DEDICATION

Dedicated to my mother, my mother, my mother

To my father

To my brother and sisters

To my husband and my sons

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I thank my God, for letting me through all difficulties. I have exprienced your guidance day by day, you are the one who let me finish my degree.
I will keep on trusting you for my future. Thank you Allah.

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CONTENTS

INTRODUCTION

The notion of invertibility of the elements is a notion that exists in all the fields of mathematics such as algebra, numerical analysis, spectral theory,....
Many problems are interpreted by an equation of the type $Ax = y$, where A is a given transformation (a matrix or a lineair application), if A is invertible then there exists a unique solution for x given by $x = A^{-1}y$.
We see that a lot of problems appear when A is non invertible, this is the reason for which some mathematicians have introduced a new notion of invertibility (generalized inverse or pseudo inverse) to solve these problems.
The generalized inverse of an element A is the element A_g satisfy the two properties $AA_gA = A$ and $A_gAA_g = A_g$, these properties which are those of the usual inverse (A^{-1}) make A_g as close to the ordinary inverse in other words we are close to getting $A_gA = AA_g = I$.
From the mathematicians who introduced the notion of generalized inverse, we mention Fredholm in 1903. He had introduced the concept of pseudo inverse of integral operators to treat the integral equations.
In 1936, J.Von Neumann [46] introduced the notion of generalized inverse for elements in a ring, then in 1948, I.Kaplansky [24] gave an extension for this notion for algebra.
In 1920, E.H. Moore [32] gave a definition of the generalized inverse of an $n \times m$ matrix A is equivalent to the existence of $n \times m$ matrix B such that $AB = P_{R(A)}$ and $BA = P_{R(B)}$ (where P is an orthogonal projection).
Unware of Moore's work, R.Penrose [34] showed in 1955 that there exists a unique matrix B satisfiying the four relations

$$AB = A,\ BAB = B,\ (AB)^* = AB,\ (BA)^* = BA.$$

A year later, in 1956 Rado [35] proved that these two definitions of Moore and Penrose are equivalent. Since then this generalized inverse is called the Moore-Penrose generalized inverse and it's denoted by A^+.
Note that the generaliezed inverse is reflexive in the sense if B is generalized inverse of A, then A is generalized inverse of B.
In 1958, Drazin [15] introduced a different kind of generalized inverse in associative rings and semigroups that does not have the reflexivity property but commutes with the element, he defined the drazin inverse of a an element of a semi group is the element b of the semi group that satisfy

$$ab = ba,\ b = ab^2,\ a^k = a^{k+1}b$$

for some non negative integer k, the drazin inverse of a is denoted by a^d.
Note that the group inverse is a special drazin inverse with $k = 1$, and it's denoted by $a^{\#}$.

Through this thesis we are interested in the $\mathbb{C}^*$ algebra of all bounded lineair operators acting on a complex Hilbert space H, so the first section of chapter one contains some basic theorems of operator theory. For the importance of the Moore-Penrose inverse and the group inverse in this thesis, we consecrate both second and third section to recall theirs algebraic and topological properties.

One of the most essential inequalities in operator theory is the Heinz inequality which is given by

$$\forall X \in \mathfrak{B}(H), \|PX + XQ\| \geq \|P^{\alpha}XQ^{1-\alpha} + P^{1-\alpha}XQ^{\alpha}\|,$$

where P and Q are positive operators, and $0 \leq \alpha \leq 1$.
It's original proof however is based on the complex analysis theory and somewhat complicated.
In 1978, McIntosh [29] proved that Heinz inequality is consequence of the following inequality

$$\forall A, B, X \in \mathfrak{B}(H), \|A^*AX + XBB^*\| \geq 2\|AXB\|. \qquad \text{(A.G.M.I)}$$

This inequality is called the arithmetic geometric mean inequality. From McIntosh inequality, we deduce the following inequality

$$\forall S \in \mathbb{S}_0(H), \forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|. \qquad \text{(C.P.R.I)}$$

Note that, this inequality was proved by Corach Porach Recht in [8, 1990]. The C.P.R.I is a key factor in their study of differential geometry of selfadjoint operators. They proved this inequality by using the integral representation of a selfadjoint operator with respect to a spectral measure.
Three years later, in 1993, J.I.Fujji, M.Fujji Furuta, and Nakamato [19] showed that Heinz inequality, A.G.M.I and C.P.R.I are equivalent, and they are equivalent to some three other inequalities. They also presented an easier proof of Heinz inequality.
In the last section of the first chapter, we discuss the above inequalities and some others norm inequalities.
Based on C.P.R.I, Seddik [38] proved that the following property characterizes exactely the class of all invertible selfadjoint operators in $\mathfrak{B}(H)$ multiplying by scalars, that is the class $\mathbb{C}^*\mathbb{S}_0(H)$

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|, \quad (S \in \mathfrak{I}(H)), \qquad \text{(S1)}$$

This was the begining of this kind of problems of the characterisation of some distinguished classes of operators in terms of operator inequalities.
In [40], Seddik has found two other characterizations of this last class given by

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| = \|S^*XS^{-1} + S^{-1}XS^*\|, \ (S \in \mathfrak{I}(H)), \qquad \text{(S2)}$$

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \geq \|S^*XS^{-1} + S^{-1}XS^*\|, \quad (S \in \mathfrak{I}(H)). \tag{S3}$$

Note that the class $\mathbb{C}^*\mathbb{S}_0(H)$ is exactly the class of all invertible normal operators in $\mathfrak{B}(H)$ the spectrum of which is included in straight line passing through the origin.

For the class $\mathcal{N}_0(H)$ of all invertible normal operators, Seddik [42] showed, that this class is characterized by each of the four following properties

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| = \|S^*XS^{-1}\| + \|S^{-1}XS^*\|, \quad (S \in \mathfrak{I}(H)), \tag{N1}$$

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \geq \|S^*XS^{-1}\| + \|S^{-1}XS^*\|, \quad (S \in \mathfrak{I}(H)), \tag{N2}$$

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \leq \|S^*XS^{-1}\| + \|S^{-1}XS^*\|, \quad (S \in \mathfrak{I}(H)), \tag{N3}$$

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \geq 2\|X\|, \quad (S \in \mathfrak{I}(H)). \tag{N4}$$

Concerning the class $\mathbb{R}^*\mathfrak{U}(H)$ of all the unitary operators multiplying with a nonzero scalar, Seddik could find in [40, 41], that this class is characterized by each of the following properties

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| = 2\|X\|, \quad (S \in \mathfrak{I}(H)), \tag{U1}$$

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \leq 2\|X\|, \quad (S \in \mathfrak{I}(H)), \tag{U2}$$

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \leq 2\|X\|, \quad (S \in \mathfrak{I}(H)), \tag{U3}$$

$$\forall X \in \mathfrak{B}(H), \|S^*XS^{-1} + S^{-1}XS^*\| \leq 2\|X\|, \quad (S \in \mathfrak{I}(H)), \tag{U4}$$

$$\forall X \in \mathfrak{B}(H), \|S^*XS^{-1} + S^{-1}XS^*\| = 2\|X\|, \quad (S \in \mathfrak{I}(H)). \tag{U5}$$

Using the two properties $(S1)$ and $(U3)$, we deduce that the class $\mathbb{R}^*\mathfrak{U}_r(H)$ of all unitary reflection operators multiplied by nonzero real numbers, is given by the following inequality

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| = 2\|X\|, \quad (S \in \mathfrak{I}(H)).$$

It is clear that the notion of the usual inverse plays a key role in all the previous characterizations, and here the question is appear what if S is not invertible, what if we replace S^{-1} by S^{+} in the above inequalities, is the characterizations still holds?
In [43], Seddik gave the extension of the above properties from the domain $\mathfrak{I}(H)$ of invertible operators to the domain $\mathfrak{R}(H)$ of operators with closed ranges. he could found that the class of all seladjoint operators with closed ranges is characterized by each of the following properties

$$\forall X \in \mathfrak{B}(H), \|SXS^{+} + S^{+}XS\| = \|S^*XS^{+} + S^{+}XS^*\|, \quad (S \in \mathfrak{R}(H)), \tag{S4}$$

$$\forall X \in \mathfrak{B}(H), \|SXS^{+} + S^{+}XS\| \geq \|S^*XS^{+} + S^{+}XS^*\|, \quad (S \in \mathfrak{R}(H)), \tag{S5}$$

$$\forall X \in \mathfrak{B}(H), \|SXS^{+} + S^{+}XS\| \geq 2\|SS^{+}XS^{+}S\|, \quad (S \in \mathfrak{R}(H)), \tag{S6}$$

$$\forall X \in \mathfrak{B}(H), \|S^2X + XS^2\| \geq 2\|SXS\|, \quad (S \in \mathfrak{R}(H)). \tag{S7}$$

Note that the above properties are extension of the properties (S1), (S2) and (S3) from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{R}(H)$.

Concerning the class $\mathcal{N}_{cr}(H)$ of all normal operators with closed ranges, the characterizations are given by

$$\forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| = \|S^*XS^+\| + \|S^+XS^*\|, \quad (S \in \mathfrak{R}(H)), \tag{N5}$$
$$\forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| \geq \|S^*XS^+\| + \|S^+XS^*\|, \quad (S \in \mathfrak{R}(H)), \tag{N6}$$
$$\forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| \leq \|S^*XS^+\| + \|S^+XS^*\|, \quad (S \in \mathfrak{R}(H)), \tag{N7}$$
$$\forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| \geq 2\|S^+SXS^+S\|, \quad (S \in \mathfrak{R}(H)) \ , \tag{N8}$$
$$\forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| \geq 2\|SXS\|, \quad (S \in \mathfrak{R}(H)). \tag{N9}$$

The second chapter contains a detailed study of all the above characterizations. Based on this chapter, we find new characterizations.
In the third chapter, we characterize some subclasses by properties that have new forms given by

$$\forall X \in \mathfrak{B}(H), \|S^*XS + SXS^*\| = 2\|SXS\|, \quad (S \in \mathfrak{B}(H)),$$
$$\forall X \in \mathfrak{B}(H), \|S^*XS + SXS^*\| \geq 2\|SXS\|, \quad (S \in \mathfrak{B}(H)),$$
$$\forall X \in \mathfrak{B}(H), \|S^*XS\| + \|SXS^*\| \leq 2\|SXS\|, \quad (S \in \mathfrak{B}(H)),$$
$$\forall X \in \mathfrak{B}(H), \|S^*XS + SXS^*\| \leq 2\|SXS\|, \quad (S \in \mathfrak{B}(H)).$$

Also, we give the extension of the properties (N1-N3) from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{I}_1(H)$, where $\mathfrak{I}_1(H)$ is the set of all operators $S \in \mathfrak{B}(H)$ such that $ind(S) \leq 1$, it's mean we replaced the usual inverse by the group inverse

$$\forall X \in \mathfrak{B}(H), \ \left\|SXS^{\#}\right\| + \left\|S^{\#}XS\right\| = \left\|S^*XS^{\#}\right\| + \left\|S^{\#}XS^*\right\|, \quad (S \in \mathfrak{I}_1(H)),$$
$$\forall X \in \mathfrak{B}(H), \ \left\|SXS^{\#}\right\| + \left\|S^{\#}XS\right\| \leq \left\|S^*XS^{\#}\right\| + \left\|S^{\#}XS^*\right\|, \quad (S \in \mathfrak{I}_1(H)),$$
$$\forall X \in \mathfrak{B}(H), \ \left\|SXS^{\#}\right\| + \left\|S^{\#}XS\right\| \geq \left\|S^*XS^{\#}\right\| + \left\|S^{\#}XS^*\right\|, \quad (S \in \mathfrak{I}_1(H)).$$

Finally, we give chracterizations of some subclasses of nonnormal operators, precisely the subclass of all isometry operators in $\mathfrak{B}(H)$ and the subclass of all quasinormal partial isometry operators $\mathfrak{B}(H)$. The first class is characterized by the following property (that has a form which missed the symmetric form):

$$\forall X \in \mathfrak{B}(H), \|X\| + \left\|S^+XS\right\| \leq 2\left\|SXS^+\right\|, \quad (S \in \mathfrak{R}(H)).$$

The second class is characterized by the following property

$$\forall X \in \mathfrak{B}(H), \|X\| + \left\|SXS^+\right\| \geq 2\left\|S^+XS\right\|, \quad (S \in \mathfrak{R}(H)).$$

In the last chapter; and with colaboration with Menkad Safa, we are interesting by the study of two classes of operators in $\mathfrak{B}(H)$, the generalized projections and hypergeneralized projections, which are extension of the orthogonal projections.
The concepts of generalized projections and hypergeneralized projections on a finite dimensional Hilbert space was introduced by GroB and Trenkler [17]. The variety characterizations of the two classes have been studied by many authors (see[1],[2]).

H.Du and Y.Li [12], was extended the concept of generalized projection on $\mathfrak{B}(H)$, where H is not necessarily finite dimensional. They proved that the properties found in [17] hold for an infinite dimensional Hilbert space.
Later in 2006, Deng, Li, and Du [9], by the spactral theoretic argument, they established the geometrical characterizations of both generalized and hypergeneralized projections.
Recently in 2012, S. Radosavljevic and D.S.Djordjevic [36], gave the same characterizations (but influenced by the work of D.S Djordjevic and J.Koliha (see[13]), who gave matrix representation of a closed range operator in $\mathfrak{B}(H)$), they gave a proof more easier than of them.
The first section contains a detailed study about those two classes of operators regards their characterizations, matrix representation and properties of product, sum and difference of generalized and hypergeneralized projections.
In the second section, we define a new class of operator which extends the notion of idempotent unlike the two above classes. Also by a similar method, we give some basic characterizations and properties of this class.

Terminology

Let $\mathfrak{B}(H)$ be the $\mathbb{C}^*$-algebra of all bounded linear operators acting on a complex Hilbert space H, and let $\mathbb{S}(H)$, $\mathcal{N}(H)$, $\mathfrak{U}(H)$ and $\mathcal{V}(H)$ denote the class of all selfadjoint operators, the class of all normal operators, the class of all unitary operators and the class of all isometry operators in $\mathfrak{B}(H)$, respectively.

