2}{\lambda_1} \end{cases} \tag{3.5}$$

$$\begin{cases} k_2 < p_2, r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{-k_2^2 + 2p_2k_2 + 1 - p_2^2}, \\ \left|\eta - \dfrac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \dfrac{\sqrt{(r_2 - \gamma_2\mu_2^2)^2 - \mu_2^2[-k_2^2 + 2p_2k_2 + 1 - p_2^2]}}{\mu_2^2}, \\ \eta < \dfrac{1-p_1}{\lambda_2} \end{cases} \tag{3.6}$$

Remark 3.1.11. *If $T_1, T_2 : H \to H$ are univariate operators, then Algorithm (3.1.6) can be replaced by the folloing Algorithm.*

Algorithm 3.1.12. *For arbitrary chosen initial points $x_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n P_K \left[g\left(y_n\right) - \rho T_1\left(y_n\right)\right] \\ y_n = P_K \left[h\left(x_n\right) - \eta T_2\left(x_n\right)\right] \end{cases}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Which is the approximate solvability of the system (3.7) :

$$\begin{cases} \langle \rho T_1\left(y^*\right) + x^* - g\left(y^*\right), g\left(v\right) - x^* \rangle \geq 0, \forall v \in H : g\left(v\right) \in K, \rho > 0 \\ \langle \eta T_2\left(x^*\right) + y^* - h\left(x^*\right), h\left(v\right) - y^* \rangle \geq 0, \forall v \in H : h\left(v\right) \in K, \eta > 0 \end{cases} \tag{3.7}$$

For the system (3.7), we use Algorithm (3.1.12) and present the following Theorem which uses less conditions than the previous Theorem.

Theorem 3.1.13. *Let (x^*, y^*) be the solution of (3.7). Suppose that $T_1, T_2, g, h : H \to H$ be both relaxed-cocoercive with constants (γ_1, r_1), (γ_2, r_2), (γ_3, r_3), (γ_4, r_4) and Lipschitz continuous with constants μ_1, μ_2, μ_3, μ_4 respectively. If*

$$\begin{cases} k_1 < 1, r_1 > \gamma_1 \mu_1^2 + \mu_1 \sqrt{-k_1^2 + 2k_1}, \\ \left| \rho - \dfrac{r_1 - \gamma_1 \mu_1^2}{\mu_1^2} \right| < \dfrac{\sqrt{\left(r_1 - \gamma_1 \mu_1^2\right)^2 - \mu_1^2 \left[-k_1^2 + 2k_1\right]}}{\mu_1^2}, \end{cases} \tag{3.8}$$

$$\begin{cases} k_2 < 1, r_2 > \gamma_2 \mu_2^2 + \mu_2 \sqrt{-k_2^2 + 2k_2}, \\ \left| \eta - \dfrac{r_2 - \gamma_2 \mu_2^2}{\mu_2^2} \right| < \dfrac{\sqrt{\left(r_2 - \gamma_2 \mu_2^2\right)^2 - \mu_2^2 \left[-k_2^2 + 2k_2\right]}}{\mu_2^2}. \end{cases} \tag{3.9}$$

Where

$$k_1 = \sqrt{1 - 2\left(r_3 - \gamma_3 \mu_3^2\right) + \mu_3^2}$$

$$k_2 = \sqrt{1 - 2\left(r_4 - \gamma_4 \mu_4^2\right) + \mu_4^2},$$

and $\alpha_n \in [0,1]$, $\sum\limits_{n=0}^{\infty} \alpha_k = \infty$, then for arbitrarily chosen initial points $x_0 \in K$, x_n and y_n obtained from Algorithm (3.1.12) converge strongly to x^* and y^* respectively.

Proof. To prove the result, we need first to evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$.

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)\, x_n + \alpha_n P_K \left[g\left(y_n\right) - \rho T_1\left(y_n\right)\right] - (1-\alpha_n)\, x^* + \alpha_n P_K \left[g\left(y^*\right) - \rho T_1\left(y^*\right)\right]\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|P_K\left[g\left(y_n\right) - \rho T_1\left(y_n\right)\right] - P_K\left[g\left(y^*\right) - \rho T_1\left(y^*\right)\right]\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|\left[g\left(y_n\right) - \rho T_1\left(y_n\right)\right] - \left[g\left(y^*\right) - \rho T_1\left(y^*\right)\right]\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|y_n - y^* - \rho\left[T_1\left(y_n\right) - T_1\left(y^*\right)\right]\| \\ &\quad + \|y_n - y^* - \left[g\left(y_n\right) - g\left(y^*\right)\right]\| \end{aligned}$$

From the relaxed (γ_1, r_1) −cocoercive on T_1, we have :

$$\begin{aligned} \|y_n - y^* - \rho [T_1(y_n) - T_1(y^*)]\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n) - T_1(y^*), y_n - y^*\rangle \\ &\quad + \rho^2 \|T_1(y_n) - T_1(y^*)\|^2 \\ &\le -2\rho\left[-\gamma_1 \|T_1(y_n) - T_1(y^*)\|^2 + r_1 \|y_n - y^*\|^2\right] \\ &\quad + \|y_n - y^*\|^2 + \rho^2 \|T_1(y_n) - T_1(y^*)\|^2 \\ &\le 2\rho\gamma_1 \|T_1(y_n) - T_1(y^*)\|^2 - 2\rho r_1 \|y_n - y^*\|^2 \\ &\quad + \|y_n - y^*\|^2 + \rho^2 \|T_1(y_n) - T_1(y^*)\|^2 \end{aligned}$$

From the μ_1-Lipschitzian definition on T_1, we have:

$$\|y_n - y^* - \rho [T_1(y_n) - T_1(y^*)]\|^2 \le \left[1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2\right] \|y_n - y^*\|^2$$

In a similar way, using the (γ_3, r_3) −cocoercivity and μ_3-Lipschitz continuity of the operator g; we have:

$$\|y_n - y^* - [g(y_n) - g(y^*)]\| \le k_1 \|y_n - y^*\|$$

As a result, we have:

$$\|x_{n+1} - x^*\| \le (1 - \alpha_n) \|x_n - x^*\| + \alpha_n\theta_1 \|y_n - y^*\| \tag{3.10}$$

where,

$$\theta_1 = k_1 + \left[1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2\right]^{\frac{1}{2}}$$

Now we evaluate $\|y_{n+1} - y^*\|$ for all $n \ge 0$.

$$\begin{aligned} \|y_n - y^*\| &= \|P_K[h(x_n) - \eta T_2(x_n)] - P_K[h(x^*) - \eta T_1(x^*)]\| \\ &\le \|[h(x_n) - \eta T_2(x_n)] - [h(x^*) - \eta T_2(x^*)]\| \\ &\le \|x_n - x^* - \eta [T_2(x_n) - T_2(x^*)]\| + \|x_n - x^* - [h(x_n) - h(x^*)]\| \end{aligned}$$

From the relaxed .(γ_2, r_2) −cocoercive on T_2, we have :

$$\begin{aligned} \|x_n - x^* - \eta [T_2(x_n) - T_2(x^*)]\|^2 &= \|x_n - x^*\|^2 - 2\eta\langle T_2(x_n) - T_2(x^*), x_n - x^*\rangle \\ &\quad + \eta^2 \|T_2(x_n) - T_2(x^*)\|^2 \\ &\le -2\eta\left[-\gamma_2 \|T_2(x_n) - T_2(x^*)\|^2 + r_2 \|x_n - x^*\|^2\right] \\ &\quad + \|x_n - x^*\|^2 + \eta^2 \|T_2(x_n) - T_2(x^*)\|^2 \\ &\le 2\eta\gamma_2 \|T_2(x_n) - T_2(x^*)\|^2 - 2\eta r_2 \|x_n - x^*\|^2 \\ &\quad + \|x_n - x^*\|^2 + \eta^2 \|T_2(x_n) - T_2(x^*)\|^2 \end{aligned}$$

By using the (γ_4, r_4) −cocoercivity and μ_4-Lipschitz continuity of the operator h, we have:

$$\|x_n - x^* - [h(x_n) - h(x^*)]\| \le k_2 \|x_n - x^*\|$$

As a result, we have:

$$\|y_n - y^*\| \leq \theta_2 \|x_n - x^*\| \tag{3.11}$$

where,

$$\theta_2 = k_2 + \left[1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2\right]^{\frac{1}{2}}$$

It is clear from the condition (3.8) and (3.9) that

$$\theta_1 < 1 \quad \text{and} \quad \theta_2 < 1.$$

It follow that from (3.21) and (3.22),

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \theta_1 \theta_2 \|x_n - x^*\|$$

wich implies that:

$$\|x_{n+1} - x^*\| \leq \prod_{k=0}^{k=n} (1 - (1 - \theta_1\theta_2)\alpha_k) \|x_0 - x^*\|$$

