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THÈME

Sur le Rayon numérique et ses applications

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Dedication

To my mother

To my father

To my Husband and my son

To my sister and my brothers

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CONTENTS

List of Symbols

Let $B(\mathbb{H})$ be the C^*-algebra of all bounded linear operators acting on a complex Hilbert space H, and let $A \in B(\mathbb{H})$.

We denote by

- $\|x\|$ The norm of x.
- $\langle x, y\rangle$ The inner product of x, and y.
- $M^{\perp}$ The orthogonal complement of M.
- $V \oplus W$ The direct sum of V, and W.
- $\oplus_{i=1}^{n} M_i$ The direct sum of M_i for all $(i = 1, 2, ..., n)$.
- **Conv(M)** The convex Hull of M.
- $\|A\|$ The norm of $A \in B(\mathbb{H})$.
- A^* The adjoint of $A \in B(\mathbb{H})$.
- $ker(A)$ The kernel of $A \in B(\mathbb{H})$.
- $R(A)$ The range of $A \in B(\mathbb{H})$.
- $A^{\frac{1}{2}}$ The square root of $A \in B(\mathbb{H})$.
- $|A|$ The absolute value of $A \in B(\mathbb{H})$.

- $\tilde{A}$ The Aluthge transform of $A \in B(\mathbb{H})$.
- $\tilde{A}_t, t \in [0,1]$ The generalized Aluthge transform of $A \in B(\mathbb{H})$.
- $Re(A)$ The real part of $A \in B(\mathbb{H})$.
- $Im(A)$ The imaginary part of $A \in B(\mathbb{H})$.
- $\sigma(A)$ The spectrum of $A \in B(\mathbb{H})$.
- $\sigma_{app}(A)$ The approximate point spectrum of $A \in B(\mathbb{H})$.
- $r(A)$ The spectral radius of $A \in B(\mathbb{H})$.
- $[A_{ij}]_{n\times n}$ An $n \times n$ Operator matrix.
- $\oplus_{i=1}^{n} A_i$ The diagonal operator matrix.
- $W(A)$ The numerical range of $A \in B(\mathbb{H})$.
- $\omega(A)$ The numerical radius of $A \in B(\mathbb{H})$.
- A^{+} The Moore-Penrose Inverse of $A \in B(\mathbb{H})$.

INTRODUCTION

The motivation behind this thesis is to prove several new numerical radius inequalities for Hilbert Space Operators.

The numerical radius of bounded linear operator $A \in B(\mathbb{H})$ is denoted by $\omega(.)$ and defined by

$$\omega(A) = \sup\{|\lambda| : \quad \lambda \in W(A)\},$$

where $W(A)$ is the Numerical Range of A. Thus the numerical radius is the smallest radius for the circular disc centred at origin which contains $W(A)$. It is well-known that $\omega(.)$ defines a norm in $\mathbb{H}$, which is equivalent to the usual operator norm $\|.\|$, we present this equivalence as follows, for all $A \in B(\mathbb{H})$

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{0.0.1}$$

These inequalities are sharp. The improvement of the second inequality in (0.0.1) received much attention from many mathematicians. In (1971), Bouldin [10] has proved that, if $A \in B(\mathbb{H})$

$$\omega(A) \leq \frac{1}{2}[\cos\alpha + (\cos^2\alpha + 1)^{\frac{1}{2}}],$$

where α is the angle between $R(A)$ and $R(A^*)$. In (2003), Kittaneh [31] improved the second inequality in (0.0.1) as follows

$$\omega(A) \leq \frac{1}{2}[\|A\| + \|A^2\|^{\frac{1}{2}}].$$

After that in (2005), Kittaneh [34] found an upper and lower bound for $\omega^2(A)$, he proved that

$$\frac{1}{4}\||A|^2| + |A|^*|^2\| \leq \omega^2(A) \leq \frac{1}{2}\||A|^2| + |A|^*|^2\|.$$

This inequality was also reformulated and generalized in [14] but in terms of Cartesian decomposition.

In (2007), Yamazaki [43] used the Aluthge transform which was defined by Aluthge [8] in (1990) to improve the result of Kittaneh [31], this result says that

$$\omega(A) \leq \frac{1}{2}[\|A\| + \omega(\tilde{A})],$$

where $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$, and U is a partial isometry. In (2013) Abu-Omar, and Kittaneh [2] have used the generalised Aluthge Transform to improve and generalise the result of Yamazaki [43], so that

$$\omega(A) \leq \frac{1}{2}[\|A\| + \omega(\tilde{A}_t)],$$

where $\tilde{A}_t = |A|^t U|A|^{1-t}$, U is a partial isometry, and $t \in [0, 1]$.

In (2008) Dragomir [12] used Buzano inequality to refine the second inequality in (0.0.1) as follows

$$\omega^2(A) \leq \frac{1}{2}[\|A\| + \omega(A^2)].$$

This result has been generalised by M. Sattari, M. S. Moslehian, and T. Yamazaki [41]. They proved that for all $r \geq 1$

$$\omega^{2r}(A) \leq \frac{1}{2}(\omega^r(A^2) + \|A\|^{2r}).$$

This thesis is devided into four chapters.

Chapter one is about prelimanary, it consists of five sections. In section 1.1, we present some basic and important properties of Hilbert space, also fundamontal properties of bounded linear operators on Hilbert space are included. We collect some basic notions as Cartesian, orthogonal, and polar decomposition, spectrum, and operator matrices that we used throughout this thesis. Sections 1.2, and 1.3 are about numerical range, and numerical radius respectively, it discusses several important and intersting well-know properties about them. In Section 1.4 we introduce the notion of Moore-Penrose Inverse, and we collect some of well-know results about them, that they will be used in the last chapter of this thesis. In Section 1.5, we present some of most important improvements of the second inequality in (0.0.1).

Numerical radius inequalities for operator matrices is presented in Chapter 2. In (1982) Halmous [22] created the notion of operator matrices, which played an important role in operator theory. In (1995) Hou, and Du [29] have proved that

$$\omega([A_{ij}]_{n\times n}) \leq \omega([\| A_{ij} \|]_{n\times n}).$$

After that in (2013) Abu-Omar, and Kittaneh improved this inequality as follows

$$\omega([A_{ij}]_{n\times n}) \leq \omega([a_{ij}]_{n\times n}),$$

where

$$a_{ij} = \omega\left(\begin{bmatrix} 0 & A_{ij} \\ A_{ji} & 0 \end{bmatrix}\right) \quad for \quad i,j = 1,2,...,n.$$

and

$$a_{ii} = \omega(A_{ii}) \quad for \quad i = 1,2,...,n.$$

Chapter 2 is divided into 2 sections. In Section 2.1 we present a new numerical radius inequalities for $n \times n$ operator matrices. First we give numerical radius inequalities for $n \times n$ operator matrices with a single non zero row, utilising this result we conclude numerical radius inequalities for arbitrary $n \times n$ operator matries. Other results for 3×3 operator matrices that involve the skew diagonal part of 2×2 operator matrices are also obtained in this section. Our results are a natural generalisation of some of numerical radius inequalities for 2×2 operator matrices which are given in [25] , and references therein, our results are sharp. In Section 2.2, we prove new numerical radius inequalities for the skew diagonal part of 3×3 operator matrices. We give several upper and lower bounds for

$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right)$ such that $A, B, C \in B(\mathbb{H})$, with an application when A, and C are positive operator, and B is a self-adjoint operator.

An inequality involving the generalised Aluthge transform is also obtained in this Section.

Chapter 3 is devoted to prove numerical radius inequalities for products and commutators of operators. In fact the numerical radius is not multiplicative, for all $A, B \in B(\mathbb{H})$ the inequality

$$\omega(AB) \le \omega(A)\omega(B) \tag{0.0.2}$$

is not true even for commuting operators, which made many researchers to find conditions under which refines the inequalty (0.0.2). The authors in [27], [21] have refined the inequality (0.0.2) but under the condition $AB = BA$ (i.e, A commute with B). They proved that for all $A, B \in B(H)$, such that $AB = BA$

$$\omega(AB) \le 2\omega(A)\omega(B)$$

For additional results, see Gustafson and Rao [21].

Further, other mathematicians proved new numerical radius inequalities for the product of two operators without commutativity conditions. In (2008) Dragomir has shown that, for all $A, B \in B(\mathbb{H})$

$$\omega(B^* A) \le \frac{1}{2}\| |A|^2 + |B|^2 \|.$$

M. Sattari, M. S. Moslehian, and T. Ymazaki [41] have established a generalisation of the result of Dragomir, as follows

$$\omega^r(B^* A) \le \frac{1}{4}\| |A^*|^{2r} + |B^*|^{2r} \| + \frac{1}{2}\omega^r(AB^*).$$

After that in (2015) Abu-Omar and Kittaneh [3] have shown that , for all $A, B \in B(\mathbb{H})$

$$\omega(AB) \le \frac{1}{4}\| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|(B)^*|^2 \| + \frac{1}{2}\omega(BA).$$

A new research appeared when they are looking for the improvement of the inequality (0.0.2) that is finding numerical radius inequalities for commutators of operators that improve the inequalities folow directly by appliying the triangle inequality of numerical radius.

Chapter 3 is divided into 2 Sections. In Section 3.1, we present numerical radius inequalities for products of three operators whithout assuming the commutativity of operators. Our results refine and generalise recent results which are obtained by Abu-Omar and Kittaneh [3] in (2015), and M, Sattari, M. S. Moslehian, and T. Yamazaki [41] in (2015). In Section 3.2, we give new numerical radius for sums and commutators of operators.

In Chapter 4, we establish a generalisation for recent results which are obtained by Abu-Omar and Kittaneh [4] in (2014) and we give new generalised numerical radius inequalities. Using an analysis that is totaly different from the ones used in Chapter 2, we obtain a new numerical radius inequalities for arbitrary $n \times n$ operators matrices, from these results, we establish a new spectral radius inequalities for sums and products of operators. A spectral radius inequality for commutators of operators is also obtained in this Chapter. An improvement of the generalised triangle inequality of the usual operator norm is given in this Chapter.

Finally, we apply some of the previous results to operators whith closed range, and we found new spectral radius inequalities for Moore-Penrose inverse and we improve some of well-known inequalities about them.

CHAPTER 1
PRELIMENARIES

Throughout this thesis H denotes a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $B(H)$ is the C^*-algebra of all bonded linear operators on H. In this Chapter we collect some of basic properties of Hilbert space and operator theory, that will be used throughout this thesis. The most data of Section 1.1 is from [16, 18, 30, 37, 39, 40].

1.1 Basic facts

Définition 1.1.1. *Let $\mathbb{H}$ be a Complex vector space.*

(1) A norm on $\mathbb{H}$ is a function $\|.\| : H \longrightarrow R$ such that for all $x, y \in \mathbb{H}$ and $\alpha \in C$

- *(a) $\|x\| \geq 0$ (strictly positive).*
- *(b) $\|x\| = 0$ if and only if $x = 0$*
- *(c) $\|\alpha x\| = |\alpha|\|x\|$ (strictly homogeneous).*
- *(d) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).*

(2) A vector space $\mathbb{H}$ on which there is a norm is called a normed vector space, or just a normed space.

(3) A ***Banach space*** *is a complet normed vector space .*

Définition 1.1.2. *Let $\mathbb{H}$ be a Complex vector space.*

(1) *An inner product on* $\mathbb{H}$ *is a function* $\langle , \rangle : \mathbb{H} \longmapsto \mathbb{H}$ *such that for all* $x, y, z \in \mathbb{H}$, *and* $\alpha, \beta \in C$

   (a) $\langle x, x\rangle \geq 0$.

   (b) $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

   (c) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

   (d) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

(2) *A complex vector space* $\mathbb{H}$ *with an inner product* $\langle , \rangle$ *is called* ***Inner product space***.

(3) *An inner product space which is complete with respect to the metric associated with the norm induced by inner product is called* ***Hilbert space***.

Any inner product salisfies an important inequality, called the Cauchy-Schwarz inequality which is due to John von Neumann (1930).

Proposition 1.1.3. *Let* $\mathbb{H}$ *be an inner product space, and let* $x, y \in \mathbb{H}$. *Then*

(a) $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$.

(b) *The function* $\|.\| : H \longmapsto C$ *defined by* $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$, *is a norm on* $\mathbb{H}$ *called the norm induced by inner product.*

An inner product $\langle x, y\rangle$ can be expressed in termes of norms as follows.

Theorem 1.1.4. ***(Polarisation identity)***
Let $\mathbb{H}$ *be an inner product space, and let* $x, y \in \mathbb{H}$. *Then*

$$\langle x, y\rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x + iy\|^2].$$

Définition 1.1.5. *Let* $\mathbb{H}$ *be an inner product space, and let* M *be a subspace of* $\mathbb{H}$. *The orthogonal complement of* M *is the set*

$$M^{\perp} = \{x \in H : \langle x, y\rangle = 0 \quad for \quad all \quad y \in M\}.$$

Définition 1.1.6. ***(Direct sum)*** *A vector space $\mathbb{H}$ is said to be direct sum of two subspaces V and W of $\mathbb{H}$, written*

$$\mathbb{H} = V \oplus W,$$

if each $x \in \mathbb{H}$ has a unique representation

$$x = v + w \qquad for \quad all \quad v \in V \quad w \in W.$$

Theorem 1.1.7. *Let $\mathbb{H}$ be an inner product space and let M be a closed subspace of $\mathbb{H}$. Then*

$$\mathbb{H} = M \oplus M^{\perp}.$$

The direct sum decomposition of $\mathbb{H}$ may be expanded to a finit of mutualy orthogonal closed subspaces $M_i \in \mathbb{H}$ for all $i = 1, 2, ..., n$, so that

$$\mathbb{H} = \oplus_{i=1}^{n} M_i.$$

Définition 1.1.8. *Let M be a subset of a Hilbert space $\mathbb{H}$.*

(a) *M is said to be convex if*

$$tx + (1-t)y \in M \qquad for \quad all \quad x, y \in M \quad and \quad t \in [0,1].$$

(b) *The convex Hull of M denoted by* ***conv(M)*** *is the smallest convex set of $\mathbb{H}$ contained M, in other word ,* ***conv(M)*** *is the intersection of all convex sets containing M.*

Définition 1.1.9. *Let $\mathbb{H}$ be a Hilbert space. An operator $A : \mathbb{H} \longmapsto \mathbb{H}$ is:*

- *Linear operator if*

  $$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y),$$

  for all $x, y \in \mathbb{H}$, and scalars α, β.

- *Bounded Linear operator if A is linear and there exist a positive number k such that*

  $$\|Ax\| \leq k\|x\| \qquad for \quad all \quad x \in \mathbb{H}.$$

Définition 1.1.10. *Let A be a bounded linear operator. The norm of A is defined by*

$$\|A\| = \sup_{\|x\|=1} \|Ax\|.$$

An equivalent definition of the operator norm is

$$\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle| \qquad for \quad all \quad x, y \in \mathbb{H}.$$

The set of all bounded linear operators $A : \mathbb{H} \longmapsto \mathbb{K}$ is denoted by $B(\mathbb{H}, \mathbb{K})$. If $\mathbb{K}$ is a Banach space, then $B(\mathbb{H}, \mathbb{K})$ is a banach space. When $\mathbb{H} = \mathbb{K}$, we will write $B(\mathbb{H})$ for $B(\mathbb{H}, \mathbb{H})$.
To each operator $A \in B(\mathbb{H})$ correspends a unique operator $A^* \in B(\mathbb{H})$ that satisfies

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \qquad for \quad all \quad x, y \in \mathbb{H}.$$

The operator A^* is called **the adjoint** of the operator A. Moreover, A^* satisfies

$$\|A\| = \|A^*\| \qquad and \qquad \|A\|^2 = \|A^* A\| = \|AA^*\|.$$

If $A = A^*$, then A is self-adjoint operator.

Theorem 1.1.11. ***(Generalized Polarization Identity)****. For each $A \in B(\mathbb{H})$ and $x,\ y \in \mathbb{H}$, we have*

$$\langle Ax, y\rangle = \frac{1}{4}[\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle] + \frac{i}{4}[\langle A(x+iy), x+iy\rangle - \langle A(x-iy), x-iy\rangle].$$

Définition 1.1.12. *An operator $A \in B(\mathbb{H})$ is called non-negative operator (or positive operator) if A is a self-adoint operator and $\langle Ax, x\rangle \geq 0$, for all $x \in \mathbb{H}$.*

Theorem 1.1.13. *Every positive operator $A \in B(\mathbb{H})$ hase a unique non-negative square root $B \in B(\mathbb{H})$ defined by $B = A^{\frac{1}{2}}$. Furthermore B commutes with each operator that commutes with A.*

If $A \in B(\mathbb{H})$. Then

- The Kernel of A is the closd subspace of $\mathbb{H}$ defined by

$$ker(A) = \{x \in \mathbb{H} : Ax = 0\}.$$

- The Range of A is the subspace of $\mathbb{H}$ defined by

$$R(A) = \{Ax : \mathbb{H} \in H\}.$$

Définition 1.1.14. *An operator $U \in B(H)$ is called a partial isometry if*

$$\|Ux\| = \|x\| \qquad for \quad all \quad x \in ker(U)^{\perp}.$$

In that case $ker(U)^{\perp}$ is called the initial space of U and $R(U)$ is called the final space. The range of U is always closed.

Theorem 1.1.15. *Let $A \in B(\mathbb{H})$. Then there exists a partial isometry $U \in B(\mathbb{H})$ such that*

$$A = U|A| \qquad where \quad |A| = (A^* A)^{\frac{1}{2}}. \tag{1.1.1}$$

Furthermore, U may be chosen such that $\overline{R(|A|)} = \overline{R(A)} = ker(A)^{\perp}$ is the initial space of U and in that case the decomposition (1.1.1) is unique and is called the Polar decomposition of A.

For every $A \in B(\mathbb{H})$. The Aluthge transform of A denoted by $\tilde{A} \in B(\mathbb{H})$ was first defined by Aluthge [8] as

$$\tilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}.$$

The generalised Aluthge transform of A denoted by $\tilde{A}_t$, is defined by

$$\tilde{A}_t = |A|^t U |A|^{1-t} \qquad for \quad t \in [0,1].$$

Theorem 1.1.16. *Let $A \in B(\mathbb{H})$. Then there exists self-adjoint operators $B, C \in B(\mathbb{H})$ such that*

$$A = B + iC. \tag{1.1.2}$$

Necessarily $B = \dfrac{A+A^*}{2}$*,and* $C = \dfrac{A-A^*}{2i}$. *The decomposition (1.1.2) is called the Cartesian decomposition of A. The operators B, and C are called real part , and imaginary part of A respectively.*

Définition 1.1.17. *[16] Let* $A \in B(\mathbb{H})$. *Then*

(a) The spectrum of A denoted by $\sigma(A)$ *is the non-empty compact set of all complex numbers* λ *defined by*

$$\sigma(A) = \{\lambda \in C : A - \lambda I \quad is \quad not \quad invertible\}.$$

(b) The spectral radius of A is the number given by

$$r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

$r(A)$ is the radius of the smallest closed disk centred at origin of the comlex plane and containing $\sigma(A)$.

The most important property of the spectral radius is the Gelfand formula

$$r(A) = \lim_{n \longrightarrow \infty} \|A^n\|^{\frac{1}{n}}.$$

It is well-Known that for all $A \in B(\mathbb{H})$

$$r(A) \leq \|A\|.$$

Proposition 1.1.18. *[16] Let* $A, B \in B(\mathbb{H})$. *Then*

(a) $\sigma(AB)\backslash\{0\} = \sigma(BA)\backslash\{0\}$.