We denote by

- $\mathfrak{I}(H)$, the group of all invertible elements in $\mathfrak{B}(H)$,
- $\mathfrak{U}_r(H) = \mathfrak{U}(H) \cap \mathbb{S}(H)$, the set of all unitary reflection operators in $\mathfrak{B}(H)$,
- $\mathbb{S}_0(H) = \mathbb{S}(H) \cap \mathfrak{I}(H)$, the set of all invertible selfadjoint operators in $\mathfrak{B}(H)$,
- $\mathcal{N}_0(H) = \mathcal{N}(H) \cap \mathfrak{I}(H)$, the set of all invertible normal operators in $\mathfrak{B}(H)$,
- $\mathfrak{R}(H)$, the set of all operators with closed ranges in $\mathfrak{B}(H)$,
- $\mathcal{N}_{cr}(H) = \mathcal{N}(H) \cap \mathfrak{R}(H)$, the set of all normal operators with closed ranges in $\mathfrak{B}(H)$,
- $\mathbb{S}_{cr}(H) = \mathbb{S}(H) \cap \mathfrak{R}(H)$, the set of all selfadjoint operators with closed ranges in $\mathfrak{B}(H)$,
- $\mathcal{F}_1(H)$, the set of all operators of rank one in $\mathfrak{B}(H)$,
- $\mathbf{S}_1$, the set of all unit bounded functionals acting on $\mathfrak{B}(H)$,
- $x \otimes y$ (where $x,\ y \in H$), the one rank operator on H defined by $(x \otimes y)\, z = \langle z, y\rangle\, x$, for every $z \in H$,
- $L \circ M = \{\sum_{i=1}^{n} \alpha_i\beta_i : (\alpha_1, ..., \alpha_n) \in L, (\beta_1, ..., \beta_n) \in M\}$, where $L \subset \mathbb{C}^n$ and $M \subset \mathbb{C}^n$,
- $R(S)$, the range of $S \in \mathfrak{B}(H)$,
- $kerS$, the kernel of $S \in \mathfrak{B}(H)$,
- $|S|$, the positive square root of the positive operator S^*S (where $S \in \mathfrak{B}(H)$),
- $D_S = [S^*, S] = S^*S - SS^*$, the self-commutator of $S \in \mathfrak{B}(H)$,
- $\{S\}' = \{X \in \mathfrak{B}(H) : SX = XS\}$, the commutator of $S \in \mathfrak{B}(H)$,
- $\sigma(S)$, the spectrum of $S \in \mathfrak{B}(H)$,
- $r(S) = \underset{\lambda\in\sigma(S)}{sup} |\lambda|$, the spectral radius of $S \in \mathfrak{B}(H)$,
- $W(S) = \{\langle Sx, x\rangle : x \in H, \|x\| = 1\}$, the numerical range of $S \in \mathfrak{B}(H)$,

- $w(S) = \sup_{\lambda \in W(S)} |\lambda|$, the numerical radius of $S \in \mathfrak{B}(H)$,
- $co\sigma(S)$, the convex hull of $S \in \mathfrak{B}(H)$,
- D_θ, the straight line passing through the origin with slope $tan\ \theta$, for $\theta \in [0, \pi[$,
- $(M)_1 = \{x \in M : \|x\| = 1\}$, where M is a subset of a normed space,

Let $\mathcal{A}$ be a complex unital normed algebra. For $A, B \in \mathcal{A}$, and $C = (C_1, ..., C_n), D = (D_1, ..., D_n)$ two n-tuples of elements in $\mathcal{A}$, we denote by:

- $V(A) = \{f(A) : f \in \mathcal{A}', f(I) = \|f\| = 1\}$,
- $\mathcal{M}_{A,B}$, the multiplication operator defined on $\mathcal{A}$ by $\mathcal{M}_{A,B}(X) = AXB$,
- $V(C) = \{(f(C_1), ..., f(C_n)) : f \in \mathcal{A}', f(I) = \|f\| = 1\}$, the joint algebraic numerical range of C,
- $\mathcal{R}_{C,D}$, the elementary operator defined on $\mathcal{A}$ by $\mathcal{R}_{C,D}(X) = \sum_{i=1}^{n} C_i X D_i$,

For a n-tuple $A = (A_1, ..., A_n)$ of commuting operators in $\mathfrak{B}(H)$, we denote by:

- Γ_A, the set of all multiplicative functionals acting on the maximal commutative Banach algebra that contains the operators $A_1, ..., A_n$,
- $\sigma(A) = \{(\varphi(A_1), ..., \varphi(A_n)) : \varphi \in \Gamma_A\}$, the joint spectrum of A.

1. PRELIMINARIES AND ARITHMETIC GEOMETRIC MEAN INEQUALITY

Through this thesis $\mathfrak{B}(H)$ denotes the $\mathbb{C}^*$- algebra of all bounded linear operators acting on the complex Hilbert space H.
This chapter contains three sections, the first section consecrates to some basic notions and theorems of operator theory, and we also give some propositions of Seddik which are useful in the second chapter. In the second section, we introduce the concept of the Moore-Penrose generalized inverse and we recall some of it's properties, we also mention the famous theorem of the reverse order law for the the Moore-Penrose inverse. The third section contains the notions of the ascent and descent of an operator, the group inverse and EP operators. We finish this chapter by the section where we discuss some famous inequalities in operator theory and the relation between them.

1.1 Basic Theorems

Theorem 1.1. *(Hahn Banach theorem). Let E be a normed vector space, let $M \subseteq E$ be a linear subspace, and g be a bounded linear functional on M. Then there is a bounded linear functional f on E which is an extension of g to E and has the same norm:* $\|f\| = \|g\|$.

Theorem 1.2. *(Closed graph theorem). Let F and G be two Banach spaces. Let A be a linear operator from F into G. Assume that the graph of A , $G(A)$ is closed in $F \times G$. Then A is continuous.*

Theorem 1.3. *([11], Theorem of Douglas). Let $A, B \in \mathfrak{B}(H)$. Then the following conditions are equivalent:*
(i) $R(A) \subset R(B)$,
$(ii) AA^* \leq \lambda BB^*$ *for some constant* $\lambda > 0$,
(iii) *there exists a bounded operator C on H so that $A = BC$.*

Proof. $(i) \Rightarrow (iii)$. Assume (i) holds.
Let $x \in H$. We have $Ax \in R(A) \subset R(B)$, then there exists $h \in (kerB)^{\perp}$ such that $Bh = Ax$.
Let B_0 be the restriction of B to $(kerB)^{\perp}$. Then $B_0^{-1} : R(B) \to (kerB)^{\perp}$ is a closed linear transformation.
Since $R(A) \subset R(B)$, then $B_0^{-1}A$ exists. This implies that $Cx = h = B_0^{-1}Ax$, for all

$x \in H$ is a closed linear transformation from H.
To prove that C is bounded it suffices to show that C has a closed graph.
Let $(x_n, h_n)_{n\geq 1}$ is a sequence of elements in the graph of C such that $x_n \to x$ and $h_n \to h$, where $x \in H, h \in (kerB)^{\perp}$. Then $Ax_n \to Ax$ and $Bh_n \to Bh$. Therefore $h = Cx$. Hence C has been shown to be bounded.
$(ii) \Rightarrow (iii)$. Assume (ii) holds. Then $\|A^*x\| \leq \lambda \|B^*x\|$, for all $x \in H$.
Let the linear map $D : R(B^*) \to R(A^*)$, defined by $D(B^*x) = A^*x$, for all $x \in H$.
We have

$$\forall x \in H, \|D(B^*x)\|^2 = \|A^*x\|^2 = \langle AA^*x, x\rangle \leq \lambda^2 \langle BB^*x, x\rangle = \lambda^2 \|B^*x\|^2$$

Hence D is bounded. Therefore D can be extended to the closure of $R(B^*)$.
Let S be a bounded operator defined on H as follows

$$S(x) = \begin{cases} \widetilde{D}(x) & if\ x \in \overline{R(B^*)}. \\ 0 & if\ x \in \overline{R(B^*)}^{\perp}. \end{cases}$$

Hence $SB^*x = \widetilde{D}B^*x = DB^*x = A^*x$, for every $x \in H$.
So $A = BC$ with $C = S^*$.
$(iii) \Rightarrow (ii)$. Assume (iii) holds. Then for all x in H :

$$\langle AA^*x, x\rangle = \|A^*x\|^2 = \|C^*B^*x\|^2 \leq \|C^*\|^2 \|B^*x\|^2 = \|C^*\|^2 \langle BB^*x, x\rangle.$$

The implication $(iii) \Rightarrow (i)$ is trivial.

□

Notation 1.1. *For $\mathcal{A}$ a complex unital normed algebra, we denote by $\mathcal{P}(\mathcal{A})$, the set of all states on $\mathcal{A}$ given by:*

$$\mathcal{P}(\mathcal{A}) = \{f \in \mathcal{A}' : f(I) = \|f\| = 1\}.$$

Definition 1.1. *For $A \in \mathcal{A}$, we define the algebraic numerical range $V(A)$ as follows*

$$V(A) = \{f(A), f \in \mathcal{P}(\mathcal{A})\}.$$

such that $f(A^*) = \overline{f(A)}$.

For every $A \in \mathcal{A}$, $V(A)$ is convex, closed and contains $\sigma(A)$ (for more details see [45]). A is Hermitian if $V(A)$ is real.

Definition 1.2. *For an operator $A \in \mathfrak{B}(H)$, the usual numerical range $W(A)$ is defined as the set*

$$W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}.$$

Proposition 1.1. *[45] Let $A \in \mathfrak{B}(H)$. Then*

(i) $V(A) = \overline{W(A)}$ *(the closure of* $W(A)$*).*

(ii) *A is selfadjoint if and only if* $W(A)$ *is real.*

Definition 1.3. • *Let* $A \in \mathfrak{B}(H)$*. A is called:*

(a) *convexoid if* $W(A) = co\sigma(A)$*,*
(b) *normaloid if* $\|A\| = r(A)$*,*
(c) *paranormal if* $\|S^2x\| \geq \|Sx\|^2$*, for every* $x \in (H)_1$*,*
(d) *quasinormal if A and* A^*A *commute.*

• *Let* $A, B \in \mathfrak{B}(H)$*. We say that A is norm-parallel to B* ($A\|B$) *if* $\|A + \lambda B\| = \|A\| + \|B\|$*, for some unit scalar* λ*.*

Proposition 1.2. *[38] Let* $A \in \mathfrak{B}(H)$*. If* $\|A - \alpha\| = r(A - \alpha)$*, for all complex* α*, then A is convexoid.*

Proposition 1.3. *Let A be a positive operator in* $\mathfrak{B}(H)$ *and let* $X \in \mathfrak{B}(H)$*. If* $A^2X = XA^2$*, then* $AX = XA$*.*

Elementary operator and injective norm on tensor product space

Let us consider E a normed space over $\mathbb{R}$. For $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ be two n-tuples of operators in $\mathfrak{B}(E)$. We denote by $\mathcal{R}_{A,B}$, the elementary operator on $\mathfrak{B}(E)$ given by

$$\mathcal{R}_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i, X \in \mathfrak{B}(E).$$

We denote by $\mathcal{E}(\mathfrak{B}(E))$, the vector space of all elementary operators acting on $\mathfrak{B}(E)$.
For $\mathcal{R} \in \mathcal{E}(\mathfrak{B}(E))$, we put $d(\mathcal{R}_{A,B}) = \sup_{\|X\|=rankX=1} \|\mathcal{R}_{A,B}(X)\|$.
For $A, B \in \mathfrak{B}(E)$, we denote $A \otimes B$ the bounded bilinear form in $\mathfrak{B}(E)' \times \mathfrak{B}(E)'$ given by

$$(A \otimes B)(f, g) = f(A)g(B), f, g \in \mathfrak{B}(E)'.$$

We denote by $\mathfrak{B}(E) \otimes \mathfrak{B}(E)$, the vector space given by $\mathfrak{B}(E) \otimes \mathfrak{B}(E) = \{\sum_{i=1}^{n} A_i \otimes B_i : n \geq 1, A_i, B_i \in \mathfrak{B}(E), i = 1, .., n\}$.
We define on $\mathfrak{B}(E) \otimes \mathfrak{B}(E)$, the standard norm of a bounded bilinear form, given as follows

$$\|\sum_{i=1}^{n} A_i \otimes B_i\|_\lambda = \sup_{f,g \in (\mathfrak{B}(E)')_1} |\sum_{i=1}^{n} f(A_i)g(B_i)|, (A_1, ..., A_n), (B_1, ..., B_n) \in \mathfrak{B}(E)$$

The norm $\|.\|_\lambda$ is called the injective norm on $\mathfrak{B}(E) \otimes \mathfrak{B}(E)$.
The next proposition shows the relation between the injective norm $\|.\|_\lambda$ and $d(.)$ of any elementary operator on $\mathfrak{B}(E)$.

Proposition 1.4. *[40] Let $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ be two n-tuples of operators in $\mathfrak{B}(E)$. The following equalities hold:*

$$\begin{aligned} d(\mathcal{R}_{A,B}) &= \|\sum_{i=1}^{n} A_i \otimes B_i\|_\lambda \\ &= \sup_{f\in(\mathfrak{B}(E)')_1} \|\sum_{i=1}^{n} f(B_i)A_i\| \\ &= \sup_{f\in(\mathfrak{B}(E)')_1} \|\sum_{i=1}^{n} f(A_i)B_i\|. \end{aligned}$$

Proof. We denote by k_1, k_2 and k_3 the obove supremum in the same order. Let $x, y \in (E)_1$, $f, g \in (\mathfrak{B}(E)')_1$ and $h \in (E')_1$.
We have $d(\mathcal{R}_{A,B}) \geq \|\sum_{i=1}^{n} A_i(x\otimes h)B_i\| \geq \|\sum_{i=1}^{n} A_i(x\otimes h)B_iy\| = \|(\sum_{i=1}^{n} h(B_iy)A_i)x\|$, then by taking the supremum over $x \in E_1$, it follows that

$$d(\mathcal{R}_{A,B}) \geq \|\sum_{i=1}^{n} h(B_iy)A_i\|.$$

So that

$$d(\mathcal{R}_{A,B}) \geq |\sum_{i=1}^{n} h(B_iy)f(A_i)| = |h(\sum_{i=1}^{n} f(A_i)B_iy)|.$$

By taking the supremum over $h \in (E')_1$, we obtain

$$d(\mathcal{R}_{A,B})\| \geq \|(\sum_{i=1}^{n} f(A_i)B_i)y\|.$$

Thus

$$d(\mathcal{R}_{A,B}) \geq \|\sum_{i=1}^{n} f(A_i)B_i\|,$$

and so

$$d(\mathcal{R}_{A,B}) \geq |g(\sum_{i=1}^{n} f(A_i) \times B_i)| = |\sum_{i=1}^{n} f(A_i) \times g(B_i)|.$$

Therefore

$$d(\mathcal{R}_{A,B}) \geq k_1.$$

It's clear that $k_1 \geq \|f(\sum_{i=1}^{n} g(B_i)A_i)\|$, then $k_1 \geq \|\sum_{i=1}^{n} g(B_i)A_i\|$.
So that

$$k_1 \geq k_2.$$

Since $k_2 \geq \|f(\sum_{i=1}^{n} g(B_i)A_i)\| = \|g(\sum_{i=1}^{n} f(A_i)B_i\|$, taking the supremum over $g \in (\mathfrak{B}(E)')_1$, it follows that

$$k_2\|\sum_{i=1}^{n} f(A_i)B_i\|.$$

Therefore

$$k_2 \geq k_3.$$

Since $k_3 \geq |h(\sum_{i=1}^n f(A_i)B_i y| = |\sum_{i=1}^n f(A_i)h(B_i y)| = |f(\sum_{i=1}^n h(B_i y)A_i|$, then

$$k_3 \geq \| \sum_{i=1}^n h(B_i y)A_i\| \geq \| \sum_{i=1}^n h(B_i y)A_i x\| = \| \sum_{i=1}^n A_i(x \otimes h)B_i y\|.$$

Therefore

$$k_3 \geq d(\mathcal{R}_{A,B}),$$

this completes the proof. □

The following proposition, is a particular case of the above proposition where the normed space is an Hilbert space.