Since $0 < \theta_1\theta_2 < 1$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$ it implies in light of [35] that $\lim_{n \to +\infty} \prod_{k=0}^{k=n} ((1 - (1 - \theta_1\theta_2)\alpha_k)) = 0$ therfore $x_n \to x^*$ and $y_n \to y^*$. □

Remark 3.1.14. *For $g = h$ and $T_1 = T_2 = T$, in Algorithm (3.1.12) we arrive at*

Algorithm 3.1.15. *For arbitrary chosen initial points $x_0 \in K$, compute the sequences $\{x_n\}$ by using*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_K [g(y_n) - \rho T(y_n)]$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

Which is the approximate solvability of the general variational inequality:

$$\langle T(x^*), h(v) - x^* \rangle \geq 0, \forall v \in H : h(v) \in K$$

which has been considered and studied by M. A. Noor [28].
For $g = h = I$ and $T_1 = T_2 = T$, in Algorithm (3.1.12) we get

Algorithm 3.1.16. *For arbitrary chosen initial points $x_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_K [y_n - \rho T(y_n)] \\ y_n = P_K [x_n - \eta T(x_n)] \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

Which is the approximate solvability of the following system of variational inequalities (SNVI):

$$\begin{cases} \langle \rho T(y^*) + x^* - y^*, v - x^* \rangle \geq 0, \forall v \in K, \rho > 0 \\ \langle \eta T(x^*) + y^* - x^*, v - y^* \rangle \geq 0, \forall v \in K, \eta > 0 \end{cases}$$

which has been considered and studied by Ram U. Verma [33].

3.2 On General convergence analysis for two-step projection methods and applications to variational problems

This section is to illustrate that the main result of the paper [Ram U. Verma, General convergence analysis for two-step projection methods and applications to variational problems, Applied Mathematics Letters. 18 (2005) 1286-1292] is incorrect by giving an counterexample. We also present and study a new iterative method (3.2.5) to correct the main result of [33].
Throughout this section we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. In this part, we consider a system of two nonlinear variational inequality (abbreviated as SNVI) problems as follows: determine elements $x^*, y^* \in K$ such that :

$$\begin{cases} \langle \rho T(y^*) + x^* - y^*, v - x^* \rangle \geq 0, \forall v \in K, \rho > 0 \\ \langle \eta T(x^*) + y^* - x^*, v - y^* \rangle \geq 0, \forall v \in K, \eta > 0 \end{cases} \tag{3.12}$$

In this note new approximation schemes (3.2.5) are discussed for solving the problem (SNVI). The results obtained in this section correct the main results in [33].

We recall:

Définition 3.2.1. *A mapping $T : H \to H$ is called r-strongly monotonic if there exist constant $r > 0$, such that :*

$$\forall x, y \in H : \|T(x) - T(y)\| \geq r \|x - y\|^2 .$$

Using Proposition 3.1.3, we can easily show that, finding the solution $(x^*, y^*) \in K \times K$ of (3.12) is equivalent to finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} x^* = (1 - \alpha_n) x^* + \alpha_n P_K [g(y^*) - \rho T(y^*)] \\ y^* = P_K [g(x^*) - \eta T(x^*)] \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.
In the following section, we show that the results of Verma in [33] are incorrect.

About [33]

Verma used the following iterative algorithm for solving the problem (SNVI)

Algorithm 3.2.2 (Algorithm 3.1 in [33])**.** *For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_K [y_n - \rho T(y_n)] \\ y_n = (1 - \beta_n) x_n + \beta_n P_K [x_n - \eta T(x_n)] \end{cases}$$

where $\alpha_n, \beta_n \in [0, 1]$ for all $n \geq 0$.

Theorem 3.2.3. *[Theorem 3.1 in [33]] Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let $T : K \to H$ be strongly r-monotonic and μ-Lipschitz continous. Suppose that $x^*, y^* \in K$ form a solution to the SNVI problem. If*

$$\begin{cases} 0 < \rho < \dfrac{2r}{\mu^2} \\ 0 < \eta < \dfrac{2r}{\mu^2} \end{cases}$$

and $\alpha_n, \beta_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$, then for arbitrarily chosen initial points $x_0, y_0 \in K$, x_n and y_n obtained from Algorithm (3.2.2) converge strongly to x^* and y^* respectively.

Commentaries

The sequence y_n does not converge to y^* because:
If we take:

$$\begin{cases} 0 < \rho < \dfrac{2r}{\mu^2} \\ 0 < \eta < \dfrac{2r}{\mu^2} \\ \alpha_n = \beta_n = \dfrac{1}{2} \end{cases}$$

It is clear that $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$, and

$$\begin{cases} x_{n+1} = \dfrac{1}{2} x_n + \dfrac{1}{2} P_K \left[y_n - \rho T(y_n) \right] \\ y_n = \dfrac{1}{2} x_n + \dfrac{1}{2} P_K \left[x_n - \eta T(x_n) \right] \end{cases}$$

By using Theorem (3.2.3), we obtian:

$$\begin{cases} x^* = \dfrac{1}{2} x^* + \dfrac{1}{2} P_K \left[y^* - \rho T(y^*) \right] \\ y^* = \dfrac{1}{2} x^* + \dfrac{1}{2} P_K \left[x^* - \eta T(x^*) \right] \end{cases}$$

Which is equivalent to,

$$\begin{cases} x^* = P_K \left[y^* - \rho T(y^*) \right] \\ 2y^* - x^* = P_K \left[x^* - \eta T(x^*) \right] \end{cases}$$

Using Proposition (3.1.3), we arrive at

$$\begin{cases} \langle \rho T(y^*) + x^* - y^*, v - x^* \rangle \geq 0, \forall v \in K, \\ \langle \eta T(x^*) + 2y^* - 2x^*, v - 2y^* + x^* \rangle \geq 0, \forall v \in K. \end{cases}$$

Which is not the same problem SNVI.

Remark 3.2.4. *Let us consider the following text quoted from the proof of (Theorem 3.1 in [33]):*
Similary, we have

$$\begin{aligned} \|y_k - y^*\| &= \|(1-\beta_k)(x_k - x^*) + \beta_n P_K[x_k - \eta T(x_k)] - \beta_n P_K[x^* - \eta T(x^*)]\| \\ &\leq (1-\beta_k)\|x_k - x^*\| + \beta_k \|[x_k - x^*] - \eta[T(x_k) - T(x^*)]\| \\ &\leq (1-\beta_k)\|x_k - x^*\| + \beta_k \left[1 - 2\eta r + (\eta\mu)^2\right]^{\frac{1}{2}} \|x_k - x^*\| \\ &= (1-\beta_k)\|x_k - x^*\| + \beta_k \sigma \|x_k - x^*\|, \end{aligned}$$

where $\sigma = \left[1 - 2\eta r + (\eta\mu)^2\right]^{\frac{1}{2}} < 1$.

This remark implies that the mistake is not an erratum.

Main result

Now we suggest and analyze the following iterative method for solving (3.12).

Algorithm 3.2.5. *For arbitrary chosen initial points* $x_0 \in K$*, compute the sequences* $\{x_n\}$ *and* $\{y_n\}$ *using*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[y_n - \rho T(y_n)] \\ y_n = P_K[x_n - \eta T(x_n)] \end{cases}$$

where $\alpha_n \in [0,1]$ *for all* $n \geq 0$.

Theorem 3.2.6. *Let* (x^*, y^*) *be the solution of (3.12). Suppose that* $T : H \to H$ *be strongly* $r-$*monotonic and* $\mu-$*Lipschitz continuous. If*

$$\begin{cases} 0 < \rho < \dfrac{2r}{\mu^2} \\ 0 < \eta < \dfrac{2r}{\mu^2} \end{cases} \tag{3.13}$$

and $\alpha_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_k = \infty$*, then for arbitrarily chosen initial points* $x_0 \in K$*,* x_n *and* y_n *obtained from Algorithm (3.2.5) converge strongly to* x^* *and* y^* *respectively.*

beginproof To prove the result, we need first to evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$.