(b) $r(AB) = r(BA)$

Let $\mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_3, ..., \mathbb{H}_n$ be Hilbert spaces and let $\mathbb{H} = \oplus_{i=1}^{n} \mathbb{H}_i$. denote by $P_{\mathbb{H}_i}$ for all $i = 1, 2, ..., n$ the projection into $\mathbb{H}_i$. Let $A \in B(\mathbb{H})$, we can express A in the form of operator matrix

$$A = [A_{ij}]_{n \times n} \qquad for \quad all \quad (i, j = 1, 2, ..., n),$$

where $A_{ij} = P_{\mathbb{H}_i}(A|\mathbb{H}_j)$ are the projections into $\mathbb{H}_i$ of the restriction of A to $\mathbb{H}_j$, so $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$. the operations on operator matrices are the obious ones with respect the same decomposition of $\mathbb{H}$. The adjoint of $A = [A_{ij}]_{n\times n}$ is the operator matrix

$$A^* = [A^*_{ji}]_{n\times n} \qquad for \quad all \quad i,j = 1,2,...,n.$$

The diagonal operator matrix $[A_{ij}]_{n\times n}$ with $A_{ij} = 0$ when $i \neq j$ is the directe sum of the operators $[A_{ii}]_{n\times n}$ for all $i = 1,2,...,n$ and denoted by $\oplus_{i=1}^n A_{ii}$.

Theorem 1.1.19. *[1] Let $A_i \in B(\mathbb{H}_i)$ for all $i = 1,2,...,n$. Then*

$$\sigma(\oplus_{i=1}^n A_i) = \cup_{i=1}^n \sigma(A_i).$$

Theorem 1.1.20. *[9] Let $A \in B(\mathbb{H})$ be a diagonal operator matrix (or $A = \oplus_{i=1}^n A_{ii}$). Then*

$$\|A\| = \max(\|A_{11}\|, \|A_{22}\|, ..., \|A_{nn})\|,$$

$$r(A) = \max(r(A_{11}), r(A_{22}), ..., r(A_{nn})).$$

Theorem 1.1.21. *[29] Let $\mathbb{H}_1, \mathbb{H}_2, ..., \mathbb{H}_n$ be Hilbert spaces and let $A = [A_{ij}]_{n\times n}$ be an operator matrix, whith $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$ for all $i, j = 1,2,...,n$ and let $T = [\|A_{ij}\|]_{n\times n}$. Then*

$$\|A\| \leq \|T\| = \|[\|A_{ij}\|]_{n\times n}\|,$$

$$r(A) \leq r(T) = r([\|A_{ij}\|]_{n\times n}).$$

Proposition 1.1.22. *[35, 9] Let $\mathbb{H}_1, \mathbb{H}_2$ be a Hilbert spaces and let $A \in B(\mathbb{H}_2, \mathbb{H}_1)$, $B \in B(H_1, H_2)$. Then*

(a) $r\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \sqrt{r(AB)}.$

(b) $\left\|\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right\| = \max\{\|A\|, \|B\|\}.$

1.2 Numerical Range

Définition 1.2.1. *[21, 22] The Numerical Range (also known as field of values) of an operator $A \in B(\mathbb{H})$ is the non empty subset of the complex numbers $\mathbb{C}$, given by*

$$W(A) = \{\langle Ax, x\rangle, \quad x \in \mathbb{H}, \quad \|x\| = 1\}.$$

The following properties are immediate

Proposition 1.2.2. *[21, 22] Let $A, B \in B(\mathbb{H})$, F subspace of $\mathbb{H}$ and $\alpha, \beta \in \mathbb{C}$. Then*

1. $W(\alpha I + \beta A) = \alpha + \beta W(A)$,
2. $W(A^*) = \{\overline{\lambda}, \lambda \in W(A)\}$,
3. $W(U^* AU) = W(A) \quad for \quad any \quad unitary \quad U \in B(\mathbb{H})$,
4. $W(A/F) \subset W(A)$,
5. $W(A+B) \subset W(A) + W(B)$,
6. $W(Re(A)) = Re(W(A))$ *and* $W(Im(A)) = Im(W(A))$.

Example 1.2.3. *[21] Let $A \in B(\mathbb{H})$ be the unilateral shift on l_2, the Hilbert space of square summable sequences. For any $x = (x_1, x_2, x_3, ...) \in \mathbb{H}, \|x\| = 1$, we have $Ax = (x_2, x_3, x_4, ...)$ and hence consider*

$$\langle Ax, x\rangle = x_1\overline{x_2} + x_2\overline{x_3} + x_3\overline{x_4} + ...$$

with

$$|x_1|^2 + |x_2|^2 + |x_3|^2 + ... = 1.$$

Notice that

$$\begin{aligned} |\langle Ax, x\rangle| &\leq |x_1||x_2| + |x_2||x_3| + |x_3||x_4| + ... \\ &\leq \frac{1}{2}[|x_1|^2 + 2|x_2|^2 + 2|x_3|^2 + ...] \\ &\leq \frac{1}{2}[2 - |x_1^2|]. \end{aligned}$$

Hence $|\langle Ax, x\rangle| < 1$ *if* $|x_1| \neq 0$. *For* $|x_1| = 0$ *and* x *containing a finite number of nonzero entries, we can show in the same way that* $|\langle Ax, x\rangle| < 1$ *by considering the minimum natural number* n *for which* $x_n \neq 0$.

Thus $W(A)$ *is contained in the open disk* $\{z : |z| < 1\}$. *We now show that it is in fact the open unit disk. Let* $z = re^{i\theta}, 0 \leq r < 1$, *be any point of this disk.*

Consider

$$x = (\sqrt{1-r^2}, r\sqrt{1-r^2}e^{i\theta}, r^2\sqrt{1-r^2}e^{-2i\theta}, ...).$$

Observe that

$$\|x\|^2 = 1 - r^2 + r^2 + r^2(1-r^2) + r^4(1-r^2) + ... = 1.$$

Furthermore

$$\langle Ax, x\rangle = r(1-r^2)e^{i\theta} + r^3(1-r^2)e^{i\theta} + ... = re^{i\theta}.$$

Thus $z \in W(A)$, *so that*

$$W(A) = \{z : |z| < 1\}.$$

Theorem 1.2.4. *[21] Let* A *be an operator on a two-dimensional space. Then* $W(A)$ *is an ellipse whose foci are the eigenvalues of* A.

Proof. Let $A = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}$, where λ_1 and λ_2 are the eigenvalues of A.

First, if $\lambda_1 = \lambda_2 = \lambda$, we have

$$A - \lambda I = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.$$

Then $W(A - \lambda I) \subseteq \{z : |z| \leq \frac{|a|}{2}\}$.

We now show that $W(A - \lambda I) = \{z : |z| \leq \frac{|a|}{2}\}$.

Let $z = re^{i\theta}$, $0 \leq r \leq \frac{\frac{|a|}{2}}{2}$, and let $x = (\overline{a}\cos\alpha, \frac{1}{|a|}\sin\alpha e^{i\theta})$, where $\sin 2\alpha = \frac{2}{|a|}r \leq 1$ and $0 \leq \alpha \leq \frac{\pi}{4}$, then

$$\langle (A - \lambda I)x, x\rangle = |a|e^{i\theta} cos\alpha \sin\alpha = |a|e^{i\theta}\frac{\sin 2\alpha}{2} = re^{i\theta},$$

so that

$$W(A - \lambda I) = \{z : |z| \leq \frac{|a|}{2}\},$$

then $W(A-\lambda I)$ is a cercle with center 0 and radius $\frac{|a|}{2}$, so $W(A)$ is a cercle with center λ and radius $\frac{|a|}{2}$, which is an ellipse.

If $\lambda_1 \neq \lambda_2$ and $a=0$, we have

$$A-\lambda I=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Let $x=\langle x_1, x_2\rangle$ be a unit vector, then

$$\langle Ax, x\rangle = \lambda_1|x_1|^2+\lambda_2|x_2|^2=\lambda_1 t+\lambda_2(1-t) \quad where \quad t=|x_1|^2.$$

So $W(A)$ is the set of combinations of λ_1 and λ_2 and is the segment joining them.

If $\lambda_1 \neq \lambda_2$ and $a \neq 0$, we have

$$A-(\frac{\lambda_1+\lambda_2}{2})I=\begin{bmatrix} \frac{\lambda_1-\lambda_2}{2} & a \\ 0 & -(\frac{\lambda_1-\lambda_2}{2}) \end{bmatrix}.$$

$$e^{-i\theta}(A-(\frac{\lambda_1+\lambda_2}{2})I)=\begin{bmatrix} r & ae^{-i\theta} \\ 0 & -r \end{bmatrix}=A' \quad where \quad \frac{\lambda_1-\lambda_2}{2}=re^{i\theta}.$$

Let $x=\langle x_1, x_2\rangle$ be a unit vector in $\mathbb{C}^2$, where $x_1=e^{i\alpha}\cos\theta,\ x_2=e^{i\beta}\sin\theta,\ \alpha\in[0,\frac{\pi}{2}]$ and, $\beta\in[0,2\pi]$. Then we get

$$A'x=\langle re^{i\alpha}\cos\theta+ae^{-i\theta}e^{i\beta}\sin\theta, -re^{i\beta}\sin\theta\rangle=\langle re^{i\alpha}\cos\theta+ae^{i\beta-\theta}\sin\theta, -re^{i\beta}\sin\theta\rangle,$$

and

$$\begin{aligned}
\langle A'x, x\rangle &= r(\cos^2\theta-\sin^2\theta)+ae^{i(\beta-\alpha-\theta)}\cos\theta\sin\theta \\
&= r\cos 2\theta+\frac{|a|}{2}\sin 2\theta[\cos(\beta-\alpha-\theta+\gamma)+i\sin(\beta-\alpha-\theta+\gamma)] \qquad where \quad \gamma=\arg(a) \\
&= v+iw,
\end{aligned}$$

with

$$v=r\cos 2\theta+\frac{|a|}{2}\sin 2\theta\cos(\beta-\alpha-\theta+\gamma),$$

and

$$w=\frac{|a|}{2}\sin 2\theta\sin(\beta-\alpha-\theta+\gamma).$$

So

$$(v - r\cos 2\theta)^2 + w^2 = \frac{|a|^2}{4}\sin^2 2\theta.$$

This is a family of circles Rewriting this last expression as

$$(v - r\cos\phi)^2 + w^2 = \frac{|a|^2}{4}\sin^2\phi, \qquad 0 \le \phi \le \pi.$$

and differentiating w.r.t. ϕ, we get

$$(v - r\cos\phi)r = \frac{|a|^2}{4}\cos\phi.$$

Eliminating ϕ between the last two equations, one obtains

$$\frac{v^2}{r^2 + (|a|^2/4)} + \frac{w^2}{(|a|^2/4)} = 1.$$

This is an ellipse with center at 0, minor axis a, and major axis $\sqrt{r^2 + (|a|^2/4)}$. □

The most important property of $W(A)$ is given in the so-called **Hausdorff-Toeplitz** theorem.

Theorem 1.2.5. *[21] The numerical range of an operator is convex.*

Proof. Let $\alpha = \langle Ax, x\rangle$, $\beta = \langle Ay, y\rangle$ two separate points, with $\|x\| = \|y\| = 1$. It's enough to prove that the segment containing α and β is contained in $W(A)$.

Let $F = Vect\langle x, y\rangle$ the subspace spanned by x and y. From Theorem 1.2.4 $w(A/F)$ is an ellipse, which contains the segment joining α and β and from Proposition 1.2.2, we get $W(A/F) \subset W(A)$. Then $W(A)$ contains the segment joining α and β. □

Theorem 1.2.6. *[21](**Spectral inclusion**)*

The spectrum of an operator $A \in B(\mathbb{H})$ is contained in the closure of its numerical range.

Proof. Let $\lambda \in \sigma_{app}(A)$ and let $\{f_n\}$ be a sequence of unit vectors with

$$\|(A - \lambda I)f_n\| \to 0.$$

By the Schwarz inequality,

$$|\langle (A-\lambda I)f_n, f_n\rangle| \leq \|(A-\lambda I)f_n\| \to 0.$$

so $\langle (A-\lambda I)f_n, f_n\rangle \to \lambda$. Thus $\lambda \in \overline{W(A)}$.

As the boundary of the spectrum is included in the approximate point spectrum, and the numerical range is convex, we conclude that the spectrum of A is contained in the closure of its numerical range.

□

Theorem 1.2.7. *[1] Let $A_i \in B(\mathbb{H}_i)$ for all $i = 1, 2, ..., n$. Then*

$$W(\oplus_{i=1}^{n} A_i) = \textbf{conv} \cup_{i=1}^{n} W(A_i).$$

Theorem 1.2.8. *[1] Let $\mathbb{H}_1, \mathbb{H}_2, ..., \mathbb{H}_n$ be Hilbert spaces and let $A = [A_{ij}]_{n\times n}$ be an operator matrix, whith $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$ for all $i, j = 1, 2, ..., n$. Then*

$$W(A_{ii}) \subset W(A) \qquad for \quad all \quad i = 1, 2, ..., n. \tag{1.2.1}$$

1.3 Numerical Radius

As the spectrum, the numerical range also links a set with each operator, this set called the set of valued-function of operators and this link generates a numerical function called Numerical Radius.

Définition 1.3.1. *[22] The numerical radius of an operator $A \in B(\mathbb{H})$ denoted by $\omega(.)$ and given by*

$$\omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|. \tag{1.3.1}$$

Obviously, for any $x \in \mathbb{H}$, we have

$$|\langle Ax, x\rangle| \leq \omega(A)\|x\|^2. \tag{1.3.2}$$

Example 1.3.2. *[21] Let T be the Unilateral Shift operator in C^n, defined by*

$$T(x_1, x_2, x_3, ..., x_n) = (0, x_1, x_2, x_3, ..., x_{n-1}).$$

The operator T is represented by the Matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & ... & ... & 0 \\ 1 & 0 & 0 & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & 0 \\ . & . & . & ... & ... & . \\ . & . & . & ... & ... & . \\ 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}$$

For all $x = (x_1, x_2, x_3, ..., x_n) \in C^n$, we have

$$|\langle Tx, x\rangle| \leq \sum_{i=1}^{n-1} |x_i||x_{i+1}|.$$

In order to find $\omega(T)$, we must calculate $\sup\{\sum_{i=1}^{n-1}|x_i||x_{i+1}|\}$ *over all unit vectors $x \in C^n$. To acheive our goal we use the method of Lagrange Multipliar. Let $y_i = |x_i|$ and consider the Lagrange function*

$$F(y_1, y_2, ..., y_n, \lambda) = y_1 y_2 + ... + y_{n-1} y_n - \lambda(\sum_{i=1}^{n} y_i^2 - 1). \tag{1.3.3}$$

From (1.3.3), we have

$$\begin{aligned} \frac{\partial F}{\partial y_1} &= y_2 - 2\lambda y_1 = 0, \\ \frac{\partial F}{\partial y_2} &= y_1 + y_3 - 2\lambda y_2 = 0, \\ &. \\ &. \\ \frac{\partial F}{\partial y_{n-1}} &= y_{n-2} + y_n - 2\lambda y_{n-1} = 0, \\ \frac{\partial F}{\partial y_n} &= y_{n-1} - 2\lambda y_n = 0. \end{aligned} \tag{1.3.4}$$

We can write the equation (1.3.4) in the form

$$BY = \lambda Y \qquad where \qquad Y = (y_1, y_2, ..., y_n),$$

and

$$B = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ . & . & . & . & \dots & \\ . & . & . & . & \dots & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Thus, λ is an eigenvalue of B, it is well known that

$$\lambda = \cos\frac{k\pi}{n+1}, \qquad k = 1, 2, \dots, n.$$

$\omega(T)$ is the maximum value of the expression for λ, we have

$$\omega(T) = \sup\{\cos\frac{k\pi}{n+1}, \qquad k = 1, 2, \dots, n\}.$$

Thus

$$\omega(T) = \cos\frac{\pi}{n+1}$$

It easy to see that $\omega(.)$ define a norm. That is for all $A, B \in B(\mathbb{H})$ and $\alpha \in \mathbb{C}$

- $\omega(A) \geq 0$ and $\omega(A) = 0$ if and only if $A = 0$.
- $\omega(\alpha A) = |\alpha|\omega(A)$.
- $\omega(A + B) \leq \omega(A) + \omega(B)$.

This norm is unitary invariant, means that

$$\omega(A) = \omega(U^* AU). \tag{1.3.5}$$

for all unitary operator U and it is equivalent to the usual operator norm, we present this equivalence in this Theorem.

Theorem 1.3.3. *[21] Let $A \in B(\mathbb{H})$, Then*

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1.3.6}$$

Proof. Let $\lambda = \langle Ax, x\rangle$, with $\|x\| = 1$ and let $y \in \mathbb{H}$, we have by the Schwarz inequality

$$|\lambda| = |\langle Ax, x\rangle| \leq \|Ax\| \leq \|A\|.$$

to prove the second inequality, we use the polarization identity, which may be verified by direct computation,

$$4\langle Ax, y\rangle = \langle A(x+y),(x+y)\rangle - \langle A(x-y),(x-y)\rangle + i\langle A(x+iy),(x+iy)\rangle - i\langle A(x-iy),(x-iy)\rangle.$$

Hence,

$$\begin{aligned} 4\langle Ax, y\rangle &\leq \omega(A)[\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2] \\ &= 4\omega(A)[\|x\|^2 + \|y\|^2]. \end{aligned}$$

Chosing $\|x\| = \|y\| = 1$, we get

$$4\langle Ax, y\rangle \leq 8\omega(A).$$

Thus,

$$\|A\| \leq 2\omega(A).$$

□

The next Theorem gives a useful characterization of the numerical radius.