Proposition 1.5. *[39] Let $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ be two n-tuples of operators in $\mathfrak{B}(H)$. The following equality holds $\| \sum_{i=1}^n A_i \otimes B_i\|_\lambda = sup\{| \sum_{i=1}^n \langle A_i x, y\rangle\langle B_i u, v\rangle : x, y, u, v \in (H)_1\}$.*

Proof. Put $k = sup\{| \sum_{i=1}^n \langle A_i x, y\rangle\langle B_i u, v\rangle : x, y, u, v \in (H)_1\}$. Let $x, y, u, v \in (H)_1$, and let $f \in \mathbf{S}_1$. From proposition 1.4, we have

$$\| \sum_{i=1}^n A_i \otimes B_i\|_\lambda \geq \| \sum_{i=1}^n f(B_i)A_i\| \geq | \sum_{i=1}^n f(B_i)\langle A_i x, y\rangle| = |f(\sum_{i=1}^n \langle A_i x, y\rangle B_i)|.$$

By taking the supremum over $f \in \mathbf{S}_1$, we obtain

$$\| \sum_{i=1}^n A_i \otimes B_i\|_\lambda \geq \| \sum_{i=1}^n \langle A_i x, y\rangle B_i\| \geq | \sum_{i=1}^n \langle A_i x, y\rangle\langle B_i u, v\rangle|.$$

Taking the supremum over $x, y, u, v \in (H)_1$, we have

$$\left\| \sum_{i=1}^n A_i \otimes B_i \right\|_\lambda \geq k.$$

It remains to prove that $k \geq \| \sum_{i=1}^n A_i \otimes B_i\|_\lambda$. It's clear that $k \geq |\langle \sum_{i=1}^n \langle A_i x, y\rangle B_i u, v\rangle|$, taking the supremum over $v \in (H)_1$ then over $u \in (H)_1$, it follows

$$k \geq \|(\sum_{i=1}^n \langle A_i x, y\rangle B_i)u\| \geq \| \sum_{i=1}^n \langle A_i x, y\rangle B_i\|.$$

Hence

$$k \geq | \sum_{i=1}^n \langle A_i x, y\rangle f(B_i)| \geq | \sum_{i=1}^n \langle f(B_i A_i x, y\rangle|,$$

taking the supremum over $v \in (H)_1$ then over $u \in (H)_1$, it follows

$$k \geq \|(\sum_{i=1}^{n} f(B_i)A_i)x\| \geq \| \sum_{i=1}^{n} f(B_i)A_i\|.$$

Finally, by taking the supremum over $f \in \mathbf{S}_1$ and by using proposition 1.4, we obtain that

$$k \geq \left\| \sum_{i=1}^{n} A_i \otimes B_i \right\|_{\lambda}.$$

□

The remaining proposition in this section, provides a lower estimate for the injective norm $\| \sum_{i=1}^{n} A_i \otimes B_i\|_{\lambda}$, where A and B are two n-tuples of commuting operators in $\mathfrak{B}(H)$ and it characterizes this norm for two n-tuples of commuting normal operators .

Proposition 1.6. *[41] Let $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ be two n-tuples of commuting operator in $\mathfrak{B}(H)$. Then $\| \sum_{i=1}^{n} A_i \otimes B_i\|_{\lambda} \geq |\sigma(A) \circ \sigma(B)|$, and $\| \sum_{i=1}^{n} A_i \otimes B_i\|_{\lambda} = |\sigma(A) \circ \sigma(B)|$ if A_i and B_i are normal operators.*

Proof. Let (ϕ, ψ) be an arbitrary pair in $\Gamma_A \times \Gamma_B$. Using Hahn–Banach theorem, we may extend ϕ and ψ to unit functionals f and g on $\mathfrak{B}(H)$, respectively. So it follows from Proposition 1.4 that

$$\| \sum_{i=1}^{n} A_i \otimes B_i\|_{\lambda} \geq | \sum_{i=1}^{n} f(A_i)g(B_i)| = | \sum_{i=1}^{n} \varphi(A_i)\psi(B_i)|.$$

Therefore

$$\| \sum_{i=1}^{n} A_i \otimes B_i\|_{\lambda} \geq |\sigma(A) \circ \sigma(B)|.$$

Now suppose that all A_i and B_i are normal operators. It suffice to prove that $|\sigma(A) \circ \sigma(B)| \geq \| \sum_{i=1}^{n} A_i \otimes B_i\|_{\lambda}$. Since $|\sigma(A) \circ \sigma(B)| \geq |\psi(\sum_{i=1}^{n} \phi(A_i)B_i)|$ and $\sum_{i=1}^{n} \varphi(A_i)B_i$ is normal, for every $(\varphi, \psi) \in \Gamma_A \times \Gamma_B$, then

$$|\sigma(A) \circ \sigma(B)| \geq \sup_{\psi \in \Gamma_B} |\psi(\sum_{i=1}^{n} \varphi(A_i)B_i)| = \|\psi(\sum_{i=1}^{n} \varphi(A_i)B_i)\|, \ \forall \varphi \in \Gamma_A.$$

Thus

$$|\sigma(A) \circ \sigma(B)| \geq | \sum_{i=1}^{n} \varphi(A_i)f(B_i)| = |\varphi(\sum_{i=1}^{n} f(B_i)A_i)|, \ \forall \varphi \in \Gamma_A; \forall f \in \mathbf{S}_1.$$

Hence

$$|\sigma(A) \circ \sigma(B)| \geq \| \sum_{i=1}^{n} f(B_i)A_i\|, \forall f \in \mathbf{S}_1.$$

So it follows from Proposition 1.4 that

$$\sigma(A) \circ \sigma(B)| \geq \| \sum_{i=1}^{n} A_i \otimes B_i \|_\lambda.$$

□

1.2 The Moore-Penrose generalized inverse and the reverse order law

1.2.1 The Moore-Penrose generalized inverse

We start by giving a little introduction about the origin of the Moore-Penrose generalized inverse.
In 1903, Fredholm introduced the concept of pseudo inverse of integral operators. Then, in 1936, J. Von Neumann [46] introduced the notion of generalized inverse for an elements of ring. Later, in 1948, I. Kaplansky [24] gave an extention of this notion for the algebras. At last the notion of Moore-Penrose inverse or Pseudo-inverse was establish par E. H. Moore [32] in 1920, he gave an explicit definition of the generalized inverse of an arbitrary matrix as follows if $A \in \mathbb{C}^{n\times m}$ which is defined to be the unique matrix A^+ such that $AA^+ = P_{R(A)}$ and $A^+A = P_{R(A^+)}$.
This work was either little noticed or its significance not realized (perhaps because of the very individualistic Moore's terminology and notation). So the subject of generalized inverses remained undeveloped for 35 years.
In 1955, R.Penrose [34] who was apparently unware of Moore's work; he gave the definition of the generalized inverse for every matrix A of real complex elements, which is the unique matrix A^+ that satisfies the four equations (that we called the Penrose equations):

$$AA^+A = A, \; A^+AA^+ = A^+, \; (AA^+)^* = AA^+, \; (A^+A)^* = A^+A.$$

A year later in 1956, Rado [35] proved that these two definitions of Moore and Penrose are equivalents. Since then this generalized inverse is called the Moore-Penrose generalized inverse.
In $\mathbb{C}^*$-algebra of all bounded linear operators acting on a complex Hilbert space H, R. Hart and M. Mbekhta [20] proved that every operator $A \in \mathfrak{B}(H)$ has a Moore-Penrose inverse if and only if A has a closed range.

Theorem 1.4. *Let $A \in \mathfrak{B}(H)$. Then*
(i) $R(A)$ is closed if and only if $R(AA^*)$ is closed. In this case $R(A) = R(AA^*)$.
(ii) $R(A)$ is closed if and only if $R(A^*)$ is closed.

Proof. (i). Suppose that $R(A)$ is closed, then $H = R(A) \oplus^\perp kerA^* = \overline{R(A^*)} \oplus^\perp kerA$. Hence A has a representation of the form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(A^*)} \\ kerA \end{bmatrix} \rightarrow \begin{bmatrix} R(A) \\ kerA^* \end{bmatrix},$$

where A_1 is invertible from $\overline{R(A^*)}$ to $R(A)$. Therefore

$$AA^* = \begin{bmatrix} A_1A_1^* & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA^* \end{bmatrix}.$$

Since A_1 is invertible, then $A_1A_1^*$ is invertible operator in $\mathfrak{B}(R(A))$. And since $R(AA^*) = R(A_1A_1^*)$, we get $R(AA^*) = R(A)$, where $R(A)$ is closed.
Conversely; suppose that $R(AA^*)$ is closed, then $H = R(AA^*) \oplus ker(AA^*)$. Since $ker(AA^*) = kerA^*$, we have

$$H = R(AA^*) \oplus ker(AA^*) \subset R(A) \oplus ker(A^*) \subset H,$$

thus $R(A) = ker(A^*)^{\perp}$. Therefore $R(A)$ is closed and $R(A) = R(AA^*)$.
(ii). Suppose that $R(A)$ is closed, then using (i) we have $R(A) = R(AA^*)$. Hence for all $x \in H$, there exists $y \in H$ such that $Ax = AA^*y$, which gives that $x - A^*y \in kerA$ and $x = (x - A^*y) + A^*y \in kerA + R(A^*)$. So $R(A^*) = (kerA)^{\perp}$. Therefore $R(A^*)$ is closed.
The inverse implication is obtained by symmetry with the adjoint. □

Theorem 1.5. *Let $A \in \mathfrak{B}(H)$. Then the following properties are equivalent:*
(i) $R(A)$ is closed,
(ii) there exists $B \in \mathfrak{B}(H)$ such that $ABA = A$,
(iii) there exists $B \in \mathfrak{B}(H)$ such that $ABA = A, BAB = B, (AB)^* = AB, (BA)^* = BA$,
(iv) there exists $B \in \mathfrak{B}(H)$ such that $ABA = A, BAB = B$.

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds. Then, by theorem 1.4, $R(A^*)$ is closed. So $H = R(A) \oplus kerA^* = R(A^*) \oplus kerA$. Thus, The general form of the operator A is given by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A^*) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA \end{bmatrix},$$

where A_1 is an invertible operator from $R(A^*)$ to $R(A)$.
Let

$$B = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(A) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A^*) \\ kerA \end{bmatrix},$$

B satisfies the equation $ABA = A$.
$(ii) \Rightarrow (iii)$. By the same choice of B, we find that B satisfies the others three equations.
$(iii) \Rightarrow (iv)$. The imlication is trivial.
$(iv) \Rightarrow (i)$. Assume that (iv) holds. Then AB is an idempotent on $R(A)$. Hence $R(A)$ is closed. □

Remarks 1.1. *1. The operator B that satisfies the condition in (ii) is called inner inverse of A.*
2. The operator B that satisfies the conditions in (iii) it is unique, and it's called Moore-Penrose inverse of A and it is denoted by A^+.
3. The operator B that satisfies the conditions in (iv) is called the reflexive inverse (or generalized inverse) of A it's mean both inner inverse and outer inverse.

Remark 1.1. *Let $A \in \mathfrak{R}(H)$. Then*

1. *AA^+ is an orthogonal projection onto $R(A)$,*
2. *A^+A is an orthogonal projection onto $R(A^*)$,*
3. *$ker A^+ = ker(AA^+) = ker A^*$,*
4. *$R(A^+) = R(A^+A) = R(A^*)$.*

The next proposition gives some properties of the Moore-Penrose inverse that show the relation between the adjoint and the Moore-Penrose inverse of an operator.

Proposition 1.7. *Let $A \in \mathfrak{R}(H)$. Then we have*
(i) $(A^+)^+ = A$,
(ii) $(A^*)^+ = (A^+)^*$,
(iii) $(A^*A)^+ = A^+(A^+)^*$,
(iv) $(AA^*)^+ = (A^+)^*A^+$,
(v) $A^* = A^+AA^* = A^*AA^+$,
(vi) $A^+ = (A^*A)^+A^* = A^*(AA^*)^+$,
(vii) $(A^*)^+ = A(A^*A)^+ = (AA^*)^+A$.

Proof. (i). Trivial.
(ii). $(A^+)^*$ must satisfies the four conditions of the Moore-Penrose:
$(A^+)^*A^*(A^+)^* = (A^+AA^+)^* = (A^+)^*$,
$A^*(A^+)^*A^* = (AA^+A)^* = A^*$,
$(A^*(A^+)^*)^* = A^+A = (A^+A)^* = A^*(A^+)^*$, and
$((A^+)^*A^*)^* = AA^+ = (AA^+)^* = (A^+)^*A^*$.
Therefore $(A^+)^*$ is the Moore-Penrose inverse of A^*.
(iii). Since $R(A^*) = R(A^*A)$ and $R(A^*)$ is closed, then $(A^*A)^+$ exists.
$A^*AA^+(A^+)^*A^*A = A^*AA^+(AA^+)^*A = A^*AA^+AA^+A = A^*A$,
$A^+(A^+)^*A^*AA^+(A^+)^* = A^+(AA^+)^*AA^+(A^+)^* = A^+AA^+(A^+)^* = A^+(A^+)^*$,
$(A^*AA^+(A^+)^*)^* = A^+AA^+A = A^+A = (A^+A)^* = (AA^+A)^*(A^+)^* = A^*AA^+(A^+)^*$, and
$(A^+(A^+)^*A^*A)^* = (A^+(A)A^+A)^* = A^+A = A^+(A^+)^*A^*A$.
Hence $A^+(A^+)^*$ is the Moore-Penrose inverse of A^*A.
(iv). Follows immediately from (iii), we replace A by A^+.
(v). Since A^+A is an orthogonal projection onto $R(A^*)$, then we obtain

$$A^* = A^+AA^* = (A^+A)^*A^* = A^*(AA^+)^* = A^*AA^+.$$

(vi). It is clear that $A^+ = A^+(A^+)^*A^*$. Using (iii), we obtain the first equality of (vi). By the same method, we obtain also the second equality of (vi).
(vii). The two inequalities follow immediately from (vi).

□

D. Djordjevic and J. Koliha [13], established the matrix representation of a linear bounded operator with a closed range operator on infinite dimensional Hilbert space H depending on the different decomposition of the space in the following lemma.

Lemma 1.1. *[13] Let $A \in \mathfrak{R}(H)$. Then the operator A has the following three matrix representation with respect to the orthogonal sum of subspaces $H = R(A) \oplus kerA^* = R(A^*) \oplus kerA$*
(a) We have

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A^*) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA^* \end{bmatrix}, \tag{1.1}$$

where A_1 is invertible. Moreover

$$A^+ = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A^*) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA \end{bmatrix}.$$

(b) We have

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A) \\ kerA^* \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA^* \end{bmatrix}, \tag{1.2}$$

where $D = A_1A_1^* + A_2A_2^*$ maps $R(A)$ onto itself and $D > 0$. Also

$$A^+ = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

(c) Alternatively

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} R(A^*) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A^*) \\ kerA \end{bmatrix}, \tag{1.3}$$

where $C = A_1^*A_1 + A_2^*A_2$ maps $R(A^*)$ onto itself and $C > 0$. Also

$$A^+ = \begin{bmatrix} C^{-1}A_1^* & C^{-1}A_2^* \\ 0 & 0 \end{bmatrix}$$

Proof. The proof of (a) is straightforward.
(b). The operator A has the following representation

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} R(A) \\ kerA^* \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA^* \end{bmatrix},$$

i.e.

$$A_1 = A \mid R(A) : R(A) \to R(A), A_2 = A \mid ker(A^*) : ker(A^*) \to R(A)$$
$$A_3 = A \mid R(A) : R(A) \to ker(A^*), A_4 = A \mid ker(A^*) : ker(A^*) \to ker(A^*)$$

Furtheremore

$$A^* = \begin{bmatrix} A_1^* & A_3^* \\ A_2^* & A_4^* \end{bmatrix} : \begin{bmatrix} R(A^*) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A^*) \\ kerA \end{bmatrix}.$$

From $A^*(ker(A^*)) = 0$, we obtain $A_3^* = 0$ and $A_4^* = 0$. So $A_3 = 0$ and $A_4 = 0$. Hence $A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$. Notice that

$$AA^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^* : R(A) \to R(A)$. From $kerA^*A = ker(A^*)$, it follows that D is injective. From $R(AA^*) = R(A)$, it follows that D is surjective. Hence D is invertible. Finally, we obtain the form of tho Moore-Penrose inverse of A, by using proposition 1.7(vi).
The proof of (c) is analogous. □

Remark 1.2. *1. If $A \in \mathfrak{B}(H)$ is an invertible operator then $A^+ = A^{-1}$.*
2. If $P \in \mathfrak{B}(H)$ is an orthogonal projection then $P^+ = P$.
3. If $S \in \mathfrak{B}(H)$ is a partial isometry then $S^+ = S^*$.
4.$(0)^+ = 0$ and $(\lambda I)^+ = \frac{1}{\lambda I}$, for $\lambda \neq 0$.

