RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
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UNIVERSITÉ DE MOSTAFA BEN BOULAÏD BATNA 2
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Présentée par:

Ghecham Wassila

THÈME:

Stabilisation de quelques systèmes d'évolution avec retards

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Mme. Lombarkia Farida,	Prof.,	Univ. de Batna 2,	Présidente
Mr. Rebiai Salah-Eddine,	Prof.,	Univ. de Batna 2,	Rapporteur
Mme. Sidi ali Fatima Zohra,	MCB.,	Univ. de Batna 2,	Co-rapporteure
Mme. Hamchi Ilhem,	MCA.,	Univ. de Batna 2,	Examinatrice
Mme. Kada Maissa,	MCA.,	Univ. de Batna2,	Examinatrice
Mr. Saoudi Khaled,	MCA.,	Univ. de Khenchela,	Examinateur

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Title: **Stabilization of some evolution systems with time delays**

Abstract:

In this thesis, we study stability problems for some evolution equations (wave equation, Schrödinger equation) with delay terms in the (linear or nonlinear) boundary or internal feedbacks. Under some assumptions, uniform decay rates for the solutions are established. Some of these results are obtained by introducing appropriate energies and by proving observability like inequalities, whereas the others are deduced from estimates for suitable Lyapunov functionals.

Keywords:

Wave equation, Schrödinger equation, stabilization, boundary feedback, internal feedback, time delay.

Contents

Introduction

The delay is defined as the time between instant of application of action on the system and the moment of its reaction.
Time delay appears in various systems such as biological, chemical, engineering and physical systems (see [31], [10], [58], [2], [3], [73], [4], [81], [5], [6], [1],[76]).
The most classic example of time delay systems is the shower presented in Figure 1 (see [85]), where the water temperature results for mixing between cold and hot water. The user wishes to obtain the desired temperature as quickly as possible, while taking into account the time produced by transport through the tap to the shower head. This time is a delay which depends on the water pressure and the length of the pipe.

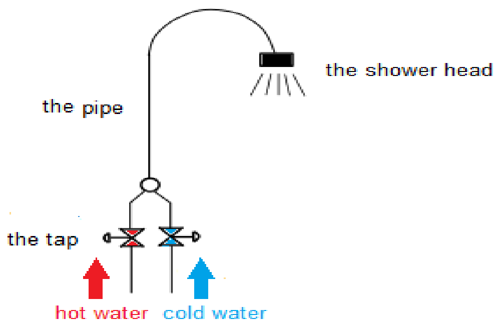

Figure 1. Sketch of a shower system

The stability analysis of control systems governed by ordinary differential equation subjected to constant or time-varying delay has been one of the main interests for many researchers in systems theory (see [37], [67], [72], [29], [82]). Two methods were proposed to derive delay dependent or delay independent stability conditions; one is based on Lyapunov-Razumikhin functionals whereas the other uses Lyapunov-Krasovskii functionals.
Stability problems for PDE systems with time delays have also been the subject of extensive studies and this since the pioneering work of Datko et al [24] on the effect of time delays in feedback stabilization of the wave equation. Below, we review some of the most relevant publications regarding stability problems for specific delayed partial differential equations, namely wave equation and Schrödinger

equation. In [24] the authors considered the following system

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + au(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) = -ku_t(1,t-\tau), & t > 0, \end{cases} \tag{1}$$

with a, k are positive constants. They proved that, if k satisfies

$$k < \frac{e^{2a}+1}{e^{2a}+1}$$

the system (1) is stable for all sufficiently small delays. However, if

$$k > \frac{e^{2a}+1}{e^{2a}+1}$$

they constructed a set D such that for each $\tau \in D$ system (1) admits an exponentially increasing solution.

Similar results were obtained in Datko [23] for the two-dimensional wave equation with damping introduced through Neumann-type boundary conditions on one edge of a square boundary and the Euler Bernoulli beam equation in one dimension with damping introduced through a specific set of boundary conditions on the right end point.

Xu et al [83] studied the stability of the one-dimensional wave equation with a constant time delay term in the boundary feedback

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) = -k\mu u_t - k(1-\mu)u_t(1,t-\tau), & t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & 0 < x < 1 \\ u_t(1,t-\tau) = f(t-\tau), & t \in (0,\tau). \end{cases} \tag{2}$$

where $k > 0$. They proved by adopting a spectral analysis approach that the system (2) is exponentially stable when $\mu > \frac{1}{2}$ and unstable if $\mu < \frac{1}{2}$. For the case $\mu = \frac{1}{2}$ they showed that the system (2) is asymptotically stable for some delays.

Nicaise and Pignotti [62] extended this result to a multidimensional wave equation with a delay term in the boundary or internal feedbacks. Under appropriate assumptions they established the exponential stability of the solution by introducing a suitable energy function and by using an observability inequality deduced from Carleman estimates for the wave equation [48]. On the contrary if one of the assumptions is not satisfied they proved the existence of a sequence of delays for which the corresponding solution is not stable.

Ammari et al [7] considered the boundary stabilization problem for the wave equation with interior delay. Under Lions geometric condition, they showed an exponential stability result provided that the delay coefficient is small enough.

Nicaise et al ([66], [64]) derived, by introducing an appropriate Lyapunov functional, necessary and explicit conditions that guarantee the exponential stability of the solution of the wave equation with

a time-varying delay term in the boundary feedback.
The asymptotic bahaviour of the solution of the wave equation with a nonlinear time-varying delay term in the nonlinear boundary or internal feedbacks has been investigated in [13], [64] and [52]. In [70], the authors considered compactly coupled wave equations with a delay term in the internal or boundary feedbacks. Using Carleman estimates for coupled wave equations due to Lasiecka and Triggiani [46], they established stability results in appropriate energy space.

In all the works mentioned above, it is assumed that the delay is concentrated at a fixed time. The case of the wave equation with distributed delay has been studied in [63] where stabilization results are given.

For the Schrödinger equation with time delay several studies have been done, see for example [30], [20], [18] and [65]. We state in particular, the reference [65] where stability and instability results were established for the multi-dimensional Schrödinger equation with a delay term in the boundary or internal feedbacks.

The aim of this thesis is to provide further results on stability of the wave and the Schrödinger equations with time delays. The main body of this work consists of six chapters. In the first chapter, we gather the main tools used throughout this thesis. We recall some basic features of semigroups of linear and nonlinear operators and their applications to abstract Cauchy problems in Hilbert spaces. We also define the stability concepts for abstract Cauchy problems we are interested in and give some of their characterizations.
In the second chapter, we study the stability of the wave equation with a delay term in the nonlinear boundary or internal feedbacks. Under suitable assumptions, global existence and uniform decay rates for the solutions are established. The proof of existence of solutions relies on a construction of suitable approximating problem for which the existence of solution will be established using nonlinear semigroup theory and then passage to the limit gives the existence of solutions to the original problem. The uniform decay rates for the solutions are obtained by proving certain integral inequalities for the energy function and by establishing a comparison theorem which relates the asymptotic behaviour of the energy and of the solutions to an appropriate dissipative ordinary differential equation.
Chapters three, four and five are devoted to the Schrödinger equation defined on an open bounded domain Ω of $\mathbb{R}^n$ with a delay term and subject to a dissipative feedback. In chapter three, we consider the case of the equation with a delay term in the nonlinear internal or boundary feedbacks. We show that it is well-posed in $L^2(\Omega)$ by adopting a nonlinear semigroup theory approach. Moreover, we prove that it is stable in $L^2(\Omega)$ with uniform decay rates described, as in chapter two, by a dissipative ordinary differential equation.
Chapter four analyzes the case of the equation with interior delay and a boundary feedback acting on the Neumann boundary condition while homogeneous boundary condition of Dirichlet type are imposed on the complementary part. Under Lions geometric assumption, exponential stability of the solution in the energy space $H^1_{\Gamma_1}(\Omega)$ is established on condition that the delay coefficient is sufficiently small. The proof uses multipliers technique and a suitable Lyapunov functional.
Chapter five deals with the case where the boundary or the internal feedback contains a delay term of distributed type. If some hypothesis are satisfied, it is proved that the solution decays exponentially in appropriate energy spaces. These results are obtained by showing some observability estimates.

In Chapter six, we consider a system of compactly coupled wave equations with distributed delay terms in the boundary or internal feedbacks. In both cases, we establish that the semigroup generating the dynamics of the closed-loop system is exponentially stable. The approach we adopt combines Carleman estimates for coupled non-conservative hyperbolic systems due to Lasiecka and Triggiani [46] and compactness-uniqueness argument.

Notation

$:=$	is the equal by definition
$\to$	designates the convergence
$\hookrightarrow$	continuous and dense injection
∇	stands for the gradient operator
Δ	is the Laplace operator
div	is the divergence operator
$\mathbb{N}$	set of the positive integers
$\mathbb{R}$	set of the real numbers
$\mathbb{C}$	set of the complex numbers
Ω	open bounded domain of $\mathbb{R}^n$
Γ	the sufficiently smooth boundary of Ω
$C^\infty(\Omega)$	the space of infinitely differentiable functions in Ω
$D(\Omega)$	the space of $C^\infty(\Omega)$ functions with compact support in Ω
$D'(\Omega)$	the distributions space on Ω
$D(\mathcal{A})$	domain of the operator $\mathcal{A}$
X'	dual space of X

$\langle .,. \rangle$	inner product
$\vert . \vert$	the absolute value for the real number or modulus for the complex number
$\Vert . \Vert$	the norm
$\mathcal{L}(X)$	bounded linear operators from X to X
Re	real part
Im	imaginary part
$\overline{z}$	complex conjugate number
$L^p(\Omega)$	class of Lebesgue measurable complex (or real) -valued functions with $\int_\Omega \vert u(x)\vert^p \, dx < \infty; 1 \leq p < \infty$
$L^\infty(\Omega)$	class of bounded measurable functions from Ω to $\mathbb{C}$ or $\mathbb{R}$ with $\vert u(x)\vert \leq C$ a.e in Ω
$L^2_{loc}([0,\infty); X)$	class of functions which are in $L^2((a,b); X)$ for all $a, b \in [0,\infty)$
$W^{k,p}(\Omega)$	Sobolev space of order k
$C([0,\infty); X)$	class of continuous functions from $[0,\infty)$ to X
$C^1([0,\infty); X)$	class of continuously differentiable functions from $[0,\infty)$ to X

Chapter 1

Preliminaries

In this chapter, we recall for later use some well know results from the theory of semigroups of linear and nonlinear operators and existence results for abstract Cauchy problems in Hilbert spaces. We also define the stability concepts for abstract evolution equations in Hilbert spaces we are interested in and provide some of its characterizations.

1.1 Semigroups of continuous linear operators

Let X be a Hilbert space.

Definition 1.1. *A one-parameter family $S(t)$ for $0 \leq t < \infty$ of $\mathcal{L}(X)$ is a C_0-(or strongly continuous) semigroup on X if*

(a) $S(t+s) = S(t)S(s)$ *for every* $t, s \geq 0$.

(b) $S(0) = I$, *(I is the identity operator in X).*

(c) $\lim_{t\to 0} \|S(t)x - x\| = 0$ *for all* $x \in X$.

Definition 1.2. *Let $S(t)$ be a C_0-semigroup defined on X. The infinitesimal generator $\mathcal{A}$ of $S(t)$ is the linear operator defined by*

$$\mathcal{A}x = \lim_{h\to 0}(S(h)x - x)/h, \quad \text{for } x \in D(\mathcal{A})$$

where $D(\mathcal{A}) = \{x \in X; \lim_{h\to 0}(S(h)x - x)/h \ \text{ exists in } X\}$.

Theorem 1.1. *(Engel and Nagel [26]) Let $S(t)$ be a semigroup. There exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that the following holds:*

$$\|S(t)\| \leq Me^{\omega t}.$$

If $\omega = 0$ and $M = 1$, then $S(t)$ is called a C_0-semigroup of contraction.

Theorem 1.2. ***(Lumer-Phillips)*** *(Pazy [69]) A linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ generates a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X if and only if $\mathcal{A}$ is m-dissipative, i.e., it satisfies*

- *Re* $\langle \mathcal{A}x, x\rangle_X \leq 0$, $\forall x \in D(\mathcal{A})$,
- $\lambda I - \mathcal{A}$ *is onto for some (hence all)* $\lambda > 0$.

Proposition 1.3. *(Curtain and Zwart [22]). Sufficient conditions for a closed, densely defined operator on X to be the infinitesimal generator of a C_0-semigroup satisfying $\|S(t)\| \leq e^{\omega t}$ are:*

$$Re\,\langle \mathcal{A}x, x\rangle \leq \omega\|x\|^2 \quad \text{for } x \in D(\mathcal{A});$$

$$Re\,\langle \mathcal{A}^*x, x\rangle \leq \omega\|x\|^2 \quad \text{for } x \in D(\mathcal{A}^*).$$

1.2 Nonlinear operators

Definition 1.3. *The operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is called*

- *monotone (dissipative) if*

$$Re\,\langle \mathcal{A}x_1 - \mathcal{A}x_2, x_1 - x_2\rangle \geq (\leq) 0, \qquad \text{for all } x_1, x_2 \in D(\mathcal{A})$$

- *strongly monotone if there is $C > 0$ for which*

$$Re\,\langle \mathcal{A}x_1 - \mathcal{A}x_2, x_1 - x_2\rangle \geq C\|x_1 - x_2\|^2, \qquad \text{for some } C > 0 \text{ and all } x_1, x_2 \in D(\mathcal{A}).$$

Definition 1.4. *A monotone operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is said to be maximal monotone if the graph of $\mathcal{A}$*

$$G(\mathcal{A}) = \{(x, \mathcal{A}x) : x \in D(\mathcal{A})\},$$

is not properly contained in the graph of any other monotone operator in X.

Proposition 1.4. *(Brezis [14]) Let $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be an operator in X. The following two assertions are equivalent.*

- $\mathcal{A}$ *is maximal monotone.*
- $\mathcal{A}$ *is monotone and Range*$(I + \mathcal{A}) = X$.

Definition 1.5. *Let $\mathcal{A}$ be an operator from X to X such that $D(\mathcal{A}) = X$. $\mathcal{A}$ is said to be hemicontinuous on X if the function $\mathbb{R} \ni t \mapsto \langle \mathcal{A}(x_1 + tx_2), w\rangle$ is continuous for all x_1, x_2 and w in X.*

Theorem 1.5. *(Barbu [11]). Let B be a monotone, everywhere defined and hemicontinuous operator from X to X. Then B is maximal monotone. If in addition B is coercive, then Range$(B) = X$.*

Proposition 1.6. *(Barbu [11]) Let B be a monotone, hemicontinuous operator from X to X. Let $\mathcal{A}$ be a maximal monotone operator from X to X. Then $\mathcal{A} + B$ is maximal monotone in $X \times X$. Moreover, if $\mathcal{A} + B$ is coercive then Range$(\mathcal{A} + B) = X$.*

1.3 Semigroups of nonlinear operators

Definition 1.6. *Let $\mathcal{G}$ be a closed subset of X. A continuous semigroup of nonlinear contractions on $\mathcal{G}$ is a family of operators $S(t)$, $0 \leq t < \infty$, from $\mathcal{G}$ to $\mathcal{G}$, satisfying the following conditions:*

(1) $S(t+s)x = S(t)S(s)x, \quad \forall x \in \mathcal{G},\ \forall\, t, s \geq 0,$

(2) $S(0)x = x, \quad \forall x \in \mathcal{G},$

(3) $\lim_{t\to 0} \|S(t)x - x\| = 0, \quad \forall\, x \in \mathcal{G},$

(4) $\|S(t)x - S(t)y| \leq \|x - y\|, \quad \forall\, x, y \in \mathcal{G}, \quad \forall t > 0.$

Definition 1.7. *Let $S(t)$ be a semigroup on $\mathcal{G}$. The infinitesimal generator $\mathcal{A}_0$ of $S(t)$ is defined by*

$$\mathcal{A}_0 x = \lim_{h\to 0}(S(h)x - x)/h, \qquad x \in D(\mathcal{A}_0)$$

where

$$D(\mathcal{A}_0) = \{x \in \mathcal{G};\ lim_{h\to 0}(S(h)x - x)/h \text{ exists in } X\ \}.$$

Theorem 1.7. *(Barbu [11]) Let $S(t)$ be a semigroup of nonlinear contractions defined on a closed convex subset $\mathcal{G}$ of X. Then the generator $\mathcal{A}_0$ of $S(t)$ is densely defined on $\mathcal{G}$.*

Theorem 1.8. *(Barbu [11], Kōmura [42]) Let $\mathcal{A}$ be a densely defined, maximal dissipative operator in X, then it generates a nonlinear contractions semigroup $S(t)$ on X.*

1.4 Abstract Cauchy problems

Let X be a Hilbert space and let $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be an operator. Consider the homogeneous Cauchy problem

$$\begin{cases} \frac{du}{dt}(t) = \mathcal{A}u(t), & t \geq 0, \\ u(0) = x. \end{cases} \tag{1.1}$$

where $x \in X$.
To define solution concepts for (1.1) and to present results pertaining their existence, we distinguish two cases depending on whether the operator $\mathcal{A}$ is linear or nonlinear.
Case I. $\mathcal{A}$ is linear

Definition 1.8. *A function $u : [0, T] \to X$ is a strong solution of (1.1) on $[0, T]$ if u is continuously differentiable on $[0, T]$ and for all $t \in [0, T]$ $u(t) \in D(\mathcal{A})$ and satisfies (1.1).*

Theorem 1.9. *(Curtain and Zwart [22]) If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t)$ on X, then for all $x \in D(\mathcal{A})$ the abstract Cauchy problem (1.1) has a unique strong solution given by*

$$u(t) = S(t)x. \tag{1.2}$$

Definition 1.9. *A function $u \in C([0,T];X)$ is a weak solution of (1.1) on $[0,T]$ if for every $y \in D(\mathcal{A}^*)$ where $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$, the function $(u(t), y)$ is absolutely continuous on $[0,T]$ and*

$$\frac{d}{dt}\langle u(t), y\rangle = \langle u(t), \mathcal{A}^* y\rangle \ \ a.e. \ \ on \ [0,T].$$

Theorem 1.10. *(Curtain and Zwart [22]) If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $S(t)$ on X, then for every $x \in X$, the problem* (1.1) *has unique weak solution given by (1.2).*

Case II. $\mathcal{A}$ is nonlinear.

Definition 1.10. *A function $u \in C([0,T];X)$ is a weak solution for (1.1) if there exist sequences $(u_n) \subset W^{1,\infty}(0,T;X)$ such that:*

- $\frac{du_n}{dt}(t) = \mathcal{A}u_n(t), \quad$ *for a.e.* $t \geq 0$, $n = 1, 2...$
- $u_n \to u \in C([0,T];X)$
- $u(0) = x$.

Definition 1.11. *The function $u \in C([0,T];X)$ is called a strong solution of (1.1) if:*

- *u is absolutely continuous on each compact subinterval of* $]0,T[$.
- *$u(t) \in D(\mathcal{A})$ for almost every $t \geq 0$.*
- *$u(0) = x$ and u satisfies $\frac{du}{dt}(t) = \mathcal{A}u(t)$, a.e. $t \geq 0$.*

Theorem 1.11. *(Djafari Rouhani and Khatibzadeh [25]) Suppose that $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is maximal dissipative and $x \in \overline{D(\mathcal{A})}$. Then there exists a unique weak solution of (1.1).*

Theorem 1.12. *(Djafari Rouhani and Khatibzadeh [25]) Suppose that $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is maximal dissipative and $x \in D(\mathcal{A})$, then the problem (1.1) has a unique strong solution $u \in W^{1,\infty}(0,T;X)$.*

1.5 Stability concepts

Consider in a Hilbert space X, the differential equation

$$\frac{du}{dt}(t) = \mathcal{A}u(t), \qquad t \geq 0, \tag{1.3}$$

where $\mathcal{A}$ is an operator from $D(\mathcal{A}) \subset X$ into X. We assume that (1.3) has a unique solution subject to the condition $u(0) = x$, which we denote by $u(., x)$. We also assume that 0 is an equilibrium point for (1.3).

Many concepts of stability have been defined for systems described by (1.3), and we are interested in the following:

Definition 1.12. *The equilibrium of (1.3) said to be*

- *uniformly exponentially stable, if there exist constants* $\delta > 0$, $M > 0$ *and* $\epsilon > 0$ *such that if* $\|x\| < \epsilon$, *then*
$$\|u(t,x)\| \leq Me^{-\delta t}\|x\|, \quad \forall\, t \geq 0,$$
- *polynomial stable if there exist constants* $\alpha > 0$, $M > 0$ *and* $\epsilon > 0$ *such that if* $\|x\| < \epsilon$, *then*
$$\|u(t,x)\| \leq \frac{M}{t^{\alpha}}\|x\|, \quad \forall\, t > 0.$$

Remark 1.1. *The previous definition is local since it shows how the state evolves after starting near the equilibrium point. If uniform exponential or polynomial stability holds for any initial state* x, *then the equilibrium is said to be globally uniformly exponentially or polynomially stable.*

In the case where $\mathcal{A}$ is a closed linear operator generating a C_0-semigroup $S(t)$ then
$$u(t,x) = S(t)x$$
and we have the following results

Proposition 1.13. *(Engel and Nagel [26]) The system (1.3) is:*

- *uniformly exponentially stable if and only if there exist constants* $\delta > 0$, $M \geq 1$ *such that*
$$\|S(t)\| \leq Me^{-\delta t}, \text{ for all } t \geq 0,$$
- *polynomially stable if and only if there exist constants* $\alpha > 0$, $M \geq 1$ *such that*
$$\|S(t)\| \leq Mt^{-\alpha}, \text{ for all } t > 0.$$

Proposition 1.14. *(Engel and Nagel [26]) For a linear* C_0*-semigroup* $(S(t))_{t\geq 0}$, *the following assertions are equivalent.*

- $(S(t))_{t\geq 0}$ *is uniformly exponentially stable.*
- *There exists* t_0 *such that* $\|S(t_0)\| < 1$.

1.6 Jensen's inequality

Theorem 1.15. *(Niculescu and Persson [68]) Let* (Ω, Σ, μ) *be a finite measure space and let* $g : \Omega \to \mathbb{R}$ *be a* $\mu-$*integrable function. If* ϕ *is a convex function given on an interval* I *that includes the image of* g, *then*
$$\frac{1}{\mu}\int_{\Omega} g\, d\mu \in I$$
and
$$\phi\left(\frac{1}{\mu}\int_{\Omega} g\, d\mu\right) \leq \frac{1}{\mu}\int_{\Omega} \phi \circ g\, d\mu.$$
provided that $\phi \circ g$ *is* μ*-integrable.*

Chapter 2

Stability of the wave equation with a delay term in the nonlinear boundary or internal feedbacks

2.1 Introduction

In this chapter, we address the problem of stability for a multi-dimensional wave equation with a delay term in the nonlinear boundary or internal feedbacks.
Let Ω be an open bounded domain of $\mathbb{R}^n$ with smooth boundary Γ which consists of two non-empty parts Γ_1 and Γ_2 such that, $\Gamma_1 \cup \Gamma_2 = \Gamma$ with $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. Let $\nu(.)$ denote the unit normal on Γ pointing towards the exterior of Ω.
In Ω, we consider the wave equation with a nonlinear delay term in the boundary conditions

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = -\alpha_1 f(u_t(x,t)) - \alpha_2 g(u_t(x,t-\tau)) & \text{on } \Gamma_2 \times (0,+\infty), \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{on } \Gamma_2 \times (0,\tau), \end{cases} \tag{2.1}$$

where

- $\frac{\partial u}{\partial \nu}$ is the normal derivative and $\tau > 0$ is the time delay.
- α_1 and α_2 are positive constants.
- u_0, u_1 and f_0 are the initial data which belong to appropriate Hilbert spaces.
- f and g are real-valued functions of class $C(\mathbb{R})$.

In absence of delay, that is $\alpha_2 = 0$, stability problems for (2.1) have received a lot of attention in the literature, (see for example [17], [86], [43], [40], [16]), and the energy estimates obtained depend

on the nonlinear function f.
Nicaise et al [64] considered the case where the delay τ depends on time and the nonlinear functions f and g are subject to the following conditions:

- $f, g \in C(\mathbb{R})$,
- $|f(s)| \leq c_1 |s|$ for all $s \in \mathbb{R}$,
- $(f(s_1) - f(s_2))(s_1 - s_2) \geq c_2 |s_1 - s_2|^2$ for all $s_1, s_2 \in \mathbb{R}$,
- $|g(s)| \leq c_3 |s|$for all $s \in \mathbb{R}$,
- $|g(s_1) - g(s_2)| \leq c_4 |s_1 - s_2|$ for all $s_1, s_2 \in \mathbb{R}$,

where $c_i, i = 1, ..., 4$, are positive constants.
Under some regularity assumptions on the delay function τ, they established a well-posedness result and an exponential stability estimate for problem (2.1). Well-posedness is proved by using nonlinear semigroup theory whereas the exponential estimate is obtained by introducing suitable energy and Lyapunov functionals. Li el al [52] investigated the case where the Laplacian is replaced by a second order differential operator with space variable coefficients, α_1, α_2 and τ depend on time, and the functions f and g satisfy the following conditions:

- $f, g \in C(\mathbb{R})$,
- $sf(s) \geq |s|^2$ for $s \in \mathbb{R}$,
- $|f(s)| \leq c_5 |s|$ for $|s| > 1$
- $s^2 + (f(s))^2 \leq c_5 (sf(s))^{1/p}$ for $|s| \leq 1$,
- $(g(s))^2 \leq sf(s)$ for $s \in \mathbb{R}$,

where c_5 and p are positive constants with $p \geq 1$. Assuming the well-posedness of the problem (which is not trivial), they obtained a uniform decay rates for the solutions by adopting a Riemann geometry methods. One of the main purposes of this chapter is to study the existence and asymptotic behaviour of the solutions of (2.1) under the following assumptions on the nonlinear functions f and g .

(**H1**) (i) f is a continuous monotone increasing function on $\mathbb{R}$;
(ii) $sf(s) > 0$ for $s \neq 0$;
(iii) $sf(s) \leq M_1 s^2$ for $|s| \geq 1$, for some $M_1 > 0$;

(**H2**) (i) g is an odd non-decreasing locally Lipschitz continuous function on $\mathbb{R}$;
(ii) $sg(s) > 0$ for $s \neq 0$;
(iii) $sg(s) \leq M_2 s^2$ for $|s| < 1$, for some $M_2 > 0$;
(iv) $sg(s) \geq M_3 s^2$ for $|s| \geq 1$, for some $M_3 > 0$;

(v) $a_1 s g(s) \le G(s) \le a_2 s f(s)$, where $G(s) = \int_0^s g(r)dr$, for some positive constants a_1 and a_2.

(**H3**) $\alpha_1 > \frac{a_2 \alpha_2}{a_1}$.

(**H4**) There exists $x_0 \in \mathbb{R}^n$ such that, with $m(x) = x - x_0$,

$$m(x).\nu(x) \le 0 \quad \text{on } \Gamma_1.$$

Remark 2.1. *As an example of functions f and g for which assumptions* (**H1**) *and* (**H2**) *hold we have*

$$f(x) = x + \frac{x}{1+x^2}$$

and

$$g(x) = 2x + \frac{2x}{(1+x^2)^2}$$

We adopt an approach due to Lasiecka and Tataru [43] to establish global existence and uniform decay rates for the solutions. The proof of existence of solutions relies on a construction of a suitable approximating problem for which the existence of solution will be established using nonlinear semi-group theory and then passage to the limit gives the existence of solutions to the original problem. The uniform decay rates for the solutions are obtained by proving certain integral inequalities for the energy function and by establishing a comparison theorem which relates the asymptotic behaviour of the energy and of the solutions to an appropriate dissipative ordinary differential equation.

Remark 2.2.
- *It follows from the mean value theorem and the monotonicity of g that $a_1 \le 1$.*
- *Assumption* (**H3**) *can be considered as a nonlinear version of the assumption (1.8) in [62].*

Regarding the existence of the solutions to the system (2.1), we have the following result.

Theorem 2.1. *Assume* (**H1**)$-$(**H3**). *Then, for each* $(u_0, u_1, f_0) \in H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2; L^2(0,\tau))$, *problem* (2.1) *has at least one solution*

$$u \in C([0,+\infty); H^1_{\Gamma_1}(\Omega)) \cap C^1([0,+\infty); L^2(\Omega)),$$

such that

$$u_t \in L^2_{loc}([0,+\infty); L^2(\Gamma_2)),\ \frac{\partial u}{\partial \nu} \in L^2_{loc}([0,+\infty); L^2(\Gamma_2)). \tag{2.2}$$

Where

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \quad on \quad \Gamma_1\}.$$

In order to state our stability result, we introduce as in [43] a real valued strictly increasing concave function $h(s)$ defined for $s \ge 0$ and satisfying

$$h(0) = 0; \tag{2.3}$$

$$h(sf(s)) \ge s^2 + f^2(s) \quad \text{for } |s| \le N, \text{ for some } N > 0, \tag{2.4}$$

and we define the following functions:

- $$\widetilde{h}(s) = h(\frac{s}{mes\ \Sigma_2}), s \geq 0,$$
  where $\Sigma_2 = \Gamma_2 \times (0, T)$ and T is a given constant.

- $$p(s) = (cI + \widetilde{h})^{-1} K s, \tag{2.5}$$
  where c and K are positive constants. Then p is a positive, continuous, strictly increasing function with $p(0) = 0$.

- $$q(s) = s - (I + p)^{-1}(s), \ \ s > 0 \tag{2.6}$$
  q is also a positive, continuous, strictly increasing function with $q(0) = 0$.

Let $E(t)$ be the energy function corresponding to the solution of (2.1) defined by

$$E(t) = \frac{1}{2} \int_{\Omega} \{|\nabla u(x,t)|^2 + |u_t(x,t)|^2\} \, dx + \frac{\xi}{2} \int_{\Gamma_2} \int_0^1 G(u_t(x, t - \rho\tau)) \, d\rho \, d\Gamma, \tag{2.7}$$

where the positive constant ξ is such that

$$\frac{2\tau\alpha_2}{a_1}(1 - a_1) < \xi < \frac{2\tau}{a_2}(\alpha_1 - \alpha_2 a_2). \tag{2.8}$$

Theorem 2.2. *Assume hypotheses* $(\mathbf{H1})-(\mathbf{H4})$. *Let u be a solution to* (2.1) *with the properties stated in Theorem 2.1. Then for some $T_0 > 0$,*

$$E(t) \leq S(\frac{t}{T_0} - 1)(E(0)) \qquad\qquad \text{for } t > T_0,$$

where $S(t)$ is the solution of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \qquad S(0) = E(0). \tag{2.9}$$

If we additionally assume that the function $f(s)$ is of a polynomial growth at the origin, the following explicit decay rates are obtained.

Corollary 2.1. *Assume in addition to* $(\mathbf{H1})-(\mathbf{H4})$ *that there exist positive constants b_1 and b_2 such that*

$$f(s)s \leq b_1 s^2 \quad \text{for all } s \in \mathbb{R}, \tag{2.10}$$
$$f(s)s \geq b_2 |s|^{p+1} \quad \text{for } |s| \leq 1, \text{ for some } p \geq 1. \tag{2.11}$$

Then

$$E(t) \leq Me^{-\alpha t} \quad \text{if } p = 1, \tag{2.12}$$
$$E(t) \leq Mt^{\frac{2}{1-p}} \quad \text{if } p > 1. \tag{2.13}$$

In this chapter, we also study the stability problem for the wave equation with a delay term in the nonlinear internal feedback. More precisely, we consider the system described by

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a(x)\left\{\alpha_1 f(u_t(x,t)) + \alpha_2 g(u_t(x,t-\tau))\right\} = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{on } \Omega \times (0,\tau), \end{cases} \tag{2.14}$$

where

α_1, α_2, u_0, u_1, f_0, τ, f and g are as above and $a(.)$ is a function in $L^\infty(\Omega)$ such that

$$a(x) \geq 0 \text{ a.e. in } \Omega \text{ and } a(x) > a_0 > 0 \text{ a.e. in } \omega, \tag{2.15}$$

where $\omega \subset \Omega$ is an open neighbourhood of Γ_2.

In the absence of delay ($i.e.$ $\alpha_2 = 0$), this problem has been considered by several authors ([32], [33], [34], [35],[38], [39], [61], [59], [12]).

For f and g linear, the problem has been treated in [62]. Benaissa et al [13] considered the case where the coefficients α_1, α_2 and the delay τ are time-dependent. Regarding the functions f and g they assumed the following.

- f is a non-decreasing function of class $C(\mathbb{R})$ and satisfies,

$$\gamma_1 |s| \leq \gamma_1 |f(s)| \leq \gamma_2 |s| \quad \text{for } |s| \leq \epsilon \text{ for some } \epsilon > 0,$$

- g is an odd non-decreasing of class $C^1(\mathbb{R})$ satisfying

$$\begin{aligned} \left|g'(s)\right| &\leq \gamma_3, \\ \gamma_4 s g(s) &\leq G(s) \leq \gamma_5 s f(s), \end{aligned}$$

  where $G(s) = \int_0^t g(r)dr$, and $\gamma_i, i = 1, ..., 5$, are positive constants.

They proved the global existence and uniqueness of solution by using Faedo-Galerkin procedure. Moreover, they obtained energy decay estimate of the solution by employing the multiplier method combined with some integral inequalities.

The second purpose of this chapter is to investigate the stability problems for (2.14) when f and g are subject to the assumptions $(\mathbf{H1}) - (\mathbf{H4})$. We use again the Lasiecka-Tataru approach to establish existence and uniform decay rates for the solutions.

The following theorem provides a result on existence and regularity of solutions to the problem (2.14).

Theorem 2.3. *Assume* $(\mathbf{H1}) - (\mathbf{H3})$. *Then, for each* $(u_0, u_1, f_0) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega; L^2(0,\tau))$, *problem (2.14) has at least one solution*

$$u \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega)).$$

To state the stability result, we recall the function h introduced after Theorem 2.1 and we define this time the functions

- $$\widehat{h}(s) = h\left(\frac{s}{mes\ Q}\right), s \geq 0,$$
  where $Q = \Omega \times (0,T)$ and T is a given constant.

- $$\widehat{p}(s) = (C''I + \widetilde{h})^{-1}(K_2\, s), \tag{2.16}$$
  where C'' and K_2 are positive constants.

- $$\widehat{q}(s) = s - (I + \widehat{p})^{-1}(s),\ \ s > 0. \tag{2.17}$$
  Obviously $\widehat{p}$ and $\widehat{q}$ have the same properties as the functions p and q given by (2.5) and (2.6) respectively.

Let $F(t)$ be the energy function corresponding to the solution of (2.14) defined by

$$F(t) = \frac{1}{2}\int_\Omega \left\{|\nabla u(x,t)|^2 + |u_t(x,t)|^2\right\} dx + \frac{\mu}{2}\int_\Omega a(x)\int_0^1 G(u_t(x,t-\tau\rho))\, d\rho\, dx, \tag{2.18}$$

where

$$2\tau a_1^{-1}\alpha_2(1-a_2) < \mu < 2\tau a_2^{-1}(\alpha_1 - a_2\alpha_2). \tag{2.19}$$

The main result can be stated as follows.

Theorem 2.4. *Assume hypotheses* (**H1**)–(**H4**). *Let (u,u_t) be a solution to (2.14) with the properties listed in Theorem 2.3. Then for some $T_0 > 0$,*

$$F(t) \leq S(\frac{t}{T_0} - 1)(F(0)) \ \ for\ \ t > T_0, \tag{2.20}$$

where $S(t)$ is the solution of the differential equation

$$\frac{d}{dt}S(t) + \widehat{q}(S(t)) = 0,\ S(0) = F(0) \tag{2.21}$$

and $\widehat{q}$ is given by (2.17).

Corollary 2.2. *Assume in addition to* (**H1**)–(**H4**) *that for some positive constants $\tilde{a}, b$,*

$$f(s)s \leq bs^2 \qquad for\ each\ real\ s, \tag{2.22}$$

$$f(s)s \geq \tilde{a} \mid s \mid^{p+1} \quad for\ \mid s \mid \leq 1,\ \ for\ some\ \ p \geq 1. \tag{2.23}$$

Then

$$F(t) \leq Ce^{-\beta t} \qquad if\ p = 1,$$

$$F(t) \leq Ct^{\frac{2}{1-p}} \qquad if\ p > 1, \tag{2.24}$$

where $C > 0$ and $\beta > 0$.

The chapter is organized as follows. Theorem 2.1, Theorem 2.2 and Corollary 2.1 is proved in Section 2.2 whereas Section 2.3 contains the proof of Theorem 2.3, Theorem 2.4 and Corollary 2.2. Some results of this chapter have been presented in the conference proceedings paper [27].

2.2 Stabilization of the wave equation with a nonlinear delay term in the boundary conditions

2.2.1 Proof of Theorem 2.1

In order to be able to manage the boundary condition with the delay term and inspired from [83] and [62], we introduce the auxiliary variable:

$$y(x,\rho,t) = u_t(x,t-\tau\rho); \quad x \in \Gamma_2, \rho \in (0,1), t > 0. \tag{2.25}$$

Then problem (2.1) is equivalent to

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ y_t(x,\rho,t) + \tau^{-1} y_\rho(x,\rho,t) = 0 & \text{on } \Gamma_2 \times (0,1) \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u(x,t)}{\partial \nu} = -\alpha_1 f(u_t(x,t)) - \alpha_2 g(y(x,1,t)) & \text{on } \Gamma_2 \times (0,+\infty), \\ y(x,\rho,0) = f_0(x,-\tau\rho) & \text{on } \Gamma_2 \times (0,1), \\ y(x,0,t) = u_t(x,t) & \text{on } \Gamma_2 \times (0,+\infty). \end{cases} \tag{2.26}$$

To prove Theorem 2.1, we adopt the following two step procedure. We first construct an auxiliary approximating problem for which the existence of the unique solution will be established by the arguments of nonlinear semigroup theory. In the second step, we obtain the solutions of problem (2.1) as the limits of the approximating equations.

Proposition 2.5. *Assume that for the continuous f and g the hypotheses $(\mathbf{H1})(iii)$ and $(\mathbf{H2})(v)$ are fulfilled. If $u \in C(0,T;H^1_{\Gamma_1}(\Omega)) \cap C^1(0,T;L^2(\Omega))$ is a solution to problem (2.1) such that:*

$$u_t \mid_{\Gamma_2} \in L^2(0,T;L^2(\Gamma_2)), \tag{2.27}$$

then the following energy identity holds for every $t > 0$

$$\begin{aligned} E(T) - E(0) = &-\alpha_1 \int_{\Sigma_2} u_t(x,t) f(u_t(x,t))\, d\Sigma_2 - \alpha_2 \int_{\Sigma_2} g(u_t(x,t-\tau)) u_t(x,t)\, d\Sigma_2 \\ &- \frac{\tau^{-1}\xi}{2} \int_{\Sigma_2} [G(u_t(x,t-\tau)) - G(u_t(x,t))]\, d\Sigma_2, \end{aligned} \tag{2.28}$$

and consequently

$$E(T) - E(0) \le -C_1 \int_{\Sigma_2} u_t(x,t) f(u_t(x,t)) + y(x,1,t) g(y(x,1,t)) d\Sigma_2, \tag{2.29}$$

where

$$C_1 = min\left\{\alpha_1 - \frac{\tau^{-1}\xi}{2}a_2 - \alpha_2 a_2, \frac{\tau^{-1}\xi}{2}a_1 + \alpha_2(a_1 - 1)\right\},$$

with ξ as in (2.8).

Proof of Proposition 2.5. It follows from (**H1**)(iii), (**H2**)(v) and (2.27) that

$$\frac{\partial u}{\partial \nu} |_{\Gamma_2} \in L^2(0, T; L^2(\Gamma_2)). \tag{2.30}$$

Then by virtue of the Lemma 2.2 in [43], it is enough to prove the identity (2.28) for smooth solutions

$$u \in C(0, T; H^2(\Omega)) \cap C^1(0, T; H^1_{\Gamma_1}(\Omega)).$$

We multiply the first equation in (2.26) by u_t and integrate by parts over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left\{|\nabla u(x,t)|^2 + u_t^2(x,t)\right\} dx = -\alpha_1 \int_{\Gamma_2} u_t(x,t) f(u_t(x,t))\, d\Gamma - \alpha_2 \int_{\Gamma_2} g(y(x,1,t))u_t(x,t)\, d\Gamma. \tag{2.31}$$

We multiply the second equation in (2.26) by $\xi\, g(y(x,\rho,t))$ and integrate over $\Gamma_2 \times (0,1)$, we obtain

$$\int_{\Gamma_2}\int_0^1 \{\xi y_t(x,\rho,t) g(y(x,\rho,t)) + \tau^{-1}\xi y_\rho(x,\rho,t) g(y(x,\rho,t))\}\, d\rho\, d\Gamma = 0.$$

We have

$$\frac{\partial G}{\partial t}(y(x,\rho,t)) = y_t(x,\rho,t).g(y(x,\rho,t)),$$
$$\frac{\partial G}{\partial \rho}(y(x,\rho,t)) = y_\rho(x,\rho,t).g(y(x,\rho,t)).$$

Consequently,

$$\begin{aligned}\xi\frac{d}{dt}\int_{\Gamma_2}\int_0^1 G(y(x,\rho,t))\, d\rho\, d\Gamma &= -\tau^{-1}\xi \int_{\Gamma_2}\int_0^1 \frac{d}{d\rho} G(y(x,\rho,t))\, d\rho\, d\Gamma \\ &= -\tau^{-1}\xi \int_{\Gamma_2} [G(y(x,1,t)) - G(y(x,0,t))]\, d\Gamma. \end{aligned}\tag{2.32}$$

From (2.31) and(2.32), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \left\{|\nabla u(x,t)|^2 + u_t^2(x,t)\right\} dx + \frac{\xi}{2}\frac{d}{dt}\int_{\Gamma_2}\int_0^1 G(y(x,\rho,t))\, d\rho\, d\Gamma = -\alpha_1 \int_{\Gamma_2} u_t(x,t) f(u_t(x,t))\, d\Gamma \\ &- \alpha_2 \int_{\Gamma_2} g(y(x,1,t))u_t(x,t)\, d\Gamma - \frac{\tau^{-1}\xi}{2}\int_{\Gamma_2} G(y(x,1,t))\, d\Gamma + \frac{\tau^{-1}\xi}{2}\int_{\Gamma_2} G(u_t(x,t))]\, d\Gamma. \end{aligned}\tag{2.33}$$

We integrate both sides of (2.33) over $(0,T)$, we obtain

$$\begin{aligned}E(T) - E(0) = &-\alpha_1 \int_{\Sigma_2} u_t(x,t) f(u_t(x,t))\, d\Sigma_2 - \alpha_2 \int_{\Sigma_2} g(u_t(x,t-\tau))u_t(x,t)\, d\Sigma_2 \\ &- \frac{\tau^{-1}\xi}{2}\int_{\Sigma_2} [G(u_t(x,t-\tau)) - G(u_t(x,t))]\, d\Sigma_2. \end{aligned}\tag{2.34}$$

From (2.25) and assumption (**H2**)*(v)*, we have

$$\begin{aligned} &G(u_t(x,t)) \leq a_2 u_t(x,t) f(u_t(x,t)), \\ &-G(y(x,1,t)) \leq -a_1 y(x,1,t) g(y(x,1,t)). \end{aligned}$$

Hence

$$\begin{aligned} E(T)-E(0) \leq &-(\alpha_1 - \frac{\tau^{-1}\xi a_2}{2}) \int_{\Sigma_2} u_t(x,t) f(u_t(x,t))\, d\Sigma_2 - \alpha_2 \int_{\Sigma_2} g(y(x,1,t)) u_t(x,t)\, d\Sigma_2 \\ &\frac{-\tau^{-1}\xi a_1}{2} \int_{\Sigma_2} y(x,1,t) g(y(x,1,t))\, d\Sigma_2. \end{aligned} \tag{2.35}$$

Let G^* be the conjugate function of the concave function G

$$G^*(s) = \sup_{t\in\mathbb{R}^+} (st - G(t)).$$

Then G^* is the Legendre transform of G, which is given by (Arnold ([8] p. 61-62))

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)] \text{ for all } s \geq 0,$$

and satisfies

$$s.t \leq G^*(s) + G(t) \text{ for all } s,t \geq 0. \tag{2.36}$$

But from the definition of G, we have

$$G^*(s) = sg^{-1}(s) - G[(g^{-1}(s)].$$

Hence

$$\begin{aligned} G^*(|g(y(x,1,t))|) &= g(y(x,1,t)) y(x,1,t) - G(y(x,1,t)) \\ &\leq (1-a_1) y(x,1,t) g(y(x,1,t)). \end{aligned} \tag{2.37}$$

Making use of (2.35) and (2.36), we get

$$\begin{aligned} E(T)-E(0) \leq &-(\alpha_1 - \frac{\tau^{-1}\xi a_2}{2}) \int_{\Sigma_2} u_t(x,t) f(u_t(x,t))\, d\Sigma_2 - \frac{\tau^{-1}\xi a_1}{2} \int_{\Sigma_2} y(x,1,t) g(y(x,1,t))\, d\Sigma_2 \\ &+ \alpha_2 \int_{\Sigma_2} (G(|u_t(x,t)|) + G^*(|g(y(x,1,t))|))\, d\Sigma_2. \end{aligned}$$

(2.37) together with assumption (**H2**)*(v)* implies

$$\begin{aligned} E(T)-E(0) \leq &-(\alpha_1 - \frac{\tau^{-1}\xi a_2}{2} - a_2\alpha_2) \int_{\Sigma_2} u_t(x,t) f(u_t(x,t))\, d\Sigma_2 \\ &-(\frac{\tau^{-1}\xi a_1}{2} - \alpha_2(1-a_1)) \int_{\Sigma_2} y(x,1,t) g(y(x,1,t))\, d\Sigma_2. \end{aligned}$$

Therefore

$$E(T) - E(0) \leq -C_1 \int_{\Sigma_2} \{u_t(x,t)f(u_t(x,t)) + y(x,1,t)g(y(x,1,t))\} \, d\Sigma_2, \tag{2.38}$$

where

$$C_1 = \min\left\{(\alpha_1 - \frac{\tau^{-1}\xi a_2}{2} - \alpha_2 a_2), (\frac{\tau^{-1}\xi a_1}{2} + \alpha_2(a_1 - 1))\right\}.$$

with ξ as in (2.8).

□

Theorem 2.6. *Assume that:*

- ∗ *The function g is Lipschitz continuous on $\mathbb{R}$, with L as a Lipschitz constant.* (2.39)
- ∗ *$f(s_1) - f(s_2) \geq r(s_1 - s_2)$ for all $s_1 - s_2 \geq 0$ and fixed $r > 0$.* (2.40)

Then, for each $(u_0, u_1, f_0) \in H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2; L^2(0,\tau))$, problem (2.1) has a unique solution

$$u \in C(0,\infty; H^1_{\Gamma_1}(\Omega)) \cap C^1(0,\infty; L^2(\Omega)).$$

Moreover if the assumptions $(\mathbf{H1})(iii), (\mathbf{H2})(ii)$ *and* $(\mathbf{H2})(v)$ *are fulfilled, then*

$$u_t \in L^2(0,\infty; L^2(\Gamma_2)), \frac{\partial u}{\partial \nu} \in L^2(0,\infty; L^2(\Gamma_2)). \tag{2.41}$$

Proof of Theorem 2.6. This follows from nonlinear semigroup theory.
Let $A : L^2(\Omega) \to L^2(\Omega)$ be the operator defined by

$$A\zeta = -\Delta\zeta \text{ with } D(A) = \left\{\zeta \in H^2(\Omega), \quad \frac{\partial \zeta}{\partial \nu} = 0 \text{ on } \Gamma_2, \quad \zeta = 0 \text{ on } \Gamma_1\right\}.$$

Let $N : L^2(\Gamma) \to L^2(\Omega)$ be the Neumann map

$$\Delta N\varphi = 0, \; N\varphi \mid_{\Gamma_1} = 0, \; \frac{\partial N\varphi}{\partial \nu} \mid_{\Gamma_2} = \varphi.$$

It is well known (see [79]) that

$$N \in \mathcal{L}(L^2(\Gamma) \to H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\epsilon}(\Omega) \equiv D(A^{\frac{3}{4}-\epsilon})),$$

and

$$N^* A^* \eta = \eta \mid_{\Gamma_2} \qquad \text{for } \eta \in D(A^{\frac{1}{2}}). \tag{2.42}$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2; L^2(0,1)).$$

Next define

$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

$$\mathcal{A}\begin{pmatrix}\zeta\\ \eta\\ \theta\end{pmatrix}=\begin{pmatrix}\eta\\ A(-\zeta-\alpha_1 Nf(\eta)-\alpha_2 Ng(\theta(.;1)))\\ -\tau^{-1}\theta_\rho\end{pmatrix}, \tag{2.43}$$

with

$$D(\mathcal{A})=\{\zeta\in H^1_{\Gamma_1}(\Omega),\eta\in H^1_{\Gamma_1}(\Omega),\theta\in L^2(\Gamma_2;H^1(0,1));-\zeta-\alpha_1 Nf(\eta)-\alpha_2 Ng(\theta(.;1))\in D(A)\}$$

Then we can rewrite (2.26) as an abstract Cauchy problem on $\mathcal{H}$

$$\begin{cases}\dfrac{dW}{dt}(t)=\mathcal{A}(W(t)),\\ W(0)=W_0,\end{cases}$$

where

$$W(t)=(u(x,t),u_t(x,t),y(x,\rho,t))^T,\ W'(t)=(u_t(x,t),u_{tt}(x,t),y_t(x,\rho,t))^T.$$

We will show that the operator $\mathcal{A}$ defined by (2.43) is maximal dissipative on the Hilbert space $\mathcal{H}$ equipped with the inner product

$$\left\langle\begin{pmatrix}\zeta\\ \eta\\ \theta\end{pmatrix},\begin{pmatrix}\tilde\zeta\\ \tilde\eta\\ \tilde\theta\end{pmatrix}\right\rangle_{\mathcal{H}}=\int_\Omega\{A^{\frac12}\zeta(x)\,.\,A^{\frac12}\tilde\zeta(x)+\eta(x)\,.\,\tilde\eta(x)\}\,dx+\delta\int_{\Gamma_2}\int_0^1\theta(x,\rho)\,.\,\tilde\theta(x,\rho)\,d\rho\,d\Gamma,$$

with

$$\tau\alpha_2 L<\delta<2\tau(\alpha_1 r-\frac{\alpha_2 L}{2}), \tag{2.44}$$

$$\frac{L}{r}<\frac{\alpha_1}{\alpha_2}. \tag{2.45}$$

First, we prove that $\mathcal{A}$ is dissipative.
Let $U=(\zeta,\eta,\theta)^T$, $V=(\tilde\zeta,\tilde\eta,\tilde\theta)^T\in D(\mathcal{A})$. Then

$$\begin{aligned}\langle\mathcal{A}U-\mathcal{A}V;U-V\rangle_{\mathcal{H}}&=-\alpha_1\int_\Omega AN(f(\eta(x))-f(\tilde\eta(x)))\ (\eta(x)-\tilde\eta(x))\,dx\\ &-\alpha_2\int_\Omega AN(g(\theta(x,1))-g(\tilde\theta(x,1)))(\eta(x)-\tilde\eta(x))\,dx\\ &-\delta\tau^{-1}\int_{\Gamma_2}\int_0^1(\theta_\rho(x,\rho)-\tilde\theta_\rho(x,\rho))(\theta(x,\rho)-\tilde\theta(x,\rho))\,d\rho\,d\Gamma\\ &=-\alpha_1\int_{\Gamma_2}(f(\eta(x))-f(\tilde\eta(x)))N^*A^*(\eta(x)-\tilde\eta(x))\,d\Gamma\\ &-\alpha_2\int_\Omega(g(\theta(x,1))-g(\tilde\theta(x,1)))N^*A^*(\eta(x)-\tilde\eta(x))\,d\Gamma\\ &-\frac{\delta\tau^{-1}}{2}\int_{\Gamma_2}|\theta(x,1)-\tilde\theta(x,1)|^2d\,\Gamma+\frac{\delta\tau^{-1}}{2}\int_{\Gamma_2}|\theta(x,0)-\tilde\theta(x,0)|^2d\,\Gamma.\end{aligned}$$

From (2.42), we have

$$\begin{aligned}\langle \mathcal{A}U - \mathcal{A}V; U - V\rangle_{\mathcal{H}} = &- \alpha_1 \int_{\Gamma_2} (f(\eta(x)) - f(\tilde{\eta}(x)))(\eta(x) - \tilde{\eta}(x))\, d\Gamma \\ &- \alpha_2 \int_{\Gamma_2} (g(\theta(x,1)) - g(\tilde{\theta}(x,1)))(\eta(x) - \tilde{\eta}(x))\, d\Gamma \\ &- \frac{\delta\tau^{-1}}{2} \int_{\Gamma_2} (\theta(x,1) - \tilde{\theta}(x,1))^2 d\,\Gamma + \frac{\delta\tau^{-1}}{2} \int_{\Gamma_2} (\theta(x,0) - \tilde{\theta}(x,0))^2 d\,\Gamma.\end{aligned}$$

Using assumptions (2.39), (2.40) and the Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned}\langle \mathcal{A}U - \mathcal{A}V; U - V\rangle_{\mathcal{H}} \le &-\alpha_1 r \int_{\Gamma_2} |\eta(x) - \tilde{\eta}(x)|^2 d\,\Gamma + \frac{\alpha_2 L}{2} \int_{\Gamma_2} |\theta(x,1) - \tilde{\theta}(x,1)|^2 d\,\Gamma \\ &+ \frac{\alpha_2 L}{2} \int_{\Gamma_2} |\eta(x) - \tilde{\eta}(x)|^2 d\,\Gamma - \frac{\delta\tau^{-1}}{2} \int_{\Gamma_2} |\theta(x,1) - \tilde{\theta}(x,1)|^2 d\,\Gamma \\ &+ \frac{\delta\tau^{-1}}{2} \int_{\Gamma_2} |\eta(x) - \tilde{\eta}(x)|^2 d\,\Gamma.\end{aligned}$$

Therefore

$$\begin{aligned}\langle \mathcal{A}U - \mathcal{A}V; U - V\rangle_{\mathcal{H}} \le &-\left(\alpha_1 r - \frac{\alpha_2 L}{2} - \frac{\delta\tau^{-1}}{2}\right) \int_{\Gamma_2} |\eta(x) - \tilde{\eta}(x)|^2 d\,\Gamma \\ &- \left(-\frac{\alpha_2 L}{2} + \frac{\delta\tau^{-1}}{2}\right) \int_{\Gamma_2} |\theta(x,1) - \tilde{\theta}(x,1)|^2 d\,\Gamma.\end{aligned}$$

From (2.44), we conclude that

$$\langle \mathcal{A}U - \mathcal{A}V; U - V\rangle_{\mathcal{H}} \le 0.$$

Thus $\mathcal{A}$ is dissipative.
In order to establish maximality, we need to prove the range condition

$$range(\lambda I - \mathcal{A}) = \mathcal{H} \;\text{ for a fixed } \lambda > 0.$$

Let $(k, l, m)^T \in \mathcal{H}$, we seek $w = (\zeta, \eta, \theta)^T \in D(\mathcal{A})$ solution of

$$(\mathcal{A} - \lambda I)w = (k, l, m)^T,$$

or equivalently

$$\lambda\,\zeta - \eta = k, \tag{2.46}$$

$$\lambda\,\eta - A[-\zeta - \alpha_1 N f(\eta_{|\Gamma_2}) - \alpha_2 N g(\theta(x,1))] = l, \tag{2.47}$$

$$\lambda\,\theta + \tau^{-1}\theta_\rho = m. \tag{2.48}$$

Hence

$$\zeta = \frac{1}{\lambda}(\eta + k). \tag{2.49}$$

From (2.48) and the last line of (2.26) we have

$$\begin{cases} \theta_\rho(x,\rho) = -\tau\lambda\theta(x,\rho) + \tau m(x,\rho), \ \rho \in (0,1), \\ \theta(x,0) = \eta(x). \end{cases}$$

The unique solution for the above initial value problem is given by

$$\theta(x,\rho) = \eta(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho m(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma, \qquad x \in \Gamma_2,\ \rho \in (0,1),$$

and in particular

$$\theta(x,1) = \eta(x)e^{-\lambda\tau} + \mathcal{Z}_0, \tag{2.50}$$

where

$$\mathcal{Z}_0 = \tau e^{-\lambda\tau}\int_0^1 m(x,\sigma)e^{\lambda\tau\sigma}d\sigma, \qquad x \in \Gamma_2.$$

Insertion of (2.49) and (2.50) into (2.47) yields

$$\lambda\eta + \frac{1}{\lambda}A\eta + \alpha_1 ANf(\eta_{|\Gamma_2}) + \alpha_2 ANg((\eta_{|\Gamma_2})e^{-\lambda\tau} + \mathcal{Z}_0) = l - \frac{1}{\lambda}Ak.$$

Set

$$\mathcal{T}\eta = \lambda\eta + \frac{1}{\lambda}A\eta + \alpha_1 AN\widehat{f}(\eta) + \alpha_2 AN\widehat{g}(\eta),$$

where

$$\widehat{f}(\eta) = f(\eta_{|\Gamma_2}),$$

and

$$\widehat{g}(\eta) = g((\eta_{|\Gamma_2})e^{-\lambda\tau} + \mathcal{Z}_0).$$

Lemma 2.1. *The operator $\mathcal{T}$ is surjective from $V = D(A^{\frac{1}{2}}) = H^1_{\Gamma_1}(\Omega)$ onto $V' = (D(A^{\frac{1}{2}}))' = (H^1_{\Gamma_1}(\Omega))'$.*

Proof of Lemma 2.1. For $\eta \in V$, let

$$B\eta = \alpha_1 AN\widehat{f}(\eta) + \alpha_2 AN\widehat{g}(\eta),$$

and

$$C\eta = \lambda\eta + \frac{1}{\lambda}A\eta.$$

Then

$$\mathcal{T}\eta = B\eta + C\eta.$$

According to Barbu [11] (Corollary 1.3, page 48), in order to establish surjectivity of the operator $\mathcal{T}$, it is sufficient to prove that B is monotone and hemicontinuous, C is maximal monotone, and $B + C$ is coercive.