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n P_K[y_n - \rho T(y_n)] - (1-\alpha_n)x^* + \alpha_n P_K[y^* - \rho T(y^*)]\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|P_K[y_n - \rho T(y_n)] - P_K[y^* - \rho T(y^*)]\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|[y_n - y^*] - \rho[T(y_n) - T(y^*)]\| \end{aligned}$$

Since T is $r-$strongly monotonic, we have :

$$\begin{aligned} \|y_n - y^* - \rho [T(y_n) - T(y^*)]\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle T(y_n) - T(y^*), y_n - y^*\rangle \\ &\quad + \rho^2 \|T(y_n) - T(y^*)\|^2 \\ &\leq -2\rho \left[r \|y_n - y^*\|^2\right] + \|y_n - y^*\|^2 + \rho^2 \|T(y_n) - T(y^*)\|^2 \end{aligned}$$

From the Lipschitzian definition on T, we have:

$$\|y_n - y^* - \rho [T(y_n) - T(y^*)]\|^2 \leq \left[1 - 2\rho r + \rho^2\mu^2\right] \|y_n - y^*\|^2$$

As a result, we have:

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n\theta_1 \|y_n - y^*\| \tag{3.14}$$

where $\theta_1 = [1 - 2\rho r + \rho^2\mu^2]^{\frac{1}{2}}$
Now we evaluate $\|y_n - y^*\|$ for all $n \geq 0$.

$$\begin{aligned} \|y_n - y^*\| &= \|P_K [x_n - \eta T(x_n)] - P_K [x^* - \eta T(x^*)]\| \\ &\leq \|[x_n - x^*] - \eta [T(x_n) - T(x^*)]\| \end{aligned}$$

Similary, Since T is $r-$strongly and $\mu-$Lipschitz continuous mapping, we obtain :

$$\|y_n - y^*\| \leq \theta_2 \|x_n - x^*\| . \tag{3.15}$$

where $\theta_2 = [1 - 2\eta r + \eta^2\mu^2]^{\frac{1}{2}}$
Notice that $\theta_1 < 1$ and $\theta_2 < 1$ from assumption (3.13), and hence from (3.14) and (3.15), we have

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n\theta_1\theta_2 \|x_n - x^*\|$$

wich implies that:

$$\|x_{n+1} - x^*\| \leq \prod_{k=0}^{k=n} (1 - (1 - \theta_1\theta_2)\alpha_k) \|x_0 - x^*\|$$

Since $0 < \theta_1\theta_2 < 1$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$ it implies in light of [35] that
$\lim_{n\to+\infty} \prod_{k=0}^{k=n} ((1 - (1 - \theta_1\theta_2)\alpha_k)) = 0$ therfore $x_n \to x^*$ and $y_n \to y^*$. endproof

3.3 On Extended General Variational Inequalities

In this notes (see [8]) we show that Lemmas 2.1 and 3.1 in [27] [Mohamed Aslam Noor, Applied Mathematical Letters 22 (2009) 182-186] are incorrect by analyzing his proofs.

Definitions and somme results in [27]

In this section we recall the relatived lemmas given in [27] which will be used in our study. Let H be a real Hilbert space whose inner product is denoted by $\langle ., . \rangle$.

Définition 3.3.1. *(Def 2.1 in [1]) Let K be any set in H. The set K is said to be hg-convex if there exist functions $g, h : H \to H$ such that*

$$tg(v) + (1-t)h(u) \in K,$$

for each $u, v \in H : h(u), g(v) \in K$ and all $t \in [0, 1]$.

Définition 3.3.2. *(Def 2.2 in [1]) A function $F : K \to \mathbb{R}$ is said to be hg-convex on K if K is hg-convex and*

$$F\left(tg(v) + (1-t)h(u)\right) \leq tF\left(g\left(v\right)\right) + (1-t)F\left(h\left(u\right)\right),$$

for each $u, v \in H : h(u), g(v) \in K$ and all $t \in [0, 1]$.

Lemma 3.3.3. *(Lemma 2.1 in [1]) Let $F : K \to \mathbb{R}$ be a differentiable hg-convex functions. Then $u \in H : h(u) \in K$ is the minimum of the hg-convex function F on K if and only if it satisfies the inequality*

$$\left\langle F'\left(h\left(u\right)\right), g\left(v\right) - h\left(u\right)\right\rangle \geq 0, \quad \forall v \in H : g\left(v\right) \in K,$$

where $F'\left(u\right)$ is the Frechet differential of F at u.

Lemma 3.3.4. *(Lemma 3.1 in [1]) The function $u \in H : h(u) \in K$ is a solution of the extended general variational inequality*

$$\langle Tu + h(u) - g(u), g(v) - h(u) \rangle \geq 0, \forall v \in H : g(v) \in K, \tag{3.16}$$

if and only if $u \in H : h(u) \in K$ satisfies the relation

$$h(u) = P_K\left[g(u) - \rho Tu\right], \tag{3.17}$$

where P_K is the projection operator and $\rho > 0$ is a constant.

On Lemma 2.1

Throughout his proof of Lemma 2.1 (the converse part) he found :

$$F(h(u)) \leq F(g(v)), \forall v \in H : g(v) \in K$$

this inequality implies not that $h(u) \in K$ is the minimum of F on K in H.

On Lemma 3.1

By a careful reading, we discovered that Lemma 3.3.4 (Lemma 3.1 in [1]) is the main tool of the paper. Unfortunately it is not true because :
If $u \in H : h(u) \in K$ be a solution of 3.16. Then we have,

$$\langle h(u) - (g(u) - Tu), g(v) - h(u)\rangle \geq 0, \forall v \in H : g(v) \in K, \tag{3.18}$$

which implies, (using the characterization of the Projection onto a Convex Set K) that

$$h(u) = P_K\left[g(u) - Tu\right],$$

which is not the required 3.17.

Remark 3.3.5. *We can not add $\rho > 0$ in (3.16).*

3.4 System of Nonlinear General Variational inequalities Involving g-relaxed cocoercive mappings

In this section, we introduce and consider a new system of nonlinear general variational inequalities involving two g-relaxed (α, β)-cocoercive operators. We suggest and analyze an iterative method for this system of general variational inequalities. We establish a convergence result for the proposed method under certain conditions. Our results can be viewed as a refinement and improvement of the previously known results for variational inequalities.
Throughout this study we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. In this part, we consider, based on the projection method, the approximate solvability of a system of g-relaxed cocoercive variational inequalities in Hilbert spaces. The results obtained extend and improve the main results in [33].
Let K be a closed convex set in H. For given non linear operator $T_1, T_2, g : H \to H$, we consider a system of nonlinear variational inequality (3.19) problem as follows: to find $x^*, y^* \in K$ such that :

$$\begin{cases} \langle \rho T_1(y^*) + x^* - g(y^*), g(v) - x^*\rangle \geq 0, \forall v \in H : g(v) \in K, \rho > 0 \\ \langle \eta T_2(x^*) + y^* - g(x^*), g(v) - y^*\rangle \geq 0, \forall v \in H : g(v) \in K, \eta > 0 \end{cases} \tag{3.19}$$

Special cases

1/ If $T_1 = T_2 = T$, then the problem (3.19) is equivalent to finding $x^* \in K$ such that

$$\langle T(x^*), g(v) - x^*\rangle \geq 0, \forall v \in H : g(v) \in K$$

which has been considered and studied by M. A. Noor [28].
2/ If $g = I$ and $T_1 = T_2 = T$, then the problem (3.19) is equivalent to the following

system of variational inequalities (SNVI): finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} \langle \rho T(y^*) + x^* - y^*, v - x^* \rangle \geq 0, \forall v \in K, \rho > 0 \\ \langle \eta T(x^*) + y^* - x^*, v - y^* \rangle \geq 0, \forall v \in K, \eta > 0 \end{cases}$$

which has been considered and studied by Ram U. Verma [33].

We recall:

Définition 3.4.1. *Let $g : H \to H$ be a map. A mapping $T : H \to H$ is called g-relaxed (α, β)-cocoercive if there exist constants $\alpha > 0$, $\beta > 0$ such that :*

$$\forall x, y \in H : \langle T(x) - T(y), g(x) - g(y) \rangle \geq -\alpha \|T(x) - T(y)\|^2 + \beta \|g(x) - g(y)\|^2$$

For $g = I$, the identity operator. Definition (3.4.1) reduces to the standard definition of relaxed (c,r)-cocoercivity .
Using Proposition (3.1.3), we can easily show that, finding the solution $(x^*, y^*) \in K \times K$ of (3.19) is equivalent to finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} x^* = (1 - \alpha_n) x^* + \alpha_n P_K [g(y^*) - \rho T_1(y^*)] \\ y^* = P_K [g(x^*) - \eta T_2(x^*)] \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

Main result

Now we suggest and analyze the following iterative method for solving (3.19).