1.2.2 The reverse order law

We know that if A and B are two invertible operators in $\mathfrak{B}(H)$, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$; is called the reverse order law for the ordinary inverse. It is well known that the reverse order law does not hold for various classes of generalized inverses. Hence, a significant number of authors treat the sufficient or equivalent conditions such that the reverse order law holds in some sense. Some authors proved that the reverse order law holds in the setting of element in rings and others in setting of matrices.
In 1974, Greville [16] gave some necessary and sufficient conditions for the Moore-Penrose inverse of a complex matrix product AB to be expressed as $(AB)^+ = B^+A^+$. This result is extended for linear bounded operators on Hilbert spaces, by Bouldin [6] and Izumino [23].
In 1981, Saichi Izumino [23] proved the following result:

Theorem 1.6. *Let $A, B \in \mathfrak{R}(H)$ such that $AB \in \mathfrak{R}(H)$. Then the following statements are equivalent:*
(i) A^+A commute with BB^* and BB^+ commute with A^*A,
(ii) $AB(AB)^+ = ABB^+A^+$ and $(AB)^+AB = B^+A^+AB$,
(iii) $(AB)^+ = B^+A^+$.

1.3 The group inverse and EP operators

Definition 1.4. *Let $A \in \mathfrak{R}(H)$. We call a group inverse of A, the operator $B \in \mathfrak{B}(H)$ satisfied*

$$(1)ABA = A, \quad (2)BAB = B, \quad (3)AB = BA$$

Remark 1.3.

- *The group inverse of an operator $A \in \mathfrak{R}(H)$ if exists, it's unique and it's denoted by $A^{\#}$.*
- *From (3) we have $R(A) = R(A^{\#})$ and $kerA = kerA^{\#}$.*
- *Since $R(A) = R(A^{\#})$, then $AA^{+}A^{\#} = A^{\#}A^{+}A = A^{\#}$.*
- *From (1) and (3), we find that $A^{\#}A = AA^{\#}$ is a projection onto $R(A)$.*

Definition 1.5. *The ascent and descent of an operator $A \in \mathfrak{B}(H)$ are defined by*

$$asc(A) = inf\{p \in \mathbb{N} \cup \{0\} : ker(A^p) = ker(A^{p+1})\},$$

$$dsc(A) = inf\{p \in \mathbb{N} \cup \{0\} : R(A^p) = R(A^{p+1})\},$$

(where $inf\emptyset = \infty$); if they are finite, they are equal and their common value is called the index of A and it is denoted by $ind(A)$.

The following theorem gives a necessary and sufficient condition which guarantee the existence of the group inverse of an operator in $\mathfrak{R}(H)$.

Theorem 1.7. *Let $A \in \mathfrak{R}(H)$. Then the following properties are equivalent:*
(i) A has a group inverse,
(ii) $H = R(A) \oplus kerA$,
(iii) $ind(A) \leq 1$.

Proof. $(i) \Rightarrow (ii)$. Suppose that B is group inverse of A. Then by definition, AB is an idempotent on $R(A)$. Hence $H = R(A) \oplus kerA$ topological direct sum.
$(ii) \Rightarrow (iii)$. Assume (ii) holds.
Let $x \in H$. Then $x = Ax + y$, where $y \in kerA$. Thus $Ax = A^2x$. Hence $R(A) = R(A^2)$. Now let $x \in kerA^2$. Then $Ax \in kerA$. Thus $Ax \in R(A) \cap kerA = \{0\}$. Therefore $kerA = kerA^2$.
$(iii) \Rightarrow (i)$. Since $kerA^2 = kerA$, then $R(A^*)^2 = R(A^*)$.
From the two hypotheses $R(A^2) = R(A), R(A^*)^2 = R(A^*)$ and using theorem of Douglas, we find that there exist two bounded operators C, D on H such that $A = A^2C = DA^2$.
Let $B = DAC$. We have
$ABA = ADACA = ADA^2CCA = AACCA = ACA = DA^2CA = DA^2 = A$,
$BAB = DACADAC = DACADA^2CC = DACAACC = DACAC = DDA^2CAC = DDAAC = DAC$.
From another side, we have
$AB = ADAC = ADA^2CC = AACC = AC$, and
$BA = DACA = DDA^2CA = DDAA = DA = DA^2C = AC$.
Then $AB = BA$. Therefore B satisfies the three equations of the group inverse. So B is the group inverse of A. □

Lemma 1.2. *Let E, F two Banach spaces and let $A \in \mathfrak{B}(E, F)$. Let M be closed subspace of F such that $R(A) \oplus M$ is closed. Then $R(A)$ is also closed.*

Proof. We equipped the vectorial space $E \setminus kerA \times M$ by the norm $\|(x,m)\| = \|x\| + \|m\|, x \in E \setminus kerA, m \in M$.
Let us define the operator $A_0 : E \setminus kerA \times M \to R(A) \oplus M$.
A_0 is definite, linear, bounded and bijective. Then A_0 is invertible operator. Hence A_0 is bounded below i.e there exists a constant $k > 0$ such that $\forall x \in E \setminus kerA, \forall m \in M, \|Ax + m\| \geq k(\|x\| + \|m\|)$. So

$$\forall x \in E, \|Ax\| \geq k\|x\|, \tag{*}$$

which proves that the operator $\overline{A} : E \setminus kerA \to R(A)$ is linear, bounded and bijective. Then, $\overline{A}$ has a bounded inverse.
Now, let us prove that $R(A)$ is closed.
Let $y_n \in R(A)$ such that $y_n = Ax_n \to y$ (where $(x_n)_{n\geq 1}$ is a sequence in H).
Applying (*) for $x = x_n - x_m$ (where $n, m \geq 1$), we find

$$\forall n, m \geq 1; \ \|x_n - x_m\| \leq \frac{1}{k}\|Ax_n - Ax_m\|.$$

By passing to the limit, we obatin that $(x_n)_{n\geq 1}$ is a Cauchy sequence, then there exists a vector $x \in E$ such that $x_n \to x$. Hence $\overline{A}x_n = Ax_n \to Ax$, so $y = Ax$. Therefore $R(A)$ is closed. □

Theorem 1.8. *[37] Let $A \in \mathfrak{B}(H)$ with $ind(A) \leq 1$. Then $R(A)$ is closed.*

Proof. Let $A \in \mathfrak{B}(H)$ with $ind(A) \leq 1$. Then from theorem 1.7, we have $H = R(A) \oplus kerA$. Since the kernel space $kerA$ is always closed, then the closedness of $R(A)$ is a consequence of lemma 1.2. □

Notation 1.2. *We denote by $\mathfrak{I}_1(H)$, the set of all operators $A \in \mathfrak{B}(H)$ such that $indA \leq 1$.*

Lemma 1.3. *[13] Let $A \in \mathfrak{I}_1(H)$. Then*
(i) Relative to the representation matrix (1.2) of A, A_1 is invertible and

$$A^{\#} = \begin{bmatrix} A_1^{-1} & A_1^{-2}A_2 \\ 0 & 0 \end{bmatrix}.$$

(ii) Relative to the representation matrix (1.3) of A, A_1 is invertible and

$$A^{\#} = \begin{bmatrix} A_1^{-1} & 0 \\ A_2A_1^{-2} & 0 \end{bmatrix}.$$

Proof. (i) A has the following matrix representation $\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ with respect to the orthogonal decomposition $H = R(A) \oplus kerA^*$ (where $A_1 : R(A) \to R(A)$). Since $ind(A) \leq 1$, then $R(A^2) = R(A)$ gives that A_1 is surjective, and $kerA^2 = kerA$ gives that A_1 is injective. Hence A_1 is invertible.

Let $\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ be the matrix representation of $A^{\#}$. $A^{\#}$ satisfied the three conditions of the group inverse, so

$$A_1B_1A_1 = A_1, B_1A_1 = B_1, B_1A_1 = A_1B_1, B_3 = B_4 = 0, B_2 = B_1^2A_2.$$

B_1 is the group inverse of A_1 and since A_1 is invertible, then $B_1 = A_1^{-1}$.
Therefore $A^{\#} = \begin{bmatrix} A_1^{-1} & A_1^{-2}A_2 \\ 0 & 0 \end{bmatrix}$.
(ii) By the same argument as above. □

Remark 1.4. *(i). If H has a finite dimension, then the ascent and descent of an operator A are finite.*
(ii). A is invertible if and only if $ind(A) = 0$.

Examples 1.1. *If $P \in \mathfrak{B}(H)$ is a projection then $P^{\#} = P$.*

Definition 1.6. *Let $A \in \mathfrak{R}(H)$. We call A is an EP stands for equal projection if $A^+A = AA^+$,or equivalently $R(A) = R(A^*)$.*

Note that any normal operator with a closed range is an EP operator, but the converse is not true even in a finite dimensional space.
The characterizations of selfadjoint, normal and EP operators on Hilbert spaces was established by Dragan S. Djordjevic and J. J. Koliha in [13], where they gave the following result.

The following lemma gives the matrix representation of some classes of operators:

Lemma 1.4. *Let $A \in \mathfrak{R}(H)$ has the following matrix representation $\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ with respect to the orthogonal decomposition $H = R(A) \oplus ker A^*$. Then the following properties hold:*
(i) A is selfadjoint if and only if $A_2 = 0$ and A_1 is selfadjoint.
(ii) A is normal if and only if $A_2 = 0$ and A_1 is normal.
(iii) A is EP if and only if $A_2 = 0$ and A_1 is invertible.
(iv) A is normal partial isometry if and only if $A_2 = 0$ and A_1 is unitary.
(v) A is selfadjoint partial isometry if and only if $A_2 = 0$ and A_1 is unitary reflection.

Proof. The proof is easy. □

Remark 1.5. *If $A \in \mathfrak{R}(H)$. Then A is an EP operator if and only if $A^+ = A^{\#}$.*

1.4 Norm inequalities equivalent to Heinz inequality

In this section, we are interested by giving some famous inequalities in operator theory and the relation between them.

In 1951, Heinz [22] proved that for every positifs operators $P, Q \in \mathfrak{B}(H)$ and for every $\alpha \in [0,1]$, we have

$$\forall X \in \mathfrak{B}(H), \|PX + XQ\| \geq \|P^{\alpha}XQ^{1-\alpha} + P^{1-\alpha}XQ^{\alpha}\|,$$

which is one of the most essential inequalities in operator theory. Its original proof is based on the complex analysis theory and is somewhat complicated.
From Heinz inequality, we may remark that for $\alpha = \frac{1}{2}$ and if we put $P = |A|$ and $Q = |B|$, we obtain the following inequality

$$\forall A, B, X \in \mathfrak{B}(H), \|A^*AX + XBB^*\| \geq 2\|AXB\|. \qquad \text{(A.G.M.I1)}$$

In literature, this inequality is called the Arithmetic-Geometric Mean Inequality.
In 1978, McIntosh [29] proved that (A.G.M.I1) holds and he deduced Heinz inequality from (A.G.M.I1). Then the two inequalities of Heniz and McIntosh are equivalent.
In 1990, independently of the two works of Heinz and McIntosh and with another motivation, Corach-Porta-Recht [8] have proved the following inequality

$$\forall S \in \mathbb{S}_0(H), \forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|. \qquad \text{(C.P.R.I)}$$

This inequality was a key factor in their study of differential geometry of selfadjoint operators. They proved this inequality by using the integral representation of a selfadjoint operator with respect to a spectral measure.
We may remark that for any invertible operator S, we have $0 \leq \underset{\|X\|=1}{inf} \|SXS^{-1} + S^{-1}XS\| \leq 2$. Using the (C.P.R.I), the above infimum gets it's maximal value 2, for S an invertible selfadjoint operator in $\mathfrak{B}(H)$.
Three years later, in 1993, J.I.Fujji, M.Fujji Furuta, and Nakamato [19], showed by elemantary calculations that Heinz inequality, (A.G.M.I1), (C.P.R.I) and some other inequalities are equivalent. They presented an easy proof for one of them and hence, they deduced Heinz inequality.

In the next proposition, we shall present other operator inequalities that are equivalent to Heinz inequality.

Proposition 1.8. *The following operator inequalities hold and are mutually equivalent:*

(1) $\forall A, B, X \in \mathfrak{B}(H),\ \|A^*AX + XBB^*\| \geq 2\,\|AXB\|\,,$

(2) $\forall S, R \in \mathfrak{R}(H), \forall X \in \mathfrak{B}(H),\ \|S^*XR^+ + S^+XR^*\| \geq 2\,\|SS^+XR^+R\|\,,$

(3) $\forall S, R \in \mathfrak{I}(H), \forall X \in \mathfrak{B}(H),\ \|S^*XR^{-1} + S^{-1}XR^*\| \geq 2\,\|X\|\,,$

(4) $\forall S, R \in \mathbb{S}_{cr}(H), \forall X \in \mathfrak{B}(H),\ \|SXR^+ + S^+XR\| \geq 2\,\|SS^+XR^+R\|\,,$

(5) $\forall S, R \in \mathbb{S}_0(H), \forall X \in \mathfrak{B}(H),\ \|SXR^{-1} + S^{-1}XR\| \geq 2\,\|X\|\,,$

(6) $\forall S, R \in \mathbb{S}_{cr}(H), \forall X \in \mathfrak{B}(H),\ \|S^2X + XR^2\| \geq 2\,\|SXR\|\,,$

(7) $\forall S, R \in \mathbb{S}(H), \forall X \in \mathfrak{B}(H),\ \|S^2X + XR^2\| \geq 2\,\|SXR\|\,,$

(1′) $\forall A, X \in \mathfrak{B}(H),\ \|A^*AX + XAA^*\| \geq 2\,\|AXA\|\,,$

(2′) $\forall S \in \mathfrak{R}(H), \forall X \in \mathfrak{B}(H),\ \|S^*XS^+ + S^+XS^*\| \geq 2\|SS^+XS^+S\|$,
(3′) $\forall S \in \mathfrak{I}(H), \forall X \in \mathfrak{B}(H),\ \|S^*XS^{-1} + S^{-1}XS^*\| \geq 2\|X\|$,
(4′) $\forall S \in \mathbb{S}_{cr}(H), \forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| \geq 2\|SXS\|$,
(5′) $\forall S \in \mathbb{S}_0(H), \forall X \in \mathfrak{B}(H),\ \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$,
(6′) $\forall S \in \mathbb{S}_{cr}(H), \forall X \in \mathfrak{B}(H),\ \|S^2X + XS^2\| \geq 2\|SXS\|$,
(7′) $\forall S \in \mathbb{S}(H), \forall X \in \mathfrak{B}(H),\ \|S^2X + XS^2\| \geq 2\|SXS\|$.

