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Thèse

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Option : ANALYSE ET CONTROLE DES SYSTEMES

EXISTENCE DE LA SOLUTION DE CERTAINS SYSTEMES D'EVOLUTION SEMI LINEAIRE

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ملخص

الهدف الرئيسي من هذه الرسالة هو دراسة مشكلة وجود واستقرار الحل الشامل لبعض الجمل شبه خطية. ترتكز الدراسة الأولى على معادلة زائدية غير خطية مع معاملات منبع و كبح، لإثبات وجود الحل الشامل لهذه الجملة نستخدم طريقة المجموعة المستقرة ثم من خلال تطبيق متراجحات كومورنيك نحصل على نتيجة الاستقرار. وتسلط الدراسة الثانية الضوء على معادلة الموجة الغير خطية مع معاملات المنبع والكبح المتغيرة، أيضا باستخدام طريقة المجموعة المستقرة و بتطبيق متراجحات كومورنيك نحصل على نتائج وجود واستقرار الحل الشامل. أخيرا، ندرس معادلة الموجة الغير الخطية والتي تحتوي علي معامل اضطراب من الدرجة الأولى و نبين أنه عندما يهيمن معامل الكبح (الخطي واللزج) على معامل الاضطراب تنخفض الطاقة المعتادة ويكون الحل شاملا.

الكلمات الافتتاحية: معامل المنبع، معامل الكبح، معامل الاضطراب، الحل الشامل، الاستقرار.

Abstract

The main objective of this thesis is to study the problem of the global existence and stability of some semi linear evolution systems. The first study focuses on the nonlinear hyperbolic equation with damping and source terms. To prove the existence of a global solution of this system, we use some assumptions for initial data. Then, by applying an integral inequality due to Komornik, we obtain the stability result. The second study highlights the nonlinear wave equation with damping and source terms of variable-exponent types. Also, by some assumptions for initial data and Komornik's integral inequality, we obtained similar results. Finally, we consider the nonlinear wave equation with damping, source and nonlinear first order perturbation terms. We show that when the damping term (linear and viscoelastic) dominates the nonlinear first order perturbation terms, the usual energy is decreases and the solution is global.
keywords: Source term, Dissipative term; Variable exponents; Perturbation term; Global solution; Stability.

CONTENTS

GENERAL INTRODUCTION

In all this work

- $T > 0$.
- Ω is a bounded domain of $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$, with a smooth boundary $\partial\Omega$.

Literature. During the last few decades, many researchers have been interested in the nonlinear wave equations with source and damping terms with constant exponents:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\,\Delta u(\tau)\,d\tau + u_t\,|u_t|^{m-2} = u\,|u|^{p-2}, & in\ \Omega \times (0,T), \\ u = 0, & on\ \partial\Omega \times (0,T), \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & in\ \Omega, \end{cases} \tag{P}$$

where $p,\ m \geq 2$ are constants and g is a given positive function defined on Ω.

In the absence of the viscoelastic terms $(g = 0)$, system (P) has been largely studied by several authors. Ball [6] showed that in the absence of the damping term $u_t\,|u_t|^{m-2}$, the source term $u\,|u|^{p-2}$ causes finite-time blow-up of solutions with negative initial energy. Haraux, Zuazua and Kopackova [23], [27] proved that in the absence of the source term, the damping term assures global existence for arbitrary initial data. In the linear damping case $m = 2$, Levine [29] established a finite-time blow-up result for negative initial energy. Georgiev and Todorova [18] extended Levine's result to the nonlinear damping case $m > 2$. They gave two results:

- if $m \geq p$ then the global solution exists for arbitrary initial data.
- if $p > m$ then solutions with sufficiently negative initial energy blow-up in finite time.

Messaoudi [30] improved the result of Georgiev and Todorova and proved a finite time blow-up result for solutions with negative initial energy only. Ikehata [24] used some assumptions for initial data, introduced by Sattinger [38] to show that the global solution exists for small enough initial energy. In addition, authors in [21], [25] and [37] have addressed this issue.

In the presence of the viscoelastic terms $(g \neq 0)$, Messaoudi in [31] and [32] proved that the solution with negative or positive initial energy blow up in finite time. A result of global existence is also obtained for any initial data. In addition, we refer to [7] , [11], [18], [19], [28] and [42] for other results in this direction.

Recently, many papers have treated the problem with variable exponents. The study of these systems is based on the use of the Lebesgue and Sobolev spaces with variable exponents, see for instance [15] and [39]. Antontsev [2] discussed the following equation

$$\begin{cases} u_{tt} = div\left(a\left|\nabla u\right|^{p(x,t)-2}\nabla u\right) + \alpha\Delta u + bu\left|u\right|^{\sigma(x,t)-2} + f, & in\ \Omega\times(0,T), \\ u = 0, & in\ \partial\Omega\times(0,T), \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & in\ \Omega, \end{cases}$$

where α is a nonnegative constant and a, b, p, σ are given functions. He proved the existence and the blow-up of weak solutions with negative initial energy. Gao [17] studied the following viscoelastic hyperbolic equation

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t\left|u_t\right|^{m(.)-2} = u\left|u\right|^{p(.)-2}, & in\ \Omega\times(0,T), \\ u = 0, & on\ \partial\Omega\times(0,T), \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & in\ \Omega. \end{cases}$$

Using the Galerkin method, the authors proved the existence of weak solutions under suitable assumptions. Another study of Sun et al. [40] where they looked into the following equation

$$\begin{cases} u_{tt} - div\left(a\left|\nabla u\right|\right) + cu_t\left|u_t\right|^{q(x,t)-1} = bu\left|u\right|^{p(x,t)-1}, & in\ \Omega\times(0,T), \\ u = 0, & on\ \partial\Omega\times(0,T), \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & in\ \Omega. \end{cases}$$

Under specific conditions on the exponents and the initial data, they established a blow-up result. Messaoudi and Talahmeh [34] considered the following equation

$$\begin{cases} u_{tt} - div\left(\left|\nabla u\right|^{m(.)-2}\nabla u\right) + \mu u_t = u\left|u\right|^{p(.)-2}, & in\ \Omega\times(0,T), \\ u = 0, & on\ \partial\Omega\times(0,T), \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & in\ \Omega, \end{cases}$$

where μ is a nonnegative constant. Under suitable conditions on the exponents, they established a blow-up result for solutions with arbitrary positive initial energy. After that, the same authors in [35] studied the following equation

$$\begin{cases} u_{tt} - div\left(\left|\nabla u\right|^{r(.)-2}\nabla u\right) + au_t\left|u_t\right|^{m(.)-2} = bu\left|u\right|^{p(.)-2}, & in\ \Omega\times(0,T), \\ u = 0, & on\ \partial\Omega\times(0,T), \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & in\ \Omega, \end{cases}$$

where a, b are positive constants and the exponents of nonlinearity m, p, r are given functions. They proved a finite time blow-up result for the solution with negative initial energy as well as for certain solutions with positive initial energy. Also, Messaoudi et al. [36] considered the following equation

$$\begin{cases} u_{tt} - \Delta u + au_t\left|u_t\right|^{m(.)-2} = bu\left|u\right|^{p(.)-2}, & in\ \Omega\times(0,T), \\ u = 0, & on\ \partial\Omega\times(0,T), \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & in\ \Omega, \end{cases}$$

and used the Galerkin method to establish the existence of a unique weak local solution. They also proved that the solutions with negative initial energy blow-up in finite time.

Another kind of system was considered in the literature: System of the wave equation which include linear or nonlinear first order perturbation order terms. Cavalcanti and Soriano [14] considered the following hyperbolic boundary problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u + au + \sum b_i \frac{\partial u}{\partial x_i} = 0, & in\ \Omega \times (0,\infty), \\ u = 0, & on\ \Gamma_1 \times (0,\infty), \\ \frac{\partial u}{\partial v} + \beta(x) \frac{\partial u}{\partial t} = 0 & on\ \Gamma_2 \times (0,\infty), \\ u(x,0) = u_0,\ \frac{\partial u}{\partial t}(x,0) = u_1, & in\ \Omega, \end{cases}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$, with restrictions on $a,\ b_i$ and β. They proved the existence, uniqueness and uniform stability of strong solutions. Cavalcanti et al. [13] studied the following degenerate hyperbolic equations with boundary damping

$$\begin{cases} K(x,t) u_{tt} - \Delta u + F(x,t,u,u_t,\nabla u) = f, & in\ \Omega \times (0,\infty), \\ u = 0, & on\ \Gamma_1 \times (0,\infty), \\ \frac{\partial u}{\partial v} + b(x) u_t = 0 & on\ \Gamma_2 \times (0,\infty), \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & in\ \Omega. \end{cases}$$

Under some assumptions, the authors proved the similar results. Cavalcanti and Guesmia [12] considered the following hyperbolic problem

$$\begin{cases} u_{tt} - \Delta u + F(x,t,u,u_t,\nabla u) = f, & in\ \Omega \times [0,+\infty[, \\ u = 0, & on\ \Gamma_1 \times [0,+\infty[, \\ u + \int_0^t g(t-s) \frac{\partial u}{\partial v}(s)\, ds = 0 & on\ \Gamma_2 \times [0,+\infty[, \\ u(x,0) = u_0,\ u_t(x,0) = u_1, & in\ \Omega. \end{cases}$$

They proved that the dissipation given by the memory term is strong enough to assure stability. Guesmia in [20] considered the following semi linear wave internal dissipative term

$$\begin{cases} u_{tt} - \Delta u + h\left(\nabla u\right) + f\left(u\right) + g\left(u_t\right) = 0, & in\ \Omega \times [0, +\infty[, \\ \\ u = 0, & on\ \partial\Omega \times [0, +\infty[, \\ \\ u\left(x, 0\right) = u_0,\ u_t\left(x, 0\right) = u_1, & in\ \Omega, \end{cases}$$

and the nonlinear boundary feedback

$$\begin{cases} u_{tt} - \Delta u + h\left(\nabla u\right) + f\left(u\right) = 0, & in\ \Omega \times [0, +\infty[, \\ u = 0, & on\ \Gamma_1 \times [0, +\infty[, \\ \partial_v u + g\left(u_t\right) = 0 & on\ \Gamma_2 \times [0, +\infty[, \\ u\left(x, 0\right) = u_0,\ u_t\left(x, 0\right) = u_1, & in\ \Omega, \end{cases}$$

where $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ are given continuous nonlinear functions. Hamchi, in [22], considered the case of linear first order perturbation and improved the result in [20] where she proved an energy decay of solution without any condition on smallness of this term. In her work, she introduced a new geometric multiplier to handle the linear first order term and used a suitable nonlinear version of a compactness uniqueness argument in [16] to absorb the lower order terms.