Monotonicity of B.
Let $\eta, v \in V$. Then

$$\begin{aligned}\langle (-B)\eta - (-B)v, \eta - v\rangle_{V'\times V} &= -\alpha_1 \langle AN(\widehat{f}(\eta) - \widehat{f}(v)), \eta - v\rangle_{V'\times V} \\ &- \alpha_2 \langle AN(\widehat{g}(\eta) - \widehat{g}(v)), \eta - v\rangle_{V'\times V} \\ &= -\alpha_1 \langle \widehat{f}(\eta) - \widehat{f}(v), N^* A^*(\eta - v)\rangle_{L^2(\Gamma_2)} \\ &- \alpha_2 \langle \widehat{g}(\eta) - \widehat{g}(v), N^* A^*(\eta - v)\rangle_{L^2(\Gamma_2)} \\ &\le -(\alpha_1 r - \alpha_2 L e^{-\lambda\tau}) \|(\eta_{|\Gamma_2}) - (v_{|\Gamma_2})\|^2_{L^2(\Gamma_2)}.\end{aligned}$$

By (2.45), we conclude that

$$\langle (-B)\eta - (-B)v, \eta - v\rangle_{V'\times V} \le 0. \tag{2.51}$$

Thus $(-B)$ is dissipative, then B is monotone.
Hemicontinuity of B.
Let $\eta, v, w \in V$. We will prove that the function

$$t \to < B(\eta + tv), w >_{V'\times V}$$

is continuous. Indeed, we have

$$\begin{aligned}|\langle B(\eta + tv) - B(\eta + t_0 v), w\rangle_{V'\times V}| &= |\langle \alpha_1 AN(\widehat{f}(\eta + tv) - \widehat{f}(\eta + t_0 v)), w\rangle_{V'\times V} \\ &+ \langle \alpha_2 AN(\widehat{g}(\eta + tv) - \widehat{g}(\eta + t_0 v)), w\rangle_{V'\times V}| \\ &\le |\langle \alpha_1 \widehat{f}(\eta + tv) - \widehat{f}(\eta + t_0 v), N^* A^* w\rangle_{L^2(\Gamma_2)}| \\ &+ |\langle \alpha_2 \widehat{g}(\eta + tv) - \widehat{g}(\eta + t_0 v), N^* A^* w\rangle_{L^2(\Gamma_2)}| \\ &\le c\|w\|_{L^2(\Gamma_2)} \{ \|\widehat{f}(\eta + tv) - \widehat{f}(\eta + t_0 v)\|_{L^2(\Gamma_2)} \\ &+ \|\widehat{g}(\eta + tv) - \widehat{g}(\eta + t_0 v)\|_{L^2(\Gamma_2)} \}.\end{aligned}$$

From assumption (2.39), we have

$$\begin{aligned}|\langle B(\eta + tv) - B(\eta + t_0 v), w\rangle_{V'\times V}| \le c\|w\|_{L^2(\Gamma_2)} \Big\{ &\|\widehat{f}(\eta + tv) - \widehat{f}(\eta + t_0 v)\|_{L^2(\Gamma_2)} \\ &+ L\|(t - t_0)(v_{|\Gamma_2})\|_{L^2(\Gamma_2)} \Big\}.\end{aligned}$$

The continuity of f allows us to deduce that

$$|\langle B(\eta + tv) - B(\eta + t_0 v), w\rangle_{V'\times V}| < \tilde{\epsilon},$$

for $|t - t_0| < \widetilde{\delta}$. This proves the continuity of the function $t \longmapsto \langle B(\eta + tv), w\rangle$.
Maximal monotonicity of C.
For $\eta, v \in V$, we have

$$\begin{aligned}\langle C\eta - Cv, \eta - v\rangle_{V'\times V} &= \langle \lambda\eta + \frac{1}{\lambda} A\eta - \lambda v - \frac{1}{\lambda} Av, \eta - v\rangle_{V'\times V} \\ &= \lambda\|\eta - v\|^2_V + \frac{1}{\lambda}\langle A(\eta - v), \eta - v\rangle_{V'\times V} \ge \lambda\|\eta - v\|^2_V.\end{aligned} \tag{2.52}$$

so that C is maximal monotone. Since we are working with V, V' framework and the operator $AN : L^2(\Gamma_2) \to V'$ is bounded, the operator C is continuous, then it is maximal monotone.
Coercivity of $B + C$.
This follows from (2.51) and (2.52). $\square$

The operator $\mathcal{T}$ is surjective from V onto V' . Then $\eta \in V$ and consequently $\zeta = \frac{\eta+k}{\lambda} \in V$. One easily shows that (ζ, η, θ) is also in $D(\mathcal{A})$. Indeed from (2.47), we have

$$A[\zeta + \alpha_1 N\widehat{f}(\eta) + \alpha_2 N g(\theta(x,1))] = l - \lambda\eta \in L^2(\Omega).$$

Hence

$$\zeta + \alpha_1 N\widehat{f}(\eta) + \alpha_2 N g(\theta(x,1)) \in D(A).$$

From nonlinear semigroup theory and the density of $D(\mathcal{A})$ in $\mathcal{H}$, we obtain existence and uniqueness of the solution

$$u \in C(0,T; H^1_{\Gamma_1}(\Omega)) \cap C^1(0,T; L^2(\Omega)),$$

for all $T > 0$.
To obtain (2.41) we first notice that for $(\zeta_0, \eta_0, f_0) \in D(\mathcal{A})$, we have

$$\eta_0\,|_\Gamma \in H^{\frac{1}{2}}(\Gamma),$$

and after using assumption (**H1**)*(iii)* and (2.39)

$$\frac{\partial \zeta_0}{\partial \nu} \in L^2(\Gamma_2)\,,\; f(\eta_0\,|_{\Gamma_2}) \in L^2(\Gamma_2), g(f_0\,|_{\Gamma_2}) \in L^2(\Gamma_2).$$

Let (u, u_t, y) denote the solution of problem (2.26) corresponding to the initial datum $(\zeta_0, \eta_0, f_0) \in D(\mathcal{A})$. Then, by the semigroup property, we have $(u(t), u_t(t), y(t)) \in D(\mathcal{A})$ and consequently

$$u_t\,|_{\Gamma_2} \in L^\infty(0,T; L^2(\Gamma_2)), \frac{\partial u}{\partial \nu}\,|_{\Gamma_2} \in L^\infty(0,T; L^2(\Gamma_2)).$$

Thus we are in a position to apply the estimate (2.38) of Proposition 2.5. Hence for all $t > 0$,

$$E(t) + C_1 \int_0^t \int_{\Gamma_2} (f(u_t)u_t + g(u_t(x, s-\tau))u_t(x, s-\tau))\, d\Gamma\, ds \le E(0). \tag{2.53}$$

Recalling assumption (2.40), we obtain

$$\begin{aligned} E(t) + C_1 r \int_0^t \int_{\Gamma_2} |u_t|^2\, d\Gamma\, ds + C_1 \int_0^t \int_{\Gamma_2} g(u_t(x, s-\tau)u_t(x, s-\tau))\, d\Gamma\, ds \\ \le |\nabla \zeta_0|^2_\Omega + |\eta_0|^2_\Omega + \int_0^1 \int_{\Gamma_2} |f_0(x, -\tau\rho)|^2\, d\Gamma\, d\rho. \end{aligned} \tag{2.54}$$

Since $D(\mathcal{A})$ is dense in $\mathcal{H}$, the above inequality can be extended to all $(\zeta_0, \eta_0, f_0) \in \mathcal{H}$.
The estimate (2.54) together with assumptions (**H2**)(ii) implies

$$u_t|_{\Gamma_2} \in L^2(0, \infty; L^2(\Gamma_2)).$$

Moreover, from hypotheses (**H1**)(iii) and (2.39), it follows that

$$f(u_t|_{\Gamma_2}) \in L^2(0,\infty; L^2(\Gamma_2)),\ g(u_t(.,t-\tau)|_{\Gamma_2}) \in L^2(0,\infty; L^2(\Gamma_2)),$$

and consequently

$$\frac{\partial u}{\partial \nu} \in L^2(0,\infty; L^2(\Gamma_2)).$$

□

Completion of the proof of Theorem 2.1.
We consider next the following approximation of system (2.1) with $l \longrightarrow +\infty$ as the parameter of approximation,

$$\begin{cases} u_{ltt}(x,t) - \Delta u_l(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ u_l(x,0) = u_0(x), u_{lt}(x,0) = u_1(x) & \text{in } \Omega, \\ u_l(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u_l(x,t)}{\partial \nu} = -\alpha_1 f_l(u_{lt}(x,t)) - \alpha_2 g_l(u_{lt}(x,t-\tau)) & \text{on } \Gamma_2 \times (0,+\infty), \\ u_{lt}(x,t-\tau) = f_0(x,t-\tau) & \text{on } \Gamma_2 \times (0,\tau), \end{cases} \tag{2.55}$$

where

$$f_l(u_{lt}(x,t)) = f(u_{lt}(x,t)) + \frac{1}{l} u_{lt}(x,t),$$

and the functions g_l are defined by

$$g_l(s) = \begin{cases} g(s),\ |s| \le l \\ g(l),\ s \ge l \\ g(-l),\ s \le -l \end{cases} \tag{2.56}$$

Notice that for each value of the parameter l, the functions g_l and f_l satisfy the hypotheses of Theorem 2.6. Thus, there exists a solution (u_l, u_{lt}) of (2.55) such that

$$u_l \in C(0,T; H^1_{\Gamma_1}(\Omega)) \cap C^1(0,T; L^2(\Omega)),$$

for any finite $T > 0$ and

$$\begin{gathered} u_{lt} \in L^2(0,\infty; L^2(\Gamma_2))\ ,\ \frac{\partial u_l}{\partial \nu} \in L^2(0,\infty; L^2(\Gamma_2))\ , \\ f_l(u_{lt}\ |_{\Gamma_2}) \in L^2(0,\infty; L^2(\Gamma_2)),\ g_l(u_t(.,t-\tau)\ |_{\Gamma_2}) \in L^2(0,\infty; L^2(\Gamma_2)). \end{gathered} \tag{2.57}$$

We prove that we can extract a subsequence from the above sequence of solutions u_l that has a limit which is a solution of the original problem (2.1). To accomplish this we need the following.

Lemma 2.2. *Under the assumptions of Theorem 2.1, we have as* $l \longrightarrow +\infty$ *and* $u_{lt}(x,t-\tau) \longrightarrow u_t(x,t-\tau)$ *weakly in* $H^1(\Omega)$

$$g_l(u_{lt}(x,t-\tau)) \longrightarrow g(u_t(x,t-\tau)) \text{ in } L^2(\Gamma_2). \tag{2.58}$$

Proof of Lemma 2.2. We write

$$\int_\Gamma |g_l(u_{lt}(x,t-\tau)) - g(u_{lt}(x,t-\tau))|^2\, d\Gamma \le 2\left[\int_{\Gamma_l} |g(u_{lt}(x,t-\tau))|^2\, d\Gamma_l + \int_{\Gamma_l} |g(l)|^2 + |g(-l)|^2\, d\Gamma_l\right], \tag{2.59}$$

where

$$\Gamma_l = \{x \in \Gamma : |u_{lt}(x,t-\tau)| \ge l\}.$$

Then, by Sobolev's Embeddings

$$H^{1/2}(\Gamma) \subset L^{\frac{2n-2}{n-2}}(\Gamma), \qquad n > 2, \tag{2.60}$$

$$H^{1/2}(\Gamma) \subset L^p(\Gamma), \quad 1 \le p < \infty, \qquad n = 2. \tag{2.61}$$

We have for $n > 2$

$$\left(\int_{\Gamma_l} l^{\frac{2n-2}{n-2}}\right)^{\frac{n-2}{2n-2}} \le \int_\Gamma \left(|(u_{lt}(x,t-\tau))|^{\frac{2n-2}{n-2}}\right)^{\frac{n-2}{2n-2}},$$

therefore

$$mes\,\Gamma_l \le \int_\Gamma \left(|(u_{lt}(x,t-\tau))|^{\frac{2n-2}{n-2}}\right)^{\frac{n-2}{2n-2}} l^{\frac{-2n+2}{n-2}}. \tag{2.62}$$

Analogously, for $n = 2$ the above inequality is valid with any exponent for l.
By assumptions $(\mathbf{H2})(iii), (\mathbf{H2})(v)$ and by (2.62),

$$\begin{aligned}\int_{\Gamma_l} |g(u_{lt}(x,t-\tau))|^2\, d\Gamma_l &\le M\int_{\Gamma_l} |u_{lt}(x,t-\tau)|^2\, d\Gamma_l\\ &\le M\left[\int_{\Gamma_l} |u_{lt}(x,t-\tau)|^{\frac{2n-2}{n-2}}\right]^{\frac{n-2}{n-1}} (mes\,\Gamma_l)^{\frac{n-1-n-2}{n-1}} \longrightarrow_{l\longrightarrow\infty} 0,\end{aligned} \tag{2.63}$$

$$\int_{\Gamma_l} |g(l)|^2\, d\Gamma_l \le Ml^2\, mes\Gamma_l \le M\left[\int_{\Gamma_l} |u_{lt}(x,t-\tau)|^{\frac{2n-2}{n-2}}\right]^{\frac{n-2}{n-1}} l^{2-\frac{2(n-1)}{n-2}} \longrightarrow_{l\to\infty} 0, \tag{2.64}$$

where

$$M = max\left\{M_2^2, (\frac{a_2}{a_1}M_1)^2\right\}. \tag{2.65}$$

Combining the results of (2.59),(2.63) and (2.64) gives (2.58). □

By using regularity properties (2.57), we are in a position to apply the estimate (2.53) for each $t > 0$, to obtain

$$E_l(t) + C_1\left\{\int_0^t \int_{\Gamma_2} \{(f(u_{lt}) + \frac{1}{l}u_{lt})u_{lt} + u_{lt}(x,s-\tau)g_l(u_{lt}(x,s-\tau))\}\, d\Gamma\, ds\right\} \le E_l(0), \tag{2.66}$$

where $E_l(t)$ is defined by (2.7) with u replaced by u_l.
Recalling assumption $(\mathbf{H2})(v)$ together with $(\mathbf{H1})(iii)$, we readily obtain

$$E_l(0) \le C\left(|u_0|_{H^1(\Omega)}, |u_1|_{L^2(\Omega)}, |f_0|_{L^2(\Gamma_2\times(0,\tau))}\right), \tag{2.67}$$

and from $(\mathbf{H1})(ii)$, combined with $(\mathbf{H2})(ii)$, (2.66) and (2.67), we infer that

$$|u_{lt}|_{L^2(0,T;L^2(\Gamma_2))} \leq C\left(|u_0|_{H^1(\Omega)}, |u_1|_{L^2(\Omega)}, |f_0|_{L^2(\Gamma_2\times(0,\tau))}\right), \tag{2.68}$$

$$|u_l|_{C(0,T;H^1_{\Gamma_1}(\Omega))} + |u_{lt}|_{C(0,T;L^2(\Omega))} \leq C. \tag{2.69}$$

Therefore, on a subsequence we have

$$u_l \longrightarrow u \quad \text{weakly in } H^1(\Omega\times(0,T)), \tag{2.70}$$

and by the trace theorem

$$u_l|_\Gamma \longrightarrow u|_\Gamma \quad \text{strongly in } \; L^\infty(0,T;L^2(\Gamma_2)), \tag{2.71}$$

$$u_{lt}|_\Gamma \longrightarrow u_t|_\Gamma \quad \text{weakly in } \; L^2(0,T;L^2(\Gamma_2)). \tag{2.72}$$

Hypotheses $(\mathbf{H1})$, $(\mathbf{H2})$ together with the compactness of the embeddings in (2.60)-(2.61) and (2.72) also give

$$g(u_{lt}(x,t-\tau)) \longrightarrow g(u_t(x,t-\tau)) \quad \text{in } L^2(0,T;L^2(\Gamma_2)), \tag{2.73}$$

$$f(u_{lt}|_\Gamma) \longrightarrow f^* \quad \text{weakly in } \; L^2(0,T;L^2(\Gamma_2)) \;\; \text{for some } \;\; f^*\in L^2(0,T;L^2(\Gamma_2)). \tag{2.74}$$

Let (u_l,u_m) be the solutions of (2.55) corresponding to the parameters l and m. Then

$$\begin{aligned}
&|\nabla(u_l-u_m)(t)|^2_{L^2(\Omega)} + |(u_{lt}-u_{mt})(t)|^2_{L^2(\Omega)} + \alpha_1\int_0^t\int_{\Gamma_2}(f(u_{lt})-f(u_{mt}))(u_{lt}-u_{mt})\,d\Gamma\,ds\\
&\leq \alpha_1(\frac{1}{l}+\frac{1}{m})\int_0^t\int_{\Gamma_2}|u_{lt}|^2\,d\Gamma\,dt + \alpha_1(\frac{1}{l}+\frac{1}{m})\int_0^t\int_{\Gamma_2}|u_{mt}|^2\,d\Gamma\,ds\\
&+\alpha_2\int_0^t\int_{\Gamma_2}|g_l(u_{lt}(x,s-\tau))-g_m(u_{mt}(x,s-\tau))|\,|u_{lt}-u_{mt}|\,d\Gamma\,ds.
\end{aligned} \tag{2.75}$$

The result (2.58) of Lemma 2.2 together with (2.72) and (2.68) implies the convergence to zero (when $l,m\to\infty$) of the last term on the RHS of (2.75).
Similarly, by (2.72) the first two terms on the RHS of (2.75) converge to zero.
Thus, we have obtained

$$u_l \longrightarrow u \in C(0,T;H^1_{\Gamma_1}(\Omega))\cap C^1(0,T;L^2(\Omega)), \tag{2.76}$$

for any finite $T>0$ and

$$\lim_{l,m\to\infty}\int_{\Sigma_2}(f(u_{lt})-f(u_{mt}))(u_{lt}-u_{mt})\,d\Sigma_2 = 0. \tag{2.77}$$

From (2.72),(2.74) and (2.77) we also get

$$\lim_{l\to\infty}\left[\int_{\Sigma_2} f(u_{lt})u_{lt}\,d\Sigma_2 - \int_{\Sigma_2} f(u_{lt})u_t\,d\Sigma_2 - \int_{\Sigma_2} f^*u_{lt}\,d\Sigma_2\right] + \lim_{m\to\infty}\int_{\Sigma_2} f(u_{mt})u_{mt}\,d\Sigma_2 = 0.$$

Hence, again using (2.72),(2.74) and changing m to l we obtain

$$2\lim_{l\to\infty}\int_{\Sigma_2} f(u_{lt})u_{lt}\,d\Sigma_2 = 2\int_{\Sigma_2} f^* u_t\,d\Sigma_2. \tag{2.78}$$

But (2.78) combined with (2.72), (2.74) and the monotonicity of f, by virtue of Lemma 13, p.42 in [11], yields

$$f^* = f(u_t|_\Gamma). \tag{2.79}$$

Passing to the limit in (2.55) and recalling (2.79) together with (2.70)- (2.73), gives

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 & \text{in } D'(\Omega\times(0,+\infty)),\\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } \Omega,\\ u(x,t) = 0 & \text{on } \Gamma_1\times(0,+\infty),\\ \frac{\partial u}{\partial\nu} = -\alpha_1 f(u_t) - \alpha_2 g(u_t(.,.-\tau)) & \text{in } L_2(0,+\infty;\Gamma_2),\\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{on } \Gamma_2\times(0,\tau), \end{cases} \tag{2.80}$$

with the regularity

$$\frac{\partial u}{\partial\nu},\ u_t \in L^2(0,T;L^2(\Gamma_2)),$$

for any finite $T>0$.
The proof of Theorem 2.1 is thus complete.

2.2.2 Proofs of Theorem 2.2 and Corollary 2.1.

Proof of Theorem 2.2.

Proposition 2.7. *Assume the hypotheses* (**H1**) – (**H4**). *Let u be a solution of (2.1) guaranteed by Theorem 2.1. Then*

$$\int_\alpha^{T-\alpha}\left\{|\nabla u(t)|^2_{L^2(\Omega)} + |u_t(t)|^2_{L^2(\Omega)}\right\}dt \le C\left\{|\nabla u|^2_{L^\infty(0,T;L^2(\Omega))} + |u_t|^2_{L^\infty(0,T;L^2(\Omega))}\right\}$$
$$+ C\left\{\int_{\Sigma_2}\{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + |g(u_t(x,t-\tau))|^2\}\,d\Sigma_2\right\} + C_T\|u\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}, \tag{2.81}$$

where the constant C does not depend on T and α, $0<\varepsilon<\frac{1}{2}$ are small enough arbitrary but fixed.

Proof of Proposition 2.7. As for the proof of Proposition 2.5 it is sufficient to establish (2.81) for smooth solution $u\in C(0,T;H^2(\Omega))\cap C^1(0,T;H^1_{\Gamma_1}(\Omega))$.
We multiply the first equation in (2.1) by $m.\nabla u$ and we integrate by parts over $Q=\Omega\times(0,T)$. This gives:

$$\int_Q u_{tt} m\nabla u\,dQ = [u_t(m\nabla u)]_0^T - \frac{1}{2}\int_{\Sigma_2} m.\nu|u_t|^2\,d\Sigma_2 + \frac{n}{2}\int_Q |u_t|^2\,dQ, \tag{2.82}$$

$$\int_Q \Delta u m \nabla u \, dQ = \frac{1}{2}\int_{\Sigma_1} \left|\frac{\partial u}{\partial \nu}\right|^2 m.\nu \, d\Sigma_1 + (\frac{n}{2}-1)\int_Q |\nabla u|^2 \, dQ$$
$$+ \int_{\Sigma_2} \frac{\partial u}{\partial \nu}(m.\nabla u)\, d\Sigma_2 - \frac{1}{2}\int_{\Sigma_2} |\nabla u|^2 m.\nu \, d\Sigma_2. \tag{2.83}$$

From (2.82), (2.83) and assumption (**H4**), we have

$$\frac{n}{2}\int_Q \{|u_t|^2 - |\nabla u|^2\}\, dQ + \int_Q |\nabla u|^2 \, dQ \leq C\left\{|\nabla u|^2_{L^\infty(0,T;L^2(\Omega))} + |u_t|^2_{L^\infty(0,T;L^2(\Omega))}\right\}$$
$$+ C\left\{\int_{\Sigma_2}\left[|u_t|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2 + |\nabla u|^2\right] d\Sigma_2\right\}. \tag{2.84}$$

Multiplying the first equation in (2.1) by u and integrating by parts, we get

$$\int_Q \{|\nabla u|^2 - |u_t|^2\}\, dQ \leq C\left\{|u|^2_{L^\infty(0,T;L^2(\Omega))} + |u_t|^2_{L^\infty(0,T;L^2(\Omega))} + \int_{\Sigma_2}\left[\frac{1}{\epsilon}\left|\frac{\partial u}{\partial \nu}\right|^2 + \epsilon |u|^2\right] d\Sigma_2\right\}, \tag{2.85}$$

where $\epsilon > 0$ can be taken arbitrary small. Combining (2.85) with (2.84) and applying trace theory, yields

$$\int_Q \{|u_t|^2 + |\nabla u|^2\}\, dQ \leq C\left\{|\nabla u|^2_{L^\infty(0,T;L^2(\Omega))} + |u_t|^2_{L^\infty(0,T;L^2(\Omega))} + \int_{\Sigma_2}\left[|u_t|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2 + |\nabla u|^2\right] d\Sigma_2\right\}. \tag{2.86}$$

But

$$|\nabla u|^2 = \left|\frac{\partial u}{\partial \nu}\right|^2 + |\nabla_\tau u|^2, \tag{2.87}$$

where $\nabla_\tau u$ is the tangential gradient of u.
Thus

$$\int_Q \{|u_t|^2 + |\nabla u|^2\}\, dQ \leq C\left\{|\nabla u|^2_{L^\infty(0,T;L^2(\Omega))} + |u_t|^2_{L^\infty(0,T;L^2(\Omega))} + \int_{\Sigma_2}\left[|u_t|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2 + |\nabla_\tau u|^2\right] d\Sigma_2\right\}. \tag{2.88}$$

From Lemma 7.2, inequality 7.5 in [44], we have

$$\int_\alpha^{T-\alpha}\int_{\Gamma_2} |\nabla_\tau u|^2 \, d\Gamma \, dt \leq C_{\varepsilon,\alpha}\left[\int_{\Sigma_2}\left[|u_t|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2\right] d\Sigma_2 + C_T \|u\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}\right]. \tag{2.89}$$

Applying (2.88) with $(0,T)$ replaced by $(\alpha, T-\alpha)$ and using (2.89) yields

$$\int_\alpha^{T-\alpha}\left\{|\nabla u(t)|^2_{L^2(\Omega)} + |u_t(t)|^2_{L^2(\Omega)}\right\} dt \leq C\left\{|\nabla u|^2_{L^\infty(0,T;L^2(\Omega))} + |u_t|^2_{L^\infty(0,T;L^2(\Omega))}\right\}$$
$$+ C\left\{\int_{\Sigma_2}\left[|u_t|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2\right] d\Sigma_2\right\} + C_T\|u\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}. \tag{2.90}$$

The sought-after estimate (2.81) follows from (2.90) and the fact that

$$\frac{\partial u}{\partial \nu}(x,t) = -\alpha_1 f(u_t(x,t)) - \alpha_2 g(u_t(x,t-\tau)) \quad \text{on } \Sigma_2.$$

□

Proposition 2.8. *Assume that the hypotheses* (**H1**) – (**H4**) *are fulfilled. Let* $T > 0$ *be sufficiently large. Then*

$$E(T) \le C_T \left[\int_{\Sigma_2} \{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + g(u_t(x,t-\tau))u_t(x,t-\tau)\}\, d\Sigma_2 \right]. \tag{2.91}$$

Proof of Proposition 2.8. Set

$$E(t) = E_s(t) + E_d(t),$$

where

$$E_s(t) = \frac{1}{2}\int_\Omega \left\{|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right\}\, dx,$$

and

$$E_d(t) = \frac{\xi}{2}\int_{\Gamma_2}\int_0^1 G(u_t(x,t-\tau\rho))\, d\rho\, d\Gamma.$$

From the mean value theorem, the monotonicity of g and change of variable, we have

$$\int_0^T E_d(t)\, dt \le C \int_0^T \int_{\Gamma_2} g(u_t(x,t-\tau))u_t(x,t-\tau)\ d\Gamma\, dt. \tag{2.92}$$

As for $E_s(t)$, we deduce from (2.33) and (2.81),

$$\begin{aligned}\int_\alpha^{T-\alpha} E_s(t)\, dt \le C_T[E(T) &+ \alpha_1 \int_{\Sigma_2} u_t(x,t) f(u_t(x,t))\, d\Sigma_2 + \alpha_2 \int_{\Sigma_2} g(u_t(x,t-\tau))u_t(x,t)\, d\Sigma_2 \\ &+ \tau^{-1}\xi \int_{\Sigma_2} G(u_t(x,t-\tau))\, d\Sigma_2 - \tau^{-1}\xi\int_{\Sigma_2} G(u_t(x,t))\, d\Sigma_2 \\ &+ \int_{\Sigma_2} \{|u_t(x,t)|^2 + \left|\frac{\partial u(x,t)}{\partial \nu}\right|^2\}\, d\Sigma_2 + \|u\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}].\end{aligned}$$

Therefore

$$\begin{aligned}\int_\alpha^{T-\alpha} E_s(t)\, dt \le C_T[E(T) &+ \int_{\Sigma_2}\{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + |g(u_t(x,t-\tau))|^2\}\, d\Sigma_2 \\ &+ \int_{\Sigma_2} G(u_t(x,t-\tau))\, d\Sigma_2 + \int_{\Sigma_2} G(u_t(x,t))\, d\Sigma_2 + \|u\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}].\end{aligned}$$

On the other hand, for a fixed α

$$\begin{aligned}\int_0^\alpha E_s(t)\, dt + \int_{T-\alpha}^T E_s(t)\, dt &\le 2\alpha E(0) \\ &\le 2\alpha\{E(T) + \int_{\Sigma_2}\{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + |g(u_t(x,t-\tau))|^2\}\, d\Sigma_2 \\ &+ \tau^{-1}\xi\int_{\Sigma_2} G(u_t(x,t-\tau))\, d\Sigma_2 - \tau^{-1}\xi \int_{\Sigma_2} G(u_t(x,t))\, d\Sigma_2\}.\end{aligned}$$

Thus,

$$\int_0^T E_s(t)\,dt \le C_T[E(T) + \int_{\Sigma_2} \{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + |g(u_t(x,t-\tau))|^2\}\,d\Sigma_2 \\ + \int_{\Sigma_2} G(u_t(x,t-\tau))\,d\Sigma_2 + \int_{\Sigma_2} G(u_t(x,t))\,d\Sigma_2 + \|u\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}]. \tag{2.93}$$

Combining (2.93) and (2.92) yields

$$\int_0^T E(t)\,dt \le C_T[E(T) + \int_{\Sigma_2} \{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + |g(u_t(x,t-\tau))|^2\}\,d\Sigma_2 \\ + \int_{\Sigma_2} G(u_t(x,t-\tau))\,d\Sigma_2 + \int_{\Sigma_2} G(u_t(x,t))\,d\Sigma_2 + \int_{\Sigma_2} g(u_t(x,t-\tau))u_t(x,t-\tau)\,d\Sigma_2 \\ + \|u\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}]. \tag{2.94}$$

Then

$$E(T) \le C_T[\int_{\Sigma_2} \{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + |g(u_t(x,t-\tau))|^2\}\,d\Sigma_2 \\ + \int_{\Sigma_2} G(u_t(x,t-\tau))\,d\Sigma_2 + \int_{\Sigma_2} G(u_t(x,t))\,d\Sigma_2 + \int_{\Sigma_2} g(u_t(x,t-\tau))u_t(x,t-\tau)\,d\Sigma_2] \\ + C_T\|u\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}.$$

Again from the mean value theorem and the monotonicity of g, we have

$$G(u_t(x,t-\tau)) \le g(u_t(x,t-\tau))u_t(x,t-\tau). \tag{2.95}$$

By using assumptions $(\mathbf{H2})(iii), (\mathbf{H2})(v)$ and (2.95), we obtain

$$E(T) \le C_T\left\{\int_{\Sigma_2} \{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + g(u_t(x,t-\tau))u_t(x,t-\tau)\}\,d\Sigma_2\right\} \\ + C_T\|u\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}. \tag{2.96}$$

To get the requested inequality (2.91) from (2.96), we need to absorb the lower order term $|u|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}$. To achieve this, we employ a compactness uniqueness argument. Suppose that (2.91) is not true. Then, there exists a sequence (u_n) of solution of problem (2.1) such that

$$E^n(T) > n\left[\int_{\Sigma_2} \{|u_{nt}(x,t)|^2 + |f(u_{nt}(x,t))|^2 + g(u_{nt}(x,t-\tau))u_{nt}(x,t-\tau)\}\,d\Sigma_2\right], \tag{2.97}$$

where $E^n(T)$ is the energy corresponding to (u_n) at the time T.
From (2.96),

$$E_n(T) \le C_T\left[\int_{\Sigma_2} \{|u_{nt}(x,t)|^2 + |f(u_{nt}(x,t))|^2 + g(u_{nt}(x,t-\tau))u_{nt}(x,t-\tau)\}\,d\Sigma_2\right] \\ + C_T\|u_n\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}. \tag{2.98}$$

(2.98) together with (2.97) implies

$$\begin{aligned} & n\left[\int_{\Sigma_2}\{|u_{nt}(x,t)|^2+|f(u_{nt}(x,t))|^2+g(u_{nt}(x,t-\tau))u_{nt}(x,t-\tau)\}\,d\Sigma_2\right] \\ & < C_T\left[\int_{\Sigma_2}\{|u_{nt}(x,t)|^2+|f(u_{nt}(x,t))|^2+g(u_{nt}(x,t-\tau))u_{nt}(x,t-\tau)\}\,d\Sigma_2\right] \\ & + C_T\|u_n\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}. \end{aligned}$$

That is

$$\begin{aligned} & \left[\int_{\Sigma_2}\{|u_{nt}(x,t)|^2+|f(u_{nt}(x,t))|^2+g(u_{nt}(x,t-\tau))u_{nt}(x,t-\tau)\}\,d\Sigma_2\right] \\ & < \frac{C_T}{(n-C_T)}\|u_n\|^2_{L^2[0,T;H^{1/2+\varepsilon}(\Omega)]}. \end{aligned} \tag{2.99}$$

Denote

$$c_n=\|u_n\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}\,,\widehat{u}_n=\frac{1}{c_n}u_n.$$

Thus

$$\|\widehat{u}_n\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}=1. \tag{2.100}$$

Dividing both sides of (2.99) by c_n^2 and using (2.100), we obtain

$$\int_{\Sigma_2}\{|\widehat{u}_{nt}(x,t)|^2+\frac{|f(u_{nt}(x,t))|^2}{c_n^2}+\frac{g(u_{nt}(x,t-\tau))u_{nt}(x,t-\tau)}{c_n^2}\}d\Sigma_2<\frac{C_T}{n-C_T}\quad for\,all\quad n>C_T. \tag{2.101}$$

Thus, (2.101) implies

$$\lim_{n\to+\infty}\int_{\Sigma_2}|\widehat{u}_{nt}(x,t)|^2\,d\Sigma_2=0, \tag{2.102}$$

$$\lim_{n\to+\infty}\int_{\Sigma_2}\frac{|f(u_{nt}(x,t))|^2}{c_n^2}\,d\Sigma_2=0, \tag{2.103}$$

$$\lim_{n\to+\infty}\int_{\Sigma_2}\frac{g(u_{nt}(x,t-\tau))u_{nt}(x,t-\tau)}{c_n^2}\,d\Sigma_2=0. \tag{2.104}$$

On the other hand, since each solution satisfies the energy estimate (2.98), we obtain after dividing both sides of such estimate by c_n^2 and invoking (2.100)

$$\frac{1}{c_n^2}E_n(T)\le C_T\{\int_{\Sigma_2}\{|\widehat{u}_{nt}(x,t)|^2+\frac{|f(u_{nt}(x,t))|^2}{c_n^2}+\frac{g(u_{nt}(x,t-\tau))u_{nt}(x,t-\tau)}{c_n^2}\}d\Sigma_2+1\}. \tag{2.105}$$

From (2.105), it follows that the sequence $(\widehat{u}_n)$ is bounded in $H^1(\Omega\times(0,T))$. Since $H^1(\Omega\times(0,T))$ is compactly embedded in $L^2(0,T;H^{1/2+\epsilon}(\Omega))$, there exists a subsequence still denoted by $(\widehat{u}_n)$ such that

$$\widehat{u}_n\to\widehat{u}\text{ strongly in }L^2(0,T;H^{1/2+\epsilon}(\Omega)).$$

Then from (2.100)

$$\|\widehat{u}\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} = 1. \tag{2.106}$$

Moreover, $\widehat{u}_n$ satisfies

$$\begin{cases} \widehat{u}_{ntt}(x,t) - \Delta \widehat{u}_n(x,t) = 0 & \text{in } \Omega \times (0,T), \\ \widehat{u}_n(x,t) = 0 & \text{on } \Sigma_1 = \Gamma_1 \times (0,T), \\ \frac{\partial \widehat{u}_n(x,t)}{\partial \nu} = -\alpha_1 \frac{f(u_{nt}(x,t))}{c_n} - \alpha_2 \frac{g(\widehat{u}_{nt}(x,t-\tau))}{c_n} & \text{on } \Sigma_2. \end{cases} \tag{2.107}$$

Passing to the limit in (2.107), and invoking (2.102)-(2.104), and assumption (**H2**)(*iii*), gives

$$\widehat{u}_t(x,t) = 0 \quad \text{on } \Sigma_2,$$

and

$$\begin{cases} \widehat{u}_{tt}(x,t) - \Delta \widehat{u}(x,t) = 0 & \text{in } \Omega \times (0,T), \\ \widehat{u}(x,t) = 0 & \text{on } \Sigma_1, \\ \frac{\partial \widehat{u}(x,t)}{\partial \nu} = 0 & \text{on } \Sigma_2. \end{cases}$$

Thus $v = \widehat{u}_t \in C(0,T; L^2(\Omega))$ satisfies

$$\begin{cases} v_{tt}(x,t) - \Delta v(x,t) = 0 & \text{in } \Omega \times (0,T), \\ v(x,t) = 0 & \text{on } \Sigma = \Gamma \times (0,T), \\ \frac{\partial v(x,t)}{\partial \nu} = 0 & \text{on } \Sigma_2. \end{cases}$$

From Holmogren's uniqueness Theorem ([54], Chap. 1, Theorem.8.2), we conclude that

$$v(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T).$$

This implies

$$\widehat{u}(x,t) = \widehat{u}(x).$$

Thus u verifies

$$\begin{cases} -\Delta \widehat{u}(x) = 0 & \text{in} \quad \Omega, \\ \widehat{u}(x) = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial \widehat{u}(x)}{\partial \nu} = 0 & \text{on} \quad \Gamma_2. \end{cases}$$

The solution of the above problem is $\widehat{u} = 0$, which contradicts (2.106). Then, the desired inequality (2.91) is proved. □

Proposition 2.9. *Assume* (**H1**) – (**H4**). *Then, the energy $E(T)$ of problem (2.1) satisfies*

$$E(T) + p(E(T)) \leq E(0),$$

where $p(.)$ is defined by (2.5), and $T > 0$ sufficiently large.

Proof of Proposition 2.9. Denote

$$\Sigma_3 = \{u_t \in L^2(\Sigma_2) : |u_t| \geq N \text{ a.e.}\},$$
$$\Sigma_4 = \Sigma_2 - \Sigma_3.$$

From assumptions (**H1**) and (**H2**), we have

$$\int_{\Sigma_3} \{u_t^2(x,t) + f^2(u_t(x,t))\}\, d\Sigma_3 \leq (a_1^{-1} M_3^{-1} a_2 + M_1) \int_{\Sigma_3} f(u_t(x,t)) u_t(x,t)\, d\Sigma_3. \tag{2.108}$$

On the other hand, from (2.4)

$$\int_{\Sigma_4} \{u_t^2(x,t) + f^2(u_t(x,t))\}\, d\Sigma_4 \leq \int_{\Sigma_4} h(f(u_t(x,t)) u_t(x,t))\, d\Sigma_4. \tag{2.109}$$

By Jensen's inequality,

$$\begin{aligned}\int_{\Sigma_4} h(f(u_t(x,t)) u_t(x,t))\, d\Sigma_4 &\leq mes\, \Sigma_2\, h\left(\frac{1}{mes\, \Sigma_2} \int_{\Sigma_2} f(u_t(x,t)) u_t(x,t)\, d\Sigma_2\right) \\ &= mes\, \Sigma_2\, \tilde{h}\left(\int_{\Sigma_2} f(u_t(x,t)) u_t(x,t)\, d\Sigma_2\right). \end{aligned} \tag{2.110}$$

Combining inequalities (2.108), (2.109), (2.110) with the result of Proposition 2.8 gives

$$\begin{aligned} E(T) \leq\ & C_T \left\{(a_1^{-1} M_3^{-1} a_2 + M_1) \int_{\Sigma_2} f(u_t(x,t)) u_t(x,t)\, d\Sigma_2 + \int_{\Sigma_2} u_t(x,t-\tau) g(u_t(x,t-\tau))\, d\Sigma_2\right\} \\ & + C_T\, mes\, \Sigma_2\, \tilde{h}\left(\int_{\Sigma_2} f(u_t(x,t)) u_t(x,t)\, d\Sigma_2\right) \\ \leq\ & C_T \left\{(a_1^{-1} M_3^{-1} a_2 + M_1) \int_{\Sigma_2} f(u_t(x,t)) u_t(x,t)\, d\Sigma_2 + \int_{\Sigma_2} u_t(x,t-\tau) g(u_t(x,t-\tau))\, d\Sigma_2\right\} \\ & + C_T\, mes\, \Sigma_2\, \tilde{h}\left(\int_{\Sigma_2} \{f(u_t(x,t)) u_t(x,t) + u_t(x,t-\tau) g(u_t(x,t-\tau))\}\, d\Sigma_2\right). \end{aligned}$$

Setting

$$K_1 = \frac{1}{C_T\, mes\, \Sigma_2};\ C' = \frac{a_1^{-1} M_3^{-1} a_2 + M_1}{mes\, \Sigma_2},$$

we obtain

$$\begin{aligned} E(T) \leq\ & \frac{C'}{K_1} \int_{\Sigma_2} f(u_t(x,t)) u_t(x,t)\, d\Sigma_2 + \frac{1}{K_1\, mes\, \Sigma_2} \int_{\Sigma_2} u_t(x,t-\tau) g(u_t(x,t-\tau))\, d\Sigma_2 \\ & + \frac{1}{K_1} \tilde{h}\left(\int_{\Sigma_2} \{f(u_t(x,t)) u_t(x,t) + u_t(x,t-\tau) g(u_t(x,t-\tau))\}\, d\Sigma_2\right), \end{aligned}$$

or

$$K_1\, E(T) \leq (cI + \tilde{h})\left(\int_{\Sigma_2} \{f(u_t(x,t)) u_t(x,t) + u_t(x,t-\tau) g(u_t(x,t-\tau))\}\, d\Sigma_2\right),$$

where
$$c = max\,\{C',\, \frac{1}{mes\,\Sigma_2}\}.$$
But from (2.29), we have
$$\int_{\Sigma_2} \{f(u_t(x,t))u_t(x,t) + u_t(x,t-\tau)g(u_t(x,t-\tau))\}\, d\Sigma_2 \le C_1^{-1}(E(0) - E(T)).$$
Therefore
$$(cI + \tilde{h})^{-1}\,(K\,E(T)) = p(E(T)) \le E(0) - E(T),$$
with $K = C_1K_1$.
Hence
$$p(E(T)) + E(T) \le E(0).$$
□

Completion of the proof of Theorem 2.2. Applying the result of Proposition 2.9 we obtain for $m = 0, 1, 2....$
$$p(E(m(T+1)) + E(m(T+1)) \le E(mT).$$
Thus, we are in a position to apply Lemma 3.3 in [43] with
$$s_m = E(mT),\ s_0 = E(0).$$
This yields
$$E(mT) \le S(m),\ m = 0, 1, 2, ...$$
where $S(t)$ is a solution of the differential equation 2.9. Let $t = mT + \tau$ and recall the evolution property, we obtain
$$E(t) \le E(mT) \le S(m) \le S(\tfrac{t-\tau}{T}) \le S(\tfrac{t}{T} - 1) \quad \text{for}\, t > T,$$
which completes the proof of Theorem 2.2.
Proof of Corollary 2.1. It is sufficient to construct a function h having the property (2.4).
From (2.10) and (2.11), we have
$$\begin{aligned}\int_{\Sigma_4} \{u_t^2(x,t) + f^2(u_t(x,t))\}\, d\Sigma_4 &\le (1+b_1^2)\int_{\Sigma_4} u_t^2(x,t)\, d\Sigma_4 \\ &\le (1+b_1^2)\int_{\Sigma_4} (b_2^{-1} f(u_t(x,t))u_t(x,t))^{\frac{2}{p+1}}\, d\Sigma_4 \\ &\le (1+b_1^2) b_2^{\frac{-2}{p+1}} \int_{\Sigma_4} (f(u_t(x,t))u_t(x,t))^{\frac{2}{p+1}}\, d\Sigma_4.\end{aligned}$$

We can take

$$h(s) = b_2^{\frac{-2}{p+1}}(1+b_1^2)s^m, \quad \text{where } m = \frac{2}{p+1} \leq 1.$$

Then

$$p(s) = (cI + \tilde{h})^{-1}(Ks);$$

Therefore

$$cp + d(b_1, b_2)s^m = Ks,$$

where d is a constant that depends on b_1 and b_2.
Also, recall that

$$q(s) = s - (I+p)^{-1}(s).$$

Since asymptotically (for s small) we have, for some constant $\alpha > 0$,

$$p(s) \sim \alpha s^{1/m} \quad \text{and therefore} \quad q(s) \sim \alpha s^{1/m},$$

by solving equation (2.9), we find

$$S(t)x = \begin{cases} c_1(t + c_2 x^{\frac{1-p}{2}})^{\frac{2}{1-p}} & \text{if } p > 1 \\ e^{-\alpha t}x & \text{if } p = 1, \end{cases} \tag{2.111}$$

where c_1 and c_2 depend only on α and p.
Finally, the estimates (2.12) and (2.13) follow from Theorem 2.2.

2.3 Stabilization of the wave equation with a nonlinear delay term in the internal feedback

2.3.1 Proof of Theorem 2.3

In order to be able to manage the boundary condition with the delay term and inspired from [83] and [62], we introduce the auxiliary variable:

$$y(x,\rho,t) = u_t(x,t-\tau\rho); \quad x \in \Omega, \rho \in (0,1), t > 0.$$

Then, the system (2.14) is equivalent to

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a(x)\left\{\alpha_1 f(u_t(x,t)) + \alpha_2 g(y(x,1,t))\right\} = 0 & \text{in } \Omega \times (0,+\infty), \\ y_t(x,\rho,t) + \tau^{-1} y_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ y(x,\rho,0) = f_0(x,-\tau\rho) & \text{in } \Omega \times (0,1), \\ y(x,0,t) = u_t(x,t) & \text{in } \Omega \times (0,+\infty). \end{cases} \tag{2.112}$$

To prove Theorem 2.3, we adopt the same approach as the one used to prove Theorem 2.1.

Proposition 2.10. *Assume that for the continuous f and g the hypotheses* (**H1**)(*iii*) *and* (**H2**)(*v*) *are fulfilled. If $u \in C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$ is a solution to problem (2.14), then the following energy identity holds for every $t > 0$*

$$\begin{aligned} F(T) - F(0) = &-\alpha_1 \int_Q a(x) u_t(x,t) f(u_t(x,t))\, dQ - \alpha_2 \int_Q a(x) g(u_t(x,t-\tau)) u_t(x,t)\, dQ \\ &- \frac{\tau^{-1}\mu}{2} \int_Q a(x)[G(u_t(x,t-\tau)) - G(u_t(x,t))]\, dQ, \end{aligned} \tag{2.113}$$

and consequently

$$F(T) - F(0) \le -C_1 \int_Q a(x) \left\{u_t(x,t) f(u_t(x,t)) + y(x,1,t) g(y(x,1,t))\right\} dQ, \tag{2.114}$$

where

$$C_1 = min\left\{\alpha_1 - \frac{\tau^{-1}\mu}{2} a_2 - \alpha_2 a_2, \frac{\tau^{-1}\mu}{2} a_1 + \alpha_2(a_1 - 1)\right\},$$

with μ as in (2.19).

Proof. By virtue of the Lemma 2.2 in [43], it is enough to prove the identity (2.113) for smooth solutions

$$u \in C(0,T;H^2(\Omega)) \cap C^1(0,T;H_0^1(\Omega)).$$

We multiply the first equation in (2.112) by u_t and integrate by parts over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega a(x)\left\{|\nabla u(x,t)|^2 + u_t^2(x,t)\right\}\,dx = -\alpha_1\int_\Omega a(x)u_t(x,t)f(u_t(x,t))\,dx - \alpha_2\int_\Omega a(x)g(y(x,1,t))u_t(x,t)\,dx. \tag{2.115}$$

We multiply the second equation in (2.112) by $\mu\, a(x)\, g(y(x,\rho,t))$ and integrate over $\Omega\times(0,1)$, we obtain

$$\int_\Omega a(x)\int_0^1 \{\mu\, y_t(x,\rho,t)g(y(x,\rho,t)) + \tau^{-1}\mu\, y_\rho(x,\rho,t)g(y(x,\rho,t))\}\,d\rho\,dx = 0.$$

We have

$$\frac{\partial G}{\partial t}(y(x,\rho,t)) = y_t(x,\rho,t).g(y(x,\rho,t)),$$
$$\frac{\partial G}{\partial \rho}(y(x,\rho,t)) = y_\rho(x,\rho,t).g(y(x,\rho,t)).$$

Consequently,

$$\mu\frac{d}{dt}\int_\Omega a(x)\int_0^1 G(y(x,\rho,t))\,d\rho\,dx = -\tau^{-1}\mu\int_\Omega a(x)\int_0^1 \frac{d}{d\rho}G(y(x,\rho,t))\,d\rho\,dx = -\tau^{-1}\mu\int_\Omega a(x)[G(y(x,1,t)) - G(y(x,0,t))]\,dx. \tag{2.116}$$

From (2.115) and(2.116) we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left\{|\nabla u(x,t)|^2 + u_t^2(x,t)\right\}\,dx + \frac{\mu}{2}\frac{d}{dt}\int_\Omega a(x)\int_0^1 G(y(x,\rho,t))\,d\rho\,dx = -\alpha_1\int_\Omega a(x)u_t(x,t)f(u_t(x,t))\,dx - \alpha_2\int_\Omega a(x)g(y(x,1,t))u_t(x,t)\,dx - \frac{\tau^{-1}\mu}{2}\int_\Omega a(x)G(y(x,1,t))\,dx + \frac{\tau^{-1}\mu}{2}\int_\Omega a(x)G(u_t(x,t))\,dx. \tag{2.117}$$

We integrate both sides of (2.117) over $(0,T)$, we obtain

$$F(T) - F(0) = -\alpha_1\int_Q a(x)u_t(x,t)f(u_t(x,t))\,dQ - \alpha_2\int_Q a(x)g(u_t(x,t-\tau))u_t(x,t)\,dQ - \frac{\tau^{-1}\mu}{2}\int_Q a(x)[G(u_t(x,t-\tau)) - G(u_t(x,t))]\,dQ. \tag{2.118}$$

From 2.25 and assumption $(\mathbf{H2})(v)$, we have

$$G(u_t(x,t)) \le a_2u_t(x,t)f(u_t(x,t)),$$
$$-G(y(x,1,t)) \le -a_1y(x,1,t)g(y(x,1,t)).$$

Hence

$$F(T)-F(0)\leq -(\alpha_1-\frac{\tau^{-1}\mu a_2}{2})\int_Q a(x)u_t(x,t)f(u_t(x,t))\,dQ-\alpha_2\int_Q a(x)g(y(x,1,t))u_t(x,t)\,dQ$$
$$\frac{-\tau^{-1}\mu a_1}{2}\int_Q a(x)y(x,1,t)g(y(x,1,t))\,dQ. \tag{2.119}$$

Let G^* be the conjugate function of the concave function G

$$G^*(s)=\sup_{t\in\mathbb{R}^+}(st-G(t)).$$

Then G^* is the Legendre transform of G, which is given by (Arnold ([8] p. 61-62))

$$G^*(s)=s(G')^{-1}(s)-G[(G')^{-1}(s)] \text{ for all } s\geq 0.$$

and satisfies

$$s.t\leq G^*(s)+G(t) \text{ for all } s,t\geq 0. \tag{2.120}$$

But from the definition of G, we have

$$G^*(s)=sg^{-1}(s)-G[(g^{-1}(s)].$$

Hence

$$\begin{aligned}G^*(|g(y(x,1,t))|)&=g(y(x,1,t))y(x,1,t)-G(y(x,1,t))\\&\leq(1-a_1)y(x,1,t)g(y(x,1,t)).\end{aligned} \tag{2.121}$$

Making use of (2.119) and (2.120), we get

$$\begin{aligned}F(T)-F(0)\leq&-(\alpha_1-\frac{\tau^{-1}\mu a_2}{2})\int_Q a(x)u_t(x,t)f(u_t(x,t))\,dQ\\&-\frac{\tau^{-1}\mu a_1}{2}\int_Q a(x)y(x,1,t)g(y(x,1,t))\,dQ\\&+\alpha_2\int_Q a(x)(G(|u_t(x,t)|)+G^*(|g(y(x,1,t))|))\,dQ.\end{aligned}$$

(2.121) together with assumption (**H2**)(v) implies

$$\begin{aligned}F(T)-F(0)\leq&-(\alpha_1-\frac{\tau^{-1}\mu a_2}{2}-a_2\alpha_2)\int_Q a(x)u_t(x,t)f(u_t(x,t))\,dQ\\&-(\frac{\tau^{-1}\mu a_1}{2}-\alpha_2(1-a_1))\int_Q a(x)y(x,1,t)g(y(x,1,t))\,dQ.\end{aligned}$$

Therefore

$$F(T)-F(0)\leq -C_1\int_Q\{u_t(x,t)f(u_t(x,t))+y(x,1,t)g(y(x,1,t))\}\,dQ, \tag{2.122}$$

where

$$C_1 = \min\left\{(\alpha_1 - \frac{\tau^{-1}\mu a_2}{2} - \alpha_2 a_2), (\frac{\tau^{-1}\mu a_1}{2} + \alpha_2(a_1 - 1))\right\},$$

with μ as in (2.19).

□

Theorem 2.11. *Assume that*

* *The function g is Lipschitz continuous on $\mathbb{R}$, with L as a Lipschitz constant.* (2.123)
* *$f(s_1) - f(s_2) \geq r(s_1 - s_2)$ for all $s_1 - s_2 \geq 0$ and fixed $r > 0$.* (2.124)

Then, for each $(u_0, u_1, f_0) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega; L^2(0, \tau))$, problem (2.14) has a unique solution

$$u \in C(0, \infty; H_0^1(\Omega)) \cap C^1(0, \infty; L^2(\Omega)).$$

Proof of Theorem 2.11. This follows from nonlinear semigroup theory.
Denote by $\widehat{\mathcal{H}}$ the Hilbert space.

$$\widehat{\mathcal{H}} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega; L^2(0, 1)),$$

where

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \quad \text{on} \quad \Gamma\}.$$

Next define

$$\tilde{\mathcal{A}} : D(\tilde{\mathcal{A}}) \subset \widehat{\mathcal{H}} \to \widehat{\mathcal{H}},$$

$$\tilde{\mathcal{A}}\begin{pmatrix} \zeta \\ \eta \\ \theta \end{pmatrix} = \begin{pmatrix} \eta \\ \Delta\zeta - a\{\alpha_1 f(\eta) + \alpha_2 g(\theta(.;1))\} \\ -\tau^{-1}\theta_\rho \end{pmatrix}, \tag{2.125}$$

with

$$D(\tilde{\mathcal{A}}) = \left\{(\zeta, \eta, \theta) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega; H^1(0, 1));\ \eta = \theta(x, 0) \text{ in } \Omega\right\}.$$

Then we can rewrite (2.112) as an abstract Cauchy problem on $\widehat{\mathcal{H}}$

$$\begin{cases} \dfrac{dW}{dt}(t) = \tilde{\mathcal{A}}(W(t)), \\ W(0) = W_0, \end{cases} \tag{2.126}$$

where

$$W(t) = (u(x,t), u_t(x,t), y(x,\rho,t))^T,\ W'(t) = (u_t(x,t), u_{tt}(x,t), y_t(x,\rho,t))^T, W_0 = (u_0, u_1, f_0)^T.$$

We will show that the operator $\tilde{\mathcal{A}}$ defined by (2.125) is maximal dissipative on the Hilbert space $\widehat{\mathcal{H}}$ equipped with the inner product

$$\left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \end{pmatrix}, \begin{pmatrix} \tilde{\zeta} \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix} \right\rangle_{\widehat{\mathcal{H}}} = \int_\Omega \left\{ \nabla\zeta(x) \,.\, \nabla\tilde{\zeta}(x) + \eta(x) \,.\, \tilde{\eta}(x) \right\} dx + \delta \int_\Omega a(x) \int_0^1 \theta(x,\rho) \,.\, \tilde{\theta}(x,\rho)\, d\rho\, dx,$$

with

$$\tau\alpha_2 L < \delta < 2\tau(\alpha_1 r - \frac{\alpha_2 L}{2}), \tag{2.127}$$

$$\frac{L}{r} < \frac{\alpha_1}{\alpha_2}. \tag{2.128}$$

First, we prove that $\tilde{\mathcal{A}}$ is dissipative.
Let $U = (\zeta, \eta, \theta)^T \in D(\tilde{\mathcal{A}})$ and $V = (\tilde{\zeta}, \tilde{\eta}, \tilde{\theta})^T \in D(\tilde{\mathcal{A}})$. Then

$$\begin{aligned} \left\langle \tilde{\mathcal{A}}U - \tilde{\mathcal{A}}V; U - V \right\rangle_{\widehat{\mathcal{H}}} &= \int_\Omega \nabla(\eta(x) - \tilde{\eta}(x))\, \nabla(\zeta(x) - \tilde{\zeta}(x))\, dx \\ &- \int_\Omega \Delta(\zeta(x) - \tilde{\zeta}(x))\ (\eta(x) - \tilde{\eta}(x))\, dx \\ &- \alpha_1 \int_\Omega a(x)(f(\eta(x)) - f(\tilde{\eta}(x)))\ (\eta(x) - \tilde{\eta}(x))\, dx \\ &- \alpha_2 \int_\Omega a(x)(g(\theta(x,1)) - g(\tilde{\theta}(x,1)))(\eta(x) - \tilde{\eta}(x))\, dx \\ &- \delta\tau^{-1} \int_\Omega a(x) \int_0^1 (\theta_\rho(x,\rho) - \tilde{\theta}_\rho(x,\rho)(\theta(x,\rho) - \tilde{\theta}(x,\rho))\, d\rho\, dx. \end{aligned}$$

From Green's second theorem, we have

$$\begin{aligned} \left\langle \tilde{\mathcal{A}}U - \tilde{\mathcal{A}}V; U - V \right\rangle_{\widehat{\mathcal{H}}} &= -\alpha_1 \int_\Omega a(x)(f(\eta(x)) - f(\tilde{\eta}(x)))(\eta(x) - \tilde{\eta}(x))\, dx \\ &- \alpha_2 \int_\Omega a(x)(g(\theta(x,1)) - g(\tilde{\theta}(x,1)))(\eta(x) - \tilde{\eta}(x))\, dx \\ &- \delta\tau^{-1} \int_\Omega a(x) \int_0^1 (\theta_\rho(x,\rho) - \tilde{\theta}_\rho(x,\rho)(\theta(x,\rho) - \tilde{\theta}(x,\rho))\, d\rho\, dx. \end{aligned}$$

Integrating by parts in ρ, we obtain

$$\begin{aligned} \int_\Omega a(x) \int_0^1 (\theta_\rho(x,\rho) - \tilde{\theta}_\rho(x,\rho)(\theta(x,\rho) - \tilde{\theta}(x,\rho))\, d\rho\, dx &= \frac{1}{2} \int_\Omega a(x)(\theta(x,1) - \tilde{\theta}(x,1))^2\, dx \\ &- \frac{1}{2} \int_\Omega a(x)(\theta(x,0) - \tilde{\theta}(x,0))^2\, dx. \end{aligned}$$

Therefore

$$\begin{aligned}\left\langle \tilde{\mathcal{A}}U - \tilde{\mathcal{A}}V; U-V\right\rangle_{\widehat{\mathcal{H}}} &= -\alpha_1 \int_\Omega a(x)(f(\eta(x)) - f(\tilde{\eta}(x)))(\eta(x)-\tilde{\eta}(x))\,dx \\ &- \alpha_2 \int_\Omega a(x)(g(\theta(x,1)) - g(\tilde{\theta}(x,1))) \times (\eta(x)-\tilde{\eta}(x))\,dx \\ &- \frac{\delta\tau^{-1}}{2}\int_\Omega a(x)(\theta(x,1)-\tilde{\theta}(x,1))^2\,dx + \frac{\delta\tau^{-1}}{2}\int_\Omega a(x)(\theta(x,0)-\tilde{\theta}(x,0))^2\,dx.\end{aligned}$$

By using assumptions (2.39), (2.40) and the Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned}\left\langle \tilde{\mathcal{A}}U - \tilde{\mathcal{A}}V; U-V\right\rangle_{\widehat{\mathcal{H}}} &\le -\alpha_1 r \int_\Omega a(x)|\eta(x)-\tilde{\eta}(x)|^2\,dx + \frac{\alpha_2 L}{2}\int_\Omega a(x)|\theta(x,1)-\tilde{\theta}(x,1)|^2\,dx \\ &+ \frac{\alpha_2 L}{2}\int_\Omega a(x)|\eta(x)-\tilde{\eta}(x)|^2\,dx - \frac{\delta\tau^{-1}}{2}\int_\Omega a(x)|\theta(x,1)-\tilde{\theta}(x,1)|^2\,dx \\ &+ \frac{\delta\tau^{-1}}{2}\int_\Omega a(x)|\eta(x)-\tilde{\eta}(x)|^2\,dx.\end{aligned}$$

Therefore

$$\begin{aligned}\left\langle \tilde{\mathcal{A}}U - \tilde{\mathcal{A}}V; U-V\right\rangle_{\widehat{\mathcal{H}}} &\le -\left(\alpha_1 r - \frac{\alpha_2 L}{2} - \frac{\delta\tau^{-1}}{2}\right)\int_\Omega a(x)|\eta(x)-\tilde{\eta}(x)|^2\,dx \\ &- \left(-\frac{\alpha_2 L}{2} + \frac{\delta\tau^{-1}}{2}\right)\int_\Omega a(x)|\theta(x,1)-\tilde{\theta}(x,1)|^2\,dx.\end{aligned}$$

From (2.127), we conclude that

$$\left\langle \tilde{\mathcal{A}}U - \tilde{\mathcal{A}}V; U-V\right\rangle_{\widehat{\mathcal{H}}} \le 0.$$

Thus $\tilde{\mathcal{A}}$ is dissipative.
In order to establish maximality, we need to prove the range condition

$$range(\lambda I - \tilde{\mathcal{A}}) = \widehat{\mathcal{H}} \ \text{ for a fixed } \lambda.$$

Let $(k,l,m)^T \in \widehat{\mathcal{H}}$ we seek a $w = (\zeta,\eta,\theta)^T \in D(\tilde{\mathcal{A}})$ solution of

$$(\lambda I - \tilde{\mathcal{A}})w = (k,l,m)^T,$$

or equivalently

$$\lambda\,\zeta - \eta = k, \tag{2.129}$$

$$\lambda\,\eta - \Delta\zeta + \alpha_1\, a\, f(\eta) + \alpha_2\, a\, g(\theta(x,1)) = l, \tag{2.130}$$

$$\lambda\,\theta + \tau^{-1}\theta_\rho = m. \tag{2.131}$$

Hence

$$\zeta = \frac{1}{\lambda}(\eta + k). \tag{2.132}$$

From (2.131) and the last line of (2.112) we have

$$\begin{cases} \theta_\rho(x,\rho) = -\tau\lambda\theta(x,\rho) + \tau m(x,\rho), \ \rho \in (0,1) \\ \theta(x,0) = \eta(x) \end{cases}$$

The unique solution for the above initial value problem is given by

$$\theta(x,\rho) = \eta(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho m(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma, \qquad x \in \Omega,\ \rho \in (0,1),$$

and in particular

$$\theta(x,1) = \eta(x)e^{-\lambda\tau} + \mathcal{Z}_0, \tag{2.133}$$

where

$$\mathcal{Z}_0 = \tau e^{-\lambda\tau}\int_0^1 m(x,\sigma)e^{\lambda\tau\sigma}d\sigma, \qquad x \in \Omega.$$

Insertion (2.132) and (2.133) into problem (2.130) yields

$$\lambda\eta - \frac{1}{\lambda}\Delta\eta + \alpha_1\,a\,f(\eta) + \alpha_2\,a\,g(\eta\,e^{-\lambda\tau} + \mathcal{Z}_0) = l + \frac{1}{\lambda}\Delta k \in L^2(\Omega).$$

Set

$$\tilde{\mathcal{T}}\eta = -\frac{1}{\lambda}\Delta\eta + \alpha_1\,a\,f(\eta) + \alpha_2\,a\,g(\eta\,e^{-\lambda\tau} + \mathcal{Z}_0) + \lambda\eta. \tag{2.134}$$

Lemma 2.3. *The operator $\tilde{\mathcal{T}}$ given by (2.134) is surjective from $L^2(\Omega)$ onto $L^2(\Omega)$*

Proof. Let

$$\tilde{\mathcal{T}}\eta = \mathcal{B}\eta + \mathcal{C}\eta\,,\ \eta \in L^2(\Omega),$$

where $\mathcal{B} : L^2(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$\mathcal{B}\eta = \alpha_1\,a\,f(\eta) + \alpha_2\,a\,g(\eta\,e^{-\lambda\tau} + \mathcal{Z}_0),$$

and $\mathcal{C} : D(\mathcal{C}) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$\mathcal{C}\eta = \lambda\eta - \frac{1}{\lambda}\Delta\eta.$$