Theorem 1.3.4. *[43] Let $A \in B(\mathbb{H})$. Then*

$$\begin{aligned} \omega(A) &= \sup_{\theta \in R} \|Re(e^{i\theta}A)\| \\ &= \sup_{\theta \in R} \|Im(e^{i\theta}A)\|. \end{aligned}$$

Proof. For all $A \in B(\mathbb{H})$, and $x \in \mathbb{H}$, we have

$$\sup_{\theta \in R} Re(e^{i\theta}\langle Ax, x\rangle) = |\langle Ax, x\rangle|.$$

Thus,

$$\sup_{\theta \in R} \|Re(e^{i\theta}A)\| = \sup_{\theta \in R} \omega(Re(e^{i\theta}A)) = \omega(A).$$

□

Theorem 1.3.5. *[9] Let $A \in B(\mathbb{H})$ be a diagonal operator matrix (or $A = \oplus_{i=1}^{n} A_{ii}$). Then*

$$\omega(A) = \max(\omega(A_{11}), \omega(A_{22}), ..., \omega(A_{nn})).$$

Theorem 1.3.6. *[29] Let $\mathbb{H}_1, \mathbb{H}_2, ..., \mathbb{H}_n$ be Hilbert spaces and let $A = [A_{ij}]_{n\times n}$ be an operator matrix, whith $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$ for all $i, j = 1, 2, ..., n$ and let $T = [\|A_{ij}\|]_{n\times n}$. Then*

$$\omega(A) \leq \omega(T) = \omega([\|A_{ij}\|]_{n\times n}). \tag{1.3.7}$$

Theorem 1.3.7. *[21] Let $A \in B(\mathbb{H})$. If $\omega(A) = \|A\|$. Then*

$$r(A) = \|A\|.$$

Proof. Let $\omega(A) = \|A\| = 1$. Then there is a sequence of unit vectors $\{f_n\}$ such that $\langle Af_n, f_n\rangle \to \lambda \in W(A)$, and $|\lambda| = 1$. From the inequality

$$|\langle Af_n, f_n\rangle| \leq \|Af_n\| \leq 1,$$

we have $\|f_n\| \to 1$. Hence

$$\|(A - \lambda I)f_n\|^2 = \|Af_n\|^2 - \langle Af_n, \lambda f_n\rangle - \langle \lambda f_n, Af_n\rangle + \|f_n\|^2 \to 0.$$

Thus, $\lambda \in \sigma_{app}(A)$ and $r(A) = 1$. □

Theorem 1.3.8. *[21] Let $A \in B(\mathbb{H})$. If $R(A) \perp R(A^*)$. Then*

$$\omega(A) = \frac{\|A\|}{2}.$$

Proof. Let $x = x_1 + x_2$ be a unit vector in $\mathbb{H} = N(A) \oplus \overline{R(A^*)}$, where $x_1 \in N(A)$ and $x_2 \in \overline{R(A^*)}$. Thus we have,

$$|\langle Ax, x\rangle| = |\langle A(x_1 + x_2), x_1 + x_2\rangle| = |\langle Ax_2, x_1\rangle|.$$

Since, $Ax_1 = 0$ and $\langle Ax_2, x_2\rangle = \langle x_2, A^* x_2\rangle = 0$, we get

$$|\langle Ax, x\rangle| \leq \|A\|\|x_1\|\|x_2\| \leq \frac{\|A\|}{2}[\|x_1\| + \|x_2\|] = \frac{\|A\|}{2},$$

then,

$$\frac{\|A\|}{2} \leq \omega(A) \leq \frac{\|A\|}{2}.$$

□

Theorem 1.3.9. *[21] Let $A \in B(\mathbb{H})$. If A is idempotent and $\omega(A) \le 1$, then A is an orthogonal projection.*

Proof. To prove this theorem it is sufficient to prove that A is null on $R(A)^{\perp}$. Let $x \in R(A)^{\perp}$ and $y = Ax$. Then for $t \ge 0$, we have

$$A(x + ty) = y + tA^2 y = (1 + t)y.$$

As $x \perp y$, we have

$$\begin{aligned} \langle A(x+ty), x+ty\rangle &= \langle (1+t)y, x+ty\rangle \\ &= \langle (1+t)y, ty\rangle \\ &= (1+t)t\|y\|^2. \end{aligned}$$

On the other hand, we have

$$(1+t)t\|y\|^2 = |\langle A(x+ty), x+ty\rangle| \le \omega(A)\|x+ty\|^2 = \|x\|^2 + t\|y\|^2$$

because $\omega(A) \le 1$. Thus

$$t\|y\|^2 \le \|x\|^2.$$

Since t is arbitrary, we conclude that $\|y\| = 0$ and $A = 0$ on $R(A)^{\perp}$. □

Theorem 1.3.10. *[19, 38] Let $A \in B(\mathbb{H})$, Then*

$$\omega(A^m) \le \omega^m(A), \qquad for \quad all \quad m = 1, 2, \ldots, \tag{1.3.8}$$

or equivalently

$$\omega(A) \le 1 \quad implie \quad \omega(A^m) \le 1, \qquad for \quad all \quad m = 1, 2, \ldots, \tag{1.3.9}$$

Proof. First we prove the equivalence between (1.3.8), and (1.3.9), clearly (1.3.8) implies (1.3.9). Conversely, suppose that (1.3.9) hold. Assume that $A \neq 0$ because if $A = 0$ there is nothing to prove. As $A \neq 0$, consider the operator $B = \dfrac{A}{\omega(A)}$, by Proposition 1.2.2 (1),

$\omega(B)=1$. Hence $\omega(B^m)=\omega(\frac{A^m}{\omega^m(A)})\leq 1$. So by Proposition 1.2.2 (1) also we obtain (1.3.8). let now prove the theorem , let m be a nonegative integer, note the two polynomial identities

$$1-z^m=\prod_{k=1}^{m}(1-r_k z), \tag{1.3.10}$$

and

$$1=\frac{1}{m}\sum_{j=1}^{m}\prod_{k=1,k\neq j}^{m}(1-r_k z), \tag{1.3.11}$$

where

$$r_k=e^{2\pi i\frac{k}{m}}.$$

The two previous equations are correct when z is replaced by any operator $B\in B(\mathbb{H})$. Now, for an arbitrary unit vector $x\in\mathbb{H}$ define the vectors,

$$x_j=\left[\prod_{k=1,k\neq j}^{m}(1-r_kB)x\right],\qquad j=1,2,...,m.$$

The following list of equations was found by using the two equations (1.3.10) and (1.3.11),

$$\begin{aligned}
\frac{1}{m}\sum_{j=1}^{m}\|x_j\|^2\left[1-r_j\langle\frac{Bx_j}{x_j},\frac{x_j}{x_j}\rangle\right] &= \frac{1}{m}\sum_{j=1}^{m}\langle(1-r_jB)x_j,x_j\rangle\\
&= \frac{1}{m}\sum_{j=1}^{m}\left\langle\left[\prod_{k=1}^{m}(1-r_kB)x\right],x_j\right\rangle\\
&= \frac{1}{m}\sum_{j=1}^{m}\left\langle(1-B^m)x,x_j\right\rangle\\
&= \langle(1-B^m)x,\frac{1}{m}\sum_{j=1}^{m}x_j\rangle\\
&= \left\langle(1-B^m)x,\frac{1}{m}\sum_{j=1}^{m}\left[\prod_{k=1,k\neq j}^{m}(1-r_kB)\right]x\right\rangle\\
&= \langle(1-B^m)x,x\rangle\\
&= 1-\langle B^mx,x\rangle.
\end{aligned}$$

In particular, setting

$$B=Ae^{i\theta},\qquad \theta\in\mathbb{R}.$$

Thus

$$\frac{1}{m}\sum_{j=1}^{m}\|x_j\|^2\left[1-r_je^{i\theta}\langle\frac{Ax_j}{x_j},\frac{x_j}{x_j}\rangle\right]=1-e^{im\theta}\langle A^mx,x\rangle.$$

As $\omega(A)\leq 1$, the real part of each term in the right-hand side of the previeus equation is positive, so that for any unit vector x and real θ, we have

$$Re(1-e^{im\theta}\langle A^mx,x\rangle)\geq 0,$$

thus,

$$|\langle A^mx,x\rangle|\leq 1.$$

Then (1.3.8) follows. □

1.4 Moore-Penrose inverse

Définition 1.4.1. *Let $A\in B(\mathbb{H})$, the Moore-Penrose Inverse of A denoted by $A^+\in B(\mathbb{H})$ is the unique solution of the following set of equations*

$$AA^+A=A, \tag{1.4.1}$$

$$A^+AA^+=A^+, \tag{1.4.2}$$

$$AA^+=(AA^+)^*,$$

$$A^+A=(A^+A)^*.$$

Notice that A^+ exists if and only if $R(A)$ is closed [23]. In this case AA^+ and A^+A are the orthogonal projections onto $R(A)$ and $R(A^*)$ respectively. The following assertions will be used in the last chapter of this thesis.

$$(A^*A)^+=A^+(A^+)^* \tag{1.4.3}$$

$$A^+=(A^*A)^+A^*=A^*(AA^*)^+ \tag{1.4.4}$$

$$A^*=A^+AA^*=A^*AA^+ \tag{1.4.5}$$

If A is a partial isometry, then $A^{+} = A^{*}$.

Many refinments and generalisations of the equation (0.0.1) have been given by many mathematicians. In the following section we give some important improvements of the equation (0.0.1).

1.5 Some improved numerical radius inequalities

In (2003) Kittaneh [31] refines the second inequality in (0.0.1) as follows.

Theorem 1.5.1. *[31] Let $A \in B(\mathbb{H})$. Then*

$$\omega(A) \leq \frac{1}{2}[\|A\| + \|A^2\|^{\frac{1}{2}}].$$

Proof. The author in [31] need the following Lemmas to prove his result, the first and the second Lemmas contains a Mixed Schwarz inequality, the third Lemma contains a norm inequality which is equivalent to some Löwer-Heinz type inequality and the fourth Lemma contains a norm inequality for sums of positive operators that is sharper than the triangle inequality.

Lemma 1.5.2. *[22] Let $A \in B(\mathbb{H})$ be a positive operator and let $x, y \in \mathbb{H}$. Then $\langle Ax, x\rangle$ defines an inner product, it follows that A satisfies the schwartz-like inequality*

$$|\langle Ax, y\rangle|^2 \leq \langle Ax, x\rangle\langle Ay, y\rangle. \tag{1.5.1}$$

Lemma 1.5.3. *[22] Let $A \in B(\mathbb{H})$ and let $x, y \in \mathbb{H}$. Then*

$$|\langle Ax, y\rangle| \leq \langle |A|x, x\rangle^{\frac{1}{2}}\langle |A^*|y, y\rangle^{\frac{1}{2}}. \tag{1.5.2}$$

Proof. Let $A = U|A|$ be the polar decomposition, whith $|A| = (A^*A)^{\frac{1}{2}}$. Since

$$AA^* = U|A||A|U^* = U|A|^2U^* = |A^*|^2,$$

it follows that,

$$|A^*| = U|A|U^*.$$

From Lemma 1.5.2, we have

$$\begin{aligned}
|\langle Ax, y\rangle|^2 &= |\langle U|A|x, y\rangle|^2 \\
&= |\langle |A|x, U^* y\rangle|^2 \\
&\leq \langle |A|x, x\rangle \langle |A|U^* y, U^* y\rangle \\
&= \langle |A|x, x\rangle \langle U|A|U^* y, y\rangle \\
&= \langle |A|x, x\rangle \langle |A^*|y, y\rangle.
\end{aligned}$$

□

Lemma 1.5.4. *[17] Let $A, B \in B(\mathbb{H})$ be positive operators. Then*

$$\|A^{\frac{1}{2}} B^{\frac{1}{2}}\| \leq \|AB\|^{\frac{1}{2}}.$$

Lemma 1.5.5. *[33] Let $A, B \in B(\mathbb{H})$ be positive operators. Then*

$$\|A+B\| \leq \frac{1}{2}[\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|A^{\frac{1}{2}} B^{\frac{1}{2}}\|^2}]$$

Proof of Theorem

Let $x \in \mathbb{H}$, by Lemma 1.5.3, and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
|\langle Ax, x\rangle| &\leq \langle |A|x, x\rangle^{\frac{1}{2}} \langle |A^*|x, x\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2}[\langle |A|x, x\rangle + \langle |A^*|x, x\rangle] \\
&= \frac{1}{2}\langle (|A| + |A^*|)x, x\rangle
\end{aligned}$$

Thus,

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\| \tag{1.5.3}$$

As $|A|$ and $|A^*|$ are positive operator and by Lemmas 1.5.4 and 1.5.5, we have

$$\||A|\| = \||A^*|\| = \|A\| \qquad and \qquad \||A||A^*|\| = \|A^2\|.$$

we have also,

$$\||A| + |A^*|\| \leq \|A\| + \|A^2\|^{\frac{1}{2}}. \tag{1.5.4}$$

The desired inequality in Theorem 1.5.1 now follows from (1.5.3) and (1.5.4).

□

The author in [31] gave an immediate consequence of Theorem 1.5.1 when $A^2 = 0$.

Corollary 1.5.1. *[31] Let $A \in B(\mathbb{H})$, is such that $A^2 = 0$. Then*

$$\omega(A) = \frac{1}{2}\|A\|. \tag{1.5.5}$$

The Aluthge transform was introduced by Aluthge in [8]. The idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one some spectral properties. It is well known the following properties of $\tilde{T}$:

(i) $||\tilde{A}|| \leq ||A||$,

(ii) $w(\tilde{A}) \leq w(A)$,

(iii) $r(\tilde{A}) = r(A)$.

The Aluthge transform has received much attentions by many mathematicians in recent years, among them is Takeaki Yamazaki [43], who proved the following important result which refines the result of Kittaneh [31].

Theorem 1.5.6. *Let $A \in B(\mathbb{H})$. Then*

$$\omega(A) \leq \frac{1}{2}[\|A\| + \omega(\tilde{A})]. \tag{1.5.6}$$

Proof. Let $A = U|A|$ be the polar decomposition of A. Then by the Generalised Polarisation identity, we have

$$\begin{aligned}
\langle e^{i\theta} Ax, x\rangle &= \langle e^{\mathrm{i}\theta}|A|x, U^* x\rangle \\
&= \frac{1}{4}(\langle |A|(e^{i\theta} + U^*)x, (e^{i\theta} + U^*)x\rangle - \langle |A|(e^{i\theta} - U^*)x, (e^{l\theta} - U^*)x\rangle) \\
&+ \frac{i}{4}(\langle |A|(e^{i\theta} + iU^*)x, (e^{i\theta} + iU^{\cdot})x\rangle - \langle |A|(e^{i\theta} - iU^*)x, (e^{i\theta} - iU^*)x\rangle).
\end{aligned}$$

Thus,

$$\begin{aligned}
Re\langle e^{i\theta}Ax,x\rangle &= \frac{1}{4}(\langle(e^{-i\theta}+U)|A|(e^{i\theta}+U^*)x,x\rangle-\langle(e^{-i\theta}-U)|A|(e^{i\theta}-U^*)x,x\rangle)\\
&\leq \frac{1}{4}\langle(e^{-i\theta}+U)|A|(e^{i\theta}+U^*)x,x\rangle\\
&\leq \frac{1}{4}||(e^{-i\theta}+U)|A|(e^{i\theta}+U^*)||\\
&= \frac{1}{4}\||A|^{\frac{1}{2}}(e^{i\theta}+U^*)(e^{-i\theta}+U)|A|^{\frac{1}{2}}\| \qquad (by \quad ||X^*X||=||XX^*||)\\
&= \frac{1}{4}||2|A|+e^{i\theta}\tilde{A}+e^{-i\theta}(\tilde{A})^*||\\
&= \frac{1}{2}|||A|+Re(e^{i\theta}\tilde{A})||\\
&\leq \frac{1}{2}||A||+\frac{1}{2}||Re(e^{i\theta}\tilde{A})||\\
&\leq \frac{1}{2}||A||+\frac{1}{2}w(\tilde{A})\\
&\quad (by \quad Theorem 1.3.4).
\end{aligned}$$

Since,

$$\begin{aligned}
|\langle Ax,x\rangle| &= \sup_{\theta\in R} Re(e^{i\theta}\langle Ax,x\rangle)\\
&\leq \frac{1}{2}||A||+\frac{1}{2}w(\tilde{A}),
\end{aligned}$$

then

$$\omega(A)\leq\frac{1}{2}||A||+\frac{1}{2}w(\tilde{A}).$$

□

Remark 1.5.7. *The result of Yamazaki [43] is sharper than the result of Kittaneh [31], this follows from the Heinz inequality [24]*

$$||A^rXB^r||\leq||AXB||^r||X||^{1-r} \qquad for \quad A,B\geq 0 \quad and \quad r\in[0,1].$$

we have,

$$w(\tilde{T})\leq||\tilde{T}||=|||T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}||\leq|||T|U|T|||^{\frac{1}{2}}||U||^{\frac{1}{2}}=||T^2||^{\frac{1}{2}}.$$

Abu-Omar and Kittaneh [2] used the generalized Aluthge transform to improve the inequality 1.5.6, they proved that

$$\omega(A) \leq \frac{1}{2}(\|A\| + \omega(\tilde{A}_t)). \tag{1.5.7}$$

The inequality (0.0.1) was also refined by Kittaneh [34], he proved that

$$\frac{1}{4}\||A|^2 + |A^*|^2\| \leq \omega^2(A) \leq \frac{1}{2}\||A|^2 + |A^*|^2\| \tag{1.5.8}$$

If $A = B + iC$ is the cartesian decomposition of A, then

$$|A|^2 + |A^*|^2 = 2(|B|^2 + |C|^2).$$

Thus, the inequalities in (1.5.8) can be written as

$$\frac{1}{2}\||B|^2 + |C|^2\| \leq \omega^2(A) \leq \||B|^2 + |C|^2\|.$$

This inequalitis was also reformulated and generalized in [14].

In (2008) Dragomir [13] used Buzano inequality to improve (0.0.1) as follows

$$\omega^2(A) \leq \frac{1}{2}[\|A\|^2 + \omega(A^2)]. \tag{1.5.9}$$

This result was also recently generalized by M, Sattari, M. S. Moslehian, and T. Yamazaki [41] in (2015), they proved that, for all $r \geq 1$

$$\omega^{2r}(A) \leq \frac{1}{2}(\omega^r(A^2) + \|A\|^{2r}).$$

CHAPTER 2
NUMERICAL RADIUS INEQUALITIES FOR OPERATOR MATRICES

The $n \times n$ operator matrices $A = [A_{ij}]_{n\times n}$ are regarded as operators on $\oplus_{i=1}^{n}\mathbb{H}_i$ (the direct sum of the complex Hilbert spaces $\mathbb{H}_1, \mathbb{H}_2, ..., \mathbb{H}_n$), where $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$ for $i, j = 1, 2, ..., n$. Here, $B(\mathbb{H}_j, \mathbb{H}_i)$ denotes the space of all bounded linear operators from $\mathbb{H}_j$ to $\mathbb{H}_i$. When $i = j$, we simply write $B(\mathbb{H}_i)$ for $B(\mathbb{H}_i, \mathbb{H}_i)$. Operator matrices played important role in operator theory. This Chapter is about new numerical radius inequalities for arbitrary $n \times n$ operator matrices. Related to this direction Hou, and Du [29] have been proved that

$$\omega([A_{ij}]_{n\times n}) \leq \omega([\|A_{ij}\|]_{n\times n}).$$

After that Abu-Omar, and Kittaneh improved this inequality as follows

$$\omega([A_{ij}]_{n\times n}) \leq \omega([a_{ij}]_{n\times n}),$$

where

$$a_{ij} = \omega\left(\begin{bmatrix} 0 & A_{ij} \\ A_{ji} & 0 \end{bmatrix}\right) \quad for \quad i, j = 1, 2, ..., n.$$

and

$$a_{ii} = \omega(A_{ii}) \quad for \quad i = 1, 2, ..., n.$$

In the first Section of this Chapter, we present numerical radius inequalities for $n \times n$ operator matrices with a single nonzero row. Then we use these inequalities to establish numerical radius inequalities for arbitrary $n \times n$ operator matrices. Moreover, we give numerical radius inequalities for 3×3 operator matrices that involve the numerical radii of the skew diagonal (or the secondary diagonal) parts of 2×2 operator matrices. Our new numerical radius inequalities for $n \times n$ operator matrices are natural generalizations of some of the numerical radius inequalities for 2×2 operator matrices given in [25], [26], [42], and references therein. In the second section, we present numerical radius inequalities for the skew diagonal (or the secondary diagonal) parts of 3×3 operator matrices, including an inequality involving the generalized Aluthge transform. Related numerical radius inequalities for 2×2 operator matrices can be found in [25], [42] and refrences therein.

2.1 Numerical radius inequalities for $n \times n$ operator matrices

To present our results, we need the following lemmas.

Lemma 2.1.1. *[28, p. 44] Let $A = [a_{ij}]_{n\times n}$ be an $n \times n$ matrix such that $a_{ij} \geq 0$ for all $i, j = 1, 2, \ldots, n$. Then*

$$\omega(A) = \frac{1}{2} r([a_{ij} + a_{ji}]).$$

Lemma 2.1.2. *[29] Let $\mathbb{H}_1, \mathbb{H}_2, \ldots, \mathbb{H}_n$ be Hilbert spaces, and let $A = [A_{ij}]_{n\times n}$ be an $n \times n$ operator matrix with $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$r(A) \leq r([\| A_{ij} \|]_{n\times n}).$$

Lemma 2.1.3. *[6] Let $A = [A_{ij}]_{n\times n}$ be $n \times n$ operator matrix with $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$\omega(A) \leq \omega([a_{ij}]),$$

where

$$a_{ij} = \omega\left(\begin{bmatrix} 0 & A_{ij} \\ A_{ji} & 0 \end{bmatrix}\right) \qquad for \quad i, j = 1, 2, \ldots, n.$$

Note that $a_{ii} = \omega(A_{ii})$ for $i = 1, 2, \ldots, n$ and the matrix $[a_{ij}]$ is real symmetric.