Main contribution. The purpose of this thesis is to study the problem of global existence and stability of solution for some semi linear hyperbolic systems. The first goal considers the study of the global existence and stability of solution for the nonlinear hyperbolic equation with damping and source terms. The second goal is also centered on the study of these problems for the nonlinear wave equation with damping and source terms of variable-exponent types. The third one is to prove that solution of the wave equation with damping, source and nonlinear first order perturbation terms is global.

Organization of the thesis. This thesis consists of three main chapters in addition to general introduction and conclusion. In the first chapter, we consider the following hyperbolic equation with source and damping terms

$$\begin{cases} u_{tt} - div\left(A\nabla u\right) + u_t \left|u_t\right|^{m-2} = u\left|u\right|^{p-2}, & in\Omega \times (0,T),\\ u\left(x,t\right) = 0, & on\partial\Omega \times (0,T),\\ u\left(x,0\right) = u_0\left(x\right), u_t\left(x,0\right) = u_1\left(x\right), & in\Omega, \end{cases}$$

where $p,\ m \geq 2$. First, we present the theorem of existence of solution. Next, we show that the energy is decreasing and we use some assumptions for initial data to prove a global existence result. Then, by applying an integral inequality due to Komornik, we obtain the stability result. In the second chapter, we consider the following nonlinear wave equation with damping and source terms of variable-exponent types

$$\begin{cases} u_{tt} - \Delta u + u_t \left|u_t\right|^{m(.)-2} = u\left|u\right|^{p(.)-2}, & in\ \Omega \times (0,T),\\ u = 0, & on\ \partial\Omega \times (0,T),\\ u\left(x,0\right) = u_0, u_t\left(x,0\right) = u_1, & in\ \Omega, \end{cases}$$

where m and p are measurable functions in Ω. First, we present the result of S. Messaoudi in [36] concerning the existence of the local solution of this system. Next, we show that the energy is decreasing and by some assumptions for initial data and Komornik's integral inequality, we obtain the global and stability results. In the third chapter, we consider the following nonlinear wave equation with damping, source and nonlinear first order perturbation terms

$$\begin{cases} u_{tt} - \Delta u + g * \Delta u + a\left(x\right) u_t + F\left(t, \nabla u\right) = \left|u\right|^{p-2} u, & in\ \Omega \times (0,T),\\ u = 0, & on\ \partial\Omega \times (0,T),\\ u\left(x,0\right) = u_0, u_t\left(x,0\right) = u_1, & in\ \Omega, \end{cases}$$

where $g,\ a$ and F are a functions satisfying some conditions to be specified later. Noting that $g * \Delta u = \int_0^t g\left(t-\tau\right)\Delta u\left(\tau\right) d\tau$. First, we show that if the damping terms (linear and viscoelastic) dominated the nonlinear first order perturbation terms then the usual energy is decreasing. After that, we use some assumptions for initial data to prove that the solution of this system is global. Finally, in the conclusion we summarize our results and give some perspectives.

CHAPTER 1

GLOBAL EXISTENCE AND STABILITY OF HYPERBOLIC EQUATION WITH SOURCE AND DAMPING TERMS

1.1 Introduction

The result of this chapter is a case of a result presented in: Congrès des Mathématiciens Algériens, CMA 2016, Batna, 08-09 Novembre 2016.

In this chapter, we consider the following system

$$\begin{cases} u_{tt} - div\left(A\nabla u\right) + u_t \left|u_t\right|^{m-2} = u\left|u\right|^{p-2}, & in\ \Omega \times (0,T)\,, \\ u = 0, & on\ \partial\Omega \times (0,T)\,, \\ u\left(x,0\right) = u_0, u_t\left(x,0\right) = u_1, & in\ \Omega, \end{cases} \tag{P}$$

with $p, m \geq 2$ and $A = \left(a_{ij}\left(x,t\right)\right)_{i,j}$ where, for all $i, j = \overline{1,n}$, a_{ij} is a symmetric function (i.e., $a_{ij} = a_{ji}$) of class $C^1\left(\bar{\Omega} \times [0,+\infty[\right)$ and there exists a constant $a_0 > 0$

such that, for all $(x,t) \in \bar{\Omega} \times [0,+\infty[$ and $\zeta \in \mathbb{R}^n$, we have

$$A\zeta.\zeta \geq a_0 |\zeta|^2 \tag{1.1}$$

and

$$A^{'}\zeta.\zeta \leq 0. \tag{1.2}$$

For the constant case i.e. $A = I_n$, many results were obtained concerning the existence, stability and blow up of solution, see for example [6], [18], [23], [24], [27] and [38]. Concerning the variable coefficient case, the first attempt in this direction was made by Benabderrahmanne and Boukhatem in [8].
Our study have been conducted to analyze the existence of global solution of wave equation with damping and source terms. Then, we employ an integral inequality, due to Komornik [26], to establish a stability result.

1.2 Preliminary results

Regarding the solution of the system (P), we can use the idea of Wu and Tsai in [41] to prove the following theorem.

Theorem 1.1. *If $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ then there exists a unique strong solution u in $(0,T)$ of* (P) *satisfying*

$$\begin{aligned}
&u \in C\left((0,T), H_0^1(\Omega) \cap H^2(\Omega)\right),\\
&u_t \in C\left((0,T), L^2(\Omega) \cap L^2\left((0,T), H_0^1(\Omega)\right)\right),\\
&u_{tt} \in L^2\left((0,T), L^2(\Omega)\right),
\end{aligned}$$

Moreover, the following alternatives hold

$$(i) \quad T = +\infty, \qquad \textit{or}$$

$$(ii) \quad T < +\infty \textit{ and } \lim_{t \to T}\left(\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2\right) = +\infty.$$

The decay of the energy of the system (P) is given in the following lemma.

Lemma 1.1. *Suppose that* (1.2) *holds. The energy E of the system* (P) *is a decreasing function. Here*

$$E(t) = \frac{1}{2}\int_\Omega A\nabla u(t)\nabla u(t)\,dx + \frac{1}{2}\|u_t(t)\|_2^2 - \frac{1}{p}\int_\Omega |u(t)|^p\,dx.$$

Proof. It is enough to multiply the first equation in (P) by u_t and integrate over Ω, to obtain

$$\int_\Omega u_{tt}(t)\, u_t(t)\, dx - \int_\Omega div\left(A\nabla u(t)\right) u_t(t)\, dx + \int_\Omega |u_t(t)|^m\, dx = \int_\Omega |u(t)|^{p-2}\, u(t)\, u_t(t)\, dx.$$

Then, we use the generalized Green formula and the boundary conditions, to find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t(t)|^2\, dx + \int_\Omega A\nabla u(t)\, \nabla u_t(t)\, dx + \int_\Omega |u_t(t)|^m\, dx = \frac{1}{p}\frac{d}{dt}\int_\Omega |u(t)|^p\, dx.$$

This implies that

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_\Omega |u_t(t)|^2\, dx + \tfrac{1}{2}\tfrac{d}{dt}\int_\Omega A\nabla u(t)\, \nabla u(t)\, dx - \tfrac{1}{2}\int_\Omega A'\nabla u(t)\, \nabla u(t)\, dx \\ &\quad + \textstyle\int_\Omega |u_t(t)|^m\, dx = \tfrac{1}{p}\tfrac{d}{dt}\int_\Omega |u(t)|^p dx. \end{aligned}$$

So

$$\tfrac{d}{dt}E(t) = \tfrac{1}{2}\textstyle\int_\Omega A'\nabla u(t)\, \nabla u(t)\, dx - \int_\Omega |u_t(t)|^m\, dx. \tag{1.3}$$

Taking into account the hypotheses on A, we find

$$\frac{d}{dt}E(t) \leq -\int_\Omega |u_t(t)|^m\, dx \leq 0. \tag{1.4}$$

□

1.3 Global existence result

To study the problem of global existence of the solution of the system (P), we define the following functions, for all $t \in (0, T)$

$$I(t) = \int_\Omega A\nabla u(t)\, \nabla u(t)\, dx - \int_\Omega |u(t)|^p\, dx$$

and

$$J(t)=\frac{1}{2}\int_{\Omega}A\nabla u(t)\nabla u(t)\,dx-\frac{1}{p}\int_{\Omega}|u(t)|^{p}\,dx.$$

The proof of the first main result of this chapter requires the following lemma.