According to Barbu [11] (Corollary 1.3, p.48), in order to establish surjectivity of the operator $\tilde{\mathcal{T}}$, it is sufficient to prove that $\mathcal{B}$ is monotone, hemicontinuous, $\mathcal{C}$ is maximal monotone and $\mathcal{B}+\mathcal{C}$ is coercive.
Monotonicity of $\mathcal{B}$.
Let $\eta, v \in L^2(\Omega)$. Then

$$\begin{aligned}\langle(-\mathcal{B})\eta - (-\mathcal{B})v, \eta - v\rangle_{L^2(\Omega)} &= -\alpha_1\int_\Omega a(x)(f(\eta) - f(v))(\eta - v)\,dx \\ &\quad - \alpha_2\int_\Omega a(x)(g(\eta e^{-\lambda\tau} + \mathcal{Z}_0) - g(v\,e^{-\lambda\tau} + \mathcal{Z}_0))(\eta - v)\,dx \\ &\le -\alpha_1 r\int_\Omega a(x)|\eta - v|^2\,dx + \alpha_2 L e^{-\lambda\tau}\int_\Omega a(x)|\eta - v|^2\,dx \\ &\le -(\alpha_1 r - \alpha_2 L e^{-\lambda\tau})\int_\Omega a(x)|\eta - v|^2\,dx.\end{aligned}$$

By (2.128), we conclude that

$$\langle (-\mathcal{B})\eta - (-\mathcal{B})v, \eta - v\rangle_{L^2(\Omega)} \leq 0.$$

Thus $(-\mathcal{B})$ is dissipative, then $\mathcal{B}$ is monotone.
Hemicontinuity of $\mathcal{B}$.
Let $\eta, v, w \in L^2(\Omega)$. We will prove that the function

$$t \to < \mathcal{B}(\eta + tv), w >_{L^2(\Omega)},$$

is continuous. Indeed, we have

$$\begin{aligned}
&|\langle \mathcal{B}(\eta + tv) - \mathcal{B}(\eta + t_0 v), w\rangle_{L^2(\Omega)}| \\
&= \left|\alpha_1 \int_\Omega a(x)(f(\eta + tv) - f(\eta + t_0 v))\, w\, dx \right.\\
&\left. + \alpha_2 \int_\Omega a(x)(g((\eta + tv)e^{-\lambda\tau} + \mathcal{Z}_0) - g((\eta + t_0 v)e^{-\lambda\tau} + \mathcal{Z}_0))\, w\, dx\right| \\
&\leq \left|\alpha_1 \int_\Omega a(x)(f(\eta + tv) - f(\eta + t_0 v))\, w\, dx\right| \\
&+ \left|\alpha_2 \int_\Omega a(x)(g((\eta + tv)e^{-\lambda\tau} + \mathcal{Z}_0) - g((\eta + t_0 v)e^{-\lambda\tau} + \mathcal{Z}_0))\, w\, dx\right| \\
&\leq c\|w\|_{L^2(\Omega)}\|a\|_\infty \Big\{ \|f(\eta + tv) - f(\eta + t_0 v)\|_{L^2(\Omega)} \\
&+ \|g((\eta + tv)e^{-\lambda\tau} + \mathcal{Z}_0) - g((\eta + t_0 v)e^{-\lambda\tau} + \mathcal{Z}_0)\|_{L^2(\Omega)} \Big\}.
\end{aligned}$$

From assumption (2.123), we have

$$\begin{aligned}
|\langle \mathcal{B}(\eta + tv) - \mathcal{B}(\eta + t_0 v), w\rangle_{L^2(\Omega)}| \leq c\|w\|_{L^2(\Omega)}\|a\|_\infty \Big\{ &\|f(\eta + tv) - f(\eta + t_0 v)\|_{L^2(\Omega)} \\
&+ L\|(t - t_0)v\|_{L^2(\Omega)} \Big\}.
\end{aligned}$$

The continuity of f allows us to deduce that

$$|\langle \mathcal{B}(\eta + tv) - \mathcal{B}(\eta + t_0 v), w\rangle_{L^2(\Omega)}| < \tilde{\epsilon},$$

for $|t - t_0| < \widetilde{\delta}$. This proves the continuity of the function $t \longmapsto \langle \mathcal{B}(\eta + tv), w\rangle$.
Maximal monotonicity of $\mathcal{C}$.
For $\eta, v \in V$, we have

$$\begin{aligned}
\langle \mathcal{C}\eta - \mathcal{C}v, \eta - v\rangle_{L^2(\Omega)} &= \langle \lambda\eta - \frac{1}{\lambda}\Delta\eta - \lambda v + \frac{1}{\lambda}\Delta v, \eta - v\rangle_{L^2(\Omega)} \\
&= \lambda\|\eta - v\|^2_{L^2(\Omega)} + \frac{1}{\lambda}\|\nabla(\eta - v)\|^2_{L^2(\Omega)} \geq 0. \qquad (2.135)
\end{aligned}$$

According to Barbu [11] (Theorem 1.3, p.40), the operator $\mathcal{C}$ continuous and monotone then it is maximal monotone.
Coercivity of $\mathcal{B}+\mathcal{C}$.

$$\begin{aligned}
\langle \tilde{\mathcal{T}}\eta, \eta\rangle_{L^2(\Omega)} &= \langle \lambda\eta - \frac{1}{\lambda}\Delta\eta + \alpha_1\, a\, f(\eta) + \alpha_2\, a\, g(\eta\, e^{-\lambda\tau} + \mathcal{Z}_0)\,\eta\rangle_{L^2(\Omega)} \\
&= \lambda\int_\Omega |\eta|^2\,dx + \frac{1}{\lambda}\int_\Omega |\nabla\eta|^2\,dx + \alpha_1\int_\Omega a\, f(\eta)\,\eta\,dx + \alpha_2\int_\Omega a\, g(\eta\, e^{-\lambda\tau} + \mathcal{Z}_0)\,\eta\,dx \\
&\geq \lambda\int_\Omega |\eta|^2\,dx + \frac{1}{c_p\lambda}\int_\Omega |\eta|^2\,dx + \alpha_1\int_\Omega a\, f(\eta)\,\eta\,dx + \alpha_2\int_\Omega a\, g(\eta\, e^{-\lambda\tau} + \mathcal{Z}_0)\,\eta\,dx,
\end{aligned}$$

where $c_p > 0$ is called Poincaré constant.
By using (2.123) and (2.124), we obtain

$$\begin{aligned}
\langle \tilde{\mathcal{T}}\eta, \eta\rangle_{L^2(\Omega)} &\geq \lambda\int_\Omega |\eta|^2\,dx + \frac{1}{c_p\lambda}\int_\Omega |\eta|^2\,dx + \alpha_1 r\|a\|_\infty\int_\Omega |\eta|^2\,dx \\
&\quad - \frac{\alpha_2\, L}{2}\|a\|_\infty\int_\Omega |\eta|^2\,dx - \frac{\alpha_2\, L\, c}{2}\|a\|_\infty\int_\Omega |\eta|^2\,dx \\
&\geq \left(\lambda + \frac{1}{c_p\lambda} + \alpha_1 r\|a\|_\infty - \frac{\alpha_2\, L}{2}\|a\|_\infty - \frac{\alpha_2\, L\, c}{2}\|a\|_\infty\right)\int_\Omega |\eta|^2\,dx,
\end{aligned}$$

where $\|\mathcal{Z}_0\|^2_{L^2(\Omega)} \leq c$.
Therefore

$$\langle \tilde{\mathcal{T}}\eta, \eta\rangle_{L^2(\Omega)} \geq C\|\eta\|^2_{L^2(\Omega)},$$

for some constant $C > 0$ as long as $\lambda > \frac{\alpha_2\, L}{2}\|a\|_\infty(1+c)$. □

The operator $\tilde{\mathcal{T}}$ is surjective from $L^2(\Omega)$ onto $L^2(\Omega)$. One easily shows that (ζ, η, θ) is also in $D(\tilde{\mathcal{A}})$. Indeed from (2.130), we have

$$-\Delta\zeta + \alpha_1\, a\, f(\eta) + \alpha_2\, a\, g(\theta(x,1)) = l - \lambda\,\eta \in L^2(\Omega).$$

Thus $\Delta\zeta \in L^2(\Omega)$. Then $\zeta \in H^2(\Omega)\cap H^1_0(\Omega)$. By using (2.129) and (2.131), we have $\eta \in H^1_0(\Omega)$ and $\theta \in L^2(\Omega, H^1(0,1))$.
From nonlinear semigroup theory and the density of $D(\tilde{\mathcal{A}})$ in $\widehat{\mathcal{H}}$, we obtain if $(u_0, u_1, f_0) \in \widehat{\mathcal{H}}$ unique existence of the solution

$$u \in C(0,T; H^1_0(\Omega)) \cap C^1(0,T; L^2(\Omega)),$$

for any finite $T > 0$. □

We consider next the following approximation of system (2.14) with $l \longrightarrow +\infty$ as the parameter of approximation,

$$\begin{cases}
u_{ltt}(x,t) - \Delta u_l(x,t) + a(x)\left\{\alpha_1 f_l(u_{lt}(x,t)) + \alpha_2 g_l(u_{lt}(x,t-\tau))\right\} = 0 & \text{in } \Omega\times(0,+\infty), \\
u_l(x,0) = u_0(x), u_{lt}(x,0) = u_1(x) & \text{in } \Omega, \\
u_l(x,t) = 0 & \text{on } \Gamma\times(0,+\infty), \\
u_{lt}(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times(0,\tau),
\end{cases} \tag{2.136}$$

where

$$f_l(u_{lt}(x,t)) = f(u_{lt}(x,t)) + \frac{1}{l}u_{lt}(x,t),$$

and the g_l are defined by

$$g_l(s) = \begin{cases} g(s), \ |s| \leq l \\ g(l), \ s \geq l \\ g(-l), \ s \leq -l \end{cases} \tag{2.137}$$

Notice that for each value of the parameter l, the functions g_l and f_l satisfy the hypotheses of Theorem 2.11.
Thus, there exists a solution (u_l, u_{lt}) of (2.136) such that

$$u_l \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)),$$

and

$$u_{lt} \in L^2(0,\infty; L^2(\Omega)) \ , \ f_l(u_{lt}) \in L^2(0,\infty; L^2(\Omega)) \ , \ g_l(u_{lt}(x,t-\tau)) \in L^2(0,\infty; L^2(\Omega)). \tag{2.138}$$

We prove that we can extract a subsequence from the above sequence of solutions u_l, that has a limit which is a solution of the original problem (2.14).
To accomplish this we need the following.

Lemma 2.4. *Under the assumptions of Theorem 2.3. We have as $l \longrightarrow +\infty$ and $u_{lt}(x,t-\tau) \longrightarrow u_t(x,t-\tau)$ weakly in $H^1(\Omega)$*

$$g_l(u_{lt}(x,t-\tau)) \longrightarrow g(u_t(x,t-\tau)) \qquad \text{in } L^2(\Omega). \tag{2.139}$$

Proof. We write

$$\int_\Omega |g_l(u_{lt}(x,t-\tau)) - g(u_{lt}(x,t-\tau))|^2 \, dx \leq 2\left[\int_{\Omega_l} |g(u_{lt}(x,t-\tau))|^2 \, d\Omega_l + \int_{\Omega_l} |g(l)|^2 + |g(-l)|^2 \, d\Omega_l\right], \tag{2.140}$$

where

$$\Omega_l = \{x \in \Omega : |u_{lt}(x,t-\tau)| \geq l\}.$$

Then, by Sobolev's Embeddings

$$H^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega), \qquad n > 2, \tag{2.141}$$

$$H^1(\Omega) \subset L^p(\Omega), \quad 1 \leq p < \infty, \qquad n = 2. \tag{2.142}$$

We have for $n > 2$

$$\left(\int_{\Omega_l} l^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \leq \int_\Omega \left(|(u_{lt}(x,t-\tau))|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}},$$

therefore

$$mes\, \Omega_l \leq \int_\Omega \left(|(u_{lt}(x,t-\tau))|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} l^{\frac{-2n}{n-2}}. \tag{2.143}$$

Analogously, for $n = 2$ the above inequality is valid with any exponent for l.
By assumptions (**H2**) (iii), (**H2**)(v) and by (2.143),

$$\int_{\Omega_l} |g(u_{lt}(x,t-\tau))|^2\, d\Omega_l \le c \int_{\Omega_l} |u_{lt}(x,t-\tau)|^2\, d\Omega_l$$

$$\le c \left[\int_{\Omega_l} |u_{lt}(x,t-\tau)|^{\frac{2n}{n-2}}\right]^{\frac{n-2}{n}} (mes\,\Omega_l)^{\frac{n-(n-2)}{n}} \longrightarrow_{l\longrightarrow\infty} 0, \tag{2.144}$$

since $n-2<n$.

$$\int_{\Omega_l} |g(l)|^2\, d\Omega_l \le c\, l^2\, mes\,\Omega_l \le c \left[\int_{\Omega_l} |u_{lt}(x,t-\tau)|^{\frac{2n}{n-2}}\right]^{\frac{n-2}{2n}} l^{2-\frac{2n}{n-2}} \longrightarrow_{l\to\infty} 0. \tag{2.145}$$

Combining the results of (2.140),(2.144) and (2.145) gives (2.139). □

We are in a position to apply energy estimate (2.114) for each $t > 0$, we obtain

$$F_l(t) + C\left\{\int_0^t \int_\Omega a(x)\{(f(u_{lt}(x,t)) + \frac{1}{l}u_{lt}(x,t))u_{lt}(x,t)) + u_{lt}(x,t-\tau)g_l(u_{lt}(x,t-\tau))\}\, dx\, ds\right\}$$
$$\le F_l(0), \tag{2.146}$$

where $F_l(t)$ is defined by (2.18) with u (respectively G) replaced by u_l (respectively G_l).

$$F_l(0) \le C\left(|u_0|_{H^1(\Omega)}, |u_1|_{L^2(\Omega)}, |f_0|_{L^2(\Omega\times(0,\tau))}\right). \tag{2.147}$$

From hypotheses (**H1**), (**H2**), (2.146) and (2.147), we infer that

$$|u_{lt}|_{L^2(0,T;L^2(\Omega))} \le C\left(|u_0|_{H^1(\Omega)}, |u_1|_{L^2(\Omega)}, |f_0|_{L^2(\Omega\times(0,\tau))}\right) \tag{2.148}$$

$$|u_l|_{C(0,T;H_0^1(\Omega))} + |u_{lt}|_{C(0,T;L^2(\Omega))} \le C. \tag{2.149}$$

Therefore, on a subsequence we have

$$u_l \longrightarrow u \quad \text{weakly in}\, H^1(\Omega\times(0,T)), \tag{2.150}$$

$$u_{lt} \longrightarrow u_t \quad \text{weakly in}\, L^2(0,T;L^2(\Omega)). \tag{2.151}$$

Hypotheses (**H1**), (**H2**) together with compactness of the embeddings (2.141)-(2.142) and (2.151) together also give.

$$g(u_{lt}(x,t-\tau)) \longrightarrow g(u_t(x,t-\tau)) \ \text{ in}\, L^2(0,T;L^2(\Omega)), \tag{2.152}$$

$$f(u_{lt}) \longrightarrow f^* \in L^2(0,T;L^2(\Omega)) \ \text{ weakly in }\ L^2(0,T;L^2(\Omega)) \ \text{ for some }\ f^* \in L^2(0,T;L^2(\Omega)). \tag{2.153}$$

Let (u_l, u_m) be the solutions of (2.136) corresponding to the parameter l and m. Then from the energy identity

$$|\nabla(u_l - u_m)(t)|^2_{L^2(\Omega)} + |(u_{lt} - u_{mt})(t)|^2_{L^2(\Omega)} + \alpha_1 \int_0^t \int_\Omega a(x)(f(u_{lt}) - f(u_{mt}))(u_{lt} - u_{mt})\, dx\, ds$$
$$\le (\frac{1}{l} + \frac{1}{m})\alpha_1 \int_0^t \int_\Omega a(x)|u_{lt}|^2\, dx \ + (\frac{1}{l} + \frac{1}{m})\alpha_1 \int_0^t \int_\Omega a(x)|u_{mt}|^2\, dx\, ds$$
$$+ \alpha_2 \int_0^t \int_\Omega a(x)|g_l(u_{lt}(x,s-\tau)) - g_m(u_{mt}(x,s-\tau))|\, |u_{lt} - u_{mt}|\, dx\, ds. \tag{2.154}$$

The result (2.139) of Lemma 2.4 together with (2.150), (2.151) and (2.148) imply the convergence to zero (when $l, m \to \infty$) of the last two terms on the right hand side (2.154).
Similarly, by (2.151) the first two terms on the right hand side of (2.154) converge to zero.
Thus, we have obtained

$$u_l \longrightarrow u \in C(0,T; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega)), \tag{2.155}$$

and

$$\lim_{l,m\to\infty} \int_0^T \int_\Omega (f(u_{lt}) - f(u_{mt}))(u_{lt} - u_{mt})\, dx\, dt = 0. \tag{2.156}$$

From (2.151),(2.153) and (2.156) we also obtain

$$\lim_{l\to\infty} \left[\int_0^T \int_\Omega f(u_{lt})u_{lt}\, dx\, dt - \int_0^T \int_\Omega f(u_{lt})u_t\, dx\, dt - \int_0^T \int_\Omega f^* u_{lt}\, dx\, dt \right]$$
$$+ \lim_{m\to\infty} \int_0^T \int_\Omega f(u_{mt})u_{mt}\, dx\, dt = 0.$$

Hence, again using (2.151),(2.153) and changing m to l we obtain

$$2 \lim_{l\to\infty} \int_0^T \int_\Omega f(u_{lt})u_{lt}\, dx\, dt = 2 \int_0^T \int_\Omega f^* u_t\, dx\, dt. \tag{2.157}$$

But (2.157) combined with (2.151),(2.153) and monotonicity of f, according to Barbu [11] (Lemma 13, p. 42), yields

$$f^* = f(u_t). \tag{2.158}$$

Passing to the limit in (2.136) and recalling (2.158) together with (2.150)- (2.152) gives

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a(x)\left\{\alpha_1 f(u_t(x,t)) + \alpha_2 g(u_t(x,t-\tau))\right\} = 0 & \text{in } D'(\Omega \times (0,+\infty)), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega \times (0,\tau). \end{cases} \tag{2.159}$$

The proof of Theorem 2.3 is thus complete.

2.3.2 Proofs of Theorem 2.4 and Corollary 2.2.

Proof of Theorem 2.4.

Proposition 2.12. *There exists a time T^* such that for all $T > T^*$, there exists a positive constant C_T such that*

$$F(T) \le C_T \left[\int_0^T \int_\Omega a(x)\{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + g(u_t(x,t-\tau))u_t(x,t-\tau)\}\, dx\, dt \right]. \tag{2.160}$$

Proof. We write the solution u of (2.14) as $u = \varphi + v$ where φ solves

$$\begin{cases} \varphi_{tt}(x,t) = \Delta\varphi(x,t) & \text{in } \Omega \times (0;+\infty), \\ \varphi(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ \varphi(x,0) = u_0(x),\ \varphi_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{2.161}$$

and v satisfies

$$\begin{cases} v_{tt}(x,t) = \Delta v(x,t) - a(x)\{\alpha_1 f(u_t(x,t)) - \alpha_2 g(u_t(x,t-\tau))\} & \text{in } \Omega \times (0;+\infty), \\ v(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ v(x,0) = v_t(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{2.162}$$

Denote by $F_\varphi(t)$ the standard energy of (2.161), that is

$$F_\varphi(t) = \frac{1}{2}\int_\Omega \left\{|\varphi_t(x,t)|^2 + |\nabla\varphi(x,t)|^2\right\} dx,$$

and $F_v(t)$ the standard energy of (2.162),

$$F_v(t) = \frac{1}{2}\int_\Omega \left\{|v_t(x,t)|^2 + |\nabla v(x,t)|^2\right\} dx.$$

It is follows from [[62] Proposition 4.2] that for all $T > T_0$, there exists a positive constant c depending on T such that

$$F_\varphi(0) \le c\int_0^T \int_\omega |\varphi_t(x,t)|^2\, dx\, dt.$$

Using (2.15) we get

$$F_\varphi(0) \le \frac{c}{a_0}\int_0^T \int_\Omega a(x)|\varphi_t(x,t)|^2\, dx\, dt. \tag{2.163}$$

On the other hand, from the mean value theorem for integrals and monotonicity of g, we have

$$G(s) = \int_0^s g(r)\, dr \le sg(s).$$

Therefore

$$\frac{\mu}{2}\int_\Omega a(x)\int_0^1 G(u_t(x,-\tau\rho))\, d\rho\, dx \le \frac{\mu}{2}\int_\Omega a(x)\int_0^1 a(x)g(u_t(x,-\tau\rho))u_t(x,-\tau\rho)\, d\rho\, dx.$$

By a change of variable, we obtain for $T > \tau$

$$\frac{\mu}{2}\int_\Omega a(x)\int_0^1 G(u_t(x,-\tau\rho))\, d\rho\, dx \le c\int_0^T \int_\Omega a(x)g(u_t(x,t-\tau))u_t(x,t-\tau)\, dx\, dt. \tag{2.164}$$

If we take $T > T^* := max\{T_0, \tau\}$, since the energy is non-increasing, from (2.163) and (2.164) we deduce that

$$\begin{aligned} F(T) &\leq F(0) = F_\varphi(0) + \frac{\mu}{2}\int_\Omega a(x)\int_0^1 G(u_t(x,-\tau\rho))\,d\rho\,dx \\ &\leq C_T\int_0^T\int_\Omega a(x)\left\{|\varphi_t(x,t)|^2 + g(u_t(x,t-\tau))u_t(x,t-\tau)\right\}\,dx\,dt \\ &\leq C_T\int_0^T\int_\Omega a(x)\left\{|u_t(x,t)|^2 + |v_t(x,t)|^2 + g(u_t(x,t-\tau))u_t(x,t-\tau)\right\}\,dx\,dt \\ &\leq C_T\int_0^T\int_\Omega a(x)\left\{|u_t(x,t)|^2 + |f(u_t(x,t))|^2 + g(u_t(x,t-\tau))u_t(x,t-\tau)\right\}\,dx\,dt \\ &+ C_T\underbrace{\int_0^T\int_\Omega a(x)|v_t(x,t)|^2\,dx\,dt}_{:=\mathcal{K}}. \end{aligned} \tag{2.165}$$

It remains to estimate the term $\mathcal{K}$.
We differentiate the energy function $F_v(t)$ with respect to t, we obtain

$$\frac{d}{dt}F_v(t) = -\int_\Omega a(x)\{\alpha_1 f(u_t(x,t))v_t(x,t) + \alpha_2 g(u_t(x,t-\tau))v_t(x,t)\}\,dx,$$

from which we get after using Cauchy-Schwarz's inequality

$$\frac{d}{dt}F_v(t) \leq C\int_\Omega a(x)\{|f(u_t(x,t))|^2 + |v_t(x,t)|^2 + |g(u_t(x,t-\tau))|^2\}\,dx, + \int_\Omega |v_t(x,t)|^2\,dx.$$

From the definition of $F_v(t)$, we obtain

$$\frac{d}{dt}F_v(t) \leq F_v(t) + C\int_\Omega a(x)\{|f(u_t(x,t))|^2 + |g(u_t(x,t-\tau))|^2\}\,dx.$$

Multiplying the last inequality by (e^{-t}) and integrating over $(0,t)$, we get

$$F_v(t) \leq Ce^t\int_0^t\int_\Omega a(x)\{|f(u_t(x,t))|^2 + |g(u_t(x,t-\tau))|^2\}\,dx\,dt.$$

We conclude for $t \in (0,T)$, that is

$$F_v(t) \leq C\int_0^T\int_\Omega a(x)\{|f(u_t(x,t))|^2 + |g(u_t(x,t-\tau))|^2\}\,dx\,dt,$$

which gives

$$\int_0^T\int_\Omega |v_t(x,t)|^2\,dx\,dt \leq C\int_0^T\int_\Omega a(x)\{|f(u_t(x,t))|^2 + |u_t(x,t)|^2 + |g(u_t(x,t-\tau))|^2\}\,dx\,dt.$$

By using assumptions (**H2**)(iii) and (**H2**)(v), we have

$$\begin{aligned} \mathcal{K} &:= \int_0^T\int_\Omega a(x)|v_t(x,t)|^2\,dx\,dt \\ &\leq C\int_0^T\int_\Omega a(x)\{|f(u_t(x,t))|^2 + |u_t(x,t)|^2 + g(u_t(x,t-\tau))u_t(x,t-\tau)\}\,dx\,dt. \end{aligned} \tag{2.166}$$

Finally, combining (2.166) and (2.165) we obtain the desired estimate given in (2.160). □

Proposition 2.13. *Assume* (**H1**) *and* (**H2**). *Then, the energy $F(T)$ of problem (2.14) satisfies*

$$F(T) + \widehat{p}(F(T) \leq F(0), \tag{2.167}$$

where $\widehat{p}(.)$ is defined by (2.16), and $T > 0$ sufficiently large.

Proof. Denote

$$Q_1 = \{u_t \in L^2(Q) : |u_t| \geq \delta \text{ a.e.}\},$$
$$Q_2 = Q - Q_1.$$

From hypotheses (**H1**) and (**H2**), we have

$$\int_{Q_1} a(x)\{u_t^2(x,t) + f^2(u_t(x,t))\}\, dQ_1 \leq (a_1^{-1} M_3^{-1} a_2 + M_1) \int_{Q_1} a(x) f(u_t(x,t)) u_t(x,t)\, dQ_1. \tag{2.168}$$

On the other side, from (2.4) and from the fact that h is concave and increasing, having in mind that

$$a(x) \leq \|a\|_\infty + 1,$$

and

$$\frac{a(x)}{1 + \|a\|_\infty} \leq a(x),$$

we deduce that

$$\begin{aligned}
\int_{Q_2} a(x)\{u_t^2(x,t) + f^2(u_t(x,t))\}\, dQ_2 &\leq \int_{Q_2} a(x) h(f(u_t(x,t)) u_t(x,t))\, dQ_2 \\
&= \int_{Q_2} (1 + \|a\|_\infty) \frac{a(x)}{1 + \|a\|_\infty} h(f(u_t(x,t)) u_t(x,t))\, dQ_2 \\
&\leq \int_{Q_2} (1 + \|a\|_\infty) h\left(\frac{a(x)}{1 + \|a\|_\infty} f(u_t(x,t)) u_t(x,t)\right)\, dQ_2 \\
&\leq \int_{Q_2} (1 + \|a\|_\infty) h(a(x) f(u_t(x,t)) u_t(x,t))\, dQ_2. \qquad (2.169)
\end{aligned}$$

By Jensen's inequality,

$$\begin{aligned}
&(1 + \|a\|_\infty) \int_{Q_2} h(a(x) f(u_t(x,t)) u_t(x,t))\, dQ_2 \\
&\leq (1 + \|a\|_\infty) mes\, Q\, h\left(\frac{1}{mes\, Q} \int_Q a(x) f(u_t(x,t)) u_t(x,t)\, dQ\right) \\
&= (1 + \|a\|_\infty) mes\, Q\, \widehat{h}\left(\int_Q a(x) f(u_t(x,t)) u_t(x,t)\, dQ\right). \qquad (2.170)
\end{aligned}$$

Combining inequalities (2.168), (2.169) and (2.170) with the result of Proposition 2.12 gives

$$\begin{aligned}F(T) &\leq C_T \left\{ (a_1^{-1} M_3^{-1} a_2 + M_1) \int_Q a(x) f(u_t(x,t)) u_t(x,t)\, dQ + \int_Q a(x) u_t(x,t-\tau) g(u_t(x,t-\tau))\, dQ \right\} \\ &+ C_T \left(1 + \|a\|_\infty\right) mes\, Q\, \widehat{h} \left(\int_Q a(x) f(u_t(x,t)) u_t(x,t)\, dQ \right) \\ &\leq C_T \left\{ (a_1^{-1} M_3^{-1} a_2 + M_1) \int_Q a(x) f(u_t(x,t)) u_t(x,t)\, dQ + \int_Q a(x) u_t(x,t-\tau) g(u_t(x,t-\tau))\, dQ \right\} \\ &+ C_T \left(1 + \|a\|_\infty\right) mes\, Q\, \widehat{h} \left(\int_Q a(x) \{ f(u_t(x,t)) u_t(x,t) + u_t(x,t-\tau) g(u_t(x,t-\tau)) \}\, dQ \right).\end{aligned} \tag{2.171}$$

Setting

$$K_1 = \frac{1}{C_T\left(1 + \|a\|_\infty\right) mes\, Q};\; C' = \frac{a_1^{-1} M_3^{-1} a_2 + M_1}{(1 + \|a\|_\infty) mes\, Q},$$

we obtain

$$\begin{aligned}F(T) \leq & \frac{C'}{K_1} \int_Q a(x) f(u_t(x,t)) u_t(x,t)\, dQ \\ &+ \frac{1}{K_1\left(1 + \|a\|_\infty\right) mes\, Q} \int_Q a(x) u_t(x,t-\tau) g(u_t(x,t-\tau))\, dQ \\ &+ \frac{1}{K_1} \widehat{h} \left(\int_Q a(x) \{ f(u_t(x,t)) u_t(x,t) + u_t(x,t-\tau) g(u_t(x,t-\tau)) \}\, dQ \right).\end{aligned}$$

Set

$$C'' = max \left\{ C',\, \frac{1}{\left(1 + \|a\|_\infty\right) mes\, Q} \right\},$$

consequently

$$K_1\, F(T) \leq (C'' I + \widehat{h}) \left(\int_Q a(x) \{ f(u_t(x,t)) u_t(x,t) + u_t(x,t-\tau) g(u_t(x,t-\tau)) \}\, dQ \right). \tag{2.172}$$

On the other hand, by using the inequality (2.114), we obtain

$$\int_Q a(x) \{ f(u_t(x,t)) u_t(x,t) + u_t(x,t-\tau) g(u_t(x,t-\tau)) \}\, dQ \leq C_1^{-1} (F(0) - F(T)). \tag{2.173}$$

By (2.172) and (2.173), we obtain

$$(C'' I + \widehat{h})^{-1} \left(K_2\, F(T)\right) = \widehat{p}(F(T) \leq F(0) - F(T),$$

where $K_2 = C_1 K_1$.
Finally,

$$\widehat{p}(F(T)) + F(T) \leq F(0).$$

□

Applying the result of Proposition 2.13 we obtain for $m = 0, 1, 2....$

$$\widehat{p}(F(m(T+1)) + F(m(T+1)) \leq F(mT).$$

Thus, we are in a position to apply [[43], Lemma 3.3, p.531] with

$$s_m = F(mT),\ s_0 = F(0).$$

This yields

$$F(mT) \leq S(m),\ m = 0, 1, 2, ...$$

Let $t = mT + \tau$ and recall the evolution property, we obtain

$$F(t) \leq F(mT) \leq S(m) \leq S(\tfrac{t-\tau}{T}) \leq S(\tfrac{t}{T} - 1) \quad \text{for}\, t > T,$$

which completes the proof of Theorem 2.4.

- **Proof of Corollary 2.2**

It is enough to construct a function h with the property (2.4).
From hypotheses (2.22) and (2.23), we have

$$\begin{aligned}
\int_{Q_2} a(x)\{u_t^2(x,t) + f^2(u_t(x,t))\}\, dQ_2 &\leq (1+b^2) \int_{Q_2} a(x) u_t^2(x,t)\, dQ_2 \\
&\leq (1+b^2) \int_{Q_2} a(x)(\tilde{a}^{-1} f(u_t(x,t)) u_t(x,t))^{\frac{2}{p+1}}\, dQ_2 \\
&\leq (1+b^2)\tilde{a}^{\frac{-2}{p+1}} \int_{Q_2} a(x)(f(u_t(x,t))u_t(x,t))^{\frac{2}{p+1}}\, dQ_2.
\end{aligned}$$

We can take

$$\widehat{h}(s) = \tilde{a}^{\frac{-2}{p+1}}(1+b^2)s^m, \quad \text{where } m = \frac{2}{p+1} \leq 1.$$

Then

$$\widehat{p}(s) = (C''I + \widehat{h})^{-1}(K_2 s).$$

Therefore

$$C''\widehat{p} + d(\tilde{a}, b)s^m = K_2 s$$

where d is a suitable constant depending on $\tilde{a}, b$.
Also, recall that

$$\widehat{q}(s) = s - (I + \widehat{p})^{-1}(s).$$

Since asymptotically (for s small) we have, for some constant $\beta > 0$,

$$\widehat{p}(s) \sim \beta s^{1/m} \quad \text{and therefore} \;\; \widehat{q}(s) \sim \beta s^{1/m},$$

by solving equation (2.21) with $\widehat{q}$ as above, we obtain

$$S(t)x = \begin{cases} c_1(t + c_2 x^{\frac{1-p}{2}})^{\frac{2}{1-p}} & \text{if } \; p > 1 \\ e^{-\beta t} x & \text{if } \; p = 1, \end{cases} \tag{2.174}$$

where c_1, c_2 depend only on β and $\widehat{p}$.

Finally, (2.24) follows from Theorem 2.4. The proof of Corollary 2.2 is complete.

Chapter 3

Stability of the Schrödinger equation with a delay term in the nonlinear boundary or internal feedbacks

3.1 Introduction

In this chapter, we study stability problems for the Schrödinger equation with a nonlinear delay term in the boundary or internal feedbacks.
Let Ω be an open bounded domain of $\mathbb{R}^n$ with smooth boundary Γ which consists of two non-empty parts Γ_1 and Γ_2 such that, $\Gamma_1 \cup \Gamma_2 = \Gamma$ with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$.
In addition to these standard hypothesis, we assume the following.

(A) There exists $x_0 \in \mathbb{R}^n$ such that, with $m(x) = x - x_0$,

$$m(x).\nu(x) \leq 0 \text{ on } \Gamma_1, \tag{3.1}$$

where $\nu(.)$ is the unit normal on Γ pointing towards the exterior of Ω.

In Ω, we consider the Schrödinger equation with a delay term in the nonlinear boundary feedback:

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 & \text{in } \Omega \times (0;+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = i\alpha_1 f(u(x,t)) + i\alpha_2 g(u(x,t-\tau)) & \text{on } \Gamma_2 \times (0,+\infty), \\ u(x,t-\tau) = f_0(x,t-\tau) & \text{on } \Gamma_2 \times (0,\tau), \end{cases} \tag{3.2}$$

where

- u_0 and f_0 are the initial data which belong to suitable spaces.
- $\frac{\partial}{\partial \nu}$ is the normal derivative.

- τ is the time delay.
- α_1 and α_2 are positive constants.
- f and g are complex-valued functions of class $C(\mathbb{C})$.

In the absence of delay ($i.e.$ $\alpha_2 = 0$), uniform decay rates have been established for the solutions of (3.2) in [57] and [51] when f is linear and in [19] and [47] for f nonlinear.
In [65], the authors examined the system (3.2) with f and g linear. They proved under the assumption

$$\alpha_1 > \alpha_2$$

that the solution decays exponentially to zero in the energy space $L^2(\Omega)$.

In this chapter, we address the uniform stability problem for (3.2) in the case where both f and g are nonlinear.
To this aim, we need to make the following assumptions.

$(H.1)$ (i) $f(s)$ is continuous complex-valued function with $f(0) = 0$.

(ii) $Re\langle f(z) - f(y), z - y\rangle \geq K|z - y|^2$ for all $z, y \in \mathbb{C}$ and $K > 0$.

(iii) $Im\{f(z)\bar{z}\} = 0$.
Thus in particular for $y = 0$, we have from (ii) that $Re\{f(z)\bar{z}\} \geq K|z|^2$ which implies in view of (iii) that $Re\{f(z)\bar{z}\} = f(z)\bar{z} \geq K|z|^2$ and consequently $f(z)\bar{z} = |f(z)\bar{z}|$.

$(H.2)$ (i) g is a Lipschitz continuous, complex-valued function; $|g(z) - g(y)| \leq L_1|z - y|$ $\forall z, y \in \mathbb{C}$ with $g(0) = 0$.

(ii) $Im\{g(z)\bar{z}\} = 0, \ \forall z \in \mathbb{C}$.

$(H.3)$ $\alpha_1 > \frac{\alpha_2 L_1}{K}$.

$(H.4)$ There exist positive constant $M > 0$, such that

$$\begin{cases} |f(z)| \leq M|z|^p, \text{ for } |z| \geq 1, \quad \forall z \in \mathbb{C}; \\ \\ \text{where } p = 5 \text{ for } n = \dim\Omega = 2, \\ \qquad\quad p = 3 \text{ for } n = \dim\Omega = 3. \end{cases} \tag{3.3}$$

Remark 3.1. *(i) Particular example of a function f satisfying assumption $(H.1)$ is:*

$$f(z) = |z|^r z + Kz, \ \ 0 < r < 1,$$

(ii) *As an example of a function g for which assumption $(H.2)$ holds we have:*

$$g(z) = \begin{cases} \tilde{\gamma}(|z|)z, & |z| \le l \\ \tilde{\gamma}(l)z, & |z| \ge l \end{cases}, z \in \mathbb{C}$$

where $\tilde{\gamma}$ is a locally Lipschitz continuous function of a real variable.

In order to state the stability result, we proceed as in the previous chapter. So, let h be the real valued strictly increasing concave function defined for $s \ge 0$ and satisfying

$$\begin{aligned} h(0) &= 0; \\ h(f(z)\overline{z}) &\ge |z| + |f(z)|^2 \quad \text{for } |z| \le \delta; \text{ for some } \delta > 0, z \in \mathbb{C} \end{aligned} \tag{3.4}$$

and define the following functions:

- $$\widetilde{h}(z) = h(\frac{z}{mes\ \Sigma_2}), z \ge 0, \tag{3.5}$$

  where $\Sigma_2 = \Gamma_2 \times (0,T)$, T is a given constant.

- $$p(z) = (C''I + \widetilde{h})^{-1}(K_3 z), \tag{3.6}$$

  where C'' and K_3 are positive constants.

- $$q(z) = z - (I+p)^{-1}(z), \ z > 0, \tag{3.7}$$

  q is also a positive, continuous, strictly increasing function with $q(0) = 0$.

Then p and q are positive, continuous, strictly increasing functions with $p(0) = q(0) = 0$.
We define the energy of a solution of (3.2) by

$$E(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2\,dx + \frac{\mu}{2}\int_{\Gamma_2}\int_0^1 |u(x,t-\tau\rho))|^2 d\rho\,dx, \tag{3.8}$$

where

$$\tau\alpha_2 L_1 < \mu < 2\tau(K\alpha_1 - \frac{L_1\alpha_2}{2}). \tag{3.9}$$

We show that if $\{\Omega, \Gamma_1, \Gamma_2\}$ satisfies (A), and the functions f and g verify assumptions $(H.1)$–$(H.4)$, then we obtain uniform decay rates of the energy of solutions. The proof of this result is, as in the previous chapter, based on certain integral inequalities for the energy functional and a comparison theorem that relates the asymptotic behaviour of the energy and of the solutions to a dissipative ordinary differential equation. This result is stated in the following theorem.

Theorem 3.1. *Let $n = dim\,\Omega = 2, 3$. Assume hypotheses $(H.1)$–$(H.4)$ and (A). Let u be the solution to (3.2). Then, for some $T_0 > 0$,*

$$E(t) \leq S(\frac{t}{T_0} - 1)(E(0)), \qquad \forall\, t \geq T_0, \tag{3.10}$$

where $S(t)$ is the solution of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0,\; S(0) = E(0), \tag{3.11}$$

where the function q is defined by (3.7).

In this chapter, we will also study the stability problem for the Schrödinger equation with a delay term in the nonlinear internal feedback. More precisely, we consider the system described by

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) + a(x)\{\alpha_1 f(u(x,t)) - \alpha_2 g(u(x,t-\tau))\} = 0 & \text{in } \Omega \times (0; +\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma \times (0, +\infty), \\ u(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega \times (0,\tau), \end{cases} \tag{3.12}$$

where α_1, α_2, u_0, f_0, f and g are of as above and $a(.)$ is a function in $L^\infty(\Omega)-$ function such that

$$a(x) \geq 0 \text{ a.e. in } \Omega \text{ and } a(x) > a_0 > 0 \text{ a.e. in } \omega, \tag{3.13}$$

where ω is an open neighborhood of Γ_2.

Stability problems for the undelayed system corresponding to (3.12) (*i.e.* $\alpha_2 = 0$) have been treated for both linear [57] and nonlinear [15] functions f.
Nicaise and Rebiai [65] obtained stability and instability results for the system (3.12) with f and g linear. In fact, they proved under the assumption

$$\alpha_1 > \alpha_2, \tag{3.14}$$

that the solution decays exponentially to zero in the energy space $L^2(\Omega)$. On the contrary, if (3.14) does not hold they constructed a sequence of delays for which the corresponding solution of (3.12) is unstable.
Here, we consider the case when f and g are nonlinear and satisfy in addition to $(H.1)$, $(H.2)$, $(H.3)$ and the following,

$(H.5)$ There exists $\widetilde{M} > 0$ such that
$$|f(z)| \leq \widetilde{M}|z|, \text{ for } |z| \geq 1.$$

Remark 1. *(i) Particular examples of a function f satisfying assumptions $(H.1)$, $(H.5)$ are:*

$$f(z) = |z|^r z + Kz, \;\; \textit{for} \;\; 0 < r < 1 \textit{ and } K > 0; \textit{ or } f(z) = |z|^2 e^{-\frac{1}{|z|^2}} z + Kz, \;\; \textit{for} \;\; K > 0.$$

Remark 2. *In [36], regional boundary stabilization of the one-dimensional Schrödinger equation with a nonlinear delay term of the form*

$$\mu u(x, t - \tau(t))|u(x, t)|^2,$$

and with bounded internal disturbance, is studied by using the backstepping method.

Before stating the stability result for (3.12), we consider the function h introduced by (3.4) and define as we have done previously, the functions:

- $$\widehat{h}(z) = h(\frac{z}{mes\ Q}), z \geq 0,$$

  where $Q = \Omega \times (0, T)$, T is a given constant.

- $$\widehat{p}(z) = (C''I + \widehat{h})^{-1}(K_2 z), \tag{3.15}$$

  where C'' and K_2 are positive constants.

- $$\widehat{q}(z) = z - (I + \widehat{p})^{-1}(z), \ z > 0. \tag{3.16}$$

Then $\widehat{p}$ and $\widehat{q}$ are positive, continuous, strictly increasing function with $\widehat{p}(0) = \widehat{q}(0) = 0$.
Let $F(t)$ be the energy of a solution of (3.12) given by

$$F(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2\, dx + \frac{\mu}{2}\int_\Omega a(x)\int_0^1 |u(x, t - \tau\rho)|^2 d\rho\, dx, \tag{3.17}$$

where

$$\tau\alpha_2 L_1 < \mu < 2\tau(K\alpha_1 - \frac{L_1\alpha_2}{2}). \tag{3.18}$$

We have the following stability result for system (3.12):

Theorem 3.2. *Assume hypotheses $(H.1)-(H.3)$, $(H.5)$ and (A). Let u be a solution to (3.12). Then for some $T_0 > 0$,*

$$F(t) \leq S(\frac{t}{T_0} - 1)(F(0)) \ \ for\ \ t > T_0,$$

where $S(t)$ is the solution of the differential equation

$$\tfrac{d}{dt}S(t) + \widetilde{q}(S(t)) = 0,\ S(0) = F(0).$$

where the function $\widetilde{q}$ is defined by (3.16).

The chapter is organized as follows. Theorem 3.1 is proved in Section 3.2 whereas Section 3.3 contains the proof of Theorem 3.2. Both sections start with the study of the well-posedness of the system under consideration.

3.2 Stability of the Schrödinger equation with a delay term in the nonlinear boundary feedback

3.2.1 Well-posedness of problem (3.2)

In order to be able to manage the boundary condition with the delay term and inspired from [62] and [83] we introduce the auxiliary variable:

$$y(x,\rho,t) = u(x,t-\tau\rho); \quad x \in \Gamma_2, \rho \in (0,1), t > 0.$$

Then, the system (3.2) is equivalent to

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 & \text{in } \Omega \times (0;+\infty), \\ y_t(x,\rho,t) + \tau^{-1} y_\rho(x,\rho,t) = 0 & \text{on } \Gamma_2 \times (0,1) \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = i\alpha_1 f(u(x,t)) + i\alpha_2 g(y(x,1,t)) & \text{on } \Gamma_2 \times (0,+\infty), \\ y(x,\rho,0) = f_0(x,-\rho\tau) & \text{on } \Gamma_2 \times (0,\tau), \\ y(x,0,t) = u(x,t) & \text{on } \Gamma_2 \times (0,+\infty). \end{cases} \tag{3.19}$$

Denote by $\mathcal{H}$ the Hilbert space.

$$\mathcal{H} = L^2(\Omega) \times L^2(\Gamma_2; L^2(0,1)).$$

We equip $\mathcal{H}$ with the inner product:

$$\left\langle \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}; \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = Re \int_\Omega u_1(x)\overline{u_2(x)}\, dx + \mu Re \int_{\Gamma_2} \int_0^1 y_1(x,\rho)\overline{y_2(x,\rho)} d\rho\, d\Gamma_2.$$

Let $A : L^2(\Omega) \to L^2(\Omega)$ be the operator defined by

$$A\zeta = -\Delta\zeta \text{ with } D(A) = \left\{ \zeta \in H^2(\Omega), \frac{\partial \zeta}{\partial \nu} = 0 \text{ on } \Gamma_2, \zeta = 0 \text{ on } \Gamma_1 \right\}.$$

Let $N : L^2(\Gamma) \to L^2(\Omega)$ be the Neumann map [45], [79], [49]

$$\chi = N\varphi \Longleftrightarrow \left\{ \Delta\chi = 0 \text{ in } \Omega;\ \chi\, |_{\Gamma_1} = 0,\ \frac{\partial \chi}{\partial \nu}\bigg|_{\Gamma_2} = \varphi \right\}, \quad \Gamma_1 \neq \emptyset. \tag{3.20}$$

It is well known that

$$N : \text{ continuous } H^s(\Gamma) \longrightarrow H^{s+\frac{3}{2}}, s \in \mathbb{R}; \tag{3.21}$$

$$N : \text{ continuous } L^2(\Gamma) \longrightarrow H^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\epsilon}(\Omega) \equiv D(A^{\frac{3}{4}-\epsilon}), \quad \forall \epsilon > 0; \tag{3.22}$$

and

$$N^*A^*\zeta = N^*A\,\zeta = \begin{cases} 0 & \text{on } \Gamma_1, \\ \zeta & \text{on } \Gamma_2, \end{cases} \quad \text{for } \zeta \in H^1_{\Gamma_1}(\Omega) = D(A^{\frac{1}{2}}). \tag{3.23}$$

Next define

$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H},$$
$$\mathcal{A}\begin{pmatrix} \zeta \\ \theta \end{pmatrix} = \begin{pmatrix} A(-i\zeta - \alpha_1 N f(\zeta) - \alpha_2 N g(\theta(.;1))) \\ -\tau^{-1}\theta_\rho \end{pmatrix}, \tag{3.24}$$

with

$$D(\mathcal{A}) = \{\zeta \in L^2(\Omega), \theta \in L^2(\Gamma_2; H^1(0,1)); -i\zeta - \alpha_1 N f(\zeta) - \alpha_2 N g(\theta(.,1)) \in D(A)\}. \tag{3.25}$$

Then we can rewrite (3.19) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \dfrac{dU}{dt}(t) = \mathcal{A}(U(t)), \\ U(0) = U_0, \end{cases} \tag{3.26}$$

where

$$U(t) = (u(x,t), y(x,\rho,t))^T,\ U'(t) = (u_t(x,t), y_t(x,\rho,t))^T,\ U_0 = (u_0, f_0)^T.$$

Theorem 3.3. *Let $n = 1, 2....$ Assume hypotheses $(H.1) - (H.3)$. Then, the following results hold true for the problem (3.2):*

(a) (Well-posedness) For any initial conditions $U_0 = (u_0, f_0)^T \in \mathcal{H}$, problem (3.2) defines a unique (nonlinear contraction semigroup) solution U satisfying

$$U(.) \in C([0,\infty); \mathcal{H}). \tag{3.27}$$

The generator $\mathcal{A}$ of the corresponding nonlinear semigroup is given explicitly in (3.24) below: it is maximal dissipative; moreover, $\overline{D(\mathcal{A})} = \mathcal{H}$.

(b) (Regularity) Let, in particular, $U_0 = (u_0, f_0)^T \in H^2(\Omega) \times L^2(\Gamma_2; H^1(0,1))$ subject to compatibility conditions:

$$U_0 \in H^2(\Omega) \times L^2(\Gamma_2; H^1(0,1)) : u_0|_{\Gamma_1} = 0;\ \left.\frac{\partial u_0}{\partial \nu}\right|_{\Gamma_2} = i\alpha_1 f(u_0) + i\alpha_2 g(f_0), \quad \text{so that } U_0 \in D(\mathcal{A}). \tag{3.28}$$

Then, the corresponding unique solution $U(.)$ guaranteed by part (a), satisfies [[11], Theorem 1.2, p. 220] ($U_t^+(.) =$ right-derivative)

$$\begin{cases} U(.) \in C([0,\infty); D(\mathcal{A})),\ D(\mathcal{A}) \subset D(A^{\frac{1}{2}}) \times L^2(\Gamma_2; H^1(0,1)) = H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2; H^1(0,1))), \\ U_t^+(.) \in C([0,\infty); \mathcal{H}); \\ U(.)|_{\Gamma_2} \in C([0,\infty); H^{\frac{1}{2}}(\Gamma_2) \times L^2(\Gamma_2; H^1(0,1))). \end{cases} \tag{3.29}$$

(c) *(Higher regularity) Assume (3.28) on U_0 and, moreover,*

(c_1) *if* $\dim\Omega = 2$, *assume assumption* (H.4).

(c_2) *if* $\dim\Omega = 3$, *assume that*

$$|f(z)| \le C_r|z|^r, \quad z \in \mathbb{C}, \text{ for some } r < 3, \ |z| \ge 1, \tag{3.30}$$

($r = 3 - \epsilon$, $\epsilon > 0$ arbitrary).

Then, in both cases (c_1) and (c_2), we have that

$$D(\mathcal{A}) \subset H^{\frac{3}{2}}(\Omega) \times L^2(\Gamma_2; H^1(0,1)), \qquad \text{so that } U(.) \in C([0,\infty); H^{\frac{3}{2}}(\Omega) \times L^2(\Gamma_2; H^1(0,1))), \tag{3.31}$$

for U_0 as in (3.28). In particular (from (3.29) , (c_1) and (c_2)):

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_2} = i\alpha_1 f(u) + i\alpha_2 g(u(.,. - \tau)) \in L^2(0,T; L^2(\Gamma_1)). \tag{3.32}$$

Proof of Theorem 3.3

To accomplish this, the following result will be needed.

Lemma 3.1. *Under assumption (H.1)(iii), (H.2)(i) and (H.3), we have*

$$D(\mathcal{A}) \subset D(A^{\frac{1}{2}}) \times L^2(\Gamma_2; H^1(0,1)) = H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2; H^1(0,1)), \tag{3.33}$$

where

$$H^1_{\Gamma_1}(\Omega) = \{\zeta \in H^1(\Omega) : \zeta|_{\Gamma_1} = 0\},$$

so that

$$\begin{aligned}(\zeta,\theta)^T \in D(\mathcal{A}) &\longrightarrow \zeta \in H^1_{\Gamma_1}(\Omega),\, \theta \in L^2(\Gamma_2; H^1(0,1)) \\ &\longrightarrow \zeta|_\Gamma = N^*A\zeta \in H^{\frac{1}{2}}(\Gamma),\, \theta \in L^2(\Gamma_2; H^1(0,1)),\ \mathcal{A}(\zeta,\theta) \in \mathcal{H}\end{aligned} \tag{3.34}$$

$$\mathcal{A} : D(A^{\frac{1}{2}}) \times L^2(\Gamma_2; H^1(0,1)) \supset D(\mathcal{A}) \longrightarrow \mathcal{H}.$$

Proof. Indeed, if $(\zeta,\theta)^T \in D(\mathcal{A})$, we obtain from (3.25)

$$\begin{aligned}&A(-i\zeta - \alpha_1 Nf(\zeta) - \alpha_2 Ng(\theta(x,1))) = L \in L^2(\Omega) \Longrightarrow \\ &- i\langle A\zeta,\zeta\rangle_{L^2(\Omega)} - \alpha_1\langle ANf(N^*A\zeta),\zeta\rangle_{L^2(\Omega)} - \alpha_2\langle ANg(\theta(x,1))),\zeta\rangle_{L^2(\Omega)} = \langle L,\zeta\rangle_{L^2(\Omega)},\end{aligned} \tag{3.35}$$

and

$$-\tau^{-1}\theta_\rho = R \in L^2(\Gamma_2, L^2(0,1)), \tag{3.36}$$

Indeed, from (3.36) we have

$$\theta(x,1) = \zeta(x) + z_0, \qquad x \in \Gamma_2, \tag{3.37}$$

where

$$z_0 = -\tau \int_0^1 R(x,\sigma)\, d\sigma.$$

Insertion (3.37) into (3.35) yields

$$\begin{aligned} &A(-i\zeta - \alpha_1 N f(\zeta) - \alpha_2 N g(\zeta(x) + z_0)) = L \in L^2(\Omega) \Longrightarrow \\ &- i\langle A\zeta, \zeta\rangle_{L^2(\Omega)} - \alpha_1 \langle f(\zeta), \zeta\rangle_{L^2(\Gamma_2)} - \alpha_2 \langle g(\zeta + z_0), \zeta\rangle_{L^2(\Gamma_2)} = \langle L, \zeta\rangle_{L^2(\Omega)}. \end{aligned} \tag{3.38}$$

Taking the imaginary part of (3.38), where A is positive self-adjoint, and recalling that

$$Im\{f(\zeta)\overline{\zeta}\} = 0,$$

we obtain

$$\langle A\zeta, \zeta\rangle_{L^2(\Omega)} = -Im\langle L, \zeta\rangle_{L^2(\Omega)} - \alpha_2 Im\langle g(\zeta + z_0), \zeta\rangle_{L^2(\Gamma_2)},$$

$$\|A^{\frac{1}{2}}\zeta\|^2_{L^2(\Omega)} \leq \|A^{-\frac{1}{2}}\, L\|^2_{L^2(\Omega)} \|A^{\frac{1}{2}}\, \zeta\|^2_{L^2(\Omega)} + \alpha_2 \|g(\zeta + z_0)\|_{L^2(\Gamma_2)} \|\zeta\|_{L^2(\Gamma_2)}.$$

By using assumption $(H.2)(i)$, we have

$$\|A^{\frac{1}{2}}\zeta\|^2_{L^2(\Omega)} \leq \|A^{-\frac{1}{2}}\, L\|_{L^2(\Omega)} \|A^{\frac{1}{2}}\, \zeta\|_{L^2(\Omega)} + \alpha_2 L_1 \|\zeta\|^2_{L^2(\Gamma_2)} + \alpha_2 L_1 c \|\zeta\|_{L^2(\Gamma_2)},$$

where $\|z_0\|_{L^2(\Gamma_2)} \leq c$.
By trace theory, we obtain

$$\|A^{\frac{1}{2}}\zeta\|^2_{L^2(\Omega)} \leq \|A^{-\frac{1}{2}}\, L\|_{L^2(\Omega)} \|A^{\frac{1}{2}}\, \zeta\|_{L^2(\Omega)} + \alpha_2 L_1 c_1 \|A^{\frac{1}{2}}\, \zeta\|^2_{L^2(\Omega)} + \alpha_2 L_1 c_2 \|A^{\frac{1}{2}}\, \zeta\|_{L^2(\Omega)}, \tag{3.39}$$

where $c_2 = c\, c_1$.
Then, (3.39) yields

$$\|A^{\frac{1}{2}}\zeta\|_{L^2(\Omega)} \leq \frac{1}{1 - \alpha_2 L_1 c_1} \Big(\|L\|_{L^2(\Omega)} + \alpha_2 L_1 c_2 \Big).$$

By using assumption $(H.3)$, we have $1 - \alpha_2 L_1 c_1 > 0$.
Thus, (3.33) is established. Then, trace theory and (3.23) yields (3.34). □

- Proof of well-posedness and regularity.

Proposition 3.4. *Assume hypotheses* $(H.1)$*,* $(H.2)$ *and* $(H.3)$*. Then, the operator* $\mathcal{A}$ *in (3.24) is maximal dissipative on* $\mathcal{H}$*.*

Proof. First, we prove that $\mathcal{A}$ is dissipative.