Our first result in this section can be stated as follows.

Theorem 2.1.4. *Let* $A_1 \in B(\mathbb{H}_1), A_2 \in B(\mathbb{H}_2,\mathbb{H}_1), ..., A_n \in B(\mathbb{H}_n,\mathbb{H}_1)$. *Then*

$$\omega\left(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(\omega(A_1) + \sqrt{\omega^2(A_1) + \sum_{j=2}^{n} \|A_j\|^2}\right).$$

Proof. Applying Lemmas 2.1.1, 2.1.3 and the identity (1.5.5), we obtain

$$\begin{aligned}
\omega\left(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) &\leq \omega\left(\begin{bmatrix} \omega(A_1) & \omega\left(\begin{bmatrix} 0 & A_2 \\ 0 & 0 \end{bmatrix}\right) & \cdots & \omega\left(\begin{bmatrix} 0 & A_n \\ 0 & 0 \end{bmatrix}\right) \\ \omega\left(\begin{bmatrix} 0 & 0 \\ A_2 & 0 \end{bmatrix}\right) & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \omega\left(\begin{bmatrix} 0 & 0 \\ A_n & 0 \end{bmatrix}\right) & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \omega\left(\begin{bmatrix} \omega(A_1) & \frac{\|A_2\|}{2} & \cdots & \frac{\|A_n\|}{2} \\ \frac{\|A_2\|}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\|A_n\|}{2} & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{1}{2} r\left(\begin{bmatrix} 2\omega(A_1) & \|A_2\| & \cdots & \|A_n\| \\ \|A_2\| & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \|A_n\| & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\left(\omega(A_1) + \sqrt{\omega^2(A_1) + \sum_{j=2}^{n} \|A_j\|^2}\right).
\end{aligned}$$

□

Based on Theorem 2.1.4, we obtain the following numerical radius inequality for arbitrary $n \times n$ operator matrices.

Corollary 2.1.1. *Let* $A = [A_{ij}]_{n\times n}$ *be an* $n \times n$ *operator matrix with* $A_{ij} \in B(\mathbb{H}_j,\mathbb{H}_i)$. *Then*

$$\omega(A) \leq \frac{1}{2}\sum_{i=1}^{n}\left(\omega(A_{ii}) + \sqrt{\omega^2(A_{ii}) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \|A_{ij}\|^2}\right).$$

Proof. For $i = 2, ..., n$, let U_i be the $n \times n$ permutation operator matrix obtained by interchanging the first and the i^{th} rows of the identity operator matrix. Then U_i is a unitary operator, and so by the triangle inequality and the identity (1.3.5), we have

$$
\begin{aligned}
\omega(A) &\leq \omega\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad + \cdots + \omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}\right) \\
&= \omega\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + \omega\left(U_2^* \begin{bmatrix} A_{22} & A_{21} & \cdots & A_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} U_2\right) \\
&\quad + \cdots + \omega\left(U_n^* \begin{bmatrix} A_{nn} & A_{n2} & \cdots & A_{n1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} U_n\right) \\
&= \omega\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} A_{22} & A_{21} & \cdots & A_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad + \cdots + \omega\left(\begin{bmatrix} A_{nn} & A_{n2} & \cdots & A_{n1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right).
\end{aligned}
$$

Now, using Theorem 2.1.4, we have

$$
\omega(A) \leq \frac{1}{2} \sum_{i=1}^{n} \left(\omega(A_{ii}) + \sqrt{\omega^2(A_{ii}) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \|A_{ij}\|^2} \right).
$$

$\square$

Other related results are given as follows.

Theorem 2.1.5. *Let* $A_1 \in B(\mathbb{H}_1), A_2 \in B(\mathbb{H}_2, \mathbb{H}_1), \cdots, A_n \in B(\mathbb{H}_n, \mathbb{H}_1)$. *Then*

$$\omega\left(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \le \frac{1}{2}\left(\|A_1\| + \sqrt{\|\sum_{j=1}^{n} A_j A_j^*\|^2}\right).$$

Proof. Let $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. Then for every $\theta \in \mathbb{R}$, we have

$$\begin{aligned} \|Re(e^{i\theta}A)\| &= r(Re(e^{i\theta}A)) \\ &= \frac{1}{2} r\left(e^{i\theta}\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + e^{-i\theta}\begin{bmatrix} A_1^* & 0 & \cdots & 0 \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\ &= \frac{1}{2} r\left(\begin{bmatrix} e^{i\theta}A_1 + e^{-i\theta}A_1^* & e^{i\theta}A_2 & \cdots & e^{i\theta}A_n \\ e^{-i\theta}A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ e^{-i\theta}A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\ &= \frac{1}{2} r\left(\begin{bmatrix} A_1^* & e^{i\theta}I & \cdots & 0 \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ A_1 & A_2 & \cdots & A_n \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right). \end{aligned}$$

It follows by the commutativity property of the spectral radius and Lemma 2.1.2 that

$$
\begin{aligned}
\|Re(e^{i\theta}A)\| &= \frac{1}{2}r\left(\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ A_1 & A_2 & \cdots & A_n \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} A_1^* & e^{i\theta}I & \cdots & 0 \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}r\left(\begin{bmatrix} e^{-i\theta}A_1^* & I & \cdots & 0 \\ \sum_{j=1}^n A_jA_j^* & e^{i\theta}A_1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\le \frac{1}{2}r\left(\begin{bmatrix} \|A_1^*\| & 1 & \cdots & 0 \\ \left\|\sum_{j=1}^n A_jA_j^*\right\| & \|A_1\| & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\left(\|A_1\| + \sqrt{\left\|\sum_{j=1}^n A_jA_j^*\right\|}\right).
\end{aligned}
$$

Now, using Lemma 1.3.4, we get

$$
\begin{aligned}
\omega(A) &= \sup_{\theta\in\mathbb{R}}\|Re(e^{i\theta}A)\| \\
&\le \frac{1}{2}\left(\|A_1\| + \sqrt{\left\|\sum_{j=1}^n A_jA_j^*\right\|}\right).
\end{aligned}
$$

□

Using Theorem 2.1.5 and an argument similar to that used in proof of Corollary 2.1.1, we have the following corollary.

Corollary 2.1.2. *Let $A = [A_{ij}]_{n\times n}$ be an $n \times n$ operator matrix with $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$
\omega(A) \le \frac{1}{2}\sum_{i=1}^n\left(\|A_{ii}\| + \sqrt{\|A_{ii}A_{ii}^* + \sum_{\substack{j=1 \\ j\neq i}}^n A_{ij}A_{ij}^*\|}\right).
$$

Theorem 2.1.6. *Let $A = [A_{ij}]_{n\times n}$ be an $n \times n$ operator matrix with $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$
\omega(A) \le \max(\omega(A_{11}), \omega(A_{22}), \cdots, \omega(A_{nn})) + \frac{1}{2}\sum_{i=1}^n\sqrt{\|\sum_{\substack{j=1 \\ j\neq i}}^n A_{ij}A_{ij}^*\|}.
$$

Proof. We have

$$
\begin{aligned}
\begin{bmatrix} 0 & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}^2
&= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ A_{21} & 0 & A_{23} & \cdots & A_{2n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^2 \\
&= \\
&. \\
&. \\
&. \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ A_{n1} & A_{n2} & \cdots & A_{n(n-1)} & 0 \end{bmatrix}^2 \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.
\end{aligned}
$$

By the triangle inequality and the identity (1.5.5), we have

$$
\begin{aligned}
\omega(A) &\leq \omega\left(\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{nn} \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad + \cdots + \omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ A_{n1} & A_{n2} & \cdots & A_{n(n-1)} & 0 \end{bmatrix}\right) \\
&= \max(\omega(A_{11}), \omega(A_{22}), \cdots, \omega(A_{nn})) + \frac{1}{2}\left\| \begin{bmatrix} 0 & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\| \\
&\quad + \frac{1}{2}\left\| \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ A_{21} & 0 & A_{23} & \cdots & A_{2n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \right\| + \cdots + \frac{1}{2}\left\| \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ A_{n1} & T_{n2} & \cdots & A_{n(n-1)} & 0 \end{bmatrix} \right\| \\
&= \max(\omega(A_{11}), \omega(A_{22}), ..., \omega(A_{nn})) + \frac{1}{2}\sqrt{\left\| \sum_{\substack{j=1 \\ j\neq 1}}^{n} A_{1j}A_{1j}^{*} \right\|} \\
&\quad + \frac{1}{2}\sqrt{\left\| \sum_{\substack{j=1 \\ j\neq 2}}^{n} A_{2j}A_{2j}^{*} \right\|} + \cdots + \frac{1}{2}\sqrt{\left\| \sum_{\substack{j=1 \\ j\neq n}}^{n} A_{nj}A_{nj}^{*} \right\|} \\
&= \max(\omega(A_{11}), \omega(A_{22}), ..., \omega(A_{nn})) + \frac{1}{2}\sum_{i=1}^{n}\sqrt{\| \sum_{\substack{j=1 \\ j\neq i}}^{n} A_{ij}A_{ij}^{*} \|}.
\end{aligned}
$$

□

The proof of the following theorem has been pointed out to us by Amer Abu-Omar. For simplicity, we state it for 3×3 operator matrices.

Theorem 2.1.7. *Let $A = [A_{ij}]_{3\times 3}$ be a 3×3 operator matrix with $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$\omega(A) \leq \frac{1}{2}\left(\sum_{i=1}^{3} a_{ii} + \sum_{1\leq i<j\leq 3}\sqrt{\left(\frac{a_{ii}-a_{jj}}{2}\right)^2 + 4a_{ij}^2}\right),$$

where,

$$a_{ij} = \omega\left(\begin{bmatrix} 0 & A_{ij} \\ A_{ji} & 0 \end{bmatrix}\right) \qquad for \quad i,j = 1,2,\ldots,n$$

and

$$a_{ii} = \omega(A_{ii}) \quad for \quad i = 1,2,\ldots,n.$$

Proof. By the triangle inequality and Lemma 2.1.3, we have

$$\begin{aligned}
\omega(A) &\leq \omega\left(\begin{bmatrix} \frac{A_{11}}{2} & A_{12} & 0 \\ A_{21} & \frac{A_{22}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} \frac{A_{11}}{2} & 0 & A_{13} \\ 0 & 0 & 0 \\ A_{31} & 0 & \frac{A_{33}}{2} \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{A_{22}}{2} & A_{23} \\ 0 & A_{32} & \frac{A_{33}}{2} \end{bmatrix}\right) \\
&\leq \omega\left(\begin{bmatrix} \frac{a_{11}}{2} & a_{12} & 0 \\ a_{21} & \frac{a_{22}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} \frac{a_{11}}{2} & 0 & a_{13} \\ 0 & 0 & 0 \\ a_{31} & 0 & \frac{a_{33}}{2} \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{a_{22}}{2} & a_{23} \\ 0 & a_{32} & \frac{a_{33}}{2} \end{bmatrix}\right) \\
&= r\left(\begin{bmatrix} \frac{a_{11}}{2} & a_{12} & 0 \\ a_{21} & \frac{a_{22}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) + r\left(\begin{bmatrix} \frac{a_{11}}{2} & 0 & a_{13} \\ 0 & 0 & 0 \\ a_{31} & 0 & \frac{a_{33}}{2} \end{bmatrix}\right) + r\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{a_{22}}{2} & a_{23} \\ 0 & a_{32} & \frac{a_{33}}{2} \end{bmatrix}\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{3} a_{ii} + \sum_{1\leq i<j\leq 3}\sqrt{\left(\frac{a_{ii}-a_{jj}}{2}\right)^2 + at_{ij}^2}\right) \quad \text{(recall that } a_{ij} = a_{ji}).
\end{aligned}$$

□

With the same notations used in Theorem 2.1.7, we have the following related result.

Theorem 2.1.8. *Let $A = [A_{ij}]_{3\times 3}$ be a 3×3 operator matrix with $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$. Then*

$$\omega(A) \leq \max(a_{11}, a_{23}) + \max(a_{22}, a_{13}) + \max(a_{33}, a_{12}).$$

Here,

$$a_{ij} = \omega\left(\begin{bmatrix} 0 & A_{ij} \\ A_{ji} & 0 \end{bmatrix}\right) \qquad for \quad i,j = 1,2,\ldots,n$$

and

$$a_{ii} = \omega(A_{ii}) \quad for \quad i = 1,2,\ldots,n.$$

Proof. Let $U = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$. Then U is a unitary operator, and so by the triangle inequality and the identity (1.3.5), we have

$$\begin{aligned}
\omega(A) &\leq \omega\left(\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & 0 \end{bmatrix}\right) \\
&= \omega\left(\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix}\right) + \omega\left(U^* \begin{bmatrix} 0 & A_{13} & 0 \\ A_{31} & 0 & 0 \\ 0 & 0 & A_{22} \end{bmatrix} U\right) \\
&= \omega\left(\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & A_{13} & 0 \\ A_{31} & 0 & 0 \\ 0 & 0 & A_{22} \end{bmatrix}\right) \\
&= \max(a_{11}, a_{23}) + \max(a_{22}, a_{13}) + \max(a_{33}, a_{12}).
\end{aligned}$$

□

Concerning Theorems 2.1.7 and 2.1.8, we note that several bounds for a_{ij} have been given in [25, 26, 42] and references therein. Moreover, when $\mathbb{H}_i = \mathbb{H}_j$, it has been shown in [6] that if A_{ij} and A_{ji} are positive, then $a_{ij} = \frac{\|A_{ij} + A_{ji}\|}{2}$.

We conclude by remarking that the numerical radius inequalities presented in this chapter are sharp. Moreover, by employing similar analysis to different partitions of operator matrices, it is possible to obtain further numerical radius inequalities.

2.2 Numerical radius inequalities for the skew diagonal parts of 3×3 operator matrices

Our first result in this section can be stated as follows.

Theorem 2.2.1. *Let $A, B, C \in B(\mathbb{H})$. Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \leq \max\left(\frac{\||A^*| + |C|\|^{\frac{1}{2}} \||A| + |C^*|\|^{\frac{1}{2}}}{2}, \omega(B)\right).$$

Proof. Let $T = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a unit vector in $\mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H}$ (i.e., $\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 = 1$). Then

$$\begin{aligned}
|\langle Tx, x\rangle| &= |\langle Ax_3, x_1\rangle + \langle Bx_2, x_2\rangle + \langle Cx_1, x_3\rangle| \\
&\leq |\langle Ax_3, x_1\rangle| + |\langle Bx_2, x_2\rangle| + |\langle Cx_1, x_3\rangle| \\
&\leq \langle |A|x_3, x_3\rangle^{\frac{1}{2}} \langle |A^*|x_1, x_1\rangle^{\frac{1}{2}} + \langle |C|x_1, x_1\rangle^{\frac{1}{2}} \langle |C^*|x_3, x_3\rangle^{\frac{1}{2}} + \omega(B)\|x_2\|^2 \\
&\qquad \text{(by Lemma 1.5.3)} \\
&\leq [\langle |A^*|x_1, x_1\rangle + \langle |C|x_1, x_1\rangle]^{\frac{1}{2}} [\langle |A|x_3, x_3\rangle + \langle |C^*|x_3, x_3\rangle]^{\frac{1}{2}} + \omega(B)\|x_2\|^2 \\
&\qquad \text{(by the Cauchy-Schwarz inequality)} \\
&= \langle (|A^*| + |C|)x_1, x_1\rangle^{\frac{1}{2}} \langle (|A| + |C^*|)x_3, x_3\rangle^{\frac{1}{2}} + \omega(B)\|x_2\|^2 \\
&\leq \||A^*| + |C|\|^{\frac{1}{2}} \||A| + |C^*|\|^{\frac{1}{2}} \|x_1\|\|x_3\| + \omega(B)\|x_2\|^2 \\
&\leq \||A^*| + |C|\|^{\frac{1}{2}} \||A| + |C^*|\|^{\frac{1}{2}} \frac{\|x_1\|^2 + \|x_3\|^2}{2} + \omega(B)\|x_2\|^2 \\
&\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&\leq \max\left(\frac{\||A^*| + |C|\|^{\frac{1}{2}} \||A| + |C^*|\|^{\frac{1}{2}}}{2}, \omega(B) \right).
\end{aligned}$$

Hence,

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \max\left(\frac{\||A^*| + |C|\|^{\frac{1}{2}} \||A| + |C^*|\|^{\frac{1}{2}}}{2}, \omega(B) \right).$$

□

As an application of Theorem 2.2.1, we have the following numerical radius inequality involving positive operators and self-adjoint operators.

Corollary 2.2.1. *Let $A, B, C \in B(\mathbb{H})$ such that A, C be positive operators and B is self-adjoint operator. Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix} \right) = \max\left(\frac{\|A + C\|}{2}, \|B\| \right).$$

Proof. Let $T = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$. Since A, C be positive operators and B is self-adjoint operator, it follows from Theorem 2.2.1 that

$$\omega(T) \leq \max\left(\frac{\|A+C\|}{2}, \|B\|\right).$$

On the other hand, it follows from [36, Theorem 2.1] that

$$\omega(T) \geq \|Re(T)\| = \left\| \begin{bmatrix} 0 & 0 & \frac{A+C}{2} \\ 0 & B & 0 \\ \frac{A+C}{2} & 0 & 0 \end{bmatrix} \right\| = \max\left(\frac{\|A+C\|}{2}, \|B\|\right).$$

Hence, $\omega(T) = \max\left(\frac{\|A+C\|}{2}, \|B\|\right)$. □

The 2×2 operator matrix version of the folowing Lemma can be founed in [26].

Lemma 2.2.2. *Let $A, B, C \in B(\mathbb{H})$ and $\theta \in \mathbb{R}$. Then*

(a) $\omega\left(\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}\right) = \max(\omega(A), \omega(B), \omega(C))$,

(b) $\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ e^{i\theta}C & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right)$,

(c) $\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ A & 0 & 0 \end{bmatrix}\right)$,

(d) $\omega\left(\begin{bmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{bmatrix}\right) = \max(\omega(A+B), \omega(A-B), \omega(C))$,

Proof. (a) This part is well-known.

(b) This part follows by applying the identity (1.3.5) to the operator $T = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$ and the unitary operator $U = \begin{bmatrix} e^{i\theta}I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & e^{-i\theta}I \end{bmatrix}$.

(c) This part follows by applying the identity (1.3.5) to the operator $T = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$ and the unitary operator $U = \begin{bmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{bmatrix}$.

(d) This part follows by applying the identity (1.3.5) to the operator $T = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$ and the unitary operator $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 & -I \\ 0 & \sqrt{2}I & 0 \\ I & 0 & I \end{bmatrix}$. In fact, we have

$$\begin{aligned} \omega\left(\begin{bmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{bmatrix}\right) &= \omega\left(U^* \begin{bmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{bmatrix} U\right) = \omega\left(\begin{bmatrix} A+B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & A-B \end{bmatrix}\right) \\ &= \max(\omega(A+B), \omega(A-B), \omega(C)). \end{aligned}$$

□

Remark 2.2.3. *Let $B \in B(\mathbb{H})$. Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & B \\ 0 & B & 0 \\ B & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ B & 0 & 0 \end{bmatrix}\right) = \omega(B).$$

Remark 2.2.4. *The results of Lemma 2.2.2 are true for the usual operator norm.*

The following lemma contains pinching inequalities for the numerical radius. It is also true for every weakly unitary invariant norm, including the usual operator norm (see, e.g., [9, p. 107]).