Lemma 1.2. *Suppose that*

$$2<p\leq 2\frac{n-1}{n-2},\qquad n\geq 3.$$

Under the assumptions of Lemma 1.1 and supposing that

$$I(0)>0$$

and

$$\beta:=\frac{C_*^p}{a_0}\left(\frac{2p}{a_0(p-2)}E(0)\right)^{\frac{p-2}{2}}<1, \tag{1.5}$$

where C_* is the best constant of the embedding $H_0^1(\Omega)\hookrightarrow L^p(\Omega)$, we have

$$I(t)>0 \; for\ all\ t\ \in(0,T).$$

Proof. By continuity, there exists T_m where $0<T_m<T$ such that

$$I(t)\geq 0,\quad \forall t\in(0,T_m). \tag{1.6}$$

Now, we will prove that this inequality is strict, we have for all $t\in(0,T_m)$

$$\begin{aligned}
J(t) &= \tfrac{1}{2}\textstyle\int_{\Omega}A\nabla u(t)\nabla u(t)\,dx-\tfrac{1}{p}\int_{\Omega}|u(t)|^{p}\,dx\\
&= \tfrac{1}{2}\textstyle\int_{\Omega}A\nabla u(t)\nabla u(t)\,dx-\tfrac{1}{p}\left[\int_{\Omega}A\nabla u(t)\nabla u(t)\,dx-I(t)\right]\\
&= \tfrac{p-2}{2p}\textstyle\int_{\Omega}A\nabla u(t)\nabla u(t)\,dx+\tfrac{1}{p}I(t).
\end{aligned}$$

Using (1.6), we obtain

$$\int_{\Omega} A\nabla u(t)\nabla u(t)\,dx \leq \frac{2p}{p-2}J(t), \qquad \forall t \in (0,T_m). \tag{1.7}$$

By assumption on A, (1.7) and the definition of E, we find

$$\|\nabla u(t)\|_2^2 \leq \frac{1}{a_0}\int_{\Omega} A\nabla u(t)\nabla u(t)\,dx \leq \frac{2p}{a_0(p-2)}E(t). \tag{1.8}$$

Using the nonincreasingness of E, we obtain

$$\|\nabla u(t)\|_2^2 \leq \frac{2p}{a_0(p-2)}E(0). \tag{1.9}$$

We have

$$\textstyle\int_{\Omega} |u(t)|^p\,dx \quad \leq C_*^p \|\nabla u(t)\|_2^p \quad = C_*^p \|\nabla u(t)\|_2^{p-2}\|\nabla u(t)\|_2^2.$$

By (1.9), we obtain

$$\int_{\Omega} |u(t)|^p\,dx \leq C_*^p\left(\frac{2p}{a_0(p-2)}E(0)\right)^{\frac{p-2}{2}}\|\nabla u(t)\|_2^2.$$

Using (1.1), we obtain

$$\int_{\Omega} |u(t)|^p\,dx \leq \beta\int_{\Omega} A\nabla u(t)\nabla u(t)\,dx. \tag{1.10}$$

Since $\beta < 1$ then

$$\int_{\Omega} |u(t)|^p\,dx < \int_{\Omega} A\nabla u(t)\nabla u(t)\,dx, \forall t \in (0,T_m).$$

This implies that

$$I(t) > 0, \quad \forall t \in (0,T_m).$$

Since

$$I\left(T_m\right) > 0$$

and

$$\frac{C_*^p}{a_0}\left(\frac{2p}{a_0\left(p-2\right)}E\left(T_m\right)\right)^{\frac{p-2}{2}} \leq \beta < 1.$$

Then, by repeating the above procedure, we extend T_m to T. $\square$

The first main result of this chapter is given in the following theorem.

Theorem 1.2. *The solution u of* (P) *is global.*

Proof. We have to prove that

$$\int_\Omega A\nabla u\left(t\right)\nabla u\left(t\right)dx + \left\|u_t\left(t\right)\right\|_2^2, \tag{1.11}$$

is bounded independently of t.
Based on the definition of E and using (1.7), we get

$$\begin{aligned} E\left(t\right) &= J\left(t\right) + \tfrac{1}{2}\left\|u_t\left(t\right)\right\|_2^2 \\ &\geq \tfrac{p-2}{2p}\int_\Omega A\nabla u\left(t\right)\nabla u\left(t\right)dx + \tfrac{1}{2}\left\|u_t\left(t\right)\right\|_2^2. \end{aligned} \tag{1.12}$$

Consequently, there exist a constant $C > 0$ such that

$$\int_\Omega A\nabla u\left(t\right)\nabla u\left(t\right)dx + \left\|u_t\left(t\right)\right\|_2^2 \leq CE\left(t\right). \tag{1.13}$$

By the nonincreasingness of E, we find

$$\int_\Omega A\nabla u\left(t\right)\nabla u\left(t\right)dx + \left\|u_t\left(t\right)\right\|_2^2 \leq CE\left(0\right).$$

$\square$

1.4 Stability result

To prove the second main result of this chapter, we establish the following lemma.

Lemma 1.3. *Suppose that*

$$2 < m \leq \frac{2n}{n-2}, \qquad n \geq 3.$$

Under the assumptions of Lemma 1.2, there exists a positive constant C such that the global solution u of (P) *satisfies*

$$\int_{\Omega} |u(t)|^m \, dx \leq CE(t), \; for \; all \; t \geq 0.$$

Proof. Let c_* be the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^m(\Omega)$. Then, we have

$$\begin{aligned} \textstyle\int_{\Omega} |u(t)|^m \, dx &\leq c_*^m \|\nabla u(t)\|_2^m \\ &= c_*^m \|\nabla u(t)\|_2^{m-2} \|\nabla u(t)\|_2^2 . \end{aligned}$$

By (1.8), we obtain

$$\int_{\Omega} |u(t)|^m \, dx \leq \frac{2pc_*^m}{a_0(p-2)} \|\nabla u(t)\|_2^{m-2} E(t) .$$

Then, by (1.9), we obtain

$$\textstyle\int_{\Omega} |u(t)|^m \, dx \leq \frac{2pc_*^m}{a_0(p-2)} \left(\frac{2p}{a_0(p-2)} E(0) \right)^{\frac{m-2}{2}} E(t) .$$

$\square$

The second main result of this chapter is given in the following theorem.

Theorem 1.3. *Under the assumptions of Lemma 1.3, there exists $C, \; \omega > 0$ such that the global solution of the system (P) satisfies*

$$\begin{cases} E(t) \leq \frac{CE(0)}{(1+t)^{2/(m-2)}}, \; \forall t \geq 0 \; if \; m > 2. \\ \\ E(t) \leq CE(0) \, e^{-\omega t}, \; \forall t \geq 0 \; if \; m = 2. \end{cases}$$

Proof. Let $T > S > 0$ and $q \geq 0$. Multiplying the first equation of (P) by $u(t) E^q(t)$ and integrating over $\Omega \times (S, T)$, to obtain

$$\int_S^T \int_\Omega E^q(t) \left[u(t) u_{tt}(t) - u(t) \, div\left(A\nabla u(t)\right) + u(t) u_t(t) \left|u_t(t)\right|^{m-2}\right] dxdt$$
$$= \int_S^T \int_\Omega E^q(t) \left|u(t)\right|^p dxdt.$$

This implies that

$$\int_S^T \int_\Omega E^q(t) \left[\left(u(t) u_t(t)\right)_t - \left|u_t(t)\right|^2 + A\nabla u(t) \nabla u(t) + u(t) u_t(t) \left|u_t(t)\right|^{m-2}\right] dxdt$$
$$= \int_S^T \int_\Omega E^q(t) \left|u(t)\right|^p dxdt.$$

We add, subtract the term

$$\int_S^T \int_\Omega E^q(t) \left[\beta A\nabla u(t) \nabla u(t) + (1+\beta) \left|u_t(t)\right|^2\right] dxdt$$

and use (1.10) to get

$$(1-\beta) \textstyle\int_S^T E^q(t) \int_\Omega \left(A\nabla u(t) \nabla u(t) + \left|u_t(t)\right|^2\right) dxdt$$
$$+ \textstyle\int_S^T E^q(t) \int_\Omega \left[\left(u(t) u_t(t)\right)_t - (2-\beta) \left|u_t(t)\right|^2\right] dxdt$$
$$+ \textstyle\int_S^T E^q(t) \int_\Omega u(t) u_t(t) \left|u_t(t)\right|^{m-2} dxdt$$
$$= - \textstyle\int_S^T E^q(t) \int_\Omega \left[\beta A\nabla u(t) \nabla u(t) - \left|u(t)\right|^p\right] dxdt \leq 0.$$

Then

$$\begin{aligned}
&(1-\beta)\int_S^T E^q(t)\int_\Omega \left(A\nabla u(t)\nabla u(t)+|u_t(t)|^2-\tfrac{2}{p}|u(t)|^p\right)dxdt\\
&\quad\le -\int_S^T E^q(t)\int_\Omega (u(t)u_t(t))_t\,dxdt\\
&\quad-\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u_t(t)|^{m-2}\,dxdt\\
&\quad+(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt.
\end{aligned}\tag{1.14}$$

Using the definition of E and the following relation

$$\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)u_t(t)\,dx\right)=qE^{q-1}(t)E'(t)\int_\Omega u(t)u_t(t)\,dx+E^q(t)\int_\Omega (u(t)u_t(t))_t\,dx,$$

inequality (1.14) turns into

$$\begin{aligned}
&2(1-\beta)\int_S^T E^{q+1}(t)\,dt\\
&\quad\le q\int_S^T E^{q-1}(t)E'(t)\int_\Omega u(t)u_t(t)\,dxdt-\int_S^T \tfrac{d}{dt}\left(E^q(t)\int_\Omega u(t)u_t(t)\,dx\right)dt\\
&\quad-\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u_t(t)|^{m-2}\,dxdt+(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt.
\end{aligned}\tag{1.15}$$