Let $U = (\zeta_1, \theta_1)^T \in D(\mathcal{A})$ and $V = (\zeta_2, \theta_2)^T \in D(\mathcal{A})$. Then

$$\begin{aligned}
Re\,\langle \mathcal{A}U - \mathcal{A}V; U - V\rangle_{\mathcal{H}} &= -Re\int_{\Omega} A(\zeta_1(x) - \zeta_2(x))\ (\overline{\zeta_1(x)} - \overline{\zeta_2(x)})\,dx \\
&- Re\int_{\Omega} \alpha_1 AN(f(\zeta_1(x)) - f(\zeta_2(x)))\ (\overline{\zeta_1(x)} - \overline{\zeta_2(x)})\,dx \\
&- Re\int_{\Omega} \alpha_2 AN(g(\theta_1(x,1)) - g(\theta_2(x,1)))(\overline{\zeta_1(x)} - \overline{\zeta_2(x)}))\,dx \\
&- \mu\tau^{-1}\,Re\int_{\Gamma_2}\int_0^1 (\theta_{1\rho}(x,\rho) - \theta_{2\rho}(x,\rho))(\overline{\theta_1(x,\rho)} - \overline{\theta_2(x,\rho)})\,d\rho\,d\Gamma_2 \\
&= -Re\int_{\Omega} A^{\frac{1}{2}}(\zeta_1(x) - \zeta_2(x))\ A^{\frac{1}{2}}(\overline{\zeta_1(x)} - \overline{\zeta_2(x)})\,dx \\
&- Re\int_{\Gamma_2} \alpha_1(f(\zeta_1(x)) - f(\zeta_2(x)))\ N^*A(\overline{\zeta_1(x)} - \overline{\zeta_2(x)})\,d\Gamma_2 \\
&- Re\int_{\Gamma_2} \alpha_2(g(\theta_1(x,1)) - g(\theta_2(x,1)))N^*A(\overline{\zeta_1(x)} - \overline{\zeta_2(x)})\,dx \\
&- \mu\tau^{-1}\,Re\int_{\Gamma_2}\int_0^1 (\theta_{1\rho}(x,\rho) - \theta_{2\rho}(x,\rho))(\overline{\theta_1(x,\rho)} - \overline{\theta_2(x,\rho)})\,d\rho\,d\Gamma_2.
\end{aligned}$$

Integrating by parts in ρ, we obtain

$$\begin{aligned}
&\int_{\Gamma_2}\int_0^1 (\theta_{1\rho}(x,\rho) - \theta_{2\rho}(x,\rho))(\overline{\theta_1(x,\rho)} - \overline{\theta_2(x,\rho)})\,d\rho\,d\Gamma_2 \\
&= -\int_{\Gamma_2}\int_0^1 (\theta_1(x,\rho) - \theta_2(x,\rho))(\overline{\theta_{1\rho}(x,\rho)} - \overline{\theta_{2\rho}(x,\rho)})\,d\rho\,d\Gamma_2 \\
&+ \int_{\Gamma_2} |\theta_1(x,1) - \theta_2(x,1))|^2 - |\theta_1(x,0) - \theta_2(x,0))|^2\,d\Gamma_2,
\end{aligned}$$

or equivalently

$$\begin{aligned}
&2Re\int_{\Gamma_2}\int_0^1 (\theta_{1\rho}(x,\rho) - \theta_{2\rho}(x,\rho))(\overline{\theta_1(x,\rho)} - \overline{\theta_2(x,\rho)})\,d\rho\,d\Gamma_2 \\
&= \int_{\Gamma_2} |\theta_1(x,1) - \theta_2(x,1))|^2 - |\theta_1(x,0) - \theta_2(x,0))|^2\,d\Gamma_2.
\end{aligned}$$

Therefore

$$\begin{aligned}
Re\,\langle \mathcal{A}U - \mathcal{A}V; U - V\rangle_{\mathcal{H}} &= -Re\int_{\Gamma_2} \alpha_1(f(\zeta_1(x)) - f(\zeta_2(x)))\ (\overline{\zeta_1(x)} - \overline{\zeta_2(x)})\,d\Gamma_2 \\
&- Re\int_{\Gamma_2} \alpha_2(g(\theta_1(x,1)) - g(\theta_2(x,1)))(\overline{\zeta_1(x)} - \overline{\zeta_2(x)})\,d\Gamma_2 \\
&- \frac{\mu\tau^{-1}}{2}\int_{\Gamma_2} |\theta_1(x,1) - \theta_2(x,1))|^2\,d\Gamma_2 + \frac{\mu\tau^{-1}}{2}\int_{\Gamma_2} |\theta_1(x,0) - \theta_2(x,0))|^2\,d\Gamma_2.
\end{aligned}$$

By using assumptions $(H.1)(ii)$, $(H.2)(i)$ and the Cauchy-Schwartz's inequality, we have

$$\begin{aligned} Re\,\langle \mathcal{A}U-\mathcal{A}V;U-V\rangle_{\mathcal{H}} \leq & -K\alpha_1\int_{\Gamma_2}|\zeta_1(x)-\zeta_2(x)|^2\,d\Gamma_2+\frac{\alpha_2L_1}{2}\int_{\Gamma_2}|\theta_1(x,1)-\theta_2(x,1)|^2\,d\Gamma_2\\ &+\frac{\alpha_2L_1}{2}\int_{\Gamma_2}|\zeta_1(x)-\zeta_2(x)|^2\,d\Gamma_2-\frac{\mu\tau^{-1}}{2}\int_{\Gamma_2}|\theta_1(x,1)-\theta_2(x,1))|^2\,d\Gamma_2\\ &+\frac{\mu\tau^{-1}}{2}\int_{\Gamma_2}|\zeta_1(x)-\zeta_2(x)|^2\,d\Gamma_2. \end{aligned}$$

Then

$$\begin{aligned} Re\,\langle \mathcal{A}U-\mathcal{A}V;U-V\rangle_{\mathcal{H}} \leq & -\left(K\alpha_1-\frac{\alpha_2L_1}{2}-\frac{\mu\tau^{-1}}{2}\right)\int_{\Gamma_2}|\zeta_1(x)-\zeta_2(x)|^2\,d\Gamma_2\\ &-\left(-\frac{\alpha_2L_1}{2}+\frac{\mu\tau^{-1}}{2}\right)\int_{\Gamma_2}|\theta_1(x,1)-\theta_2(x,1)|^2\,d\Gamma_2. \end{aligned}$$

From (3.9), we conclude that

$$Re\,\langle \mathcal{A}U-\mathcal{A}V;U-V\rangle_{\mathcal{H}}\leq 0.$$

Thus, $\mathcal{A}$ is dissipative.
In order to establish maximality, we need to prove the range condition $range(I-\mathcal{A})=\mathcal{H}$. In other words, given any $(L,R)^T\in\mathcal{H}$, we need to establish the existence of an element $U=(\zeta,\theta)^T\in D(\mathcal{A})$ such that

$$(I-\mathcal{A})U=(L,R)^T, \tag{3.40}$$

or equivalently

$$\zeta-A(-i\zeta-N\alpha_1f(\zeta)-N\alpha_2g(\theta(x,1)))=L, \tag{3.41}$$

$$\theta(x,\rho)+\tau^{-1}\theta_\rho(x,\rho)=R. \tag{3.42}$$

Indeed, from (3.42) and the last line of (3.19) we have

$$\begin{aligned} &\theta_\rho(x,\rho)=-\tau\theta(x,\rho)+\tau R(x,\rho), && x\in\Gamma_2,\rho\in(0,1),\\ &\theta(x,0)=\zeta(x), && x\in\Gamma_2. \end{aligned}$$

The unique solution of the above initial value problem is given by

$$\theta(x,\rho)=\zeta(x)e^{-\tau\rho}+\tau e^{-\tau\rho}\int_0^\rho R(x,\sigma)e^{\tau\sigma}\,d\sigma,\qquad x\in\Gamma_2,\rho\in(0,1),$$

and in particular

$$\theta(x,1)=\zeta(x)e^{-\tau}+Z_0,\qquad x\in\Gamma_2, \tag{3.43}$$

where

$$Z_0 = \tau e^{-\tau} \int_0^1 R(x,\sigma) e^{\tau\sigma}\, d\sigma.$$

Insertion of (3.43) into problem (3.41) yields

$$\zeta + iA\zeta + AN\alpha_1 f(\zeta) + AN\alpha_2 g(\zeta e^{-\tau} + Z_0) = L \in L^2(\Omega). \tag{3.44}$$

Set

$$\mathcal{T}\zeta = \zeta + iA\zeta + AN\alpha_1 f(\zeta) + AN\alpha_2 g(\zeta e^{-\tau} + Z_0). \tag{3.45}$$

Lemma 3.2. *The operator $\mathcal{T}$ is surjective from $V = D(A^{\frac{1}{2}}) = H^1_{\Gamma_1}(\Omega)$ onto $V' = (D(A^{\frac{1}{2}}))' = (H^1_{\Gamma_1}(\Omega))'$.*

Proof. We need to prove that given any $l \in V'$ there exists an element $\zeta \in V$ such that

$$\mathcal{T}\zeta = l. \tag{3.46}$$

Set

$$\mathcal{T}\zeta = B\zeta + C\zeta, \tag{3.47}$$

where

$$B\zeta = AN\alpha_1 f(\zeta) + AN\alpha_2 g(\zeta e^{-\tau} + Z_0),$$

and

$$C\zeta = \zeta + iA\zeta.$$

We will prove that $B+C : V \longrightarrow V'$ is maximal monotone. According to Corollary 1.1 in [[11], p.33] it is sufficient to prove boundedness, hemicontinuity and monotonicity of B and maximal monotonicity of C.
Since we are working with V, V' framework, $AN : L^2(\Gamma_2) \longrightarrow V'$ is bounded.
Monotonicity of B.
Let $\zeta, v \in V$. Then

$$\begin{aligned} Re\langle(-B)\zeta - (-B)v, \zeta - v\rangle_{V'\times V} &= -\alpha_1 Re\langle AN(f(\zeta) - f(v)), \zeta - v\rangle_{V'\times V} \\ &- \alpha_2 Re\langle AN(g(\zeta e^{-\tau} + Z_0) - g(ve^{-\tau} + Z_0)), \zeta - v\rangle_{V'\times V} \\ &= -\alpha_1 Re\langle f(\zeta) - f(v), N^*A(\zeta - v)\rangle_{L^2(\Gamma_2)} \\ &- \alpha_2 Re\langle g(\zeta e^{-\tau} + Z_0) - g(ve^{-\tau} + Z_0)), N^*A(\zeta - v)\rangle_{L^2(\Gamma_2)}. \end{aligned}$$

From assumptions $(H.1)(ii)$, $(H.2)(i)$, we have

$$\begin{aligned} Re\langle(-B)\zeta - (-B)v, \zeta - v\rangle_{V'\times V} &\le -\alpha_1 K\|\zeta - v\|^2_{L^2(\Gamma_2)} + \alpha_2 L_1 e^{-\tau}\|\zeta - v\|^2_{L^2(\Gamma_2)} \\ &\le -(\alpha_1 K - \alpha_2 L_1 e^{-\tau})\|\zeta - v\|^2_{L^2(\Gamma_2)}. \end{aligned}$$

Using assumption $(H.3)$, we conclude that

$$Re\langle(-B)\zeta - (-B)v, \zeta - v\rangle_{V'\times V} \le 0.$$

Thus $(-B)$ is dissipative, then B is monotone.
Hemicontinuity of B.
Let $u, v, w \in V$. We show that the function $t \longmapsto \langle B(\zeta + tv), w\rangle$ is continuous.

$$\begin{aligned}|\langle B(\zeta+tv)-B(\zeta+t_0v),w\rangle_{V'\times V}| &= |\langle \alpha_1 AN(f(\zeta+tv))-f(\zeta+t_0v))),w\rangle_{V'\times V}\\ &+\langle \alpha_2 AN(g((\zeta+tv)e^{-\tau}+\mathcal{Z}_0))-g((\zeta+t_0v)e^{-\tau}+\mathcal{Z}_0))),w\rangle_{V'\times V}|\\ &\le |\langle \alpha_1 f(\zeta+tv))-f(\zeta+t_0v)),N^*A^*w\rangle_{L^2(\Gamma_2)}|\\ &+|\langle \alpha_2 g((\zeta+tv)e^{-\tau}+\mathcal{Z}_0))-g((\zeta+t_0v)e^{-\tau}+\mathcal{Z}_0)),N^*A^*w\rangle_{L^2(\Gamma_2)}|\\ &\le c\|w\|_{L^2(\Gamma_2)}\{\|f(\zeta+tv)-f(\zeta+t_0v)\|_{L^2(\Gamma_2)}\\ &+\|g((\zeta+tv)e^{-\tau})-g((\zeta+t_0v)e^{-\tau})\|_{L^2(\Gamma_2)}\}.\end{aligned}$$

From assumption $(H.2)(i)$, we have

$$|\langle B(\zeta+tv)-B(\zeta+t_0v),w\rangle_{V'\times V}|\le c\|w\|_{L^2(\Gamma_2)}\{\|f(\zeta+tv)-f(\zeta+t_0v)\|_{L^2(\Gamma_2)}+L_1e^{-\tau}\|(t-t_0)v\|_{L^2(\Gamma_2)}\}.$$

Since f is continuous, we conclude that

$$|\langle B(\zeta+tv)-B(\zeta+t_0v),w\rangle_{V'\times V}|<\tilde{\epsilon},$$

for $|t-t_0|<\tilde{\delta}$. This proves continuity of the function $t \longmapsto \langle B(\zeta+tv),w\rangle$.
Maximal monotonicity of C.
For $\zeta, v \in V$, we have

$$\begin{aligned}Re\langle C\zeta - Cv, \zeta - v\rangle_{V'\times V} &= Re\langle \zeta + iA\zeta - v - iAv, \zeta - v\rangle_{V'\times V}\\ &= \|\zeta - v\|_V^2 \ge 0.\end{aligned}$$

Thus C is monotone.
The operator C is continuous and monotone then it is maximal monotone.
We have proved that $\mathcal{T}$ is maximal monotone $V \longrightarrow V'$, consequently according to Minty's Theorem [[11] Theorem 1.2, p.39] for any $\lambda > 0$ the operator

$$\lambda J + \mathcal{T} \ \text{ is surjective } \quad \text{from } V \ \text{ onto } V', \tag{3.48}$$

where J is the canonical injection V onto V', and hence can be taken to be $J = A$.
In the sequel, to establish the surjectivity result (3.46), we employ an approximation argument using (3.48) with $\lambda \searrow 0$.

Approximation argument.
Let $\lambda > 0$ and let $l \in V'$. By the surjectivity property (3.48), there exists $\zeta_\lambda \in V$ such that (with $J = A$):

$$\lambda J\zeta_\lambda + \mathcal{T}\zeta_\lambda = \lambda J\zeta_\lambda + \zeta_\lambda + iA\zeta_\lambda + \alpha_1 ANf(\zeta_\lambda) + \alpha_2 ANg(\zeta_\lambda e^{-\tau} + Z_0) = l \in V'. \tag{3.49}$$

Next we take the duality pairing $V \times V'$ of (3.49), where $J = A$, with any $\psi \in V$,

$$\begin{aligned}\langle \lambda J\zeta_\lambda + \mathcal{T}\zeta_\lambda, \psi\rangle_{L^2(\Omega)} &= \langle \lambda J\zeta_\lambda + \zeta_\lambda, \psi\rangle_{L^2(\Omega)} + i\langle A\zeta_\lambda, \psi\rangle_{L^2(\Omega)} + \alpha_1\langle ANf(\zeta_\lambda), \psi\rangle_{L^2(\Omega)} \\ &\quad + \alpha_2\langle ANg(\zeta_\lambda e^{-\tau} + Z_0), \psi\rangle_{L^2(\Omega)} \\ &= \langle l, \psi\rangle_{L^2(\Omega)}, \qquad \psi \in V. \end{aligned} \tag{3.50}$$

Setting, $\psi = \zeta_\lambda \in V$, taking the Im part of (3.50), we obtain

$$\langle A\zeta_\lambda, \zeta_\lambda\rangle_{L^2(\Omega)} = \|A^{\frac{1}{2}}\zeta_\lambda\|^2_{L^2(\Omega)} = Im\langle l, \zeta_\lambda\rangle_{L^2(\Omega)} - \alpha_2 Im\langle ANg(\zeta_\lambda e^{-\tau} + Z_0), \zeta_\lambda\rangle_{L^2(\Omega)}. \tag{3.51}$$

Since $V = D(A^{\frac{1}{2}})$, $\|\zeta_\lambda\|_V = \|A^{\frac{1}{2}}\zeta_\lambda\|_{L^2(\Omega)}$, then

$$\begin{aligned}\|\zeta_\lambda\|^2_V &= Im\langle l, \zeta_\lambda\rangle_{L^2(\Omega)} - \alpha_2 Im\langle ANg(\zeta_\lambda e^{-\tau} + Z_0), \zeta_\lambda\rangle_{L^2(\Omega)} \\ &\le \|l\|_{V'}\|\zeta_\lambda\|_V + \alpha_2\|g(\zeta_\lambda e^{-\tau} + Z_0)\|_{L^2(\Gamma_2)}\|\zeta_\lambda\|_V.\end{aligned}$$

Using assumption $(H.2)(i)$, we have

$$\|\zeta_\lambda\|^2_V \le \|l\|_{V'}\|\zeta_\lambda\|_V + \alpha_2 L_1 e^{-\tau}\|\zeta_\lambda\|^2_V + \alpha_2 L_1 c\|\zeta_\lambda\|_V, \tag{3.52}$$

where $\|Z_0\|_{L^2(\Gamma_2)} \le c$.
Then, (3.52) yields

$$\begin{cases} \|\zeta_\lambda\|_V \le \frac{1}{1-\alpha_2 L_1 e^{-\tau}}\left(\|l\|_{V'} + \alpha_2 L_1 c\right), \quad \forall\lambda > 0, \text{ hence} \\ \zeta_\lambda \longrightarrow \text{some } \zeta \in V \text{ weakly in } V, \text{as } \lambda \searrow 0 \end{cases} \tag{3.53}$$

where $1 - \alpha_2 L_1 e^{-\tau} > 0$ by assumption $(H.3)$. For a subsequence, still denote by ζ_λ.
Next, take any element $\alpha_1 ANf(\zeta_\lambda) + \alpha_2 ANg(\zeta_\lambda e^{-\tau} + Z_0)$ call it ζ^*_λ:

$$\zeta^*_\lambda = \alpha_1 ANf(\zeta_\lambda) + \alpha_2 ANg(\zeta_\lambda e^{-\tau} + Z_0) \in V'. \tag{3.54}$$

By (3.45) and (3.48), for such element ζ^*_λ in (3.54) we can write

$$\zeta^*_\lambda = -\lambda J\zeta_\lambda - iA\zeta_\lambda - \zeta_\lambda \in V',$$

and hence

$$\begin{cases} \|\zeta^*_\lambda\|_{V'} = \| - \lambda J\zeta_\lambda + iA\zeta_\lambda + \zeta_\lambda\|_{V'} \le c_1\|\zeta_\lambda\|_V \le \frac{c_1}{1-\alpha_2 L_1 e^{-\tau}}\left(\|l\|_{V'} + \alpha_2 L_1 c\right), \quad \forall\lambda > 0, \text{ hence} \\ \zeta^*_\lambda \longrightarrow \text{some } \zeta^* \in V' \text{ weakly in } V', \text{ as } \lambda \searrow 0 \end{cases} \tag{3.55}$$

for a subsequence, still denote by ζ^*_λ.

The limit process. Using (3.53), we obtain with $J = A$ and $\psi \in V = D(A^{\frac{1}{2}})$, as $\lambda \searrow 0$

$$\begin{aligned}|\langle \lambda J\zeta_\lambda, \psi\rangle_{L^2(\Omega)}| = |\langle \lambda A^{\frac{1}{2}}\zeta_\lambda, A^{\frac{1}{2}}\psi\rangle_{L^2(\Omega)}| &\le \lambda\|\zeta_\lambda\|_V \ \|\psi\|_V \\ &\le \lambda\frac{1}{1-\alpha_2 L_1 e^{-\tau}}\left(\|l\|_{V'} + \alpha_2 L_1 c\right)\|\psi\|_V \longrightarrow 0;\end{aligned} \tag{3.56}$$

$$\begin{aligned}\langle \zeta_\lambda, \psi\rangle_{L^2(\Omega)} + i\langle A\zeta_\lambda, \psi\rangle_{L^2(\Omega)} &= \langle \zeta_\lambda, A^{-1}\psi\rangle_V + i\langle A^{\frac{1}{2}}\zeta_\lambda, A^{\frac{1}{2}}\psi\rangle_{L^2(\Omega)} \\ &= \langle \zeta_\lambda, A^{-1}\psi\rangle_V + i\langle \zeta_\lambda, \psi\rangle_V \\ &\longrightarrow \langle \zeta, \psi\rangle_{L^2(\Omega)} + i\langle A\zeta, \psi\rangle_{L^2(\Omega)}. \end{aligned} \tag{3.57}$$

Moreover, using (3.54) and the last line of (3.55), we obtain with $\psi \in V$, hence $A\psi \in V'$, as $\lambda \searrow 0$:

$$\langle \alpha_1 ANf(\zeta_\lambda) + \alpha_2 ANg(\zeta_\lambda e^{-\tau} + Z_0), \psi\rangle_{L^2(\Omega)} = \langle \zeta_\lambda^*, \psi\rangle_{L^2(\Omega)} = \langle \zeta_\lambda^*, A\psi\rangle_{V'} \longrightarrow \langle \zeta^*, \psi\rangle_{L^2(\Omega)}. \tag{3.58}$$

Thus, letting $\lambda \searrow 0$ in (3.50) and invoking (3.56)-(3.58), we obtain

$$\langle I\zeta + iA\zeta, \psi\rangle_{L^2(\Omega)} + \langle \zeta^*, \psi\rangle_{L^2(\Omega)} = \langle l, \psi\rangle_{L^2(\Omega)} \qquad \forall\, \psi \in V, \tag{3.59}$$

where $\zeta \in V$ and $\zeta^* \in V'$ are defined in (3.53) and (3.55), respectively.

To obtain (3.46) from (3.59), we need to show that the limit $\zeta^* \in V'$ obtained in (3.55) and the limit $\zeta \in V$ obtained in (3.53) are linked to each other by the relationship

$$\zeta^* = \alpha_1 ANf(\zeta) + \alpha_2 ANg(\zeta e^{-\tau} + Z_0). \tag{3.60}$$

To establish (3.60), we apply Lemma 1.3 in ([11], p. 42). To this end, it suffices to establish that for $\zeta_\lambda,\ \zeta_\nu \in V,\ \zeta_\lambda^*,\ \zeta_\nu^* \in V'$, we have

$$\lim_{\lambda,\nu \searrow 0} \langle \zeta_\lambda - \zeta_\nu, \zeta_\lambda^* - \zeta_\nu^*\rangle_{L^2(\Omega)} = 0, \tag{3.61}$$

such that

$$\begin{cases} \zeta_\lambda^* = \alpha_1 ANf(\zeta_\lambda) + \alpha_2 ANg(\zeta_\lambda e^{-\tau} + Z_0) \\ \\ \zeta_\nu^* = \alpha_1 ANf(\zeta_\nu) + \alpha_2 ANg(\zeta_\nu e^{-\tau} + Z_0) \end{cases} \tag{3.62}$$

after which, then, (3.60) is proved.
Here, by definition of ζ_λ and ζ_ν, we have

$$\lambda J\zeta_\lambda + \zeta_\lambda + iA\zeta_\lambda + \alpha_1 ANf(\zeta_\lambda) + \alpha_2 ANg(\zeta_\lambda e^{-\tau} + Z_0) = l \in V',\ \lambda > 0 \tag{3.63}$$

$$\nu J\zeta_\nu + \zeta_\nu + iA\zeta_\nu + \alpha_1 ANf(\zeta_\nu) + \alpha_2 ANg(\zeta_\nu e^{-\tau} + Z_0) = l \in V',\ \nu > 0, \tag{3.64}$$

Subtracting the second relation from the first, taking the duality pairing $V \times V'$ with $(\zeta_\lambda, \zeta_\nu)$ and finally taking the Re part of the resulting expression yields

$$\begin{aligned}\langle \lambda J\zeta_\lambda - \nu J\zeta_\nu, \zeta_\lambda - \zeta_\nu\rangle_{L^2(\Omega)} + \|\zeta_\lambda - \zeta_\nu\|^2_{L^2(\Omega)} + \alpha_1 Re\langle AN(f(\zeta_\lambda) - f(\zeta_\nu)), \zeta_\lambda - \zeta_\nu\rangle_{L^2(\Omega)} \\ + \alpha_2 Re\langle AN(g(\zeta_\lambda e^{-\tau} + Z_0) - g(\zeta_\nu e^{-\tau} + Z_0)), \zeta_\lambda - \zeta_\nu\rangle_{L^2(\Omega)} = 0 \end{aligned} \tag{3.65}$$

Arguing as in (3.56), we obtain from the first line of (3.53),

$$\begin{aligned}|\langle \lambda J\zeta_\lambda - \nu J\zeta_\nu, \zeta_\lambda - \zeta_\nu\rangle_{L^2(\Omega)}| &\leq \lambda\|A^{\frac{1}{2}}(\lambda\zeta_\lambda - \nu\zeta_\nu)\|_{L^2(\Omega)}\ \|A^{\frac{1}{2}}(\zeta_\lambda - \zeta_\nu)\|_{L^2(\Omega)} \\ &\leq (\lambda\|\zeta_\lambda\|_V + \nu\|\zeta_\nu\|_V)(\|\zeta_\lambda\|_V + \|\zeta_\nu\|_V) \\ &\leq (\lambda+\nu)\frac{1}{1-\alpha_2 L_1 e^{-\tau}}\,(\|l\|_{V'} + \alpha_2 L_1 c)\left[\frac{2}{1-\alpha_2 L_1 e^{-\tau}}\,(\|l\|_{V'} + \alpha_2 L_1 c)\right] \longrightarrow 0; \end{aligned} \tag{3.66}$$

as $\lambda, \nu \searrow 0$.
Thus, returning to (3.65), we obtain

$$\lim_{\lambda,\nu\searrow 0}\{\|\zeta_\lambda-\zeta_\nu\|^2_{L^2(\Omega)}+\alpha_1 Re\langle AN(f(\zeta_\lambda)-f(\zeta_\nu)),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)}$$
$$+\alpha_2 Re\langle AN(g(\zeta_\lambda e^{-\tau}+Z_0)-g(\zeta_\nu e^{-\tau}+Z_0)),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)}\}=0.$$

Therefore

$$\lim_{\lambda,\nu\searrow 0}\{\|\zeta_\lambda-\zeta_\nu\|^2_{L^2(\Omega)}+Re\langle B\zeta_\lambda-B\zeta_\nu,\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)}\}=0. \tag{3.67}$$

But, by monotonicity of the operator B, each term in (3.67) is non-negative and hence each of the two terms in (3.67) has limit equal to zero.
In particular, we obtain by $(H.1)(iii)$ and $(H.2)(ii)$,

$$\begin{aligned}
&\lim_{\lambda,\nu\searrow 0}\{Im\langle\alpha_1 AN(f(\zeta_\lambda)-f(\zeta_\nu)),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)}\\
&+Im\langle\alpha_2 AN(g(\zeta_\lambda e^{-\tau}+Z_0)-g(\zeta_\nu e^{-\tau}+Z_0)),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)}\}\\
&=\lim_{\lambda,\nu\searrow 0}\alpha_1 Im\langle f(\zeta_\lambda)-f(\zeta_\nu),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Gamma_2)}+\alpha_2 Im\langle g(\zeta_\lambda e^{-\tau}+Z_0)-g(\zeta_\nu e^{-\tau}+Z_0),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Gamma_2)}\\
&=\lim_{\lambda\searrow 0}\alpha_1 Im\langle f(\zeta_\lambda),\zeta_\lambda\rangle_{L^2(\Gamma_2)}+\lim_{\nu\searrow 0}\alpha_1 Im\langle f(\zeta_\nu),\zeta_\nu\rangle_{L^2(\Gamma_2)}\\
&-\lim_{\lambda,\nu\searrow 0}\alpha_1 Im\langle f(\zeta_\lambda),\zeta_\nu\rangle_{L^2(\Gamma_2)}-\lim_{\lambda,\nu\searrow 0}\alpha_1 Im\langle f(\zeta_\nu),\zeta_\lambda\rangle_{L^2(\Gamma_2)}\\
&+\lim_{\lambda\searrow 0}\alpha_2 e^{\tau} Im\langle g(\zeta_\lambda e^{-\tau}+Z_0),\zeta_\lambda e^{-\tau}+Z_0\rangle_{L^2(\Gamma_2)}+\lim_{\nu\searrow 0}\alpha_2 e^{\tau} Im\langle g(\zeta_\nu e^{-\tau}+Z_0),\zeta_\nu e^{-\tau}+Z_0\rangle_{L^2(\Gamma_2)}\\
&-\lim_{\lambda,\nu\searrow 0}\alpha_2 e^{\tau} Im\langle g(\zeta_\lambda e^{-\tau}+Z_0),\zeta_\nu e^{-\tau}+Z_0\rangle_{L^2(\Gamma_2)}-\lim_{\lambda,\nu\searrow 0}\alpha_2 e^{\tau} Im\langle g(\zeta_\nu e^{-\tau}+Z_0),\zeta_\lambda e^{-\tau}+Z_0\rangle_{L^2(\Gamma_2)}\\
&=0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
0=&\lim_{\lambda,\nu\searrow 0}\{Re\langle\alpha_1 AN(f(\zeta_\lambda)-f(\zeta_\nu)),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)}\\
&+Re\langle\alpha_2 AN(g(\zeta_\lambda e^{-\tau}+Z_0)-g(\zeta_\nu e^{-\tau}+Z_0)),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)}\}\\
&=\lim_{\lambda,\nu\searrow 0}\{\langle\alpha_1 AN(f(\zeta_\lambda)-f(\zeta_\nu)),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)}\\
&+\langle\alpha_2 AN(g(\zeta_\lambda e^{-\tau}+Z_0)-g(\zeta_\nu e^{-\tau}+Z_0)),\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)}\}\\
&=\lim_{\lambda,\nu\searrow 0}\langle\zeta^*_\lambda-\zeta^*_\nu,\zeta_\lambda-\zeta_\nu\rangle_{L^2(\Omega)},
\end{aligned} \tag{3.68}$$

and (3.61) follows from (3.68). Thus, (3.60) is proved.
Inserting (3.60) into (3.59), we obtain

$$\langle I\zeta+iA\zeta,\psi\rangle_{L^2(\Omega)}+\langle\alpha_1 ANf(\zeta)+\alpha_2 ANg(\zeta e^{-\tau}+Z_0),\psi\rangle_{L^2(\Omega)}=\langle l,\psi\rangle_{L^2(\Omega)}\qquad\forall\psi\in V, \tag{3.69}$$

and the surjectivity of T in (3.46) is deduced from (3.69). □

Let $l \in L^2(\Omega) \subset V'$. By Lemma 3.2, there exist $\zeta \in V = D(A^{\frac{1}{2}})$ such that (3.46) holds true. We show that $(\zeta, \theta)^T \in D(\mathcal{A})$.
By (3.41), we have

$$iA\zeta + AN\alpha_1 f(\zeta) + AN\alpha_2 g(\theta(x,1))) = A(i\zeta + N\alpha_1 f(\zeta) + N\alpha_2 g(\theta(x,1))) = l - \zeta \in L^2(\Omega). \quad (3.70)$$

Hence

$$i\zeta + N\alpha_1 f(\zeta) + N\alpha_2 g(\theta(x,1)) \in D(A), \qquad \zeta \in V \subset L^2(\Omega).$$

Invoking (3.25), we see that $W = (\zeta, \theta)^T \in D(\mathcal{A})$, as desired.
Thus, the surjectivity of $(I - \mathcal{A}) : D(\mathcal{A})$ onto $\mathcal{H}$ in (3.40) is established.

□

We have proved that the operator $\mathcal{A}$ in (3.24) is maximal dissipative on $\mathcal{H}$. Then, the claims (a) and (b) of Theorem 3.3 follow from the theory of m-dissipative operators [[11] p.33; p.71].

- Proof of Theorem 3.3 (c) (higher regularity).

We know from Lemma 3.1 that, for any dimension, we have :

$$(\zeta, \theta)^T \in D(\mathcal{A}) \longrightarrow \zeta|_\Gamma \in H^{\frac{1}{2}}(\Gamma),\, \theta \in L^2(\Gamma_2; H^1(0,1)).$$

By (3.37), we have

$$\theta(x,1) = \zeta + z_0 \in L^2(\Gamma_2).$$

Using assumption $(H.2)(i)$, we obtain

$$\|g(\theta(.,1))\|_{L^2(\Gamma_2)} = \left(\int_{\Gamma_2} |g(\theta(.,1))|^2 \, d\Gamma_2 \right)^{\frac{1}{2}} \leq L_1 \|\theta(.,1)\|_{L^2(\Gamma_2)} < \infty.$$

Consequently,

$$\theta(x,1) \in L^2(\Gamma_2) \longrightarrow g(\theta(x,1)) \in L^2(\Gamma_2) \longrightarrow Ng(\theta(x,1))|_{\Gamma_2}) \in H^{\frac{3}{2}}(\Omega), \quad (3.71)$$

after recalling the regularity of N in (3.22).
Case (c_1): $dim\,\Omega = 2$, $dim\,\Gamma_2 = 1$.
In this case, we have

$$H^{\frac{1}{2}}(\Gamma_2) \subset L^{10}(\Gamma_2), \quad (3.72)$$

as it follows from the usual embedding [[80], p.206, p.328]

$$W^{s,p'} \subset W^{t,q},\ 0 \leq t \leq s < \infty;\ 1 < p' \leq q < \infty,$$

with $s = \frac{1}{2}, p' = 2, t = 0, q = 10$, so that $s - \frac{n}{p'} \geq t - \frac{n}{q}, n = 1$, as required.
Thus, we have by combining (3.34) and (3.72):

$$(\zeta, \theta)^T \in D(\mathcal{A}) \longrightarrow \zeta|_{\Gamma_2} \in L^{10}(\Gamma_2),\, \theta \in L^2(\Gamma_2; H^1(0,1)).$$

$$\|f(\zeta)\|^2_{L^2(\Gamma_2)} = \int_{\Gamma_2} |f(\zeta)|^2 \, d\Gamma_2 = \int_{\Gamma_A=\{\Gamma_2, |\zeta|\geq 1\}} |f(\zeta)|^2 \, d\Gamma_A + \int_{\Gamma_B=\{\Gamma_2, |\zeta|\leq 1\}} |f(\zeta)|^2 \, d\Gamma_B.$$

From the continuity of f and assumption $(H.4)$, we have

$$\|f(\zeta)\|^2_{L^2(\Gamma_2)} \leq M^2 \int_{\Gamma_2} |\zeta|^{10} \, d\Gamma_2 + \int_{\Gamma_2} C^2 \, d\Gamma_2 < \infty.$$

Then,

$$f(\zeta|_{\Gamma_2}) \in L^2(\Gamma_2), \quad \text{and } Nf(\zeta|_{\Gamma_2}) \in H^{\frac{3}{2}}(\Omega). \tag{3.73}$$

Thus, returning to

$$A(-i\zeta - \alpha_1 Nf(\zeta) - \alpha_2 Ng(\theta(x,1)) = L \in L^2(\Omega),$$

in (3.35) with $N^* A\zeta = \zeta|_{\Gamma_1}$ by (3.34), we obtain via (3.71), (3.73) and $A^{-1}L \in D(A) \subset H^2(\Omega)$:

$$-i\zeta = A^{-1}L + \alpha_1 Nf(\zeta) + \alpha_2 Ng(\theta(x,1)) \in H^{\frac{3}{2}}(\Omega), \tag{3.74}$$

as desired. The sought-after conclusion $D(\mathcal{A}) \subset H^{\frac{3}{2}}(\Omega) \times L_2(\Gamma_2; H^1(0,1))$ in (3.31) then holds true.
Case (c_2): $dim\,\Omega = 3$, $dim\,\Gamma_2 = 2$. We start again with

$$(\zeta, \theta)^T \in D(\mathcal{A}) \longrightarrow \zeta|_{\Gamma_2} \in H^{\frac{1}{2}}(\Gamma_2),\, \theta \in L^2(\Gamma_2; H^1(0,1)).$$

In this case, we have

$$H^{\frac{1}{2}}(\Gamma_2) \subset L^4(\Gamma_2), \text{ so that } \zeta|_{\Gamma_2} \in L^4(\Gamma_2), \tag{3.75}$$

which follows from the embedding

$$W^{s,p} \subset W^{t,q}, \; 0 \leq t \leq s < \infty; \, 1 < p \leq q < \infty,$$

with $s = \frac{1}{2}, p = 2, t = 0, q = 4$, so that $s - \frac{n}{p} \geq t - \frac{n}{q}$, with $n = 2$ as required.
Then, invoking assumption (3.30), we obtain by (3.75),

$$\int_{\Gamma_2} |f(\zeta|_{\Gamma_2})|^{\frac{4}{r}} \, d\Gamma_2 \leq C_r \int_{\Gamma_2} |\zeta|_{\Gamma_2}|^4 \, d\Gamma_2 < \infty, \text{ i.e. } f(\zeta|_{\Gamma_2}) \in L^{\frac{4}{r}}(\Gamma_2), \tag{3.76}$$

(3.76) together with (3.21) implies via L^p-elliptic theory [[78], Chapter 3] that

$$\text{Dirichlet trace of } Nf(\zeta|_{\Gamma_2}) \text{ over } \Gamma_2 = [Nf(\zeta|_{\Gamma_2})]_{\Gamma_2} \in W^{1,\frac{4}{r}}(\Gamma_2). \tag{3.77}$$

Moreover, since $W^{1,\frac{4}{r}}(\Gamma_2) \subset L^{\frac{4}{r-2}}(\Gamma_2)$ which follows from the embedding

$$W^{s,p} \subset W^{t,q}, \; 0 \leq t \leq s < \infty; \, 1 < p \leq q < \infty,$$

with $s = 1, p = \frac{4}{r}, t = 0, q = \frac{4}{r-2}, r = 3 - \epsilon$, then

$$[Nf(\zeta|_{\Gamma_2})]_{\Gamma_2} \in L^{\frac{4}{r-2}}(\Gamma_2). \tag{3.78}$$

Next, with reference to (3.20) and (3.35), we have $A^{-1}L \in H^2(\Omega)$ for $L \in L^2(\Omega)$, hence by trace theory

$$[A^{-1}L]_{\Gamma_2} \in H^{\frac{3}{2}}(\Gamma_2) \subset L_a(\Gamma_2) \text{ for any } 2 \le a < \infty; \text{ in particular, } [A^{-1}L]_{\Gamma_2} \subset L^{\frac{4}{r-2}}(\Gamma_2), \tag{3.79}$$

from the embedding

$$W^{s,p} \subset W^{t,q},\ 0 \le t \le s < \infty;\ 1 < p \le q < \infty,$$

with $s = \frac{3}{2}, p = 2, t = 0, q = a$, so that $s - \frac{n}{p} \ge t - \frac{n}{q}$, with $n = 2$, which is true.
By (3.71), we have $Ng(\theta(x,1) \in H^{\frac{3}{2}}(\Omega)$, hence by trace theory

$$[Ng(\theta(x,1)|_{\Gamma_2})]_{\Gamma_2} \in H^1(\Gamma_2) \subset L_a(\Gamma_2) \text{ for any } 2 \le a < \infty; \text{ in particular, } [Ng(\theta(x,1)|_{\Gamma_2})]_{\Gamma_2} \subset L^{\frac{4}{r-2}}(\Gamma_2), \tag{3.80}$$

again invoking the embedding

$$W^{s,p} \subset W^{t,q},\ 0 \le t \le s < \infty;\ 1 < p \le q < \infty,$$

with $s = 1, p = 2, t = 0, q = a$, so that $s - \frac{n}{p} \ge t - \frac{n}{q}$, with $n = 2$, which is true.
Applying the Dirichlet trace to identity (3.74) and using (3.78), (3.79) and (3.80) in the resulting identity yields

$$-i\zeta|_{\Gamma_2} = [A^{-1}L]_{\Gamma_2} + [Nf(\zeta|_{\Gamma_2})]_{\Gamma_2} + [Ng(\theta(x,1)|_{\Gamma_2})]_{\Gamma_2} \in L^{\frac{4}{r-2}}(\Gamma_2),\ \ r = 3 - \epsilon. \tag{3.81}$$

This completes the first step of the bootstrap argument: the original regularity $\zeta|_{\Gamma_2} \in L^4(\Gamma_2)$ in (3.75) has been boosted to the new regularity $\zeta|_{\Gamma_2} \in L^{\frac{4}{r-2}}(\Gamma_2)$, where $\frac{4}{r-2} > 4$ for $r = 3 - \epsilon$.

The regularity $f(\zeta|_{\Gamma_2})$ has also been improved. Indeed, with $\zeta|_{\Gamma_2}$ in (3.81), we invoke assumption (3.30) with $r = 3 - \epsilon$ (i.e. $r < 3$) on f and obtain

$$\begin{cases} \int_{\Gamma_2} |f(\zeta|_{\Gamma_2})|^s \, d\Gamma_2 \le C_r \int_{\Gamma_2} |\zeta|_{\Gamma_2}|^{sr} \, d\Gamma_2 < \infty, \quad \text{with } sr = \frac{4}{r-2}, \\ \text{thus, } f(\zeta|_{\Gamma_2}) \in L^{\frac{4}{r(r-2)}}(\Gamma_2) \subsetneqq L^{\frac{4}{r}}(\Gamma_2). \end{cases} \tag{3.82}$$

Finite repetition of the bootstrap. After finitely many steps, the bootstrap argument will lead to the desired integrability

$$f(\zeta|_{\Gamma_2}) \in L^2(\Gamma_2) \tag{3.83}$$

We just quantify the second step . In fact, with $f(\zeta|_{\Gamma_2}) \in L^{p'}(\Gamma_2) \equiv W^{0,p'}(\Gamma_2)$ by (3.82), the same L_p-elliptic theory , with $p' = \frac{4}{r(r-2)}$ now yields via the definition (3.21) for N:

$$[Nf(\zeta|_{\Gamma_2})]_{\Gamma_2} \in W^{1,p'}(\Gamma_2) \subset L_q(\Gamma_2) \subset L^2(\Gamma_2),\ 2 < q = \frac{4}{r(r-2)-2}, \tag{3.84}$$

for the Neumann-Dirichlet map, with $q = \frac{4}{r(r-2)-2} > 2$ for $r = 3 - \epsilon$.
To obtain (3.84), we have used the embedding:

$$W^{s,p} \subset W^{t,q'},\ 0 \le t \le s < \infty;\ 1 < p \le q' < \infty,$$

with $s=1$, $p=p'$; $t=0$, so that $s-\frac{n}{p}\geq t-\frac{n}{q'}$, $n=2$, which is true, as required, for $q'\leq\frac{4}{r(r-2)-2}$, in particular for $q'=q$, hence , from the identity in(2.88), as well as from (3.79) and (3.80) with $a=q>2$, we obtain

$$-i\zeta|_{\Gamma_2}=[A^{-1}L]_{\Gamma_2}+[Nf(\zeta|_{\Gamma_2})]_{\Gamma_2}+[Ng(\theta(x,1)|_{\Gamma_2})]_{\Gamma_2}\in L^q(\Gamma_2),\;\; q=\frac{4}{r(r-2)-2}. \tag{3.85}$$

Thus, the original regularity of $\zeta|_{\Gamma_2}$ in (3.81) at the beginning of the second step (end of first step) has been further boosted to the new regularity of $\zeta|_{\Gamma_2}$ in (3.85), as $\frac{4}{r(r-2)-2}>\frac{4}{(r-2)}$ for $r=3-\epsilon$. After finitely many steps, one achieves (3.83). Finally, (3.83) then yields via the L_2-elliptic theory (3.22) that $(\zeta,\theta)^T\in D(\mathcal{A})$ implies, since $A^{-1}L\in H^2(\Omega)$ and $Ng(\theta(x,1)|_{\Gamma_2})\in H^{\frac{3}{2}}(\Omega)$ via (3.71):

$$Nf(\zeta|_{\Gamma_2})\in H^{\frac{3}{2}}(\Omega),\;\text{ hence }\; -i\zeta|_{\Gamma_2}=[A^{-1}L]_{\Gamma_2}+[Nf(\zeta|_{\Gamma_2})]_{\Gamma_2}+[Ng(\theta(x,1)|_{\Gamma_2})]_{\Gamma_2}\in H^{\frac{3}{2}}(\Omega). \tag{3.86}$$

The sought-after conclusion $D(\mathcal{A})\subset H^{\frac{3}{2}}(\Omega)\times L_2(\Gamma_2,H^1(0,1))$ in (3.31) then holds true in case (c_2). The proof of Theorem 3.3 is complete.

3.2.2 Proof of Theorem 3.1

- We first show that the energy $E(t)$ of every solution of (3.2) is decreasing.

Proposition 3.5. *The energy corresponding to any strong solution of the problem (3.2) is decreasing and there exists $C>0$ such that*

$$\frac{d}{dt}E(t)\leq -C\left\{\int_{\Gamma_2}f(u(x,t))\overline{u}(x,t)\,d\Gamma_2+\int_{\Gamma_2}|u(x,t-\tau)|^2\,d\Gamma_2\right\}, \tag{3.87}$$

where

$$C=min\left\{\alpha_1-\frac{\alpha_2L_1}{2K}-\frac{\mu\tau^{-1}}{2K},-\frac{\alpha_2L_1}{2}+\frac{\mu\tau^{-1}}{2}\right\}.$$

Proof. We multiply the first equation in problem (3.19) by $\overline{u}(x,t)$, integrate over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|u(x,t)|^2\;dx-i\int_{\Gamma_2}\frac{\partial u(x,t)}{\partial\nu}\,\overline{u}(x,t)\,d\Gamma+i\int_\Omega|\nabla u(x,t)|^2\,dx=0.$$

We take the real part and insert the boundary conditions of problem (3.19), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|u(x,t)|^2\;dx=\alpha_1 Re\int_{\Gamma_2}f(u(x,t))\,\overline{u}(x,t)\,d\Gamma_2+\alpha_2 Re\int_{\Gamma_2}g(y(x,1,t))\,\overline{u}(x,t)\,d\Gamma_2. \tag{3.88}$$

We multiply the second equation in (3.19) by $\mu\,\overline{y}(x,\rho,t)$ and integrate over $\Gamma_2\times(0,1)$, to obtain

$$\mu\int_{\Gamma_2}\int_0^1\{y_t(x,\rho,t)\overline{y}(x,\rho,t)+\tau^{-1}\mu y_\rho(x,\rho,t)\overline{y}(x,\rho,t)\}\,d\rho\,d\Gamma_2=0.$$

Therefore

$$\frac{\mu}{2}\frac{d}{dt}\int_{\Gamma_2}\int_0^1|y(x,\rho,t)|^2\;d\rho\,d\Gamma_2=-\frac{\tau^{-1}\mu}{2}\int_{\Gamma_2}|y(x,1,t)|^2d\Gamma_2+\frac{\tau^{-1}\mu}{2}\int_{\Gamma_2}|y(x,0,t)|^2\,d\Gamma_2. \tag{3.89}$$

From (3.88) and (3.89), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\int_{\Omega}|u(x,t)|^2\,dx+\mu\int_{\Gamma_2}\int_0^1|y(x,\rho,t)|^2\,d\rho\,d\Gamma_2\right\}\\&=Re\int_{\Gamma_2}\{\alpha_1 f(u(x,t))\,\overline{u}(x,t)+\alpha_2 g(y(x,1,t))\,\overline{u}(x,t)\}\,d\Gamma_2\\&-\frac{\tau^{-1}\mu}{2}\int_{\Gamma_2}|y(x,1,t)|^2\,d\Gamma_2+\frac{\tau^{-1}\mu}{2}\int_{\Gamma_2}|y(x,0,t)|^2\,d\Gamma_2.\end{aligned}\tag{3.90}$$

By using assumptions $(H.2)(i)$ and Cauchy-Schwartz's inequality, we have

$$\begin{aligned}\frac{d}{dt}E(t)\le&-\alpha_1 Re\int_{\Gamma_2}f(u(x,t))\,\overline{u}(x,t)\,d\Gamma_2+\frac{\alpha_2L_1}{2}\int_{\Gamma_2}|y(x,1,t)|^2\,d\Gamma_2+\frac{\alpha_2L_1}{2}\int_{\Gamma_2}|u(x,t)|^2\,d\Gamma_2\\&-\frac{\tau^{-1}\mu}{2}\int_{\Gamma_2}|y(x,1,t)|^2\,d\Gamma_2+\frac{\tau^{-1}\mu}{2}\int_{\Gamma_2}|u(x,t)|^2\,d\Gamma_2.\end{aligned}$$

By using assumptions $(H.1)(ii)$ and (iii), we have

$$\begin{aligned}\frac{d}{dt}E(t)\le&-\alpha_1K^{-1}\int_{\Gamma_2}f(u(x,t))\,\overline{u}(x,t)\,d\Gamma_2+\frac{\alpha_2L_1}{2}\int_{\Gamma_2}|y(x,1,t)|^2\,d\Gamma_2+\frac{\alpha_2L_1K^{-1}}{2}\int_{\Gamma_2}f(u(x,t))\,\overline{u}(x,t)\,d\Gamma_2\\&-\frac{\tau^{-1}\mu}{2}\int_{\Gamma_2}|y(x,1,t)|^2\,d\Gamma_2+\frac{\tau^{-1}\mu K^{-1}}{2}\int_{\Gamma_2}f(u(x,t))\,\overline{u}(x,t)\,d\Gamma_2.\end{aligned}$$

Therefore

$$\frac{d}{dt}E(t)\le-\left(\alpha_1K^{-1}-\frac{\alpha_2L_1}{2K}-\frac{\tau^{-1}\mu}{2K}\right)\int_{\Gamma_2}f(u(x,t))\,\overline{u}(x,t)\,d\Gamma_2-\left(-\frac{\alpha_2L_1}{2}+\frac{\tau^{-1}\mu}{2}\right)\int_{\Gamma_2}|y(x,1,t)|^2\,d\Gamma_2.\tag{3.91}$$

(3.91) can be rewritten as

$$\frac{d}{dt}E(t)\le-C\left\{\int_{\Gamma_2}f(u(x,t))\,\overline{u}(x,t)\,d\Gamma_2+\int_{\Gamma_2}|y(x,1,t)|^2\,d\Gamma_2\right\},\tag{3.92}$$

where

$$C=min\left\{\alpha_1-\frac{\alpha_2L_1}{2K}-\frac{\mu\tau^{-1}}{2K},-\frac{\alpha_2L_1}{2}+\frac{\mu\tau^{-1}}{2}\right\}.$$

which is positive due to (3.9). □

- Next, we establish an observability inequality for problem (3.2).

Theorem 3.6. *Let $n=1,2,....$ Assume hypothesis $(H.1)$ on f, $(H.2)$ on g and (A) on $\{\Omega,\Gamma_1,\Gamma_2\}$ then the solution of problem (3.2) satisfies the following inequality: there exists a constant $C_T>0$ such that*

$$\begin{aligned}E(t)\le E(0)\le C_T\bigg\{&\int_{\Sigma_2}|u(x,t)|^2\,d\Sigma_2+\int_{\Sigma_2}|u(x,t-\tau)|^2\,d\Sigma_2\\&+\int_{\Sigma_2}f(u(x,t))\overline{u}(x,t)\,d\Sigma_2+\|f(u)\|^2_{H_a^{-1}(\Sigma_2)}\bigg\}.\end{aligned}\tag{3.93}$$

Proof. Set

$$E(t) = E_s(t) + E_d(t),$$

where

$$E_s(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2\, dx,$$

and

$$E_d(t) = \frac{\mu}{2}\int_{\Gamma_2}\int_0^1 |u(x,t-\tau\rho)|^2\, d\rho\, dx.$$

We have from [[50] and [51]], for $T > 0$

$$E_s(0) \leq C_T\bigg\{\|u\|^2_{L^2(\Sigma_2)} + \int_0^T\int_{\Gamma_2}\left|\frac{\partial u}{\partial \nu}\right||u|\, d\Gamma_2\, dt + \left\|\frac{\partial u}{\partial \nu}\right\|^2_{H_a^{-1}(\Sigma_2)} + \|u\|^2_{H^{-1}(Q)}\bigg\}, \tag{3.94}$$

where $H_a^{-1}(\Sigma_2)$ is the dual space of the space $H_a^1(\Sigma_2) = H^{\frac{1}{2}}(0,T;L^2(\Gamma_2)) \cap L^2(0,T;H^1(\Gamma_2))$. We now impose the boundary conditions in (3.2). Then (3.94) becomes

$$\begin{aligned} E_s(0) \leq C_T\bigg\{&\|u\|^2_{L^2(\Sigma_2)} + \int_0^T\int_{\Gamma_2} |i\alpha_1 f(u(x,t)) + i\alpha_2 g(u(x,t-\tau))||u(x,t)|\, d\Gamma_2\, dt \\ &+ \|i\alpha_1 f(u) + i\alpha_2 g(u(.,.-\tau))\|^2_{H_a^{-1}(\Sigma_2)} + \|u\|^2_{H^{-1}(Q)}\bigg\}. \end{aligned} \tag{3.95}$$

Therefore

$$\begin{aligned} E_s(0) \leq C_T\bigg\{&\|u\|^2_{L^2(\Sigma_2)} + \int_0^T\int_{\Gamma_2} \left[|f(u(x,t))||u(x,t)| + |g(u(x,t-\tau))||u(x,t)|\right]\, d\Gamma_2\, dt \\ &+ \|f(u)\|^2_{H_a^{-1}(\Sigma_2)} + \|g(u(.,.-\tau))\|^2_{H_a^{-1}(\Sigma_2)} + \|u\|^2_{H^{-1}(Q)}\bigg\}. \end{aligned} \tag{3.96}$$

$E_d(t)$ can be rewritten, via a change of variable, as follows

$$E_d(t) = \frac{\mu}{2\tau}\int_{\Gamma_2}\int_{t-\tau}^t |u(x,s)|^2\, ds\, d\Gamma_2.$$

Hence

$$E_d(0) \leq \frac{\mu}{2\tau}\int_{\Gamma_2}\int_{-\tau}^0 |u(x,s)|^2\, ds\, d\Gamma_2. \tag{3.97}$$

By another change of variable in (3.97), we have for $T \geq \tau$

$$E_d(0) \leq C_T\int_0^T\int_{\Gamma_2} |u(x,t-\tau)|^2\, d\Gamma_2\, dt. \tag{3.98}$$

Combining (3.96) and (3.98) we obtain for any $T \geq \tau$

$$\begin{aligned} E(0) \leq C_T\bigg\{&\|u\|^2_{L^2(\Sigma_2)} + \int_0^T\int_{\Gamma_2} \left[|f(u(x,t))||u(x,t)| + |g(u(x,t-\tau))||u(x,t)|\right] d\Gamma_2\, dt \\ &+ \|f(u)\|^2_{H_a^{-1}(\Sigma_2)} + \|g(u(.,.-\tau))\|^2_{H_a^{-1}(\Sigma_2)} + \|u(.,.-\tau)\|^2_{L^2(\Sigma_2)} + \|u\|^2_{H^{-1}(Q)}\bigg\}. \end{aligned} \tag{3.99}$$

By using assumptions $(H.1)$, $(H.2)(i)$ and Cauchy-Schwartz's inequality, we obtain

$$E(0) \leq C_T\Big\{\|u\|^2_{L^2(\Sigma_2)} + \|u(.,.-\tau)\|^2_{L^2(\Sigma_2)} + \int_0^T\int_{\Gamma_2} f(u(x,t))\bar{u}(x,t)\,d\Gamma_2\,dt + \|f(u)\|^2_{H_a^{-1}(\Sigma_2)} + \|u\|^2_{H^{-1}(Q)}\Big\}.$$

From Proposition 3.5, we deduce

$$E(t) \leq E(0) \leq C_T\Big\{\|u\|^2_{L^2(\Sigma_2)} + \|u(.,.-\tau)\|^2_{L^2(\Sigma_2)} + \int_0^T\int_{\Gamma_2} f(u(x,t))\bar{u}(x,t)\;d\Gamma_2\,dt + \|f(u)\|^2_{H_a^{-1}(\Sigma_2)} + \|u\|^2_{H^{-1}(Q)}\Big\}. \tag{3.100}$$

The next step is to further clean estimate (3.100) by absorbing the term $\|u\|^2_{H^{-1}(Q)}$, via a nonlinear, compactness-uniqueness argument.

Lemma 3.3. *There exists a constant C_T (dependent on $E(0)$), such that the solution of problem (3.2) satisfies:*

$$\|u\|^2_{H^{-1}(Q)} \leq C_T(E(0))\Big\{\|u\|^2_{L^2(\Sigma_2)} + \|u(.,.-\tau)\|^2_{L^2(\Sigma_2)} + \int_0^T\int_{\Gamma_2} f(u(x,t))\bar{u}(x,t)\;d\Gamma_2\,dt + \|f(u)\|^2_{H_a^{-1}(\Sigma_2)}\Big\}. \tag{3.101}$$

Proof. It is based, on a compactness-uniqueness argument.

Step 1. Let (u_n) be a sequence of solutions to problem (3.2). Then

$$E_n(t) + C\left\{\int_0^t\int_{\Gamma_2} f(u_n(x,s))\overline{u}_n(x,s)\,d\Gamma_2\,ds + \int_0^t\int_{\Gamma_2} |u_n(x,s-\tau)|^2\,d\Gamma_2\,ds\right\} \leq E_n(0),\;\; 0 \leq t \leq T, \tag{3.102}$$

where $E_n(0)$ the energy of the initial data (u_n^0, f_n^0), it remains uniformly (in n) bounded by say, $E_n(0) \leq M$.
Hence, by (3.102),

$$E_n(t) \leq M,\;\; 0 \leq t \leq T, \tag{3.103}$$

We assume that (u_n) is such that (3.101) is violated; that is

$$\lim_{n\longrightarrow\infty} \frac{\|u_n\|^2_{H^{-1}(Q)}}{\|u_n\|^2_{L^2(\Sigma_2)} + \|u_n(.,-\tau)\|^2_{L^2(\Sigma_2)} + \int_0^T\int_{\Gamma_2} f(u_n)\overline{u}_n\,d\Gamma_2\,dt + \|f(u_n)\|^2_{H_a^{-1}(\Sigma_2)}} = \infty. \tag{3.104}$$

It follows from (3.103) that

$$u_n \longrightarrow \text{some } u,\;\text{weak star in } L^\infty(0,T;L^2(\Omega)), \tag{3.105}$$

$$u_{nt} \longrightarrow \text{some } u_t,\;\text{weak star in } L^\infty(0,T;H^{-2}(\Omega)), \tag{3.106}$$

$$u_n \longrightarrow u,\;\text{weakly in } L^2(0,T;L^2(\Omega)), \tag{3.107}$$

$$u_n \longrightarrow u,\;\text{strongly in } H^{-1}(Q), \tag{3.108}$$

and hence

$$\|u_n\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_{nt}\|^2_{L^\infty(0,T;H^{-2}(\Omega))} \leq Const \quad \text{for all } n \in \mathbb{N}. \tag{3.109}$$

The passage from (3.105) to (3.108) invokes a well-known compactness (in time and space). Since the injection $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is compact, (3.109) implies (see [9] and [75]) that for $0 < T < +\infty$ the injection

$$Z \hookrightarrow L^\infty(0,T;H^{-1}(\Omega)),$$

is also compact, where Z is the Banach space equipped with the norm on the left-hand side of (3.109), is also compact. As a consequence there is a subsequence still denoted by (u_n) such that

$$u_n \longrightarrow u \in L^\infty(0,T;H^{-1}(\Omega)) \quad \text{strongly.}$$

Thus

$$u_n \longrightarrow u, \text{ strongly in } \ H^{-1}(Q), \tag{3.110}$$

is proved.
Because of (3.108), the numerator in (3.104) is uniformly bounded. This implies that each (positive) term in the denominator in (3.104) must tend to zero, as $n \to \infty$:

$$\|u_n\|_{L^2(\Sigma_2)} \to 0; \ \ \|u_n(.,.-\tau)\|_{L^2(\Sigma_2)} \to 0; \ \ \int_0^T \int_{\Gamma_2} f(u_n)\overline{u}_n \, d\Gamma_2 \, dt \to 0; \ \ \|f(u_n)\|_{H_a^{-1}(\Sigma_2)} \to 0. \tag{3.111}$$

To continue, we need the following lemma.

Lemma 3.4. *Let (u_n) be a sequence of solution of problem (3.2) such that, as $n \to 0$,*

$$u_n \to 0 \quad \textit{in } L^2(\Sigma_2); \qquad \int_{\Sigma_2} f(u_n)\overline{u}_n \, d\Sigma_2 \to 0, \tag{3.112}$$

as asserted in (3.111). Then

$$\int_{\Sigma_2} f(u_n)\phi \, d\Sigma_2 \to 0, \qquad \forall \phi \in C^\infty(\Sigma_2). \tag{3.113}$$

Proof of (3.113). To prove (3.113) as a consequence of (3.112), given $u_n(x,t)$ and $\delta > 0$, we divide Σ_2 accordingly as follows:

$$\Sigma_A(\delta,n) \equiv \{(x,t) \in \Sigma_2 : |u_n(x,t)| \leq \delta\}; \qquad \Sigma_B(\delta,n) \equiv \{(x,t) \in \Sigma_2 : |u_n(x,t)| \geq \delta\}, \tag{3.114}$$

with $\delta > 0$ to be selected below. Then, for $\phi \in C^\infty(\Sigma_2)$:

$$\int_{\Sigma_2} f(u_n)\phi \, d\Sigma_2 = \int_{\Sigma_A(\delta,n)} f(u_n)\phi \, d\Sigma_A + \int_{\Sigma_B(\delta,n)} f(u_n)\phi \, d\Sigma_B. \tag{3.115}$$

Since f is continuous and $f(0) = 0$ by $(H.1)$, then given $\epsilon > 0$, there is $\delta_0 = \delta_0(\epsilon) > 0$ such that $|f(u_n(x,t))| < \epsilon$, for $(x,t) \in \Sigma_A(\delta_0, n)$.
Thus,

$$\left| \int_{\Sigma_A(\delta_0,n)} f(u_n)\phi \, d\Sigma_A \right| \leq \epsilon \|\phi\|_{L^\infty(\Sigma_2)}. \tag{3.116}$$

Moreover, from the definition of $\Sigma_B(\delta_0, n)$ and the assumption $(H.1)$:

$$\begin{aligned} \left| \int_{\Sigma_B(\delta_0,n)} f(u_n)\phi \, d\Sigma_B \right| &= \left| \int_{\Sigma_B(\delta_0,n)} f(u_n)\frac{\overline{u}_n}{\overline{u}_n}\phi \, d\Sigma_B \right| \\ &\leq \frac{\|\phi\|_{L^\infty(\Sigma_2)}}{\delta_0} \int_{\Sigma_B(\delta_0,n)} f(u_n)\overline{u}_n \, d\Sigma_B \\ &\leq \frac{\|\phi\|_{L^\infty(\Sigma_2)}}{\delta_0} \int_{\Sigma_2(\delta_0,n)} f(u_n)\overline{u}_n \, d\Sigma_2 \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.117}$$

by (3.112). Thus, given $\epsilon > 0$, $\delta_0 > 0$, there exists $N = N_{\epsilon,\delta_0}$, such that for all $n > N$ we have the integral in (3.117) is less than $(\delta_0\epsilon)$, and hence

$$\left| \int_{\Sigma_B(\delta_0,n)} f(u_n)\phi \, d\Sigma_B \right| \leq \epsilon \|\phi\|_{L^\infty(\Sigma_2)}, \qquad n > N_{\epsilon,\delta_0}. \tag{3.118}$$

Combining (3.116) and (3.118) in (3.115) yields (3.113), as desired. □

Next, we specialize to $\phi \in C^\infty(\Sigma_2)$ such that $\phi\,|_{t=0} = \phi\,|_{t=T} = 0$ and $\phi = 0$ on Σ_1. Integrating by parts (second Green Theorem) and using problem (3.2) for u_n yields:

$$\begin{aligned} 0 = \langle i(u_n)_t + \Delta u_n, \phi \rangle_{L^2(Q)} = -i \int_0^T \int_\Omega u_n \phi_t \, dx \, dt + i \int_{\Sigma_2} \alpha_1 f(u_n)\phi \, d\Sigma_2 + i \int_{\Sigma_2} \alpha_2 g(u_n(x, t-\tau))\phi \, d\Sigma_2 \\ - \int_0^T \int_{\Gamma_2} u_n \frac{\partial \phi}{\partial \nu} \, d\Sigma_2 + \int_0^T \int_\Omega u_n \Delta\phi \, dx \, dt. \end{aligned} \tag{3.119}$$

On the RHS of (3.119), we invoke the weak convergence (3.107) on its first and last integral terms; as well as the convergence to zero of its second integral term by (3.113), $(H.2)$ and the second statement of (3.111) on its third integral and the firsts statement of (3.111) on its penultimate integral term. The final result is

$$-i \int_0^T \int_\Omega u\phi_t \, dx \, dt + \int_0^T \int_\Omega u\Delta\phi \, dx \, dt = 0 \tag{3.120}$$

for $u = 0$ on Σ and $\frac{\partial u}{\partial \nu} = 0$ on Σ_2 (the first claim follows from $u_n = 0$ on Σ_0 and $u_n \to 0$ on $L^2(\Sigma_2)$; the second by $\frac{\partial u_n}{\partial \nu} = i\alpha_1 f(u_n) + i\alpha_2 g(u_n(x, t-\tau)) \to 0$ (by using the last statement of (3.111) for f and $(H.2)$ and the second statement of (3.111) for g)).
Thus specializing further $\phi \in D(Q)$, we see that (3.120) is the weak formulation of the following problem

$$\begin{cases} iu_t + \Delta u = 0 & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Sigma_2. \end{cases} \tag{3.121}$$

By Holmogren's uniqueness Theorem ([54], Chap. 1, Theorem.8.2), we conclude that (3.121) implies then that $u = 0$ in Q. Thus, all convergences in (3.105), (3.107), (3.108) are to the limit $u = 0$.