Lemma 2.2.5. *Let $A, B, C, D, E, F, G, H, K \in B(\mathbb{H})$. Then*

$$\omega\left(\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & K \end{bmatrix}\right) \geq \omega\left(\begin{bmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & K \end{bmatrix}\right)$$

and

$$\omega\left(\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & K \end{bmatrix}\right) \geq \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & E & 0 \\ G & 0 & 0 \end{bmatrix}\right).$$

Theorem 2.2.6. *Let $A, B, C \in B(\mathbb{H})$. Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \le \frac{\min(\|A+C\|, \|A-C\|)}{2} + \min(\omega(A), \omega(C)) + \omega(B).$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & 0 & -I \\ 0 & \sqrt{2}I & 0 \\ I & 0 & I \end{bmatrix}$. Then U is a unitary operator, and so by the identity (1.3.5), we have

$$\begin{aligned}
\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) &= \omega\left(U^* \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix} U\right) \\
&= \frac{1}{2}\omega\left(\begin{bmatrix} A+C & 0 & A-C \\ 0 & 2B & 0 \\ -(A-C) & 0 & -(A+C) \end{bmatrix}\right) \\
&= \frac{1}{2}\omega\left(\begin{bmatrix} A+C & 0 & A+C \\ 0 & 0 & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix} + \begin{bmatrix} 0 & 0 & -2C \\ 0 & 0 & 0 \\ 2C & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2B & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&\le \frac{1}{2}\omega\left(\begin{bmatrix} A+C & 0 & A+C \\ 0 & 0 & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix}\right) + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & -2C \\ 0 & 0 & 0 \\ 2C & 0 & 0 \end{bmatrix}\right) \\
&\quad + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2B & 0 \\ 0 & 0 & 0 \end{bmatrix}\right).
\end{aligned}$$

Since

$$\begin{bmatrix} A+C & 0 & A+C \\ 0 & 0 & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we have

$$\begin{aligned}
\omega\left(\begin{bmatrix} A+C & 0 & A+C \\ 0 & 0 & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix}\right) &= \frac{1}{2}\left\|\begin{bmatrix} A+C & 0 & A+C \\ 0 & 0 & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix}\right\| \\
&= \|A+C\|.
\end{aligned}$$

Here, we used the fact that $\|T\| = \|T^*T\|^{\frac{1}{2}}$ for every operator T. By Lemma 2.2.2 (b) and (c) and Remark 2.2.3, we have

$$\omega\left(\begin{bmatrix} 0 & 0 & -2C \\ 0 & 0 & 0 \\ 2C & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & 2C \\ 0 & 0 & 0 \\ 2C & 0 & 0 \end{bmatrix}\right) = 2\omega(C).$$

Hence,

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \le \frac{\|A+C\|}{2} + \omega(C) + \omega(B). \tag{2.2.1}$$

In the inequality (2.2.1) replacing C by $-C$, we have

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ -C & 0 & 0 \end{bmatrix}\right) \le \frac{\|A-C\|}{2} + \omega(C) + \omega(B).$$

By Lemma 2.2.2 (b) and (c), we have

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ -C & 0 & 0 \end{bmatrix}\right).$$

Thus,

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \le \frac{\|A-C\|}{2} + \omega(C) + \omega(B), \tag{2.2.2}$$

From the inequalities (2.2.1) and (2.2.2), we have

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \le \frac{\min(\|A+C\|, \|A-C\|)}{2} + \omega(C) + \omega(B). \tag{2.2.3}$$

In the inequality (2.2.3), interchanging A and C, we have

$$\omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ A & 0 & 0 \end{bmatrix}\right) \le \frac{\min(\|A+C\|, \|A-C\|)}{2} + \omega(A) + \omega(B).$$

By Lemma 2.2.2 (c), we have

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ A & 0 & 0 \end{bmatrix}\right).$$

Thus,

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \le \frac{\min(\|A+C\|, \|A-C\|)}{2} + \omega(A) + \omega(B). \tag{2.2.4}$$

From the inequalities (2.2.3) and (2.2.4), we have

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \leq \frac{\min(\|A+C\|,\|A-C\|)}{2} + \min(\omega(A),\omega(C)) + \omega(B).$$

□

Theorem 2.2.7. *Let $A,B,C \in B(\mathbb{H})$. Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \leq \frac{\omega(A+C)+\omega(A-C)}{2} + \omega(B).$$

Proof. By proof of Theorem 2.2.6, we have

$$\begin{aligned}
\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) &= \frac{1}{2}\omega\left(\begin{bmatrix} A+C & 0 & A-C \\ 0 & 2B & 0 \\ -(A-C) & 0 & -(A+C) \end{bmatrix}\right) \\
&= \frac{1}{2}\omega\left(\begin{bmatrix} A+C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(A+C) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2B & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A-C \\ 0 & 0 & 0 \\ -(A-C) & 0 & 0 \end{bmatrix}\right) \\
&\leq \frac{1}{2}\omega\left(\begin{bmatrix} A+C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(A+C) \end{bmatrix}\right) + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2B & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&\quad + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & A-C \\ 0 & 0 & 0 \\ -(A-C) & 0 & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\omega(A+C) + \omega(B) + \frac{1}{2}\omega(A-C) \text{ (by Lemma 2.2.2 (a), (b), and (d))}.
\end{aligned}$$

□

Now, we give a lower bounded estimase that complements the upper bounds given in Theorems 2.2.1, 2.2.6, and 2.2.7.

Theorem 2.2.8. *Let $A,B,C \in B(\mathbb{H})$. Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \geq \frac{\max(\omega(A+C),\omega(A-C))}{2} - \min(\omega(A),\omega(C)) - \omega(B).$$

Proof. Let U be as in the proof of Theorem 2.2.6. Then

$$\begin{aligned} U^*\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix} U &= \frac{1}{2}\begin{bmatrix} A+C & 0 & A-C \\ 0 & 2B & 0 \\ -(A-C) & 0 & -(A+C) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} A+C & 0 & A+C \\ 0 & 2B & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & -2C \\ 0 & 0 & 0 \\ 2C & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2B & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

and so

$$\frac{1}{2}\begin{bmatrix} A+C & 0 & A+C \\ 0 & 2B & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix} = U^*\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix} U - \frac{1}{2}\begin{bmatrix} 0 & 0 & -2C \\ 0 & 0 & 0 \\ 2C & 0 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2B & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, by the identity (1.3.5), we have

$$\begin{aligned} &\frac{1}{2}\omega\left(\begin{bmatrix} A+C & 0 & A+C \\ 0 & 2B & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix}\right) \\ \leq\ &\omega\left(U^*\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix} U\right) + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & -2C \\ 0 & 0 & 0 \\ 2C & 0 & 0 \end{bmatrix}\right) + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2B & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ =\ &\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & -2C \\ 0 & 0 & 0 \\ 2C & 0 & 0 \end{bmatrix}\right) + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2B & 0 \\ 0 & 0 & 0 \end{bmatrix}\right). \end{aligned}$$

Clearly, it follows by Lemmas 2.2.5 that

$$\omega\left(\begin{bmatrix} A+C & 0 & A+C \\ 0 & 2B & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix}\right) \geq \omega(A+C).$$

Thus,

$$\begin{aligned} \frac{1}{2}\omega(A+C) &\leq \frac{1}{2}\omega\left(\begin{bmatrix} A+C & 0 & A+C \\ 0 & 2B & 0 \\ -(A+C) & 0 & -(A+C) \end{bmatrix}\right) \\ &\leq \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & -2C \\ 0 & 0 & 0 \\ 2C & 0 & 0 \end{bmatrix}\right) + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2B & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ &\leq \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \omega(C) + \omega(B). \qquad (2.2.5) \end{aligned}$$

Replacing C by $-C$ in the inequality (2.2.5), we have

$$\frac{1}{2}\omega(A-C) \le \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ -C & 0 & 0 \end{bmatrix}\right) + \omega(C) + \omega(B).$$

By Lemma 2.2.2 (b), we have

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ -C & 0 & 0 \end{bmatrix}\right).$$

Thus,

$$\frac{1}{2}\omega(A-C) \le \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \omega(C) + \omega(B). \tag{2.2.6}$$

From the inequalities (2.2.5) and (2.2.6), it follows that

$$\frac{\max(\omega(A+C), \omega(A-C))}{2} \le \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \omega(C) + \omega(B). \tag{2.2.7}$$

In the inequality (2.2.7), interchanging A and C, we have

$$\frac{\max(\omega(A+C), \omega(A-C))}{2} \le \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ A & 0 & 0 \end{bmatrix}\right) + \omega(A) + \omega(B).$$

By Lemma 2.2.2 (c), we have

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ A & 0 & 0 \end{bmatrix}\right).$$

Thus,

$$\frac{\max(\omega(A+C), \omega(A-C))}{2} \le \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \omega(A) + \omega(B). \tag{2.2.8}$$

From the inequalities (2.2.7) and (2.2.8), we have

$$\frac{\max(\omega(A+C), (A-C))}{2} \le \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \min(\omega(A), \omega(C)) + \omega(B).$$

Hence,

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \geq \frac{\max(\omega(A+C),\omega(A-C))}{2} - \min(\omega(A),\omega(C)) - \omega(B).$$

□

Theorem 2.2.9. *Let* $A, B, C \in B(\mathbb{H})$. *Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & \frac{1}{2}B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \leq \frac{\omega(A+C)+\omega(A-C)+\omega(B)}{2},$$

and

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & \frac{1}{2}B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \geq \frac{\omega(A+B+C)-\max(\omega(A+C),\omega(B))}{2}.$$

Proof. The first inequality follows from Theorem 2.2.7. To prove the second inequality, we apply the Remark 2.2.3, we have

$$\begin{aligned} \omega(A+B+C) &= \omega\left(\begin{bmatrix} 0 & 0 & A+B+C \\ 0 & A+B+C & 0 \\ A+B+C & 0 & 0 \end{bmatrix}\right) \\ &\leq \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & \frac{1}{2}B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & \frac{1}{2}B & 0 \\ A & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & B \\ 0 & A+C & 0 \\ B & 0 & 0 \end{bmatrix}\right), \end{aligned}$$

By Lemma 2.2.2 (c), we have

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & \frac{1}{2}B & 0 \\ C & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & \frac{1}{2}B & 0 \\ A & 0 & 0 \end{bmatrix}\right),$$

and by Lemma 2.2.2 (d), we have

$$\omega\left(\begin{bmatrix} 0 & 0 & B \\ 0 & A+C & 0 \\ B & 0 & 0 \end{bmatrix}\right) = \max(\omega(B),\omega(A+C)).$$

Thus,

$$\omega(A+B+C) \leq 2\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & \frac{1}{2}B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \max(\omega(B),\omega(A+C)),$$

and so

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & \frac{1}{2}B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \geq \frac{\omega(A+B+C) - \max(\omega(A+C), \omega(B))}{2}.$$

□

Let $A = U|A|$, $B = V|B|$, and $C = W|C|$ be the polar decompsitions of the operators A, B, and C, respectively. Then we conclude with the following numerical radius inequalities involving the generelized Aluthdge transform.

Theorem 2.2.10. *Let* $A, B, C \in B(\mathbb{H})$. *Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max(\|A\|, \|B\|, \|C\|) + \frac{1}{4}[\||C|^t|A^*|^{1-t}\| + \||A|^t|C^*|^{1-t}\|] + \frac{1}{2}\omega(\tilde{B}_t)$$

for all $t \in [0,1]$.

Proof. Let $T = \left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right)$ and $t \in [0,1]$. Then

$$\|T\| = \max\|(\|A\|, \|B\|, \|C\|).$$

The polar decomposition of T is given by

$$\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & U \\ 0 & V & 0 \\ W & 0 & 0 \end{bmatrix}\begin{bmatrix} |C| & 0 & 0 \\ 0 & |B| & 0 \\ 0 & 0 & |A| \end{bmatrix}.$$

The generalized Aluthge transform of T is given by

$$\begin{aligned}
\tilde{T}_t &= |T|^t \begin{bmatrix} 0 & 0 & U \\ 0 & V & 0 \\ W & 0 & 0 \end{bmatrix} |T|^{1-t} \\
&= \begin{bmatrix} |C|^t & 0 & 0 \\ 0 & |B|^t & 0 \\ 0 & 0 & |A|^t \end{bmatrix} \begin{bmatrix} 0 & 0 & U \\ 0 & V & 0 \\ W & 0 & 0 \end{bmatrix} \begin{bmatrix} |C|^{1-t} & 0 & 0 \\ 0 & |B|^{1-t} & 0 \\ 0 & 0 & |A|^{1-t} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & |C|^t U|A|^{1-t} \\ 0 & |B|^t V|B|^{1-t} & 0 \\ |A|^t W|C|^{1-t} & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & |C|^t U|A|^{1-t} \\ 0 & \tilde{B}_t & 0 \\ |A|^t W|C|^{1-t} & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Now,

$$\begin{aligned}
\omega(\tilde{T}_t) &= \omega\left(\begin{bmatrix} 0 & 0 & |C|^t U|A|^{1-t} \\ 0 & \tilde{B}_t & 0 \\ |A|^t W|C|^{1-t} & 0 & 0 \end{bmatrix}\right) \\
&\leq \omega\left(\begin{bmatrix} 0 & 0 & |C|^t U|A|^{1-t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{B}_t & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&\quad + \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ |A|^t W|C|^{1-t} & 0 & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}[\||C|^t U|A|^{1-t}\| + \||A|^t W|C|^{1-t}\|] + \omega(\tilde{B}_t).
\end{aligned}$$

Since

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ |A|^t W|C|^{1-t} & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & |C|^t U|A|^{1-t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we get

$$\begin{aligned}
\omega\left(\begin{bmatrix} 0 & 0 & |C|^t U|A|^{1-t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) &= \frac{1}{2}\left\|\begin{bmatrix} 0 & 0 & |C|^t U|A|^{1-t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\||C|^t U|A|^{1-t}\| = \frac{1}{2}\||C|^t UU^*|A^*|^{1-t}U\| \leq \frac{1}{2}\||C|^t|A^*|^{1-t}\|,
\end{aligned}$$

and

$$\begin{aligned}
\omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ |A|^t W|C|^{1-t} & 0 & 0 \end{bmatrix}\right) &= \frac{1}{2}\left\|\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ |A|^t W|C|^{1-t} & 0 & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\||A|^t W|C|^{1-t}\| \\
&= \frac{1}{2}\||A|^t WW^*|C^*|^{1-t}W\| \\
&\leq \frac{1}{2}\||A|^t|C^*|^{1-t}\|.
\end{aligned}$$

By the inequality (1.5.7), we have

$$\begin{aligned}
\omega(T) &\leq \frac{1}{2}(\|T\| + \omega(\tilde{T}_t)) \\
&\leq \frac{1}{2}\max(\|A\|, \|B\|, \|C\|) + \frac{1}{4}[\||C|^t|A^*|^{1-t}\| + \||A|^t|C^*|^{1-t}\|] + \frac{1}{2}\omega(\tilde{B}_t).
\end{aligned}$$

□

CHAPTER 3
NEW NUMERICAL RADIUS INEQUALITIES FOR PRODUCTS AND COMMUTATORS OF OPERATORS

The numerical radius is not submultiplicative, in fact, the inequality

$$\omega(AB) \leq \omega(A)\omega(B),$$

is not true even for commuting operators $A, B \in B(\mathbb{H})$. It follows readily from the inequalities (0.0.1) that

$$\omega(AB) \leq 4\omega(A)\omega(B).$$

A simple example shows that $\omega(AB)$ can exceed the product $\omega(A)\omega(B)$ for all $A, B \in B(\mathbb{H})$ Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

be the right shift. Then by the example 1.3.2, we have $\omega(A) = \cos(\frac{\pi}{5}) = 0,809$ and $\omega(A^2) = \omega(A^3) = 0,5$, so that

$$0,5 = \omega(A^3) \geq \omega(A)\omega(A^2).$$

The authors of [21] have proved the following results about the numerical radius of product of operators assuming that $AB = BA$.

Theorem 3.0.11. *Let $A, B \in B(\mathbb{H})$ such that $AB = BA$. Then*

(1) $\omega(AB) \le 2\omega(A)\omega(B)$,

(2) If A is unitary operator, then

$$\omega(AB) \le \omega(B).$$

(3) If A is a normal operator, then

$$\omega(AB) \le \omega(A)\omega(B).$$

We say that A and B double commute if $AB = BA$ and $AB^* = B^*A$.

Theorem 3.0.12. *Let $A, B \in B(\mathbb{H})$ if A and B double commute . Then*

$$\omega(AB) \le \|A\|\omega(B).$$

The previous results are also known in [27]. For more results in this direction see [21, 22, 27].

In the first section of this Chapter, we prove new numerical radius inequalities for products of three bounded operators without assuming the commutativity of the operators. In the second Section we present new numerical radius for sums and commutators of operators.

3.1 Numerical radius inequalities for products of operators

More recently Dragomir [12] has shown that

$$\omega(B^*A) \le \frac{1}{2}\||A|^2 + |B|^2\|. \tag{3.1.1}$$

M. Sattari, M. S. Moslehian, and T. Ymazaki [41] have established an improvement of the inequality (3.1.1) and a generalisation of the inequality (0.0.1), these results says that for all $A, B \in B(\mathbb{H})$,

$$\omega^r(B^*A) \le \frac{1}{4}\||A^*|^{2r} + |B^*|^{2r}\| + \frac{1}{2}\omega^r(AB^*). \tag{3.1.2}$$

and

$$\omega^{2r}(A) \le \frac{1}{2}(\omega^r(A^2) + \|A\|^{2r}). \tag{3.1.3}$$

Abu-Omar and Kittaneh [3] have shown that , for all $A, B \in B(\mathbb{H})$

$$\omega(AB) \leq \frac{1}{4}\| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|(B)^*|^2\| + \frac{1}{2}\omega(BA). \tag{3.1.4}$$

In this Section, we prove new numerical radius inequalities for products of three bounded operators without assuming the commutativity of the operators, using this results we give some improvements of the inequalities (3.1.2), (3.1.3) and (3.1.4) and we establish new bounds for $\omega(A)$ and $\omega(\tilde{A}_t)$.

To present our result we need the following Lemmas

Lemma 3.1.1. *[25] Let $X, Y \in B(\mathbb{H})$. Then*

1. $\omega\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} = \omega\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} = \omega\begin{pmatrix} 0 & X \\ e^{i\theta}Y & 0 \end{pmatrix}$.
2. $\omega\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} = \omega(X)$.

Lemma 3.1.2. *[6] Let $\mathbb{H}_1, \mathbb{H}_2$ be Hilbert spaces, and let $A = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$, with $X \in B(\mathbb{H}_2, \mathbb{H}_1)$, and $Y \in B(\mathbb{H}_1, \mathbb{H}_2)$. Then*

$$\frac{1}{2}\sqrt{\||X|^2 + |Y^*|^2 + 2m(YX)} \leq \omega(A) \leq \frac{1}{2}\sqrt{\||X|^2 + |Y^*|^2 + 2\omega(YX)},$$

where $m(YX)$ is the nonnegative number defined by

$$m(YX) = \inf|\langle YXx, x\rangle|.$$

Lemma 3.1.3. *[36] Let $A, B \in B(\mathbb{H})$. Then*

$$\|A + B\| \leq 2\omega\begin{pmatrix} 0 & A \\ B^* & 0 \end{pmatrix} \leq \|A\| + \|B\|.$$

Lemma 3.1.4. *[7]*

If $a, b \in \mathbb{C}$. Then

$$|a|^2 + |b|^2 = \frac{|a+b|^2 + |a-b|^2}{2}.$$

Let $A \in B(\mathbb{H})$, from Theorem 1.3.4, we have

$$\omega \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\|A\|. \tag{3.1.5}$$

and

$$\omega \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} = \omega(A). \tag{3.1.6}$$

Our main result can be stated as follows. The following Theorem gives a new estimate for $\omega^r(ABC)$ for all $r \geq 1$.

Theorem 3.1.5. *Let $A, B, C \in B(\mathbb{H})$. Then*

$$\omega^r(ABC) \leq \omega^{2r}\left(\begin{bmatrix} 0 & A \\ BC & 0 \end{bmatrix}\right) \leq \frac{1}{4}\||A|^{2r} + |(BC)^*|^{2r}\| + \frac{1}{2}\omega^r(BCA),$$

for all $r \geq 1$.