Let C be a positive generic constant. We estimate the terms in the right of (1.15) as follows

$$\begin{aligned}
&q\int_S^T E^{q-1}(t)E'(t)\int_\Omega u(t)u_t(t)\,dxdt\\
&\quad\le q\int_S^T E^{q-1}(t)\left(-E'(t)\right)\left[\tfrac{1}{2}\int_\Omega |u(t)|^2\,dx+\tfrac{1}{2}\int_\Omega |u_t(t)|^2\,dx\right]dt\\
&\quad\le C\int_S^T E^{q-1}(t)\left(-E'(t)\right)\left[\int_\Omega |\nabla u(t)|^2\,dx+\int_\Omega |u_t(t)|^2\,dx\right]dt\\
&\quad\le C\int_S^T E^{q-1}(t)\left(-E'(t)\right)\left[\int_\Omega \tfrac{1}{a_0}A\nabla u(t)\nabla u(t)\,dx+\int_\Omega |u_t(t)|^2\,dx\right]dt.
\end{aligned}$$

Based on the definition of E and using (1.7) we get

$$\begin{aligned} q\int_S^T E^{q-1}(t)\,E'(t)\int_\Omega u(t)\,u_t(t)\,dxdt &\leq C\int_S^T E^q(t)\left(-E'(t)\right)dt \\ &\leq CE^{q+1}(S)-CE^{q+1}(T)\,. \end{aligned}$$

That is

$$q\int_S^T E^{q-1}(t)\,E'(t)\int_\Omega u(t)\,u_t(t)\,dxdt \leq CE^q(0)\,E(S)\,. \tag{1.16}$$

For the second term, we have

$$\begin{aligned} &-\int_S^T \tfrac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dt \\ &\quad\leq \left|E^q(S)\int_\Omega u(x,S)\,u_t(x,S)\,dx - E^q(T)\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \\ &\quad\leq E^q(S)\left|\int_\Omega u(x,S)\,u_t(x,S)\,dx\right| + E^q(T)\left|\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \\ &\quad\leq CE^{q+1}(S)+CE^{q+1}(T)\,. \end{aligned}$$

That is

$$-\int_S^T \frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dt \leq CE^q(0)\,E(S)\,. \tag{1.17}$$

For the third term, we use the following Young inequality

$$XY \leq \frac{\epsilon}{r_1}X^{r_1} + \frac{1}{r_2\epsilon^{r_2/r_1}}Y^{r_2},\ X,\ Y\geq 0,\ \epsilon>0\ and\ \frac{1}{r_1}+\frac{1}{r_2} = 1. \tag{1.18}$$

With $r_1 = m$ and $r_2 = \frac{m}{m-1}$, to find

$$|u(t)|\,|u_t(t)|^{m-1} \leq \epsilon C\,|u(t)|^m + C_\epsilon\,|u_t(t)|^m\,.$$

By (1.4) and Lemma 1.3 , we obtain

$$-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m-2}\,dxdt$$

$$\leq \int_S^T E^q(t)\left[\epsilon C\int_\Omega |u(t)|^m\,dx + C_\epsilon\int_\Omega |u_t(t)|^m\,dx\right]dt$$

$$\leq \epsilon C\int_S^T E^q(t)\int_\Omega |u(t)|^m\,dxdt + C_\epsilon\int_S^T E^q(t)\left(-E'(t)\right)dt.$$

That is

$$-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m-2}\,dxdt \leq \epsilon C\int_S^T E^{q+1}(t) + C_\epsilon E^q(0)\,E(S)\,. \tag{1.19}$$

For the last term of (1.15), we have

$$(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \leq C\int_S^T E^q(t)\left(\int_\Omega |u_t(t)|^m\right)^{2/m}dxdt.$$

This implies that

$$(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \leq C\int_S^T E^q(t)\left(-E'(t)\right)^{2/m}dt. \tag{1.20}$$

But if we use Young inequality (1.18) with $r_1 = \frac{q+1}{q}$ and $r_2 = q+1$, we obtain

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/m}dt \leq \epsilon C\int_S^T E^{q+1}(t)\,dt + \frac{1}{\epsilon_C}\int_S^T \left(-E'(t)\right)^{2(q+1)/m}dt.$$

We take $q = \frac{m}{2} - 1$ to find

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/m}dt \leq \epsilon C\int_S^T E^{q+1}(t)\,dt + C_\epsilon\int_S^T \left(-E'(t)\right)dt.$$

This implies that

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/m}dt \leq \epsilon C\int_S^T E^{q+1}(t)\,dt + C_\epsilon E(S)\,. \tag{1.21}$$

Replacing (1.21) in (1.20) to find

$$(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \le \epsilon C\int_S^T E^{q+1}(t)\,dt + C_\epsilon E(S). \tag{1.22}$$

By substituting (1.16), (1.17), (1.19) and (1.22) in (1.15) (with $q=\frac{m}{2}-1$), we arrive to

$$2(1-\beta)\int_S^T E^{\frac{m}{2}}(t)\,dt \le \epsilon C\int_S^T E^{\frac{m}{2}}(t) + C_\epsilon E^{\frac{m}{2}-1}(0)\,E(S).$$

Choosing ϵ small enough, to find

$$\int_S^T E^{\frac{m}{2}}(t)\,dt \le CE^{\frac{m}{2}-1}(0)\,E(S).$$

By taking $T\longrightarrow\infty$, we get

$$\int_S^\infty E^{\frac{m}{2}}(t) \le CE^{\frac{m}{2}-1}(0)\,E(S).$$

Komornik's integral inequality [26] then yields the desired result. □

CHAPTER 2

GLOBAL EXISTENCE AND STABILITY OF A NONLINEAR WAVE EQUATION WITH VARIABLE EXPONENT NONLINEARITIES

2.1 Introduction

This chapter is the subject of an article written in collaboration with Dr. Hamchi (University of Batna 2) and Pr. Messaoudi (University of Dhahran Arabie Saoudite). This article was published in Applicable Analysis Journal.

In this chapter, we consider the following problem

$$\begin{cases} u_{tt} - \Delta u + u_t \left|u_t\right|^{m(.)-2} = u \left|u\right|^{p(.)-2}, & in\ \Omega \times (0,T), \\ u = 0, & on\ \partial\Omega \times (0,T), \\ u(x,0) = u_0, u_t(x,0) = u_1, & in\ \Omega, \end{cases} \tag{P}$$

where $p(.)$ and $m(.)$ are given measurables functions defined on Ω.

In the case when $m,\ p$ are constants, there have been many results about the existence and blow-up properties of the solutions, we refer the readers to the bibliography given in [9], [10], [32] and [33]. In the recent years, much attention have been paid to the study of problem with variable exponents. More details on these problems can be found in [1], [2], [3], [4] and [5].
In this chapter, we use some assumptions for initial data to prove the global existence of solution. Then, we employ an integral inequality, due to Komornik [26], to establish a stability result.

2.2 Preliminary results

Let $q : \Omega \longrightarrow [1, \infty]$ be a measurable function. We define the Lebesgue space with variable exponent $q\,(.)$ by

$$L^{q(.)}\,(\Omega) := \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable and } \int_\Omega |\lambda u\,(x)|^{q(x)}\, dx < \infty \text{ for some } \lambda > 0\}.$$

Equipped with the following Luxembourg-type norm

$$\|u\|_{q(.)} := \inf\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{q(x)} dx \leq 1\}$$

is a Banach space.
To obtain our results, we need the following Lemma [39]

Lemma 2.1. *If*

$$1 \leq q_1 := ess \inf_{x\in\Omega} q\,(x) \leq q\,(x) \leq q_2 := ess \sup_{x\in\Omega} q\,(x) < \infty,$$

then

$$\min\{\|u\|_{q(.)}^{q_1}, \|u\|_{q(.)}^{q_2}\} \leq \int_\Omega |u\,(.)|^{q(x)}\, dx \leq \max\{\|u\|_{q(.)}^{q_1}, \|u\|_{q(.)}^{q_2}\},$$

for any $u \in L^{q(.)}\,(\Omega)$.

For the existence of the local solution of problem (P), we refer the reader to [36]. Their result is given in the following theorem.

Theorem 2.1. *Suppose that* $m,\ p \in C\left(\bar{\Omega}\right)$ *with*

$$2 \leq m_1 \leq m(x) \leq m_2 < \tfrac{2n}{n-2}, \quad if\ n \geq 3.$$

$$m(x) \geq 2, \qquad\qquad if\ n = 1, 2,$$

and

$$2 \leq p_1 \leq p(x) \leq p_2 < 2\tfrac{n-1}{n-2}, \quad if\ n \geq 3,$$

$$p(x) \geq 2, \qquad\qquad if\ n = 1, 2,$$

where

$$m_1 := ess\inf_{x\in\Omega} m(x), \qquad m_2 := ess\sup_{x\in\Omega} m(x)$$

and

$$p_1 := ess\inf_{x\in\Omega} p(x), \qquad p_2 := ess\sup_{x\in\Omega} p(x).$$

Assume further that $m(.)\,,\ p(.)$ *verify the log-Hölder continuity condition*

$$|q(x) - q(y)| \leq -\tfrac{A}{\log|x-y|} \quad \textit{for all } x,\ y \in \Omega,$$

$$with\ |x-y| < \delta,\ A > 0,\ 0 < \delta < 1.$$

Then, for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$*, problem* (P) *has a unique local solution*

$$\begin{aligned}
&u \in L^\infty\left((0,T), H_0^1(\Omega)\right),\\
&u_t \in L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{m(.)}\left(\Omega \times (0,T)\right),\\
&u_{tt} \in L^2\left((0,T), H^{-1}(\Omega)\right),
\end{aligned}$$

Now, if $(u_0, u_1) \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$*, problem* (P) *has a unique strong solution*

$$\begin{aligned}
&u \in L^\infty\left((0,T), H_0^1(\Omega) \cap H^2(\Omega)\right),\\
&u_t \in L^\infty\left((0,T), H_0^1(\Omega) \cap L^{m(.)}\left(\Omega \times (0,T)\right)\right),\\
&u_{tt} \in L^2\left((0,T), L^2(\Omega)\right),
\end{aligned}$$

for some $T > 0$.