Step 2. Denote

$$c_n = \|u_n\|_{H^{-1}(Q)}, \quad \widehat{u}_n = \frac{1}{c_n} u_n.$$

Thus

$$\|\widehat{u}_n\|^2_{H^{-1}(Q)} = 1. \tag{3.122}$$

Dividing the numerator and the denominator of the fraction in (3.104) by c_n^2, and using (3.122), we obtain

$$\lim_{n \longrightarrow \infty} \frac{1}{\|\widehat{u}_n\|^2_{L^2(\Sigma_2)} + \frac{\|u_n(.,-\tau)\|^2_{L^2(\Sigma_2)}}{c_n^2} + \frac{\int_0^T \int_{\Gamma_2} f(u_n)\overline{u}_n \, d\Gamma_2 \, dt}{c_n^2} + \frac{\|f(u_n)\|^2_{H_a^{-1}(\Sigma_2)}}{c_n^2}} = \infty. \tag{3.123}$$

Thus, (3.123) implies

$$\|\widehat{u}_n\|^2_{L^2(\Sigma_2)} \to 0; \qquad \frac{\int_0^T \int_{\Gamma_2} f(u_n)\overline{u}_n \, d\Gamma_2 \, dt}{c_n^2} \to 0; \qquad \frac{\|f(u_n)\|^2_{H_a^{-1}(\Sigma_2)}}{c_n^2} \to 0. \tag{3.124}$$

On the other hand, since each solution satisfies the energy estimate (3.100), we obtain after dividing both sides of such estimate by c_n^2 and invoking (3.122)

$$\begin{aligned} \frac{1}{c_n^2} E_n(t) \leq C_T \Bigg\{ & \|\widehat{u}_n\|^2_{L^2(\Sigma_2)} + \frac{\|u_n(.,.-\tau)\|^2_{L^2(\Sigma_2)}}{c_n^2} + \int_0^T \frac{\int_{\Gamma_2} f(u_n(x,t))\overline{u}_n(x,t)}{c_n^2} \, d\Sigma_2 \\ & + \frac{\|f(u_n)\|^2_{H_a^{-1}(\Sigma_2)}}{c_n^2} + 1 \Bigg\}, \end{aligned} \tag{3.125}$$

as $f(u_n)\overline{u}_n = |f(u_n)|\,|u_n|$ by (H.1). Invoking on the RHS of (3.125) the convergence statements in (3.124), we then arrive at

$$\|\widehat{u}_n(t)\|_{L^2(\Omega)} \leq Const, \quad 0 \leq t \leq T, n = 1, 2, ... \tag{3.126}$$

Hence, as in (3.105), (3.106), (3.107) and (3.108), we deduce from (3.126) that

$$\widehat{u}_n \longrightarrow \text{some } \widehat{u}, \text{ weak star in } L^\infty(0,T; L^2(\Omega)), \tag{3.127}$$

$$\widehat{u}_{nt} \longrightarrow \text{some } \widehat{u}_t, \text{ weak star in } L^\infty(0,T; H^{-2}(\Omega)), \tag{3.128}$$

$$\widehat{u}_n \longrightarrow \widehat{u}, \text{ weakly in } L^2(0,T; L^2(\Omega)), \tag{3.129}$$

$$\widehat{u}_n \longrightarrow \widehat{u}, \text{ strongly in } H^{-1}(Q) \text{ so that } \|\widehat{u}\|_{H^{-1}(Q)} = 1, \tag{3.130}$$

by (3.122).

Next, we divide problem (3.2) for u_n by c_n, thus obtaining

$$\begin{cases} (\widehat{u}_n)_t - i\Delta \widehat{u}_n = 0 & \text{in } Q, \\ \widehat{u}_n = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial \widehat{u}_n}{\partial \nu} = i\alpha_1 \frac{f(u_n)}{c_n} + i\alpha_2 \frac{g(u_n(x,t-\tau))}{c_n} & \text{on } \Sigma_2. \end{cases} \tag{3.131}$$

Proceeding as in Step 1. from (3.111) to (3.121) we then arrive at the following conclusion that the limit function $\widehat{u}$ satisfies the following problem:

$$\begin{cases} \widehat{u}_t - i\Delta\widehat{u} = 0 & \text{in} \quad Q, \\ \widehat{u} = 0 & \text{on} \quad \Sigma, \\ \dfrac{\partial\widehat{u}}{\partial\nu} = 0 & \text{on} \quad \Sigma_2. \end{cases} \tag{3.132}$$

Via Holmogren's uniqueness Theorem, problem (3.132) implies then that $\widehat{u} = 0$ in Q and this contradicts $\|\widehat{u}\|_{H^{-1}(Q)} = 1$ in (3.130). Thus, the contradiction hypothesis (3.104) is false. Hence (3.101). □

As a corollary, using (3.101) in the RHS of estimate (3.100), we obtain (3.93). □

- Estimate of the term $f(u)$ in $H_a^{-1}(\Sigma_2)$.

Because of assumption $(H.1)$, whereby $f(z)\overline{z} = |f(z)\overline{z}|$, we set

$$\Lambda_{\Sigma_2}(u) = \int_0^T \int_{\Gamma_2} f(u)\overline{u}\, d\Gamma_2\, dt = \int_0^T \int_{\Gamma_2} |f(u)\overline{u}|\, d\Gamma_2\, dt = |f(u)\overline{u}|_{L^1(\Sigma_2)}, \tag{3.133}$$

where $L^1(\Sigma_2) = L^1(0,T;L^1(\Gamma_2))$.
Denote

$$\Sigma_{1A} \equiv \{(x,t) \in \Sigma_2 : |u(x,t)| \geq 1\}; \qquad \Sigma_{1B} \equiv \{(x,t) \in \Sigma_2 : |u(x,t)| < 1\}. \tag{3.134}$$

Proposition 3.7. *Let $n = dim\Omega = 2, 3$. Assume hypotheses $(H.1)$, $(H.2)$ and $(H.4)$. Let u be the solution of (3.2) guaranteed by Theorem 3.3. Then there exists a positive constant C_p (depending on p in $(H.4)$) such that the following estimate holds true for problem (3.2):*

$$\|f(u)\|^2_{H_a^{-1}(\Sigma_2)} \leq \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}(\Lambda_{\Sigma_2}(u)) + 2\|f(u)\|^2_{L^2(\Sigma_{1B})}, \tag{3.135}$$

where: $p = 5$ for dim $\Omega = 2$; $p = 3$ for dim $\Omega = 3$, as in assumption $(H.4)$; $E(0)$ and $\Lambda_{\Sigma_2}(u)$ are defined by (3.8) and (3.133), respectively. Moreover, $\tilde{C}_p = C_p(\frac{1}{C})^{\frac{p-1}{p+1}}$. with M defined by $(H.4)$.

Proof. We need the following result.

Lemma 3.5. *(Lasiecka and Triggiani [47])*
Let dim $\Omega = 2, 3$. Under assumptions $(H.1)$ and $(H.4)$, the following estimate holds true, where M is defined in $(H.4)$:

$$\|f(u)\|_{H_a^{-1}(\Sigma_2)} \leq \|f(u)\|_{L^2(\Sigma_{1B})} + c_p M^{\frac{1}{p+1}} \|f(u)\overline{u}\|^{\frac{p}{p+1}}_{H_a^{-1}(\Sigma_2)}, \tag{3.136}$$

with p as specified below (3.135), c_p the constant defined in [[47] p. 518].

Recalling $\Lambda_{\Sigma_2}(u)$ from (3.133), we rewrite inequality (3.136) as follows, after squaring both sides:

$$\|f(u)\|^2_{H_a^{-1}(\Sigma_2)} \leq 2\|f(u)\|^2_{L^2(\Sigma_{1B})} + C_p\|f(u)\overline{u}\|^{\frac{2p}{p+1}}_{H_a^{-1}(\Sigma_2)} \tag{3.137}$$

$$= 2\|f(u)\|^2_{L^2(\Sigma_{1B})} + C_p[\Lambda_{\Sigma_2}(u)]^{\frac{p-1}{p+1}}[\Lambda_{\Sigma_2}(u)], \tag{3.138}$$

where $C_p = 2(c_p M^{\frac{1}{p+1}})^2$.
Next, we have from Proposition 3.5 in the notation of (3.133) for $\Lambda_{\Sigma_2}(u)$ as

$$E(T) + C\left\{\Lambda_{\Sigma_2}(u) + \int_0^T \int_{\Gamma_2} |u(x,t-\tau)|^2\, d\Gamma_2\, dt\right\} \leq E(0), \text{ thus } \Lambda_{\Sigma_2}(u) \leq \frac{1}{C}E(0). \tag{3.139}$$

Using the inequality of (3.139) in (3.137) yields:

$$\|f(u)\|^2_{H_a^{-1}(\Sigma_2)} \leq 2\|f(u)\|^2_{L^2(\Sigma_{1B})} + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}[\Lambda_{\Sigma_2}(u)], \tag{3.140}$$

where $\tilde{C}_p = C_p(\frac{1}{C})^{\frac{p-1}{p+1}}$, and (3.140) proves (3.135), as desired. Proposition 3.7 is established. □

- Estimates of $u \in L^2(\Sigma_2)$ and $f(u)\overline{u} \in L^2(\Sigma_2)$.

Lemma 3.6. *Assume* $(H.1)$ *and recall (3.133). Then, the solution of problem (3.2) guaranteed by Theorem 3.3 satisfies:*

$$\int_{\Sigma_{1A}} |u(x,t)|^2\, d\Sigma_{1A} \leq K^{-1}\int_{\Sigma_{1A}} f(u(x,t))\overline{u}(x,t)\, d\Sigma_{1A} \leq K^{-1}\Lambda_{\Sigma_2}(u), \tag{3.141}$$

and

$$\int_{\Sigma_{1B}} |u(x,t)|^2\, d\Sigma_{1B} + \int_{\Sigma_{1B}} |f(u(x,t))|^2\, d\Sigma_{1B} \leq (mes\,\Sigma_2)\tilde{h}\left(\int_{\Sigma_2} f(u(x,t))\overline{u}(x,t)\, d\Sigma_2\right)$$

$$= (mes\,\Sigma_2)\tilde{h}(\Lambda_{\Sigma_2}(u)), \tag{3.142}$$

where $\tilde{h}$ *is the concave strictly increasing function defined in (3.5).*

Proof. Inequality (3.141) is an application of assumption $(H.1)$ in the notation of (3.133).
We invoke the property (3.4) for h where the constant δ can be taken to be $\delta = 1$ and by using assumption $(H.1)$, we have

$$\int_{\Sigma_{1B}} |u(x,t)|^2\, d\Sigma_{1B} + \int_{\Sigma_{1B}} |f(u(x,t))|^2\, d\Sigma_{1B} \leq \int_{\Sigma_{1B}} h(f(u(x,t))\overline{u}(x,t))\, d\Sigma_{1B} \leq \int_{\Sigma_2} h(f(u(x,t))\overline{u}(x,t))\, d\Sigma_2. \tag{3.143}$$

By Jensen's inequality [[53], p. 38],

$$\int_{\Sigma_2} h(f(u(x,t))\overline{u}(x,t))\, d\Sigma_2 \leq (mes\,\Sigma_2)h\left(\frac{1}{mes\,\Sigma_2}\int_{\Sigma_2} f(u(x,t))\overline{u}(x,t)\, d\Sigma_2\right) \tag{3.144}$$

$$\leq (mes\,\Sigma_2)\tilde{h}(\Lambda_{\Sigma_2}(u)) = (mes\,\Sigma_2)\tilde{h}(\Lambda_{\Sigma_2}(u)), \tag{3.145}$$

where $\tilde{h}$ is defined in (3.5) and $\Lambda_{\Sigma_2}(u)$ is defined in (3.133). Then, (3.145) establishes (3.142). □

- Completion of the Proof of Theorem 3.1

Lemma 3.7. *Assume assumptions* $(H.1)$ *and* $(H.4)$*. Then the problem (3.2) satisfies:*

$$\int_{\Sigma_{1B}} |u(x,t)|^2 \, d\Sigma_{1B} + \|f(u)\|^2_{H_a^{-1}(\Sigma_2)} \leq \left(K^{-1} + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}\right)\Lambda_{\Sigma_2}(u) + 2(mes\,\Sigma_2)\tilde{h}(\Lambda_{\Sigma_2}(u)). \tag{3.146}$$

Proof.

$$\int_{\Sigma_{1B}} |u(x,t)|^2 \, d\Sigma_{1B} + \|f(u)\|^2_{H_a^{-1}(\Sigma_2)} = \int_{\Sigma_{1A}} |u(x,t)|^2 \, d\Sigma_{1A} + \int_{\Sigma_{1B}} |u(x,t)|^2 \, d\Sigma_{1B} + \|f(u)\|^2_{H_a^{-1}(\Sigma_2)}.$$

By (3.135), we have

$$\begin{aligned}\int_{\Sigma_{1B}} |u(x,t)|^2 \, d\Sigma_{1B} + \|f(u)\|^2_{H_a^{-1}(\Sigma_2)} \leq \int_{\Sigma_{1A}} |u(x,t)|^2 \, d\Sigma_{1A} + \int_{\Sigma_{1B}} |u(x,t)|^2 \, d\Sigma_{1B} + 2\int_{\Sigma_{1B}} |f(u(x,t)|^2 \, d\Sigma_{1B} \\ + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}(\Lambda_{\Sigma_2}(u)).\end{aligned}$$

By (3.141) and (3.142), we obtain

$$\begin{aligned}\int_{\Sigma_{1B}} |u(x,t)|^2 \, d\Sigma_{1B} + \|f(u)\|^2_{H_a^{-1}(\Sigma_2)} &\leq K^{-1}\Lambda_{\Sigma_2}(u) + 2(mes\,\Sigma_2)\tilde{h}(\Lambda_{\Sigma_2}(u)) \\ &+ \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}(\Lambda_{\Sigma_2}(u)).\end{aligned} \tag{3.147}$$

Then, (3.147) establishes (3.146). □

Lemma 3.8. *Assume* $(H.1)$ *and* $(H.4)$*. Then, the energy* $E(T)$ *of problem (3.2) satisfies the inequality*

$$E(T) + p(E(T)) \leq E(0), \tag{3.148}$$

where $p(.)$ *defined by (3.6).*

Proof. Returning to (3.93), recalling (3.133), and using (3.146), we have

$$\begin{aligned}E(t) \leq E(0) \leq C_T\left[K^{-1} + 1 + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}\right]\Lambda_{\Sigma_2}(u) + C_T\int_{\Sigma_2} |u(x,t-\tau)|^2 \, d\Sigma_2 \\ + 2C_T(mes\,\Sigma_2)\tilde{h}(\Lambda_{\Sigma_2}(u)).\end{aligned}$$

Since $h(.)$, hence $\tilde{h}(.)$, is strictly increasing, where we have

$$\begin{aligned}E(t) \leq E(0) \leq C_T\left[K^{-1} + 1 + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}\right]\Lambda_{\Sigma_2}(u) + C_T\int_{\Sigma_2} |u(x,t-\tau)|^2 \, d\Sigma_2 \\ + 2C_T(mes\,\Sigma_2)\tilde{h}\left(\Lambda_{\Sigma_2}(u) + \int_{\Sigma_2} |u(x,t-\tau)|^2 \, d\Sigma_2\right).\end{aligned} \tag{3.149}$$

Setting

$$K_1 = \frac{1}{2C_T mes\,\Sigma_2}, \quad K_2 = \frac{K^{-1} + 1 + \tilde{C}_p(E(0))^{\frac{p-1}{p+1}}}{2mes\,\Sigma_2}. \tag{3.150}$$

Consequently, for $t = T$

$$E(T) \le \frac{K_2}{K_1}\Lambda_{\Sigma_2}(u) + \frac{1}{2K_1 mes\,\Sigma_2}\int_{\Sigma_2} |u(x,t-\tau)|^2\, d\Sigma_2 + \frac{1}{K_1}\tilde{h}\Big(\Lambda_{\Sigma_2}(u) + \int_{\Sigma_2} |u(x,t-\tau)|^2\, d\Sigma_2\Big). \tag{3.151}$$

Set

$$C'' = max\{K_2, \frac{1}{2mes\,\Sigma_2}\}.$$

□

Therefore

$$K_1 E(T) \le (C''I + \tilde{h})\Big(\Lambda_{\Sigma_2}(u) + \int_{\Sigma_2} |u(x,t-\tau)|^2\, d\Sigma_2\Big). \tag{3.152}$$

On the other hand, integrate the inequality (3.87) over $(0,T)$, we obtain

$$\Lambda_{\Sigma_2}(u) + \int_{\Sigma_2} |u(x,t-\tau)|^2\, d\Sigma_2 \le C^{-1}(E(0) - E(T)). \tag{3.153}$$

By (3.152) and (3.153), we have

$$(C''I + \tilde{h})^{-1}(K_3 E(T)) = p(E(T)) \le E(0) - E(T), \tag{3.154}$$

where $K_3 = CK_1$.
Finally,

$$p(E(T)) + E(T) \le E(0) \tag{3.155}$$

and Lemma 3.8 is established.
Applying the result of Proposition 3.8 we obtain for $m = 0, 1, 2....$

$$p(E(m(T+1)) + E(m(T+1)) \le E(mT).$$

Thus, we are in a position to apply [[43], Lemma 3.3, p.531] with

$$s_m = E(mT),\ s_0 = E(0).$$

This yields

$$E(mT) \le S(m),\ m = 0, 1, 2, ...$$

Let $t = mT + \tau$ and recall the evolution property, we obtain

$$E(t) \le E(mT) \le S(m) \le S(\tfrac{t-\tau}{T}) \le S(\tfrac{t}{T} - 1) \quad \text{for}\, t > T,$$

which completes the proof of Theorem 3.1.

3.3 Stability of the Schrödinger equation with a delay term in the nonlinear internal feedback

3.3.1 Well-posedness of problem (3.12)

Inspired from [62], we introduce the auxiliary variable:

$$y(x,\rho,t) = u(x,t-\tau\rho); \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Then, problem (3.12) is equivalent to

$$\begin{cases} u_t(x,t) = i\Delta u(x,t) - a(x)\{\alpha_1 f(u(x,t)) - \alpha_2 g(y(x,1,t))\} & \text{in } \Omega \times (0;+\infty), \\ y_t(x,\rho,t) + \tau^{-1} y_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ y(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } \Omega \times (0,1), \\ y(x,0,t) = u(x,t) & \text{in } \Omega \times (0,\infty). \end{cases} \tag{3.156}$$

Let $\widehat{\mathcal{H}}$ denote the Hilbert space.

$$\widehat{\mathcal{H}} = L^2(\Omega) \times L^2(\Omega; L^2(0,1)),$$

equipped with the inner product:

$$\left\langle \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}; \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right\rangle_{\widehat{\mathcal{H}}} = Re \int_\Omega u_1(x)\overline{u_1(x)}\, dx + \mu Re \int_\Omega a(x) \int_0^1 y_1(x,\rho)\overline{y_2(x,\rho)}\, d\rho\, dx.$$

Set

$$U(t) = (u,y)^T,\ U_0 = (u_0, f_0)^T.$$

Then problem (3.156) can be formulated as an abstract Cauchy problem in $\widehat{\mathcal{H}}$

$$\begin{cases} \dfrac{dU}{dt}(t) = \tilde{\mathcal{A}}(U(t)), \\ U(0) = U_0, \end{cases} \tag{3.157}$$

where the operator $\tilde{\mathcal{A}}$ is defined by

$$\tilde{\mathcal{A}} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} i\Delta u - a\alpha_1 f(u) - a\alpha_2 g(y(.,1)) \\ -\tau^{-1} y_\rho \end{pmatrix}, \tag{3.158}$$

with

$$D(\tilde{\mathcal{A}}) = \{(u,y) \in H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega; H^1(0,1)), u = y(.,0) \text{ in } \Omega\}. \tag{3.159}$$

Theorem 3.8. *Assume* $(H.1)-(H.3)$ *and* $(H.5)$*. Then, For every* $U_0 \in \widehat{\mathcal{H}}$*, the problem (3.157) has a unique (nonlinear contraction semigroup) solution* U *whose regularity depends on the initial datum* U_0 *as follows:*

$$\begin{aligned} U(.) &\in C([0,+\infty);\widehat{\mathcal{H}}) \text{ if } U_0 \in \widehat{\mathcal{H}}, \\ U(.) &\in C^1([0,+\infty);\widehat{\mathcal{H}}) \cap C([0,+\infty);D(\tilde{\mathcal{A}})) \text{ if } U_0 \in D(\tilde{\mathcal{A}}). \end{aligned}$$

Proof. According to nonlinear semigroup theory, we need only to show that $\tilde{\mathcal{A}}$ defined by (3.158) and (3.159) is maximal dissipative on $\widehat{\mathcal{H}}$.
Let $U = (\zeta, \theta)^T \in D(\tilde{\mathcal{A}}); V = (\tilde{\zeta}, \tilde{\theta})^T \in D(\tilde{\mathcal{A}})$.
Then

$$\begin{aligned} Re\left\langle \tilde{\mathcal{A}}U - \tilde{\mathcal{A}}V; U - V\right\rangle_{\widehat{\mathcal{H}}} &= Re\int_\Omega i(\nabla\zeta(x) - \nabla\tilde{\zeta}(x))\overline{(\zeta(x)-\tilde{\zeta}(x))}\,dx - \alpha_1 Re\int_\Omega a(x)(f(\zeta(x)) - f(\tilde{\zeta}(x))) \\ &\overline{(\zeta(x)-\tilde{\zeta}(x))}\,dx - \alpha_2 Re\int_\Omega a(x)(g(\theta(x,1)) - g(\tilde{\theta}(x,1)))\overline{(\zeta(x)-\tilde{\zeta}(x))}\,dx \\ &- \mu\tau^{-1} Re\int_\Omega a(x)\int_0^1 (\theta_\rho(x,\rho) - \tilde{\theta}_\rho(x,\rho))\overline{(\theta(x,\rho)-\tilde{\theta}(x,\rho))}\,d\rho\,dx. \end{aligned}$$

Applying Green's theorem, we get

$$\begin{aligned} Re\left\langle \tilde{\mathcal{A}}U - \tilde{\mathcal{A}}V; U - V\right\rangle_{\widehat{\mathcal{H}}} =& Re\int_\Gamma \frac{\partial(\zeta(x)-\tilde{\zeta}(x))}{\partial\nu}\overline{(\zeta(x)-\tilde{\zeta}(x))}\,dx - Re\, i\int_\Omega \nabla(\zeta(x)-\tilde{\zeta(x)})\nabla\overline{(\zeta(x)-\tilde{\zeta}(x))}\,dx \\ &- \alpha_1 Re\int_\Omega a(x)(f(\zeta(x)) - f(\tilde{\zeta}(x)))\overline{(\zeta(x)-\tilde{\zeta}(x))}\,dx \\ &- \alpha_2 Re\int_\Omega a(x)(g(\theta(x,1)) - g(\tilde{\theta}(x,1)))\overline{(\zeta(x)-\tilde{\zeta}(x))}\,dx \\ &- \mu\tau^{-1} Re\int_\Omega a(x)\int_0^1 (\theta_\rho(x,\rho) - \tilde{\theta}_\rho(x,\rho))\overline{(\theta(x,\rho)-\tilde{\theta}(x,\rho))}\,d\rho\,dx. \end{aligned} \tag{3.160}$$

Integrating by parts in ρ the last term on the right-hand side of (3.160), we obtain

$$\begin{aligned} &\int_\Omega a(x)\int_0^1 (\theta_\rho(x,\rho) - \tilde{\theta}_\rho(x,\rho))\overline{(\theta(x,\rho)-\tilde{\theta}(x,\rho))}\,d\rho\,dx \\ &= -\int_\Omega a(x)\int_0^1 (\theta(x,\rho) - \tilde{\theta}(x,\rho))\overline{(\theta(x,\rho)_\rho-\tilde{\theta}_\rho(x,\rho))}\,d\rho\,dx \\ &+ \int_\Omega a(x)|\theta(x,1) - \tilde{\theta}(x,1))|^2\,dx - \int_\Omega a(x)|\theta(x,0) - \tilde{\theta}(x,0))|^2\,dx, \end{aligned}$$

or equivalently

$$\begin{aligned} &2Re\int_\Omega a(x)\int_0^1 (\theta_\rho(x,\rho) - \tilde{\theta}_\rho(x,\rho))\overline{(\theta(x,\rho)-\tilde{\theta}(x,\rho))}\,d\rho\,dx \\ &= \int_\Omega a(x)|\theta(x,1) - \tilde{\theta}(x,1))|^2\,dx - \int_\Omega a(x)|\theta(x,0) - \tilde{\theta}(x,0))|^2\,dx. \end{aligned}$$

Therefore

$$\begin{aligned}Re\left\langle \tilde{\mathcal{A}}U-\tilde{\mathcal{A}}V;U-V\right\rangle_{\widehat{\mathcal{H}}} &= -\alpha_1 Re\int_\Omega a(x)(f(\zeta(x))-f(\tilde{\zeta}(x)))\overline{(\zeta(x)-\tilde{\zeta}(x))}\,dx\\ &- \alpha_2 Re\int_\Omega a(x)(g(\theta(x,1))-g(\tilde{\theta}(x,1)))\overline{(\zeta(x)-\tilde{\zeta}(x))}\,dx\\ &- \frac{\mu\tau^{-1}}{2}\int_\Omega a(x)|\theta(x,1)-\tilde{\theta}(x,1))|^2\,dx+\frac{\mu\tau^{-1}}{2}\int_\Omega a(x)|\theta(x,0)-\tilde{\theta}(x,0))|^2\,dx.\end{aligned}$$

From assumptions $(H.1)(ii)$, $(H.2)$ and the Cauchy-Schwartz's inequality, we have

$$\begin{aligned}Re\left\langle \tilde{\mathcal{A}}U-\tilde{\mathcal{A}}V;U-V\right\rangle_{\widehat{\mathcal{H}}} &\le -K\alpha_1\int_\Omega a(x)|\zeta(x)-\tilde{\zeta}(x)|^2\,dx+\frac{\alpha_2L_1}{2}\int_\Omega a(x)|\theta(x,1)-\tilde{\theta}(x,1)|^2\,dx\\ &+\frac{\alpha_2L_1}{2}\int_\Omega a(x)|\zeta(x)-\tilde{\zeta}(x)|^2\,dx-\frac{\mu\tau^{-1}}{2}\int_\Omega a(x)|\theta(x,1)-\tilde{\theta}(x,1))|^2\,dx\\ &+\frac{\mu\tau^{-1}}{2}\int_\Omega a(x)|\zeta(x)-\tilde{\zeta}(x)|^2\,dx.\end{aligned}$$

Then

$$\begin{aligned}Re\left\langle \tilde{\mathcal{A}}U-\tilde{\mathcal{A}}V;U-V\right\rangle_{\widehat{\mathcal{H}}} &\le -(K\alpha_1-\frac{\alpha_2L_1}{2}-\frac{\mu\tau^{-1}}{2})\int_\Omega a(x)|\zeta(x)-\tilde{\zeta}(x)|^2\,dx\\ &-(-\frac{\alpha_2L_1}{2}+\frac{\mu\tau^{-1}}{2})\int_\Omega a(x)|\zeta(x)-\tilde{\zeta}(x)|^2\,dx.\end{aligned}$$

Recalling (3.18), we conclude that

$$Re\left\langle \tilde{\mathcal{A}}U-\tilde{\mathcal{A}}V;U-V\right\rangle_{\widehat{\mathcal{H}}}\le 0.$$

This shows the dissipativity of $\tilde{\mathcal{A}}$.
In order to establish maximality, we need to prove the range condition: $range(I-\tilde{\mathcal{A}})=\widehat{\mathcal{H}}$. In other words, given any $(l,m)^T\in\widehat{\mathcal{H}}$, we need to establish the existence of an element $W=(\zeta,\theta)^T\in D(\tilde{\mathcal{A}})$ such that

$$(I-\tilde{\mathcal{A}})W=(l,m)^T,$$

or equivalently

$$\zeta-i\Delta\zeta+a(.)\alpha_1 f(\zeta)+a(.)\alpha_2 g(\theta(.,1))=l, \tag{3.161}$$

$$\theta(x,\rho)+\tau^{-1}\theta_\rho(.,\rho)=m. \tag{3.162}$$

From (3.162) and the last line of (3.156) we have

$$\begin{aligned}&\theta_\rho(x,\rho)=-\tau\theta(x,\rho)+\tau m, && x\in\Omega,\rho\in(0,1),\\ &\theta(x,0)=\zeta(x), && x\in\Omega.\end{aligned}$$

The unique solution of the above initial value problem is given by

$$\theta(x,\rho) = \zeta(x)e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho m(x,\sigma)e^{\tau\sigma}\,d\sigma, \qquad x\in\Omega, \rho\in(0,1),$$

and in particular

$$\theta(x,1) = \zeta(x)e^{-\tau} + Z_0, \qquad x\in\Omega, \tag{3.163}$$

where

$$Z_0 = \tau e^{-\tau}\textstyle\int_0^1 m(x,\sigma)e^{\tau\sigma}\,d\sigma.$$

Insertion of (2.64) into problem (3.161) results in the equation

$$\zeta - i\Delta\zeta + a(.)\alpha_1 f(\zeta) + a(.)\alpha_2 g(\zeta e^{-\tau} + Z_0) = l,$$

which we rewrite as

$$\tilde{\mathcal{T}}\zeta = l,$$

where

$$\tilde{\mathcal{T}}\zeta = \zeta + \tilde{\mathcal{T}}_1\zeta,$$

and

$$\tilde{\mathcal{T}}_1\zeta = -i\Delta\zeta + a(.)\alpha_1 f(\zeta) + a(.)\alpha_2 g(\zeta e^{-\tau} + Z_0). \tag{3.164}$$

Lemma 3.9. *The operator $\tilde{\mathcal{T}}_1$ defined by (3.164) with $D(\tilde{\mathcal{T}}_1) = H^2(\Omega)\cap H_0^1(\Omega)$, is maximal monotone on $L^2(\Omega)$.*

Proof. Set

$$\tilde{\mathcal{T}}_1\zeta = \mathcal{B}\zeta + \mathcal{C}\zeta,$$

where

$$\mathcal{B} : L^2(\Omega) \longrightarrow L^2(\Omega) \text{ defined by } \mathcal{B}\zeta = a(.)\alpha_1 f(\zeta) + a(.)\alpha_2 g(\zeta e^{-\tau} + Z_0),$$
$$\mathcal{C} : D(\mathcal{C}) = H^2(\Omega)\cap H_0^1(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega) \text{ defined by } \mathcal{C}\zeta(x) = -i\Delta\zeta(x).$$

Clearly, $\mathcal{C}$ is maximal monotone on $L^2(\Omega)$. Then according to Corollary 1.1 in [11], it is sufficient to prove boundedness, hemicontinuity and monotonicity of $\mathcal{B}$. To this end, let $\zeta\in L^2(\Omega)$. We have

$$\begin{aligned}&\int_\Omega |a(x)\left\{\alpha_1 f(\zeta(x)) + \alpha_2 g(\zeta(x)e^{-\tau} + Z_0)|^2\right\}\,dx\\ &\le \|a\|^2_{L^\infty(\Omega)} cst\left\{\int_\Omega |f(\zeta(x))|^2\,dx + \int_\Omega |g(\zeta(x)e^{-\tau} + Z_0)|^2\,dx\right\}.\end{aligned} \tag{3.165}$$

Since f is continuous; $|f(\zeta(x))|^2 \leq K_1^2$ for $|\zeta(x)| \leq 1$, and for $|\zeta(x)| \geq 1$, assumption (H.5) implies $|f(\zeta(x))|^2 \leq \widetilde{M}^2|\zeta(x)|^2$. Consequently

$$|f(\zeta(x))|^2 \leq K_1^2 + \widetilde{M}^2|\zeta(x)|^2, \ \ \forall x \in \Omega. \tag{3.166}$$

From assumption (H.2), we have

$$|g(\zeta(x)e^{-\tau} + Z_0(x))|^2 \leq L_1^2|\zeta(x)e^{-\tau} + Z_0(x))|^2. \tag{3.167}$$

Substituting (3.166) and (3.167) in (3.165), we obtain

$$\begin{aligned}
&\int_\Omega |a(x)\left\{\alpha_1 f(\zeta(x)) + \alpha_2 g(\zeta(x)e^{-\tau} + Z_0(x)\right\}|^2\, dx \\
&\leq \|a\|^2_{L^\infty(\Omega)} c\left\{\int_\Omega |f(\zeta(x))|^2\, dx + \int_\Omega |g(\zeta(x)e^{-\tau} + Z_0(x)|^2\, dx\right\} \\
&\leq \|a\|^2_{L^\infty(\Omega)} c\left\{\int_\Omega K_1^2\, dx + C^2\|\zeta\|^2_{L^2(\Omega)} + L_1^2 \int_\Omega |\zeta(x)e^{-\tau} + Z_0(x)|^2\, dx\right\} \\
&\leq \|a\|^2_{L^\infty(\Omega)} c\left\{\int_\Omega K_1^2\, dx + C^2\|\zeta\|^2_{L^2(\Omega)} + L_1^2 e^{-2\tau}\|\zeta\|^2_{L^2(\Omega)} + L_1^2\|Z_0\|^2_{L^2(\Omega)}\right\} \\
&\leq \|a\|^2_{L^\infty(\Omega)} c\left\{\int_\Omega K_1^2\, dx + (C^2 + L_1^2 e^{-2\tau})\|\zeta\|^2_{L^2(\Omega)} + L_1^2\|Z_0\|^2_{L^2(\Omega)}\right\} < \infty.
\end{aligned}$$

Therefore $\mathcal{B}$ is well defined and bounded.
To prove that $\mathcal{B}$ is hemicontinuous, we need to prove that for all $u, v, w \in L^2(\Omega)$

$$\lim_{t\to 0} \langle \mathcal{B}(u + x_n v), w\rangle_{L^2(\Omega)} = \langle \mathcal{B}u, w\rangle_{L^2(\Omega)},$$

or equivalently

$$\lim_{n > \infty} \langle \mathcal{B}(u + x_n v), w\rangle_{L^2(\Omega)} = \langle \mathcal{B}u, w\rangle_{L^2(\Omega)}, \tag{3.168}$$

for every sequence $(x_n)_n \subset \mathbb{R}$ such that $x_n \longrightarrow 0$ when $n \longrightarrow \infty$.
Let $F_n = a\alpha_1 f(u + x_n v) + a\alpha_2 g((u + x_n v)e^{-\tau} + Z_0)\overline{w}, \ n \in \mathbb{N}$, thus $(F_n)_n \subset L^1(\Omega)$. In fact,

$$\begin{aligned}
|F_n(x)| &= |a(x)\alpha_1 f(u(x) + x_n v(x)) + a(x)\alpha_2 g((u(x) + x_n v(x))e^{-\tau} + Z_0(x))||\overline{w}(x)| \\
&\leq |a(x)|\left[\alpha_1 K_1 + (\alpha_1\widetilde{M} + \alpha_2 L_1 e^{-\tau})|u(x) + x_n v(x)| + \alpha_2 L_1\|Z_0\|^2_{L^2(\Omega)}\right]|\overline{w}(x)| \\
&\leq \alpha_1 K_1|a(x)|\,|\overline{w}(x)| + (\alpha_1\widetilde{M} + \alpha_2 L_1 e^{-\tau})|a(x)|\,|u(x)|\,|\overline{w}(x)| \\
&+ (\alpha_1\widetilde{M} + \alpha_2 L_1 e^{-\tau})|a(x)|\,|x_n|\,|v(x)|\,|\overline{w}(x)| + \alpha_2 L_1\|Z_0\|^2_{L^2(\Omega)}|a(x)|\,|\overline{w}(x)|,
\end{aligned}$$

a.e. in Ω.
Since $a \in L^\infty(\Omega)$ and $|x_n| \leq N$, for all $n \in \mathbb{N}$, then $F_n \in L^1(\Omega)$, for all n $\in \mathbb{N}$.
Moreover, if

$$\begin{aligned}
R(x) = {} & \alpha_1 K_1|a(x)|\,|\overline{w}(x)| + (\alpha_1\widetilde{M} + \alpha_2 L_1 e^{-\tau})|a(x)|\,|u(x)|\,|\overline{w}(x)| \\
& + M_1|a(x)|\,|v(x)|\,|\overline{w}(x)| + \alpha_2 L_1\|Z_0\|^2_{L^2(\Omega)}|a(x)|\,|\overline{w}(x)|,
\end{aligned}$$

where $M_1 = N(\alpha_1 \widetilde{M} + \alpha_2 L_1 e^{-\tau})$, then $R \in L^1(\Omega)$ and $|F_n(x)| \leq R(x)$ a.e. in Ω.
From the continuity of f and g, we have

$$\begin{aligned}\lim_{n\to\infty} a(x)\left\{\alpha_1 f(u(x)+x_n v(x)) + \alpha_2 g((u(x)+x_n v(x))e^{-\tau} + Z_0(x)\right\}\overline{w}(x)\\ = a(x)\left\{\alpha_1 f(u(x)) + \alpha_2 g((u(x)e^{-\tau} + Z_0(x))\right\}\overline{w}(x),\end{aligned}$$

a.e. in Ω.
Recalling Lebesgue's dominated convergence theorem we deduce that

$$\begin{aligned}\int_\Omega |a(x)\left\{\alpha_1 f(u(x)+x_n v(x)) + \alpha_2 g((u(x)+x_n v(x))e^{-\tau} + Z_0(x)\right\}\overline{w}(x)\\ - a(x)\left\{\alpha_1 f(u(x)) + \alpha_2 g((u(x)e^{-\tau} + Z_0(x)\right\}\overline{w}(x)|\,dx \longrightarrow 0.\end{aligned}$$

Thus,

$$\begin{aligned}\Bigg|\int_\Omega [a(x)\left\{\alpha_1 f(u(x)+x_n v(x)) + \alpha_2 g((u(x)+x_n v(x))e^{-\tau} + Z_0(x))\right\}\overline{w}(x)\\ - a(x)\left\{\alpha_1 f(u(x)) + \alpha_2 g((u(x)e^{-\tau} + Z_0(x))\right\}\overline{w}(x)]\,dx\Bigg| \longrightarrow 0,\end{aligned}$$

and consequently,

$$\begin{aligned}Re\int_\Omega [a(x)\left\{\alpha_1 f(u(x)+x_n v(x)) + \alpha_2 g((u(x)+x_n v(x))e^{-\tau} + Z_0(x))\right\}\overline{w}(x)]\,dx\\ \longrightarrow Re\int_\Omega a(x)\left\{\alpha_1 f(u(x)) + \alpha_2 g((u(x)e^{-\tau} + Z_0(x))\right\}\overline{w}(x)\,dx,\end{aligned}$$

which proves (3.168).
Now, we show that $\mathcal{B}$ is monotone ($-\mathcal{B}$ is dissipative). Indeed, for $u, v \in L^2(\Omega)$, we have

$$\begin{aligned}&Re\langle(-\mathcal{B})u - (-\mathcal{B})v, u-v\rangle_{L^2(\Omega)\times L^2(\Omega)} = Re\int_\Omega -\alpha_1 a(x)(f(u)-f(v))\overline{(u-v)}\,dx\\ &+ Re\int_\Omega -\alpha_2 a(x)(g(ue^{-\tau}+Z_0(x)) - g(ve^{-\tau}+Z_0(x)))\overline{(u-v)}\,dx\\ &\leq -\alpha_1 K\int_\Omega a(x)|u-v|^2\,dx + \alpha_2\int_\Omega a(x)|g(ue^{-\tau}+Z_0(x)) - g(ve^{-\tau}+Z_0(x))|\,|u-v|\,dx\\ &\leq -\alpha_1 K\int_\Omega a(x)|u-v|^2\,dx + \alpha_2 e^{-\tau}L_1\int_\Omega a(x)|u-v|^2\,dx\\ &\leq -(\alpha_1 K - \alpha_2 e^{-\tau}L_1)\int_\Omega a(x)|u-v|^2\,dx,\end{aligned}$$

and the desired conclusion follows from assumption (H.3). □

The operator $\tilde{\mathcal{T}}_1$ with is maximal monotone on $L^2(\Omega)$ and consequently $\tilde{\mathcal{T}}$ is surjective. Therefore $range(I - \tilde{\mathcal{A}}) = \widehat{\mathcal{H}}$. This completes the proof of the maximal dissipativity of $\tilde{\mathcal{A}}$. □

3.3.2 Proof of Theorem 3.2

We first show that the energy $F(t)$ of every solution of (3.12) is decreasing.

Proposition 3.9. *The energy corresponding to any strong solution of the problem (3.12) is decreasing and there exists* $\tilde{C} > 0$ *such that*

$$\frac{d}{dt}F(t) \leq -\tilde{C}\int_\Omega \left\{a(x)f(u(x,t))\overline{u}(x,t) + a(x)|u(x,t-\tau)|^2\right\}\,dx, \tag{3.169}$$

where

$$\tilde{C} = min\left\{\alpha_1 - \frac{\alpha_2 L_1}{2K} - \frac{\mu\tau^{-1}}{2K}, -\frac{\alpha_2 L_1}{2} + \frac{\mu\tau^{-1}}{2}\right\}.$$

Proof. Differentiating $F(t)$ defined by (3.17) in time, we obtain

$$\begin{aligned}\frac{d}{dt}F(t) &= Re\int_\Omega u_t(x,t)\overline{u}(x,t)\,dx + \mu Re\int_\Omega\int_0^1 a(x)u_t(x,t-\tau\rho)\overline{u}(x,t-\tau\rho)\,d\rho\,dx\\ &= Re\int_\Omega (i\Delta u(x,t))\overline{u}(x,t)\,dx - Re\int_\Omega a(x)\alpha_1 f(u(x,t))\overline{u}(x,t)\,dx - Re\int_\Omega a(x)\alpha_2 g(u(x,t-\tau))\overline{u}(x,t)\,dx\\ &+ \mu Re\int_\Omega\int_0^1 a(x)u_t(x,t-\tau\rho)\overline{u}(x,t-\tau\rho)\,d\rho\,dx.\end{aligned}$$

Applying Green's second theorem, we get

$$\begin{aligned}\frac{d}{dt}F(t) &= -\alpha_1 Re\int_\Omega a(x)f(u(x,t))\overline{u}(x,t)\,dx - \alpha_2 Re\int_\Omega a(x)g(u(x,t-\tau))\overline{u}(x,t)\,dx\\ &+ \mu Re\int_\Omega\int_0^1 a(x)u_t(x,t-\tau\rho)\overline{u}(x,t-\tau\rho)\,d\rho\,dx.\end{aligned} \tag{3.170}$$

Now observe that

$$u_t(x,t-\tau\rho) = -\tau^{-1}u_\rho(x,t-\tau\rho),$$

and

$$\frac{d}{d\rho}|u(x,t-\tau\rho)|^2 = 2Re(u_\rho(x,t-\tau\rho)\overline{u}(x,t-\tau\rho)). \tag{3.171}$$

Inserting (3.171) into (3.170), we obtain

$$\begin{aligned}\frac{d}{dt}F(t) &= -\alpha_1 Re\int_\Omega a(x)f(u(x,t))\overline{u}(x,t)\,dx - \alpha_2 Re\int_\Omega a(x)g(u(x,t-\tau))\overline{u}(x,t)\,dx\\ &- \frac{\mu\tau^{-1}}{2}\int_\Omega\int_0^1 a(x)\frac{d}{d\rho}|u(x,t-\tau\rho)|^2\,d\rho\,dx\\ &= -\alpha_1 Re\int_\Omega a(x)f(u(x,t))\overline{u}(x,t)\,dx - \alpha_2 Re\int_\Omega a(x)g(u(x,t-\tau))\overline{u}(x,t)\,dx\\ &- \frac{\mu\tau^{-1}}{2}\int_\Omega a(x)(|u(x,t-\tau)|^2 - |u(x,t)|^2\,dx.\end{aligned}$$

By using assumption (H.2) and Cauchy-Schwartz's inequality, we have

$$\frac{d}{dt}F(t) \leq -\alpha_1 Re \int_\Omega a(x)f(u(x,t))\overline{u}(x,t)\,dx + \frac{\alpha_2 L_1}{2}\int_\Omega a(x)|u(x,t-\tau)|^2\,dx + \frac{\alpha_2 L_1}{2}\int_\Omega a(x)|u(x,t)|^2\,dx$$
$$- \frac{\mu\tau^{-1}}{2}\int_\Omega a(x)|u(x,t-\tau)|^2\,dx + \frac{\mu\tau^{-1}}{2}\int_\Omega a(x)|u(x,t)|^2\,dx. \tag{3.172}$$

Recalling assumption (H.1)(ii) and (H.1)(iii), we rewrite (3.172) as

$$\frac{d}{dt}F(t) \leq -\alpha_1 \int_\Omega a(x)f(u(x,t))\overline{u}(x,t)\,dx + \frac{\alpha_2 L_1}{2}\int_\Omega a(x)|u(x,t-\tau)|^2\,dx + \frac{\alpha_2 L_1}{2K}\int_\Omega a(x)f(u(x,t))\overline{u}(x,t)\,dx$$
$$- \frac{\mu\tau^{-1}}{2}\int_\Omega a(x)|u(x,t-\tau)|^2\,dx + \frac{\mu\tau^{-1}}{2K}\int_\Omega a(x)f(u(x,t))\overline{u}(x,t)\,dx.$$

Therefore

$$\frac{d}{dt}F(t) \leq -(\alpha_1 - \frac{\alpha_2 L_1}{2K} - \frac{\mu\tau^{-1}}{2K})\int_\Omega a(x)f(u(x,t))\overline{u}(x,t)\,dx - (-\frac{\alpha_2 L_1}{2} + \frac{\mu\tau^{-1}}{2})\int_\Omega a(x)|u(x,t-\tau)|^2\,dx,$$

this last inequality can be written

$$\frac{d}{dt}F(t) \leq -\tilde{C}\int_\Omega \left\{a(x)f(u(x,t))\overline{u}(x,t) \;+ a(x)|u(x,t-\tau)|^2\right\}\,dx,$$

where

$$\tilde{C} = min\left\{\alpha_1 - \frac{\alpha_2 L_1}{2K} - \frac{\mu\tau^{-1}}{2K}, -\frac{\alpha_2 L_1}{2} + \frac{\mu\tau^{-1}}{2}\right\},$$

which is positive due to the (3.18).

□

Next, we establish an observability inequality for problem (3.12).

Proposition 3.10. *Let $T > 0$ be sufficiently large. Then there exists a positive constant $C(T)$ depending on T such that*

$$F(T) \leq C(T)\int_0^T \int_\Omega a(x)\left\{|u(x,t)|^2 + |f(u(x,t))|^2 + |u(x,t-\tau)|^2\right\}\,dx\,dt. \tag{3.173}$$

Proof. We write the solution u of (3.12) as $u = \varphi + v$ where φ solves

$$\begin{cases} \varphi_t(x,t) = i\Delta\varphi(x,t) & \text{in } \Omega\times(0;+\infty), \\ \varphi(x,t) = 0 & \text{on } \Gamma\times(0,+\infty), \\ \varphi(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.174}$$

and v satisfies

$$\begin{cases} v_t(x,t) = i\Delta v(x,t) - a(x)\{\alpha_1 f(u(x,t)) - \alpha_2 g(u(x,t-\tau))\} & \text{in } \Omega\times(0;+\infty), \\ v(x,t) = 0 & \text{on } \Gamma\times(0,+\infty), \\ v(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{3.175}$$

Denote by

$$F_\varphi(t) = \int_\Omega |\varphi(x,t)|^2\,dx,$$

the energy corresponding to the solution of (3.174). Then, it follows from [[57] Proposition 3.1] that for all $T > 0$, there exists a positive constant c depending on T such that

$$F_\varphi(0) = \|u_0\|^2_{L^2(\Omega)} \le c\int_0^T \int_\omega |\varphi(x,t)|^2\,dx\,dt.$$

Here and throughout the rest of the section, c is a positive constant different at different occurences. Using (3.13), we get

$$\|u_0\|^2_{L^2(\Omega)} \le \frac{c}{a_0}\int_0^T \int_\Omega a(x)|\varphi(x,t)|^2\,dx\,dt. \tag{3.176}$$

On the other hand, we have for $T > \tau$

$$\frac{\mu}{2}\int_\Omega a(x)\int_0^1 |u(x,-\tau\rho)|^2\,d\rho\,dx \le c\int_0^T \int_\Omega a(x)|u(x,t-\tau)|^2\,dx\,dt. \tag{3.177}$$

Since the energy is non-increasing, we deduce from (3.176) and (3.177) that

$$\begin{aligned}
F(T) \le F(0) &= \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{\mu}{2}\int_\Omega a(x)\int_0^1 |u(x,-\tau\rho)|^2\,d\rho\,dx \\
&\le c\int_0^T \int_\Omega a(x)\left\{|\varphi(x,t)|^2 + |u(x,t-\tau)|^2\right\}\,dx\,dt \\
&\le c\int_0^T \int_\Omega a(x)\left\{|u(x,t)|^2 + |v(x,t)|^2 + |u(x,t-\tau)|^2\right\}\,dx\,dt \\
&\le c\int_0^T \int_\Omega a(x)\left\{|u(x,t)|^2 + |f(u(x,t))|^2 + |u(x,t-\tau)|^2\right\}\,dx\,dt + c\int_0^T \int_\Omega a(x)|v(x,t)|^2\,dx\,dt.
\end{aligned} \tag{3.178}$$

We will now estimate the last integral on the right-hand side of (3.178). For this purpose, we define the linear map

$$\begin{aligned}
\mathcal{M} : L^2(\Omega) \times L^1(0,T;L^2(\Omega)) &\longrightarrow L^\infty(0,T;L^2(\Omega)), \\
(z_0,\tilde{f}) &\longmapsto \mathcal{M}(z_0,\tilde{f}) = z,
\end{aligned}$$

where z is the solution of the problem

$$\begin{cases} iz_t(x,t) + \Delta z(x,t) = \tilde{f} & \text{in } \Omega \times (0;+\infty), \\ z(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ z(x,0) = z_0(x) & \text{in } \Omega, \end{cases} \tag{3.179}$$

$\mathcal{M}$ is continuous. Indeed, since z is solution of (3.179), then

$$z(t) = S(t)z_0 + \int_0^t S(t-s)\tilde{f}(s)\,ds.$$

where $S(t)$ is the contraction semigroup generated by the maximal monotone operator $\mathcal{C}$. Thus,

$$\begin{aligned}\|z(t)\|^2_{L^2(\Omega)} &= \|S(t)z_0 + \int_0^t S(t-s)\tilde{f}(s)ds\|^2_{L^2(\Omega)}\\ &\leq c\|S(t)z_0\|^2_{L^2(\Omega)} + c\|\int_0^t S(t-s)\tilde{f}(s)ds\|^2_{L^2(\Omega)}\\ &\leq c_1\|z_0\|^2_{L^2(\Omega)} + c_2\left(\int_0^t \|\tilde{f}(s)\|^2_{L^2(\Omega)}ds\right)\\ &\leq \left\{\|z_0\|^2_{L^2(\Omega)} + \|\tilde{f}(s)\|_{L^1(0,T;L^2(\Omega))}\right\} = C\|(z_0,\tilde{f})\|^2_{L^2(\Omega)\times L^1(0,T;L^2(\Omega))}.\end{aligned}$$

We rewrite the above estimate for the v-problem (3.175) with $\tilde{f}(t) = a(x)\{\alpha_1 f(u(x,t)) + \alpha_2 g(u(x,t-\tau))\}$, and $z_0 = 0$. We obtain after using Hölder inequality and the continuity of $\mathcal{M}$

$$\begin{aligned}\int_0^T\int_\Omega a(x)|v(x,t)|^2\,dx\,dt &\leq \|a\|_{L^\infty(\Omega)}\|v(x,t)\|^2_{L^2(0,T;L^2(\Omega))}\\ &\leq \|a\|_{L^\infty(\Omega)}\|v(x,t)\|^2_{L^\infty(0,T;L^2(\Omega))}\\ &\leq \|a\|_{L^\infty(\Omega)}\|a(x)\{\alpha_1 f(u(x,t)) + \alpha_2 g(u(x,t-\tau))\}\|^2_{L^1(0,T;L^2(\Omega))}\\ &\leq c\left(\int_0^T\left[\int_\Omega a(x)|\alpha_1 f(u(x,t)) + \alpha_2 g(u(x,t-\tau))|^2\,dx\right]^{\frac{1}{2}}dt\right)^2\\ &\leq T^2c\int_0^T\int_\Omega a(x)|\alpha_1 f(u(x,t)) + \alpha_2 g(u(x,t-\tau))|^2\,dx\,dt\\ &\leq T^2c\int_0^T\int_\Omega \left[a(x)|f(u(x,t))|^2 + a(x)|g(u(x,t-\tau))|^2\right]\,dx\,dt\\ &\leq c_6(T)\int_0^T\int_\Omega a(x)\left[|f(u(x,t))|^2 + |u(x,t)|^2 + |u(x,t-\tau)|^2\right]\,dx\,dt.\end{aligned} \tag{3.180}$$

Inserting (3.180) into (3.178) we obtain the desired estimate (3.173). □

In the next step, we prove an estimate for a nonlinear function of the energy $F(T)$.

Lemma 3.10. *The energy $F(t)$ of problem (3.12) satisfies*

$$F(T) + \widehat{p}(F(T)) \leq F(0), \tag{3.181}$$

where $\widehat{p}(.)$ is defined by (3.15), and $T > 0$ is sufficiently large.

Proof. Denote

$$\begin{aligned}Q_1 &= \{u \in L^2(Q) : |u| \geq \delta \text{ a.e.}\},\\ Q_2 &= Q - Q_1.\end{aligned}$$

From hypotheses (H.1) and (H.5), we have

$$\int_{Q_1} a(x)\{|u(x,t)|^2 + |f(u(x,t))|^2\}\, dQ_1 \le (K^{-1} + \widetilde{M}) \int_{Q_1} a(x) f(u(x,t))\overline{u}(x,t)\, dQ_1, \tag{3.182}$$

on the other side, from (3.4) and from the fact that h is concave and increasing, having in mind that

$$a(x) \le \|a\|_\infty + 1,$$

and

$$\frac{a(x)}{1+\|a\|_\infty} \le a(x),$$

we deduce that

$$\begin{aligned}
\int_{Q_2} a(x)\{|u(x,t)|^2 + |f(u(x,t))|^2\}\, dQ_2 &\le \int_{Q_2} a(x) h(f(u(x,t))\overline{u}(x,t))\, dQ_2 \\
&= \int_{Q_2} (1+\|a\|_\infty) \frac{a(x)}{1+\|a\|_\infty} h(f(u(x,t))\overline{u}(x,t))\, dQ_2 \\
&\le \int_{Q_2} (1+\|a\|_\infty) h\left(\frac{a(x)}{1+\|a\|_\infty} f(u(x,t))\overline{u}(x,t)\right) dQ_2 \\
&\le \int_{Q_2} (1+\|a\|_\infty) h(a(x) f(u(x,t))\overline{u}(x,t))\, dQ_2.
\end{aligned} \tag{3.183}$$

By Jensen's inequality,

$$\begin{aligned}
(1+\|a\|_\infty) \int_{Q_2} h(a(x) f(u(x,t))\overline{u}(x,t))\, dQ_2 &\le (1+\|a\|_\infty) mes\, Q\, h\left(\frac{1}{mes\, Q} \int_Q a(x) f(u(x,t))\overline{u}(x,t)\, dQ\right) \\
&= (1+\|a\|_\infty) mes\, Q\, \widehat{h}\left(\int_Q a(x) f(u(x,t))\overline{u}(x,t)\, dQ\right).
\end{aligned} \tag{3.184}$$

Combining inequalities (3.182), (3.183), and (3.184) with the result of Proposition 3.10 gives

$$\begin{aligned}
F(T) &\le C(T) \left\{(K^{-1} + \widetilde{M}) \int_Q a(x) f(u(x,t))\overline{u}(x,t)\, dQ + \int_Q a(x)|u(x,t-\tau)|^2\, dQ\right\} \\
&\quad + C(T)(1+\|a\|_\infty)\, mes\, Q\, \widehat{h}\left(\int_Q a(x) f(u(x,t))\overline{u}(x,t)\, dQ\right) \\
&\le C(T) \left\{(K^{-1} + \widetilde{M}) \int_Q a(x) f(u(x,t))\overline{u}(x,t)\, dQ + \int_Q a(x)|u(x,t-\tau)|^2\, dQ\right\} \\
&\quad + C(T)(1+\|a\|_\infty)\, mes\, Q\, \widehat{h}\left(\int_Q \{a(x) f(u(x,t))\overline{u}(x,t) + a(x)|u(x,t-\tau)|^2\}\, dQ\right).
\end{aligned} \tag{3.185}$$

Setting

$$K_1 = \frac{1}{C(T)(1+\|a\|_\infty)\, mes\, Q};\ C' = \frac{K^{-1} + \widetilde{M}}{(1+\|a\|_\infty) mes\, Q}$$

$$C'' = max\,\{C',\, \frac{1}{(1+\|a\|_\infty) mes\, Q}\}.$$

We obtain from (3.185)

$$K_1\, F(T) \leq (C''I + \widehat{h}) \left(\int_Q a(x)\{f(u(x,t))\overline{u}(x,t) + |u(x,t-\tau)|^2\}\, dQ \right).$$

But

$$\int_Q a(x)\{f(u(x,t))\overline{u}(x,t) + |u(x,t-\tau)|^2\}\, dQ \leq \tilde{C}^{-1}(F(0) - F(T)).$$

Hence

$$(C''I + \widehat{h})^{-1}\,(K_2\, F(T)) = \widehat{p}(F(T)) \leq F(0) - F(T),$$

where $K_2 = \tilde{C}K_1$.
Therefore

$$\widehat{p}(F(T)) + F(T) \leq F(0).$$

□

The sought-after stability result follows now, as in [43], from inequality (3.181) and Lemma 3.3 in ([43], p.531).
This completes the proof of Theorem 3.2.

Chapter 4

Boundary stabilization of the Schrödinger equation with interior delay

4.1 Introduction

In this chapter, we study stability problem for the Schrödinger equation with interior delay term and boundary feedback. To this end, let Ω be an open bounded domain of $\mathbb{R}^n, n \geq 2$ with boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that, $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. In addition to these standard hypothesis, we assume the following.

(A) There exists $x_0 \in \mathbb{R}^n$ such that, with $m(x) = x - x_0$,

$$m(x).v(x) \leq 0, \qquad x \in \Gamma_1, \tag{4.1}$$

$$m(x).v(x) \geq \delta > 0, \qquad x \in \Gamma_2, \tag{4.2}$$

where $\nu(.)$ is the unit normal on Γ pointing towards the exterior of Ω.

In Ω, we consider the following Schrödinger equation with interior delay term and dissipative boundary feedback:

$$u_t(x,t) - i\Delta u(x,t) + \alpha u(x,t-\tau) = 0 \qquad \text{in } \Omega \times (0;+\infty), \tag{4.3}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{4.4}$$

$$u(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{4.5}$$

$$\frac{\partial u}{\partial \nu}(x,t) = -\beta u_t(x,t) \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{4.6}$$

$$u(x,t-\tau) = f_0(x,t-\tau) \qquad \text{in } \Omega \times (0,\tau_2), \tag{4.7}$$

where

- u_0 and f_0 are the initial data which belong to a suitable spaces.
- $\frac{\partial}{\partial \nu}$ is the normal derivative.
- $\tau > 0$ is the time delay.
- α and β are a positive constants.

Boundary stabilization of the Schrödinger equation is the subject of the papers [57] and [51]. In [57], the authors proved that the solution of (4.3)-(4.7) wih $\alpha = 0$ decays exponentially in the energy space $H^1_{\Gamma_1}(\Omega)$ by adopting the classical multipliers method. Lasiecka et al [51] used $L^2(\Omega)$- Carleman estimates for the general linear Schrödinger equation to provide a uniform stabilization result for (4.3) wih $\alpha = 0$ in the energy space $L^2(\Omega)$ by means of a feedback control $\frac{\partial u}{\partial \nu}(x,t) = -u(x,t)$.
Nicaise and Rebiai [65] established stability and instability results for the Schrödinger equation with a delay term in the boundary or internal feedbacks. In this chapter, we study the stability of the Schrödinger equation with interior delay and a dissipative boundary feedback as described in (4.3)-(4.7). For the wave equation, this problem has been investigated by Ammari et al [7]. We use multipliers technique and a suitable Lyapunov functional to prove that the solution of (4.3)-(4.7) decays exponentially in the energy space $H^1_{\Gamma_1}(\Omega)$.
We define the energy associated to problem (4.3)-(4.7) by

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2\,dx + \frac{\xi\tau}{2}\int_\Omega\int_0^1 |\nabla u(x,t-\tau\rho)|^2\,d\rho\,dx, \tag{4.8}$$

where ξ is a strictly positive constant.
The main result of this chapter can be stated as follows.

Theorem 4.1. *For any $\beta > 0$ there exist positive constants α_0, M, C such that*

$$E(t) \le Me^{-Ct}E(0), \tag{4.9}$$

for any regular solution of problem (4.3)-(4.7) with $0 \le \alpha < \alpha_0$. The constants α_0, M and C are independent of the initial data but they depend on β and on the geometry of Ω.

Theorem 4.1 is proved in section 4.3. In section 4.2, we will study the well-posedness of system the $(4.3) - (4.7)$ by using semigroup theory.