Algorithm 3.4.2. *For arbitrary chosen initial points $x_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_K [g(y_n) - \rho T_1(y_n)] \\ y_n = P_K [g(x_n) - \eta T_2(x_n)] \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

If $g = I$ and $T_1 = T_2 = T$, then algorithm (3.4.2) reduces to the following

Algorithm 3.4.3. *For arbitrary chosen initial points $x_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using*

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_K [y_n - \rho T(y_n)] \\ y_n = P_K [x_n - \eta T(x_n)] \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

which is mainly due to Verma [33] (see Algorithm 2.3 in [33])

Theorem 3.4.4. *Let (x^*, y^*) be the solution of (3.19). Suppose that $T_1, T_2 : H \to H$ be both g-relaxed-cocoercive with constants (γ_1, r_1), (γ_2, r_2) and Lipschitz continuous with constants μ_1, μ_2 respectively. Let $g : H \to H$ be $\mu-$Lipschitz continuous. If*

$$\begin{cases} \theta_1 = [\mu^2 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1\mu^2 + \rho^2\mu_1^2]^{\frac{1}{2}} < 1, \\ \theta_2 = [\mu^2 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2\mu^2 + \eta^2\mu_2^2]^{\frac{1}{2}} < 1 \end{cases} \tag{3.20}$$

and $\alpha_n \in [0,1]$, $\sum\limits_{n=0}^{\infty} \alpha_k = \infty$, then for arbitrarily chosen initial points $x_0 \in K$, x_n and y_n obtained from Algorithm (3.4.2) converge strongly to x^* and y^* respectively.

Proof. To prove the result, we need first to evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$.

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T_1(y_n)] - (1-\alpha_n)x^* + \alpha_n P_K[g(y^*) - \rho T_1(y^*)]\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|P_K[g(y_n) - \rho T_1(y_n)] - P_K[g(y^*) - \rho T_1(y^*)]\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|[g(y_n) - \rho T_1(y_n)] - [g(y^*) - \rho T_1(y^*)]\| \end{aligned}$$

From the g-relaxed $(\gamma_1, r_1)-$cocoercive on T_1, we have :

$$\begin{aligned} \|g(y_n) - g(y^*) - \rho[T_1(y_n) - T_1(y^*)]\|^2 &= \|g(y_n) - g(y^*)\|^2 - 2\rho\langle T_1(y_n) - T_1(y^*), g(y_n) - g(y^*)\rangle \\ &\quad + \rho^2\|T_1(y_n) - T_1(y^*)\|^2 \\ &\leq -2\rho\left[-\gamma_1\|T_1(y_n) - T_1(y^*)\|^2 + r_1\|g(y_n) - g(y^*)\|^2\right] \\ &\quad + \|g(y_n) - g(y^*)\|^2 + \rho^2\|T_1(y_n) - T_1(y^*)\|^2 \\ &\leq 2\rho\gamma_1\|T_1(y_n) - T_1(y^*)\|^2 - 2\rho r_1\|g(y_n) - g(y^*)\|^2 \\ &\quad + \|g(y_n) - g(y^*)\|^2 + \rho^2\|T_1(y_n) - T_1(y^*)\|^2 \end{aligned}$$

From the Lipschitzian definition on T_1 and g , we have:

$$\|g(y_n) - g(y^*) - \rho[T_1(y_n) - T_1(y^*)]\|^2 \leq \left[\mu^2 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1\mu^2 + \rho^2\mu_1^2\right]\|y_n - y^*\|^2$$

As a result, we have:

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| \tag{3.21}$$

Now we evaluate $\|y_n - y^*\|$ for all $n \geq 0$.

$$\begin{aligned} \|y_n - y^*\| &= \|P_K[g(x_n) - \eta T_2(x_n)] - P_K[g(x^*) - \eta T_1(x^*)]\| \\ &\leq \|[g(x_n) - \eta T_2(x_n)] - [g(x^*) - \eta T_2(x^*)]\| \end{aligned}$$

Similary, Since T_2 is a g-relaxed (γ_2, r_2) −cocoercive and g is μ-Lipschitz continuous mapping, we obtain :

$$\|y_n - y^*\| \leq \theta_2 \|x_n - x^*\| . \tag{3.22}$$

Notice that $\theta_1 < 1$ and $\theta_2 < 1$ from assumption (3.23), and hence from (3.21) and (3.22), we have

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \theta_1 \theta_2 \|x_n - x^*\|$$

wich implies that:

$$\|x_{n+1} - x^*\| \leq \prod_{k=0}^{k=n} (1 - (1 - \theta_1 \theta_2) \alpha_k) \|x_0 - x^*\|$$

Since $0 < \theta_1\theta_2 < 1$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$ it implies in light of [35] that
$\lim_{n \to +\infty} \prod_{k=0}^{k=n} ((1 - (1 - \theta_1\theta_2) \alpha_k)) = 0$ therfore $x_n \to x^*$ and $y_n \to y^*$. □

If $g = I$ and $T_1 = T_2 = T$, then the following theorem can be directly obtained from Theorem (3.4.4)

Theorem 3.4.5. *Let (x^*, y^*) be the solution of (3.19). Suppose that $T : H \to H$ be (γ, r)-relaxed-cocoercive and μ-Lipschitz continuous mappings. If*

$$\begin{cases} \theta_1 = [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]^{\frac{1}{2}} < 1, \\ \theta_2 = [1 + 2\eta\gamma\mu^2 - 2\eta r + \eta^2\mu^2]^{\frac{1}{2}} < 1 \end{cases} \tag{3.23}$$

and $\alpha_n \in [0, 1]$, $\sum_{n=0}^{\infty} \alpha_k = \infty$, then for arbitrarily chosen initial points $x_0 \in K$, x_n and y_n obtained from Algorithm (3.4.2) converge strongly to x^* and y^* respectively.

Chapter 4

A new parameterized logarithmic kernel function for linear optimization with a double barrier term yielding the best known iteration bound

4.1 An introduction to kernel functions and its roles

After the groundbreaking paper of Karmarkar [20], Kernel functions play an important role in the complexity analysis of the interior point methods (IPMs) for linear optimization (LO).

In 2001, Peng et al. [32] designed a new paradigm of primal-dual algorithms based on the socalled self-regular proximity functions for LO. They improved iteration bound and achieved the best known complexity results for large and small-update methods. Subsequently, in 2004 Bai et al. [5] proposed new kernel function with an exponential barrier term, and introduced the first new kernel function with a trigonometric barrier term. These functions enjoy useful properties and determine new search directions for primal-dual interior point algorithms. Based on these functions, they obtained the best known complexity results for large-update methods, namely, $\mathbf{O}\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$ and good numerical results.

In 2008, El Ghami et al. [14] proposed parameterized kernel function with a logarithmic barrier term. This function generalized the kernel functions given in [15, 36].

In 2018, Bouafia et al. [10] proposed a parameterized logarithmic kernel function for primal-dual IPMs. They obtained the best known complexity results for large and small-update methods, they took the middle between Peng [32] and Elghami's [14] barrier as a barrier term. The objective of this chapter is to introduce a new class of kernel functions which differs from the existing kernel functions in which it has a double barrier term (logarithmic-exponential barrier term) . This chapter is organized as follows. In Sect 2, we recall the preliminaries. In Sect 3 and 4, we define a new kernel function and give its properties which are essential for the complexity analysis. The estimate of the step size and the decrease behavior of the new barrier function are discussed in Sect 5. Also we derive the complexity result for both large-update and small-update methods. Some numerical results are provided in Section 6. Finally, we end up the chapter by a conclusion

4.2 Preliminaries

In this section we recall some basic concepts and the generic IPMs, we consider linear optimization (LO) problem in the standard format:

$$\min < c, x >: Ax = b, x \geq 0, \tag{P}$$

where $A \in \mathbb{R}^{m\times n}$, $rank(A) = m$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, and its dual problem

$$\max < b, y >: A^T y + s = c,\ s \geq 0. \tag{D}$$

A new polynomial-time method for solving LO is proposed by Karmarkar [20]. After that, this method was developped in the literature which play an important role for solving linear optimization problem and its variants are now called IPMs. For more details about the subject, we can refer to Bai et al. [4], Peng et al. [31], Roos et al. [36] and Ye [40]. Without loss of generality, we assume that (P) and (D) satisfy the interior point condition (IPC), i.e., there exist (x^0, y^0, s^0) such that

$$Ax^0 = b,\ x^0 > 0,\ A^T y^0 + s^0 = c,\ s^0 > 0. \tag{4.1}$$

It is well known that finding an optimal solution of (P) and (D) is equivalent to solving the following system

$$Ax = b,\ x \geq 0,\ A^T y + s = c,\ s \geq 0,\ xs = 0. \tag{4.2}$$

The basic idea of primal-dual IPMs is to replace the third equation in (4.2), the so-called complementarity condition for (P) and (D), by the parameterized equation $xs = \mu e$, with $\mu > 0$. Thus we consider the system

$$Ax = b,\ x \geq 0,\ A^T y + s = c,\ s \geq 0,\ xs = \mu e. \tag{4.3}$$

Surprisingly enough, if the IPC is satisfied, then there exists a solution, for each $\mu > 0$, and this solution is unique. It is denoted as $(x(\mu), y(\mu), s(\mu))$, and we call $x(\mu)$ the μ-center of (P) and $(y(\mu), s(\mu))$ the μ-center of (D). The set of μ-centers (with μ running through all positive real numbers) gives a homotype path, which is called the central path of (P) and (D). The relevance of the central path for LO was recognized first by Megiddo [26] and Sonnevend [37]. If $\mu \to 0$, then the limit of the central path exists, and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for (P) and (D). From a theoretical point of view, the IPC can be assumed without loss of generality. In fact, we may, and will, assume that $x^0 = s^0 = e$. In practice, this can be realized by embedding the given problems (P) and (D) into a homogeneous self-dual problem, which has two additional variables and two additional constraints. For this and the other properties mentioned above, see [36].