2.3 Global existence result

In order to state and prove our result, we define the following functionals, for all $t \in (0,T)$

$$I(t) = \|\nabla u(t)\|_2^2 - \int_\Omega |u(t)|^{p(x)}\, dx,$$

$$J(t) = \frac{1}{2}\|\nabla u(t)\|_2^2 - \int_\Omega \frac{|u(t)|^{p(x)}}{p(x)} dx,$$

and

$$E(t) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2 .$$

The proof of the first main result in this chapter requires the following two lemmas.

Lemma 2.2. *Under the assumptions of Theorem 2.1, E is a decreasing function.*

Proof. It is enough to multiply the first equation in (P) by u_t, integrate it over Ω, to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t(t)|^2\, dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u(t)|^2\, dx + \int_\Omega |u_t(t)|^{m(x)}\, dx = \frac{d}{dt}\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)} dx.$$

This implies that

$$\frac{d}{dt}E(t) = -\int_\Omega |u_t(t)|^{m(x)}\, dx \leq 0, \quad \forall t \in (0,T). \tag{2.1}$$

□

Lemma 2.3. *Under the assumptions of Theorem 2.1 such that*

$$I(0) > 0$$

and

$$\beta := \max\left\{ C_*^{p_1}\left(\frac{2p_1}{p_1-2}E(0)\right)^{\frac{p_1-2}{2}}, C_*^{p_2}\left(\frac{2p_1}{p_1-2}E(0)\right)^{\frac{p_2-2}{2}} \right\} < 1, \tag{2.2}$$

where C_* is the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$. We have

$$I(t) > 0 \; for \; all \; t \in (0,T).$$

Proof. By continuity, there exists T_m, where $0 < T_m \leq T$, such that

$$I(t) \geq 0, \quad \forall t \in (0,T_m). \tag{2.3}$$

Now, we will prove that this inequality is strict. We have for all $t \in (0,T)$

$$\begin{aligned} J(t) &= \tfrac{1}{2}\|\nabla u(t)\|_2^2 - \textstyle\int_\Omega \tfrac{|u(t)|^{p(x)}}{p(x)}dx \\ &\geq \tfrac{1}{2}\|\nabla u(t)\|_2^2 - \tfrac{1}{p_1}\left[\|\nabla u(t)\|_2^2 - I(t)\right] \\ &\geq \tfrac{p_1-2}{2p_1}\|\nabla u(t)\|_2^2 + \tfrac{1}{p_1}I(t). \end{aligned}$$

Using (2.3), we obtain

$$\|\nabla u(t)\|_2^2 \leq \frac{2p_1}{p_1-2}J(t), \forall t \in (0,T_m). \tag{2.4}$$

By the definition of E, we find

$$\|\nabla u(t)\|_2^2 \leq \frac{2p_1}{p_1-2}E(t). \tag{2.5}$$

Using the nonincreasingness of E, we obtain

$$\|\nabla u(t)\|_2^2 \leq \frac{2p_1}{p_1-2}E(0), \forall t \in (0,T_m). \tag{2.6}$$

On the other hand, by Lemma 2.1, we have

$$\int_\Omega |u(t)|^{p(x)}dx \leq \max\left\{\|u(t)\|_{p(.)}^{p_1}, \|u(t)\|_{p(.)}^{p_2}\right\}.$$

Hence, we have

$$\begin{aligned}\int_\Omega |u(t)|^{p(x)}\,dx &\le \max\left\{C_*^{p_1}\|\nabla u(t)\|_2^{p_1}, C_*^{p_2}\|\nabla u(t)\|_2^{p_2}\right\}\\ &\le \max\left\{C_*^{p_1}\|\nabla u(t)\|_2^{p_1-2}, C_*^{p_2}\|\nabla u(t)\|_2^{p_2-2}\right\}\|\nabla u(t)\|_2^2.\end{aligned}$$

By (2.6), we obtain

$$\int_\Omega |u(t)|^{p(x)}\,dx \le \beta\|\nabla u(t)\|_2^2, \forall t \in (0, T_m). \tag{2.7}$$

Since $\beta < 1$ then

$$\int_\Omega |u(t)|^{p(x)}\,dx < \|\nabla u(t)\|_2^2, \forall t \in (0, T_m).$$

This implies that

$$I(t) > 0, \quad \forall t \in (0, T_m),$$

hence

$$I(T_m) > 0$$

and

$$\max\left\{C_*^{p_1}\left(\frac{2p_1}{p_1-2}E(T_m)\right)^{\frac{p_1-2}{2}}, C_*^{p_2}\left(\frac{2p_1}{p_1-2}E(T_m)\right)^{\frac{p_2-2}{2}}\right\} \le \beta < 1.$$

Then, by repeating the above procedure, we extend T_m to T. □

The first main result in this chapter is given in the following theorem.

Theorem 2.2. *Under the assumptions of Lemma 2.3, the local solution u of* (P) *is global.*

Proof. Based on the definition of E and using (2.4) we get

$$\begin{aligned} E(t) &= J(t) + \tfrac{1}{2}\|u_t(t)\|_2^2 \\ &\geq \tfrac{p_1-2}{2p_1}\|\nabla u(t)\|_2^2 + \tfrac{1}{2}\|u_t(t)\|_2^2 . \end{aligned}$$

Consequently, a constant $C > 0$ exists such that

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq CE(t). \tag{2.8}$$

By using the nonincreasingness of E, we find

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq CE(0).$$

This implies that the local solution u of (P) is global and bounded. □

2.4 Stability result

To prove the second main result in this chapter, we establish the following lemma.

Lemma 2.4. *Under the assumptions of Lemma 2.3, there exists a positive constant C, such that the global solution u of* (P) *satisfies*

$$\int_\Omega |u(t)|^{m(x)}\,dx \leq CE(t), \ for\ all\ t \geq 0.$$

Proof. Let c_* be the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{m(.)}(\Omega)$; then, we have

$$\begin{aligned} \textstyle\int_\Omega |u(t)|^{m(x)}\,dx &\leq \max\left\{\|u(t)\|_{m(.)}^{m_1}, \|u(t)\|_{m(.)}^{m_2}\right\} \\ &\leq \max\{c_*^{m_1}\|\nabla u(t)\|_2^{m_1}, c_*^{m_2}\|\nabla u(t)\|_2^{m_2}\} \\ &\leq \max\left\{c_*^{m_1}\|\nabla u(t)\|_2^{m_1-2}, c_*^{m_2}\|\nabla u(t)\|_2^{m_2-2}\right\}\|\nabla u(t)\|_2^2 . \end{aligned}$$

By (2.5), we obtain the desired result

$$\int_\Omega |u(t)|^{m(x)}\,dx \leq \max\left\{c_*^{m_1}\|\nabla u(t)\|_2^{m_1-2}, c_*^{m_2}\|\nabla u(t)\|_2^{m_2-2}\right\}\frac{2p_1}{p_1-2}E(t).$$

□

The following theorem gives the stability of the global solution u.

Theorem 2.3. *Under the assumptions of Lemma 2.3, there exists constants* $C,\ \omega > 0$ *such that the global solution of* (P) *satisfies*

$$\begin{cases} E(t) \leq \frac{CE(0)}{(1+t)^{2/(m_2-2)}},\ \forall t \geq 0\ if\ m_2 > 2. \\ \\ E(t) \leq CE(0)\, e^{-\omega t},\ \forall t \geq 0\ if\ m_2 = 2. \end{cases}$$

Proof. Let $T > S > 0$ and $q \geq 0$. Multiplying (P) by $u(t)\, E^q(t)$ and integrating over $\Omega \times (S,T)$, we obtain

$$\int_S^T \int_\Omega E^q(t) \left[u(t)\, u_{tt}(t) - u(t)\, \Delta u(t) + u(t)\, u_t(t)\, |u_t(t)|^{m(x)-2}\right] dxdt$$
$$= \int_S^T \int_\Omega E^q\, |u(t)|^{p(x)}\, dxdt.$$

This implies that

$$\int_S^T \int_\Omega E^q(t) \left[(u(t)\, u_t(t))_t - |u_t(t)|^2 + |\nabla u(t)|^2 + u(t)\, u_t(t)\, |u_t(t)|^{m(x)-2}\right] dxdt$$
$$= \int_S^T \int_\Omega E^q(t)\, |u(t)|^{p(x)}\, dxdt.$$