4.2 Well-posedness

We introduce the auxiliary variable:

$$z(x,\rho,t) = u(x,t-\tau\rho); \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Then, the system (4.3)-(4.7) is equivalent to

$$
\begin{aligned}
&u_t(x,t) - i\Delta u(x,t) + \alpha z(x,1,t) = 0 && \text{in } \Omega \times (0;+\infty), && (4.10)\\
&z_t(x,\rho,t) + \tau^{-1} z_\rho(x,\rho,t) = 0 && \text{in } \Omega \times (0,1) \times (0,+\infty) && (4.11)\\
&u(x,0) = u_0(x) && \text{in } \Omega, && (4.12)\\
&z(x,0,t) = u(x,t) && \text{in } \Omega \times (0,+\infty), && (4.13)\\
&u(x,t) = 0 && \text{on } \Gamma_1 \times (0,+\infty), && (4.14)\\
&\frac{\partial u}{\partial \nu}(x,t) = -\beta u_t(x,t) && \text{on } \Gamma_2 \times (0,+\infty), && (4.15)\\
&z(x,\rho,0) = f_0(x,-\tau\rho) && \text{in } \Omega \times (0,1). && (4.16)
\end{aligned}
$$

Denote by $\mathcal{H}$ the Hilbert space.

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2((0,1); H^1_{\Gamma_1}(\Omega)).$$

We equip $\mathcal{H}$ with the inner product:

$$\left\langle \begin{pmatrix} u_1 \\ z_1 \end{pmatrix}; \begin{pmatrix} u_2 \\ z_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = Re \int_\Omega \nabla u_1(x) \overline{\nabla u_2(x)}\, dx + \xi\tau Re \int_\Omega \int_0^1 \nabla z_1(x,\rho) \overline{\nabla z_2(x,\rho)}\, d\rho\, dx.$$

Define in $\mathcal{H}$ the linear operator $\mathcal{A}$ by

$$\mathcal{A}(u,y)^T = (i\Delta u - \alpha z(.,1), -\tau^{-1} z_\rho)^T, \tag{4.17}$$

with domain $D(\mathcal{A})$ defined by

$$
\begin{aligned}
D(\mathcal{A}) = &\{(u,z) \in H^1_{\Gamma_1}(\Omega) \times H^1((0,1); H^1_{\Gamma_1}(\Omega)) : i\Delta u(x) - \alpha z(x,1) \in H^1_{\Gamma_1}(\Omega),\\
&\frac{\partial u}{\partial \nu}(x) = -i\beta \Delta u(x) + \alpha\beta z(x,1) \text{ on } \Gamma_2,\ u(x) = z(x,0) \text{ in } \Omega\}.
\end{aligned} \tag{4.18}
$$

Then we can rewrite (4.10)-(4.16) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \dfrac{dU}{dt}(t) = \mathcal{A}U(t), \\ U(0) = U_0, \end{cases} \tag{4.19}$$

where

$$U(t) = (u(.,t), z(.,.,t))^T,\ U'(t) = (u_t(.,t), z_t(.,.,t))^T,\ U_0 = (u_0, f_0)^T.$$

Theorem 4.2. *For every $U_0 \in \mathcal{H}$, the problem (4.19) has a unique solution U whose regularity depends on the initial datum U_0 as follows:*

$$
\begin{aligned}
U(.) &\in C([0,+\infty); \mathcal{H}) \text{ if } U_0 \in \mathcal{H},\\
U(.) &\in C^1([0,+\infty); \mathcal{H}) \cap C([0,+\infty); D(\mathcal{A})) \text{ if } U_0 \in D(\mathcal{A}).
\end{aligned}
$$

Proof. The well-posedness of the problem (4.10)-(4.16) or its abstract version (4.19) follows via Lumer Phillips Theorem (see for instance Theorem I.4.3 of [69]).
We show that there exists a positive constant c such that $\mathcal{A} - cI$ is dissipative.
Let $(u, z)^T \in D(\mathcal{A})$, then

$$Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = Re \int_\Omega \nabla(i\Delta u(x) - \alpha z(x,1))\overline{\nabla u(x)}\, dx - \xi Re \int_\Omega \int_0^1 \nabla z_\rho(x,\rho)\overline{\nabla z(x,\rho)}\, d\rho\, dx.$$

Applying Green's Theorem, we obtain

$$\begin{aligned} Re\langle \mathcal{A}U, U\rangle &= Re \int_{\Gamma_2} (i\Delta u(x) - \alpha z(x,1))\overline{\frac{\partial u(x)}{\partial \nu}}\, d\Gamma - Re \int_\Omega (i\Delta u(x) - \alpha z(x,1))\overline{\Delta u(x)}\, dx \\ &- \xi Re \int_\Omega \int_0^1 \nabla z_\rho(x,\rho)\overline{\nabla z(x,\rho)}\, d\rho\, dx. \end{aligned}$$

Integrating par parts in ρ, we obtain

$$\int_\Omega \int_0^1 \nabla z_\rho(x,\rho)\overline{\nabla z(x,\rho)}\, d\rho\, dx = -\int_\Omega \int_0^1 \nabla z(x,\rho)\overline{\nabla z_\rho(x,\rho)}\, d\rho\, dx + \int_\Omega |\nabla z(x,1)|^2\, dx - \int_\Omega |\nabla z(x,0)|^2\, dx,$$

or equivalently

$$2Re \int_\Omega \int_0^1 \nabla z_\rho(x,\rho)\overline{\nabla z(x,\rho)}\, d\rho\, dx = \int_\Omega |\nabla z(x,1)|^2\, dx - \int_\Omega |\nabla z(x,0)|^2\, dx. \tag{4.20}$$

Therefore

$$\begin{aligned} Re\langle \mathcal{A}U, U\rangle &= Re \int_{\Gamma_2} (i\Delta u(x) - \alpha z(x,1))\overline{\frac{\partial u(x)}{\partial \nu}}\, d\Gamma - Re \int_\Omega (i\Delta u(x) - \alpha z(x,1))\overline{(\Delta u(x))}\, dx \\ &- \frac{\xi}{2} \int_\Omega |\nabla z(x,1)|^2\, dx + \frac{\xi}{2} \int_\Omega |\nabla z(x,0)|^2\, dx. \end{aligned} \tag{4.21}$$

Recalling (4.18), (4.21) can be rewritten as follows yields

$$\begin{aligned} Re\langle \mathcal{A}U, U\rangle &= -\beta \int_{\Gamma_2} |\Delta u(x)|^2\, d\Gamma + \alpha\beta \int_{\Gamma_2} i\Delta u(x)\overline{z(x,1)}\, d\Gamma - \alpha\beta Re \int_{\Gamma_2} iz(x,1)\overline{\Delta u(x)}\, d\Gamma \\ &- \alpha^2\beta \int_{\Gamma_2} |z(x,1)|^2\, d\Gamma - Re \int_\Omega i|\Delta u(x)|^2\, dx + \alpha Re \int_\Omega z(x,1))\overline{\Delta u(x)}\, dx \\ &- \frac{\xi}{2} \int_\Omega |\nabla z(x,1)|^2\, dx + \frac{\xi}{2} \int_\Omega |\nabla u(x)|^2\, dx. \end{aligned}$$

Applying Green's Theorem, we obtain

$$\begin{aligned} Re\langle \mathcal{A}U, U\rangle &= -\beta \int_{\Gamma_2} |\Delta u(x)|^2\, d\Gamma + \alpha\beta Re \int_{\Gamma_2} i\Delta u(x)\overline{z(x,1)}\, d\Gamma - \alpha\beta Re \int_{\Gamma_2} iz(x,1)\overline{\Delta u(x)}\, d\Gamma \\ &- \alpha^2\beta \int_{\Gamma_2} |z(x,1)|^2\, d\Gamma + \alpha Re \int_{\Gamma_2} z(x,1)\overline{\frac{\partial u(x)}{\partial \nu}}\, d\Gamma - \alpha Re \int_\Omega \nabla z(x,1)\overline{\nabla u(x)}\, dx \\ &- \frac{\xi}{2} \int_\Omega |\nabla z(x,1)|^2\, dx + \frac{\xi}{2} \int_\Omega |\nabla u(x)|^2\, dx, \end{aligned} \tag{4.22}$$

which together (4.18) implies

$$Re\langle \mathcal{A}U, U\rangle = -\beta \int_{\Gamma_2} |\Delta u(x)|^2 \, d\Gamma + \alpha\beta Re \int_{\Gamma_2} i\Delta u(x)\overline{z(x,1)} \, d\Gamma - \alpha Re \int_{\Omega} \nabla z(x,1)\overline{\nabla u(x)} \, dx$$
$$- \frac{\xi}{2} \int_{\Omega} |\nabla z(x,1)|^2 \, dx + \frac{\xi}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx.$$

Using Cauchy-Schwartz's inequality, we obtain

$$Re\langle \mathcal{A}U, U\rangle \leq -\beta \int_{\Gamma_2} |\Delta u(x)|^2 \, d\Gamma + \frac{\beta^2}{2\epsilon} \int_{\Gamma_2} |\Delta u(x)|^2 \, d\Gamma + \frac{\alpha^2 \epsilon}{2} \int_{\Gamma_2} |z(x,1)|^2 \, d\Gamma + \frac{\alpha}{2} \int_{\Omega} |\nabla z(x,1)|^2 \, dx$$
$$+ \frac{\alpha}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx - \frac{\xi}{2} \int_{\Omega} |\nabla z(x,1)|^2 \, dx + \frac{\xi}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx,$$

where $\epsilon > 0$.
From the trace Theorem, we have

$$Re\langle \mathcal{A}U, U\rangle \leq -\beta \int_{\Gamma_2} |\Delta u(x)|^2 \, d\Gamma + \frac{\beta^2}{2\epsilon} \int_{\Gamma_2} |\Delta u(x)|^2 \, d\Gamma + \frac{\alpha^2 \epsilon C_0}{2} \int_{\Omega} |\nabla z(x,1)|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |\nabla z(x,1)|^2 \, dx$$
$$+ \frac{\alpha}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx - \frac{\xi}{2} \int_{\Omega} |\nabla z(x,1)|^2 \, dx + \frac{\xi}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx,$$

where C_0 is such that

$$\int_{\Gamma_2} |u|^2 \, d\Gamma \leq C_0 \int_{\Omega} |\nabla u|^2 \, dx.$$

Therefore

$$Re\langle \mathcal{A}U, U\rangle \leq \left(-\beta + \frac{\beta^2}{2\epsilon} \right) \int_{\Gamma_2} |\Delta u(x)|^2 \, d\Gamma + \left(\frac{\alpha^2 \epsilon C_0}{2} + \frac{\alpha}{2} - \frac{\xi}{2} \right) \int_{\Omega} |\nabla z(x,1)|^2 \, dx$$
$$+ \left(\frac{\alpha}{2} + \frac{\xi}{2} \right) \int_{\Omega} |\nabla u(x)|^2 \, dx.$$

Choose

$$\epsilon \geq \frac{\beta}{2}. \tag{4.23}$$

Then, we have from (4.62)

$$Re\langle \mathcal{A}U, U\rangle \leq \left(\frac{\alpha}{2} + \frac{\xi}{2} \right) \int_{\Omega} |\nabla u(x)|^2 \, dx,$$

from which we deduce that there exists $c \geq \frac{\alpha}{2} + \frac{\xi}{2}$ such that

$$Re\langle \mathcal{A}U - cU, U\rangle \leq 0.$$

This shows that $\mathcal{A} - cI$ is dissipative.
Next, we show that $(\lambda I - \mathcal{A})$ is surjective for some $\lambda > 0$.

Given a vector $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}$, we need $\begin{pmatrix} u \\ z \end{pmatrix} \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A}) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

this is equivalent to

$$\lambda u(x) - i\Delta u(x) + \alpha z(x,1) = f(x), \tag{4.24}$$
$$\lambda z(x,\rho) + \tau^{-1} z_\rho(x,\rho) = g(x,\rho). \tag{4.25}$$

Indeed, from (4.25) and (4.13), we have

$$\begin{aligned} &z_\rho(x,\rho) = -\lambda\tau z(x,\rho) + \tau g(x,\rho), && x \in \Omega,\ \rho \in (0,1), \\ &z(x,0) = u(x), && x \in \Omega. \end{aligned}$$

The unique solution of the above initial value problem is given by

$$z(x,\rho) = u(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho} \int_0^\rho g(x,\sigma)e^{\lambda\tau\sigma}\, d\sigma, \qquad x \in \Omega,\ \rho \in (0,1),$$

and in particular

$$z(x,1) = u(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau} \int_0^1 g(x,\sigma)e^{\lambda\tau\sigma}\, d\sigma, \qquad x \in \Omega. \tag{4.26}$$

Inserting (4.26) into (4.24), we obtain

$$\lambda u(x) - i\Delta u(x) + \alpha u(x)e^{-\lambda\tau} = f(x) - \alpha\tau e^{-\lambda\tau} \int_0^1 g(x,\sigma)e^{\lambda\tau\sigma}\, d\sigma. \tag{4.27}$$

The problem (4.27) can be reformulated as follows

$$\left\langle \lambda u(x) - i\Delta u(x) + \alpha u(x)e^{-\lambda\tau}, v \right\rangle = \left\langle f(x) - \alpha\tau e^{-\lambda\tau} \int_0^1 g(x,\sigma)e^{\lambda\tau\sigma}\, d\sigma, v \right\rangle, \qquad \forall v \in H^1_{\Gamma_1}(\Omega), \tag{4.28}$$

or equivalently

$$\begin{aligned} \lambda \int_\Omega \nabla u(x)\overline{\nabla v(x)}\, dx - i \int_\Omega \nabla(\Delta u(x))\overline{\nabla v(x)}\, dx + \alpha e^{-\lambda\tau} \int_\Omega \nabla u(x)\overline{\nabla v(x)}\, dx \\ = \int_\Omega \nabla f(x)\overline{\nabla v(x)}\, dx - \alpha\tau e^{-\lambda\tau} \int_\Omega \int_0^1 \nabla g(x,\sigma)e^{\lambda\tau\sigma}\, d\sigma \overline{\nabla v(x)}\, dx, \qquad \forall v \in H^1_{\Gamma_1}(\Omega). \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} (\lambda + \alpha e^{-\lambda\tau}) \int_\Omega \nabla u(x)\overline{\nabla v(x)}\, dx + i \int_\Omega \Delta u(x)\overline{\Delta v(x)}\, dx - i \int_{\Gamma_2} \Delta u(x)\overline{\frac{\partial v(x)}{\partial \nu}}\, d\Gamma \\ = \int_\Omega \nabla f(x)\overline{\nabla v(x)}\, dx - \alpha\tau e^{-\lambda\tau} \int_\Omega \int_0^1 \nabla g(x,\sigma)e^{\lambda\tau\sigma}\, d\sigma \overline{\nabla v(x)}\, dx, \qquad \forall v \in H^1_{\Gamma_1}(\Omega). \end{aligned} \tag{4.29}$$

From (4.18) and (4.26), we have

$$-i\Delta u(x) = \frac{1}{\beta}\frac{\partial u(x)}{\partial \nu} - \alpha\left(u(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 g(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma\right), \quad \text{on } \Gamma_2. \tag{4.30}$$

Inserting (4.30) in (4.29), we obtain

$$\begin{aligned}
&(\lambda + \alpha e^{-\lambda\tau})\int_\Omega \nabla u(x)\overline{\nabla v(x)}\,dx + i\int_\Omega \Delta u(x)\overline{\Delta v(x)}\,dx + \int_{\Gamma_2}\frac{\partial u(x)}{\partial\nu}\overline{\frac{\partial v(x)}{\partial\nu}}\,d\Gamma \\
&-\alpha e^{-\lambda\tau}\int_{\Gamma_2} u(x)\overline{\frac{\partial v(x)}{\partial\nu}}\,d\Gamma = \alpha\tau e^{-\lambda\tau}\int_0^1 g(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma\overline{\frac{\partial v(x)}{\partial\nu}}\,d\Gamma + \int_\Omega \nabla f(x)\overline{\nabla v(x)}\,dx \\
&-\alpha\tau e^{-\lambda\tau}\int_\Omega\int_0^1 \nabla g(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma\overline{\nabla v(x)}\,dx, \quad \forall v \in H^1_{\Gamma_1}(\Omega),
\end{aligned}$$

or equivalently

$$a(u,v) = \langle k, v\rangle, \qquad \forall v \in H^1_{\Gamma_1}(\Omega), \tag{4.31}$$

where

$$a(u,v) = (\lambda + \alpha e^{-\lambda\tau})\langle u,v\rangle_{H^1_{\Gamma_1}(\Omega)} + i\langle \Delta u, \Delta v\rangle_{L^2(\Omega)} + \frac{1}{\beta}\left\langle \frac{\partial u}{\partial\nu}, \frac{\partial v}{\partial\nu}\right\rangle_{L^2(\Gamma_2)} - \alpha e^{-\lambda\tau}\left\langle u, \frac{\partial v}{\partial\nu}\right\rangle_{L^2(\Gamma_2)}, \tag{4.32}$$

and

$$\langle k, v\rangle = \alpha\tau e^{-\lambda\tau}\left\langle \int_0^1 g(.,\sigma)e^{\lambda\tau\sigma}\,d\sigma, \frac{\partial v}{\partial\nu}\right\rangle_{L^2(\Gamma_2)} + \langle f, v\rangle_{H^1_{\Gamma_1}(\Omega)} - \alpha\tau e^{-\lambda\tau}\left\langle \int_0^1 g(.,\sigma)e^{\lambda\tau\sigma}\,d\sigma, v\right\rangle_{H^1_{\Gamma_1}(\Omega)}.$$

The form $a(u,v)$ is not continuous on $H^1_{\Gamma_1}(\Omega)$.
We introduce the space

$$\mathcal{Z} = \left\{u \in H^1_{\Gamma_1}(\Omega) \,|\, \Delta u \in L^2(\Omega), \frac{\partial u}{\partial\nu} \in L^2(\Gamma_2)\right\},$$

equipped with the norm

$$\|u\|^2_{\mathcal{Z}} = \|u\|^2_{H^1_{\Gamma_1}(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial\nu}\right\|^2_{L^2(\Gamma_2)}.$$

We use the complex version of Lax Millgram Theorem applied on the Banach space $\mathcal{Z}$ (see [77], p.344). We show that for any fixed $k \in H^1_{\Gamma_1}(\Omega) \subset \mathcal{Z}'$, there exists a unique $u \in \mathcal{Z}$ satisfying

$$a(u,v) = k(v), \qquad \forall v \in \mathcal{Z},$$

with $a(u,v)$ given by (4.32). So we need to prove that $a(u,v)$ is continuous and coercive on $\mathcal{Z}$.
Observe that

$$a(u,v) = (\lambda + \alpha e^{-\lambda\tau})\langle u,v\rangle_{H^1_{\Gamma_1}(\Omega)} + i\langle \Delta u, \Delta v\rangle_{L^2(\Omega)} + \frac{1}{\beta}\left\langle \frac{\partial u}{\partial\nu}, \frac{\partial v}{\partial\nu}\right\rangle_{L^2(\Gamma_2)} - \alpha e^{-\lambda\tau}\left\langle u, \frac{\partial v}{\partial\nu}\right\rangle_{L^2(\Gamma_2)}.$$

By the triangle inequality,

$$|a(u,v)| \leq |(\lambda + \alpha e^{-\lambda\tau})\,\langle u,v\rangle_{H^1_{\Gamma_1}(\Omega)}\,| + |\,\langle \Delta u, \Delta v\rangle_{L^2(\Omega)}\,| + \left|\frac{1}{\beta}\left\langle \frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Gamma_2)}\right| + \left|\alpha e^{-\lambda\tau}\left\langle u, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Gamma_2)}\right|.$$

Applying Cauchy-Schwarz's inequality to each inner product on the right-hand side of the previous estimate, we obtain

$$\begin{aligned}|a(u,v)| \leq{}& (\lambda + \alpha e^{-\lambda\tau})\|u\|_{H^1_{\Gamma_1}(\Omega)}\|v\|_{H^1_{\Gamma_1}(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\|\Delta v\|_{L^2(\Omega)} + \frac{1}{\beta}\left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\Gamma_2)}\left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\Gamma_2)} \\ &+ \alpha e^{-\lambda\tau}\|u\|_{L^2(\Gamma_2)}\left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\Gamma_2)}.\end{aligned}$$

From the trace theorem, we have

$$\begin{aligned}|a(u,v)| \leq{}& (\lambda + \alpha e^{-\lambda\tau})\|u\|_{H^1_{\Gamma_1}(\Omega)}\|v\|_{H^1_{\Gamma_1}(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\|\Delta v\|_{L^2(\Omega)} + \frac{1}{\beta}\left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\Gamma_2)}\left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\Gamma_2)} \\ &+ \alpha e^{-\lambda\tau}C_0\|u\|_{H^1_{\Gamma_1}(\Omega)}\left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\Gamma_2)},\end{aligned} \tag{4.33}$$

(4.33) implies that

$$|a(u,v)| \leq Const\,\|u\|_{\mathcal{Z}}\|v\|_{\mathcal{Z}}.$$

Thus, $a(u,v)$ is continuous on $\mathcal{Z}$.
For the coercivity of $a(u,v)$, observe that

$$a(u,u) = (\lambda + \alpha e^{-\lambda\tau})\|u\|^2_{H^1_{\Gamma_1}(\Omega)} + i\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{\beta}\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma_2)} - \alpha e^{-\lambda\tau}\left\langle u, \frac{\partial u}{\partial \nu}\right\rangle_{L^2(\Gamma_2)}.$$

Recalling the inequality $|z| \geq \frac{1}{2}|x| + \frac{1}{2}|y|$ for any complex $z = x + iy$, we obtain

$$\begin{aligned}|a(u,u)| \geq{}& \frac{1}{2}\left|(\lambda + \alpha e^{-\lambda\tau})\|u\|^2_{H^1_{\Gamma_1}(\Omega)} + \frac{1}{\beta}\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma_2)} - \alpha e^{-\lambda\tau}Re\left\langle u, \frac{\partial u}{\partial \nu}\right\rangle_{L^2(\Gamma_2)}\right| \\ &+ \frac{1}{2}\left|\|\Delta u\|^2_{L^2(\Omega)} - \alpha e^{-\lambda\tau}Im\left\langle u, \frac{\partial u}{\partial \nu}\right\rangle_{L^2(\Gamma_2)}\right|.\end{aligned}$$

Therefore

$$\begin{aligned}|a(u,u)| \geq{}& \frac{1}{2}\left[(\lambda + \alpha e^{-\lambda\tau})\|u\|^2_{H^1_{\Gamma_1}(\Omega)} + \frac{1}{\beta}\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma_2)} - \alpha e^{-\lambda\tau}\left|Re\left\langle u, \frac{\partial u}{\partial \nu}\right\rangle_{L^2(\Gamma_2)}\right|\right] \\ &+ \frac{1}{2}\left[\|\Delta u\|^2_{L^2(\Omega)} - \alpha e^{-\lambda\tau}\left|Im\left\langle u, \frac{\partial u}{\partial \nu}\right\rangle_{L^2(\Gamma_2)}\right|\right].\end{aligned}$$

Applying Cauchy-Schwarz's inequality, we get

$$|a(u,u)| \geq \frac{1}{2}\left[(\lambda+\alpha e^{-\lambda\tau})\|u\|^2_{H^1_{\Gamma_1}(\Omega)} + \frac{1}{\beta}\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma_2)} - \alpha^2\epsilon\|u\|^2_{L^2(\Gamma_2)} - \frac{1}{4\epsilon}\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma_2)}\right]$$
$$+\frac{1}{2}\left[\|\Delta u\|^2_{L^2(\Omega)} - \alpha^2\epsilon\|u\|^2_{L^2(\Gamma_2)} - \frac{1}{4\epsilon}\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma_2)}\right],$$

and consequently,

$$|a(u,u)| \geq \frac{1}{2}\left[\left(\lambda+\alpha e^{-\lambda\tau} - 2\alpha^2 C_0\epsilon\right)\|u\|^2_{H^1_{\Gamma_1}(\Omega)} + \left(\frac{1}{\beta}-\frac{1}{2\epsilon}\right)\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma_2)} + \|\Delta u\|^2_{L^2(\Omega)}\right].$$

Therefore

$$|a(u,u)| \geq C\|u\|^2_{\mathcal{Z}},$$

for some constant $C>0$ as long as $\lambda > 2\alpha^2 C_0\epsilon$.
We conclude from the complex version of the Bowder-Minty's Theorem (see [71], p.364, Theorem 10.49) that, for all $k \in \mathcal{Z}'$ where $\mathcal{Z}'$ denotes the dual space of $\mathcal{Z}$, there is a solution $u \in \mathcal{Z}$ to $a(u,v) = k(v)$.
Moreover we observe that

$$D(\mathcal{A}) \subset \mathcal{Z} \times H^1((0,1), H^1_{\Gamma_1}(\Omega)) \subset H^1_{\Gamma_1}(\Omega) \times L^2((0,1), H^1_{\Gamma_1}(\Omega)) \subset \mathcal{Z}' \times (H^1((0,1), H^1_{\Gamma_1}(\Omega)))',$$

and hence for all $k \in H^1_{\Gamma_1}(\Omega)$, the functional defined by $\langle k, v\rangle_{H^1_{\Gamma_1}(\Omega)}$ belongs to $\mathcal{Z}'$.
Hence, there is a unique solution $u \in \mathcal{Z} \subset H^1_{\Gamma_1}(\Omega)$ to a variational form $a(u,v) = \langle k, v\rangle_{H^1_{\Gamma_1}(\Omega)}$ for all $v \in \mathcal{Z}$.
Furthermore, from the above variational form, by restricting to function v with zero Neuman data, one recovers the equation

$$\lambda u - i\Delta u + \alpha u e^{-\lambda\tau} = f - \alpha\tau e^{-\lambda\tau}\int_0^1 g(x,\sigma)e^{\lambda\tau\sigma}\, d\sigma. \tag{4.34}$$

Since $f - \alpha\tau e^{-\lambda\tau}\int_0^1 g(x,\sigma)e^{\lambda\tau\sigma}\, d\sigma \in H^1_{\Gamma_1}(\Omega)$ and $u \in H^1_{\Gamma_1}(\Omega)$, we obtain $\Delta u \in H^1_{\Gamma_1}(\Omega)$.
With the above regularity, we go back to variational form after integration by parts:

$$(\lambda+\alpha e^{-\lambda\tau})\langle u, v\rangle_{H^1_{\Gamma_1}(\Omega)} - i\langle\nabla(\Delta u), \nabla v\rangle_{L^2(\Omega)} + i\left\langle \Delta u, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Gamma_2)} + \frac{1}{\beta}\left\langle\frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Gamma_2)}$$
$$-\alpha e^{-\lambda\tau}\left\langle u, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Gamma_2)} = \langle k, v\rangle_{H^1_{\Gamma_1}(\Omega)},$$

which combined with (4.34) gives

$$i\left\langle \Delta u, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Gamma_2)} + \frac{1}{\beta}\left\langle\frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Gamma_2)}$$
$$-\alpha e^{-\lambda\tau}\left\langle u, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Gamma_2)} - \alpha\tau e^{-\lambda\tau}\left\langle\int_0^1 g(x,\sigma)e^{\lambda\sigma}\, d\sigma, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Gamma_2)} = 0, \qquad v \in \mathcal{Z}.$$

The above identity implies that

$$i\Delta u + \frac{1}{\beta}\frac{\partial u}{\partial \nu} - \alpha e^{-\lambda\tau} u - \alpha\tau e^{-\lambda\tau}\int_0^1 g(x,\sigma)e^{\lambda\sigma}\,d\sigma = 0, \qquad \text{on } \Gamma_2,$$

therefore

$$\frac{\partial u}{\partial \nu} = -\beta(i\Delta u - \alpha z(x,1)), \qquad \text{on } \Gamma_2.$$

Moreover, $\Delta u \in H^1_{\Gamma_1}(\Omega)$ implies that

$$\Delta u|_{\Gamma_2} \in H^{1/2}(\Gamma_2),$$

and $z(x,1) \in H^1_{\Gamma_1}(\Omega)$ implies that

$$z(x,1)|_{\Gamma_2} \in H^{1/2}(\Gamma_2),$$

and thus

$$\frac{\partial u}{\partial \nu} \in H^{1/2}(\Gamma_2),$$

as well.
Trace theory tells us that $(u,z) \in H^2(\Omega) \times H^1((0,1), H^1_{\Gamma_1}(\Omega))$, and so we know that the regularity of $D(\mathcal{A})$ is at least $H^2(\Omega) \times H^1((0,1), H^1_{\Gamma_1}(\Omega))$.
So we have found $(u,z) \in D(\mathcal{A})$ which satisfies (4.24) and (4.25). Consequently, $(\lambda I - \mathcal{A})$ is surjective and therefore $(\lambda I - (\mathcal{A} - cI))$ is also surjective. Finally, the Lumer-Phillips leads to the fact that $\mathcal{A} - cI$ generates a strongly continuous semigroup of contraction in $\mathcal{H}$, hence $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{H}$.

□

4.3 Proof of Theorem 4.1

We prove Theorem 4.1 for smooth initial data. The general case follows by a density argument.

Proposition 4.3. *For any solution of problem* (4.3) − (4.7) *and for every* $\epsilon > 0$*, the following estimate holds:*

$$\frac{d}{dt}E(t) \le \left(-\beta + \frac{\beta^2}{2\epsilon}\right)\int_{\Gamma_2} |u_t(x,t)|^2 + \left(\frac{\alpha^2\epsilon C_0}{2} + \frac{\alpha}{2} - \frac{\xi}{2}\right)\int_\Omega |\nabla u(x,t-\tau)|^2\,dx + \left(\frac{\alpha}{2} + \frac{\xi}{2}\right)\int_\Omega |\nabla u(x)|^2\,dx. \tag{4.35}$$

Proof. Differentiating $E(t)$ defined by (4.8), we obtain

$$\frac{d}{dt}E(t) = Re\int_\Omega \nabla u_t(x,t)\overline{\nabla u(x,t)}\,dx + \xi\tau Re\int_\Omega\int_0^1 \nabla u_t(x,t-\tau\rho)\overline{\nabla u(x,t-\tau\rho)}\,d\rho\,dx.$$

Applying Green's Theorem and recalling the boundary conditions in (4.3) – (4.7), we obtain

$$\frac{d}{dt}E(t) = -\beta\int_{\Gamma_2}|u_t(x,t)|^2\,d\Gamma - Re\int_\Omega (i\Delta u(x,t) - \alpha u(x,t-\tau))\overline{\Delta u(x,t)}\,dx$$
$$+\,\xi\tau Re\int_\Omega\int_0^1 \nabla u_t(x,t-\tau\rho)\overline{\nabla u(x,t-\tau\rho)}\,d\rho\,dx. \tag{4.36}$$

Now observe that

$$\nabla u_t(x,t-\tau\rho) = -\tau\nabla u_\rho(x,t-\tau\rho), \tag{4.37}$$

and

$$\frac{d}{d\rho}|\nabla u(x,t-\tau\rho)|^2 = 2Re\nabla u_\rho(x,t-\tau\rho)\overline{\nabla u(x,t-\tau\rho)}. \tag{4.38}$$

Insertion of (4.38) into (4.36) yields

$$\frac{d}{dt}E(t) = -\beta\int_{\Gamma_2}|u_t(x,t)|^2\,d\Gamma + \alpha Re\int_\Omega u(x,t-\tau))\overline{\Delta u(x,t)}\,dx - \frac{\xi}{2}\int_\Omega\int_0^1\frac{d}{d\rho}|\nabla u(x,t-\tau\rho)|^2\,d\rho\,dx$$
$$= -\beta\int_{\Gamma_2}|u_t(x,t)|^2\,d\Gamma + \alpha Re\int_\Omega u(x,t-\tau))\overline{\Delta u(x,t)}\,dx - \frac{\xi}{2}\int_\Omega|\nabla u(x,t-\tau)|^2\,dx$$
$$+\frac{\xi}{2}\int_\Omega|\nabla u(x,t)|^2\,dx.$$

Applying Green's Theorem and recalling the boundary conditions in problem (4.3) – (4.7), we have

$$\frac{d}{dt}E(t) \le -\beta\int_{\Gamma_2}|u_t(x,t)|^2\,d\Gamma + \alpha\beta Re\int_{\Gamma_2}u(x,t-\tau)u_t(x,t)\,d\Gamma - \alpha Re\int_\Omega\nabla u(x,t-\tau)\overline{\nabla u(x,t)}\,dx$$
$$-\frac{\xi}{2}\int_\Omega|\nabla u(x,t-\tau)|^2\,dx + \frac{\xi}{2}\int_\Omega|\nabla u(x,t)|^2\,dx.$$

From Cauchy-Schwarz's inequality, we have

$$\frac{d}{dt}E(t) \le -\beta\int_{\Gamma_2}|u_t(x,t)|^2\,d\Gamma + \frac{\alpha^2\epsilon}{2}\int_{\Gamma_2}|u(x,t-\tau)|^2\,d\Gamma + \frac{\beta^2}{2\epsilon}\int_{\Gamma_2}|u_t(x,t)|^2\,d\Gamma + \frac{\alpha}{2}\int_\Omega|\nabla u(x,t-\tau)|^2\,dx$$
$$+\frac{\alpha}{2}\int_\Omega|\nabla u(x,t)|^2\,dx - \frac{\xi}{2}\int_\Omega|\nabla u(x,t-\tau)|^2\,dx + \frac{\xi}{2}\int_\Omega|\nabla u(x,t)|^2\,dx.$$

From the trace theorem, we obtain

$$\frac{d}{dt}E(t) \le \left(-\beta + \frac{\beta^2}{2\epsilon}\right)\int_{\Gamma_2}|u_t(x,t)|^2\,d\Gamma + \left(\frac{\alpha^2 C_0\epsilon}{2} + \frac{\alpha}{2} - \frac{\xi}{2}\right)\int_\Omega|\nabla u(x,t-\tau)|^2\,dx$$
$$+\left(\frac{\alpha}{2}+\frac{\xi}{2}\right)\int_\Omega|\nabla u(x,t)|^2\,dx.$$

□

Let us introduce the Lyapunov functional

$$\mathcal{E}(t) = E(t) + \gamma_1 \mathcal{E}_2(t) + \gamma_2 S(t), \tag{4.39}$$

with

$$\mathcal{E}_2(t) = Im \int_\Omega u(x,t) m(x).\overline{\nabla u(x,t)}\, dx, \tag{4.40}$$

and

$$S(t) = \tau \int_\Omega \int_0^1 e^{-\tau\rho} |\nabla u(x, t-\tau\rho)|^2 \, d\rho \, dx, \tag{4.41}$$

where γ_1 and γ_2 are suitable positive small constants that will be precised later.

Lemma 4.1. *For γ_1 small enough, there exist positive constants C_1, a_1 and a_2 such that*

$$a_1 E(t) \leq \mathcal{E}(t) \leq a_2 E(t), \qquad \forall\, 0 < \gamma_1, \gamma_2 \leq C_1. \tag{4.42}$$

Proof. From the definition of $\mathcal{E}_2(t)$, we have

$$\gamma_1 \mathcal{E}_2(t) \leq \gamma_1 \left| \int_\Omega u\, m.\overline{\nabla u}\, dx \right|,$$

Applying Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$\mathcal{E}(t) \leq a_2 E(t), \tag{4.43}$$

for suitable positive constant $a_2 = max \left\{ 1 + \gamma_1 R(C_p + 1), 1 + \frac{2\gamma_2}{\xi} \right\}$, where $R = \|m\|_\infty$ and C_p is the Poincaré's constant.
On the other hand, we also have from Cauchy-Schwarz and Poincaré's inequalities

$$\mathcal{E}_2(t) \geq -\gamma_1 \frac{R(C_p+1)}{2} \int_\Omega |\nabla u|^2 \, dx.$$

From the definition of $S(t)$, we deduce

$$\gamma_2 S(t) \geq \gamma_2 \tau e^{-\tau} \int_\Omega \int_0^1 |\nabla u(x, t-\tau\rho)|^2 \, d\rho \, dx.$$

Therefore

$$\mathcal{E}(t) \geq E(t) - \gamma_1 \frac{R(C_p+1)}{2} \int_\Omega |\nabla u|^2 \, dx + \gamma_2 \tau e^{-\tau} \int_\Omega \int_0^1 |\nabla u(x, t-\tau\rho)|^2 \, d\rho \, dx.$$

Consequently

$$\mathcal{E}(t) \geq (1 - \gamma_1 R(1 + C_p)) \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \left(1 + \frac{2\gamma_2 e^{-\tau}}{\xi} \right) \frac{\xi\tau}{2} \int_\Omega \int_0^1 |\nabla u(x, t-\tau\rho)|^2 \, d\rho \, dx,$$

and for

$$\gamma_1 \leq \frac{1}{R(1+C_p)}, \tag{4.44}$$

we obtain

$$\mathcal{E}(t) \geq a_1 E(t), \tag{4.45}$$

for suitable positive constant a_1. By (4.43) and (4.45) we have (4.42). □

Proposition 4.4. *For any solution of problem* (4.3) – (4.7) *and for every* $\epsilon > 0$, *we have*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_2(t) \leq & \left(-\frac{3}{2} + R^2\alpha + \frac{n^2\alpha C_p}{2}\right)\int_\Omega |\nabla u|^2\,dx + \left(\frac{\beta^2 R^2}{\delta} + \frac{C_0(R^2+n^2\beta^2)}{2}\right)\int_{\Gamma_2}|u_t|^2\,d\Gamma \\ & + \frac{3\alpha C_p}{2}\int_\Omega |\nabla u(x,t-\tau)|^2\,dx. \end{aligned}\tag{4.46}$$

Proof. Differentiating $\mathcal{E}_2(t)$ in (4.40) we obtain

$$\frac{d}{dt}\mathcal{E}_2(t) = Im\int_\Omega u_t\,(m.\overline{\nabla u})\,dx + Im\int_\Omega u\,(m.\overline{\nabla u_t})\,dx. \tag{4.47}$$

Using Green's Theorem, we get

$$\begin{aligned} Im\int_\Omega u\,(m.\overline{\nabla u_t})\,dx &= Im\int_\Gamma (m.\nu)\,u\,\overline{u_t}\,d\Gamma - Im\int_\Omega (m.\nabla u)\overline{u_t}\,dx - div\,m\,Im\int_\Omega u\,\overline{u_t}\,dx \\ &= Im\int_{\Gamma_2}(m.\nu)\,u\,\overline{u_t}\,d\Gamma + Im\int_\Omega (m.\overline{\nabla u})\,u_t\,dx - n\,Im\int_\Omega u\,\overline{u_t}\,dx. \end{aligned}\tag{4.48}$$

On the other hand, from (4.3)-(4.7), we get after another use of Green's Theorem

$$\begin{aligned} Im\int_\Omega u\,\overline{u_t}\,dx &= -Re\int_\Omega \Delta u\,\overline{u}\,dx - \alpha\,Im\int_\Omega u\,\overline{u(x,t-\tau)}\,dx \\ &= -Re\int_\Gamma \frac{\partial u}{\partial\nu}\,\overline{u}\,d\Gamma + \int_\Omega |\nabla u|^2\,dx - \alpha\,Im\int_\Omega u\,\overline{u(x,t-\tau)}\,dx \\ &= \beta\int_{\Gamma_2} u_t\,\overline{u}\,d\Gamma + \int_\Omega |\nabla u|^2\,dx - \alpha\,Im\int_\Omega u\,\overline{u(x,t-\tau)}\,dx, \end{aligned}$$

and

$$Im\int_\Omega (m.\nabla u)\overline{u_t}\,dx = -Re\int_\Omega \Delta u(m.\overline{\nabla u})\,dx - \alpha\,Im\int_\Omega (m.\nabla u)\overline{u(x,t-\tau)}\,dx.$$

Thus

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2(t) = & 2Re\int_\Omega \Delta u(m.\overline{\nabla u})\,dx - n\int_\Omega |\nabla u|^2\,dx - Re\int_{\Gamma_2}(i\,m.\nu + n\beta)u\overline{u_t}\,d\Gamma \\ & + n\alpha\,Im\int_\Omega u\,\overline{u(x,t-\tau)}\,dx - 2\alpha\,Im\int_\Omega (m.\nabla u)\overline{u(x,t-\tau)}\,dx. \end{aligned}\tag{4.49}$$

Applying Green's Theorem to the first term on the right-hand side of (4.49), we obtain

$$Re\int_\Omega \Delta u(m.\overline{\nabla u})\,dx = Re\int_\Gamma \frac{\partial u}{\partial \nu}\,(m.\overline{\nabla u})\,d\Gamma - Re\int_\Omega \nabla(m.\overline{\nabla u})\nabla u\,dx.$$

Therefore

$$Re\int_\Omega \Delta u(m.\overline{\nabla u})\,dx = Re\int_\Gamma \frac{\partial u}{\partial \nu}\,(m.\overline{\nabla u})\,d\Gamma - \int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Gamma |\nabla u|^2(m.\nu)\,d\Gamma + \frac{n}{2}\int_\Omega |\nabla u|^2\,dx. \tag{4.50}$$

Combining (4.49) and (4.50) together with (4.1) and (4.2), we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_2(t) &= -2\int_\Omega |\nabla u|^2\,dx + 2Re\int_\Gamma \frac{\partial u}{\partial \nu}\,(m.\overline{\nabla u})\,d\Gamma - \int_\Gamma |\nabla u|^2(m.\nu)\,d\Gamma - Re\int_{\Gamma_2}(i\,m.\nu + n\beta)u\overline{u_t}\,d\Gamma \\
&\quad + n\alpha\,Im\int_\Omega u\,\overline{u(x,t-\tau)}\,dx - 2\alpha\,Im\int_\Omega (m.\nabla u)\overline{u(x,t-\tau)}\,dx \\
&= -2\int_\Omega |\nabla u|^2\,dx + Re\int_{\Gamma_1}(m.\nu)\left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma - 2\beta Re\int_{\Gamma_2} u_t\,(m.\overline{\nabla u})\,d\Gamma - \int_{\Gamma_2}|\nabla u|^2(m.\nu)\,d\Gamma \\
&\quad - Re\int_{\Gamma_2}(i\,m.\nu + n\beta)u\overline{u_t}\,d\Gamma + n\alpha\,Im\int_\Omega u\,\overline{u(x,t-\tau)}\,dx - 2\alpha\,Im\int_\Omega (m.\nabla u)\overline{u(x,t-\tau)}\,dx \\
&\le -2\int_\Omega |\nabla u|^2\,dx - 2\beta Re\int_{\Gamma_2} u_t\,(m.\overline{\nabla u})\,d\Gamma - \delta\int_{\Gamma_2}|\nabla u|^2\,d\Gamma - Re\int_{\Gamma_2}(i\,m.\nu + n\beta)\,u\,\overline{u_t}\,d\Gamma \\
&\quad + n\alpha\,Im\int_\Omega u\,\overline{u(x,t-\tau)}\,dx - 2\alpha\,Im\int_\Omega (m.\nabla u)\overline{u(x,t-\tau)}\,dx.
\end{aligned} \tag{4.51}$$

From the trace theorem, we have

$$\begin{aligned}
\left|\int_{\Gamma_2}(i\,m.\nu + n\beta)\,u\,\overline{u_t}\,d\Gamma\right| &\le C_0\frac{(R^2+n^2\beta^2)}{2}\int_{\Gamma_2}|u_t|^2\,d\Gamma + \frac{1}{2C_0}\int_{\Gamma_2}|u|^2\,d\Gamma \\
&\le C_0\frac{(R^2+n^2\beta^2)}{2}\int_{\Gamma_2}|u_t|^2\,d\Gamma + \frac{1}{2}\int_\Omega|\nabla u|^2\,dx.
\end{aligned} \tag{4.52}$$

We also have

$$\left|2\beta u_t\,(m.\overline{\nabla u})\right| \le \frac{2\beta^2R^2}{\delta}|u_t|^2 + \frac{\delta}{2}|\nabla u|^2, \qquad \text{on } \Gamma_2\times(0,\infty). \tag{4.53}$$

Combining (4.51)-(4.53), we conclude that

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_2(t) &\le -2\int_\Omega |\nabla u|^2\,dx + \frac{2\beta^2R^2}{\delta}\int_{\Gamma_2}|u_t|^2\,d\Gamma + C_0\frac{(R^2+n^2\beta^2)}{2}\int_{\Gamma_2}|u_t|^2\,d\Gamma + \frac{1}{2}\int_{\Omega_2}|\nabla u|^2\,dx \\
&\quad + n\alpha\,Im\int_\Omega u\,\overline{u(x,t-\tau)}\,dx - 2\alpha\,Im\int_\Omega (m.\nabla u)\overline{u(x,t-\tau)}\,dx.
\end{aligned}$$

Applying Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_2(t) &\le -2\int_\Omega |\nabla u|^2\,dx + \frac{2\beta^2R^2}{\delta}\int_{\Gamma_2}|u_t|^2\,d\Gamma + C_0\frac{(R^2+n^2\beta^2)}{2}\int_{\Gamma_2}|u_t|^2\,d\Gamma + \frac{1}{2}\int_{\Omega_2}|\nabla u|^2\,dx \\
&\quad + \frac{n^2\alpha\,C_p}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\alpha\,C_p}{2}\int_\Omega |\nabla u(x,t-\tau)|^2\,dx + R^2\alpha\int_\Omega |\nabla u|^2\,dx + \alpha\,C_p\int_\Omega |\nabla u(x,t-\tau)|^2\,dx.
\end{aligned} \tag{4.54}$$

The estimate (4.46) is therefore proved. □

Proposition 4.5. *For any solution of problem (4.3)-(4.7), the following estimate holds:*

$$\frac{d}{dt}\mathcal{S}(t) \leq -e^{-\tau}\int_\Omega |\nabla u(x,t-\tau)|^2\,dx + \int_\Omega |\nabla u|^2\,dx - \tau e^{-\tau}\int_\Omega\int_0^1 |\nabla u(x,t-\tau\rho)|^2\,d\rho\,dx. \tag{4.55}$$

Proof. Differentiating $S(t)$ defined by (4.41), we obtain

$$\frac{d}{dt}S(t) = 2\tau Re\int_\Omega\int_0^1 e^{-\tau\rho}\nabla u_t(x,t-\tau\rho)\overline{\nabla u(x,t-\tau\rho)}\,d\rho\,dx.$$

Using (4.37) and (4.38), we obtain

$$\frac{d}{dt}S(t) = -\int_\Omega\int_0^1 e^{-\tau\rho}\frac{d}{d\rho}|\nabla u(x,t-\tau\rho)|^2\,d\rho\,dx.$$

Integrating by parts in ρ, yields

$$\begin{aligned}\frac{d}{dt}S(t) &= -e^{-\tau}\int_\Omega |\nabla u(x,t-\tau)|^2\,dx + \int_\Omega |\nabla u|^2\,dx - \tau\int_\Omega\int_0^1 e^{-\tau\rho}|\nabla u(x,t-\tau\rho)|^2\,d\rho\,dx\\ &\leq -e^{-\tau}\int_\Omega |\nabla u(x,t-\tau)|^2\,dx + \int_\Omega |\nabla u|^2\,dx - \tau e^{-\tau}\int_\Omega\int_0^1 |\nabla u(x,t-\tau\rho)|^2\,d\rho\,dx.\end{aligned}$$

The estimate (4.55) is proved. □

Now, we can deduce an estimate for the Lyapunov functional $\mathcal{E}(t)$.

Proposition 4.6. *For any $\beta > 0$ there exist α_0, m and c such that for any solution of problem (4.3)-(4.7) with $0 \leq \alpha < \alpha_0$ we have*

$$\mathcal{E}(t) \leq m\,e^{-ct}\mathcal{E}(0), \qquad t > 0. \tag{4.56}$$

The constants α_0, m and c are independent of the initial data but they depend on β and the geometry of Ω.

Proof. Differentiating the Lyapunov functional $\mathcal{E}$ and using Proposition 4.3, Proposition 4.4 and Proposition 4.5, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) \leq &\left\{-\beta + \frac{\beta^2}{2\epsilon} + \gamma_1\left(\frac{2\beta^2R^2}{\delta} + \frac{C_0(R^2+n^2\beta^2)}{2}\right)\right\}\int_{\Gamma_2}|u(x,t)|^2\,d\Gamma\\ &+\left\{\frac{\alpha}{2} + \frac{\xi}{2} + \gamma_1\left(-\frac{3}{2} + R^2\alpha + \frac{n^2\alpha C_p}{2}\right) + \gamma_2\right\}\int_\Omega |\nabla u(x,t)|^2\,dx\\ &+\left\{\frac{\alpha^2\epsilon\, C_0}{2} + \frac{\alpha}{2} - \frac{\xi}{2} + \gamma_1\frac{3\,\alpha\, C_p}{2} - \gamma_2 e^{-\tau}\right\}\int_\Omega |\nabla u(x,t-\tau)|^2\,dx\\ &-\gamma_2\tau e^{-\tau}\int_\Omega\int_0^1 |\nabla u(x,t-\tau\rho)|^2\,d\rho\,dx.\end{aligned} \tag{4.57}$$

For a fixed β we want to choose $\gamma_1, \gamma_2 < C_1$ and a sufficiently small in order to obtain

$$\frac{d}{dt}\mathcal{E}(t) \leq -CE(t), \tag{4.58}$$

from which follows (4.56).

To deduce (4.58) from (4.57), we need

$$-\beta + \frac{\beta^2}{2\epsilon} + \gamma_1 \left(\frac{2\beta^2 R^2}{\delta} + \frac{C_0(R^2 + n^2\beta^2)}{2} \right) \leq 0 \tag{4.59}$$

$$\frac{\alpha}{2} + \frac{\xi}{2} + \gamma_1 \left(-\frac{3}{2} + R^2\alpha + \frac{n^2\alpha C_p}{2} \right) + \gamma_2 < 0 \tag{4.60}$$

$$\frac{\alpha^2\epsilon\, C_0}{2} + \frac{\alpha}{2} - \frac{\xi}{2} + \gamma_1 \frac{3\,\alpha\, C_p}{2} - \gamma_2 e^{-\tau} \leq 0. \tag{4.61}$$

A sufficient condition for the last inequality is

$$\frac{\alpha^2\epsilon\, C_0}{2} + \frac{\alpha}{2} - \frac{\xi}{2} + \gamma_1 \frac{3\,\alpha\, C_p}{2} \leq 0. \tag{4.62}$$

This conditions (4.59), (4.60) and (4.62) are equivalent to

$$\gamma_1 \left[\beta^2 \left(\frac{2R^2}{\delta} + \frac{C_0 n^2}{2} \right) + \frac{C_0 R^2}{2} \right] \leq \beta - \frac{\beta^2}{2\epsilon} \tag{4.63}$$

$$\alpha \left(\frac{1}{2} + \gamma_1 R^2 + \gamma_1 \frac{n^2 C_p}{2}) \right) + \frac{\xi}{2} < \frac{3}{2}\gamma_1 - \gamma_2 \tag{4.64}$$

$$\alpha \left(\frac{\alpha\,\epsilon\, C_0}{2} + \frac{1}{2} + \gamma_1 \frac{3\, C_p}{2} \right) - \frac{\xi}{2} \leq 0. \tag{4.65}$$

By the assumption

$$0 < \beta < 2\epsilon, \tag{4.66}$$

(4.63) is satisfied for

$$\gamma_1 \leq \left(\beta - \frac{\beta^2}{2\epsilon} \right) \left(\beta^2 \left(\frac{2R^2}{\delta} + \frac{C_0 n^2}{2} \right) + \frac{C_0 R^2}{2} \right)^{-1}. \tag{4.67}$$

A necessary condition (4.64) is

$$\frac{2}{3}\gamma_2 < \gamma_1. \tag{4.68}$$

Then we now fix γ_1 and γ_2 fulfilling the above requirements and look at (4.64) and (4.65) as conditions on α and ξ. An analysis shows that the set of pairs (α, ξ) fulfilling this constraints is not empty (see Figure 2.).

It can be seen from this figure that for α and ξ small enough, (4.64) and (4.65) are valid. Note further that due to (4.63) if β goes to ∞ or to 0, then γ_1 must tend to zero, and therefore γ_2 as well and the maximal value α_0 of α goes to zero. □

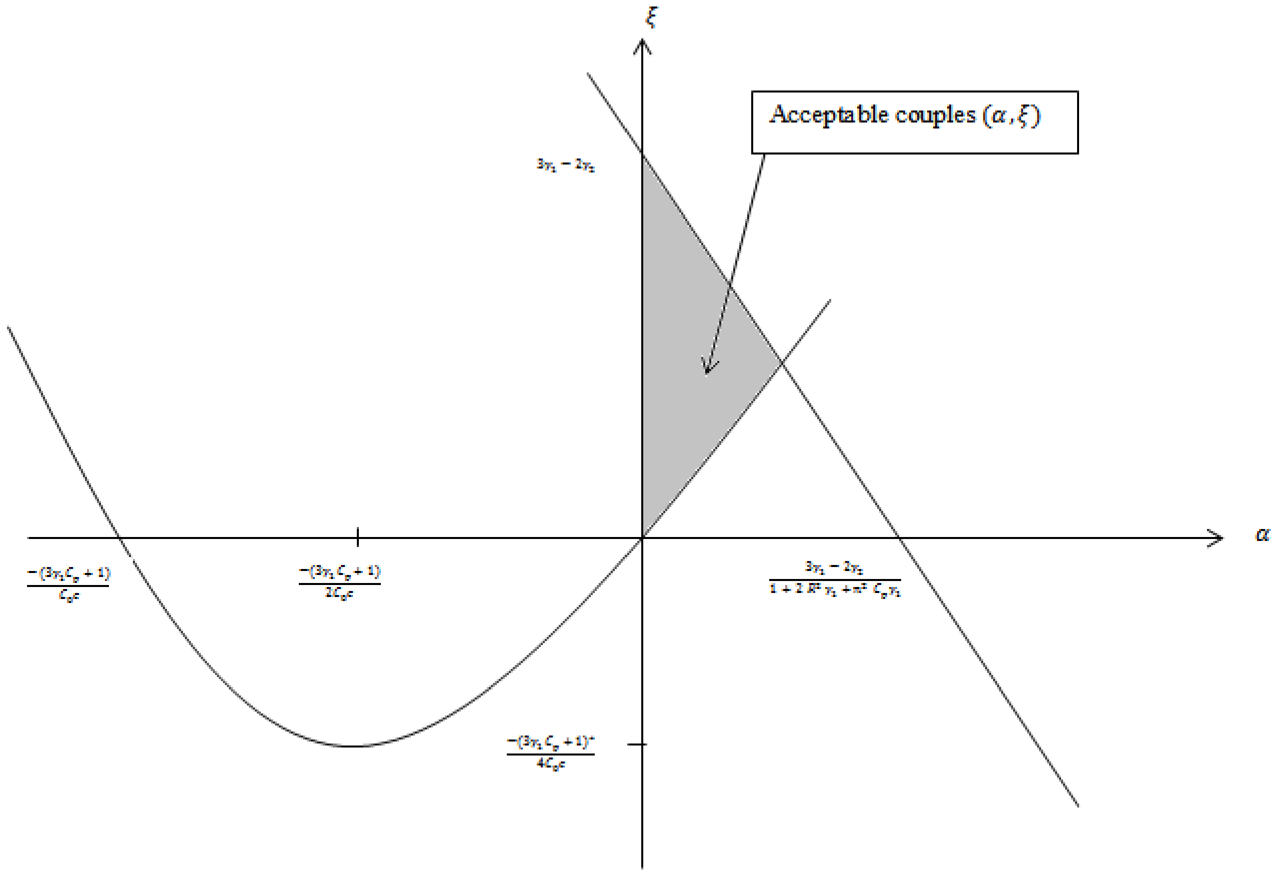

Figure 2.

From Proposition 4.6 and the energy equivalence (4.42) we deduce estimate (4.9).

Remark 4.1. *We can make explicit the relation between β and α_0 by choosing the constants ξ, γ_1 and γ_2 in the definitions (4.8) and (4.39) of the energy function $E(.)$ and of the Lyapunov functional $\mathcal{E}(.)$ in such a way that the conditions (4.63)-(4.65) are satisfied.*
For instance, we fix

$$\xi = 2\alpha$$

Now, choose

$$\gamma_1 = min\left\{\frac{1}{3C_p + \epsilon_0}, \frac{1}{R(1 + C_p)}, \left(\beta - \frac{\beta^2}{2\epsilon}\right)\left(\beta^2\left(\frac{2R^2}{\delta} + \frac{C_0 n^2}{2}\right) + \frac{C_0 R^2}{2}\right)^{-1}\right\},$$

and

$$\gamma_2 = \frac{\gamma_1}{2}$$

The above choices of γ_1 and γ_2 conditions (4.63)-(4.65) are satisfied for any $\alpha > 0$.

The remaining conditions are satisfied for all

$$0 \leq \alpha < \alpha_0,$$

with

$$\alpha_0 = min\left\{\frac{1-3C_p\gamma_1}{C_0\epsilon}, \frac{2\gamma_1}{3+\gamma_1(2R^2+n^2C_p)}\right\},$$

that is

$$\alpha_0 = min\Bigg\{K_1, K_2, K_3, K_4, K_5, K_6\Bigg\},$$

where

$$K_1 = \frac{1}{C_0(3C_p+\epsilon_0)},\ K_2 = \frac{1-\frac{3C_p}{R(1+C_p)}}{C_0\epsilon},\ K_3 = \frac{1-3C_p\left(\beta-\frac{\beta^2}{2\epsilon}\right)\left(\beta^2\left(\frac{2R^2}{\delta}+\frac{C_0n^2}{2}\right)+\frac{C_0R^2}{2}\right)^{-1}}{C_0\epsilon},$$

$$K_4 = \frac{2}{3(3C_p+\epsilon_0)+2R^2+n^2C_p},\ K_5 = \frac{2}{3R(1+C_p)+2R^2+n^2C_p},\ K_6 = \frac{2\left(\beta-\frac{\beta^2}{2\epsilon}\right)\left(\beta^2\left(\frac{2R^2}{\delta}+\frac{C_0n^2}{2}\right)+\frac{C_0R^2}{2}\right)^{-1}}{3+\left(\beta-\frac{\beta^2}{2\epsilon}\right)\left(\beta^2\left(\frac{2R^2}{\delta}+\frac{C_0n^2}{2}\right)+\frac{C_0R^2}{2}\right)^{-1}(2R^2+n^2C_p)}$$

Observe that $\alpha_0 \to 0$ if $\beta \to 0$ and also, if $\beta \to +\infty$.

Chapter 5

Stabilization of the Schrödinger equation with boundary or internal distributed time delay

5.1 Introduction

In this chapter, we study stability problems for the Schrödinger equation with a distributed delay term in the boundary or internal feedbacks.
Let Ω be an open bounded domain of $\mathbb{R}^n$ with smooth boundary Γ which consists of two non-empty parts Γ_1 and Γ_2 such that, $\Gamma_1 \cup \Gamma_2 = \Gamma$ with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$.
In addition to these standard hypothesis, we assume the following.

(A) There exists $x_0 \in \mathbb{R}^n$ such that, with $m(x) = x - x_0$,

$$m(x).\nu(x) \leq 0 \text{ on } \Gamma_1, \tag{5.1}$$

where $\nu(.)$ is the unit normal to Γ pointing towards the exterior of Ω.

In Ω, we consider the following system described by the Schrödinger equation with distributed delay term in the boundary feedback:

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 & \text{in } \Omega \times (0;+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = i\alpha_0 u(x,t) + i\int_{\tau_1}^{\tau_2} \alpha(s) u(x,t-s)\,ds & \text{on } \Gamma_2 \times (0,+\infty), \\ u(x,-t) = f_0(x,-t) & \text{on } \Gamma_2 \times (0,\tau_2), \end{cases} \tag{5.2}$$

where

- u_0 and f_0 are the initial data which belong to suitable spaces.
- $\frac{\partial}{\partial \nu}$ is the normal derivative.

- τ_1 and τ_2 are two real numbers with $0 \leq \tau_1 < \tau_2$.
- α_0 is a positive constant.
- $\alpha : [\tau_1, \tau_2] \to \mathbb{R}$ is an L^∞ function, $\alpha \geq 0$ almost everywhere.

In the absence of delay (*i.e.* $\alpha = 0$), Lasiecka et al [47] have shown that the solution of decays exponentially to zero in the energy space $L^2(\Omega)$. In the presence of delay concentrated at a time τ that is the boundary condition on Γ_2 in (5.2) is replaced by

$$\frac{\partial u}{\partial \nu}(x,t) = i\alpha_0 u(x,t) + i\alpha_1 u(x, t-\tau), \qquad \text{on } \Gamma_2 \times (0, +\infty),$$

where τ is the time delay, Nicaise and Rebiai [65] have shown that the solution decays exponentially in an appropriate energy space provided that

$$\alpha_0 > \alpha_1 \tag{5.3}$$

On the contrary, if (5.3) does not hold they constructed a sequence of delays for which the corresponding solution of (5.2) is unstable. In [21], the authors developed an observer-predictor scheme to stabilize the 1-d Schrödinger equation with distributed input time delay.
One of the purposes of this chapter is to investigate the stability of system (5.2). To this aim, assume as in [63]

$$\alpha_0 > \int_{\tau_1}^{\tau_2} \alpha(s)\, ds, \tag{5.4}$$

which guarantees the existence of a positive constant c_0 such that

$$\alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\, ds - \frac{c_0}{2}(\tau_2 - \tau_1) > 0, \tag{5.5}$$

and define the energy of a solution of system (5.2) by

$$E(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2 dx + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s) + c_0)\int_0^1 |u(x, t - \rho s)|^2\, d\rho\, ds\, d\Gamma. \tag{5.6}$$

Then we have the following stability result for system (5.2).