The IPMs follow the central path approximately. We briefly describe the usual approach. Without loss of generality, we assume that $(x(\mu), y(\mu), s(\mu))$ is known for some positive μ. For example, due to the above assumption, we may assume this for $\mu = 1$, with $x(1) = s(1) = e$. We then decrease μ to $\mu = (1 - \theta)\mu$ for some fixed $\theta \in]0, 1[$, and we solve the following Newton system:

$$A\Delta x = 0,\ A^T\Delta y + \Delta s = 0,\ s\Delta x + x\Delta s = \mu e - xs. \tag{4.4}$$

This system uniquely defines a search direction $(\Delta x, \Delta y, \Delta s)$. By taking a step along the search direction, with the step size defined by some line search rules, we construct a new triple (x, y, s). If necessary, we repeat the procedure until we find iterates that are "close" to $(x(\mu), y(\mu), s(\mu))$. Then μ is again reduced by the factor $1-\theta$, and we apply Newton's method targeting the new μ-centers, and so on. This process is repeated until μ is small enough, say until $n\mu \le \epsilon$, at this stage, we have found an ϵ-solution of problems (P) and (D). The result of a Newton step with step size α is denoted as

$$x_+ = x + \alpha\Delta x,\ s_+ = s + \alpha\Delta s,\ y_+ = y + \alpha\Delta y, \tag{4.5}$$

where the step size α satisfies $0 < \alpha \le 1$. Now we introduce the scaled vector v and the scaled search directions d_x and d_s as follows:

$$v = \sqrt{\frac{xs}{\mu}},\ d_x = \frac{v\Delta x}{x},\ d_s = \frac{v\Delta s}{s}. \tag{4.6}$$

System (4.4) can be rewritten as follows:

$$\overline{A}d_x = 0,\ \overline{A}^T\Delta y + d_s = 0,\ d_x + d_s = v^{-1} - v, \tag{4.7}$$

where $\overline{A} = \frac{1}{\mu}AV^{-1}X$, $V = diag(v)$, $X = diag(x)$. Note that the right-hand side of the third equation in (4.7) is equal to the negative gradient of the logarithmic barrier function $\Phi(v)$, i.e., $d_x + d_s = -\nabla\Phi(v)$, system (4.7) can be rewritten as follows:

$$\overline{A}d_x = 0,\ \overline{A}^T\Delta y + d_s = 0,\ d_x + d_s = -\nabla\Phi(v), \tag{4.8}$$

where the barrier function $\Phi(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ is defined as follows:

$$\Phi(v) = \Phi(x, s; \mu) = \sum_{i=1}^{n} \psi(v_i), \tag{4.9}$$

$$\psi(v_i) = \frac{v_i^2 - 1}{2} - \log v_i. \tag{4.10}$$

We use $\Phi(v)$ as the proximity function to measure the distance between the current iterate and the μ-center for given $\mu > 0$. We also define the norm-based proximity measure, $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$, as follows

$$\delta(v) = \frac{1}{2}||\nabla\Phi(v)|| = \frac{1}{2}||d_x + d_s||, \tag{4.11}$$

We call $\psi(t)$ the kernel function of the logarithmic barrier function $\Phi(v)$. In this study, we replace $\psi(t)$ by a new kernel function $\psi_{New}(t)$ and $\Phi(v)$ by a new barrier function $\Phi_{New}(v)$, which will be defined in Sect. 4.3. Note that the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$ if and only if $v = e$. It is clear from the above description that the closeness of (x, s) to $(x(\mu), s(\mu))$ is measured by the value of $\Phi(v)$ with $\tau > 0$ as a threshold value. If $\Phi(v) \leq \tau$, then we start a new outer iteration by performing a μ-update; otherwise, we enter an inner iteration by computing the search directions at the current iterates with respect to the current value of μ and apply (4.5) to get new iterates. If necessary, we repeat the procedure until we find iterates that are in the neighborhood of $(x(\mu), s(\mu))$. Then μ is again reduced by the factor $1 - \theta$ with $0 < \theta < 1$, and we apply Newton's method targeting the new μ-centers, and so on. This process is repeated until μ is small enough, say until $n\mu < \epsilon$; at this stage, we have found an ϵ-approximate solution of LO. The parameters τ, θ and the step size α should be chosen in such a way that the algorithm is optimized in the sense that the number of iterations required by algorithm is as small as possible. The choice of the so-called barrier update parameter θ plays an important role in both theory and practice of IPMs . Usually, if θ is a constant independent of the dimension n of the problem, for instance, $\theta = \frac{1}{2}$, then we call the algorithm a large-update (or long-step) method. If θ depends on the dimension of the problem, such as $\theta = \frac{1}{\sqrt{n}}$, then the algorithm is called a small-update (or short-step) method.

In most cases, the best complexity result obtained for small-update IPMs is $\mathbf{O}\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$. For large-update methods the best obtained bound is $\mathbf{O}\left(\sqrt{n}\log n \log\frac{n}{\varepsilon}\right)$, which until now has been the best known bound for such methods [5, 32].

In this part, we define a new kernel function and propose primal-dual interior point methods which improve all the results of the complexity bound for large-update methods based on a logarithmic-exponential kernel function for LO. Another interesting choice is m dependent with n, which minimizes the iteration complexity bound. In fact, if we take $m = \log n$, we obtain the best known complexity bound for large-update methods namely $\mathbf{O}\left(\sqrt{n}\log(n)\log\left(\frac{n}{\varepsilon}\right)\right)$. This bound improves the so far obtained complexity results for large-update methods based on a logarithmic kernel function given by El Ghami et al. [5].

Generic Primal-dual IPMs for LO

Input:
a dimension of the problem $n, n \in \mathbb{N}$
a proximity function $\Phi_{New}(v)$,
a threshold parameter $\tau > 1$
an accuracy parameter $\epsilon > 0$,
a fixed barrier update parameter $\theta, 0 < \theta < 1$,
begin
$x = e$, $s = e$, $\mu = 1$, $v = e$.
while $n\mu \geq \epsilon$ do
begin (outer iteration)
$\mu = (1-\theta)\mu$,
while $\Phi(x, s;\mu) > \tau$ do
begin (inner iteration)
solve the system (4.8) , $\Phi(v)$ replaced by $\Phi_{New}(v)$ to obtain $(\Delta x, \Delta y, \Delta s)$,
choose a suitable step size α,
$x = x + \alpha\Delta x, y = y + \alpha\Delta y, s = s + \alpha\Delta s$
$v = \sqrt{\frac{xs}{\mu}}$,
end (inner iteration)
end (outer iteration)
end .