We add and subtract the term

$$\int_S^T \int_\Omega E^q(t) \left[\beta\, |\nabla u(t)|^2 + (1+\beta)\, |u_t(t)|^2\right] dxdt$$

and use (2.7) to get

$$(1-\beta) \int_S^T E^q(t) \int_\Omega \left(|\nabla u(t)|^2 + |u_t(t)|^2\right) dxdt$$
$$+ \int_S^T E^q(t) \int_\Omega \left[(u(t)\, u_t(t))_t - (2-\beta)\, |u_t(t)|^2\right] dxdt$$
$$+ \int_S^T E^q(t) \int_\Omega u(t)\, u_t(t)\, |u_t(t)|^{m(x)-2}\, dxdt$$
$$= -\int_S^T E^q(t) \int_\Omega \left[\beta\, |\nabla u(t)|^2 - |u(t)|^{p(x)}\right] dxdt \leq 0.$$

Then

$$\begin{aligned}
&(1-\beta)\int_S^T E^q(t)\int_\Omega\left(|\nabla u(t)|^2+|u_t(t)|^2-2\tfrac{|u(t)|^{p(x)}}{p(x)}\right)dxdt\\
&\leq -\int_S^T E^q(t)\int_\Omega (u(t)\,u_t(t))_t\,dxdt\\
&+(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt\\
&-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}\,dxdt.
\end{aligned}\tag{2.9}$$

Using the definition of E and the following relation

$$\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)=qE^{q-1}(t)\,E'(t)\int_\Omega u(t)\,u_t(t)\,dx+E^q(t)\int_\Omega (u(t)\,u_t(t))_t\,dx,$$

inequality (2.9) turns into

$$\begin{aligned}
&2(1-\beta)\int_S^T E^{q+1}(t)\,dt\\
&\leq q\int_S^T E^{q-1}(t)\,E'(t)\int_\Omega u(t)\,u_t(t)\,dxdt-\int_S^T \tfrac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dt\\
&-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}\,dxdt+(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt.
\end{aligned}\tag{2.10}$$

Let C be a positive generic constant. We estimate the terms in the right-hand side of (2.10) as follows

$$\begin{aligned}
&q\int_S^T E^{q-1}(t)\,E'(t)\int_\Omega u(t)\,u_t(t)\,dxdt\\
&\leq q\int_S^T E^{q-1}(t)\left(-E'(t)\right)\left[\tfrac{1}{2}\int_\Omega |u(t)|^2\,dx+\tfrac{1}{2}\int_\Omega |u_t(t)|^2\,dx\right]dt\\
&\leq C\int_S^T E^{q-1}(t)\left(-E'(t)\right)\left[\int_\Omega |\nabla u(t)|^2\,dx+\int_\Omega |u_t(t)|^2\,dx\right]dt.
\end{aligned}$$

Thus, by (2.8), we find

$$\begin{aligned} q\int_S^T E^{q-1}(t)\,E'(t)\int_\Omega u(t)\,u_t(t)\,dxdt &\le C\int_S^T E^q(t)\left(-E'(t)\right)dt \\ &\le CE^{q+1}(S)-CE^{q+1}(T) \qquad (2.11) \\ &\le CE^q(0)\,E(S). \end{aligned}$$

For the second term, we have

$$\begin{aligned} &-\int_S^T \tfrac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dt \\ &\le \left|E^q(S)\int_\Omega u(x,S)\,u_t(x,S)\,dx - E^q(T)\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \\ &\le E^q(S)\left|\int_\Omega u(x,S)\,u_t(x,S)\,dx\right| + E^q(T)\left|\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \\ &\le CE^{q+1}(S)+CE^{q+1}(T) \\ &\le CE^q(0)\,E(S). \end{aligned} \tag{2.12}$$

For the third term, we use the following Young inequality

$$XY \le \frac{\epsilon}{r_1}X^{r_1} + \frac{1}{r_2\epsilon^{r_2/r_1}}Y^{r_2},\ X,\ Y\ge 0,\ \epsilon>0\ and\ \frac{1}{r_1}+\frac{1}{r_2} = 1.$$

With $r_1(x)=m(x)$, $r_2(x)=\frac{m(x)}{m(x)-1}$ and $\epsilon<1$, we find

$$|u(t)|\,|u_t(t)|^{m(x)-1} \le \epsilon C\,|u(t)|^{m(x)} + C_\epsilon\,|u_t(t)|^{m(x)}.$$

By (2.1) and Lemma 2.4 , we have

$$\begin{aligned}
&-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m(x)-2}\,dxdt \\
&\quad\le \int_S^T E^q(t)\left[\epsilon C\int_\Omega |u(t)|^{m(x)}\,dx + C_\epsilon\int_\Omega |u_t(t)|^{m(x)}\,dx\right]dt \\
&\quad\le \epsilon C\int_S^T E^q(t)\int_\Omega |u(t)|^{m(x)}\,dxdt + C_\epsilon\int_S^T E^q(t)\left(-E'(t)\right)dt \\
&\quad\le \epsilon C\int_S^T E^{q+1}(t) + C_\epsilon E^q(0)\,E(S).
\end{aligned} \tag{2.13}$$

For the last term of (2.10), we have

$$\begin{aligned}
&(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \\
&\quad= (2-\beta)\int_S^T E^q(t)\left[\int_{\Omega_-}|u_t(t)|^2\,dx + \int_{\Omega_+}|u_t(t)|^2\,dx\right]dt \\
&\quad\le C\int_S^T E^q(t)\left[\left(\int_{\Omega_-}|u_t(t)|^{m_2}\right)^{2/m_2}dx + \left(\int_{\Omega_+}|u_t(t)|^{m_1}\right)^{2/m_1}dx\right]dt \\
&\quad\le C\int_S^T E^q(t)\left[\left(\int_{\Omega}|u_t(t)|^{m(x)}\right)^{2/m_2}dx + \left(\int_{\Omega}|u_t(t)|^{m(x)}\right)^{2/m_1}dx\right]dt.
\end{aligned}$$

This implies

$$\begin{aligned}
(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \quad &\le C\int_S^T E^q(t)\left(-E'(t)\right)^{2/m_2}dt \\
&+C\int_S^T E^q(t)\left(-E'(t)\right)^{2/m_1}dt.
\end{aligned} \tag{2.14}$$

We use Young's inequality with $r_1 = \frac{q+1}{q}$ and $r_2 = q+1$, we have

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/m_2}dt \le \frac{q\epsilon}{q+1}\int_S^T E^{q+1}(t)\,dt + \frac{1}{(q+1)\,\epsilon^q}\int_S^T\left(-E'(t)\right)^{2(q+1)/m_2}dt.$$

We take $q = \frac{m_2}{2} - 1$ to find

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/m_2}dt \le \epsilon C\int_S^T E^{q+1}(t)\,dt + C_\epsilon\int_S^T\left(-E'(t)\right)dt.$$

This implies

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/m_2} dt \leq \epsilon C \int_S^T E^{q+1}(t)\,dt + C_\epsilon E(S). \tag{2.15}$$

On the other hand, we have

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/m_1} dt \leq \epsilon C \int_S^T E^{q+1}(t)\,dt + C_\epsilon E(S). \tag{2.16}$$

Indeed,

- if $m_1 = 2$ then

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/m_1} dt \leq CE(S) \leq \epsilon C \int_S^T E^{q+1}(t)\,dt + C_\epsilon E(S).$$

- if $m_1 > 2$, we use Young's inequality with $r_1 = \frac{m_1}{m_1-2}$ and $r_2 = \frac{m_1}{2}$, to obtain

$$\begin{aligned}\int_S^T E^q(t)\left(-E'(t)\right)^{2/m_1} &\leq \epsilon C \int_S^T E^{qm_1/(m_1-2)}(t)\,dt + C_\epsilon \int_S^T \left(-E'(t)\right) dt \\ &\leq \epsilon C \int_S^T E^{qm_1/(m_1-2)}(t)\,dt + C_\epsilon E(S).\end{aligned}$$

We notice that $\frac{qm_1}{m_1-2} = q + 1 + \frac{m_2-m_1}{m_1-2}$, then

$$\begin{aligned}\int_S^T E^q(t)\left(-E'(t)\right)^{2/m_1} &\leq \epsilon C \left(E(S)\right)^{\frac{m_2-m_1}{m_1-2}} \int_S^T E^{q+1}(t)\,dt + C_\epsilon E(S) \\ &\leq \epsilon C \int_S^T E^{q+1}(t)\,dt + C_\epsilon E(S).\end{aligned}$$

We insert (2.15) and (2.16) in (2.14) to obtain

$$(2-\beta)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \leq \epsilon C \int_S^T E^{q+1}(t)\,dt + C_\epsilon E(S). \tag{2.17}$$

By substituting (2.11), (2.12), (2.13) and (2.17) in (2.10) (with $q = \frac{m_2}{2} - 1$), we arrive to

$$2(1-\beta)\int_S^T E^{\frac{m_2}{2}}(t)\,dt \leq \epsilon C \int_S^T E^{\frac{m_2}{2}}(t) + C_\epsilon E^{\frac{m_2}{2}-1}(0)\,E(S).$$

Choosing $\epsilon < 1$ small enough

$$\int_S^T E^{\frac{m_2}{2}}(t)\,dt \leq C E^{\frac{m_2}{2}-1}(0)\,E(S)\,.$$

By taking $T \longrightarrow \infty$, we get

$$\int_S^\infty E^{\frac{m_2}{2}}(t) \leq C E^{\frac{m_2}{2}-1}(0)\,E(S)\,.$$

Komornik's integral inequality [26] then yields the desired result. $\square$

CHAPTER 3

GLOBAL EXISTENCE OF THE WAVE EQUATION WITH NONLINEAR FIRST ORDER PERTURBATION TERM

3.1 Introduction

This chapter is the subject of an article written in collaboration with Dr. Hamchi. This article was submitted.