Proof. (1) ⇒ (2). Assume (1) holds. Let $S, R \in \mathfrak{R}(H), X \in \mathfrak{B}(H)$. Since $S^* = S^*SS^+$ and $R^* = R^+RR^*$, then from (1) it follows that

$$\begin{aligned}\left\|S^*XR^+ + S^+XR^*\right\| &= \left\|S^*S\left(S^+XR^+\right) + \left(S^+XR^+\right)RR^*\right\| \\ &\geq 2\left\|SS^+XR^+R\right\|.\end{aligned}$$

Hence (2) holds.
(2) ⇒ (3). trivial.
(3) ⇒ (1). Let P, Q be two invertible positive operators. Then from (3), it follows that

$$\forall X \in \mathfrak{B}(H), \left\|P^2X + XQ^2\right\| \geq 2\|PXQ\|.$$

Now let $A, B \in \mathfrak{B}(H)$. Then $P = |A|, Q = |B|$ are two positive operators. It's clear that the two operators $P + \epsilon I$ and $Q + \epsilon I$ are normal and invertible, for every $\epsilon > 0$. So, using the last inequality, we obtain

$$\forall \epsilon > 0, \forall X \in \mathfrak{B}(H), \left\|(P+\epsilon I)^2X + X(Q+\epsilon I)^2\right\| \geq 2\|(P+\epsilon I)X(Q+\epsilon I)\|.$$

By letting $\epsilon \to 0$, we deduce (1).
Hence the operator inequalities (1),(2) and (3) are equivalent.
(4) ⇒ (2). Trivial.
(1) ⇒ (4). Assume (1) holds. Let $S, R \in \mathbb{S}_{cr}(H), X \in \mathfrak{B}(H)$. Since $S = S^*SS^+$ and $R = R^+RR^*$, then from (1) it follows that

$$\begin{aligned}\left\|SXR^+ + S^+XR\right\| &= \left\|S^*S\left(S^+XR^+\right) + \left(S^+XR^+\right)RR^*\right\| \\ &\geq 2\left\|SS^+XR^+R\right\|.\end{aligned}$$

Hence (4) holds.
The implications (1) ⇒ (7) ⇒ (6) and (6) ⇒ (5) are trivial.
(5) ⇒ (3). Assume (5) holds. Let $S, R \in \mathfrak{I}(H), X \in \mathfrak{B}(H)$ and let $S = UP$ and $R = VQ$ be the polar decomposition of S and R. Then we obtain

$$\begin{aligned}\left\|S^*XR^{-1} + S^{-1}XR^*\right\| &= \|PU^*XQ^{-1}V^* + P^{-1}U^*XQV^*\| \\ &= \|(PU^*XQ^{-1} + P^{-1}U^*XQ)V^*\| \\ &= \|P(U^*X)Q^{-1} + P^{-1}(U^*X)Q\|.\end{aligned}$$

Now applying (5), it follows that

$$\left\|S^*XR^{-1}+S^{-1}XR^*\right\| \geq 2\|U^*X\| = 2\|X\|.$$

Hence (3) holds. Thus the operator inequalities $(1)-(7)$ are equivalent.

From pair operators to single operators, we deduce that the operator inequalities $(1')-(7')$ are also equivalent.

The implication $(1) \Rightarrow (1')$ is trivial.

$(1') \Rightarrow (1)$. This follows immediately by using the Berberian technics.

Let $A, B, X \in \mathfrak{B}(H)$. Hence $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \in \mathfrak{B}(H \oplus H)$, then by using $(1')$ we get that

$$\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^*\right\| \geq 2\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ 0 \end{bmatrix}\right.$$

So that we obtain

$$\left\|\begin{bmatrix} 0 & A^*AX+XBB^* \\ 0 & 0 \end{bmatrix}\right\| \geq 2\left\|\begin{bmatrix} 0 & AXB \\ 0 & 0 \end{bmatrix}\right\|.$$

Hence $\|A^*AX+XBB^*\| \geq 2\,\|AXB\|$.

Then the inequalities $(1)-(7)$, and $(1')-(7')$ are mutually equivalent.

It remains to prove that one of them holds. we shall prove that $(5')$ holds. The proof is given in two steps.

Step 1. Let $S \in \mathbb{S}_0(H)$, and $X \in \mathbb{S}(H)$.

Since X is selfadjoint, then $\|X\| = r(X)$. So there exists $\lambda \in \sigma(X)$ such that $|\lambda| = \|X\|$.

We have $\sigma(X) = \sigma(SXS^{-1}) \subset V(SXS^{-1})$, then $\lambda \in V(SXS^{-1})$. So there exists $f \in \mathbf{S}_1$ such that $\lambda = f(SXS^{-1})$.

Since X is selfadjoint then λ is real. Hence we have

$$\begin{aligned} 2\lambda &= \lambda + \bar{\lambda} \\ &= f(SXS^{-1}) + \overline{(f(SXS^{-1}))} \\ &= f(SXS^{-1}) + f(SXS^{-1})^* \\ &= f(SXS^{-1}) + f(S^{-1}XS) \\ &= f(SXS^{-1}+S^{-1}XS). \end{aligned}$$

Hence

$$2\|X\| = 2|\lambda| = |f(SXS^{-1}+S^{-1}XS)| \leq \|SXS^{-1}+S^{-1}XS\|.$$

Step 2. Let $S \in \mathbb{S}_0(H)$, and $X \in \mathfrak{B}(H)$.

Since $\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \in \mathbb{S}_0(H \oplus H)$, $\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \in \mathbb{S}(H \oplus H)$. Then by using step 1 we have that

$$\left\|\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}^{-1} + \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}^{-1}\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\right\| \geq 2\left\|\begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}\right\|.$$

So that we obtain

$$\left\| \begin{bmatrix} 0 & SXS^{-1} + S^{-1}XS \\ SX^*S^{-1} + S^{-1}X^*S & 0 \end{bmatrix} \right\| \geq 2 \left\| \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \right\|.$$

Hence it follows that $\|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|$.

□

By triangular inequality, it follows from the (A.G.M.I1) that the following inequality holds

$$\forall A, B, X \in \mathfrak{B}(H), \|A^*AX\| + \|XBB^*\| \geq 2\|AXB\| \qquad \text{(A.G.M.I2)}$$

In the next proposition, we shall give some operator inequalities equivalent to (A.G.M.I2). For the proof, we need the following elementary characterization

$$\forall S \in \mathfrak{B}(H),\ (S \in \mathcal{N}(H)) \Leftrightarrow (\forall X \in \mathfrak{B}(H), \|SX\| = \|S^*X\|). \qquad \text{(N)}$$

Proposition 1.9. *The following operator inequalities hold and are mutually equivalent:*

(1) $\forall A, B, X \in \mathfrak{B}(H),\ \|A^*AX\| + \|XBB^*\| \geq 2\,\|AXB\|,$

(2) $\forall S, R \in \mathfrak{R}(H), \forall X \in \mathfrak{B}(H),\ \|S^*XR^+\| + \|S^+XR^*\| \geq 2\,\|SS^+XR^+R\|,$

(3) $\forall S, R \in \mathfrak{I}(H), \forall X \in \mathfrak{B}(H),\ \|S^*XR^{-1}\| + \|S^{-1}XR^*\| \geq 2\,\|X\|,$

(4) $\forall S, R \in \mathcal{N}_{cr}(H), \forall X \in \mathfrak{B}(H),\ \|SXR^+\| + \|S^+XR\| \geq 2\,\|SS^+XR^+R\|,$

(5) $\forall S, R \in \mathcal{N}_0(H), \forall X \in \mathfrak{B}(H),\ \|SXR^{-1}\| + \|S^{-1}XR\| \geq 2\,\|X\|,$

(6) $\forall S, R \in \mathcal{N}_{cr}(H), \forall X \in \mathfrak{B}(H),\ \|S^2X\| + \|XR^2\| \geq 2\,\|SXR\|,$

(7) $\forall S, R \in \mathcal{N}(H), \forall X \in \mathfrak{B}(H), \|S^2X\| + \|XR^2\| \geq 2\,\|SXR\|,$

(1′) $\forall A, X \in \mathfrak{B}(H),\ \|A^*AX\| + \|XAA^*\| \geq 2\,\|AXA\|,$

(2′) $\forall S \in \mathfrak{R}(H), \forall X \in \mathfrak{B}(H),\ \|S^*XS^+\| + \|S^+XS^*\| \geq 2\,\|SS^+XS^+S\|,$

(3′) $\forall S \in \mathfrak{I}(H), \forall X \in \mathfrak{B}(H),\ \|S^*XS^{-1}\| + \|S^{-1}XS^*\| \geq 2\,\|X\|,$

(4′) $\forall S \in \mathcal{N}_{cr}(H), \forall X \in \mathfrak{B}(H),\ \|SXS^+\| + \|S^+XS\| \geq 2\,\|SS^+XS^+S\|,$

(5′) $\forall S \in \mathcal{N}_0(H), \forall X \in \mathfrak{B}(H),\ \|SXS^{-1}\| + \|S^{-1}XS\| \geq 2\,\|X\|,$

(6′) $\forall S \in \mathcal{N}_{cr}(H), \forall X \in \mathfrak{B}(H),\ \|S^2X\| + \|XS^2\| \geq 2\,\|SXS\|,$

(7′) $\forall S \in \mathcal{N}(H), \forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| \geq 2\,\|SXS\|.$

Proof. (1) $\Rightarrow$ (2). Using the same argument as used in the implication ((1) $\Rightarrow$ (2)) of the proposition 1.8, we find that (2) holds.
(2) $\Rightarrow$ (3). trivial.
(3) $\Rightarrow$ (1). Using the same argument as used in the implication ((3) $\Rightarrow$ (1)) of the proposition 1.8, we find that (1) holds.
Hence the operator inequality (1), (2) and (3) are equivalent.
(1) $\Rightarrow$ (4). Assume (1) holds. Let $S, R \in \mathcal{N}_{cr}(H), X \in \mathfrak{B}(H)$. Since S is normal then from (N), we obtain that

$$\|SXR^+\| + \|S^+XR\| = \|S^*XR^+\| + \|S^+XR^*\|.$$

Since $S^* = S^*SS^+$ and $R^* = R^+RR^*$, then from the above equality and from (1), it follows that

$$\begin{aligned}\left\|SXR^+\right\| + \left\|S^+XR\right\| &= \left\|S^*S\left(S^+XR^+\right)\right\| + \left\|\left(S^+XR^+\right)RR^*\right\| \\ &\geq 2\left\|SS^+XR^+R\right\|.\end{aligned}$$

Hence (4) holds.

(4) $\Rightarrow$ (6). Assume (4) holds. Let $S, R \in \mathcal{N}_{cr}(H), X \in \mathfrak{B}(H)$. Then from (4) and since $SS^+S = S,\ RR^+R = R,$ and both of $S^+S,\ RR^+$ are orthogonal projections, it follows that

$$\begin{aligned}\left\|S^2X\right\| + \left\|XR^2\right\| &\geq \left\|S\left(SXR\right)R^+\right\| + \left\|S^+\left(SXR\right)R\right\| \\ &\geq 2\left\|SS^+\left(SXR\right)R^+R\right\| \\ &= 2\left\|SXR\right\|.\end{aligned}$$

Thus (6) holds.

(6) $\Rightarrow$ (5). This implication is trivial.

(5) $\Rightarrow$ (1). Assume (5) holds. Then the following inequality holds

$$\forall S, R \in \mathcal{N}_0(H), \forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XR^2\right\| \geq 2\left\|SXR\right\|.$$

Let $A, B, X \in \mathfrak{B}(H)$. Put $P = |A|,\ Q = |B^*|$. It is clear that the two operators $P+\epsilon I$ and $Q+\epsilon I$ are normal and invertible, for every $\epsilon > 0$. So, using the last inequality, we obtain

$$\forall \epsilon > 0,\ \left\|(P+\epsilon I)^2 X\right\| + \left\|X\left(Q+\epsilon I\right)^2\right\| \geq 2\left\|(P+\epsilon I)X\left(Q+\epsilon I\right)\right\|.$$

By letting $\epsilon \to 0$, we deduce (1).

(1) $\Rightarrow$ (7). This follows immediately by using (N).

(7) $\Rightarrow$ (6). This implication is trivial.

Therefore the operator inequalities (1), (4), (5), (6) and (7) are equivalent.

Hence the operator inequalities (1) $-$ (7) are equivalent.
From pair operators to single operators, we deduce that the operator inequalities $(1') - (5')$ are also equivalent.

(1) $\Rightarrow$ $(1')$. This implication is trivial.

$(1')$ $\Rightarrow$ (1). This follows immediately by using the Berberian technics.

Therefore the inequalities $(1)-(7)$, and $(1')-(7')$ are mutually equivalent. It remains to prove that one of them holds. It is clear that (1) is an immediate consequence of $(A.G.M.I1)$. But here, we shall give a direct proof of (1) independently of $(A.G.M.I1)$ by using the numerical arithmetic-geometric mean inequality. Let $A, B, X \in \mathfrak{B}(H)$.

Then we have

$$\begin{aligned}
\frac{1}{2}\left(\|A^*AX\| + \|XBB^*\|\right) &\geq \sqrt{\|A^*AX\|\,\|XBB^*\|} \\
&\geq \sqrt{\|BB^*X^*A^*AX\|} \\
&\geq \sqrt{r(BB^*X^*A^*AX)} \\
&= \sqrt{r\left(B^*X^*A^*AXB\right)} \\
&= \|AXB\|\,.
\end{aligned}$$

□

2. CHARACTERIZATIONS OF SOME DISTINGUISHED CLASSES OF OPERATORS IN TERMS OF OPERATOR INEQUALITIES

From Proposition 1.8 and with single operator case, we may introduce the following properties generated by operator inequalities

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\left\|X\right\|, (S \in \mathfrak{I}(H)), \tag{PS1}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geq 2\left\|SXS\right\|, \quad (S \in \mathfrak{R}(H)), \tag{PS2}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^+ + S^+XS\right\| \geq 2\left\|SS^+XS^+S\right\|, (S \in \mathfrak{R}(H)), \tag{PS3}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geq 2\left\|SXS\right\|, \quad (S \in \mathfrak{B}(H)). \tag{PS4}$$

Using Proposition 1.8, the property $(PS1)$ is satisfied for every $S \in \mathbb{C}^*\mathbb{S}_0(H)$, each of the two properties $(PS2)$, $(PS3)$ is satisfied for every $S \in \mathbb{C}\mathbb{S}_{cr}(H)$, and the property $(PS4)$ is satisfied for every $S \in \mathbb{C}\mathbb{S}(H)$. Note that Corach-Porta-Recht [8], proved with another motivation and independently of $(A.G.M.I1)$ that the property $(PS1)$ is valid for every $S \in \mathbb{S}_0(H)$.

From Proposition 1.9 and with single operator case, we may introduce the following properties generated by operator inequalities

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\left\|X\right\|, (S \in \mathfrak{I}(H)), \tag{PN1}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|, \quad (S \in \mathfrak{R}(H)), \tag{PN2}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^+\right\| + \left\|S^+XS\right\| \geq 2\left\|SS^+XS^+S\right\|, (S \in \mathfrak{R}(H)), \tag{PN3}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|, (S \in \mathfrak{B}(H)). \tag{PN4}$$

Using Proposition 1.9, the property $(PN1)$ is satisfied for every $S \in \mathcal{N}_0(H)$, each of the two properties $(PN2)$, $(PN3)$ is satisfied for every $S \in \mathcal{N}_{cr}(H)$, and the property $(PN4)$ is satisfied for every $S \in \mathcal{N}(H)$.

So it is interesting to ask to describe the subclasses characterized by the above properties that are generated by operator inequalities related to the known arithmetic-geometric mean inequality.
This kind of problem was introduced by Seddik [38], by considering the Corach-Porta-Recht inequality. It was proved that the property $(PS1)$ characterizes exactly the subclass $\mathbb{C}^*\mathbb{S}_0(H)$ (the subclass of all rotations of invertible selfadjoint operators in $\mathfrak{B}(H)$). This was the beginning of the characterization of some distinguished classes of operators in term of operator inequalities.

In this chapter, we will give all the characterizations found of some subclasses by the author Seddik.

2.1 Characterizations of the class of normal operators

In [43], Seddik has given characterizations of the class of normal invertible operators in terms of operators inequalities, to give his main result of the characterizations we need the following results which are useful in the proof of the main result.
The following proposition gives some preliminary characterizations of normal operators:

Proposition 2.1. *Let $S \in \mathfrak{B}(H)$. Then the following properties are equivalent:*
(i) S is normal,
(ii) $\forall x \in H, \|Sx\| = \|S^*x\|$,
(iii) $\forall X \in \mathfrak{B}(H), \|SX\| = \|S^*X\|$,
(iv) $\forall X \in \mathfrak{B}(H), \|XS\| = \|XS^*\|$.