Generic algorithm

4.3 The new kernel function and its properties

In this section, a new kernel function with its properties are provided. We call $\psi(t)$: $\mathbb{R}_+^* \longrightarrow \mathbb{R}_+$ a kernel function if it is twice differentiable and satisfies the following conditions

$$\psi'(1) = \psi(1) = 0,\ \psi''(t) > 0,\ \lim_{t\to 0^+} \psi(t) = \lim_{t\longrightarrow +\infty} \psi(t) = +\infty.$$

In the other words, a kernel function is a univariate strictly convex function which is defined for all positive real t and is minimal at $t = 1$ whereas the minimal value equals 0 (see El Ghami et al. [5]). Now, let us introduce our new kernel function, which are used in the above Generic Algorithm.

$$\psi(t) = t^2 - 1 - \log(t) + \frac{e^{m(\frac{1}{t}-1)} - 1}{m},\ m > 0. \tag{4.12}$$

Kernel function	Large update	References
$\frac{1}{2}(t^2-1)-\log(t)$	$\mathbf{O}\left(n\log\frac{n}{\varepsilon}\right)$	[5]
$\frac{1}{2}(t-\frac{1}{t})^2$	$\mathbf{O}\left(n^{\frac{2}{3}}\log\frac{n}{\varepsilon}\right)$	[30]
$\frac{1}{1+p}(t^{1+p}-1)-\log(t), p\in[0,1]$	$\mathbf{O}\left(n\log\frac{n}{\varepsilon}\right)$	[14]
$\frac{1}{2}(t^2-1)+e^{(\frac{1}{t}-1)}-1$	$\mathbf{O}\left(\sqrt{n}(\log n)^2\log\frac{n}{\varepsilon}\right)$	[5]
$\frac{1}{2}(t^2-1)+\frac{t^{1-q}-1}{q-1}, q>1$	$\mathbf{O}\left(qn^{\frac{q+1}{2q}}\log\frac{n}{\varepsilon}\right)$	[32]
$t-1+\frac{t^{1-q}-1}{q-1}, q>1$	$\mathbf{O}\left(qn\log\frac{n}{\varepsilon}\right)$	[4]
$\frac{1}{2}(t^2-1-\log(t))+\frac{t^{1-q}-1}{2(q-1)}, q>1$	$\mathbf{O}\left(qn^{\frac{q+1}{2q}}\log\frac{n}{\varepsilon}\right)$	[10]
$\frac{p}{2}(t^2-1)+e^{p(\frac{1}{t}-1)}-1, p\geq 1$	$\mathbf{O}\left(\sqrt{np^5}(\log pn)^2\log\frac{n}{\varepsilon}\right)$	[11]
$t^2-1-\log(t)+\frac{e^{m(\frac{1}{t}-1)}-1}{m}, m>0$	$\mathbf{O}\left(m\sqrt{n}(1+\frac{\log(n)}{m})^2\log\frac{n}{\varepsilon}\right)$	New

Table 1. Examples of kernel functions and its iteration bound for large-update methods.

It can be easily seen that as $t\to 0^+$ or $t\to+\infty$, then $\psi(t)\to+\infty$. Therefore, $\psi(t)$ is indeed a kernel function. As we need the first three derivatives of $\psi(t)$, we list them here:

$$\psi'(t) = 2t - t^{-1} - t^{-2}e^{m(\frac{1}{t}-1)}, \tag{4.13}$$
$$\psi''(t) = 2 + t^{-2} + (m+2t)t^{-4}e^{m(\frac{1}{t}-1)}, \tag{4.14}$$
$$\psi'''(t) = -[2t^{-3} + (6t^2+6mt+m^2)t^{-6}e^{m(\frac{1}{t}-1)}]. \tag{4.15}$$

4.4 Eligibility of the new kernel function

Next lemma serves to prove that the new kernel function (4.12) is efficient.

Lemma 4.4.1. *Let $\psi(t)$ be as defined in (4.12) and $t>0$. Then,*

$$\psi''(t) > 2, \tag{4.16}$$
$$\psi'''(t) < 0, \tag{4.17}$$
$$t\psi''(t) - \psi'(t) > 0, \tag{4.18}$$
$$t\psi''(t) + \psi'(t) > 0, \tag{4.19}$$

Proof. It is easy to see that (4.16) and (4.17) follow from (4.14) and (4.15) respectively. To prove (4.18) and (4.19), we have from (4.13) and (4.14) the following

$$t\psi^{''}(t) - \psi^{'}(t) = 2t^{-1} + (m+3t)t^{-3}e^{m(\frac{1}{t}-1)} > 0.$$

and

$$t\psi^{''}(t) + \psi^{'}(t) = 4t + (m+t)t^{-3}e^{m(\frac{1}{t}-1)} > 0,$$

the right-hand side of the above equality is positive, which proves (4.19). □

The last property (4.19) in lemma 4.4.1 is equivalent to convexity of composed functions $t \to \psi(e^t)$ and this holds if only if

$$\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}\left(\psi(t_2) + \psi(t_2)\right), \text{ for any } t_1, t_2 \geq 0. \tag{4.20}$$

This property is known in the literature, and it was demonstrated by several researchers (see [20, 40]).

Lemma 4.4.2. *For $\psi(t)$, we have*

$$(t-1)^2 \leq \psi(t) \leq \frac{1}{4}\left[\psi^{'}(t)\right]^2, \; t > 0. \tag{4.21}$$

$$\psi(t) \leq \frac{1}{2}\left[5+m\right](t-1)^2, \; t > 1. \tag{4.22}$$

Proof. For (4.21), using (4.16), we have

$$\psi(t) = \int_1^t \int_1^x \psi^{''}(y)dydx \geq \int_1^t \int_1^x 2dydx = (t-1)^2.$$

$$\begin{aligned}\psi(t) &= \int_1^t \int_1^x \psi^{''}(y)dydx \leq \int_1^t \int_1^x \frac{1}{2}\psi^{''}(x)\psi^{''}(y)dydx \\ &= \frac{1}{2}\int_1^t \psi^{''}(x)\psi^{'}(x)dx \\ &= \frac{1}{2}\int_1^t \psi^{'}(x)d\psi^{'}(x) = \frac{1}{4}\left[\psi^{'}(t)\right]^2.\end{aligned}$$

Since $\psi(1) = \psi^{'}(1) = 0$, $\psi^{'''}(t) < 0$, $\psi^{''}(1) = 5+m$, and by using Taylor's Theorem, we have

$$\begin{aligned}\psi(t) &= \psi(1) + \psi^{'}(1)(t-1) + \frac{1}{2}\psi^{''}(1)(t-1)^2 + \frac{1}{6}\psi^{'''}(\xi)(t-1)^3 \\ &\leq \frac{1}{2}\psi^{''}(1)(t-1)^2 \\ &= \frac{1}{2}\left[5+m\right](t-1)^2,\end{aligned}$$

for some ξ, $1 \leq \xi \leq t$. This completes the proof. □

Now, we analysis the generic algorithm by Following the steps presented in [5].

Step 1. (steps 1 and 3 in [5]) we derive some bounds for $\sigma(t)$ and $\rho(t)$.
Let $\sigma : [0, \infty[\to [1, +\infty[$ be the inverse function of $\psi(t)$ for $t \geq 1$ and $\rho : [0, \infty[\to]0, 1]$ be the inverse function of $-\frac{1}{2}\psi'(t)$ for all $t \in]0, 1]$. Then we have the following results.

Proposition 4.4.3. *For $\psi(t)$, we have*

$$1 + \sqrt{\frac{2s}{5+m}} \leq \sigma(s) \leq 1 + \sqrt{s},\ s \geq 0. \tag{4.23}$$

$$\rho(z) \geq \frac{1}{1 + \frac{1}{m}\log(2z+1)},\ z > 0. \tag{4.24}$$

Proof. For (4.23), let $s = \psi(t)$, $t \geq 1$, i.e., $\sigma(s) = t, t \geq 1$.
By (4.21), we have $s \geq (t-1)^2$, this implies that $t = \sigma(s) \leq 1 + \sqrt{s}$.
By (4.22), we have:

$$s = \psi(t) \leq \frac{1}{2}(5+m)(t-1)^2,\ t \geq 1, \text{ so } t = \sigma(s) \geq 1 + \sqrt{\frac{2s}{5+m}}.$$

For (4.24), let $z = -\frac{1}{2}\psi'(t)$, $t \in]0, 1]$. By the definition of $\psi'(t)$, we have:

$$\begin{aligned} 2z &= -2t + t^{-1} + t^{-2}e^{m(\frac{1}{t}-1)} \\ &\geq -2 + t^{-1} + t^{-2}e^{m(\frac{1}{t}-1)} \\ &\geq -1 + t^{-2}e^{m(\frac{1}{t}-1)} \\ &\geq -1 + e^{m(\frac{1}{t}-1)}. \end{aligned}$$

which implies $t = \rho(z) \geq \frac{1}{1+\frac{1}{m}\log(2z+1)}$. This completes the proof. □

Step 2. Derive a lower bound for δ in term of Φ

Proposition 4.4.4. *Let $\delta(v)$ be as defined in (4.11). Then we have*

$$\delta(v) \geq \sqrt{\Phi(v)}. \tag{4.25}$$

Proof. Using (4.21), we have

$$\Phi(v) = \sum_{i=1}^{n} \psi(v_i) \leq \sum_{i=1}^{n} \frac{1}{4}\left[\psi'(v_i)\right]^2 = \frac{1}{4}||\nabla\Phi(v)||^2 = \delta(v)^2,$$

so $\delta(v) \geq \sqrt{\Phi(v)}$. □

4.5 An Estimation for the step size

Step 3. In this section, we compute a default step size α, we have

$$x_+ = x + \alpha\Delta x,\ y_+ = y + \alpha\Delta y,\ s_+ = s + \alpha\Delta s.$$