Proof. Let $x \in \mathbb{H}$ be a unit vector, we have

$$\begin{aligned}
Re\langle e^{i\theta}ABCx, x\rangle &= Re\langle e^{i\theta}BCx, A^*x\rangle \\
&= \frac{1}{4}\|(e^{i\theta}BC + A^*)x\|^2 - \frac{1}{4}\|(e^{i\theta}BC - A^*)x\|^2 \\
&\quad (by \quad the \quad polarisation \quad identity) \\
&\leq \frac{1}{4}\|(e^{i\theta}BC + A^*)x\|^2 \\
&\leq \frac{1}{4}\|e^{i\theta}BC + A^*\|^2 \\
&\leq \omega^2\begin{pmatrix} 0 & e^{i\theta}BC \\ A & 0 \end{pmatrix} \\
&\quad (by \quad Lemma \quad 3.1.3) \\
&= \omega^2\left(\begin{bmatrix} 0 & A \\ BC & 0 \end{bmatrix}\right) \\
&\quad (by \quad Lemma \quad 3.1.1(1)) \\
&\leq \frac{1}{4}\||A|^2 + |(BC)^*|^2\| + \frac{1}{2}\omega(BCA) \\
&\quad (by \quad Lemma \quad 3.1.2)
\end{aligned}$$

By taking the supremum over $x \in \mathbb{H}$ whith $\|x\| = 1$, we have

$$\omega(ABC) \leq \omega^2\left(\begin{bmatrix} 0 & A \\ BC & 0 \end{bmatrix}\right) \leq \frac{1}{4}\||A|^2 + |(BC)^*|^2\| + \frac{1}{2}\omega(BCA).$$

From the convexity of the function t^r, and the concavity of $t^{\frac{1}{r}}$ for all $r \geq 1$, we have

$$
\begin{aligned}
\omega^r(ABC) \leq \omega^{2r}\left(\begin{bmatrix} 0 & A \\ BC & 0 \end{bmatrix}\right) &\leq \left[\frac{1}{4}\||A|^2 + |(BC)^*|^2\| + \frac{1}{2}\omega(BCA)\right]^r \\
&= \left[\frac{1}{2}\|\frac{|A|^2 + |(BC)^*|^2}{2}\| + \frac{1}{2}\omega(BCA)\right]^r \\
&\leq \frac{1}{2}\|\frac{|A|^2 + |(BC)^*|^2}{2}\|^r + \frac{1}{2}\omega^r(BCA) \\
&\leq \frac{1}{2}\|(\frac{|A|^{2r} + |(BC)^*|^{2r}}{2})^{\frac{1}{r}}\|^r + \frac{1}{2}\omega^r(BCA) \\
&= \frac{1}{2}\|(\frac{|A|^{2r} + |(BC)^*|^{2r}}{2})\| + \frac{1}{2}\omega^r(BCA).
\end{aligned}
$$

□

In the following Corollary, we present our improvement of the inequality (1.5.3).

Corollary 3.1.1. *Let $A, B \in B(\mathbb{H})$, and $r \geq 1$. Then*

$$\omega^r(B^*A) \leq \omega^{2r}\begin{pmatrix} 0 & B^* \\ A & 0 \end{pmatrix} \leq \frac{1}{4}\||B^*|^{2r} + |(A)^*|^{2r}\| + \frac{1}{2}\omega^r(AB^*),$$

for all $r \geq 1$.

Proof. Leting $C = I$ in Theorem 3.1.5. □

Our result in Theorem 3.1.5 refined and generalised the inequality (2.2.8).

Corollary 3.1.2. *Let $A, B \in B(\mathbb{H})$. Then*

$$\omega^r(AB) \leq \omega^{2r}\left(\begin{bmatrix} 0 & \sqrt{\frac{\|B\|}{\|A\|}}A \\ \sqrt{\frac{\|A\|}{\|B\|}}B & 0 \end{bmatrix}\right) \leq \frac{1}{4}\|\frac{\|B\|}{\|A\|}|A|^{2r} + \frac{\|A\|}{\|B\|}|(B)^*|^{2r}\| + \frac{1}{2}\omega^r(BA),$$

for all $r \geq 1$.

Proof. Leting $A = \sqrt{\frac{\|B\|}{\|A\|}}A$, $B = \sqrt{\frac{\|A\|}{\|B\|}}B$, and $C = I$ in Theorem 3.1.5. □

Corollary 3.1.3. *Let $A \in B(\mathbb{H})$. Then*

$$\omega^r(A) \leq \frac{\||A|^{2r} + I\|}{2}$$

for all $r \geq 1$.

A new bounds of $\omega^r(A)$, and $\omega^r(\tilde{A}_t)$ for all $r \geq 1$ can be stated as follows.

Corollary 3.1.4. *Let $A \in B(\mathbb{H})$, $t \in [0,1]$ and $r \geq 1$. Then*

$$\omega^r(A) \leq \omega^{2r}\begin{pmatrix} 0 & |A|^t \\ |A|^{1-t}U^* & 0 \end{pmatrix} \leq \frac{1}{4}\||A|^{2tr} + |A|^{2(1-t)r}\| + \frac{1}{2}\omega^r(\tilde{A}_t),$$

and

$$\omega^r(\tilde{A}_t) \leq \omega^{2r}\begin{pmatrix} 0 & |A|^t \\ U|A|^{1-t} & 0 \end{pmatrix} \leq \frac{1}{4}\||A|^{2tr} + |A^*|^{2(1-t)r}\| + \frac{1}{2}\omega^r(A).$$

In particular if $r = 1, t = \frac{1}{2}$, we have

$$\omega(A) \leq \omega^2\begin{pmatrix} 0 & |A|^{\frac{1}{2}} \\ |A|^{\frac{1}{2}}U^* & 0 \end{pmatrix} \leq \frac{1}{2}[\|A\| + \omega(\tilde{A})],$$

Proof. From the polar decomposition of A and the definition of Aluthge transform of A, we have

$$A^* = |A|U^* = |A|^t|A|^{1-t}U^*, \qquad and \qquad (\tilde{A}_t)^* = |A|^{1-t}U^*|A|^t.$$

From Theorem 3.1.5, we have

$$\begin{aligned}
\omega^r(A) = \omega^r(A^*) &= \omega(|A|^t|A|^{1-t}U^*) \\
&\leq \omega^{2r}\begin{pmatrix} 0 & |A|^t \\ |A|^{1-t}U^* & 0 \end{pmatrix} \\
&\leq \frac{1}{4}\||A|^{2tr} + (|A|^{1-t}U^*U|A|^{1-t})^r\| + \frac{1}{2}\omega^r(|A|^{1-t}U^*|A|^t) \\
&= \frac{1}{4}\||A|^{2tr} + |A|^{2(1-t)r}\| + \frac{1}{2}\omega^r((\tilde{A}_t)^*) \\
&= \frac{1}{4}\||A|^{2tr} + |A^*|^{2(1-t)r}\| + \frac{1}{2}\omega^r(\tilde{A}_t).
\end{aligned}$$

This proves the first inequality

Similarly it follows from the definition of the Aluthge transform and Theorem 3.1.5 that

$$\begin{aligned}
\omega^r(\tilde{A}_t) &= \omega^r(|A|^t U|A|^{1-t}) \\
&\le \omega^{2r}\begin{pmatrix} 0 & |A|^t \\ U|A|^{1-t} & 0 \end{pmatrix} \\
&\le \frac{1}{4}\||A|^{2tr} + (U|A|^{(1-t)}|A|^{(1-t)}U^*)^r\| + \frac{1}{2}\omega^r(U|A|^{1-t}|A|^t) \\
&= \frac{1}{4}\||A|^{2tr} + (U|A|^{2(1-t)}U^*)^r\| + \frac{1}{2}\omega^r(U|A|^{1-t}|A|^t) \\
&= \frac{1}{4}\||A|^{2tr} + |A^*|^{2(1-t)r}\| + \frac{1}{2}\omega^r(U|A|^{1-t}|A|^t) \\
&\le \frac{1}{4}\||A|^{2tr} + |A^*|^{2r(1-t)}\| + \frac{1}{2}\omega^r(A).
\end{aligned}$$

This proves the second inequality and completes the proof of the Corollary. □

We are in a position to give our improvement of the inequality (1.3.3).

Corollary 3.1.5. *Let $A \in B(\mathbb{H})$. Then*

$$\omega^{2r}(A) \le \frac{1}{4}\||A|^{2r} + |A^*|^{2r}\| + \frac{1}{2}\omega^r(A^2) \le \frac{1}{2}[\|A\|^{2r} + \omega^r(A^2)],$$

for all $r \ge 1$.

Proof. In Theorem 3.1.5 let $B = A, C = I$, we have

$$\omega^r(A^2) \le \omega^{2r}\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

By Lemma 3.1.1 , we have

$$\omega^{2r}\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} = \omega^{2r}(A).$$

Thus,

$$\begin{aligned}
\omega^{2r}(A) &\le \frac{1}{4}\||A|^{2r} + |A^*|^{2r}\| + \frac{1}{2}\omega^r(A^2) \\
&\le \frac{1}{2}[\|A\|^{2r} + \omega^r(A^2)].
\end{aligned}$$

□

3.2 Numerical radius inequalities for commutators of operators

Our aim in this Section is to give a new numerical radius inequalities for commutators of operators. The following Theorem is our main result

Theorem 3.2.1. *Let $A, B, C, D, E, F \in B(\mathbb{H})$. Then*

$$\omega^r(ABC \pm DEF) \leq \frac{1}{2}\left[\omega^{2r}\left(\begin{bmatrix} 0 & A+D \\ BC+EF & 0 \end{bmatrix}\right) + \omega^{2r}\left(\begin{bmatrix} 0 & A-D \\ BC-EF & 0 \end{bmatrix}\right)\right],$$

for all $r \geq 1$.

Proof. Let $x \in \mathbb{H}$ be a unit vector, we have

$$\begin{aligned}
Re\langle e^{i\theta}(ABC \pm DEF)x, x\rangle &\leq Re\langle e^{i\theta}ABCx, x\rangle + Re\langle e^{i\theta}DEFx, x\rangle \\
&= Re\langle e^{i\theta}BCx, A^*x\rangle + Re\langle e^{i\theta}EFx, D^*x\rangle \\
&= \frac{1}{4}\|(e^{i\theta}BC + A^*)x\|^2 - \frac{1}{4}\|(e^{i\theta}BC - A^*)x\|^2 \\
&+ \frac{1}{4}\|(e^{i\theta}EF + D^*)x\|^2 - \frac{1}{4}\|(e^{i\theta}EF - D^*)x\|^2 \\
&\quad (by \quad the \quad polarisation \quad identity) \\
&\leq \frac{1}{4}\|(e^{i\theta}BC + A^*)x\|^2 + \frac{1}{4}\|(e^{i\theta}EF + D^*)x\|^2 \\
&= \frac{1}{4}\left[\langle|e^{i\theta}BC + A^*|^2x, x\rangle + \langle|e^{i\theta}EF + D^*|^2x, x\rangle\right] \\
&= \frac{1}{4}\langle(|e^{i\theta}BC + A^*|^2 + |e^{i\theta}EF + D^*|^2)x, x\rangle \\
&\leq \frac{1}{8}\langle|e^{i\theta}BC + A^* + e^{i\theta}EF + D^*|^2 + |e^{i\theta}BC + A^* - (e^{i\theta}EF + D^*)|^2x, x\rangle \\
&\quad (by \quad Lemma \quad 3.1.4) \\
&\leq \frac{1}{8}\||e^{i\theta}BC + A^* + e^{i\theta}EF + D^*|^2 + |e^{i\theta}BC + A^* - (e^{i\theta}EF + D^*)|^2\| \\
&\leq \frac{1}{2}[\frac{1}{4}\||e^{i\theta}(BC + EF) + (A^* + D^*)|^2\| + \frac{1}{4}\||e^{i\theta}(BC - EF) + (A^* - D^*)|^2\|] \\
&= \frac{1}{2}[\frac{1}{4}\|e^{i\theta}(BC + EF) + (A^* + D^*)\|^2 + \frac{1}{4}\|e^{i\theta}(BC - EF) + (A^* - D^*)\|^2] \\
&\leq \frac{1}{2}\omega^2\left(\begin{bmatrix} 0 & A^* + D^* \\ e^{i\theta}(BC + EF)^* & 0 \end{bmatrix}\right) \\
&+ \frac{1}{2}\omega^2\left(\begin{bmatrix} 0 & A^* - D^* \\ e^{i\theta}(BC - EF)^* & 0 \end{bmatrix}\right) \\
&\quad (by \quad Lemma \quad 3.1.3) \\
&= \frac{1}{2}\left[\omega^2\left(\begin{bmatrix} 0 & A + D \\ BC + EF & 0 \end{bmatrix}\right) + \omega^2\left(\begin{bmatrix} 0 & A - D \\ BC - EF & 0 \end{bmatrix}\right)\right]\|x\|^2 \\
&\quad (by \quad Lemma \quad 3.1.1).
\end{aligned}$$

By taking the supremum over $x \in \mathbb{H}$ whith $\|x\| = 1$, we have

$$\omega(ABC \pm DEF) \leq \frac{1}{2}\left[\omega^2\left(\begin{bmatrix} 0 & A + D \\ BC + EF & 0 \end{bmatrix}\right) + \omega^2\left(\begin{bmatrix} 0 & A - D \\ BC - EF & 0 \end{bmatrix}\right)\right].$$

From the convexity of the function t^r, we have

$$\begin{aligned} \omega^r(ABC \pm DEF) &\leq \left(\frac{1}{2}\left[\omega^2\left(\begin{bmatrix} 0 & A+D \\ BC+EF & 0 \end{bmatrix}\right) + \omega^2\left(\begin{bmatrix} 0 & A-D \\ BC-EF & 0 \end{bmatrix}\right)\right]\right)^r \\ &\leq \frac{1}{2}\left[\omega^{2r}\left(\begin{bmatrix} 0 & A+D \\ BC+EF & 0 \end{bmatrix}\right) + \omega^{2r}\left(\begin{bmatrix} 0 & A-D \\ BC-EF & 0 \end{bmatrix}\right)\right]. \end{aligned}$$

□

Corollary 3.2.1. *Let $A, B \in B(\mathbb{H})$. Then*

$$\omega^r(AB \pm BA) \leq \frac{1}{2}[\omega^{2r}(A+B) + \omega^{2r}(A-B)],$$

for all $r \geq 1$.

Proof. In Theorem 3.2.1 let $E = A, D = B, C = F = I$, we have

$$\begin{aligned} \omega^r(AB \pm BA) &\leq \frac{1}{2}\left[\omega^{2r}\left(\begin{bmatrix} 0 & A+B \\ B+A & 0 \end{bmatrix}\right) + \omega^{2r}\left(\begin{bmatrix} 0 & A-B \\ -(A-B) & 0 \end{bmatrix}\right)\right] \\ &\leq \frac{1}{2}[\omega^{2r}(A+B) + \omega^{2r}(A-B)] \\ &\quad (by \quad Lemma \quad 3.1.1). \end{aligned}$$

□

Corollary 3.2.2. *Let $A, B \in B(\mathbb{H})$. Then*

$$\|AA^* \pm BB^*\|^r \leq \frac{1}{2}[\|A+B\|^{2r} + \|A-B\|^{2r}],$$

for all $r \geq 1$.

Proof. Letting $B = A^*, D = B, E = B^*, C = F = I$ in Teorem 3.2.1, we have

$$\begin{aligned} \|AA^* \pm BB^*\|^r &= \omega^r(AA^* + BB^*) \\ &\leq \frac{1}{2}\left[\omega^{2r}\begin{pmatrix} 0 & A+B \\ A^*+B^* & 0 \end{pmatrix} + \omega^{2r}\begin{pmatrix} 0 & A-B \\ A^*-B^* & 0 \end{pmatrix}\right] \\ &= \frac{1}{2}[\|A+B\|^{2r} + \|A-B\|^{2r}]. \end{aligned}$$

□

Corollary 3.2.3. *Let $A, B \in B(\mathbb{H})$. Then*

$$\omega^r(A \pm B) \leq \frac{\|A-B\|^{2r}}{4^r} + \frac{\||A+B|^2 + 2^{2r}I\|}{4},$$

for all $r \geq 1$.

Proof. Leting in Theorem 3.2.1 $B = C = E = F = I, D = B.$, we have

$$\omega^r(A \pm B) \leq \frac{1}{2}\left[\omega^{2r}\left(\begin{bmatrix} 0 & A+B \\ 2I & 0 \end{bmatrix}\right) + \omega^{2r}\left(\begin{bmatrix} 0 & A-B \\ 0 & 0 \end{bmatrix}\right)\right]$$

From Lemmas 3.1.2, and the inequality (3.1.5), we have

$$\begin{aligned} \omega^r(A \pm B) &\leq \frac{1}{2}\left[\omega^{2r}\left(\begin{bmatrix} 0 & A+B \\ 2I & 0 \end{bmatrix}\right) + \omega^{2r}\left(\begin{matrix} 0 & A-B \\ 0 & 0 \end{matrix}\right]\right)\right] \\ &\leq \frac{1}{2}\left[\frac{\|A-B\|^{2r}}{4^r} + \frac{1}{4}\||A+B|^{2r} + 2^{2r}I\| + \frac{1}{2}\omega^r(2(A+B))\right]. \end{aligned}$$

Thus,

$$\omega^r(A \pm B) \leq \frac{\|A-B\|^{2r}}{4^r} + \frac{\||A+B|^2 + 2^{2r}I\|}{4}.$$

□

Theorem 3.2.2. *Let $A_i, B_i, C_i \in B(\mathbb{H})$ for all $i = 1, \dots, n$. Then*

$$\omega^r(\sum_{i=1}^n A_iB_iC_i) \leq \sum_{i=1}^n \omega^{2r}\left(\begin{bmatrix} 0 & A_i \\ B_iC_i & 0 \end{bmatrix}\right) \leq \sum_{i=1}^n \frac{1}{4}\||A_i|^{2r} + |(B_iC_i)^*|^{2r}\| + \frac{1}{2}\omega^r(B_iC_iA_i),$$

for all $r \geq 1$.

Proof. for all $r \geq 1$, We have

$$\begin{aligned} \omega^r(\sum_{i=1}^n A_iB_iC_i) &\leq \sum_{i=1}^n \omega^r(A_iB_iC_i) \\ &\leq \sum_{i=1}^n \omega^{2r}\left(\begin{bmatrix} 0 & A_i \\ B_iC_i & 0 \end{bmatrix}\right) \\ &\leq \sum_{i=1}^n \left[\frac{1}{4}\||A_i|^2 + |(B_iC_i)^*|^2\| + \frac{1}{2}\omega(B_iC_iA_i)\right]^r \\ &\quad (by \quad Theorem \quad 2.1.5) \\ &\leq \sum_{i=1}^n \frac{1}{4}\||A_i|^{2r} + |(B_iC_i)^*|^{2r}\| + \frac{1}{2}\omega^r(B_iC_iA_i) \\ &\quad (the \quad convexity \quad of \quad the \quad function \quad t^r, \quad and \quad the \quad concavity \quad of \quad t^{\frac{1}{r}}) \end{aligned}$$

□

Corollary 3.2.4. *Let $A, B \in B(\mathbb{H})$. Then*

$$\omega^r(AB + BA) \leq 2\omega^{2r}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right),$$

for all $r \geq 1$.