In this chapter, we study the system

$$\begin{cases} u_{tt} - \Delta u + g * \Delta u + a(x) u_t + F(t, \nabla u) = |u|^{p-2} u, & in\, \Omega \times (0,T), \\ u = 0, & on\, \partial\Omega \times (0,T), \\ u(x,0) = u_0, u_t(x,0) = u_1, & in\, \Omega, \end{cases} \tag{P}$$

where

$$g * \upsilon = \int_0^t g(t-\tau)\, \upsilon(\tau)\, d\tau \qquad for\ all\ t \geq 0.$$

Under the following assumptions
(H_1) **Assumption on the viscoelastic damping term** g is a $C^1(\mathbb{R}_+)$ positive decreasing function satisfying

$$1 - \int_0^{\infty} g(s)\, ds = l > 0.$$

(H_2) **Assumption on the linear damping term**

$$0 < a_1 := \inf_{x \in \Omega} a(x) \leq a(x) \leq a_2 := \sup_{x \in \Omega} a(x) < \infty, \quad \forall x \in \Omega.$$

(H_3) **Assumption on the nonlinear perturbation term** F is a $C^1(\mathbb{R}^+ \times \mathbb{R}^n)$ function satisfying

$$|F(t, U)| \leq \sqrt{2a_1\, g(t)}\, |U|, \quad \forall t \geq 0, \quad \forall\, U \in \mathbb{R}^n.$$

(H_4) **Assumption on the source term**

$$p > 2 \quad if\; n = 1, 2$$

and

$$2 \leq p < \tfrac{2(n-1)}{n-2} \quad if\; n \geq 3.$$

In the absence of the nonlinear first order perturbation term, the global existence and nonexistence problems of the wave equation with polynomial source and damping terms have been largely studied by several authors, see for instance [7], [11], [18],[28], [30], [31] and [32].
Concerning the wave equation which include linear or nonlinear first perturbation order terms, an uniform decay rate estimates was obtained under strong hypothesis on these terms, see [12], [14] and [16]. Hamchi, in [22], considered the case of linear first order perturbation and improved the result in [20] where she has proved an energy decay of solution without any condition on smallness of this term.
In this chapter, we show that if the damping terms dominated the nonlinear first order perturbation terms then the usual energy is decreasing. After that, by some assumptions for initial data we prove our main result.

3.2 Preliminary results

As in [7] and [41], the existence result of (P) can be established by Galerkin method combined with the well known contraction mapping theorem. This result is given in the following theorem.

Theorem 3.1. *If $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ then there exists a unique maximal solution u in $(0,T)$ of (P) satisfying*

$$\begin{aligned}
&u \in C\left((0,T), H_0^1(\Omega) \cap H^2(\Omega)\right), \\
&u_t \in C\left((0,T), L^2(\Omega) \cap L^2\left((0,T), H_0^1(\Omega)\right)\right), \\
&u_{tt} \in L^2\left((0,T), L^2(\Omega)\right),
\end{aligned}$$

Moreover, the following alternatives hold

$$(i) \quad T = +\infty, \qquad\qquad or$$

$$(ii) \quad T < +\infty \; and \; \lim_{t \to T}\left(\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2\right) = +\infty.$$

Consider the usual energy functional for the solution u of the system (P) defined for all $t \in (0,T)$ by

$$E(t) = \tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{1}{2}\left(1 - \int_0^t g(\tau)\,d\tau\right)\|\nabla u(t)\|_2^2 + \tfrac{1}{2}(g \circ \nabla u)(t) - \tfrac{1}{p}\|u(t)\|_p^p,$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2\,d\tau.$$

Lemma 3.1. *E is a decreasing function.*

Proof. If we multiply the first equation in (P) by $u_t(t)$, integrate it over Ω, we obtain

$$\int_\Omega a(x)|u_t(t)|^2\,dx + \int_\Omega F(t, \nabla u(t))\,u_t(t)\,dx = I_1 + I_2 + I_3 + I_4, \tag{3.1}$$

where

$$\begin{aligned} I_1 &= -\int_\Omega u_{tt}(t)\, u_t(t)\, dx, \\ I_2 &= \int_\Omega \Delta u(t)\, u_t(t)\, dx, \\ I_3 &= \int_\Omega |u(t)|^{p-2} u(t)\, u_t(t)\, dx, \end{aligned}$$

and

$$I_4 = -\int_\Omega (g * \Delta u)(t)\, u_t(t)\, dx.$$

We have

$$I_1 = -\frac{1}{2}\frac{d}{dt}\|u_t(t)\|_2^2. \tag{3.2}$$

By the Green formula and the bounded conditions, we obtain

$$I_2 = -\int_\Omega \nabla u(t)\, \nabla u_t(t)\, dx \quad = -\tfrac{1}{2}\tfrac{d}{dt}\|\nabla u(t)\|_2^2, \tag{3.3}$$

and

$$I_3 = \frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p. \tag{3.4}$$

For I_4, we use the Green formula and the bounded conditions, to find

$$I_4 = -\int_\Omega \int_0^t g(t-\tau)\, \Delta u(\tau)\, u_t(t)\, d\tau dx = \int_0^t g(t-\tau)\left[\int_\Omega \nabla u(\tau)\, \nabla u_t(t)\, dx\right] d\tau.$$

We add and subtract the term

$$\int_0^t g(t-\tau)\left[\int_\Omega \nabla u(t)\, \nabla u_t(t)\, dx\right] d\tau,$$

to get

$$
\begin{aligned}
I_4 &= -\int_0^t g(t-\tau)\left[\int_\Omega (\nabla u(t) - \nabla u(\tau))\nabla u_t(t)\,dx\right] d\tau \\
&\quad + \int_0^t g(t-\tau)\left[\int_\Omega \nabla u(t)\nabla u_t(t)\,dx\right] d\tau \\
&= -\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\int_\Omega |\nabla u(t) - \nabla u(\tau)|^2\,dx d\tau \\
&\quad +\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\int_\Omega |\nabla u(t)|^2\,dx d\tau.
\end{aligned}
$$

By a variable change, we find

$$
I_4 = -\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\|\nabla u(t) - \nabla u(\tau)\|_2^2\,d\tau + \frac{1}{2}\int_0^t g(\tau)\frac{d}{dt}\|\nabla u(t)\|_2^2\,d\tau. \tag{3.5}
$$

If we apply the following Leibniz integral rule

$$
\frac{d}{dt}\left(\int_{A(t)}^{B(t)} f(t,\tau)\,d\tau\right) = \int_{A(t)}^{B(t)} \frac{\partial f(t,\tau)}{\partial t}d\tau + f(t,B(t))\frac{d}{dt}B(t) - f(t,A(t))\frac{d}{dt}A(t), \tag{3.6}
$$

with

$$
A(t) = 0, \quad B(t) = t \quad \text{and} \quad f(t,\tau) = -\frac{1}{2}g(t-\tau)\|\nabla u(t) - \nabla u(\tau)\|_2^2,
$$

we obtain

$$
\begin{aligned}
-\frac{1}{2}\frac{d}{dt}(g\circ\nabla u)(t) &= -\frac{1}{2}\frac{d}{dt}\int_0^t g(t-\tau)\|\nabla u(t) - \nabla u(\tau)\|_2^2\,d\tau \\
&= -\frac{1}{2}\left(g'\circ\nabla u\right)(t) - \frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\|\nabla u(t) - \nabla u(\tau)\|_2^2\,d\tau.
\end{aligned}
$$

This implies that

$$
-\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\|\nabla u(t) - \nabla u(\tau)\|_2^2\,d\tau = -\frac{1}{2}\frac{d}{dt}(g\circ\nabla u)(t) + \frac{1}{2}\left(g'\circ\nabla u\right)(t). \tag{3.7}
$$

Now, if we apply (3.6) with

$$A(t) = 0, \quad B(t) = t \; and \quad f(t,\tau) = \frac{1}{2} g(\tau) \|\nabla u(t)\|_2^2,$$

we obtain

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\int_0^t g(\tau)\|\nabla u(t)\|_2^2 \, d\tau &= \tfrac{1}{2}\int_0^t \tfrac{d}{dt}\left[g(\tau)\|\nabla u(t)\|_2^2\right] d\tau + \tfrac{1}{2} g(t)\|\nabla u(t)\|_2^2 \\ &= \tfrac{1}{2}\int_0^t g(\tau)\tfrac{d}{dt}\|\nabla u(t)\|_2^2 \, d\tau + \tfrac{1}{2} g(t)\|\nabla u(t)\|_2^2 . \end{aligned}$$

This implies that

$$\tfrac{1}{2}\int_0^t g(\tau)\tfrac{d}{dt}\|\nabla u(t)\|_2^2 \, d\tau = \tfrac{1}{2}\tfrac{d}{dt}\int_0^t g(\tau)\|\nabla u(t)\|_2^2 \, d\tau - \tfrac{1}{2} g(t)\|\nabla u(t)\|_2^2 . \quad (3.8)$$

We insert (3.7) and (3.8) in (3.5) to obtain

$$\begin{aligned} I_4 = \; & -\tfrac{1}{2}\tfrac{d}{dt}(g \circ \nabla u)(t) + \tfrac{1}{2}\left(g' \circ \nabla u\right)(t) + \tfrac{1}{2}\tfrac{d}{dt}\int_0^t g(\tau)\|\nabla u(t)\|_2^2 \, d\tau \\ & -\tfrac{1}{2} g(t)\|\nabla u(t)\|_2^2 . \end{aligned} \quad (3.9)$$