Theorem 5.1. *Assume* (*A*) *and* (5.4). *Then, there exist constants* $M \geq 1$ *and* $\delta > 0$ *such that*

$$E(t) \leq M e^{-\delta t} E(0).$$

The proof of this result is based on an energy estimate at the $L^2(\Omega)$ level for a fully Schrödinger equation with gradient and potential terms stated in [50], Theorem 2.6.1 and established in [51], Section 10. This result can be summarized as follows: Assume that the hypothesis (A) holds and let u be a smooth solution of the partial differential equation in (5.2) satisfying

$$u(x,t) = 0 \;\text{ on } \Gamma_1 \times (0, T)$$

Then there exists a constant $c > 0$ depending on T such that

$$\int_\Omega |u(x,0)|^2\, dx \le c\Big\{ \|u\|^2_{L^2(0,T;L^2(\Gamma_2))} + \int_0^T \int_{\Gamma_2} \left|\frac{\partial u(x,t)}{\partial \nu}\right| |u(x,t)|\, d\Gamma\, dt + \left\|\frac{\partial u}{\partial \nu}\right\|^2_{H_a^{-1}((0,T)\times\Gamma_2)} + \|u\|^2_{H^{-1}((0,T)\times\Omega)}\Big\}. \tag{5.7}$$

In (5.7), $H_a^{-1}((0,T)\times\Gamma_2)$ is the dual space of the space

$$H_a^1((0,T)\times\Gamma_2) = H^{\frac{1}{2}}(0,T;L^2(\Gamma_2)) \cap L^2(0,T;H^1(\Gamma_2)), \tag{5.8}$$

with respect to the pivot space $L^2((0,T)\times\Gamma_2)$.

Remark 3. *Theorem 5.1 remains true if m is replaced by a real-valued vector field $m \in (C^2(\overline{\Omega}))^n$ such that m is coercive in $\overline{\Omega}$, that is there exists $\beta > 0$ such that the Jacobian matrix J of m satisfies*

$$Re(J(x)v.\overline{v}) \ge \beta |v|^2,\ \forall x \in \overline{\Omega},\ v \in \mathbb{C}^n.$$

In this chapter, we also study the stability problem for the Schrödinger equation with distributed delay in the internal feedback. More precisely, we consider the system described by

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) + a(x)\left\{\alpha_0\, u(x,t) + \int_{\tau_1}^{\tau_2} \alpha(s)\, u(x,t-s)\, ds\right\} = 0 & \text{in } \Omega \times (0;+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ u(x,-t) = f_0(x,-t) & \text{on } \Omega \times (0,\tau_2). \end{cases} \tag{5.9}$$

In (5.9), $a(.)$ is an $L^\infty(\Omega)$-function that satisfies

$$a(x) \ge 0 \text{ a.e. in } \Omega \text{ and } a(x) > a_0 > 0 \text{ a.e. in } \omega, \tag{5.10}$$

where $\omega \subset \Omega$ is an open neighborhood of Γ_2.
In the absence of delay (*i.e.* $\alpha = 0$), Machtyngier and Zuazua [57] have shown that the $L^2(\Omega)$-energy of the solution decays exponentially to zero. Their proof relies on an observability inequality established previously by the first author in [56]. If the delay is concentrated at time τ, i.e. if instead of the partial differential equation in (5.9) we have

$$u_t(x,t) - i\Delta u(x,t) + a(x)\left\{\alpha_0\, u(x,t) + \alpha_1\, u(x,t-s)\, ds\right\} = 0 \qquad \text{in } \Omega \times (0;+\infty), \tag{5.11}$$

then system (5.9) is exponentially stable in the case $\alpha_0 > \alpha_1$ and may be unstable otherwise (Nicaise and Rebiai [65]).
The second purpose of this chapter is to investigate the stability of system (5.9). To this aim, assume as in [63]

$$\alpha_0 > \int_{\tau_1}^{\tau_2} \alpha(s)\, ds, \tag{5.12}$$

which imply that there exist a positive constant c_1 such that

$$\alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\, ds - \frac{c_1}{2}(\tau_2 - \tau_1) > 0, \tag{5.13}$$

then the energy of a solution of system (5.9) defined by

$$F(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2\, dx + \frac{1}{2}\int_\Omega a(x) \int_{\tau_1}^{\tau_2} s(\alpha(s) + c_1) \int_0^1 |u(x, t-\rho s)|^2\, d\rho\, ds\, dx, \tag{5.14}$$

decays exponentially to zero.
Our stability result concerning system (5.9) can be stated as follows.

Theorem 5.2. *Assume* (A) *and* (5.12)*. Then, there exist constants* $M_1 \geq 1$ *and* $\delta_1 > 0$ *such that*

$$F(t) \leq M_1 e^{-\delta_1 t} F(0).$$

The chapter is organized as follows. Theorem 5.1 is proved in Section 5.2 whereas Section 5.3 contains the proof of Theorem 5.2. Both sections start with the study of the well-posedness of the system under consideration.

5.2 Stability of the Schrödinger equation with distributed delay in the boundary feedback

5.2.1 Well-posedness of system (5.2)

In this subsection, we will establish the well-posedness of system (2.1) using linear semigroup theory. In order to be able to manage the boundary condition with the delay term and inspired from [62], [63], [83], we introduce the auxiliary variable:

$$y(x,\rho,t,s) = u(x, t-\rho s); \quad x \in \Gamma_2, \rho \in (0,1), s \in (\tau_1,\tau_2), t > 0.$$

Then, system (5.2) is equivalent to

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 & \text{in } \Omega \times (0;+\infty), \\ y_t(x,\rho,t,s) + s^{-1} y_\rho(x,\rho,t,s) = 0 & \text{on } \Gamma_2 \times (0,1) \times (0,+\infty) \times (\tau_1,\tau_2), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ y(x,0,t,s) = u(x,t) & \text{on } \Gamma_2 \times (0,+\infty) \times (\tau_1,\tau_2), \\ u(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = i\alpha_0 u(x,t) + i\int_{\tau_1}^{\tau_2} \alpha(s) y(x,1,t,s)\, ds & \text{on } \Gamma_2 \times (0,+\infty), \\ y(x,\rho,0,s) = f_0(x,\rho,s) & \text{on } \Gamma_2 \times (0,1) \times (0,\tau_2). \end{cases} \tag{5.15}$$

Denote by $\mathcal{H}$ the Hilbert space.

$$\mathcal{H} = L^2(\Omega) \times L^2(\Gamma_2 \times (0,1) \times (\tau_1,\tau_2)),$$

equipped $\mathcal{H}$ with the inner product:

$$\left\langle \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}; \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = Re \int_{\Omega} u_1(x)\overline{u_2(x)}\, dx + Re \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} s\alpha(s) \int_0^1 y_1(x,\rho,s)\overline{y_2(x,\rho,s)}\, d\rho\, ds\, d\Gamma.$$

Define in $\mathcal{H}$ the linear operator $\mathcal{A}$ by

$$\mathcal{A}(u,y)^T = (i\Delta u, -s^{-1}y_\rho)^T, \tag{5.16}$$

with

$$\begin{aligned} D(\mathcal{A}) =& \{(u,y) \in H^{3/2}(\Omega) \cap H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times (\tau_1,\tau_2); H^1(0,1)) : \Delta u \in L^2(\Omega), \\ & \frac{\partial u}{\partial \nu}(x) = i\alpha_0 u(x) + i\int_{\tau_1}^{\tau_2} \alpha(s) y(x,1,s)\, ds \text{ on } \Gamma_2,\ u(x) = y(x,0,s) \text{ on } \Gamma_2\}. \end{aligned} \tag{5.17}$$

Then we can rewrite (5.15) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \dfrac{dU}{dt}(t) = \mathcal{A}U(t), \\ U(0) = U_0, \end{cases} \tag{5.18}$$

where

$$U(t) = (u(.,t), y(.,.,t,.))^T,\, U'(t) = (u_t(.,t), y_t(.,.,t,.))^T,\, U_0 = (u_0, f_0)^T.$$

Theorem 5.3. *Assume that*

$$\alpha_0 \geq \int_{\tau_1}^{\tau_2} \alpha(s)\, ds. \tag{5.19}$$

Then, for every $U_0 \in \mathcal{H}$, system (5.18) has a unique solution U whose regularity depends on the initial datum U_0 as follows:

$$\begin{aligned} U(.) &\in C([0,+\infty); \mathcal{H}) \text{ if } U_0 \in \mathcal{H}, \\ U(.) &\in C^1([0,+\infty); \mathcal{H}) \cap C([0,+\infty); D(\mathcal{A})) \text{ if } U_0 \in D(\mathcal{A}). \end{aligned}$$

Proof. Clearly $\mathcal{A}$ is closed and densely defined. We show that $\mathcal{A}$ is dissipative.
Let $U = (u,y)^T \in D(\mathcal{A})$. Then

$$Re\langle \mathcal{A}U, U\rangle = Re \int_{\Omega} i\Delta u(x)\overline{u(x)}\, dx - Re \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} \alpha(s) \int_0^1 y_\rho(x,\rho,s)\overline{y(x,\rho,s)}\, d\rho\, ds\, d\Gamma.$$

Applying Green's Theorem, we obtain

$$Re\langle \mathcal{A}U, U\rangle = Re \int_{\Gamma_2} i\,\frac{\partial u(x)}{\partial \nu}\overline{u(x)}\, d\Gamma - Re \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} \alpha(s) \int_0^1 y_\rho(x,\rho,s)\overline{y(x,\rho,s)}\, d\rho\, ds\, d\Gamma. \tag{5.20}$$

For the last term on the right-hand side of (5.20), we have after, integrating by parts in ρ,

$$\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\int_0^1 y_\rho(x,\rho,s)\overline{y(x,\rho,s)}\,d\rho\,ds\,d\Gamma = -\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\int_0^1 y(x,\rho,s)\overline{y_\rho(x,\rho,s)}\,d\rho\,ds\,d\Gamma + \int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|y(x,1,s)|^2\,ds\,d\Gamma - \int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|y(x,0,s)|^2\,ds\,d\Gamma,$$

or equivalently

$$2Re\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\int_0^1 y_\rho(x,\rho,s)\overline{y(x,\rho,s)}\,d\rho\,ds\,d\Gamma = \int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|y(x,1,s)|^2\,ds\,d\Gamma - \int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|y(x,0,s)|^2\,ds\,d\Gamma.$$

Therefore

$$Re\langle\mathcal{A}U,U\rangle = Re\int_{\Gamma_2} i\,\frac{\partial u(x)}{\partial\nu}\overline{u(x)}\,d\Gamma - \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|y(x,1,s)|^2\,ds\,d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|y(x,0,s)|^2\,ds\,d\Gamma. \tag{5.21}$$

Insertion of the boundary conditions on Γ_2 in (5.15) into (5.21) yields

$$Re\langle\mathcal{A}U,U\rangle = -\alpha_0\int_{\Gamma_2}|u(x)|^2\,d\Gamma - Re\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)y(x,1,s)\,ds\,\overline{u(x)}\,d\Gamma - \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|y(x,1,s)|^2\,ds\,d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|u(x)|^2\,ds\,d\Gamma. \tag{5.22}$$

For the second term on the right-hand side of (5.22), we have via the Cauchy-Schwartz's inequality,

$$\left|\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)y(x,1,s)\,ds\,\overline{u(x)}\,d\Gamma\right| \le \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|y(x,1,s)|^2\,ds\,d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\,ds\,|u(x)|^2\,d\Gamma. \tag{5.23}$$

Combining (5.22) with (5.23), we obtain

$$Re\langle\mathcal{A}U,U\rangle \le \left(-\alpha_0 + \int_{\tau_1}^{\tau_2}\alpha(s)\,ds\right)\int_{\Gamma_2}|u(x)|^2\,d\Gamma. \tag{5.24}$$

(5.24) together with assumption (5.19) implies that $\mathcal{A}$ is dissipative.
Now we show that for a fixed $\lambda > 0$ and $(g,h)\in\mathcal{H}$, there exists $U=(u,y)\in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix},$$

or equivalently

$$\lambda u - \mathbf{i}\Delta u = g, \tag{5.25}$$

$$\lambda y + s^{-1}y_\rho = h. \tag{5.26}$$

Suppose that we have found u with the appropriate regularity, then we can determine y. Indeed from (5.26) and the last line of (5.15), we have

$$\begin{cases} y_\rho(x,\rho,s) = -\lambda s y(x,\rho,s) + s h(x,\rho,s), \text{ for } x \in \Gamma_2, \rho \in (0,1), s \in (\tau_1,\tau_2), \\ y(x,0,s) = u(x). \end{cases}$$

The unique solution of the above initial value problem is given by

$$y(x,\rho,s) = u(x)e^{-\lambda\rho s}u(x) + se^{-\lambda\rho s}\int_0^\rho h(x,\sigma,s)e^{\lambda\sigma s}d\sigma, \quad x \in \Gamma_2, \rho \in (0,1), s \in (\tau_1,\tau_2),$$

and in particular

$$y(x,1,s) = u(x)e^{-\lambda s} + y_0(x,s), \quad x \in \Gamma_2, s \in (\tau_1,\tau_2), \tag{5.27}$$

where $y_0 \in L^2(\Gamma_2 \times (\tau_1,\tau_2))$ is defined by

$$y_0(x,s) = se^{-\lambda s}\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma.$$

Problem (5.25) can be reformulated as

$$\int_\Omega (\lambda u - \mathbf{i}\Delta u)\overline{w}\,dx = \int_\Omega g\overline{w}\,dx, \qquad \text{for all } w \in H^1_{\Gamma_1}(\Omega). \tag{5.28}$$

Integrating by parts, we obtain

$$\begin{aligned} \int_\Omega (\lambda u - \mathbf{i}\Delta u)\overline{w}\,dx &= \int_\Omega (\lambda u\overline{w} + \mathbf{i}\nabla u\nabla\overline{w})\,dx - \mathbf{i}\int_{\Gamma_2}\frac{\partial u}{\partial\nu}\overline{w}\,d\Gamma \\ &= \int_\Omega (\lambda u\overline{w} + \mathbf{i}\nabla u\nabla\overline{w})\,dx + \int_{\Gamma_2}(\alpha_0 u\overline{w} + (\int_{\tau_1}^{\tau_2}\alpha(s)y(x,1,s)\,ds)\overline{w})\,d\Gamma \\ &= \int_\Omega (\lambda u\overline{w} + \mathbf{i}\nabla u\nabla\overline{w})\,dx + \int_{\Gamma_2}(\alpha_0 u\overline{w} + (\int_{\tau_1}^{\tau_2}\alpha(s)(u(x)e^{-\lambda s} + y_0(x,s))\,ds)\overline{w})\,d\Gamma, \end{aligned}$$

where we have used (5.27). Therefore (5.28) can be rewritten as

$$\begin{aligned} &\int_\Omega (\lambda u\overline{w} + \mathbf{i}\nabla u\nabla\overline{w})\,dx + \int_{\Gamma_2}\alpha_0 u\overline{w}\,d\Gamma + \int_{\Gamma_2}(\int_{\tau_1}^{\tau_2}\alpha(s)e^{-\lambda s}\,ds)u\overline{w}\,d\Gamma \\ &= \int_\Omega g\overline{w}\,dx - \int_{\Gamma_2}(\int_{\tau_1}^{\tau_2}\alpha(s)y_0(x,s)\,ds)\overline{w}\,d\Gamma, \quad \text{for all } w \in H^1_{\Gamma_1}(\Omega). \end{aligned}$$

Multiplying both sides of this equation by $1-\mathbf{i}$, we get

$$\begin{aligned} &(1-\mathbf{i})\int_\Omega (\lambda u\overline{w} + \mathbf{i}\nabla u\nabla\overline{w})\,dx + (1-\mathbf{i})\int_{\Gamma_2}\alpha_0 u\overline{w}\,d\Gamma + (1-\mathbf{i})\int_{\Gamma_2}(\int_{\tau_1}^{\tau_2}\alpha(s)e^{-\lambda s}\,ds)u\overline{w}\,d\Gamma \\ &= (1-\mathbf{i})\int_\Omega g\overline{w}\,dx - (1-\mathbf{i})\int_{\Gamma_2}(\int_{\tau_1}^{\tau_2}\alpha(s)y_0(x,s)\,ds)\overline{w}\,d\Gamma, \quad \text{for all } w \in H^1_{\Gamma_1}(\Omega). \end{aligned} \tag{5.29}$$

As the left-hand side of (5.29) is coercive on $H^1_{\Gamma_1}(\Omega)$ (in the sense that if we denote the left-hand side by $b(u,w)$, then $Re\, b(u,u) \geq Const\,\|u\|^2_{H^1_{\Gamma_1}(\Omega)}$ for all $u \in H^1_{\Gamma_1}(\Omega)$, and since the right-hand side

defines a continuous linear form on $H^1_{\Gamma_1}(\Omega)$ because $(g,h) \in \mathcal{H}$, the Lax-Milgram Theorem guarantees the existence and uniqueness of a solution $u \in H^1_{\Gamma_1}(\Omega)$ of (5.29).
If we consider $w \in \mathcal{D}(\Omega)$ in (5.29), then u solves

$$\lambda u - \mathbf{i}\Delta u = g, \tag{5.30}$$

in $\mathcal{D}'(\Omega)$ and thus $\Delta u \in L^2(\Omega)$. Using Green's Theorem and recalling (5.30), we obtain

$$\mathbf{i}\int_{\Gamma_2} \frac{\partial u}{\partial \nu}\overline{w}\, d\Gamma + \int_{\Gamma_2} \alpha_0 u \overline{w}\, d\Gamma + \int_{\Gamma_2} (\int_{\tau_1}^{\tau_2} \alpha(s) e^{-\lambda s}\, ds) u \overline{w}\, d\Gamma = \int_{\Gamma_2} (\int_{\tau_1}^{\tau_2} \alpha(s) y_0(x,s)\, ds)\overline{w}\, d\Gamma, \quad \text{for all } w \in H^1_{\Gamma_1}(\Omega),$$

from which it follows that

$$\mathbf{i}\frac{\partial u}{\partial \nu} + \alpha_0 u + (\int_{\tau_1}^{\tau_2} \alpha(s) e^{-\lambda s}\, ds) u = \int_{\tau_1}^{\tau_2} \alpha(s) y_0(x,s)\, ds.$$

Hence

$$\frac{\partial u}{\partial \nu} = \mathbf{i}\alpha_0 u + \mathbf{i}\int_{\tau_1}^{\tau_2} \alpha(s) y(.,1,s)\, ds \quad \text{on } \Gamma_1,$$

and this implies that $\frac{\partial u}{\partial \nu} \in L^2(\Gamma_2)$, and by [55] Theorem 2.7.4, we deduce that $u \in H^{3/2}(\Omega)$. So we have found $(u,y) \in D(\mathcal{A})$ which satisfies (5.25) and (5.26). By the Lumer-Phillips Theorem, $\mathcal{A}$ is the generator of a C_0-semigroup of contractions on $\mathcal{H}$.

□

5.2.2 Proof of Theorem 5.1

We prove the Theorem 5.1 for smooth initial data. The general case follows by a density argument. First, we show that the energy function $E(t)$ defined by (5.6) is decreasing.

Proposition 5.4. *The energy corresponding to any regular solution of problem* (5.2), *is decreasing and there exists a positive constant K such that*

$$\frac{d}{dt}E(t) \leq -K\int_{\Gamma_2} \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2\, ds \right\} d\Gamma, \tag{5.31}$$

where

$$K = \min\left\{ \alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\, ds - \frac{c_0}{2}(\tau_2 - \tau_1), \frac{c_0}{2} \right\}.$$

Proof. Differentiating $E(t)$ defined by (5.6) with respect to time, we obtain

$$\begin{aligned} \frac{d}{dt}E(t) &= Re\int_{\Omega} u_t(x,t)\overline{u(x,t)}\, dx + Re\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\, d\rho\, ds\, d\Gamma \\ &= Re\int_{\Omega} (i\Delta u(x,t))\overline{u(x,t)}\, dx + Re\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\, d\rho\, ds\, d\Gamma. \end{aligned}$$

Applying Green's Theorem, we get

$$\frac{d}{dt}E(t) = Re\int_{\Gamma_2} i\frac{\partial u}{\partial \nu}\,\overline{u}\,d\Gamma + Re\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\,d\rho\,ds\,d\Gamma. \tag{5.32}$$

Now, we have

$$-su_t(x,t-\rho s) = u_\rho(x,t-\rho s).$$

Therefore

$$\begin{aligned}
&Re\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\,d\rho\,ds\,d\Gamma\\
&= -Re\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} (\alpha(s)+c_0)\int_0^1 u_\rho(x,t-\rho s)\overline{u(x,t-\rho s)}\,d\rho\,ds\,d\Gamma\\
&= -\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} (\alpha(s)+c_0)\int_0^1 \frac{d}{d\rho}|u(x,t-\rho s)|^2\,d\rho\,ds\,d\Gamma,
\end{aligned}$$

from which follows, after integration by parts in ρ

$$\begin{aligned}
&Re\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\,d\rho\,ds\,d\Gamma\\
&= -\frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\alpha(s)+c_0)|u(x,t-s)|^2\,ds\,d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\alpha(s)+c_0)|u(x,t)|^2\,ds\,d\Gamma.
\end{aligned} \tag{5.33}$$

Inserting (5.33) and the boundary conditions on Γ_2 in (5.2) into (5.32), we obtain

$$\begin{aligned}
\frac{d}{dt}E(t) &= -\alpha_0\int_{\Gamma_2}|u(x,t)|^2\,d\Gamma - Re\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)u(x,t-s)\,ds\,\overline{u(x,t)}\,d\Gamma\\
&\quad -\frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\alpha(s)+c_0)|u(x,t-s)|^2\,ds\,d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\alpha(s)+c_0)|u(x,t)|^2\,ds\,d\Gamma.
\end{aligned} \tag{5.34}$$

For the second integral on the right-hand side of (5.34), we have from the Cauchy-Schwartz's inequality,

$$\begin{aligned}
&\left|\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)u(x,t-s)\,ds\,\overline{u(x,t)}\,d\Gamma\right|\\
&\le \int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\mid u(x,t-s)\mid\,ds\mid u(x,t)\mid\,d\Gamma\\
&\le \int_{\Gamma_2}\left(\int_{\tau_1}^{\tau_2}\alpha(s)|u(x,t-s)|^2\,ds\right)^{\frac{1}{2}}\left(\int_{\tau_1}^{\tau_2}\alpha(s)\,ds\right)^{\frac{1}{2}}\mid u(x,t)\mid\,d\Gamma\\
&\le \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|u(x,t-s)|^2\,ds\,d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)|u(x,t)|^2\,ds\,d\Gamma.
\end{aligned} \tag{5.35}$$

So, from (5.34) and (5.35), we obtain

$$\frac{d}{dt}E(t) \le \left(-\alpha_0 + \int_{\tau_1}^{\tau_2}\alpha(s)\,ds + \frac{c_0}{2}(\tau_2-\tau_1)\right)\int_{\Gamma_2}|u(x,t)|^2\,d\Gamma - \frac{c_0}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}|u(x,t-s)|^2\,ds\,d\Gamma,$$

which implies
$$\frac{d}{dt}E(t) \le -K \int_{\Gamma_2} \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2 \, ds \right\} d\Gamma,$$
where
$$K = \min \left\{ \alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\, ds - \frac{c_0}{2}(\tau_2 - \tau_1), \frac{c_0}{2} \right\},$$
which is positive due to the assumption (5.5). □

Now we give an observability inequality which we will use to prove the exponential decay of the energy $E(t)$.

Proposition 5.5. *For any regular solution of problem (5.2), there exists a positive constant C depending on T such that for all $T > \tau_2$*
$$E(0) \le C \int_0^T \int_{\Gamma_2} \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2 \, ds \right\} d\Gamma \, dt. \tag{5.36}$$

Proof. We rewrite
$$E(t) = E_s(t) + E_d(t),$$
where
$$E_s(t) = \frac{1}{2} \int_\Omega |u(x,t)|^2 \, dx,$$
and
$$E_d(t) = \frac{1}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} s(\alpha(s) + c_0) \int_0^1 |u(x, t - \rho s)|^2 \, d\rho \, ds \, d\Gamma.$$
In particular,
$$E_d(0) = \frac{1}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} s(\alpha(s) + c_0) \int_0^1 |u(x, -\rho s)|^2 \, d\rho \, ds \, d\Gamma. \tag{5.37}$$
By a change of variable in (5.37) we obtain, for $T \ge \tau_2$,
$$\begin{aligned} E_d(0) &= \frac{1}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} (\alpha(s) + c_0) \int_0^s |u(x,t-s)|^2 \, dt \, ds \, d\Gamma \\ &\le \frac{1}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} (\alpha(s) + c_0) \int_0^T |u(x,t-s)|^2 \, dt \, ds \, d\Gamma \\ &\le C \int_0^T \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2 \, ds \, d\Gamma \, dt. \end{aligned} \tag{5.38}$$
Here and throughout the rest of this chapter C is some positive constant different at different occurrences.
From Theorem 2.6.1 of [50], we have the following estimate
$$\begin{aligned} E_s(0) \le C \Bigg\{ & \|u\|^2_{L^2(0,T;L^2(\Gamma_2))} + \int_0^T \int_{\Gamma_2} \left| \frac{\partial u(x,t)}{\partial \nu} \right| |u(x,t)| \, d\Gamma \, dt \\ & + \left\| \frac{\partial u}{\partial \nu} \right\|^2_{H_a^{-1}((0,T)\times\Gamma_2)} + \|u\|^2_{H^{-1}((0,T)\times\Omega)} \Bigg\}, \end{aligned}$$

for $T > 0$ and for a suitable constant C depending on T.
Inserting the boundary conditions on Γ_2 in (5.2) into the above estimate, we obtain

$$E_s(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2 \, ds \right\} d\Gamma \, dt + C\|u\|^2_{H^{-1}((0,T)\times\Omega)}, \tag{5.39}$$

since the $H_a^{-1}((0,T)\times\Gamma_2)$-norm is dominated by the $L^2((0,T)\times\Gamma_2)$-norm.
Combining (5.38) with (5.39), we obtain for any $T > \tau_2$

$$E(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2 \, ds \right\} d\Gamma \, dt + C\|u\|^2_{H^{-1}((0,T)\times\Omega)}, \tag{5.40}$$

for a suitable constant C depending on T. Naturally, (5.40) implies a fortiori

$$E(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2 \, ds \right\} d\Gamma \, dt + C\|u\|^2_{L^\infty((0,T);H^{-1}(\Omega))}. \tag{5.41}$$

To obtain the desired estimate (5.36) we need to absorb the lower order terms $\|u\|^2_{L^\infty((0,T);H^{-1}(\Omega))}$ on the right-hand side of (5.41). To achieve this, we employ as in Nicaise and Pignotti [62] and Nicaise and Rebiai [65] a compactness-uniqueness argument.
Suppose that (5.36) is not true. Then, there exists a sequence u_n of solution of problem (5.2) with

$$u_n(x,0) = u_n^0(x),\ u_n(x,-t) = f_n^0(x,-t),$$

such that

$$E^n(0) > n \int_0^T \int_{\Gamma_2} \left\{ |u_n(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u_n(x,t-s)|^2 \, ds \right\} d\Gamma \, dt, \tag{5.42}$$

where $E^n(0)$ is the energy corresponding to u_n at the time $t = 0$.
From (5.41), we have

$$E_n(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ |u_n(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u_n(x,t-s)|^2 \, ds \right\} d\Gamma \, dt + C\|u_n\|^2_{L^\infty((0,T);H^{-1}(\Omega))}. \tag{5.43}$$

(5.42) together with (5.43), yields

$$\begin{aligned} &n \int_0^T \int_{\Gamma_2} \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2 \, ds \right\} d\Gamma \, dt \\ &< C \int_0^T \int_{\Gamma_2} \left\{ |u_n(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u_n(x,t-s)|^2 \, ds \right\} d\Gamma \, dt + C\|u_n\|^2_{L^\infty((0,T);H^{-1}(\Omega))}. \end{aligned}$$

That is

$$(n-C) \int_0^T \int_{\Gamma_2} \left\{ |u_n(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u_n(x,t-s)|^2 \, ds \right\} d\Gamma \, dt < C\|u_n\|^2_{L^\infty((0,T);H^{-1}(\Omega))}.$$

Renormalizing, we obtain a sequence u_n of solution of problem (5.2) verifying

$$\|u_n\|^2_{L^\infty((0,T);H^{-1}(\Omega))} = 1, \tag{5.44}$$

and

$$\int_0^T \int_{\Gamma_2} \left\{ |u_n(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u_n(x,t-s)|^2 \, ds \right\} d\Gamma \, dt < \frac{C}{n-C} \quad \forall \, n > C. \tag{5.45}$$

From (5.43), (5.44), and (5.45), it follows that the sequence (u_n^0, f_n^0) is bounded in $\mathcal{H}$. Then there is a subsequence still denote by (u_n^0, f_n^0) that converges weakly to $(u^0, f^0) \in \mathcal{H}$. Let ψ be the solution of problem (5.2) with initial condition (u^0, f^0).
We have

$$\psi \in C(0,T); L^2(\Omega)),$$

from Theorem 5.3 and

$$\int_0^T \int_{\Gamma_2} |\psi(x,t)|^2 \, d\Gamma \, dt + \int_0^T \int_{\Gamma_2} \left| \frac{\partial \psi(x,t)}{\partial \nu} \right|^2 d\Gamma \, dt \leq C,$$

from Proposition 5.4 for some $C > 0$. It then follows that

$$\begin{aligned} u_n &\longrightarrow \psi \;\text{ in } L^\infty(0,T; L^2(\Omega)) \;\text{ weak star}, \\ u_{nt} &\longrightarrow \psi_t \;\text{ in } L^\infty(0,T; H^{-2}(\Omega)) \;\text{ weak star}, \end{aligned}$$

and hence

$$\|u_n\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_{nt}\|^2_{L^\infty(0,T;H^{-2}(\Omega))} \leq C \qquad \forall n \in \mathbb{N}. \tag{5.46}$$

Since the injection $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is compact, (5.46) implies (see [9] and [75]) that for $0 < T < +\infty$ the injection

$$Z \hookrightarrow L^\infty(0,T; H^{-1}(\Omega))$$

is also compact, where Z is the Banach space equipped with the norm on the left hand side of (5.46). As a consequence there is a subsequence still denoted by u_n such that

$$u_n \longrightarrow \psi \;\text{ in } L^\infty(0,T; H^{-1}(\Omega)) \;\text{ strongly}. \tag{5.47}$$

Hence by (5.44) we obtain

$$\|u\|^2_{L^\infty((0,T);H^{-1}(\Omega))} = 1. \tag{5.48}$$

On the other hand, we have from (5.45) and (5.47),

$$\psi(x,t) = 0 \quad \text{on} \quad \Gamma_2 \times (0,T).$$

Thus ψ satisfies

$$\begin{cases} \psi_t(x,t) - i\Delta\psi(x,t) = 0 & \text{in} \quad \Omega \times (0,T), \\ \psi(x,t) = 0 & \text{on} \quad \Gamma \times (0,T), \\ \dfrac{\partial \psi(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma_2 \times (0,T). \end{cases}$$

Therefore, from Holmgren's uniqueness Theorem (see [54], Chap.1, Thm. 8.2), we conclude that

$$\psi(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T),$$

which contradicts (5.48). Then, the desired inequality (5.36) is proved. □

Completion of the proof of Theorem 5.1

From (5.31), we have

$$E(T) - E(0) \leq -K \int_0^T \int_{\Gamma_2} \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2 \, ds \right\} d\Gamma \, dt,$$

and the observability inequality (5.36) leads to

$$\begin{aligned} E(T) \leq E(0) &\leq C \int_0^T \int_{\Gamma_2} \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2 \, ds \right\} d\Gamma \, dt. \\ &\leq CK^{-1}(E(0) - E(T)), \end{aligned}$$

so

$$E(T) \leq \frac{C}{K+C} E(0). \tag{5.49}$$

Since we $0 < C/(K+C) < 1$, the desired conclusion follows now from (5.49).

5.3 Stability of the Schrödinger equation with distributed delay term in the internal feedback

5.3.1 Well-posedness of system (5.9)

We introduce the auxiliary variable:

$$y(x,\rho,t,s)=u(x,t-\rho s);\quad x\in\Omega,\ \rho\in(0,1),s\in(\tau_1,\tau_2),t>0.$$

Then, system (5.9) is equivalent to

$$\begin{cases} u_t(x,t)-i\Delta u(x,t)+a(x)\left\{\alpha_0\,u(x,t)+\int_{\tau_1}^{\tau_2}\alpha(s)\,y(x,1,t,s)\,ds\right\}=0 & \text{in } \Omega\times(0;+\infty),\\ y_t(x,\rho,t,s)+s^{-1}y_\rho(x,\rho,t,s)=0 & \text{in } \Omega\times(0,1)\times(0,+\infty)\times(\tau_1,\tau_2),\\ u(x,0)=u_0(x) & \text{in } \Omega,\\ y(x,0,t,s)=u(x,t) & \text{in } \Omega\times(0,+\infty)\times(\tau_1,\tau_2),\\ u(x,t)=0 & \text{on } \Gamma\times(0,+\infty),\\ y(x,\rho,0,s)=f_0(x,\rho,s) & \text{in } \Omega\times(0,1)\times(0,\tau_2). \end{cases} \tag{5.50}$$

Denote by $\widehat{\mathcal{H}}$ the Hilbert space

$$\widehat{\mathcal{H}}=L^2(\Omega)\times L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)),$$

equipped with the inner product:

$$\left\langle\begin{pmatrix}u_1\\y_1\end{pmatrix};\begin{pmatrix}u_2\\y_2\end{pmatrix}\right\rangle_{\widehat{\mathcal{H}}}=Re\int_\Omega u_1(x)\overline{u_2(x)}\,dx+Re\int_\Omega a(x)\int_{\tau_1}^{\tau_2}s\alpha(s)\int_0^1 y_1(x,\rho,s)\overline{y_2(x,\rho,s)}\,d\rho\,ds\,dx.$$

Define in $\widehat{\mathcal{H}}$ the linear operator $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}}(u,y)^T=(i\Delta u-a\alpha_0\,u-a\int_{\tau_1}^{\tau_2}\alpha(s)\,y(.,1,s)\,ds,-s^{-1}y_\rho)^T,$$

with

$$D(\tilde{\mathcal{A}})=\{(u,y)\in H^2(\Omega)\cap H_0^1(\Omega)\times L^2(\Omega\times(\tau_1,\tau_2);H^1(0,1)):u(x)=y(x,0,s)\text{ in }\Omega\},$$

Then we can rewrite (5.50) as an abstract Cauchy problem in $\widehat{\mathcal{H}}$

$$\begin{cases}\dfrac{dU}{dt}(t)=\tilde{\mathcal{A}}U(t),\\ U(0)=U_0,\end{cases} \tag{5.51}$$

where

$$U(t)=(u(.,t),y(.,.,t,.))^T,\ U'(t)=(u_t(.,t),y_t(.,.,t,.))^T,\ U_0=(u_0,f_0)^T.$$

Clearly $\tilde{\mathcal{A}}$ is closed and densely defined. Proceeding as in Subsection 5.2.1, we prove that if

$$\alpha_0 \geq \int_{\tau_1}^{\tau_2} \alpha(s)\, ds \tag{5.52}$$

then $\tilde{\mathcal{A}}$ is maximal dissipative. Therefore $\tilde{\mathcal{A}}$ generates a strongly continuous semigroup of contractions on $\widehat{\mathcal{H}}$ and consequently we have the following well-posedness result for system (5.51).

Theorem 5.6. *Assume (5.52). Then for every $U_0 \in \widehat{\mathcal{H}}$, system (5.51) has a unique solution U whose regularity depends on the initial datum U_0 as follows:*

$$\begin{aligned} U(.) &\in C([0,+\infty); \widehat{\mathcal{H}}) \ \text{ if } U_0 \in \widehat{\mathcal{H}}, \\ U(.) &\in C^1([0,+\infty); \widehat{\mathcal{H}}) \cap C([0,+\infty); D(\tilde{\mathcal{A}})) \ \text{ if } U_0 \in D(\tilde{\mathcal{A}}). \end{aligned}$$

5.3.2 Proof of Theorem 5.2

We prove Theorem 5.2 for smooth initial data. The general case follows by a density argument. First, we show that the energy function $F(t)$ defined by (5.14) is decreasing.

Proposition 5.7. *The energy corresponding to any regular solution of problem* (5.9)*, is decreasing and there exists a positive constant K_1 such that,*

$$\frac{d}{dt}F(t) \leq -K_1 \int_\Omega a(x) \left\{ |u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2\, ds \right\} dx, \tag{5.53}$$

where

$$K_1 = \min\left\{ \alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\, ds - \frac{c_1}{2}(\tau_2 - \tau_1), \frac{c_1}{2} \right\}.$$

Proof. Differentiating $F(t)$ defined by (5.14) with respect to time, we obtain

$$\begin{aligned} \frac{d}{dt}F(t) &= Re \int_\Omega u_t(x,t)\overline{u(x,t)}\, dx + Re \int_\Omega a(x) \int_{\tau_1}^{\tau_2} s(\alpha(s)+c_1) \int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\, d\rho\, ds\, dx \\ &= Re \int_\Omega (i\Delta u(x,t))\overline{u(x,t)}\, dx - \alpha_0 \int_\Omega a(x)|u(x,t)|^2\, dx - Re \int_\Omega a(x) \int_{\tau_1}^{\tau_2} \alpha(s) u(x,t-s)\, ds\, \overline{u(x,t)}\, dx \\ &\quad + Re \int_\Omega a(x) \int_{\tau_1}^{\tau_2} s(\alpha(s)+c_1) \int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\, d\rho\, ds\, dx. \end{aligned}$$

Applying Green's Theorem, we get

$$\begin{aligned} \frac{d}{dt}F(t) &= Re \int_\Gamma i\frac{\partial u}{\partial \nu}(x,t)\, \overline{u(x,t)}\, d\Gamma - \alpha_0 \int_\Omega a(x)|u(x,t)|^2\, dx - Re \int_\Omega a(x) \int_{\tau_1}^{\tau_2} \alpha(s) u(x,t-s)\, ds\, \overline{u(x,t)}\, dx \\ &\quad + Re \int_\Omega a(x) \int_{\tau_1}^{\tau_2} s(\alpha(s)+c_1) \int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\, d\rho\, ds\, dx. \end{aligned} \tag{5.54}$$

Now, we have

$$-s u_t(x,t-\rho s) = u_\rho(x,t-\rho s),$$

Therefore

$$\begin{aligned}
&Re\int_\Omega a(x)\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_1)\int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\,d\rho\,ds\,dx\\
&=-Re\int_\Omega a(x)\int_{\tau_1}^{\tau_2}(\alpha(s)+c_1)\int_0^1 u_\rho(x,t-\rho s)\overline{u(x,t-\rho s)}\,d\rho\,ds\,dx\\
&=-\int_\Omega a(x)\int_{\tau_1}^{\tau_2}(\alpha(s)+c_1)\int_0^1\frac{d}{d\rho}|u(x,t-\rho s)|^2\,d\rho\,ds\,dx,
\end{aligned}$$

from which follows, after integration by parts in ρ

$$\begin{aligned}
&Re\int_\Omega a(x)\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_1)\int_0^1 u_t(x,t-\rho s)\overline{u(x,t-\rho s)}\,d\rho\,ds\,dx\\
&=-\frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}(\alpha(s)+c_1)|u(x,t-s)|^2\,ds\,dx+\frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}(\alpha(s)+c_1)|u(x,t)|^2\,ds\,dx. \qquad (5.55)
\end{aligned}$$

Inserting (5.55) and the boundary condition on Γ_2 in (5.9) into (5.54), we find

$$\begin{aligned}
\frac{d}{dt}F(t)=&-\alpha_0\int_\Omega a(x)|u(x,t)|^2\,dx-Re\int_\Omega a(x)\int_{\tau_1}^{\tau_2}\alpha(s)u(x,t-s)\,ds\,\overline{u(x,t)}\,dx\\
&-\frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}(\alpha(s)+c_1)|u(x,t-s)|^2\,ds\,dx+\frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}(\alpha(s)+c_1)|u(x,t)|^2\,ds\,dx.
\end{aligned} \qquad (5.56)$$

For the second integral on the right-hand side of (5.56), we have the following estimate deduced from the Cauchy-Schwartz's inequality,

$$\begin{aligned}
&\left|\int_\Omega a(x)\int_{\tau_1}^{\tau_2}\alpha(s)u(x,t-s)\,ds\,\overline{u(x,t)}\,dx\right|\\
&\leq\int_\Omega a(x)\int_{\tau_1}^{\tau_2}\alpha(s)\mid u(x,t-s)\mid\,ds\mid u(x,t)\mid\,dx\\
&\leq\int_\Omega a(x)\left(\int_{\tau_1}^{\tau_2}\alpha(s)|u(x,t-s)|^2\,ds\right)^{\frac{1}{2}}\left(\int_{\tau_1}^{\tau_2}\alpha(s)\,ds\right)^{\frac{1}{2}}\mid u(x,t)\mid\,dx\\
&\leq\frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}\alpha(s)|u(x,t-s)|^2\,ds\,dx+\frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}\alpha(s)|u(x,t)|^2\,ds\,dx. \qquad (5.57)
\end{aligned}$$

(5.56) together with (5.57) gives

$$\frac{d}{dt}F(t)\leq\left(-\alpha_0+\int_{\tau_1}^{\tau_2}\alpha(s)\,ds+\frac{c_1}{2}(\tau_2-\tau_1)\right)\int_\Omega a(x)|u(x,t)|^2\,dx-\frac{c_1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}|u(x,t-s)|^2\,ds\,dx,$$

which in turn implies

$$\frac{d}{dt}F(t)\leq -K_1\int_\Omega a(x)\left\{|u(x,t)|^2+\int_{\tau_1}^{\tau_2}|u(x,t-s)|^2ds\right\}\,dx,$$

where

$$K_1 = \min\left\{\alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\,ds - \frac{c_1}{2}(\tau_2-\tau_1), \frac{c_1}{2}\right\},$$

K_1 is positive because of the assumption (5.13). □

Now we give an observability inequality that will be used to establish the exponential decay of the energy function $F(t)$.

Proposition 5.8. *For any regular solution of problem (5.9), there exists a positive constant C_0 depending on T such that for all $T > \tau_2$*

$$F(0) \le C_0 \int_0^T \int_\Omega a(x)\left\{|u(x,t)|^2 + \int_{\tau_1}^{\tau_2} |u(x,t-s)|^2\,ds\right\} dx\,dt. \tag{5.58}$$

Proof. Following [57], we write the solution u of (5.9) as $u = z + v$ where z solves

$$\begin{cases} z_t(x,t) - i\Delta z(x,t) = 0 & \text{in } \Omega \times (0;+\infty), \\ z(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ z(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{5.59}$$

and v satisfies

$$\begin{cases} v_t(x,t) = i\Delta v(x,t) - a(x)\left\{\alpha_0\,u(x,t) + \int_{\tau_1}^{\tau_2} \alpha(s)\,u(x,t-s)\,ds\right\} = 0 & \text{in } \Omega \times (0;+\infty), \\ v(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ v(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Let us denote by

$$\mathcal{E}_z(t) = \frac{1}{2}\int_\Omega |z(x,t)|^2\,dx,$$

the energy corresponding to the solution of (5.59). Then, it follows from [57] (Proposition 3.1) that for all $T > 0$, there exists a positive constant c depending on T such that

$$\mathcal{E}_z(0) \le c \int_0^T \int_\omega |z(x,t)|^2\,dx\,dt.$$

Using (5.10) we get

$$\mathcal{E}_z(0) \le \frac{c}{a_0} \int_0^T \int_\Omega a(x)|z(x,t)|^2\,dx\,dt.$$

On the other hand we have for $T \ge \tau_2$

$$\frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_1)\int_0^1 |u(x,-\rho s)|^2\,d\rho\,ds\,dx \le c\int_0^T \int_\Omega a(x)\int_{\tau_1}^{\tau_2} (\alpha(s)+c_1)|u(x,t-s)|^2\,ds\,dx\,dt.$$

Hence for $T \geq \tau_2$

$$\begin{aligned} F(0) &= \mathcal{E}_z(0) + \frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_1)\int_0^1 |u(x,-\rho s)|^2\,d\rho\,ds\,dx \\ &\leq c\int_0^T\int_\Omega a(x)\left\{|z(x,t)|^2 + \int_{\tau_1}^{\tau_2}|u(x,t-s)|^2\,ds\right\}dx\,dt \\ &\leq c\int_0^T\int_\Omega a(x)\left\{|u(x,t)|^2 + |v(x,t)|^2 + \int_{\tau_1}^{\tau_2}|u(x,t-s)|^2\,ds\right\}dx\,dt. \end{aligned} \tag{5.60}$$

By classical energy estimates on the Schrödinger equation we have

$$\|v\|^2_{L^\infty((0,T);L^2(\Omega))} \leq c\int_0^T\int_\Omega a(x)\left\{|u(x,t)|^2 + \int_{\tau_1}^{\tau_2}|u(x,t-s)|^2\,ds\right\}dx\,dt. \tag{5.61}$$

Combining (5.60) and (5.61), we obtain (5.58). □

Completion of the proof of Theorem 5.2

From (5.53), we have

$$F(T) - F(0) \leq -K_1\int_0^T\int_\Omega a(x)\left\{|u(x,t)|^2 + \int_{\tau_1}^{\tau_2}|u(x,t-s)|^2\,ds\right\}dx\,dt.$$

and the observability inequality (5.58) leads to

$$\begin{aligned} F(T) \leq F(0) &\leq C_0\int_0^T\int_\Omega a(x)\left\{|u(x,t)|^2 + \int_{\tau_1}^{\tau_2}|u(x,t-s)|^2\,ds\right\}dx\,dt. \\ &\leq C_0K_1^{-1}(F(0) - F(T)), \end{aligned}$$

so

$$F(T) \leq \frac{C_0}{K_1 + C_0}F(0). \tag{5.62}$$

Since we have $0 < C_0/(K_1 + C_0) < 1$, the desired conclusion follows now from (5.62).

Chapter 6

Stabilization of coupled wave equations with boundary or internal distributed delay

6.1 Introduction

In this chapter, we study stability problems for compactly coupled wave equations with distributed delay terms in the boundary or internal feedbacks. To this end, let Ω be an open bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that, $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$.
Furthermore we assume that there exists a scalar function $\Phi \in C^2(\bar{\Omega})$ such that
$(H.1)$ Φ is strictly convex in $\overline{\Omega}$; that is, there exists $\lambda > 0$ such that

$$H(x)\Theta.\Theta \geq \lambda \left|\Theta(x)\right|^2 \qquad \forall\, x \in \overline{\Omega},\, \Theta \in \mathbb{R}^n,$$

where H is the Hessian matrix of Φ.
$(H.2)$ $h(x).\nu(x) \leq 0$ on Γ_1, where $h(x) = \nabla\Phi(x)$ and ν is the unit normal on Γ pointing towards the exterior of Ω.
In Ω, we consider the following coupled system of two wave equations with distributed delay terms in the boundary conditions:

$$u_{tt}(x,t) - \Delta u(x,t) + l(u(x,t) - v(x,t)) = 0 \qquad \text{in } \Omega \times (0; +\infty), \tag{6.1}$$

$$v_{tt}(x,t) - \Delta v(x,t) + l(v(x,t) - u(x,t) = 0 \qquad \text{in } \Omega \times (0; +\infty), \tag{6.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \qquad \text{in } \Omega, \tag{6.3}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x) \qquad \text{in } \Omega, \tag{6.4}$$

$$u(x,t) = v(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0, +\infty), \tag{6.5}$$

$$\frac{\partial u}{\partial \nu}(x,t) = -\int_{\tau_1}^{\tau_2} \alpha(s) u_t(x,t-s)\,ds - \alpha_0 u_t(x,t) \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{6.6}$$

$$\frac{\partial v}{\partial \nu}(x,t) = -\int_{\tau_1}^{\tau_2} \beta(s) v_t(x,t-s)\,ds - \beta_0 v_t(x,t) \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{6.7}$$

$$u_t(x,-t) = f_0(x,-t) \qquad \text{on } \Gamma_2 \times (0,\tau), \tag{6.8}$$

$$v_t(x,-t) = g_0(x,-t) \qquad \text{on } \Gamma_2 \times (0,\tau), \tag{6.9}$$

where

- l, α_0 and β_0 are positive constants,
- u_0, u_1, v_0, v_1, f_0 and g_0 are the initial data,
- $\frac{\partial}{\partial \nu}$ is the normal derivative,
- τ_1 and τ_2 are two real numbers with $0 \leq \tau_1 < \tau_2$,
- $\alpha, \beta : [\tau_1, \tau_2] \to (\mathbb{R})$ are nonnegative almost everywhere functions of class L^∞.

For the case of one-dimensional spatial domain Ω, u and v may represent the displacements of two vibrating objects measured from their equilibrium positions, the coupling terms $\pm l(u-v)$ are the distributed springs linking the two vibrating objects [60].
It is well known that if $\alpha = \beta = 0$, i.e. in the absence of delay, then the solution (u,v) of (6.1)-(6.9) decays exponentially in the energy space $H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)\times$ $H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)$ ([60], [41]).

In the presence of delay concentrated at a time τ that is the boundary conditions (6.6) and (6.7) are replaced by

$$\frac{\partial u}{\partial \nu}(x,t) = -\alpha_1 u_t(x,t-\tau)\,ds - \alpha_0 u_t(x,t) \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{6.10}$$

$$\frac{\partial v}{\partial \nu}(x,t) = -\beta_1 v_t(x,t-\tau)\,ds - \beta_0 v_t(x,t) \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{6.11}$$

the solution of $(6.1)-(6.5), (6.10), (6.11), (6.8)$ and (6.9) decays exponentially in an appropriate energy space provided that $\alpha_0 > \alpha_1$ and $\beta_0 > \beta_1$ [70]. One of the purposes of this chapter is to investigate the uniform exponential stability of the system $(6.1-(6.9)$. To this aim, assume as in [63]

$$\alpha_0 > \int_{\tau_1}^{\tau_2} \alpha(s)\,ds, \tag{6.12}$$

and

$$\beta_0 > \int_{\tau_1}^{\tau_2} \beta(s)\,ds. \tag{6.13}$$

and define the energy of a solution of (6.1)-(6.9) by

$$E(t) = \frac{1}{2}\int_{\Omega}\left[\mid \nabla u(x,t)\mid^2 + u_t^2(x,t) + \mid \nabla v(x,t)\mid^2 + v_t^2(x,t) + l(u(x,t)-v(x,t))^2\right] dx$$
$$+ \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t^2(x,t-\rho s)\,d\rho\,ds\,d\Gamma$$
$$+ \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\beta(s)+c_1)\int_0^1 v_t^2(x,t-\rho s)\,d\rho\,ds\,d\Gamma. \tag{6.14}$$

In (6.14), c_0 and c_1 are positive constants such that

$$\alpha_0 - \int_{\tau_1}^{\tau_2}\alpha(s)\,ds - \frac{c_0}{2}(\tau_2-\tau_1) > 0, \tag{6.15}$$

and

$$\beta_0 - \int_{\tau_1}^{\tau_2}\beta(s)\,ds - \frac{c_1}{2}(\tau_2-\tau_1) > 0. \tag{6.16}$$

Then we have the following stability result for system $(6.1)-(6.9)$

Theorem 6.1. *Assume* $(H.1)$, $(H.2)$, (6.12) *and* (6.13). *Then the coupled wave equations system* $(6.1)-(6.9)$ *is uniformly exponentially stable, i.e., there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq Me^{-\omega t}E(0).$$

The proof of this result is based on Carleman estimates (see Appendix 6.4) for a system of coupled non-conservative hyperbolic systems established by Lasiecka and Triggiani [46] and will be given in Section 6.2.

In this chapter, we also study the stability problem for a system of two coupled wave equations with distributed delay in the internal feedback. More precisely, we consider the system described by

$$u_{tt} - \Delta u + l(u-v) + a(x)(\alpha_0 u_t + \int_{\tau_1}^{\tau_2}\alpha(s)u_t(x,t-s)\,ds) = 0 \quad \text{in } \Omega\times(0;+\infty), \tag{6.17}$$
$$v_{tt} - \Delta v + l(v-u) + b(x)(\beta_0 v_t + \int_{\tau_1}^{\tau_2}\beta(s)v_t(x,t-s)\,ds) = 0 \quad \text{in } \Omega\times(0;+\infty), \tag{6.18}$$
$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{6.19}$$
$$v(x,0) = v_0(x), v_t(x,0) = v_1(x) \quad \text{in } \Omega, \tag{6.20}$$
$$u(x,t) = v(x,t) = 0 \quad \text{on } \Gamma_1\times(0,+\infty), \tag{6.21}$$
$$\frac{\partial u}{\partial \nu}(x,t) = \frac{\partial v}{\partial \nu}(x,t) = 0 \quad \text{on } \Gamma_2\times(0,+\infty), \tag{6.22}$$
$$u_t(x,-t) = f_0(x,-t) \quad \text{in } \Omega\times(0,\tau_2), \tag{6.23}$$
$$v_t(x,-t) = g_0(x,-t) \quad \text{in } \Omega\times(0,\tau_2). \tag{6.24}$$

In $(6.17)-(6.24)$, $a(.)$ and $b(.)$ are two $L^\infty(\Omega)$ functions which satisfy

$$a(x) \geq 0 \text{ a.e in } \Omega,\ a(x) > a_0 > 0 \text{ a.e in } \omega_1,$$

and

$$b(x) \geq 0 \text{ a.e in } \Omega, \;\; b(x) > b_0 > 0 \text{ a.e in } \omega_2,$$

where $\omega_1 \subset \omega_2 \subset \Omega$ are open neighbourhoods of Γ_2.

If the delay is concentrated at time τ, *i.e.* if instead of $(6.17) - (6.18)$ we have

$$u_{tt} - \Delta u + l(u-v) + a(x)(\alpha_0 u_t + \alpha_1 u_t(x, t-s)) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \;, \tag{6.25}$$

$$v_{tt} - \Delta v + l(v-u) + b(x)(\beta_0 v_t + \beta_1 v_t(x, t-s)) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \;, \tag{6.26}$$

then system $(6.25), (6.26), (6.19) - (6.24)$ is exponentially stable in the case $\alpha_0 > \alpha_1$ and $\beta_0 > \beta_1$ and may be unstable otherwise [74].
The second purpose of this chapter is to investigate the uniform exponential stability of the system $(6.17 - (6.24)$. To this aim, assume as in [63]

$$\alpha_0 > \int_{\tau_1}^{\tau_2} \alpha(s)\, ds \;\; \text{and} \quad \beta_0 > \int_{\tau_1}^{\tau_2} \beta(s)\, ds, \tag{6.27}$$

then the energy of system $(6.17) - (6.24)$ defined by

$$\begin{aligned} F(t) = & \frac{1}{2} \int_{\Omega} \left\{ | \nabla u(x,t) |^2 + u_t^2(x,t) + | \nabla v(x,t) |^2 + v_t^2(x,t) + l(u(x,t) - v(x,t))^2 \right\} dx \\ & + \frac{1}{2} \int_{\Omega} a(x) \int_{\tau_1}^{\tau_2} s(\alpha(s) + \widetilde{\alpha}_0) \int_0^1 u_t^2(x, t - \rho s)\, d\rho\, ds\, dx \\ & + \frac{1}{2} \int_{\Omega} b(x) \int_{\tau_1}^{\tau_2} s(\beta(s) + \widetilde{\beta}_0) \int_0^1 v_t^2(x, t - \rho s)\, d\rho\, ds\, dx, \end{aligned} \tag{6.28}$$

decays exponentially to zero. In (6.28), $\widetilde{\alpha}_0$ and $\widetilde{\beta}_0$ are positive constants such that

$$\alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\, ds - \frac{\widetilde{\alpha}_0}{2}(\tau_2 - \tau_1) > 0, \tag{6.29}$$

and

$$\beta_0 - \int_{\tau_1}^{\tau_2} \beta(s)\, ds - \frac{\widetilde{\beta}_0}{2}(\tau_2 - \tau_1) > 0. \tag{6.30}$$

Our stability result concerning system $(6.17) - (6.24)$ can be stated as follows.

Theorem 6.2. *Assume* $(H.1)$, $(H.2)$ *and* (6.27). *Then there exist constants* $M \geq 1$ *and* $\delta > 0$ *such that*

$$F(t) \leq M e^{-\delta t} F(0), \tag{6.31}$$

for all solutions of $(6.17) - (6.24)$.