Proof. Easy. □

Lemma 2.1. *[38] Let P,Q be in $\mathfrak{B}(H)$ such that $P > 0$ and $Q > 0$. If we have*

$$\forall X \in \mathfrak{B}(H), \|PXP^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|, \tag{1}$$

then $\{P\}' = \{Q\}'$.

Proof. Assume (1) holds.
If we replace X by QXQ^{-1} in (1), we have

$$\forall X \in \mathfrak{B}(H), \|X\| + \|PQXQ^{-1}P^{-1}\| \geq 2\|QXQ^{-1}\|, \tag{*}$$

Let UM be the polar decomposition of PQ (U is unitary and $M = (QP^2Q)^{\frac{1}{2}}$).
Then, from $(*)$, we obtain

$$\forall X \in \mathfrak{B}(H), \|X\| + \|MXM^{-1}\| \geq 2\|QXQ^{-1}\|. \tag{**}$$

Let $X \in \mathbb{S}(H)$ such that $MX = XM$, and let α be a complex number. Then, by $(**)$, we get

$$\|X - \alpha I\| \geq \|Q(X - \alpha I)Q^{-1}\|.$$

Since $(X - \alpha I)$ is normal, we have

$$r(X - \alpha I) = \|X - \alpha I\| \geq \|Q(X - \alpha I)Q^{-1}\| \geq r(Q(X - \alpha I)Q^{-1}) = r(X - \alpha I);$$

so that

$$\|QXQ^{-1} - \alpha I\| = \|Q(X - \alpha I)Q^{-1}\| = r(Q(X - \alpha I)Q^{-1}) = r(QXQ^{-1} - \alpha I).$$

Then, by proposition 1.2, we have QXQ^{-1} is convexoid. Hence $W(QXQ^{-1}) = co\sigma(QXQ^{-1}) = co\sigma(X)$. Since X is selfadjoint, then $co\sigma(X) \subset \mathbb{R}$. So $W(QXQ^{-1}) \subset \mathbb{R}$. Thus QXQ^{-1} is selfadjoint. Hence $QXQ^{-1} = Q^{-1}XQ$, then $Q^2X = XQ^2$. Since Q is positive, then from proposition 1.3, we conclude that $QX = XQ$.

Now let $X \in \mathfrak{B}(H)$ and put $X = X_1 + iX_2$(where $X_1 = ReX$ and $X_2 = ImX$), such that $MX = MX$. Then, we have $MX_1 = X_1M$ and $MX_2 = X_2M$. From the above step it follows that $QX_1 = X_1Q$ and $QX_2 = X_2Q$, thus $QX = XQ$. We conclude that $\{M\}' \subset \{Q\}'$. So we have $MQ = QM$. Then $M^2Q = QM^2$, thus $QP^2Q^2 = Q^2P^2Q$. Hence $P^2Q = QP^2$. Since P is positive, then from proposition 1.3, we find that $PQ = QP$.
Let $X \in \mathbb{S}(H)$ such that $PX = XP$, and let α be a complex number.
Since $PX = XP$ and $PQ = QP$, then $QXQ^{-1} \in \{P\}'$. So from $(*)$, we obtain

$$\|X - \alpha I\| \geq \|Q(X - \alpha I)Q^{-1}\|.$$

By the same argument as used before, it follows that $QX = XQ$, so that $\{P\}' \subset \{Q\}'$. From (1) and by the same method, we obtain that $\{Q^{-1}\}' \subset \{P^{-1}\}'$. Then $\{Q\}' \subset \{P\}'$. Finally we have $\{P\}' = \{Q\}'$. □

Lemma 2.2. *[38] Let $\epsilon > 0$, $\alpha_1, ..., \alpha_n$, $\beta_1, ..., \beta_n$ ($n \in \mathbb{N}^*$) such that $0 < \alpha_1 < ... < \alpha_n \leq 1$, $\{\alpha_1, ..., \alpha_n\} = \{\beta_1, ..., \beta_n\}$ and $\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \epsilon$, for all i, j.
Then we have $|\alpha_i - \beta_i| < \epsilon$, for all i.*

Proof. We have $\frac{\alpha_i}{\alpha_j} < 1$, if $i < j$.
Let $i, j \in \{1, ..., n\}$ such that $i < j$. Thus $2 - \epsilon < \frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} < 1 + \frac{\beta_j}{\beta_i}$. Then $1 - \epsilon < \frac{\beta_j}{\beta_i}$. Hence $\beta_i - \beta_j < \epsilon\beta_i < \epsilon$. Therefore $\beta_i - \beta_j < \epsilon$.
There are three cases.
Let $i \in \{1, ..., n\}$ such that $\alpha_i \neq \beta_i$ (in the case $\alpha_i = \beta_i$, of course we have $|\alpha_i - \beta_i| = 0 < \epsilon$).
Case 1. $i = 1$. There exists $j \geq 2$, such that $\beta_j = \alpha_1$, so we have $|\beta_1 - \alpha_1| = \beta_1 - \beta_j < \epsilon$, since $j > 1$.
Case 2. $i = n$. There exists $j < n$, such that $\beta_j = \alpha_n$, so we have $|\beta_n - \alpha_n| = \beta_j - \beta_n < \epsilon$, since $n > j$.
Case 3. $1 < i < n$. If $\beta_i > \alpha_i$, then there exists $j > i$, such that $\beta_j \leq \alpha_i$, so we have $|\beta_i - \alpha_i| = \beta_i - \beta_j < \epsilon$, since $j > i$. If $\beta_i < \alpha_i$, then there exists $j < i$, such that $|\beta_i - \alpha_i| = \beta_j - \beta_i < \epsilon$, since $i > j$. □

Lemma 2.3. *[38] Let $P, Q \in \mathfrak{B}(H)$ such that $P > 0$, $Q > 0$ and $\sigma(P) \subseteq \sigma(Q)$. Then the following properties are equivalents:
(i) $\forall X \in \mathfrak{B}(H)$, $\|PXP^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|$,
(ii) $P = Q$.*

Proof. We may assume, without loss of the generality, that $\|P\| = \|Q\| = 1$.
$(i) \Rightarrow (ii)$. Decompose P and Q using the spectral measure

$$P = \int \lambda dE_\lambda, Q = \int \lambda dF_\lambda,$$

and consider

$$P_n = \int h_n(\lambda)dE_\lambda = h_n(P), Q_n = \int h_n(\lambda)dF_\lambda = h_n(Q),$$

where $h_n(\lambda)$ is a function of the form:

$$h_n(\lambda) = \frac{k}{n}, \text{ for } \frac{k}{n} \leq \lambda < \frac{k+1}{n}, \text{ and } k = 0, 1, 2, ...$$

Hence P_n and Q_n are positive invertible operators with finite spectrums.
By the spectral theorem and by the form of $h_n(\lambda)$, we have $h_n(\lambda) \to \lambda$ uniformly. Thus $P_n \longrightarrow P$ and $Q_n \longrightarrow Q$ uniformly.
We have also

$$\sigma(P_n) = \sigma(h_n(P)) = h_n(\sigma(P)) \subseteq h_n(\sigma(Q)) = \sigma(h_n(Q)) = \sigma(Q_n).$$

On the other hand, we have $P_n \in \{P\}^{''}$ and $Q_n \in \{Q\}^{''}$. By (i) and lemma 2.1 we have $\{P\}^{'} = \{Q\}^{'}$.
Let $X \in \{P\}^{'}$. Since $P_n \in \{P\}^{''}$, thus $X \in \{P_k\}^{'}$, for all $k \geq 1$. So $\{P\}^{'} \subset \cap_{k=1}^{\infty}\{P_k\}^{'}$.
Now let $X \in \cap_{k=1}^{\infty}\{P_k\}^{'}$, then $X \in \{P_k\}^{'}$, for all $k \geq 1$. Since $P_n \in \{P\}^{''}$, thus $X \in \{P\}^{'}$.
Hence $\cap_{k=1}^{\infty}\{P_k\}^{'} \subset \{P\}^{'}$. Therefore $\{P\}^{'} = \cap_{k=1}^{\infty}\{P_k\}^{'}$.
By the same argument as used before, we obtain that $\{Q\}^{'} = \cap_{k=1}^{\infty}\{Q_k\}^{'}$.
Since $\sigma(P_n)$ is finite. Put $\sigma(P_n) = \{\alpha_1, ..., \alpha_p\}$ such that $0 < \alpha_1 < ... < \alpha_p \leq 1$. Then there exist p orthogonal projections $E_1, ..., E_p$ such that $E_iE_j = 0$ if $i \neq j$, $\sum\limits_{i=1}^{p} E_i = I$ and $\sum\limits_{i=1}^{p} \alpha_i E_i = P_n$.
We have $\{P\}^{'} = \{Q\}^{'}$, thus $PQ = QP$. Since $P_n \in \{P\}^{''}$, then $P_nQ = QP_n$. And since $Q_n \in \{Q\}^{''}$, then $P_nQ_n = Q_nP_n$.
Since $\sigma(P_n) \subseteq \sigma(Q_n)$, $P_nQ_n = Q_nP_n$ and Q_n is selfadjoint, then there exist p positive numbers $\beta_1, ..., \beta_p$ such that $\sigma(P_n) = \{\alpha_1, ..., \alpha_p\} \subseteq \{\beta_1, ..., \beta_p\} = \sigma(Q_n)$ and $\sum\limits_{i=1}^{p} \beta_i E_i = Q_n$.
We have $0 < \alpha_1 < ... < \alpha_p \leq 1$, then $card(\sigma(P_n)) = p$. And $card(\sigma(Q_n)) \leq p$. Since $\sigma(P_n) \subseteq \sigma(Q_n)$, so that $\sigma(P_n) = \{\alpha_1, ..., \alpha_p\} = \{\beta_1, ..., \beta_p\} = \sigma(Q_n)$.
We have $P_n \longrightarrow P$ and $Q_n \longrightarrow Q$ uniformly, thus $P_n^{-1} \longrightarrow P^{-1}$ and $Q_n^{-1} \longrightarrow Q^{-1}$ uniformly.
Then there exists a constant $M > 1$ such that

$$\|P_n^{-1}\| \leq M, \ \|Q_n^{-1}\| \leq M, \text{ for every } n \geq 1.$$

Let $\epsilon > 0$. Then there exists an integer $N \geq 1$ such that

$$\forall n \geq N : \|P_n - P\| < \tfrac{\epsilon}{4M}, \|Q_n - Q\| < \tfrac{\epsilon}{4M}, \|P_n^{-1} - P^{-1}\| < \tfrac{\epsilon}{4M} \text{ et } \|Q_n^{-1} - Q^{-1}\| < \tfrac{\epsilon}{4M}.$$

Let $X \in (\mathfrak{B}(H))_1$ and let $n \geq N$. Then, we have

$$\begin{aligned}\|P_nXP_n^{-1}\| &= \|P_nXP_n^{-1} + PXP_n^{-1} - PXP_n^{-1} + PXP^{-1} - PXP^{-1}\| \\ &= \|(P_n - P)XP_n^{-1} + PX(P_n^{-1} - P^{-1}) + PXP^{-1}\| \\ &\geq \|PXP^{-1}\| - \|P_n - P\|\|P_n^{-1}\| - \|P_n^{-1} - P^{-1}\| \\ &\geq \|PXP^{-1}\| - \frac{\epsilon}{2}.\end{aligned}$$

Using the same argument, we obtain also

$$\|Q_nXQ_n^{-1}\| \geq \|QXQ^{-1}\| - \frac{\epsilon}{2}.$$

So, we deduce

$$\forall X \in (\mathfrak{B}(H))_1, \forall n \geq N\colon\ \|P_nXP_n^{-1}\| + \|Q_n^{-1}XQ_n\| \geq 2 - \epsilon.$$

Then

$$\forall n > N,\, \forall X \in \mathfrak{B}(H),\, \|P_nXP_n^{-1}\| + \|Q_n^{-1}XQ_n\| \geq (2-\epsilon)\left\|X\right\|.$$

Let $n > N$ and $X_{ij} = E_iXE_j$. For $X \in \mathfrak{B}(H)$, we have:

$$\|P_nX_{ij}P_n^{-1}\| + \|Q_n^{-1}X_{ij}Q_n\| \geq (2-\epsilon)\left\|X_{ij}\right\|.$$

Since

$$\begin{aligned}P_nX_{ij}P_n^{-1} &= \sum_{n=1}^{p} \alpha_n E_n E_i X E_j \sum_{n=1}^{p} \alpha_n^{-1} E_n \\ &= \alpha_i E_i E_i X E_j \alpha_j^{-1} E_j \\ &= \alpha_i E_i X E_j \alpha_j^{-1} \\ &= \frac{\alpha_i}{\alpha_j} E_i X E_j,\end{aligned}$$

$$\begin{aligned}Q_n^{-1}X_{ij}Q_n &= \sum_{n=1}^{p} \beta_n^{-1} E_n E_i X E_j \sum_{n=1}^{p} \beta_n E_n \\ &= \beta_i^{-1} E_i E_i X E_j \beta_j E_j \\ &= \beta_i^{-1} E_i X E_j \beta_j \\ &= \frac{\beta_j}{\beta_i} E_i X E_j,\end{aligned}$$

then using the last inequaliy, we get

$$\left(\tfrac{\alpha_i}{\alpha_j} + \tfrac{\beta_j}{\beta_i}\right)\left\|X_{ij}\right\| \geq (2-\epsilon)\left\|X_{ij}\right\|.$$

Which gives

$$\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \epsilon.$$

From above, it follows that
(a) $\alpha_1 < \alpha_2 < ... < \alpha_p$,
(b) $\{\alpha_1, ..., \alpha_p\} = \{\beta_1, ..., \beta_p\}$,
(c) $\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \epsilon$, for $i, j = 1, ..., p$.
Then from lemma 2.2, we obtain that $|\alpha_i - \beta_i| < \epsilon$, for all i. Therefore

$$\|P_n - Q_n\| = \max_{1 \leq i \leq p} |\alpha_i - \beta_i| < \epsilon.$$

By letting $n \longrightarrow \infty$, we obtain $\|P - Q\| < \epsilon$. Now, letting $\epsilon \longrightarrow 0$, we deduce $P = Q$.
$(ii) \Rightarrow (i)$. Suppose that $P = Q$. Then we have

$$\forall X \in \mathfrak{B}(H),\ \|PXP^{-1}\| + \|P^{-1}XP\| \geq 2\,\|X\|,$$

which is true from Corach Porta Recht inequality. □

The characterization of the class $\mathcal{N}_0(H)$ of all invertible normal operators in $\mathfrak{B}(H)$ is given in the following theorem:

Theorem 2.1. *[41,43] Let $S \in \mathfrak{I}(H)$. Then the following properties are equivalent:*
(i) $S \in \mathcal{N}_0(H)$,
(ii) $\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| = \|S^*XS^{-1}\| + \|S^{-1}XS^*\|$,
(iii) $\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \geq \|S^*XS^{-1}\| + \|S^{-1}XS^*\|$,
(iv) $\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \geq 2\|X\|$,
(v) $\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \leq \|S^*XS^{-1}\| + \|S^{-1}XS^*\|$.