Using (4.6), we have

$$x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}\left(v + \alpha d_x\right),$$
$$s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}\left(v + \alpha d_s\right).$$

So, we have $v_+ = \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{\left(v + \alpha d_x\right)\left(v + \alpha d_s\right)}$. Define for $\alpha > 0$, $f(\alpha) = \Phi(v_+) - \Phi(v)$. Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed μ. By (4.19), we have

$$\Phi(v_+) = \Phi\left(\sqrt{\left(v + \alpha d_x\right)\left(v + \alpha d_s\right)}\right) \leq \frac{1}{2}\left(\Phi((v + \alpha d_x)) + \Phi((v + \alpha d_s))\right).$$

Therefore, we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) = \frac{1}{2}\left(\Phi((v + \alpha d_x)) + \Phi((v + \alpha d_s))\right) - \Phi(v). \tag{4.26}$$

Obviously, $f(0) = f_1(0) = 0$. Taking the first two derivatives of $f_1(\alpha)$ with respect to α, we have

$$f_1^{'}(\alpha) = \sum_{i=1}^{n}\left(\psi^{'}(v_i + \alpha d_{x_i})d_{x_i} + \psi^{'}(v_i + \alpha d_{s_i})d_{s_i}\right),$$
$$f_1^{''}(\alpha) = \sum_{i=1}^{n}\left(\psi^{''}(v_i + \alpha d_{x_i})d_{x_i}^2 + \psi^{''}(v_i + \alpha d_{s_i})d_{s_i}^2\right).$$

Using (4.6) and (4.11), we have

$$f_1^{'}(0) = \frac{1}{2} < \nabla\Phi(v), (d_x + d_s) > = -\frac{1}{2} < \nabla\Phi(v), \nabla\Phi(v) > = -2\delta(v)^2.$$

For convenience, we denote $v_1 = min(v)$, $\delta = \delta(v)$, $\Phi = \Phi(v)$.

Remark 4.5.1. *Throughout this study, we assume that* $\tau \geq 1$. *Using Lemma 4.4.4 and the assumption that* $\Phi(v) \geq \tau$, *we have* $\delta(v) \geq 1$.

From Lemmas 4.1-4.4 in [5], we have the following Lemma 4.5.2-4.5.5, because $\psi(t)$ is kernel function and $\psi^{''}(t)$ is monotonically decreasing.

Lemma 4.5.2. *[Bai et al.[5]] Let $f_1(\alpha)$ be as defined in (4.26) and $\delta(v)$ be as defined in (4.11). Then we have $f_1''(\alpha) \leq 2\delta^2\psi''(v_{min} - 2\alpha\delta)$. Since $f_1(\alpha)$ is convex, we will have $f_1'(\alpha) \leq 0$ for all α less than or equal to the value where $f_1(\alpha)$ is minimal, and vice versa.*

The previous Lemma leads to the following three Lemmas:

Lemma 4.5.3. *[Bai et al.[5]] $f_1'(\alpha) \leq 0$ certainly holds if α satisfies the inequality*

$$\psi'(v_{min}) - \psi'(v_{min} - 2\alpha\delta) \leq 2\delta \tag{4.27}$$

Lemma 4.5.4. *[Bai et al.[5]] The largest step size $\overline{\alpha}$ holding (4.27) is given by $\overline{\alpha} = \frac{\rho(\delta)-\rho(2\delta)}{2\delta}$.*

Lemma 4.5.5. *[Bai et al. [5]] Let $\overline{\alpha}$ be as defined in Lemma 4.5.4. Then $\overline{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}$.*

Now, we are in position to prove the following Lemma

Lemma 4.5.6. *Let ρ and $\overline{\alpha}$ be as defined in Lemma 4.5.5. If $\Phi(v) \geq \tau \geq 1$, then we have $\overline{\alpha} \geq \frac{1}{2+[1+\frac{1}{m}\log(4\sqrt{\Phi(v)}+1)]^2[1+(m+2)(4\sqrt{\Phi(v)}+1)]}$.*

Proof. Using Lemma 4.5.5, (4.14), (4.24), and (4.25) we have

$$\begin{aligned}
\overline{\alpha} &\geq \frac{1}{\psi''(\rho(2\delta))} \\
&= \frac{1}{2 + [\rho(2\delta)]^{-2} + (m + 2\rho(2\delta))[\rho(2\delta)]^{-4} e^{m(\frac{1}{\rho(2\delta)}-1)}} \\
&\geq \frac{1}{2 + [1 + \frac{1}{m}\log(4\delta + 1)]^2 + (m+2)[\rho(2\delta)]^{-2}[\rho(2\delta)]^{-2} e^{m(\frac{1}{\rho(2\delta)}-1)}} \\
&\geq \frac{1}{2 + [1 + \frac{1}{m}\log(4\delta + 1)]^2 + (m+2)[1 + \frac{1}{m}\log(4\delta+1)]^2[4\delta + 1]} \\
&\geq \frac{1}{2 + [1 + \frac{1}{m}\log(4\delta + 1)]^2[1 + (m+2)(4\delta+1)]} \\
&\geq \frac{1}{2 + [1 + \frac{1}{m}\log(4\sqrt{\Phi(v)} + 1)]^2[1 + (m+2)(4\sqrt{\Phi(v)}+1)]}
\end{aligned}$$

This completes the proof. □

Denoting

$$\widetilde{\alpha} = \frac{1}{2 + [1 + \frac{1}{m}\log(4\sqrt{\Phi(v)} + 1)]^2[1 + (m+2)(4\sqrt{\Phi(v)}+1)]}, \tag{4.28}$$

we have that $\widetilde{\alpha}$ is the default step size and that $\widetilde{\alpha} \leq \overline{\alpha}$.

Step 4. finding a positive constants κ and γ

Lemma 4.5.7. *[Lemma 4.5 in [5]] If the step size α satisfies $\alpha \le \overline{\alpha}$, then*

$$f(\alpha) \le -\alpha\delta^2.$$

Proposition 4.5.8. *Let $\Phi_0 \ge \Phi(v) \ge 1$ and let $\widetilde{\alpha}$ be the default step size as defined in (4.28). Then, we have*

$$f(\widetilde{\alpha}) \le -\kappa\left[(\Phi)_0\right]^{1-\gamma} \tag{4.29}$$

with $\kappa = \frac{1}{18m[1+\frac{1}{m}\log(4\sqrt{\Phi_0}+1)]^2}$ *and* $\gamma = \frac{1}{2}$.

Proof. from (4.25), (4.28) and by using Lemma 4.5.7 (Lemma 4.5 in [5]) with $\alpha = \widetilde{\alpha}$ we have

$$\begin{aligned}
\widetilde{\alpha}\delta^2 &= \frac{\delta^2}{2+[1+\frac{1}{m}\log(4\sqrt{\Phi(v)}+1)]^2[1+(m+2)(4\sqrt{\Phi(v)}+1)]}\\
&\ge \frac{\Phi(v)}{2+[1+\frac{1}{m}\log(4\sqrt{\Phi(v)}+1)]^2[1+(m+2)(4\sqrt{\Phi(v)}+1)]}\\
&\ge \frac{\Phi(v)}{2\sqrt{\Phi(v)}+[1+\frac{1}{m}\log(4\sqrt{\Phi(v)}+1)]^2[1+(m+2)5]\sqrt{\Phi(v)}}\\
&\ge \frac{\sqrt{\Phi(v)}}{2+[1+\frac{1}{m}\log(4\sqrt{\Phi_0}+1)]^2[1+(m+2)5]}\\
&\ge \frac{\sqrt{\Phi(v)}}{18m[1+\frac{1}{m}\log(4\sqrt{\Phi_0}+1)]^2}
\end{aligned}$$

This completes the proof. □

step 5. Calculate the uniform upper bound $(\Phi)_0$ for $\Phi(v)$.