The proof of Theorem 6.2 is given in Section 6.3.
This chapter was published in [28]

6.2 Exponential stability of coupled wave equations with distributed delay terms in the boundary feedbacks

6.2.1 Well-posedness of system (6.1) – (6.9)

Adapting an idea of [62], we introduce new variables by setting

$$\begin{aligned} y(x,\rho,t,s) &= u_t(x,t-\rho s), \quad x\in\Gamma_2,\ \rho\in(0,1),\ s\in(\tau_1,\tau_2), t>0,\\ z(x,\rho,t,s) &= v_t(x,t-\rho s), \quad x\in\Gamma_2,\ \rho\in(0,1),\ s\in(\tau_1,\tau_2), t>0. \end{aligned}$$

Then system (6.1) – (6.9) is equivalent to

$$u_{tt}(x,t)-\Delta u(x,t)+l(u(x,t)-v(x,t))=0 \qquad \text{in } \Omega\times(0;+\infty), \tag{6.32}$$
$$y_t(x,\rho,t,s)+s^{-1}y_\rho(x,\rho,t,s)=0 \quad \text{on } \Gamma_2\times(0,1)\times(0,+\infty)\times(\tau_1,\tau_2), \tag{6.33}$$
$$v_{tt}(x,t)-\Delta v(x,t)+l(v(x,t)-u(x,t)=0 \qquad \text{in } \Omega\times(0;+\infty), \tag{6.34}$$
$$z_t(x,\rho,t,s)+s^{-1}z_\rho(x,\rho,t,s)=0 \quad \text{on } \Gamma_2\times(0,1)\times(0,+\infty)\times(\tau_1,\tau_2) \tag{6.35}$$
$$u(x,0)=u_0(x), u_t(x,0)=u_1(x) \qquad \text{in } \Omega, \tag{6.36}$$
$$v(x,0)=v_0(x), v_t(x,0)=v_1(x) \qquad \text{in } \Omega, \tag{6.37}$$
$$u(x,t)=v(x,t)=0 \qquad \text{on } \Gamma_1\times(0,+\infty), \tag{6.38}$$
$$\frac{\partial u}{\partial\nu}(x,t)=-\int_{\tau_1}^{\tau_2}\alpha(s)y(x,1,t,s)\,ds-\alpha_0u_t(x,t) \qquad \text{on } \Gamma_2\times(0,+\infty), \tag{6.39}$$
$$\frac{\partial v}{\partial\nu}(x,t)=-\int_{\tau_1}^{\tau_2}\beta(s)z(x,1,t,s)\,ds-\beta_0v_t(x,t) \qquad \text{on } \Gamma_2\times(0,+\infty), \tag{6.40}$$
$$y(x,0,t,s)=u_t(x,t) \qquad \text{on } \Gamma_2\times(0,+\infty)\times(\tau_1,\tau_2), \tag{6.41}$$
$$z(x,0,t,s)=v_t(x,t) \qquad \text{on } \Gamma_2\times(0,+\infty)\times(\tau_1,\tau_2), \tag{6.42}$$
$$y(x,\rho,0,s)=f_0(x,\rho,s), z(x,\rho,0,s)=g_0(x,\rho,s) \quad \text{on} \quad \Gamma_2\times(0,1)\times(0,\tau_2). \tag{6.43}$$

Denote by

$$U=(u,u_t,y,v,v_t,z)^T \quad .$$

Then system (6.43) can be formulated as an abstract Cauchy problem

$$\begin{cases} U'(t)=\mathcal{A}U(t),\\ U(0)=(u_0,u_1,g,v_0,v_1,h)^T, \end{cases} \tag{6.44}$$

in the Hilbert space

$$\mathcal{H}=(H^1_{\Gamma_1}(\Omega)\times L^2(\Omega)\times L^2(\Gamma_2\times(0,1)\times(\tau_1,\tau_2)))^2.$$

with the inner product

$$\left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \phi \\ \chi \\ \psi \end{pmatrix}; \begin{pmatrix} \widetilde{\zeta} \\ \widetilde{\eta} \\ \widetilde{\theta} \\ \widetilde{\phi} \\ \widetilde{\chi} \\ \widetilde{\psi} \end{pmatrix} \right\rangle = \int_\Omega (\nabla\zeta(x).\nabla\widetilde{\zeta}(x) + \eta(x)\tilde{\eta}(x))\,dx + \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s\alpha(s)\int_0^1 \theta(x,\rho,s)\tilde{\theta}(x,\rho,s)\,d\rho\,ds\,d\Gamma$$
$$+ \int_\Omega (\nabla\varphi(x).\nabla\tilde{\varphi}(x) + \chi(x)\tilde{\chi}(x))\,dx + \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s\beta(s)\int_0^1 \psi(x,\rho,s)\tilde{\psi}(x,\rho,s)\,d\rho\,ds\,d\Gamma$$
$$+ l\int_\Omega (\zeta(x)-\varphi(x))(\tilde{\zeta}(x)-\tilde{\varphi}(x))\,dx.$$

The linear operator $\mathcal{A}$ is defined by

$$\mathcal{A}(\zeta,\eta,\theta,\phi,\chi,\psi)^T = (\eta, \Delta\zeta + l(\zeta-\phi), -s^{-1}\theta_\rho, \chi, \Delta\phi + l(\phi-\zeta), -s^{-1}\psi_\rho)^T, \tag{6.45}$$

and

$$D(\mathcal{A}) = \Big\{(\zeta,\eta,\theta,\phi,\chi,\psi) \in ((E(\Delta,L^2(\Omega))) \cap H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2\times(\tau_1,\tau_2);H^1(0,1)))^2 :$$
$$\frac{\partial\zeta(x)}{\partial\nu} = -\alpha_0\eta(x) - \int_{\tau_1}^{\tau_2}\alpha(s)\theta(x,1,s)\,ds, \eta(x) = \theta(x,0,s) \quad \text{on } \Gamma_2,$$
$$\frac{\partial\phi(x)}{\partial\nu} = -\beta_0\chi(x) - \int_{\tau_1}^{\tau_2}\beta(s)\psi(x,1,s)\,ds, \chi(x) = \psi(x,0,s) \quad \text{on } \Gamma_2\Big\}, \tag{6.46}$$

where

$$E(\Delta, L^2(\Omega)) = \{u \in H^1(\Omega); \Delta u \in L^2(\Omega)\}.$$

Clearly $\mathcal{A}$ is closed and densely defined. Its adjoint $\mathcal{A}^*$ which is given by

$$\mathcal{A}^*(f,g,h,k,L,m)^T = (-g, -\Delta f + l(f-k), s^{-1}h_\rho, -L, -\Delta k + l(k-f), s^{-1}m_\rho)^T, \tag{6.47}$$

with domain

$$D(\mathcal{A}^*) = \Big\{(\zeta,\eta,\theta,\varphi,\chi,\psi) \in (E(\Delta,L^2(\Omega)) \cap H^1_{\Gamma_1}(\Omega)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times H^1(0,1) \times (\tau_1,\tau_2))$$
$$\times (E(\Delta,L^2(\Omega)) \cap H^1_{\Gamma_1}(\Omega)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times H^1(0,1) \times (\tau_1,\tau_2));$$
$$\frac{\partial\zeta(x)}{\partial\nu} = \alpha_0\eta(x) - \int_{\tau_1}^{\tau_2}\alpha(s)\theta(x,0,s)\,ds,\ \eta(x) = -\theta(x,1,s) \ \text{ on } \Gamma_2;$$
$$\frac{\partial\varphi(x)}{\partial\nu} = \beta_0\chi(x) - \int_{\tau_1}^{\tau_2}\beta(s)\psi(x,0,s)\,ds,\ \chi(x) = -\psi(x,1,s) \text{ on } \Gamma_2\Big\}. \tag{6.48}$$

Proposition 6.3. *The operator $\mathcal{A}$ and $\mathcal{A}^*$ defined by* (6.45), (6.46) *and* (6.47), (6.48) *respectively are dissipative.*

Proof. We first show that $\mathcal{A}$ is dissipative.
Let $U = (\zeta, \eta, \theta, \varphi, \chi, \psi)^T \in D(\mathcal{A})$. Then

$$\begin{aligned}
\langle \mathcal{A}U, U \rangle &= \left\langle \left(\eta, \Delta\zeta - l(\zeta - \varphi), -s^{-1}\theta_\rho, \chi, \Delta\varphi - l(\varphi - \zeta), -s^{-1}\psi_\rho\right)^T, U \right\rangle \\
&= \int_\Omega \nabla\zeta(x)\nabla\eta(x)\,dx + \int_\Omega \eta(x)\Delta\zeta(x)\,dx - l\int_\Omega \eta(x)(\zeta(x) - \varphi(x))\,dx \\
&- \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s\alpha(s)\int_0^1 \theta(x,\rho,s)s^{-1}\theta_\rho(x,\rho,s)\,d\rho\,ds\,d\Gamma + \int_\Omega \nabla\varphi(x)\nabla\chi(x)\,dx + \int_\Omega \chi(x)\Delta\varphi(x)\,dx \\
&- l\int_\Omega \chi(x)(\varphi(x) - \zeta(x))\,dx - \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s\beta(s)\int_0^1 \psi(x,\rho,s)s^{-1}\psi_\rho(x,\rho,s)\,d\rho\,ds\,d\Gamma \\
&+ l\int_\Omega (\zeta(x) - \varphi(x))(\eta(x) - \chi(x))\,dx.
\end{aligned}$$

From Green's Theorem, we obtain

$$\begin{aligned}
\langle \mathcal{A}U, U \rangle = &\int_{\Gamma_2} \eta(x)\frac{\partial\zeta(x)}{\partial\nu}\,d\Gamma - \int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\int_0^1 \theta(x,\rho,s)\theta_\rho(x,\rho,s)\,d\rho\,ds\,d\Gamma \\
&+ \int_{\Gamma_2} \chi(x)\frac{\partial\varphi(x)}{\partial\nu}\,d\Gamma - \int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\beta(s)\int_0^1 \psi(x,\rho,s)\psi_\rho(x,\rho,s)\,d\rho\,ds\,d\Gamma. \qquad (6.49)
\end{aligned}$$

Integrating by parts in ρ, the second qnd fourth terms on the right-hand side of (6.49), we obtain

$$\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\int_0^1 \theta(x,\rho,s)\theta_\rho(x,\rho,s)\,d\rho\,ds\,d\Gamma = \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\{\theta^2(x,1,s) - \theta^2(x,0,s)\}\,ds\,d\Gamma \qquad (6.50)$$

$$\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\beta(s)\int_0^1 \psi(x,\rho,s)\psi_\rho(x,\rho,s)\,d\rho\,ds\,d\Gamma = \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\beta(s)\{\psi^2(x,1,s) - \psi^2(x,0,s)\}\,ds\,d\Gamma. \qquad (6.51)$$

Recalling (6.46), we have after inserting (6.50) and (6.51) into (6.49),

$$\begin{aligned}
\langle \mathcal{A}U, U \rangle &= \int_{\Gamma_2} \eta(x)[-\alpha_0\eta(x) - \int_{\tau_1}^{\tau_2}\alpha(s)\theta(x,1,s)\,ds]\,d\Gamma - \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)[\theta^2(x,1,s) - \theta^2(x,0,s)]\,ds\,d\Gamma \\
&+ \int_{\Gamma_2} \chi(x)[-\beta_0\chi(x) - \int_{\tau_1}^{\tau_2}\beta(s)\psi(x,1,s)\,ds]\,d\Gamma - \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\beta(s)[\psi^2(x,1,s) - \psi^2(x,0,s)]\,ds\,d\Gamma. \\
&= -\alpha_0\int_{\Gamma_2}\eta^2(x)\,d\Gamma - \int_{\Gamma_2}\eta(x)\int_{\tau_1}^{\tau_2}\alpha(s)\theta(x,1,s)\,ds\,d\Gamma - \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\theta^2(x,1,s)\,ds\,d\Gamma \\
&+ \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\eta^2(x)\,ds\,d\Gamma - \beta_0\int_{\Gamma_2}\chi^2(x)\,d\Gamma - \int_{\Gamma_2}\chi(x)\int_{\tau_1}^{\tau_2}\beta(s)\psi(x,1,s)\,ds\,d\Gamma \\
&- \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\beta(s)\psi^2(x,1,s)\,ds\,d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\beta(s)\chi^2(x)\,ds\,d\Gamma.
\end{aligned}$$

Using Cauchy-Schwarz's inequality, we get

$$\left|\int_{\Gamma_2}\eta(x)\int_{\tau_1}^{\tau_2}\alpha(s)\theta(x,1,s)\,ds\,d\Gamma\right| \le \frac{1}{2}\int_{\Gamma_2}\eta^2(x)\left(\int_{\tau_1}^{\tau_2}\alpha(s)\,ds\right)d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\alpha(s)\theta^2(x,1,s)\,ds\,d\Gamma,$$

and

$$\left|\int_{\Gamma_2} \chi(x) \int_{\tau_1}^{\tau_2} \beta(s)\psi(x,1,s)\,ds\,d\Gamma\right| \leq \frac{1}{2}\int_{\Gamma_2} \chi^2(x)\left(\int_{\tau_1}^{\tau_2} \beta(s)\,ds\right) d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} \beta(s)\psi^2(x,1,s)\,ds\,d\Gamma,$$

then

$$\langle \mathcal{A}U, U\rangle \leq \left(-\alpha_0 + \int_{\tau_1}^{\tau_2} \alpha(s)\,ds\right)\int_{\Gamma_2} \eta^2(x)\,d\Gamma + \left(-\beta_0 + \int_{\tau_1}^{\tau_2} \beta(s)\,ds\right)\int_{\Gamma_2} \chi^2(x)\,d\Gamma.$$

From (6.12) and (6.13), we conclude that $Re\langle \mathcal{A}U, U\rangle \leq 0$, thus $\mathcal{A}$ is dissipative.
The dissipativity of $\mathcal{A}^*$ is proved in a similar manner.

□

Therefore $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{H}$ (see [26], p.15, Corollary 4.4), and consequently we have the following well-posedness result.

Theorem 6.4. *For every $U_0 \in \mathcal{H}$, the problem (6.44) has a unique solution U such that:*

$$U(.) \in C([0,+\infty);\mathcal{H}).$$

In addition, if we assume $U_0 \in D(\mathcal{A})$, then we have

$$U(.) \quad \in \quad C^1([0,+\infty);\mathcal{H}) \cap C([0,+\infty);D(\mathcal{A})).$$

6.2.2 Proof of Theorem 1.1

We prove Theorem 6.1 for regular initial data, the general case follows by a density argument. We proceed in several steps.
Step 1. We prove that the energy function $E(t)$ defined by (6.14) is decreasing.

Proposition 6.5. *The energy corresponding to any regular solution of system* (6.1) – (6.9)*, is decreasing and there exists a positive constant K such that,*

$$\frac{d}{dt}E(t) \leq -K\int_{\Gamma_2}\left\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds + v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\right\} d\Gamma, \tag{6.52}$$

where

$$K = \min\left\{\alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\,ds - \frac{c_0}{2}(\tau_2-\tau_1), \frac{c_0}{2}, \beta_0 - \int_{\tau_1}^{\tau_2} \beta(s)\,ds - \frac{c_1}{2}(\tau_2-\tau_1), \frac{c_1}{2}\right\}.$$

Proof. Differentiating $E(t)$ with respect to time, we obtain

$$\begin{aligned}\frac{d}{dt}E(t) = &\int_\Omega \{\nabla u.\nabla u_t + u_{tt}u_t + \nabla v.\nabla v_t + v_{tt}v_t + l(u-v)(u_t-v_t)\}\,dx \\ &+ \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)\,d\rho\,ds\,d\Gamma \\ &+ \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\beta(s)+c_1)\int_0^1 v_t(x,t-\rho s)v_{tt}(x,t-\rho s)\,d\rho\,ds\,d\Gamma.\end{aligned}$$

Applying Green's Theorem, we get

$$\begin{aligned}\frac{d}{dt}E(t) = \int_{\Gamma_2} \frac{\partial u}{\partial \nu}(x,t)u_t(x,t)\,d\Gamma + \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)\,d\rho\,ds\,d\Gamma \\ + \int_{\Gamma_2} \frac{\partial v}{\partial \nu}(x,t)v_t(x,t)d\Gamma + \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\beta(s)+c_1)\int_0^1 v_t(x,t-\rho s)v_{tt}(x,t-\rho s)\,d\rho\,ds\,d\Gamma. \quad (6.53)\end{aligned}$$

Now, we have

$$\begin{aligned}-su_t(x,t-\rho s) &= u_\rho(x,t-\rho s),\\ -sv_t(x,t-\rho s) &= v_\rho(x,t-\rho s),\end{aligned}$$

which lead to

$$\begin{aligned}s^2u_{tt}(x,t-\rho s) &= u_{\rho\rho}(x,t-\rho s),\\ s^2v_{tt}(x,t-\rho s) &= v_{\rho\rho}(x,t-\rho s).\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)\,d\rho\,ds\,d\Gamma\\ &+ \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\beta(s)+c_1)\int_0^1 v_t(x,t-\rho s)v_{tt}(x,t-\rho s)\,d\rho\,ds\,d\Gamma\\ &= \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} (-s)^{-3}s(\alpha(s)+c_0)\int_0^1 u_\rho(x,t-\rho s)u_{\rho\rho}(x,t-\rho s)\,d\rho\,ds\,d\Gamma\\ &+ \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} (-s)^{-3}s(\beta(s)+c_1)v_\rho(x,t-\rho s)v_{\rho\rho}(x,t-\rho s)\,d\rho\,ds\,d\Gamma.\end{aligned}$$

Integrating by parts in ρ, we get

$$\begin{aligned}&\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\alpha(s)+c_0)\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)\,d\rho\,ds\,d\Gamma\\ &+ \int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\beta(s)+c_1)\int_0^1 v_t(x,t-\rho s)v_{tt}(x,t-\rho s)\,d\rho\,ds\,d\Gamma\\ &= \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\{(\alpha(s)+c_0)(u_t^2(x,t)-u_t^2(x,t-s))+(\beta(s)+c_1)(v_t^2(x,t)-v_t^2(x,t-s))\}\,ds\,d\Gamma. \quad (6.54)\end{aligned}$$

Inserting (6.54) and the boundary conditions (6.6) and (6.7) into (6.53), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) = &-\alpha_0\int_{\Gamma_2} u_t^2(x,t)\,d\Gamma - \int_{\Gamma_2} u_t(x,t)\int_{\tau_1}^{\tau_2}\alpha(s)u_t(x,t-s)\,ds\,d\Gamma\\ &- \beta_0\int_{\Gamma_2} v_t^2(x,t)\,d\Gamma - \int_{\Gamma_2} v_t(x,t)\int_{\tau_1}^{\tau_2}\beta(s)v_t(x,t-s)\,ds\,d\Gamma\\ &- \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\alpha(s)+c_0)u_t^2(x,t-s)\,ds\,d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\alpha(s)+c_0)u_t^2(x,t)\,ds\,d\Gamma\\ &- \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\beta(s)+c_1)v_t^2(x,t-s)\,ds\,d\Gamma + \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\beta(s)+c_1)v_t^2(x,t)\,ds\,d\Gamma. \quad (6.55)\end{aligned}$$

From Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
\left| \int_{\Gamma_2} u_t(x,t) \int_{\tau_1}^{\tau_2} \alpha(s) u_t(x,t-s)\, ds\, d\Gamma \right| &\leq \int_{\Gamma_2} \mid u_t(x,t) \mid \int_{\tau_1}^{\tau_2} \alpha(s) \mid u_t(x,t-s) \mid \, ds\, d\Gamma \\
&\leq \int_{\Gamma_2} \mid u_t(x,t) \mid \left(\int_{\tau_1}^{\tau_2} \alpha(s)\, ds \right)^{\frac{1}{2}} \left(\int_{\tau_1}^{\tau_2} \alpha(s) u_t^2(x,t-s)\, ds \right)^{\frac{1}{2}} d\Gamma \\
&\leq \frac{1}{2} \int_{\Gamma_2} u_t^2(x,t) \left(\int_{\tau_1}^{\tau_2} \alpha(s)\, ds \right) d\Gamma + \frac{1}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} \alpha(s) u_t^2(x,t-s)\, ds\, d\Gamma, \qquad (6.56)
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_{\Gamma_2} v_t(x,t) \int_{\tau_1}^{\tau_2} \beta(s) v_t(x,t-s)\, ds\, d\Gamma \right| &\leq \frac{1}{2} \int_{\Gamma_2} v_t^2(x,t) \left(\int_{\tau_1}^{\tau_2} \beta(s)\, ds \right) d\Gamma \\
&\quad + \frac{1}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} \beta(s) v_t^2(x,t-s)\, ds\, d\Gamma. \qquad (6.57)
\end{aligned}$$

Combining (6.55) together with (6.56) and (6.57), we obtain

$$\begin{aligned}
\frac{d}{dt} E(t) \leq &\left(-\alpha_0 + \int_{\tau_1}^{\tau_2} \alpha(s)\, ds + \frac{c_0}{2}(\tau_2 - \tau_1) \right) \int_{\Gamma_2} u_t^2(x,t)\, d\Gamma - \frac{c_0}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\, ds\, d\Gamma \\
&+ \left(-\beta_0 + \int_{\tau_1}^{\tau_2} \beta(s)\, ds + \frac{c_1}{2}(\tau_2 - \tau_1) \right) \int_{\Gamma_2} v_t^2(x,t) d\Gamma - \frac{c_1}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\, ds\, d\Gamma,
\end{aligned}$$

which implies

$$\frac{d}{dt} E(t) \leq -K \int_{\Gamma_2} \left\{ u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\, ds + v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\, ds \right\} d\Gamma,$$

where K is a positive constant defined by,

$$K = \min \left\{ \alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\, ds - \frac{c_0}{2}(\tau_2 - \tau_1), \frac{c_0}{2}, \beta_0 - \int_{\tau_1}^{\tau_2} \beta(s)\, ds - \frac{c_1}{2}(\tau_2 - \tau_1), \frac{c_1}{2} \right\}.$$

□

Step 2. Now, we establish an observability estimate for the problem (6.1) – (6.9) that will be used to prove the exponential decay of the energy $E(t)$.

Proposition 6.6. *For any regular solution of system* (6.1) – (6.9)*, there exists a positive constant* C *depending on* T *such that*

$$E(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\, ds + v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\, ds \right\} d\Gamma\, dt. \qquad (6.58)$$

Proof. We rewrite

$$E(t) = E_s(t) + E_d(t),$$

where

$$E_s(t) = \frac{1}{2}\int_\Omega \left\{|\nabla u(x,t)|^2 + u_t^2(x,t) + |\nabla v(x,t)|^2 + v_t^2(x,t) + l(u(x,t)-v(x,t))^2\right\} dx,$$

and

$$E_d(t) = \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} [s(\alpha(s)+c_0)\int_0^1 u_t^2(x,t-\rho s)\,d\rho + s(\beta(s)+c_1)\int_0^1 v_t^2(x,t-\rho s)\,d\rho]\,ds\,d\Gamma.$$

In particular,

$$E_d(0) = \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} \left[s(\alpha(s)+c_0)\int_0^1 u_t^2(x,-\rho s)\,d\rho + s(\beta(s)+c_1)\int_0^1 v_t^2(x,-\rho s)\,d\rho\right] ds\,d\Gamma,$$

$E_d(0)$ can be rewritten via a change of variable as,

$$E_d(0) = \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} \left[(\alpha(s)+c_0)\int_0^s u_t^2(x,t-s)dt + (\beta(s)+c_1)\int_0^s v_t^2(x,t-s)\,dt\right] ds\,d\Gamma.$$

From the above equality, we obtain

$$E_d(0) \le C\int_0^T\int_{\Gamma_2}\left\{\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\right\} d\Gamma\,dt, \tag{6.59}$$

for $T \ge \tau_2$. Here and throughout the rest of the chapter C is some positive constant different at different occurrences.
From Proposition 3.5 of [46], we have for T sufficiently large and for any $\epsilon > 0$,

$$\begin{aligned} E_s(0) \le C\int_0^T\int_{\Gamma_2}&\left\{\left|\frac{\partial u}{\partial\nu}(x,t)\right|^2 + u_t^2(x,t) + \left|\frac{\partial v}{\partial\nu}(x,t)\right|^2 + v_t^2(x,t)\right\} d\Gamma\,dt \\ &+ C\left\{\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}\right\}. \end{aligned}$$

Inserting the boundary conditions (6.6) and (6.7) into the above inequality, we get

$$\begin{aligned} E_s(0) \le & C\int_0^T\int_{\Gamma_2}\left\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds + v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\right\} d\Gamma\,dt \\ &+ C\left\{\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}\right\}. \end{aligned} \tag{6.60}$$

Combining (6.59) with (6.60), we obtain

$$\begin{aligned} E(0) \le & C\int_0^T\int_{\Gamma_2}\left\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds + v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\right\} d\Gamma\,dt \\ &+ C\left\{\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}\right\}. \end{aligned} \tag{6.61}$$

To obtain the desired estimate (6.58) we need to absorb the lower order terms $\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}$ and $\|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}$ on the right-hand side of (6.61). We do this by a compactness-uniqueness

argument.
Suppose that (6.58) is not true. Then, there exists a sequence (u_n, v_n) of solution of problem (6.1)-(6.9) with,

$$u_n(x,0) = u_n^0(x),\ u_{nt}(x,0) = u_n^1(x),\ u_n(x,-t) = f_n^0(x,-t),$$
$$v_n(x,0) = v_n^0(x),\ v_{nt}(x,0) = v_n^1(x),\ v_n(x,-t) = g_n^0(x,-t),$$

such that

$$E^n(0) > n\int_0^T \int_{\Gamma_2} \left\{ u_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} u_{nt}^2(x,t-s)\,ds + v_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} v_{nt}^2(x,t-s)\,ds \right\} d\Gamma\,dt, \quad (6.62)$$

where $E^n(0)$ is the energy corresponding to $(u_n^0, u_n^1, v_n^0, v_n^1)$.
From (6.61), we have

$$\begin{aligned} E^n(0) \leq & C\int_0^T \int_{\Gamma_2} \left\{ u_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} u_{nt}^2(x,t-s)\,ds + v_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} v_{nt}^2(x,t-s)\,ds \right\} d\Gamma\,dt \\ & + C\left\{ \|u_n\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v_n\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} \right\}. \end{aligned} \quad (6.63)$$

(6.62) together with (6.63), implies

$$\begin{aligned} & n\int_0^T \int_{\Gamma_2} \left\{ u_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} u_{nt}^2(x,t-s)\,ds + v_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} v_{nt}^2(x,t-s)\,ds \right\} d\Gamma\,dt \\ & < C\int_0^T \int_{\Gamma_2} \left\{ u_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} u_{nt}^2(x,t-s)\,ds + v_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} v_{nt}^2(x,t-s)\,ds \right\} d\Gamma\,dt \\ & + C\left\{ \|u_n\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v_n\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} \right\}, \end{aligned}$$

that is

$$\begin{aligned} (n-C)\int_0^T \int_{\Gamma_2} & \left\{ u_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} u_{nt}^2(x,t-s)\,ds + v_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} v_{nt}^2(x,t-s)\,ds \right\} d\Gamma\,dt \\ & < C\left\{ \|u_n\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v_n\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} \right\}. \end{aligned} \quad (6.64)$$

Renormalizing, we obtain a sequence (u_n, v_n) of solution of problem $(6.1) - (6.9)$ satisfying

$$\|u_n\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))} + \|v_n\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))} = 1, \quad (6.65)$$

and

$$\begin{aligned} & \int_0^T \int_{\Gamma_2} \left\{ u_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} u_{nt}^2(x,t-s)\,ds + v_{nt}^2(x,t) + \int_{\tau_1}^{\tau_2} v_{nt}^2(x,t-s)\,ds \right\} d\Gamma\,dt \\ & \quad < \frac{C}{n-C} \quad for\, all \quad n > C. \end{aligned} \quad (6.66)$$

From (6.63), (6.65), and (6.66), it follows that the sequence $(u_n^0, u_n^1, f_n^0, v_n^0, v_n^1, g_n^0)$ is bounded in $\mathcal{H}$. Then there is a subsequence still denoted by $(u_n^0, u_n^1, f_n^0, v_n^0, v_n^1, g_n^0)$ that converges weakly to $(u^0, u^1, f^0, v^0, v^1, g^0) \in \mathcal{H}$. Let (u, v) be the solution of problem (6.1)-(6.9) with initial condition $(u^0, u_n^1, f^0, v^0, v^1, g^0)$. We have from Theorem 6.1

$$(u,v) \in C([0,+\infty); H^1_{\Gamma_1}(\Omega)) \times C([0,+\infty); H^1_{\Gamma_1}(\Omega)).$$

Then

$$(u_n, v_n) \longrightarrow (u,v) \text{ weakly in } L^2(0,T; H^1_{\Gamma_1}(\Omega)) \times L^2(0,T; H^1_{\Gamma_1}(\Omega)).$$

Since $H^1_{\Gamma_1}(\Omega)$ is compactly embedded in $H^{1/2+\varepsilon}(\Omega)$, there exists a subsequence, still denoted by (u_n, v_n) such that,

$$(u_n, v_n) \longrightarrow (u,v) \text{ strongly in } L^2(0,T; H^{1/2+\varepsilon}(\Omega)) \times L^2(0,T; H^{1/2+\varepsilon}(\Omega)).$$

So, (6.65) leads to

$$\|u\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))} = 1. \tag{6.67}$$

Moreover, by (6.64)

$$\int_0^T \int_{\Gamma_2} \left\{ u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\, ds + v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\, ds \right\} d\Gamma\, dt = 0.$$

Then

$$u_t(x,t) = v_t(x,t) = 0 \quad \text{on} \quad \Gamma_2 \times (0,T),$$

and

$$\frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_2 \times (0,T).$$

Set $\varphi := u_t, \psi := v_t$. Therefore (φ, ψ) satisfies

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) + l(\varphi(x,t) - \psi(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ \psi_{tt}(x,t) - \Delta\psi(x,t) + l(\psi(x,t) - \varphi(x,t)) = 0 & \text{in} \quad \Omega \times (0,T), \\ \varphi(x,t) = \psi(x,t) = 0 & \text{on} \quad \Gamma \times (0,T), \\ \dfrac{\partial \varphi(x,t)}{\partial \nu} = \dfrac{\partial \psi(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma_2 \times (0,T). \end{cases} \tag{6.68}$$

(6.68) implies

$$\begin{cases} (\varphi+\psi)_{tt}(x,t) - \Delta(\varphi+\psi)(x,t) = 0 & \text{in} \quad \Omega \times (0,T), \\ (\varphi+\psi)(x,t) = 0 & \text{on} \quad \Gamma \times (0,T), \\ \dfrac{\partial(\varphi+\psi)(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma_2 \times (0,T). \end{cases}$$

From Holmgren's uniqueness Theorem (see [54], p.92 Chap.I, Thm.8.2), we conclude that

$$\varphi(x,t) + \psi(x,t) = 0,$$

and the problem (6.68) can be rewritten as

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) + 2l\varphi(x,t) = 0 & \text{in} \quad \Omega \times (0,T), \\ \varphi(x,t) = 0 & \text{on} \quad \Gamma \times (0,T), \\ \dfrac{\partial \varphi(x,t)}{\partial \nu} = 0 & \text{on} \quad \Gamma_2 \times (0,T). \end{cases}$$

Invoking Theorem 3.1 of [84], we readily obtain for T large enough

$$\varphi(x,t) = \psi(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T).$$

This implies

$$u(x,t) = u(x), v(x,t) = v(x).$$

Thus (u,v) verifies

$$\begin{cases} -\Delta u(x) + l(u(x) - v(x)) = 0 & \text{in} \quad \Omega, \\ -\Delta v(x) + l(v(x) - u(x)) = 0 & \text{in} \quad \Omega, \\ u(x) = v(x) = 0 & \text{on} \quad \Gamma, \\ \dfrac{\partial u(x)}{\partial \nu} = \dfrac{\partial v(x)}{\partial \nu} = 0 & \text{on} \quad \Gamma_2. \end{cases}$$

The unique solution of the above problem is $(u,v) = (0,0)$ in Ω, a contradiction to (6.67). The proof of Proposition 6.6 is complete. □

Step 3.
From (6.53), we have

$$E(T) - E(0) \leq -K \int_0^T \int_{\Gamma_2} \left\{ u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds + v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds \right\} d\Gamma\,dt,$$

which combined with the observability inequality (6.58) gives

$$\begin{aligned} E(T) \leq E(0) &\leq C \int_0^T \int_{\Gamma_2} \left\{ u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds + v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds \right\} d\Gamma\,dt \\ &\leq CK^{-1}(E(0) - E(T)), \end{aligned}$$

so

$$E(T) \leq \frac{C}{K+C} E(0). \tag{6.69}$$

Since $0 < C/(K+C) < 1$, the desired conclusion follows now from (6.69). (see [26], p. 299, Proposition 1.7).

6.3 Exponential stability of coupled wave equations with distributed delay terms in the internal feedbacks

6.3.1 Well-posedness of system (6.17) – (6.24)

As we have done previously, let us define:

$$y(x,\rho,t,s) = u_t(x,t-\rho s); \quad z(x,\rho,t,s) = v_t(x,t-\rho s); \quad x \in \Omega,\ \rho \in (0,1),\ s \in (\tau_1,\tau_2),\ t > 0.$$

Then, the system (6.17)-(6.24) is equivalent to

$$u_{tt} - \Delta u + l(u-v) + a(x)(\alpha_0 u_t(x,t) + \int_{\tau_1}^{\tau_2} \alpha(s) y(x,1,t,s)\,ds) = 0 \text{ in } \Omega \times (0;+\infty), \tag{6.70}$$

$$v_{tt} - \Delta v + l(v-u) + b(x)(\beta_0 v_t(x,t) + \int_{\tau_1}^{\tau_2} \beta(s) z(x,1,t,s)\,ds) = 0 \text{ in } \Omega \times (0;+\infty), \tag{6.71}$$

$$y_t(x,\rho,t,s) + s^{-1} y_\rho(x,\rho,t,s) = 0 \quad \text{in } \Omega \times (0,1) \times (0,+\infty) \times (\tau_1,\tau_2), \tag{6.72}$$

$$z_t(x,\rho,t,s) + s^{-1} z_\rho(x,\rho,t,s) = 0 \quad \text{in } \Omega \times (0,1) \times (0,+\infty) \times (\tau_1,\tau_2), \tag{6.73}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{6.74}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x) \quad \text{in } \Omega, \tag{6.75}$$

$$u(x,t) = v(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,+\infty), \tag{6.76}$$

$$\frac{\partial u}{\partial \nu}(x,t) = \frac{\partial v}{\partial \nu}(x,t) = 0 \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{6.77}$$

$$y(x,0,t,s) = u_t(x,t) \quad \text{in } \Omega \times (0,+\infty) \times (\tau_1,\tau_2), \tag{6.78}$$

$$z(x,0,t,s) = v_t(x,t) \quad \text{in } \Omega \times (0,+\infty) \times (\tau_1,\tau_2), \tag{6.79}$$

$$y(x,\rho,0,s) = f_0(x,\rho,t), z(x,\rho,0,s) = g_0(x,\rho,t) \quad \text{in} \quad \Omega \times (0,1) \times (0,\tau_2). \tag{6.80}$$

Denote by $\widehat{\mathcal{H}}$ the Hilbert space

$$\widehat{\mathcal{H}} = (H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)))^2,$$

endowed with the inner product:

$$\left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \varphi \\ \chi \\ \psi \end{pmatrix} ; \begin{pmatrix} \widetilde{\zeta} \\ \widetilde{\eta} \\ \widetilde{\theta} \\ \widetilde{\varphi} \\ \widetilde{\chi} \\ \widetilde{\psi} \end{pmatrix} \right\rangle = \int_\Omega (\nabla\zeta(x).\nabla\widetilde{\zeta}(x) + \eta(x)\tilde{\eta}(x))\,dx + \int_\Omega a(x) \int_{\tau_1}^{\tau_2} s\alpha(s) \int_0^1 \theta(x,\rho,s)\tilde{\theta}(x,\rho,s)\,d\rho\,ds\,dx$$

$$+ \int_\Omega (\nabla\varphi(x).\nabla\tilde{\varphi}(x) + \chi(x)\tilde{\chi}(x))\,dx + \int_\Omega b(x) \int_{\tau_1}^{\tau_2} s\beta(s) \int_0^1 \psi(x,\rho,s)\widetilde{\psi}(x,\rho,s)\,d\rho\,ds\,dx$$

$$+ l \int_\Omega (\zeta(x) - \varphi(x))(\tilde{\zeta}(x) - \tilde{\varphi}(x))\,dx.$$

The system (6.70)-(6.80) can be rewritten as as an abstract Cauchy problem in $\widehat{\mathcal{H}}$

$$\begin{cases} \dfrac{dU}{dt}(t) = \tilde{\mathcal{A}}U(t), \\ \\ U(0) = U_0, \end{cases} \tag{6.81}$$

where

$$U(t) = (u(x,t), u_t(x,t), y(x,\rho,t), v(x,t), v_t(x,t), z(x,\rho,t))^T,$$
$$U_0 = (u_0, u_1, f_0, v_0, v_1, g_0)^T.$$

and

$$\begin{aligned} \tilde{\mathcal{A}}(\zeta,\eta,\theta,\phi,\chi,\psi)^T =& (\eta, \Delta\zeta - l(\zeta-\varphi) - a(x)(\alpha_0\eta + \int_{\tau_1}^{\tau_2} \alpha(s)\theta(.,1,.,s)\,ds), -s^{-1}\theta_\rho, \\ & \chi, \Delta\varphi - l(\varphi-\zeta) - b(x)(\beta_0\chi + \int_{\tau_1}^{\tau_2} \beta(s)\psi(.,1,.,s)\,ds), -s^{-1}\psi_\rho)^T, \end{aligned} \tag{6.82}$$

with domain

$$\begin{aligned} D(\tilde{\mathcal{A}}) =& \Big\{ (\zeta,\eta,\theta,\varphi,\chi,\psi) \in ((H^2(\Omega)\cap H^1_{\Gamma_1}(\Omega)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega\times(\tau_1,\tau_2); H^1(0,1)))^2 \\ & \frac{\partial\zeta}{\partial\nu}(x) = 0 \text{ on } \Gamma_2,\ \eta(x) = \theta(x,0,s) \text{ in } \Omega;\ \frac{\partial\varphi}{\partial\nu}(x) = 0 \text{ on } \Gamma_2,\ \chi(x) = \psi(x,0,s) \text{ in } \Omega \Big\}. \end{aligned}$$

Arguing as in Subsection 6.2.1, we can show that $\tilde{\mathcal{A}}$ generates a strongly continuous semigroup on $\widehat{\mathcal{H}}$. Hence, the following well-posedness result.

Theorem 6.7. *For every $U_0 \in \widehat{\mathcal{H}}$, the problem (6.81) has a unique solution U whose regularity depends on the initial datum U_0 as follows:*

$$\begin{aligned} U(.) &\in C([0,+\infty);\widehat{\mathcal{H}}) \ \text{ if } U_0 \in \widehat{\mathcal{H}}, \\ U(.) &\in C^1([0,+\infty);\widehat{\mathcal{H}}) \cap C([0,+\infty); D(\tilde{\mathcal{A}})) \ \text{ if } U_0 \in D(\tilde{\mathcal{A}}). \end{aligned}$$

6.3.2 Proof of Theorem 6.2

We prove Theorem 6.2 for regular initial data, and the general case follows by a density argument. We proceed in three steps.

Step 1. We prove that the energy function $F(t)$ is decreasing.

Proposition 6.8. *The energy corresponding to any regular solution of system* (6.17) – (6.24), *is decreasing and there exists a positive constant L such that,*

$$\frac{d}{dt}F(t) \le -L\int_\Omega \left\{ a(x)\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds\} + b(x)\{v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\} \right\} dx, \tag{6.83}$$

where

$$L = \min\left\{\alpha_0 - \int_{\tau_1}^{\tau_2} \alpha(s)\,ds - \frac{\tilde{\alpha}_0}{2}(\tau_2-\tau_1), \frac{\tilde{\alpha}_0}{2}, \beta_0 - \int_{\tau_1}^{\tau_2} \beta(s)\,ds - \frac{\tilde{\beta}_0}{2}(\tau_2-\tau_1), \frac{\tilde{\beta}_0}{2}\right\}.$$

Proof. Differentiating $F(t)$ with respect to time, applying Green's Theorem, we obtain

$$\begin{aligned}\frac{d}{dt}F(t) = &\int_{\Gamma_2} \frac{\partial u}{\partial \nu}(x,t)u_t(x,t)\,d\Gamma - \alpha_0 \int_\Omega a(x)u_t^2(x,t)\,dx - \int_\Omega a(x)\int_{\tau_1}^{\tau_2} \alpha(s)u_t(x,t)u_t(x,t-s)\,ds\,dx\\ &+ \int_\Omega a(x)\int_{\tau_1}^{\tau_2} s(\alpha(s)+\tilde{\alpha}_0)\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)\,d\rho\,ds\,dx + \int_{\Gamma_2} \frac{\partial v}{\partial \nu}(x,t)v_t(x,t)\,d\Gamma\\ &- \beta_0 \int_\Omega b(x)v_t^2(x,t)\,dx - \int_\Omega b(x)\int_{\tau_1}^{\tau_2} \beta(s)v_t(x,t)v_t(x,t-s)\,ds\,dx + \int_\Omega b(x)\int_{\tau_1}^{\tau_2} s(\beta(s)+\tilde{\beta}_0)\\ &\int_0^1 v_t(x,t-\rho s)v_{tt}(x,t-\rho s)\,d\rho\,ds\,dx. \end{aligned} \tag{6.84}$$

Now, we observe that

$$\begin{aligned} -su_t(x,t-\rho s) &= u_\rho(x,t-\rho s),\\ -sv_t(x,t-\rho s) &= v_\rho(x,t-\rho s),\\ s^2u_{tt}(x,t-\rho s) &= u_{\rho\rho}(x,t-\rho s),\\ s^2v_{tt}(x,t-\rho s) &= v_{\rho\rho}(x,t-\rho s). \end{aligned}$$

Therefore

$$\begin{aligned} &\int_\Omega a(x)\int_{\tau_1}^{\tau_2} s(\alpha(s)+\tilde{\alpha}_0)\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)\,d\rho\,ds\,dx\\ &+ \int_\Omega b(x)\int_{\tau_1}^{\tau_2} s(\beta(s)+\tilde{\beta}_0)\int_0^1 v_t(x,t-\rho s)v_{tt}(x,t-\rho s)\,d\rho\,ds\,dx\\ &= \int_\Omega a(x)\int_{\tau_1}^{\tau_2} (-s)^{-3}s(\alpha(s)+\tilde{\alpha}_0)\int_0^1 u_\rho(x,t-\rho s)u_{\rho\rho}(x,t-\rho s)\,d\rho\,ds\,dx\\ &+ \int_\Omega \int_{\tau_1}^{\tau_2} (-s)^{-3}s(\beta(s)+\tilde{\beta}_0)v_\rho(x,t-\rho s)v_{\rho\rho}(x,t-\rho s)\,d\rho\,ds\,dx, \end{aligned}$$

from which follows, after integration by parts in ρ

$$\begin{aligned} &\int_\Omega a(x)\int_{\tau_1}^{\tau_2} s(\alpha(s)+\tilde{\alpha}_0)\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)\,d\rho\,ds\,dx\\ &+ \int_\Omega b(x)\int_{\tau_1}^{\tau_2} s(\beta(s)+\tilde{\beta}_0)\int_0^1 v_t(x,t-\rho s)v_{tt}(x,t-\rho s)\,d\rho\,ds\,dx\\ &= \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\Big\{a(x)(\alpha(s)+\tilde{\alpha}_0)(u_t^2(x,t)-u_t^2(x,t-s))\\ &+ b(x)(\beta(s)+\tilde{\beta}_0)(v_t^2(x,t)-v_t^2(x,t-s))\Big\}\,ds\,dx. \end{aligned} \tag{6.85}$$

Inserting (6.85) and the boundary conditions (6.76) and (6.77) into (6.84), we obtain

$$\begin{aligned}\frac{d}{dt}F(t) &= -\alpha_0 \int_\Omega a(x)u_t^2(x,t)\,dx - \int_\Omega a(x)u_t(x,t)\int_{\tau_1}^{\tau_2}\alpha(s)u_t(x,t-s)\,ds\,dx\\ &- \beta_0 \int_\Omega b(x)v_t^2(x,t)\,dx - \int_\Omega b(x)v_t(x,t)\int_{\tau_1}^{\tau_2}\beta(s)v_t(x,t-s)\,ds\,dx\\ &- \frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}(\alpha(s)+\tilde{\alpha}_0)u_t^2(x,t-s)\,ds\,dx + \frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}(\alpha(s)+\tilde{\alpha}_0)u_t^2(x,t)\,ds\,dx\\ &- \frac{1}{2}\int_\Omega b(x)\int_{\tau_1}^{\tau_2}(\beta(s)+\tilde{\beta}_0)v_t^2(x,t-s)\,ds\,dx + \frac{1}{2}\int_\Omega b(x)\int_{\tau_1}^{\tau_2}(\beta(s)+\tilde{\beta}_0)v_t^2(x,t)\,ds\,dx. \quad (6.86)\end{aligned}$$

From Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\left|\int_\Omega a(x)u_t(x,t)\int_{\tau_1}^{\tau_2}\alpha(s)u_t(x,t-s)\,ds\,dx\right| &\le \int_\Omega a(x)\mid u_t(x,t)\mid \int_{\tau_1}^{\tau_2}\alpha(s)\mid u_t(x,t-s)\mid\,ds\,dx\\ &\le \int_\Omega a(x)\mid u_t(x,t)\mid\left(\int_{\tau_1}^{\tau_2}\alpha(s)\,ds\right)^{\frac{1}{2}}\left(\int_{\tau_1}^{\tau_2}\alpha(s)u_t^2(x,t-s)\,ds\right)^{\frac{1}{2}}dx\\ &\le \frac{1}{2}\int_\Omega a(x)u_t^2(x,t)\left(\int_{\tau_1}^{\tau_2}\alpha(s)\,ds\right)dx + \frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}\alpha(s)u_t^2(x,t-s)\,ds\,dx, \quad (6.87)\end{aligned}$$

and

$$\begin{aligned}\left|\int_\Omega b(x)v_t(x,t)\int_{\tau_1}^{\tau_2}\beta(s)v_t(x,t-s)\,ds\,dx\right| &\le \frac{1}{2}\int_\Omega b(x)v_t^2(x,t)\left(\int_{\tau_1}^{\tau_2}\beta(s)\,ds\right)dx\\ &+ \frac{1}{2}\int_\Omega b(x)\int_{\tau_1}^{\tau_2}\beta(s)v_t^2(x,t-s)\,ds\,dx. \quad (6.88)\end{aligned}$$

Combining (6.86) with (6.87) and (6.88), we obtain

$$\begin{aligned}\frac{d}{dt}F(t) \le &\left(-\alpha_0 + \int_{\tau_1}^{\tau_2}\alpha(s)\,ds + \frac{\tilde{\alpha}_0}{2}(\tau_2-\tau_1)\right)\int_\Omega a(x)u_t^2(x,t)\,dx - \frac{\tilde{\alpha}_0}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2}u_t^2(x,t-s)\,ds\,dx\\ &+ \left(-\beta_0 + \int_{\tau_1}^{\tau_2}\beta(s)\,ds + \frac{\tilde{\beta}_0}{2}(\tau_2-\tau_1)\right)\int_\Omega b(x)v_t^2(x,t)dx - \frac{\tilde{\beta}_0}{2}\int_\Omega b(x)\int_{\tau_1}^{\tau_2}v_t^2(x,t-s)\,ds\,dx,\end{aligned}$$

which implies

$$\frac{d}{dt}F(t) \le -L\int_\Omega \left\{a(x)\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2}u_t^2(x,t-s)\,ds\} + b(x)\{v_t^2(x,t) + \int_{\tau_1}^{\tau_2}v_t^2(x,t-s)\,ds\}\right\}dx,$$

where L is given by

$$L = \min\left\{\alpha_0 - \int_{\tau_1}^{\tau_2}\alpha(s)\,ds - \frac{\tilde{\alpha}_0}{2}(\tau_2-\tau_1), \frac{\tilde{\alpha}_0}{2}, \beta_0 - \int_{\tau_1}^{\tau_2}\beta(s)\,ds - \frac{\tilde{\beta}_0}{2}(\tau_2-\tau_1), \frac{\tilde{\beta}_0}{2}\right\}.$$

□

Step 2. To obtain the exponential decay result for the energy function $F(0)$, we need the following obsevability estimate for system (6.17) – (6.24).

Proposition 6.9. *There exists a time T^* such that for all $T > T^*$, there exists a positive constant C_1 (depending on T) such that*

$$F(0) \leq C_1 \int_0^T \int_\Omega \left(a(x) \left\{ u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds \right\} + b(x) \left\{ v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds \right\} \right) dx\,dt. \tag{6.89}$$

For any regular solution of system (6.17)-(6.24).

Proof. We rewrite

$$F(t) = F_s(t) + F_d(t),$$

where

$$F_s(t) = \frac{1}{2} \int_\Omega \left\{ |\nabla u(x,t)|^2 + u_t^2(x,t) + \mid \nabla v(x,t) \mid^2 + v_t^2(x,t) + l(u(x,t) - v(x,t))^2 \right\} dx,$$

and

$$F_d(t) = \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} [a(x)s(\alpha(s) + c_0) \int_0^1 u_t^2(x,t-\rho s)\,d\rho + b(x)s(\beta(s) + \tilde{\beta}_0) \int_0^1 v_t^2(x,t-\rho s)\,d\rho]\,ds\,dx.$$

We decompose the solution (u,v) of (6.17) – (6.24) as follows

$$u = \varphi + \widehat{\varphi}, v = \psi + \widehat{\psi},$$

where (φ, ψ) solves

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) + l(\varphi(x,t) - \psi(x,t)) = 0 & \text{in } \Omega \times (0,+\infty), \\ \psi_{tt}(x,t) - \Delta\psi(x,t) + l(\psi(x,t) - \varphi(x,t)) = 0 & \text{in } \Omega \times (0,+\infty), \\ \varphi(x,0) = u_0(x,t); \varphi_t(x,0) = u_1(x) & \text{in } \Omega, \\ \psi(x,0) = v_0(x,t); \psi_t(x,0) = v_1(x) & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu}(x,t) = \frac{\partial\psi}{\partial\nu}(x,t) = 0 & \text{on } \Gamma_2 \times (0,+\infty), \\ \varphi(x,t) = \psi(x,t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \end{cases} \tag{6.90}$$

and $(\widehat{\varphi}, \widehat{\psi})$ is the solution of:

$$\begin{cases} \widehat{\varphi}_{tt} - \Delta\widehat{\varphi} + l(\widehat{\varphi} - \widehat{\psi}) + a(x)(\alpha_0 u_t(x,t) + \int_{\tau_1}^{\tau_2} \alpha(s) u_t(x, t-s)\, ds) = 0 & \text{in } \Omega \times (0, +\infty), \\ \widehat{\psi}_{tt} - \Delta\widehat{\psi} + l(\widehat{\psi} - \widehat{\varphi}) + b(x)(\beta_0 v_t + \int_{\tau_1}^{\tau_2} \beta(s) v_t(x, t-s)\, ds) = 0 & \text{in } \Omega \times (0, +\infty), \\ \widehat{\varphi}(x, 0) = \widehat{\varphi}_t(x, 0) = 0 & \text{in } \Omega \\ \widehat{\psi}(x, 0) = \widehat{\psi}_t(x, 0) = 0 & \text{in } \Omega, \\ \frac{\partial\widehat{\varphi}}{\partial\nu}(x,t) = \frac{\partial\widehat{\psi}}{\partial\nu}(x,t) = 0 & \text{on } \Gamma_2 \times (0, +\infty), \\ \widehat{\varphi}(x,t) = \widehat{\psi}(x,t) = 0 & \text{on } \Gamma_1 \times (0, +\infty). \end{cases} \tag{6.91}$$

Denote by $\Lambda(t)$ the standard energy of (6.90), that is

$$\Lambda(t) = \frac{1}{2} \int_\Omega \left\{ \varphi_t^2(x,t) + |\nabla\varphi(x,t)|^2 + \psi_t^2(x,t) + |\nabla\psi(x,t)|^2 + l(\varphi(x,t) - \psi(x,t))^2 \right\} dx,$$

and by $\tilde{\Lambda}(t)$ the standard energy of (6.91),

$$\tilde{\Lambda}(t) = \frac{1}{2} \int_\Omega \left\{ \widehat{\varphi}_t^2(x,t) + |\nabla\widehat{\varphi}(x,t)|^2 + \widehat{\psi}_t^2(x,t) + |\nabla\widehat{\psi}(x,t)|^2 + l(\widehat{\varphi}(x,t) - \widehat{\psi}(x,t))^2 \right\} dx,$$

then

$$F(0) = \Lambda(0) + F_d(0). \tag{6.92}$$

But for system (6.90), we have the following obsevability estimate that can be deduced from Proposition 2.2.1 of [46].

$$\Lambda(0) \le C_1 \int_0^T \int_\omega \left\{ \varphi_t^2(x,t) + \psi_t^2(x,t) \right\} dx\, dt,$$

for all times $T > T_0$.
Therefore

$$\begin{aligned} \Lambda(0) &\le C_1 \int_0^T \left\{ \int_{\omega_1} \varphi_t^2(x,t)\, dx + \int_{\omega_2} \psi_t^2(x,t)\, dx \right\} dt \\ &\le C_1 \int_0^T \int_{\omega_2} \left\{ \varphi_t^2(x,t)\, dx + \psi_t^2(x,t) \right\} dx\, dt \\ &\le C_1 \int_0^T \int_\Omega \left\{ a(x)\varphi_t^2(x,t) + b(x)\psi_t^2(x,t) \right\} dx\, dt, \end{aligned} \tag{6.93}$$

since $a(x) > a_0$ in ω_1 and $b(x) > b_0$ in ω_2 and $\omega_1 \subset \omega_2 \subset \Omega$.
On the other hand, by a change of variable, we have for $T > \tau_2$

$$F_d(0) \le C \int_\Omega \left\{ a(x) \int_{\tau_1}^{\tau_2} \int_0^T u_t^2(x, t-s)\, dt\, ds + b(x) \int_{\tau_1}^{\tau_2} \int_0^T v_t^2(x, t-s)\, dt\, ds \right\} dx. \tag{6.94}$$

If we take $T > T^* := max\{T_0, \tau_2\}$, we get from (6.93) and (6.94)

$$
\begin{aligned}
F(0) \leq C_1 \int_0^T \int_\Omega \Big\{ & a(x)\varphi_t^2(x,t) + b(x)\psi_t^2(x,t) + a(x)\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds \\
& + b(x)\int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds \Big\}\,dx\,dt \\
\leq C_1 \int_0^T \int_\Omega \Big\{ & a(x)(\widehat{\varphi}_t^2(x,t) + u_t^2(x,t)) + b(x)(\widehat{\psi}_t^2(x,t) + v_t^2(x,t)) + a(x)\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds \\
& + b(x)\int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds \Big\}\,dx\,dt.
\end{aligned}
$$

It remains to estimate the term

$$
\int_0^T \int_\Omega \left\{ a(x)\widehat{\varphi}_t^2(x,t) + b(x)\widehat{\psi}_t^2(x,t) \right\}\,dx\,dt.
$$

We differentiate the energy function $\tilde{\Lambda}(t)$ with respect to t, we obtain

$$
\begin{aligned}
\frac{d}{dt}\tilde{\Lambda}(t) = & -\int_\Omega a(x)\left\{ \alpha_0 \widehat{\varphi}_t(x,t)u_t(x,t) + \int_{\tau_1}^{\tau_2} \alpha(s)\widehat{\varphi}_t(x,t)u_t(x,t-s)\,ds \right\}\,dx \\
& - \int_\Omega b(x)\Big\{ \alpha_1 \widehat{\psi}_t(x,t)v_t(x,t) + \int_{\tau_1}^{\tau_2} \beta(s)\widehat{\psi}_t(x,t)v_t(x,t-s)\,ds \Big\}\,dx,
\end{aligned}
$$

from which we get after using Cauchy-Schwartz's inequality

$$
\begin{aligned}
\frac{d}{dt}\tilde{\Lambda}(t) \leq C\Big(& \int_\Omega a(x)\{\widehat{\varphi}_t^2(x,t) + u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds\}\,dx + \int_\Omega b(x)\{\widehat{\psi}_t^2(x,t) + v_t^2(x,t) \\
& + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\}\,dx \Big) + \int_\Omega \{\widehat{\varphi}_t^2(x,t) + \widehat{\psi}_t^2(x,t)\}\,dx.
\end{aligned}
$$

From the definition of $\tilde{\Lambda}(t)$, we obtain

$$
\begin{aligned}
\frac{d}{dt}\tilde{\Lambda}(t) \leq \tilde{\Lambda}(t) + C\Big(& \int_\Omega a(x)\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds\}\,dx \\
& + \int_\Omega b(x)\{v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\}\,dx \Big).
\end{aligned}
$$

Multiplying the last inequality by (e^{-t}) and integrating over $(0,t)$, we get

$$
\begin{aligned}
\tilde{\Lambda}(t) \leq Ce^t \Big(& \int_0^t \int_\Omega a(x)\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds\}\,dx\,dt \\
& + \int_0^t \int_\Omega b(x)\{v_t^2(x,t) + \int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\}\,dx\,dt \Big).
\end{aligned}
$$

We conclude for $t \in (0,T)$, that is

$$\tilde{\Lambda}(t) \leq C\Big(\int_0^T\int_\Omega a(x)\{u_t^2(x,t)+\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds\}\,dx\,dt + \int_0^T\int_\Omega b(x)\{v_t^2(x,t)+\int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\}\,dx\,dt\Big),$$

which gives

$$\int_0^T\int_\Omega \{\widehat{\varphi}_t^2(x,t)+\widehat{\psi}_t^2(x,t)\}\,dx\,dt \leq C\Big(\int_0^T\int_\Omega a(x)\{u_t^2(x,t)+\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds\}\,dx\,dt + \int_0^T\int_\Omega b(x)\{v_t^2(x,t)+\int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\}\,dx\,dt\Big).$$

Consequently, we have

$$F(0) \leq C_1\int_0^T\int_\Omega \Big\{a(x)\{u_t^2(x,t)+\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds\}+b(x)\{v_t^2(x,t)+\int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\}\Big\}\,dx\,dt. \tag{6.95}$$

□

Step 3. From (6.83)

$$F(t)-F(0) \leq -L\int_0^T\int_\Omega \Big\{a(x)\{u_t^2(x,t)+\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)\,ds\}+b(x)\{v_t^2(x,t)+\int_{\tau_1}^{\tau_2} v_t^2(x,t-s)\,ds\}\Big\}\,dx\,dt,$$

which together with (6.95) leads to

$$F(t) \leq \frac{C_1L^{-1}}{1+C_1L^{-1}}E(0). \tag{6.96}$$

Since $0 < \frac{C_1L^{-1}}{1+C_1L^{-1}} < 1$, then the energy of solutions of system (6.17) – (6.24) decays exponentially.

6.4 Appendix

Carleman estimates for coupled non-conservative hyperbolic equations

Let Ω be an open bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that, $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$.
Let $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\phi(x,t) \equiv |x-x_0|^2 - c|t-\frac{T}{2}|^2,$$

where $x_0 \in \mathbb{R}^n$, $T > 0$, $0 < c < 1$, are selected so that the following two properties are achieved:

(i)

$$cT > 2\max_{y\in\Omega}|y-x_0|;$$

(ii) there exists a subinterval $[t_0, t_1] \subset (0, T)$ such that

$$\phi(x,t) > 1 \text{ for } t \in [t_0, t_1]; x \in \Omega;$$

$$\phi(x,0) < -\delta < 0;\ \phi(x,T) < -\delta < 0 \text{ uniformly in } x \in \Omega,$$

for a suitable constant $\delta > 0$.

Consider the following coupled system of two second-order hyperbolic equations in the unknowns $w(t,x)$ and $z(t,x)$:

$$\begin{cases} w_{tt} = \Delta w + F_1(w) + P_1(z) & \text{in } (0,T] \times \Omega \equiv Q, \\ z_{tt} = \Delta z + F_2(z) + P_2(w) & \text{in } Q, \end{cases}$$

defined on a bounded domain $\Omega \in \mathbb{R}^n$ with smooth boundary Γ, where F_1, F_2, P_1, P_2 are (linear) differential operators of order one in all variables $t, x_1, ..., x_n$, with $L^\infty(Q)$- coefficients, thus satisfying the point wise bounds

$$\begin{aligned} |F_1(w)|^2 + |P_2(w)|^2 &\le c_T\left[w_t^2 + |\nabla w|^2 + w^2\right] & \forall\, t, x \in Q, \\ |F_2(z)|^2 + |P_1(z)|^2 &\le c_T\left[w_t^2 + |\nabla z|^2 + w^2\right] & \forall\, t, x \in Q. \end{aligned}$$

Proposition 6.10. *(Lasiecka and Triggiani [46]) Let w and z be solutions of the above problem in the following class*

$$\begin{cases} w, z \in H^1(Q) = L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)) \\ w_t, \frac{\partial w}{\partial \nu}, z_t, \frac{\partial z}{\partial \nu} \in L^2(0,T; L^2(\Gamma)). \end{cases}$$

then the following inequality holds true for τ sufficiently large:

- *there exists a positive constant $k_{\phi,\tau} > 0$ such that*

$$\begin{aligned} k_{\phi,\tau} E(0) \le \int_0^T \int_\Gamma \left[\left(\frac{\partial w}{\partial \nu}\right)^2 + w_t^2 + \left(\frac{\partial z}{\partial \nu}\right)^2 + z_t^2\right] d\Gamma\, dt \\ + const_{T,\tau,\epsilon_0}\left\{\|w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)}\right\}, \end{aligned}$$

or equivalently,

$$\begin{aligned} k_{\phi,\tau}\left[E(0) + E(T)\right] \le \int_0^T \int_\Gamma \left[\left(\frac{\partial w}{\partial \nu}\right)^2 + w_t^2 + \left(\frac{\partial z}{\partial \nu}\right)^2 + z_t^2\right] d\Gamma\, dt \\ + const_{T,\tau,\epsilon_0}\left\{\|w\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)} + \|z\|^2_{H^{\frac{1}{2}+\epsilon_0}(Q)}\right\}, \end{aligned}$$

- *If, moreover, w and/or z satisfy the boundary condition*

$$w|_{\Sigma_1} \equiv 0, \text{ and / or, respectively, } z|_{\Sigma_1} \equiv 0, \qquad \Sigma_1 = (0,T] \times \Gamma_1,$$

where Γ_1 *is defined by*

$$\Gamma_1 = \{x \in \Gamma : \nabla\phi.\nu(x) \leq 0\}\,;$$

then the corresponding integral term for w *and/or for* z *replaces* Γ *with* Γ_2*.*

Conclusion

In this thesis we have proved some uniform stabilization results for two types of partial differential equations: wave and Schrödinger equations with time delay.
In Chapter two, we established global existence and uniform decay rates for the solutions of the multidimensional wave equations with a delay term in the nonlinear boundary or internal feedbacks. The proof of existence of solutions relies on a construction of a suitable approximating problem for which the existence of solution will be established using nonlinear semigroup theory and then passage to the limit gives the existence of solutions to the original problem. The uniform decay rates for the solutions are obtained by proving certain integral inequalities for the energy function and by establishing a comparison theorem which relates the asymptotic behaviour of the energy and of the solutions to an appropriate dissipative ordinary differential equation.
Chapters three, four and five are devoted to the Schrödinger equation defined on an open bounded domain Ω of $\mathbb{R}^n$ with a delay term and subject to a dissipative feedback. In Chapter three, we considered the case where the equation contains a delay term in the nonlinear internal or boundary feedbacks. We proved that it is $L^2(\Omega)$-wellposed and $L^2(\Omega)$-stable with uniform decay rates described, as in chapter two, by a dissipative ordinary differential equation. In Chapter four, we analyzed the case of the equation with interior delay and a boundary feedback. Using multipliers technique and a suitable Lyapunov functional, we proved exponential stability of the solution in the energy space $H^1_{\Gamma_1}(\Omega)$ on condition that the delay coefficient is sufficiently small.
In Chapter five we dealt with the case where the boundary or the internal feedback contains a delay term of distributed type. By introducing suitable energy functionals and by using some observability estimates, we showed that the solution decays exponentially in appropriate energy space.
In Chapter six, we considered a system of compactly coupled wave equations with distributed delay terms in the boundary or internal feedbacks. In both cases, we established that the semigroup generating the dynamics of the closed-loop system is exponentially stable. The approach we adopted combines Carleman estimates for coupled non-conservative hyperbolic systems and compactness-uniqueness argument.

There are several extensions of the results obtained in this thesis. For example the following questions can be considered for future work:

- Nonlinear boundary stabilization of the wave equation with nonlinear interior delay.
- Nonlinear boundary stabilization of the Schrödinger equation with nonlinear interior delay.

- Internal stabilization of the wave equation with a delay term in the boundary conditions.

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Titre**: Stabilisation de quelques systèmes d'évolution avec retards**

Résumé:

Dans cette thèse, nous avons étudié le problème de stabilité pour quelques équations d'évolution (équation d'onde, équation de Schrödinger) avec des termes de retards dans les feedbacks (linéaire ou non linéaire) frontière ou interne. Sous certaines hypothèses, les taux de décroissance uniformes pour les solutions sont établis. Certaines de ces résultats sont obtenues en introduisant des fonctions d'énergies appropriées et en prouvant des inégalités d'observabilité, tandis que les autres sont déduits à partir d'estimations des fonctions de Lyapunov appropriées.

Mots Clés:

Equation d'onde, équation de Schrödinger, stabilisation, feedback frontière, feedback interne, retard de temps.

العنوان: **استقرار بعض أنظمة التطور بوجود تأخر زمني**

الملخص:

تتضمن هذه الأطروحة دراسة استقرار بعض معادلات التطور (معادلة الموجة، معادلة شرودنجر) مع حدوث تأخر في ردود الأفعال (الخطية أو غير الخطية) الحدية أو الداخلية. و بوضع بعض الفرضيات، نحقق معدلات التضاؤل المنتظمة للحلول. و نتحصل على بعض هذه النتائج بإدراج الطاقة الملائمة و بإثبات بعض متراجحات الملاحظة، بينما البعض الآخر يتم استنتاجها من تقديرات وظائف ليابونوف الملائمة.

الكلمات المفتاحية:

معادلة الموجة، معادلة شرودنجر، إعادة الاستقرار، رد فعل الحدي، رد فعل الداخلي، تأخرزمني.