Proof. $(i) \Rightarrow (ii)$. This implication follows immediately from proposition 2.1.
$(ii) \Rightarrow (iii)$. The implication is trivial.
The implication $(iii) \Rightarrow (iv)$ follows immediately from proposition 1.8(3′).
$(iv) \Rightarrow (i)$. Let $S = UP$ and $S^* = U^*Q$ be the polar decomposition of S and S^*(where $P = |S|$ and $Q = |S^*|$). Using the polar decomposition of S and S^* in (iv) we obtain

$$\forall X \in \mathfrak{B}(H), \|PXP^{-1}\| + \|Q^{-1}XQ\| \geq 2\|X\|. \qquad (*)$$

Since $P^2 = S^*S$ and $Q^2 = SS^*$, then $\sigma(P^2) = \sigma(Q^2)$. By the spectral theorem, it follows that $\sigma(P) = \sigma(Q)$. Using the inequality (*) and the condition $\sigma(P) = \sigma(Q)$, and applying lemma 2.3, it follows that $P = Q$. Therefore $S^*S = SS^*$.
$(i) \Rightarrow (v)$. Follows immediately from proposition 2.1.
$(v) \Rightarrow (i)$. Assume (v) holds. If we replace X by $x \otimes y$ (where $x, y \in (H)_1$) in (v), we obtain

$$\forall x, y \in (H)_1, \|Sx\|\|(S^{-1})^*y\| + \|S^{-1}x\|\|S^*y\| \leq \|S^*x\|\|(S^{-1})^*y\| + \|S^{-1}x\|\|Sy\|.$$

Hence

$$\forall x, y \in (H)_1, (\|Sx\| - \|S^*x\|)\|(S^{-1})^*y\| \leq (\|Sy\| - \|S^*y\|)\|S^{-1}x\|. \qquad (**)$$

If there exists $y \in (H)_1$, such that $\|Sy\| \leq \|S^*y\|$, then $\forall x \in (H)_1, \|Sx\| \leq \|S^*x\|$. Else $\forall y \in (H)_1, \|Sy\| \geq \|S^*y\|$.
Thus

$$(\forall x \in (H)_1, \|Sx\| \geq \|S^*x\|) \text{ or } (\forall x \in (H)_1, \|Sx\| \leq \|S^*x\|).$$

Assume that the inequality $\|Sx\| \geq \|S^*x\|$ holds for every $x \in (H)_1$.
Since the relation $\frac{1}{\|T^{-1}\|} \leq \|Tx\| \leq \|T\|$ holds for every $T \in \mathfrak{I}(H)$ and for every $x \in (H)_1$, then from $(**)$, it follows that

$$\forall x, y \in (H)_1, \|Sx\| - \|S^*x\| \leq k(\|Sy\| - \|S^*y\|),$$

where $k = \|S\|\|S^{-1}\|$. So we have

$$\forall x, y \in (H)_1, \|Sx\| + k\|S^*y\| \leq \|S^*x\| + k\|Sy\|.$$

Hence

$$\forall x \in (H)_1, \sup_{\|y\|=1} (\|Sx\| + k\|S^*y\|) \leq \sup_{\|y\|=1} (\|S^*x\| + k\|Sy\|).$$

Thus

$$\forall x \in (H)_1, \|Sx\| + k\|S\| \leq \|S^*x\| + k\|S\|.$$

So it follows that the inequality $\|Sx\| \leq \|S^*x\|$ holds for every vector x in $(H)_1$.
Hence, the equality $\|Sx\| = \|S^*x\|$ holds for every vector x in $(H)_1$. Therefore $S \in \mathcal{N}_0(H)$.
With the second assumption and by the same argument, we find also that $S \in \mathcal{N}_0(H)$. □

Remark 2.1. *From the above propositions, it follows that the class $\mathcal{N}_0(H)$ is given by*

$$\begin{aligned}
\mathcal{N}_0(H) &= \{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| = \|S^*XS^{-1}\| + \|S^{-1}XS^*\|\} \\
&= \{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \geq \|S^*XS^{-1}\| + \|S^{-1}XS^*\|\} \\
&= \{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \leq \|S^*XS^{-1}\| + \|S^{-1}XS^*\|\} \\
&= \{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \geq 2\|X\|\}.
\end{aligned}$$

In [44] Seddik extend the properties $(N1), (N2)$, and $(N4)$ in the above theorem from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{R}(H)$. (he did a version of the above inequalities for Moore-Penrose invertible operators).
For the extentions, we need the following lemma which is useful in the proof of his principal theorem.

Lemma 2.4. *[44] Let $S \in \mathfrak{B}(H)$. If S injective with closed range (or surjective) and satisfies the following inequality*

$$\forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| \geq 2\|SXS\|, \tag{2}$$

then S is normal.

Proof. Assume that $S \neq 0$ and all 2×2 matrices used in this proof are given with respect to the orthogonal direct sum $H = R(S) \oplus kerS^*$. Then $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$.
Put $P = |S|$, $Q = |S^*|$, $P_1 = |S_1|$, $P_2 = |S_2|$, $Q_1 = (S_1S_1^* + S_2S_2^*)^{1/2}$.
So we have $S^*S = P^2 = \begin{bmatrix} P_1^2 & S_1^*S_2 \\ S_2^*S_1 & P_2^2 \end{bmatrix}$, $SS^* = Q^2 = \begin{bmatrix} Q_1^2 & 0 \\ 0 & 0 \end{bmatrix}$. It is clear that Q_1 is invertible then $Q^+ = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$.
Since S is injective with a closed range, then $S^+S = I, kerP = kerS = \{0\}$, and $R(P) = R(S^*S)$ is closed (since $R(S^*)$ is also closed). Thus $kerP = 0$ and $R(P) = (kerP)^\perp = H$. So, P is invertible.
Since S is injective with closed range, then $ker(S^2) \subset ker(S^+S^2) = kerS = \{0\}$. Thus S^2 is also injective with closed range. Hence $(S^2)^+S^2 = I$.
If we replace X by S^+XS^+ in (2), it follows that

$$\forall X \in \mathfrak{B}(H), \|S^2S^+XS^+\| + \|S^+XS\| \geq 2\|SS^+X\|. \qquad (*)$$

The proof is given in four steps.
Step 1. Prove that $(S^2)^+S = S^+$.
If we replace X by XS^+ in (2), we obtain

$$\forall X \in \mathfrak{B}(H), \|S^2XS^+\| + \|XS\| \geq 2\|SX\|.$$

It is known that S^+ is the unique solution of the following four equations: $SXS = S; XSX = X; (XS)^* = XS; (SX)^* = SX$. It is easy to see that $(S^2)^+S$ satisfies the first three equations.
Now we prove that $(S^2)^+S$ also satisfies the last equation. Since, the operator $S(S^2)^+S$ is a projection, it suffices to prove that its norm is less than or equal to one. By taking $X = (S^2)^+S$ in $(*)$, we obtain

$$2 \geq \|S^2(S^2)^+S(S)^+\| + \|(S^2)^+SS\| \geq 2\|S(S^2)^+S\|.$$

Hence $\|S(S^2)^+S\| \leq 1$. Therefore $(S^2)^+S = S^+$.
Step 2. Prove that $(S^+)^2 = (S^2)^+$.
Since $S^2(S^2)^+ = SS^+S^2(S^2)^+$, then $S^2(S^2)^+ = S^2(S^2)^+SS^+$. So from step 1, we obtain $S^2(S^2)^+ = S^2(S^+)^2$. Since S^2 is injective, we have $(S^+)^2 = (S^2)^+$.
Step 3. Prove that $kerS^* = \{0\}$.
Since $(S^+)^2 = (S^2)^+$, then by theorem of the reverse order law the two operators S^*S and SS^+ commute. Hence $S^*S = P^2 = \begin{bmatrix} S_1^*S_1 & 0 \\ 0 & S_2^*S_2 \end{bmatrix}$. So that $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$.
Since $kerS^* \neq \{0\}$, then $\sigma(Q^2) = \sigma(Q_1^2) \cup \{0\}$. From the fact that $\sigma(P^2) = \sigma(Q^2) - \{0\}$, we have $\sigma(P^2) = \sigma(Q_1^2)$. Then $\sigma(P_1^2) \cup \sigma(P_2^2) = \sigma(Q_1^2)$. Hence $\sigma(P_1^2) \subset \sigma(Q_1^2)$. Thus $\sigma(P_1) \subset \sigma(Q_1)$. Using the polar decomposition of S and S^* in the inequality $(*)$, we obtain the following inequality

$$\forall X \in \mathfrak{B}(H), \|S^2S^+X(P)^{-1}\| + \|Q^+XQ\| \geq 2\|SS^+X\|.$$

By taking $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$ (resp. $X = \begin{bmatrix} 0 & X_2 \\ 0 & 0 \end{bmatrix}$), where $X_1 \in \mathfrak{B}(R(S))$ (resp. $X_2 \in \mathfrak{B}(kerS^*; R(S))$ in the last inequality and since $S^2S^+ = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}$, we deduce the two following inequalities

$$\forall X_1 \in \mathfrak{B}(R(S)), \|P_1X_1P_1^{-1}\| + \|Q_1^{-1}X_1Q_1\| \geq 2\|X_1\|. \tag{**}$$

$$\forall X_2 \in \mathfrak{B}(kerS^*; R(S)), \|P_1X_2P_2^{-1}\| \geq 2\|X_2\|. \tag{***}$$

By taking $X_2 = x \otimes y$ (where $x \in (R(S))_1, y \in kerS^*$) in $(***)$, we obtain

$$\forall x \in (R(S))_1, \forall y \in kerS^*, \|P_1x\|\|P_2^{-1}y\| \geq 2\|y\|.$$

So we have

$$\forall x \in (R(S))_1, \forall y \in (kerS^*)_1, \|P_1x\| \geq 2\|P_2y\|.$$

Thus $\|P_2y\| \leq \frac{k}{2}$, for every $y \in (kerS^*)_1$ (where $k = inf\|P_1x\| > 0$). Then $\langle P_2^2y, y\rangle \leq \frac{k^2}{4}$, for every $y \in (kerS^*)_1$. So we obtain $\sigma(P_2^2) \subset (0, \frac{k^2}{4}]$ and $\sigma(P_1^2) \subset [k^2, \infty)$. Since $\sigma(P_1) \subset \sigma(Q_1)$, then using Lemma 2.3 with $(**)$, we obtain $P_1 = Q_1$. Hence $\sigma(Q_1^2) = \sigma(P_1^2) = \sigma(P_1^2) \cup \sigma(P_2^2)$. Then $\sigma(P_2^2) \subset \sigma(P_1^2)$, that is impossible since $(0, \frac{k^2}{4}] \cap [k^2, \infty) = \emptyset$. Therefore $kerS^* = \{0\}$.
Step 4. Prove that S is normal.
Since $kerS^* = \{0\}$, then $R(S) = H$. So that S is invertible and satisfies the inequality (2). Hence S satisfies the following inequality

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \geq 2\|X\|$$

Therefore S is normal (using theorem 2.1).
With the second assumption S surjective. S^* injective with closed range satisfying also the inequality (2), so that S^* is normal. Hence S is normal. □

Theorem 2.2. *[44] Let $S \in \mathfrak{R}(H)$. Then the following properties are equivalent:*
(i) $S \in \mathcal{N}_{cr}(H)$,
(ii) $\forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| = \|S^*XS^+\| + \|S^+XS^*\|$,
(iii) $\forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| \geq \|S^*XS^+\| + \|S^+XS^*\|$,
(iv) $\forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| \geq 2\|SS^+XS^+S\|$,
(v) $\forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| \geq 2\|SXS\|$.

Proof. The proof is trivial if $S = 0$. Assume now that $S \neq 0$.
$(i) \Rightarrow (ii)$. This implication follows immediately using proposition 2.1.
The implication $(ii) \Rightarrow (iii)$ is trivial.
$(iii) \Rightarrow (vi)$. This implication follows immediately from proposition 1.9 $(3')$.
$(iv) \Rightarrow (v)$. Assume (iv) holds. If we replace X by SXS. Then the following inequality holds

$$\forall X \in \mathfrak{B}(H), \|S^2XSS^+\| + \|S^+SXS^2\| \geq 2\|SS^+SXSS^+S\|.$$

From this inequality and since $\|SS^+\| = \|S^+S\| = 1$, the property (v) follows immediately.
$(v) \Rightarrow (i)$. Assume (v) holds. The implication is trivial if $S = 0$. Assume now that $S \neq 0$.
Let $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ kerS^* \end{bmatrix}$ and let $S^* = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S^*) \\ kerS \end{bmatrix}$.
Put $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ kerS^* \end{bmatrix}$. By a simple computation, we obtain

$$S^2X = \begin{bmatrix} S_1^2X_1 & 0 \\ 0 & 0 \end{bmatrix}, XS^2 = \begin{bmatrix} X_1S_1^2 & X_1S_1S_2 \\ 0 & 0 \end{bmatrix}, SXS = \begin{bmatrix} S_1X_1S_1 & S_1X_1S_2 \\ 0 & 0 \end{bmatrix}.$$

Hence, we have

$$\|S^2X\| = \|S_1^2X_1\|, \|XS^2\|^2 = \|X_1S_1(S_1S_1^* + S_2S_2^*)S_1^*X_1^*\| = \|X_1S_1K\|^2,$$

$$\|SXS\|^2 = \|S_1X_1(S_1S_1^* + S_2S_2^*)X_1^*S_1^*\| = \|S_1X_1K\|^2,$$

(where K is the positive square root of the positive operator $S_1S_1^* + S_2S_2^*$). Then using (v), we obtain the following inequality

$$\forall X_1 \in \mathfrak{B}(R(S)), \|S_1^2X_1\| + \|X_1S_1K\| \geq 2\|S_1X_1K\|. \qquad (*)$$

On the other hand, if we put $X = x \otimes y$ (for $x; y \in H$) in (v), we obtain

$$\forall x, y \in (H), \|y\|\|S^2x\| + \|x\|\|(S^*)^2y\| \geq 2\|Sx\|\|S^*y\|. \qquad (**)$$

We shall prove (i) in three steps.
Step 1. Prove that S_1 or T_1 is bounded below.
Assume that it is not the case. S_1 and T_1 are not bounded below.
With the condition S_1 is not bounded below, we may choose a sequence (u_n) in H such that

$$S^2u_n \to 0 \ and \ \|Su_n\| = 1, \ for \ n \geq 1.$$

For every $n \geq 1$, there exist $x_n \in R(S^*)$ and $z_n \in kerS$ such that $u_n = x_n + z_n$. Thus, we obtain

$$S^2x_n = S^2u_n \to 0, \|Sx_n\| = \|Su_n\| = 1, \|x_n\| = \|S^+Su_n\| \leq \|S^+\|, \text{ for } n \geq 1.$$

With the second condition "T_1 is not bounded below", by the same argument, we may choose a bounded sequence (y_n) in H satisfying

$$(S^*)^2y_n \to 0 \ and \ \|S^*y_n\| = 1, \text{ for } n \geq 1.$$

Applying (**) for $x = x_n$ and $y = y_n$, we obtain

$$\forall n \geq 1, \|y_n\|\|S^2x_n\| + \|x_n\|\|(S^*)^2y_n\| \geq 2.$$

Letting $n \to \infty$, we have $0 \geq 2$, which is impossible. Therefore S_1 or T_1 is bounded below.