Lemma 4.5.9. *Let $\sigma : [0,\infty[\to [1,+\infty[$ be the inverse function of $\psi(t)$ for $t \ge 1$. Then we have*

$$\Phi(\beta v) \le n\psi\left(\beta\sigma\left(\frac{\Phi(v)}{n}\right)\right),\ v \in \mathbb{R}^*,\ \beta \ge 1.$$

Proof. Using (4.17) and (4.18), and Lemma 2.4 in [5], we can get the result. This completes the proof. □

Proposition 4.5.10. *Let $0 \le \theta < 1$, $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Phi(v) \le \tau$, then we have*

$$\Phi(v_+) \le \frac{2\sqrt{2\tau n}+2\tau+\theta n}{(1-\theta)}.$$

Proof. Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\sigma\left(\frac{\Phi(v)}{n}\right) \geq 1$, then $\frac{\sigma\left(\frac{\Phi(v)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. And for $t \geq 1$, we have $\psi(t) \leq t^2 - 1$.
Using Lemma 4.5.9 with $\beta = \frac{1}{\sqrt{1-\theta}}$, (4.23), and $\Phi(v) \leq \tau$, we have

$$
\begin{aligned}
\Phi(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi(v)}{n}\right)\right) \\
&\leq n\left(\left[\frac{\sigma\left(\frac{\Phi(v)}{n}\right)}{\sqrt{1-\theta}}\right]^2 - 1\right) = \frac{n}{(1-\theta)}\left(\left[\sigma\left(\frac{\Phi(v)}{n}\right)\right]^2 - (1-\theta)\right) \\
&\leq \frac{n}{(1-\theta)}\left(\left[1 + \sqrt{2\frac{\Phi(v)}{n}}\right]^2 - (1-\theta)\right) \\
&\leq \frac{n}{(1-\theta)}\left(2\sqrt{\frac{2\tau}{n}} + \frac{2\tau}{n} + \theta\right) = \frac{2\sqrt{2\tau n} + 2\tau + \theta n}{(1-\theta)}.
\end{aligned}
$$

This completes the proof. □

Denote

$$
(\Phi)_0 = \frac{2\sqrt{2\tau n} + 2\tau + \theta n}{(1-\theta)} = L\left(n, \theta, \tau\right), \tag{4.30}
$$

then $(\Phi)_0$ is an upper bound for $\Phi(v_+)$ during the process of the algorithm.

Step 6. (An upper bound for the total iteration bound)

Lemma 4.5.11. *Let K be the total number of inner iterations in the outer iteration. Then we have*

$$
K \leq 36m[1 + \frac{1}{m}\log(4\sqrt{\Phi_0} + 1)]^2(\Phi_0)^{\frac{1}{2}}
$$

Proof. By Lemma 1.3.2 in [31], we have :
$K \leq \frac{[(\Phi)_0]^\gamma}{\kappa\gamma} = 36m[1 + \frac{1}{m}\log(4\sqrt{\Phi_0} + 1)]^2(\Phi_0)^{\frac{1}{2}}$. This completes the proof. □

The number of outer iterations is bounded above by $\frac{\log\left(\frac{n}{\epsilon}\right)}{\theta}$ (see [36] Lemma II.17, page 116). By multiplying the number of outer iterations by the number of inner iterations, we get an upper bound for the total number of iterations, namely,

$$
36m[1 + \frac{\log(4\sqrt{\Phi_0} + 1)}{m}]^2(\Phi_0)^{\frac{1}{2}}\frac{\log\left(\frac{n}{\epsilon}\right)}{\theta}. \tag{4.31}
$$

Step 7. For large-update methods with $\tau = \mathbf{O}(n)$ and $\theta = \Theta(1)$, we get

$$\Phi_0 = \mathbf{O}(n)$$
and by choosing $m = \log(n)$ the iteration bound becomes
$$\mathbf{O}\left(\sqrt{n}\log(n)\log\left(\frac{n}{\epsilon}\right)\right) \text{ iterations complexity .}$$

In case of a small-update methods, we have $\tau = \mathbf{O}(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$. Substitution of these values into (4.31) does not give the best possible bound. A better bound is obtained as follows.
By (4.22), (4.23) with $\psi(t) \leq \frac{1}{2}[m+5](t-1)^2$, $t > 1$. We have

$$\begin{aligned}
\Phi(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi(v)}{n}\right)\right)\\
&\leq \frac{n(m+5)}{2}\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi(v)}{n}\right)-1\right)^2\\
&= \frac{n(m+5)}{2(1-\theta)}\left(\sigma\left(\frac{\Phi(v)}{n}\right)-\sqrt{1-\theta}\right)^2\\
&\leq \frac{n(m+5)}{2(1-\theta)}\left(1+\sqrt{\frac{\Phi(v)}{n}}-\sqrt{1-\theta}\right)^2\\
&\leq \frac{(m+5)}{2(1-\theta)}\left(\theta\sqrt{n}+\sqrt{\tau}\right)^2
\end{aligned}$$

where we also used that $1-\sqrt{1-\theta} = \frac{\theta}{1+\theta} \leq \theta$ and $\Phi(v) \leq \tau$, using this upper bound for $(\Phi)_0$, we get

$$\Phi_0 = \mathbf{O}(m)$$
and the iteration bound becomes
$$\mathbf{O}\left(m^{\frac{3}{2}}\sqrt{n}\log\left(\frac{n}{\epsilon}\right)\right) \text{ iterations complexity .}$$

4.6 Numerical results

In this section, we deal with the numerical implementation of this algorithm applied to the large dimension problem. Here we used Iter which means the iterations number produced by the algorithm. The implementation is manipulated in Matlab. Our tolerance is $\epsilon = 10^{-4}$. For our kernel we take $m = \log(n)$.

Example 4.6.1. *We consider the following (LO) problem (see [10])*

$$n = 2k,\ A(i,j) = \begin{cases} 0 \text{ if } i \neq j \text{ and } j \neq i+k \\ 1 \text{ if } i = j \text{ or } j = i+k \end{cases}$$

$c(i) = -1, c(i+k) = 0, b(i) = 2$, *and the interior point condition (IPC),* $x^0(i) = x^0(i+k) = 1, y^0(i) = -2, s^0(i) = 1, s^0(i+k) = 2$ *for* $i = 1, ..., k$. *To prove the effectiveness of our new kernel function* ψ *and evaluate its effect on the behavior of the algorithm, we conducted comparative numerical tests between it and Elghami's kernel [5],* $\psi_{Gh} = \frac{1}{2}(t^2 - 1) - \log(t)$ *. We summarize this numerical study in Tables 2, 3 and 4.*

Kernel functions	Large update	Outer It.	Inner It.	Time(s)
ψ_{Gh}	$\mathbf{O}\left(n \log\left(\frac{n}{\epsilon}\right)\right)$	5	6219	0.4998
ψ	$\mathbf{O}\left(\sqrt{n} \log(n) \log\left(\frac{n}{\epsilon}\right)\right)$	5	3943	0.4785

Table 2. Comparison for $k = 25, n = 50$.

Kernel functions	Large update	Outer It.	Inner It.	Time(s)
ψ_{Gh}	$\mathbf{O}\left(n \log\left(\frac{n}{\epsilon}\right)\right)$	5	11977	2.9003
ψ	$\mathbf{O}\left(\sqrt{n} \log(n) \log\left(\frac{n}{\epsilon}\right)\right)$	5	5830	1.9039

Table 3. Comparison for $k = 50, n = 100$.

Kernel functions	Large update	Outer It.	Inner It.	Time(s)
ψ_{Gh}	$\mathbf{O}\left(n \log\left(\frac{n}{\epsilon}\right)\right)$	5	17675	15.4944
ψ	$\mathbf{O}\left(\sqrt{n} \log(n) \log\left(\frac{n}{\epsilon}\right)\right)$	5	7355	8.2340

Table 4. Comparison for $k = 75, n = 150$.

Conclusion

In this thesis, The concepts of E-convex sets, E-convex functions are developed and discussed it's basic properties. this studies allows to treat the limit analysis by Γ-convergence and scale convergence of E-convex integral functionals. Also we presented and corrected some works on E-convexity, on General and System of General Variational Inequalities.
Finally we propose a new double barrier function and primal-dual interior point algorithms for LO and analyze the large-update and small-update versions of the primal-dual interior point algorithm that are based on the parameterized kernel function (4.12) with a logarithmic-exponential barier term. Another interesting choice is m dependent with n, which minimizes the iteration complexity bound. In fact, if we take $m = \log n$, we obtain the best known complexity bound for large-update methods namely $\mathbf{O}\left(\sqrt{n}\log(n)\log\left(\frac{n}{\varepsilon}\right)\right)$. This bound improves the so far obtained complexity results for large-update methods based on a logarithmic kernel function given by El Ghami et al. [5].

As a perspective, it would be interesting to study with time-dependent variable the Γ-convergence of E-convex integral functionals. Also in linear programming it would be interesting to search for a kernel function with a trigonometric barier term, primal-dual interior point algorithms for LO, analyze the large-update and small-update versions of the primal-dual interior point algorithm .

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