Proof. leting $A_1 = A_2 = A, B_1 = B_2 = B, C = F = I$ in Theorem 3.2.2 , and by Lemma 3.1.1, we have

$$\omega^r(AB + BA) \leq \omega^{2r}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + \omega^{2r}\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right) = 2\omega^{2r}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right).$$

□

Chapter 4
Generalized numerical radius, spectral radius and norm inequalities with applications to Moore-Penros inverse

In this chapter we establish some generalized results. In Section 4.1, we present a generalisation of some of recent results which are obtained by kittaneh [4, 5] and we give new generalised numerical radius inequalities for bounded operators, whith an analysis which is totaly different from ones used in Chapter 2 we prove new numerical radius inequality for $n \times n$ operator matrices. Using this result we present new spectral radius inequalities for sums, products and commutators of operarors in Section 4.2. In Section 4.3 we improve the generalised triangle inequality for the usual operator norm. At the end of this Chapter we apply some of our results to Moore-Penrose inverse to give new spectral radius inequalities for Moore-Penrose inverse and to improve some of well-known inequalities about them.

4.1 Generalized numerical radius inequalities for Hilbert space operators

In (2014) Abu-Omar and Kittaneh [4] refined the triangle inequality of the numerical radius as follows.

Lemma 4.1.1. *[4] Let $A_1, A_2 \in B(\mathbb{H})$. Then*

$$\omega(A_1 + A_2) \leq \sqrt{\omega^2(A_1) + \omega^2(A_2) + \|A_1\|\|A_2\| + \omega(A_2^* A_1)}.$$

Our result is to prove the general case.

Theorem 4.1.2. *Let $A_i \in B(\mathbb{H})$ for all $i = 1, \ldots, n$. Then*

$$\omega(\sum_{i=1}^{n} A_i) \leq \sqrt{\sum_{i=1}^{n} \omega^2(A_i) + \sum_{i=1}^{n-1} \|A_i\| \sum_{j=i+1}^{n} \|A_j\| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega(A_j^* A_i)}. \tag{4.1.1}$$

Proof. By recurrence, from Lemma (4.1.1) the formula (4.1.1) is true when $n = 2$.

Now assume that

$$\omega(\sum_{i=1}^{n} A_i) \leq \sqrt{\sum_{i=1}^{n} \omega^2(A_i) + \sum_{i=1}^{n-1} \|A_i\| \sum_{j=i+1}^{n} \|A_j\| + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega(A_j^* A_i)}. \tag{4.1.2}$$

for all integer $n \geq 1$.

Then, we have

$$\begin{aligned} \omega(\sum_{i=1}^{n+1} A_i) &= \omega(A_1 + \sum_{i=2}^{n+1} A_i) \\ &\leq \sqrt{\omega^2(A_1) + \omega^2(\sum_{i=2}^{n+1} A_i) + \|A_1\|\|\sum_{i=2}^{n} A_i\| + \omega(\sum_{i=2}^{n} A_i A_1)} \\ &\quad (By \quad Lemma \quad 4.1.1). \end{aligned}$$

Thus, by equation (4.1.2), we have

$$\omega(\sum_{i=1}^{n+1} A_i)$$
$$\leq$$
$$\sqrt{\omega^2(A_1) + \sum_{i=2}^{n+1} \omega^2(A_i) + \sum_{i=2}^{n} \|A_i\| \sum_{j=i+1}^{n+1} \|A_j\| + \sum_{i=2}^{n} \sum_{j=i+1}^{n+1} \omega(A_j^* A_i) + \|A_1\| \sum_{i=2}^{n+1} \|A_i\| + \sum_{i=2}^{n+1} \omega(A_i^* A_1)}$$
$$=$$
$$\sqrt{\sum_{i=1}^{n+1} \omega^2(A_i) + \sum_{i=1}^{n} \|A_i\| \sum_{j=i+1}^{n+1} \|A_j\| + \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \omega(A_j^* A_i)}.$$

Therefore by the principle of mathematical induction, our formula holds for all integers n greater than 1. □

A direct result of Theorem 4.1.2 is given in this Corollary.

Corollary 4.1.1. *Let $A_1, A_2, A_3 \in B(\mathbb{H})$. Then*

$$\omega(A_1 + A_2 + A_3) \leq \sqrt{\sum_{i=1}^{3} \omega^2(A_i) + \sum_{i=1}^{2} \sum_{j=i+1}^{3} \omega(A_j^* A_i) + \sum_{i=1}^{2} \|A_i\| \sum_{j=i+1}^{3} \|A_j\|}.$$

In the same context, the authors of [5] have found a special case of Lemma 4.1.1 that is A, B are doubly commutes.

Lemma 4.1.3. *[5] Let $A_1, A_2 \in B(\mathbb{H})$ such that A_1 doubly commutes with A_2. Then*

$$\omega(A_1 + A_2) \leq \sqrt{\omega^2(A_1) + \omega^2(A_2) + \omega(A_1 A_2) + \omega(A_1^* A_2)}.$$

Now let us prove the general case

Theorem 4.1.4. *Let $A_i \in B(\mathbb{H})$ such that A_i doubly commutes with A_j for all $i, j = 1, ..., n$. Then*

$$\omega(\sum_{i=1}^{n} A_i) \leq \sqrt{\sum_{i=1}^{n} \omega^2(A_i) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega(A_i A_j) + \omega(A_i^* A_j)}. \tag{4.1.3}$$

Proof. By recurrence, from Lemma (4.1.3) the formula (4.1.3) is true when $n = 2$. Now assume that

$$\omega(\sum_{i=1}^{n} A_i) \leq \sqrt{\sum_{i=1}^{n} \omega^2(A_i) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega(A_i A_j) + \omega(A_i^* A_j)}. \tag{4.1.4}$$

for all integer $n \geq 1$.

We have

$$\begin{aligned}
\omega(\sum_{i=1}^{n+1} A_i) &= \omega(A_1 + \sum_{i=2}^{n+1} A_i) \\
&\leq \sqrt{\omega^2(A_1) + \omega^2(\sum_{i=2}^{n+1} A_i) + \omega(A_1 \sum_{i=2}^{n+1} A_i) + \omega(A_1^* \sum_{i=2}^{n+1} A_i)} \\
&\quad (By \quad Lemma \quad 4.1.3) \\
&\leq \sqrt{\omega^2(A_1) + \sum_{i=2}^{n+1} \omega^2(A_i) + \sum_{i=2}^{n} \sum_{j=i+1}^{n+1} [\omega(A_i A_j) + \omega(A_i^* A_j)] + \omega(A_1 \sum_{i=2}^{n+1} A_i) + \omega(A_1^* \sum_{i=2}^{n+1} A_i)} \\
&\quad (By \quad equation \quad (4.1.4)) \\
&= \sqrt{\sum_{i=1}^{n+1} \omega^2(A_i) + \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} [\omega(A_i A_j) + \omega(A_i^* A_j)]}.
\end{aligned}$$

Therefore by the principle of mathematical induction, our formula holds for all integers n great than 1 □

In the following theorem we give a new bounds for arbitrary $n \times n$ operator matrix

Theorem 4.1.5. *Let $\mathbb{H}_1, \mathbb{H}_2, ..., \mathbb{H}_n$ be Hilbert spaces and let $A = [A_{ij}]_{n\times n}$, be an $n \times n$ operator matrix, with $A_{ij} \in B(\mathbb{H}_j, \mathbb{H}_i)$ for all $i, j = 1, 2, ..., n$. Then*

$$\omega(A) \leq \frac{1}{2}\left[\sum_{i=1}^{n} \omega(A_{ii}) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sqrt{(\frac{\omega(A_{ii})-\omega(A_{jj})}{n-1})^2 + \| |A_{ij}| + |A_{ji}^*| \| \| |A_{ij}^*| + |A_{ji}| \|} \right].$$

Proof. Let $\mathbb{H} = \oplus_{i=1}^{n} \mathbb{H}_i$, and let $X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}$ be a unit vector in $\mathbb{H}$

(i.e., $\|x\|^2=\sum_{i=1}^n \|x_i\|^2=1$). Then

$$\left|\left\langle \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \right\rangle\right| = \left|\left\langle \begin{bmatrix} \sum_{j=1}^n A_{1j}x_j \\ \sum_{j=1}^n A_{2j}x_j \\ . \\ . \\ . \\ \sum_{j=1}^n A_{nj}x_j \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \right\rangle\right|$$
$$\le \sum_{i=1}^n |\langle A_{ii}x_i, x_i\rangle| + \sum_{i=1}^n \sum_{j=i+1}^{n-1} |\langle A_{ij}x_j, x_i\rangle|.$$

From the inequality (1.3.2) and Lemma 1.5.3, we have

$$\begin{aligned} |\langle Ax,x\rangle| &\le \sum_{i=1}^n \omega(A_{ii})\|x_i\|^2 + \sum_{i=1}^n \sum_{j=i+1}^{n-1} \langle |A_{ij}|x_i,x_i\rangle^{\frac12}\langle |A_{ij}|^* x_j,x_j\rangle^{\frac12} + \langle |A_{ji}|x_j,x_j\rangle^{\frac12}\langle |A_{ji}|^* x_i,x_i\rangle^{\frac12} \\ &= \sum_{i=1}^n \omega(A_{ii})\|x_i\|^2 + \sum_{i=1}^n \sum_{j=i+1}^{n-1} \left[\langle |A_{ij}x_i,x_i\rangle + \langle |A_{ji}^*|x_i,x_i\rangle\right]^{\frac12}\left[\langle |A_{ij}^*|x_j,x_j\rangle + \langle |A_{ji}|x_j,x_j\rangle\right]^{\frac12} \\ &\quad (by \quad the \quad Cauchy-Schwarz \quad inequality) \\ &= \sum_{i=1}^n \omega(A_{ii})\|x_i\|^2 + \sum_{i=1}^n \sum_{j=i+1}^{n-1} \left[\langle (|A_{ij}|+|A_{ji}^*|)x_i,x_i\rangle\right]^{\frac12}\left[\langle (|A_{ij}^*|+|A_{ji}|)x_j,x_j\rangle\right]^{\frac12} \\ &\le \sum_{i=1}^n \omega(A_{ii})\|x_i\|^2 + \sum_{i=1}^n \sum_{j=i+1}^{n-1} \||A_{ij}|+|A_{ji}^*|\|^{\frac12}\||A_{ij}^*|+|A_{ji}|\|^{\frac12}\|x_i\|\|x_j\| \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[\frac{\omega(A_{ii})\|x_i\|^2+\omega(A_{jj})\|x_j\|^2}{n} + \||A_{ij}|+|A_{ji}^*|\|^{\frac12}\||A_{ij}^*|+|A_{ji}|\|^{\frac12}\|x_i\|\|x_j\|\right]. \end{aligned}$$

As x is a unit vector in $\oplus_{i=1}^n \mathbb{H}_i$. Then $\|x_i\|^2+\|x_j\|^2\le 1$, for all $i,j=1,2,\dots,n, i\ne j$. and $\|x_j\|^2\le\sqrt{1-\|x_i\|^2}$.

Thus, with same condition of i,j, we have

$$\begin{aligned} \omega(A) &\le \sup_{\|x_i\|^2+\|x_j\|^2\le 1} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{\omega(A_{ii})\|x_i\|^2+\omega(A_{jj})\|x_i\|\sqrt{1-\|x_i\|^2}}{2} \\ &\quad + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \||A_{ij}|+|A_{ji}^*|\|^{\frac12}\||A_{ij}^*|+|A_{ji}|\|^{\frac12}\|x_i\|\sqrt{1-\|x_i\|^2}. \end{aligned}$$

By maximizing the function

$$f(x)=\frac{\omega(A_{ii})-\omega(A_{jj})}{n}x^2+\||A_{ij}|+|A_{ji}^*|\|^{\frac12}\||A_{ij}^*|+|A_{ji}|\|^{\frac12}x\sqrt{1-x^2}+\frac1n\omega(A_{jj}),$$

we find an upper bound for $|\langle Ax, x\rangle|$ is

$$\frac{1}{2}\left[\frac{\omega(A_{ii})+\omega(A_{jj})}{n}+\sqrt{(\frac{\omega(A_{ii})-\omega(A_{jj})}{n})^2+\||A_{ij}|+|A^*_{ji}|\|\||A^*_{ij}|+|A_{ji}|\|}\right].$$

Thus

$$\begin{aligned}\omega(A) &\leq \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{2}\left[\frac{\omega(A_{ii})+\omega(A_{jj})}{n}+\sqrt{(\frac{\omega(A_{ii})-\omega(A_{jj})}{n})^2+\||A_{ij}|+|A^*_{ji}|\|\||A^*_{ij}|+|A_{ji}|\|}\right]\\ &= \frac{1}{2}\left[\sum_{i=1}^{n}\omega(A_{ii})+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sqrt{(\tfrac{\omega(A_{ii})-\omega(A_{jj})}{n})^2+\||A_{ij}|+|A^*_{ji}|\|\||A^*_{ij}|+|A_{ji}|\|}\right].\end{aligned}$$

□

In this Theorem we find an upper and lower bounds of $\omega(AB)$, where A, B are bounded linear operators.

Theorem 4.1.6. *Let $A, B \in B(\mathbb{H})$. Then*

$$2m(B)\omega(A)-\|B\|\|A\| \leq \omega(AB) \leq \|B\|^{\frac{1}{2}}\||B^*|^{\frac{1}{2}}A^*\|.$$

where

$$m(B)=\inf_{\|x\|=1}|\langle Bx, x\rangle|.$$

Proof. To prove the first inequality we need the following extention of Schwartz inequality, which obtained by Buzano [11], if a, b, x are vectors in an inner product space. Then

$$|\langle a, x\rangle||\langle x, b\rangle| \leq \frac{\|a\|\|b\|+|\langle a, b\rangle|}{2}\|x\|^2. \tag{4.1.5}$$

In the equation (4.1.5) let $a = Bx$ and $b = Ax$, we have

$$\begin{aligned}|\langle Bx, x\rangle||\langle x, A^*x\rangle| &\leq \frac{\|Bx\|\|Ax\|+|\langle Bx, A^*x\rangle|}{2}\\ &\leq \frac{\|B\|\|A\|+|\langle ABx, x\rangle|}{2}\\ &\leq \frac{\|B\|\|A\|+\omega(AB)}{2}.\end{aligned}$$

In the other hand, we have

$$|\langle Bx,x\rangle||\langle x,A^*x\rangle| = |\langle Bx,x\rangle||\langle Ax,x\rangle| \geq m(B)|\langle Ax,x\rangle|,$$

Thus,

$$m(B)|\langle Ax,x\rangle| \leq m(B)\sup_{\|x\|=1}|\langle Ax,x\rangle| \leq \frac{\|B\|\|A\|+\omega(AB)}{2}.$$

Consequently

$$m(B)\omega(A) \leq \frac{\|B\|\|A\|+\omega(AB)}{2}.$$

To prove the second inequality, we use Lemma (1.5.3), we have

$$\begin{aligned}
|\langle ABx,x\rangle| &= |\langle Bx,A^*x\rangle| \\
&\leq \langle |B|x,x\rangle^{\frac{1}{2}}\langle |B^*|A^*x,A^*x\rangle^{\frac{1}{2}} \\
&= \langle |B|x,x\rangle^{\frac{1}{2}}\langle |B^*|^{\frac{1}{2}}A^*x,|B^*|^{\frac{1}{2}}A^*x\rangle^{\frac{1}{2}} \\
&= \langle |B|x,x\rangle^{\frac{1}{2}}[\||B^*|^{\frac{1}{2}}A^*x\|^2]^{\frac{1}{2}} \\
&= \langle |B|x,x\rangle^{\frac{1}{2}}\||B^*|^{\frac{1}{2}}A^*x\| \\
&\leq \||B|\|^{\frac{1}{2}}\||B^*|^{\frac{1}{2}}A^*\| \\
&= \|B\|^{\frac{1}{2}}\||B^*|^{\frac{1}{2}}A^*\|.
\end{aligned}$$

By (1.3.1), we have

$$\omega(AB) \leq \|B\|^{\frac{1}{2}}\||B^*|^{\frac{1}{2}}A^*\|.$$

This completes the proof. □

Remark 4.1.7. *In Theorem 4.1.6 see that*

$$\omega(AB) \leq \|B\|^{\frac{1}{2}}\||B^*|^{\frac{1}{2}}A^*\| \leq \|A\|\|B\|.$$

Theorem 4.1.8. *Let $A_i \in B(\mathbb{H})$ for all $i = 1,\ldots,n$. Then*

$$2^{n-1}\prod_{i=1}^{n-1}m(A_i)\omega(A_n) - [2^{n-1}-1]\prod_{i=1}^{n}\|A_i\| \leq \omega(\prod_{i=1}^{n}A_i) \leq \|A_1\|^{\frac{1}{2}}\|\prod_{i=2}^{n}A_i|(A_1)^*|(\prod_{i=1}^{n}A_i)^*\|^{\frac{1}{2}} \tag{4.1.6}$$

for all $n \geq 1$.

Proof. We start with the first inequality.

By recurence, from Theorem 4.1.6 the formula (4.1.6) is true when $n = 2$.

Now assume that for all $n \geq 2$

$$2^{n-1}\prod_{i=1}^{n-1} m(A_i)\omega(A_n) - [2^{n-1}-1]\prod_{i=1}^{n}\|A_i\| \leq \omega(\prod_{i=1}^{n} A_i). \tag{4.1.7}$$

Then, we have

$$\begin{aligned}
\omega(\prod_{i=1}^{n+1} A_i) &= \omega(A_1\prod_{i=2}^{n+1} A_i) \\
&\geq 2m(A_1)\omega(\prod_{i=2}^{n+1} A_i) - \|A_1\|\|\prod_{i=2}^{n+1} A_i\| \\
&\quad (by \quad theorem \quad 4.1.6) \\
&\geq 2m(A_1)\left[2^{n+1-1}\prod_{i=2}^{n+1-1} m(A_i)\omega(A_{n+1}) - [2^{n-1}-1]\prod_{i=2}^{n+1}\|A_i\|\right] - \|A_1\|\prod_{i=1}^{n+1}\|A_i\| \\
&\quad (by \quad the \quad inequality \quad (4.1.7)) \\
&= 2^n\prod_{i=1}^{n} m(A_i)\omega(A_{n+1}) - 2m(A_1)[2^{n-1}-1]\prod_{i=2}^{n+1}\|A_i\| - \|A_1\|\prod_{i=1}^{n+1}\|A_i\| \\
&= 2^n\prod_{i=1}^{n} m(A_i)\omega(A_{n+1}) - 2\|A_1\|[2^{n-1}-1]\prod_{i=2}^{n+1}\|A_i\| - \|A_1\|\prod_{i=1}^{n+1}\|A_i\| \\
&= 2^n\prod_{i=1}^{n} m(A_i)\omega(A_{n+1}) - [2^{n-1}-1]\prod_{i=1}^{n+1}\|A_i\|.
\end{aligned}$$

Therefore by the principle of mathematical induction, our formula holds for all integers n greater than 1.