By substituting (3.2), (3.3), (3.4) and (3.9) in (3.1) and using (H_2), we obtain

$$\begin{aligned} a_1\|u_t(t)\|_2^2 + \int_\Omega F(t, \nabla u(t))\, u_t(t)\, dx \; & \leq -\tfrac{1}{2}\tfrac{d}{dt}\|u_t(t)\|_2^2 - \tfrac{1}{2}\tfrac{d}{dt}(g \circ \nabla u)(t) \\ & -\tfrac{1}{2}\tfrac{d}{dt}\left[\left(1 - \int_0^t g(\tau)\, d\tau\right)\|\nabla u(t)\|_2^2\right] \\ & +\tfrac{1}{2}\left(g' \circ \nabla u\right)(t) - \tfrac{1}{2} g(t)\|\nabla u(t)\|_2^2 \\ & +\tfrac{1}{p}\tfrac{d}{dt}\|u(t)\|_p^p . \end{aligned}$$

Using the definition of E, we arrive to

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)\,d\tau\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u(t)\|_p^p\right] \\ &\leq \frac{1}{2}\left(g' \circ \nabla u\right)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - a_1\|u_t(t)\|_2^2 - \int_\Omega F(t, \nabla u(t))\,u_t(t)\,dx.\end{aligned}$$

Since $g' \leq 0$ then

$$\frac{d}{dt}E(t) \leq -\frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - a_1\|u_t(t)\|_2^2 + \int_\Omega |F(t, \nabla u(t))|\,|u_t(t)|\,dx.$$

We use the following Young inequality

$$XY \leq \frac{\epsilon}{2}X^2 + \frac{1}{2\epsilon}Y^2, \; for\ all\ X,\ Y \geq 0\ and\ \epsilon > 0,$$

with

$$X = |F(t, \nabla u(t))| \qquad and \qquad Y = |u_t(t)|,$$

to find

$$\frac{d}{dt}E(t) \leq \frac{1}{2}\int_\Omega \left[\epsilon\,|F(t, \nabla u(t))|^2 - g(t)\,|\nabla u(t)|^2\right] + \left(\frac{1}{2\epsilon} - a_1\right)\|u_t(t)\|_2^2.$$

We take $\epsilon = \frac{1}{2a_1}$ to obtain

$$\frac{d}{dt}E(t) \leq \frac{1}{2}\int_\Omega \left[\frac{1}{2a_1}|F(t, \nabla u(t))|^2 - g(t)\,|\nabla u(t)|^2\right].$$

Thus, by (H_3), we find

$$\frac{d}{dt}E(t) \leq 0, \quad \forall\, t \in (0, T).$$

□

3.3 Global existence result

To study the global existence of the solution of the system (P), we define the following functionals for all $t \in (0,T)$

$$I(t) = \left(1 - \int_0^t g(\tau)\, d\tau\right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - \|u(t)\|_p^p,$$

and

$$J(t) = \tfrac{1}{2}\left(1 - \int_0^t g(\tau)\, d\tau\right) \|\nabla u(t)\|_2^2 + \tfrac{1}{2}(g \circ \nabla u)(t) - \tfrac{1}{p}\|u(t)\|_p^p.$$

The proof of our main result in this chapter requires the following lemma.

Lemma 3.2. *If*

$$I(0) > 0,$$

and

$$\beta := \frac{C_*^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{3.10}$$

where C_* is the best constant of the embedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega).$$

Then

$$I(t) > 0 \; for \; all \; t \; \in (0,T).$$

Proof. By continuity, there exists T_m where $0 < T_m \leq T$ such that

$$I(t) \geq 0, \quad \forall t \in (0, T_m). \tag{3.11}$$

Using the definition of I and J, we obtain

$$\tfrac{p-2}{2p}\left(1-\int_0^t g(\tau)\,d\tau\right)\|\nabla u(t)\|_2^2 = J(t)-\tfrac{1}{p}I(t)-\tfrac{p-2}{2p}(g\circ\nabla u)(t).$$

By (3.11) and (H_1), we get

$$\left(1-\int_0^t g(\tau)\,d\tau\right)\|\nabla u(t)\|_2^2 \le \frac{2p}{p-2}J(t).$$

But

$$l\,\|\nabla u(t)\|_2^2 \le \left(1-\int_0^t g(\tau)\,d\tau\right)\|\nabla u(t)\|_2^2. \tag{3.12}$$

Then

$$l\,\|\nabla u(t)\|_2^2 \le \frac{2p}{p-2}J(t). \tag{3.13}$$

Noting that

$$J(t) = E(t)-\frac{1}{2}\|u_t(t)\|_2^2 \le E(t).$$

Then, we have

$$l\,\|\nabla u(t)\|_2^2 \le \frac{2p}{p-2}E(t).$$

By the nonincreasingness of E, we find

$$\|\nabla u(t)\|_2^2 \le \frac{2p}{l(p-2)}E(0). \tag{3.14}$$

On the other hand

$$\|u(t)\|_p^p \;\le C_*^p\,\|\nabla u(t)\|_2^p \;= \tfrac{C_*^p}{l}\,\|\nabla u(t)\|_2^{p-2}\,l\,\|\nabla u(t)\|_2^2.$$

Then, by (3.12), we get

$$\|u(t)\|_p^p \leq \frac{C_*^p}{l} \|\nabla u(t)\|_2^{p-2} \left(1 - \int_0^t g(\tau)\, d\tau\right) \|\nabla u(t)\|_2^2 .$$

By (3.14), we have

$$\|u(t)\|_p^p \leq \frac{C_*^p}{l} \left(\frac{2p}{l(p-2)} E(0)\right)^{\frac{p-2}{2}} \left(1 - \int_0^t g(\tau)\, d\tau\right) \|\nabla u(t)\|_2^2 .$$

This implies that

$$\|u(t)\|_p^p \leq \beta \left(1 - \int_0^t g(\tau)\, d\tau\right) \|\nabla u(t)\|_2^2 .$$

Since $\beta < 1$ then

$$\|u(t)\|_p^p < \left(1 - \int_0^t g(\tau)\, d\tau\right) \|\nabla u(t)\|_2^2 ,\ \forall t \in (0, T_m) .$$

Hence

$$\begin{aligned} I(t) &= \left(1 - \textstyle\int_0^t g(\tau)\, d\tau\right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - \|u(t)\|_p^p \\ &\geq \left(1 - \textstyle\int_0^t g(\tau)\, d\tau\right) \|\nabla u(t)\|_2^2 - \|u(t)\|_p^p > 0. \end{aligned}$$

That is

$$I(t) > 0, \quad \forall t \in (0, T_m) .$$

Finally, we can see that

$$I(T_m) > 0$$

and

$$\frac{C_*^p}{l} \left(\frac{2p}{l(p-2)} E(T_m)\right)^{\frac{p-2}{2}} \leq \beta.$$

Then, by repeating the above procedure, we extend T_m to T. □

The main result of this chapter is given in the following theorem.

Theorem 3.2. *The solution u of* (P) *is bounded and global.*

Proof. Based on (3.13), we get

$$\tfrac{(p-2)l}{2p}\left\|\nabla u\left(t\right)\right\|_2^2+\tfrac{1}{2}\left\|u_t\left(t\right)\right\|_2^2\leq J\left(t\right)+\tfrac{1}{2}\left\|u_t\left(t\right)\right\|_2^2=E\left(t\right).$$

Consequently, it exists a constant $C>0$ such that

$$\left\|\nabla u\left(t\right)\right\|_2^2+\left\|u_t\left(t\right)\right\|_2^2\leq CE\left(t\right).$$

By the nonincreasingness of E, we find

$$\left\|\nabla u\left(t\right)\right\|_2^2+\left\|u_t\left(t\right)\right\|_2^2\leq CE\left(0\right).$$

This implies that u is bounded.
The last estimate and the continuation principle give the desired result. □

CONCLUSION

In this thesis, we have studied the existence and the stability of global solution of some semi linear wave systems.

First, we have studied global existence of hyperbolic equations with damping and source terms using some assumptions for initial data. Then, by applying an integral inequality due to Komornik, we have obtained the stability result.

Many researches have been interested by nonlinear wave equations with variable-exponent, on which studies focus on blow up of solution. However, the global solution have not been studied for this type of equations. This issue have been the subject of a detailed study. Our results have been proved using some assumptions for initial data. Then, by applying an integral inequality due to Komornik, we have obtained the stability result.

In the literature, we have observed that the wave equations having damping and nonlinear first order perturbation terms have been largely studied. Thus, these equations may be considered with a source term. Actually, we have proved the global existence of solutions of the nonlinear wave equation with damping, source, and nonlinear first order perturbation terms.

PERSPECTIVES

As a perspective, it would be interesting to study the existence of the global solution of the system:

$$u_{tt} - div\left(|\nabla u|^{r(.)-2}\nabla u\right) + au_t\,|u_t|^{m(.)-2} = bu\,|u|^{p(.)-2}\,.$$

Given the fact that Messaoudi and Talahmeh [35] have proved a finite time blow-up result for the solution.
In the same context, the stability of wave equation with variable exponent in the case when the source term dominates the damping term also can be explored.
As an extension of the system studied in the chapter 2, on which we have considered the variables exponents as $p\,(x)$ and $m\,(x)$,the case of time varying exponents:

$$u_{tt} - \Delta u + u_t\,|u_t|^{m(x,t)-2} = u\,|u|^{p(x,t)-2}\,.$$

can be investigated. Actually, these type of systems were studied by several authors. For instance, Antontsev [2] and Sun et al. [40] have studied the existence and blow-up of the solution for these systems.
In the chapter 3 we have studied the global existence of the solution of the wave equation with nonlinear source, damping and first order perturbation term. Subsequently, the stability of this system is another challenge.

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