For the second inequality, by Theorem 4.1.6, we have

$$\omega(\prod_{i=1}^{n} A_i) = \omega(A_1\prod_{i=2}^{n} A_i) \leq \|A_1\|^{\frac{1}{2}}\|\prod_{i=2}^{n} A_i|(A_1)^*|(\prod_{i=1}^{n} A_i)^*\|^{\frac{1}{2}}.$$

□

4.2 Generalized spectral radius inequalities for Hilbert space operators

Theorem 4.2.1. *Let $A_i, B_i \in B(\mathbb{H})$ for all $i = 1, ..., n$, then*

$$r(\sum_{i=1}^{n} A_i B_i) \leq \frac{1}{2}\left[\sum_{i=1}^{n} \omega(B_i A_i) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n+1}\sqrt{(\frac{\omega(B_i A_i) - \omega(B_j A_j)}{n-1})^2 + \||B_i A_j| + |A_i^* B_j^*|\| \||A_j^* B_i^*| + |B_j A_i|\|}\right].$$

Proof. We have

$$\begin{aligned}
r(\sum_{i=1}^{n} A_i B_i) &= r\begin{pmatrix} \sum_{i=1}^{n} A_i B_i & 0 & .. & 0 \\ 0 & 0 & ... & 0 \\ . & . & .. & . \\ . & . & .. & . \\ . & . & .. & . \end{pmatrix} \\
&= r\begin{pmatrix} A_1 & A_2 & .. & A_n \\ 0 & 0 & ... & 0 \\ . & . & .. & . \\ . & . & .. & . \\ . & . & .. & . \end{pmatrix}\begin{pmatrix} B_1 & 0 & .. & 0 \\ B_2 & 0 & .. & 0 \\ . & . & .. & . \\ . & . & .. & . \\ B_n & . & .. & . \end{pmatrix} \\
&= r\begin{pmatrix} B_1 & 0 & .. & 0 \\ B_2 & 0 & .. & 0 \\ . & . & .. & . \\ . & . & .. & . \\ B_n & . & .. & . \end{pmatrix}\begin{pmatrix} A_1 & A_2 & .. & A_n \\ 0 & 0 & ... & 0 \\ . & . & .. & . \\ . & . & .. & . \\ 0 & . & .. & 0 \end{pmatrix} \\
&= r\begin{pmatrix} B_1 A_1 & B_1 A_2 & .. & B_1 A_n \\ B_2 A_1 & B_2 A_2 & .. & B_2 A_n \\ . & . & .. & . \\ . & . & .. & . \\ B_n A_1 & B_n A_2 & .. & B_n A_n \end{pmatrix} \\
&\leq \omega\begin{pmatrix} B_1 A_1 & B_1 A_2 & ... & B_1 A_n \\ B_2 A_1 & B_2 A_2 & ... & B_2 A_n \\ . & . & .. & . \\ . & . & .. & . \\ B_n A_1 & B_n A_2 & .. & B_n A_n \end{pmatrix}.
\end{aligned}$$

Hence, by Theorem 4.1.5, we have

$$r(\sum_{i=1}^{n} A_i B_i) \leq \frac{1}{2}\left[\sum_{i=1}^{n} \omega(B_i A_i) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n+1}\sqrt{(\frac{\omega(B_i A_i) - \omega(B_j A_j)}{n-1})^2 + \||B_i A_j| + |A_i^* B_j^*|\| \||A_j^* B_i^*| + |B_j A_i|\|}\right].$$

□

A directe result of Theorem 4.2.1, when $n = 2$ is given in this Corollary

Corollary 4.2.1. *Let $A_1, A_2, B_1, B_2 \in B(H)$. Then*

$$r(A_1B_1 + A_2B_2) \le \frac{1}{2}\left[\omega(B_1A_1) + \omega(B_2A_2) + \sqrt{(\omega(B_1A_1) - \omega(B_2A_2))^2 + \||B_1A_2| + |A_1^*B_2^*|\| \||A_2^*B_1^*| + |B_2A_1|\|}\right].$$

The following list of corollaries is an immediate consequence of Corollary 4.2.1.

Corollary 4.2.2. *Let $A, B \in B(\mathbb{H})$. Then*

$$r(AB \pm BA) \le \frac{1}{2}\left[\omega(BA) + \omega(AB) + \sqrt{(\omega(BA) - \omega(AB))^2 + \||B^2| + |(A^2)^*|\| \||(B^2)^*| + |A^2|\|}\right].$$

Proof. In Corollary 4.2.1 we put $A_1 = B_2 = A, B_1 = A_2 = \pm B$, we have

$$r(AB \pm BA) \le \frac{1}{2}\left[\omega(BA) + \omega(AB) + \sqrt{(\omega(BA) - \omega(AB))^2 + \||B^2| + |(A^2)^*|\| \||(B^2)^*| + |A^2|\|}\right].$$

□

Remark 4.2.2. *In Corollary 4.2.2 if P, Q are two projection , we have*

$$r(PQ \pm QP) \le \frac{1}{2}\left[\omega(QP) + \omega(PQ) + \sqrt{(\omega(QP) - \omega(PQ))^2 + \||Q| + |P^*|\| \||Q^*| + |P|\|}\right].$$

Corollary 4.2.3. *Let $A, B \in B(\mathbb{H})$. Then*

$$r(A + B) \le \frac{1}{2}\left[\omega(A) + \omega(B) + \sqrt{(\omega(A) - \omega(B))^2 + \|I + |(BA)^*|\| \|I + |BA|\|}\right].$$

Proof. In Corollary 4.2.1 we put $A_1 = A, B_1 = A_2 = I$, and $B_2 = B$, we have

$$r(A + B) \le \frac{1}{2}\left[\omega(A) + \omega(B) + \sqrt{(\omega(A) - \omega(B))^2 + \|I + |(BA)^*|\| \|I + |BA|\|}\right].$$

□

Remark 4.2.3. *In Corrolary 4.2.3 if $A = P$ and $B = I - P$ such that P is a projection, we have*

$$1 = r(I) = r(P + (I - P) \le \frac{1}{2}\left[\omega(P) + \omega(I - P) + \sqrt{(\omega(P) - \omega(I - P))^2 + 1}\right].$$

Corollary 4.2.4. *Let $A, B \in B(\mathbb{H})$. Then*

$$r(AB) \leq \frac{1}{4}\left[\omega(BA) + \omega(AB) + \sqrt{(\omega(BA) - \omega(AB))^2 + \||B| + |(ABA)^*|\|\||B^*| + |ABA|\|}\right].$$

Proof. In Corollary 4.2.1 we put $A_1 = A, B_1 = \frac{1}{2}B, A_2 = I$, and $B_2 = \frac{1}{2}AB$, we have

$$\begin{aligned} r(AB) &\leq \frac{1}{2}\left[\omega(\frac{1}{2}BA) + \omega(\frac{1}{2}AB) + \sqrt{(\omega(\frac{1}{2}BA) - \omega(\frac{1}{2}AB))^2 + \||\frac{1}{2}B| + |(\frac{1}{2}ABA)^*|\|\||\frac{1}{2}B^*| + |\frac{1}{2}ABA|\|}\right] \\ &= \frac{1}{4}\left[\omega(BA) + \omega(AB) + \sqrt{(\omega(BA) - \omega(AB))^2 + \||B| + |(ABA)^*|\|\||B^*| + |ABA|\|}\right]. \end{aligned}$$

□

4.3 Generalized norm inequalities for Hilbert space operators

In the following Theorem we present our improvement of the triangle inequality for the usual operator norm.

Theorem 4.3.1. *Let $A_1, A_2 \in B(\mathbb{H})$. Then*

$$\|A_1 + A_2\| \leq \sqrt{\|A_1\|^2 + \|A_2\|^2 + 2\omega(A_2^* A_1)}.$$

Proof. We have,

$$\begin{aligned} \|(A_1 + A_2)x\|^2 &= \langle (A_1 + A_2)x, (A_1 + A_2)x\rangle \\ &= \langle A_1 x, A_1 x\rangle + \langle A_1 x, A_2 x\rangle + \langle A_2 x, A_1 x\rangle + \langle A_2 x, A_1 x\rangle \\ &= \|A_1 x\|^2 + \|A_2 x\|^2 + 2Re\langle A_1 x, A_2 x\rangle \\ &\leq \|A_1 x\|^2 + \|A_2 x\|^2 + 2|\langle A_1 x, A_2 x\rangle| \\ &= \|A_1 x\|^2 + \|A_2 x\|^2 + 2|\langle A_2^* A_1 x, x\rangle|. \end{aligned}$$

Thus,

$$\|A_1 + A_2\|^2 \leq \|A_1\|^2 + \|A_2\|^2 + 2\omega(A_2^* A_1),$$

and so,

$$\|A_1 + A_2\| \le \sqrt{\|A_1\|^2 + \|A_2\|^2 + 2\omega(A_2^* A_1)}.$$

□

Corollary 4.3.1. *Let $A = B + iC$ be Cartesian decomposition of $A \in B(\mathbb{H})$. Then*

$$\|A\| \le \sqrt{\|B\|^2 + \|C\|^2 + 2\omega(CB)} \le \sqrt{\|B\|^2 + \|C\|^2 + 2\omega(B)\omega(C)}.$$

Proof. From Theorem 4.3.1, we have

$$\|A\| = \|B + iC\| \le \sqrt{\|B\|^2 + \|C\|^2 + 2\omega(C^* B)}.$$

As B, C are self-adjoint, we have

$$\omega(C^* B) = \omega(CB) \le \|CB\| \le \|C\|\|B\| = \omega(C)\omega(B).$$

Thus

$$\|A\| = \|B + iC\| \le \sqrt{\|B\|^2 + \|C\|^2 + 2\omega(CB)} \le \sqrt{\|B\|^2 + \|C\|^2 + 2\omega(B)\omega(C)}.$$

□

We are in a position to give the general case of Theorem 4.3.1.

Theorem 4.3.2. *Let $A_i \in B(\mathbb{H})$ for all $i = 1, \ldots, n$. Then*

$$\|\sum_{i=1}^{n} A_i\| \le \sqrt{\sum_{i=1}^{n} \|A_i\|^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2\omega(A_j^* A_i)}. \tag{4.3.1}$$

Proof. Let $x \in \mathbb{H}$. Then

$$\begin{aligned}
\|\sum_{i=1}^{n} A_i x\|^2 &= \langle \sum_{i=1}^{n} A_i x, \sum_{i=1}^{n} A_i x \rangle \\
&= \sum_{i=1}^{n} \langle A_i x, A_i x \rangle + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \langle A_i x, A_j x \rangle + \langle A_j x, A_i x \rangle \\
&= \sum_{i=1}^{n} \langle A_i x, A_i x \rangle + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2Re(\langle A_i x, A_j x \rangle) \\
&\leq \sum_{i=1}^{n} \langle A_i x, A_i x \rangle + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2|\langle A_i x, A_j x \rangle| \\
&= \sum_{i=1}^{n} \|A_i x\|^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2|\langle A_j^* A_i x, x \rangle|.
\end{aligned}$$

Thus,

$$\|\sum_{i=1}^{n} A_i\|^2 \leq \sum_{i=1}^{n} \|A_i\|^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2\omega(A_j^* A_i).$$

and so,

$$\|\sum_{i=1}^{n} A_i\| \leq \sqrt{\sum_{i=1}^{n} \|A_i\|^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2\omega(A_j^* A_i)}.$$

□

4.4 Applications to Moore-Penrose inverse

Theorem 4.4.1. *Let $A \in B(\mathbb{H})$. Then*

$$\omega(A^2) \leq \|A\|^{\frac{1}{2}} \|(\tilde{A})^* |A^*|^{\frac{1}{2}}\| \leq \|A\|^2.$$

Proof.

$$
\begin{aligned}
\omega(A^2) &= \omega(AA) \\
&\leq \|A\|^{\frac{1}{2}}\||A^*|^{\frac{1}{2}}A^*\| \\
&\quad (by \quad Theorem \quad 4.1.6) \\
&= \|A\|^{\frac{1}{2}}\||A^*|^{\frac{1}{2}}U^*|A^*|\| \\
&\quad (from \quad Polar \quad decomposition) \\
&= \|A\|^{\frac{1}{2}}\||A^*|^{\frac{1}{2}}U^*|A^*|^{\frac{1}{2}}|A^*|^{\frac{1}{2}}\| \\
&= \|A\|^{\frac{1}{2}}\|(\tilde{A})^*|A^*|^{\frac{1}{2}}\| \\
&\leq \|A\|^2.
\end{aligned}
$$

□

Theorem 4.4.2. *Let $A \in B(\mathbb{H})$ be a non zero operator with closed range. Then*

$$1 \leq \|A\|^{\frac{1}{2}}\||A^*|^{\frac{1}{2}}(A^+)^*\| \leq \|A\|\|A^+\|.$$

Proof. From , we have

$$\omega(A^+A) = 1.$$

By Theorem 4.1.6, we have

$$1 = \omega(A^+A) \leq \|A\|^{\frac{1}{2}}\||A^*|^{\frac{1}{2}}(A^+)^*\| \leq \|A\|\|A^+\|.$$

□

Theorem 4.4.3. *Let $A \in B(\mathbb{H})$ be a nonzero operator with closed range. Then*

$$m(A) \leq \frac{1}{2}\left[\|A\| + \frac{1}{\|A^+\|}\right] \leq \|A\|.$$

Proof. By the equation (1.4.4), we have

$$A^+ = A^*(AA^*)^+$$

From Theorem 4.1.6, we have

$$\begin{aligned}
\omega(A^+) &= \omega(A^*(AA^*)^+) \\
&\geq 2m(A^*)\omega((AA^*)^+) - \|A^*\|\|(AA^*)^+\| \\
&= 2m(A)\|A^+\|^2 - \|A\|\|A^+\|^2 \\
&= \|A^+\|^2\,[2m(A)\| - \|A\|]\,.
\end{aligned}$$

As $A \neq 0$, $A^+ \neq 0$ also. Then

$$2m(A)\| - \|A\| \leq \frac{\omega(A^+)}{\|A^+\|^2} \leq \frac{1}{\|A^+\|}.$$

Thus,

$$m(A) \leq \frac{1}{2}\left[\|A\| + \frac{1}{\|A^+\|}\right].$$

As $\|A\|\|A^+ \geq 1$, then $\dfrac{1}{\|A^+\|} \leq \|A\|$

Thus,

$$\frac{1}{2}\left[\|A\| + \frac{1}{\|A^+\|}\right] \leq \|A\|.$$

□

Theorem 4.4.4. *Let $A \in B(\mathbb{H})$ be a nonzero operator with closed range. Then*

$$\|AA^+ + A^+A\| \leq 1 + \frac{\sqrt{\||(A^+)^2| + |(A^2)^*|\|\||((A^+)^2)^*| + |A^2|\|}}{2}.$$

Proof. In Corollary 4.2.2 we put $B = A^+$, we have

$$r(AA^+ + A^+A) \leq \frac{1}{2}\left[\omega(AA^+) + \omega(A^+A) + \sqrt{\||(A^+)^2| + |(A^2)^*|\|\||((A^+)^2)^*| + |A^2|\|}\right].$$

We have $AA^+ + A^+A$ is a self-adjoint operator and AA^+, A^+A are orthogonal projections , then

$$r(AA^+ + A^+A) = \omega(AA^+ + A^+A) = \|AA^+ + A^+A\|,$$

and

$$\omega(AA^+) = \omega(A^+A) = 1.$$

Thus,

$$\|AA^{+}+A^{+}A\| \leq 1+\frac{\sqrt{\||(A^{+})^{2}|+|(A^{2})^{*}|\|\||((A^{+})^{2})^{*}|+|A^{2}|\|}}{2}.$$

□

Theorem 4.4.5. *Let $A \in B(\mathbb{H})$ be a nonzero operator with closed range. Then*

$$2 \leq \sqrt{\||A^{+}|+|A^{*}|\|\||(A^{+})^{*}|+|A|\|}.$$

Proof. In corollary 4.2.4 we put $B = A^{+}$, we have

$$\begin{aligned} 1 &= r(AA^{+}) \\ &\leq \frac{1}{4}\left[\omega(A^{+}A)+\omega(AA^{+})+\sqrt{(\omega(A^{+}A)-\omega(AA^{+}))^{2}+\||A^{+}|+|(AA^{+}A)^{*}|\|\||(A^{+})^{*}|+|AA^{+}A|\|}\right] \\ &= \frac{1}{4}\left[1+1+\sqrt{\||A^{+}|+|(AA^{+}A)^{*}|\|\||(A^{+})^{*}|+|AA^{+}A|\|}\right]. \end{aligned}$$

Thus,

$$2 \leq \sqrt{\||A^{+}|+|(AA^{+}A)^{*}|\|\||(A^{+})^{*}|+|AA^{+}A|\|}.$$

From the inequality (1.4.1), we have $A = AA^{+}A$, and so

$$2 \leq \sqrt{\||A^{+}|+|A^{*}|\|\||(A^{+})^{*}|+|A|\|}.$$

□

Theorem 4.4.6. *Let $A \in B(\mathbb{H})$ be a nonzero operator with closed range. Then*

$$r(A^{+}) \leq \frac{1}{4}\left[\omega(A^{+})+\|A^{+}\|+\sqrt{(\|A^{+}\|-\omega(A^{+}))^{2}+\|AA^{+}+|((A^{+})^{2})^{*}|\|\|AA^{+}+|(A^{+})^{2}|\|}\right].$$

Proof. By (1.4.2),

$$A^{+}AA^{+} = A^{+}.$$

In corollary 4.2.4 we put $A = A^{+}$ and $B = AA^{+}$, we have

$$\begin{aligned} r(A^{+}) &\leq \frac{1}{4}\left[\omega(A^{+})+\omega(AA^{+}A^{+})+\sqrt{(\omega(A^{+})-\omega(AA^{+}A^{+}))^{2}+\|AA^{+}+|((A^{+})^{2})^{*}|\|\|AA^{+}+|(A^{+})^{2}|\|}\right] \\ &= \frac{1}{4}\left[\omega(A^{+})+\omega(AA^{+}A^{+})+\sqrt{(\omega(AA^{+}A^{+})-\omega(A^{+}))^{2}+\|AA^{+}+|((A^{+})^{2})^{*}|\|\|AA^{+}+|(A^{+})^{2}|\|}\right]. \end{aligned}$$

We have,

$$\omega(AA^+A^+) \le \|AA^+A^+\| \le \|AA^+\|\|A^+\| = \|A^+\|.$$

Thus,

$$r(A^+) \le \frac{1}{4}\left[\omega(A^+) + \|A^+\| + \sqrt{(\|A^+\| - \omega(A^+))^2 + \|AA^+ + |((A^+)^2)^*|\|\|AA^+ + |(A^+)^2|\|}\right].$$

□

Conclusion

- In this thesis we prove new numerical radius inequalities for Hilbert space operators
  - we present numerical radius inequalities for operator matrices, products, and commutators of operators,
  - Inequalities involving the Generalized Aluthge Transform are also obtained,
  - we generalise some results,
  - we apply some of our results to Moore-Penrose inverse to give new spectral radius inequalities for Moore-Penrose inverse and to improve some of well-known inequalities about them.

Prospects

In the future we examinate 3 problems

- Problem 1: The improvement of the inequality

$$\frac{1}{2}\|A\| \leq \omega(A) \qquad for \quad all \quad A \in B(H),$$

- Problem 2: Numerical radius inequalities for products of operators under conditions around operators,
  - normal operators.
  - hyponormal operators.
  - projections.
  - commuting operators.
- Problem 3: More results about numerical radius inequalities for Moore-Penrose Inverse.

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Abstract

In this thesis we prove new numerical radius inequalities for Hilbert space operators, we present numerical radius inequalities for operator matrices, products, and commutators of operators, we give also inequalities involving the generalized Aluthge transform. Some of generalized results are given in this thesis, we apply some of our results to Moore-Penrose inverse to give new spectral radius inequalities for Moore-Penrose inverse and to improve some of well-known inequalities about them

Keywords

Numerical radius, Spectral radius, Operator matrix, spectrum, commutators, operator norm, Aluthge transform, inequality.

Résume

Dans cette thèse, nous prouvons nouvelles inégalités sur le rayon numérique des opérateurs dans un espace de Hilbert, nous présentons des inégalités sur le rayon numérique pour matrice d´opérateurs, produits et commutateurs des opérateurs, nous donnons des inégalités impliquant la transformation d´Aluthge généralisée. Certains des résultats généralisés sont donnés dans cette thèse , nous appliquons certains de nos résultats à l´inverse de Moore-Penrose pour donner de nouvelles inégalités de rayon spectral Ã l´inverse de Moore-Penrose et pour améliorer certaines inégalités bien connues Ã leur sujet.

Mots-clés

Rayon numérique, Rayon spectral, matrice dâopérateur, spectre, commutateurs, norme d´opérateur, transformation d´Aluthge, inégalité.