Step 2. Prove that S_1 or T_1 is surjective.
Assume that T_1 is bounded below. Then there exists a constant $k > 0$ such that

$$\forall x \in H, \|(S^*)^2 x\| \geq k\|S^* x\|.$$

So we have $S^2(S^*)^2 \geq k^2 SS^*$. From theorem of Douglas, we obtain $R(S^2) \supset R(S)$. Thus $R(S^2) = R(S)$. So S_1 is surjective.
Also, if S_1 is bounded below, then by the same argument, we deduce that T_1 is surjective.
Step 3. Prove that S is normal.
Assume that S_1 is surjective (on $R(S)$). Then S_1^2 is also surjective on $R(S)$. Since $R(S) \neq 0$, thus $S_1 S_1^+ = I_1 = S_1^2(S_1^2)^+$ (where I_1 is the identity operator on $R(S)$), $S_1 S_1^+$ and $S_1^2(S_1^2)^+$ are nonzero orthogonal projections.
By putting $X_1 = (S_1^2)^+$ in $(*)$, we obtain

$$\|S_1^2(S_1^2)^+\| + \|(S_1^2)^+ S_1 K\| \geq 2\|S_1(S_1^2)^+ K\|.$$

Hence, $\|S_1^2(S_1^2)^+\| = 1, \|(S_1^2)^+ S_1 K\| = \|(S_1^2)^+ S_1^2 S_1^+ K\| \leq \|S_1^+ K\|$, and $\|S_1(S_1^2)^+ K\| \geq \|S_1^+ S_1^2(S_1^2)^+ K\| = \|S_1^+ K\|$. Thus $1 \geq \|S_1^+ K\|$. Hence

$$1 \geq \|S_1^+ K\|^2 = \|S_1^+ K^2 (S_1^+)^*\| = \|S_1^+ S_1 + (S_1^+ S_2)(S_1^+ S_2)^*\| \geq \|S_1^+ S_1\| = 1.$$

Hence $\|S_1^+ S_1 + (S_1^+ S_2)(S_1^+ S_2)^*\| = 1$. Since $S_1^+ S_1$ is an orthogonal projection, then by a simple computation, we obtain that $S_1^+ S_1 S_1^+ S_2 = 0$. Hence $S_2 = S_1 S_1^+ S_1 S_1^+ S_2 = 0$. So, we obtain that S_1 is a surjective operator (as element in $\mathfrak{B}(R(S))$ and satisfies the following inequality

$$\forall X_1 \in \mathfrak{B}(R(S)), \|S_1^2 X_1\| + \|X_1 S_1^2\| \geq 2\|S_1 X S_1\|.$$

Utilizing Lemma 2.4, we obtain that S_1 is normal. Hence S is normal. With the second assumption "T_1 surjective", and since S^* satisfies (v), by using the same argument as used with the first assumption, we obtain also that S^* is normal. Thus S is normal □

The extension of the property $(N3)$ in theorem 2.1 from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{R}(H)$, was found by Menkad [31], as follows

Theorem 2.3. *Let $S \in \mathfrak{R}(H)$. Then the following properties are equivalent:*
(i) $S \in \mathcal{N}_{cr}(H)$,
(ii) $\forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| \leq \|S^*XS^+\| + \|S^+XS^*\|$.

Proof. The proof is trivial if $S = 0$. Assume now that $S \neq 0$.
The implication $(i) \Rightarrow (ii)$ follows immediately from proposition 2.1.
$(ii) \Rightarrow (i)$. If we put $X = x \otimes y$ (where $x,\ y \in H$) in (ii), we obtain

$$\forall x, y \in H,\ \|Sx \otimes (S^+)^* y\| + \|S^+ x \otimes S^* y\| \leq \|S^* x \otimes (S^+)^* y\| + \|S^+ x \otimes Sy\|. \quad (**)$$

Let $y \in kerS^*$ and since $kerS^+ = kerS^*$, from (**), we obtain

$$\forall x \in H,\ \|Sx\|\|(S^+)^* y\| = 0.$$

Since $S \neq 0$, choose x such that $Sx \neq 0$ in this last inequality, we obtain $Sy = 0$. Hence $kerS^* \subset kerS$. The same argument shows that $kerS \subset kerS^*$. Consequently, $kerS^* = kerS$, and so S is an EP operator. Hence $S = S_1 \oplus 0$ with respect to the orthogonal direct sum $H = R(S) \oplus kerS$, and where S_1 is an invertible operator in $\mathfrak{B}(R(S))$. By choosing $X = X_1 \oplus 0$ (where $X_1 \in \mathfrak{B}(R(S))$), in (ii) we have

$$\forall X \in \mathfrak{B}(R(S))_1, \|S_1X_1S_1^{-1}\| + \|S_1^{-1}X_1S_1\| \leq \|S_1^*X_1S_1^{-1}\| + \|S_1^{-1}X_1S_1^*\|.$$

Hence using theorem 2.1, we obtain S_1 is normal. Therefore S is normal. □

Remark 2.2. *From the above theorems, it follows that the class $\mathcal{N}_{cr}(H)$ is characterized by*

$$\begin{aligned}\mathcal{N}_{cr}(H) &= \{S \in \mathfrak{R}(H) : \forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| = \|S^*XS^+\| + \|S^+XS^*\|\} \\ &= \{S \in \mathfrak{R}(H) : \forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| \geq \|S^*XS^+\| + \|S^+XS^*\|\} \\ &= \{S \in \mathfrak{R}(H) : \forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| \leq \|S^*XS^+\| + \|S^+XS^*\|\} \\ &= \{S \in \mathfrak{R}(H) : \forall X \in \mathfrak{B}(H), \|SXS^+\| + \|S^+XS\| \geq 2\|S^+SXS^+S\|\} \\ &= \{S \in \mathfrak{R}(H) : \forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| \geq 2\|SXS\|\}.\end{aligned}$$

2.2 Characterizations of the class of selfadjoint operator multiplied by scalars

In 2001, Seddik [38] could find a characterization of nonzero scalars of invertible self-adjoint operators based on the C.P.R.I. Note that this class of operators is the class of all invertible normal operators in $\mathfrak{B}(H)$ the spectrum of which is included in a straight line passing through the origin. We start by the following lemma which is useful in investigating his theorem.

Lemma 2.5. *[38] Let $\lambda, \mu \in \mathbb{C}^*$ such that $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} \in \mathbb{R}$ and $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \geq 2$. Then there exists a scalar $\theta \in [0, \pi[$ $\lambda, \mu \in D_\theta$.*

Proof. Let $r_1e^{i\theta_1}$ and $r_2e^{i\theta_2}$ be the polar decomposition of λ and μ. Then we have:

$$\frac{\lambda}{\mu} + \frac{\mu}{\lambda} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} + \frac{r_2e^{i\theta_2}}{r_1e^{i\theta_1}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} + \frac{r_2}{r_1}e^{i(\theta_2-\theta_1)}.$$

So, we have

$$\frac{\lambda}{\mu} + \frac{\mu}{\lambda} = (\frac{r_1}{r_2} + \frac{r_2}{r_1})\cos(\theta_1 - \theta_2) + i(\frac{r_1}{r_2} - \frac{r_2}{r_1})\sin(\theta_1 - \theta_2).$$

Since $\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\in\mathbb{R}$, then $(\frac{r_1}{r_2}-\frac{r_2}{r_1})\sin(\theta_1-\theta_2)=0$. Thus $\frac{r_1}{r_2}-\frac{r_2}{r_1}=0$ or $\sin(\theta_1-\theta_2)=0$.
There are two cases.
Case 1: $\frac{r_1}{r_2}-\frac{r_2}{r_1}=0$.
Then $r_1=r_2$, thus

$$2\leq\left|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right|=2\left|cos(\theta_1-\theta_2)\right|\leq 2,$$

which implies that $|cos(\theta_1-\theta_2)|=1$, and so $\theta_1-\theta_2\equiv k_1\pi$, for some $k_1\in\mathbb{Z}$.
Case 2: $\sin(\theta_1-\theta_2)=0$.
Then $\theta_1-\theta_2\equiv k_2\pi$, for some $k_2\in\mathbb{Z}$.
Both cases ensure that there exists some k such that $\theta_1-\theta_2\equiv k\pi$. So we have $\lambda=r_1e^{i\theta_1}=r_1e^{i(\theta_2+k\pi)}=^{+}_{-}r_1e^{i\theta_2}$. Therefore $\lambda,\mu\in D_{\theta_1}$. □

Theorem 2.4. *[38,42] Let $S\in\mathfrak{I}(H)$. Then the following properties are equivalent:*
(i) $S\in\mathbb{C}^*\mathbb{S}_0(H)$.
(ii) $\forall X\in\mathfrak{B}(H),\|SXS^{-1}+S^{-1}XS\|=\|S^*XS^{-1}+S^{-1}XS^*\|$,
(iii) $\forall X\in\mathfrak{B}(H),\|SXS^{-1}+S^{-1}XS\|\geq\|S^*XS^{-1}+S^{-1}XS^*\|$,
(iv) $\forall X\in\mathfrak{B}(H)$, $\|SXS^{-1}+S^{-1}XS\|\geq 2\|X\|$.

Proof. $(i)\Rightarrow(ii)\Rightarrow(iii)$. The implications are trivial.
$(iii)\Rightarrow(iv)$. This implication follows immediately using proposition 1.8(3′).
$(iv)\Rightarrow(i)$. Assume that (iv) holds.
Using theorem 2.1, we find that S is normal. Then, by the spectral measure of S; there exists a sequence (S_n) of invertible normal operators with finite spectrum such that

(a) $S_n\longrightarrow S$ uniformly,

(b) for all $\lambda\in\sigma(S)$, there exists a sequence (λ_n) such that $\lambda_n\in\sigma(S_n)$, for all n and $\lambda_n\longrightarrow\lambda$.

By the same argument as used in lemma 2.3, we find that

$$\forall n>N,\forall X\in\mathfrak{B}(H),\left\|S_nXS_n^{-1}+S_n^{-1}XS_n\right\|\geq(2-\epsilon)\left\|X\right\|. \qquad (*)$$

Let $\lambda,\mu\in\sigma(S)$. For all $n\geq 1$, there exist λ_n, $\mu_n\in\sigma(S_n)$, $\lambda_n\longrightarrow\lambda$, $\mu_n\longrightarrow\mu$.
Let $n>N$ and since S_n is normal with finite spectrum, there exist p orthogonal projections $E_1,...,E_p$ such that $E_kE_j=0$, if $k\neq j$, $\sum_{k=1}^p E_k=I$, and $S_n=\sum_{k=1}^p\alpha_kE_k$ (with respect to the decomposition $H=H_1\oplus H_2\oplus...\oplus H_p$), where $\sigma(S_n)=\{\alpha_1,...,\alpha_p\}$ and $\alpha_1=\lambda_n$, $\alpha_2=\mu_n$.
We may choose an orthonormal set $\{e_1,e_2\}$ such that $e_1\in H_1,e_2\in H_2$.
Decompose orthogonaly H_1,H_2 as follows $H_1=[\{e_1\}]\oplus K_1,H_2=[\{e_2\}]\oplus K_2$.
Put $L=[\{e_1,e_2\}]$. Then $H=L\oplus M$, where $M=K_1\oplus K_2\oplus_{i=3}^p H_i$.
Put $A_n=S_n|L$. Hence $A_n=\begin{bmatrix}\lambda_n & 0\\ 0 & \mu_n\end{bmatrix}\in\mathfrak{B}(L)$.

Let $Y = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in \mathfrak{B}(L)$ (where $x_1, x_2, x_3, x_4 \in \mathbb{C}$).
By taking $X = Y \oplus 0 \in \mathfrak{B}(L \oplus M)$ in $(*)$, we find

$$\left\| A_n Y A_n^{-1} + A_n^{-1} Y A_n \right\| \geq (2-\epsilon) \left\| Y \right\|.$$

Using an elementary matricielle computation, we deduce

$$\forall Y \in \mathfrak{B}(L), \left\| \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 0 \end{bmatrix} \circ Y \right\| \geq (2-\epsilon) \|Y\|, \qquad (**)$$

where $\gamma_n = \dfrac{\lambda_n}{\mu_n} + \dfrac{\mu_n}{\lambda_n}$. So, it follows that

$$\forall Y \in \mathfrak{B}(L), \left\| \begin{bmatrix} \frac{1}{2} & \delta_n \\ \delta_n & \frac{1}{2} \end{bmatrix} \circ Y \right\| \leq \frac{\|Y\|}{2-\epsilon}, \qquad (***)$$

where $\delta_n = \frac{1}{\gamma_n}$.
By taking $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in (**), we obtain

$$|\gamma_n| \geq 2 - \epsilon.$$

Letting $n \to 0$, we have $\left| \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right| \geq 2 - \epsilon$.
Now, letting $\epsilon \to 0$, we deduce $\left| \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \right| \geq 2$.
On the other hand, if we put $Y = \begin{pmatrix} 1 & ia \\ ia & 1 \end{pmatrix}$, such that $a > 0$, in (***), we obtain

$$\begin{aligned} \left\| \begin{pmatrix} \frac{1}{2} & \delta_n \\ \delta_n & \frac{1}{2} \end{pmatrix} \circ \begin{pmatrix} 1 & ia \\ ia & 1 \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} \frac{1}{2} & \delta_n ia \\ \delta_n ia & \frac{1}{2} \end{pmatrix} \right\|^2 \\ &= \frac{1}{4} + a^2 \left|\delta_n\right|^2 + a \left|Im(\delta_n)\right| \\ &\leq \frac{\|X\|^2}{(2-\epsilon)^2} \\ &= \frac{1+a^2}{(2-\epsilon)^2}. \end{aligned}$$

Thus

$$\tfrac{1}{4} + a^2 \left|\delta_n\right|^2 + a \left|Im(\delta_n)\right| \leq \tfrac{1+a^2}{(2-\epsilon)^2}.$$

Letting $n \to +\infty$, then

$$\tfrac{1}{4} + a^2 \left|\delta\right|^2 + a \left|Im(\delta)\right| \leq \tfrac{1+a^2}{(2-\epsilon)^2}.$$

Now, let $\epsilon \longrightarrow 0$, we obtain

$$\tfrac{1}{4} + a^2 |\delta|^2 + a\,|Im(\delta)| \leq \tfrac{1+a^2}{4}.$$

This implies $a\,|\delta|^2 + |Im(\delta)| \leq a$. Letting $a \longrightarrow 0$, then $Im(\delta) = 0$. Hence $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} \in \mathbb{R}$. Since $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} \in \mathbb{R}$ and $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \geq 2$, then using lemma 2.5 there exists a scalar $\theta \in [0, \pi[$ such that $\lambda, \mu \in D_\theta$. Hence $\sigma(S) \subset D_\theta$. If we put $M = e^{-i\theta}S$, then M is an invertible normal operator with real spectrum. So we have $S = e^{i\theta}M$, where M is an invertible selfadjoint operator. Therefore (i) holds. □

Remark 2.3. *It follows from the above theorem that*

$$\begin{aligned}\mathbb{C}^*\mathbb{S}_0(H) &= \{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| = \|S^*XS^{-1} + S^{-1}XS^*\|\} \\ &= \{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \geq \|S^*XS^{-1} + S^{-1}XS^*\|\} \\ &= \{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|\}.\end{aligned}$$

For the class $\mathbb{CS}_{cr}(H)$, Seddik [44] extend the properties $(S1 - S3)$ in the above theorem from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{R}(H)$. So he gets the following theorem that characterizes the class $\mathbb{CS}_{cr}(H)$:

Theorem 2.5 (44)**.** *Let $S \in \mathfrak{R}(H)$. Then the following properties are equivalent:*
(i) $S \in \mathbb{CS}_{cr}(H)$,
(ii) $\forall X \in \mathfrak{B}(H), \|SXS^+ + S^+XS\| = \|S^*XS^+ + S^+XS^*\|$,
(iii) $\forall X \in \mathfrak{B}(H), \|SXS^+ + S^+XS\| \geq \|S^*XS^+ + S^+XS^*\|$,
(iv) $\forall X \in \mathfrak{B}(H), \|SXS^+ + S^+XS\| \geq 2\|SS^+XS^+S\|$,
(v) $\forall X \in \mathfrak{B}(H), \|S^2X + XS^2\| \geq 2\|SXS\|$.

Proof. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial.
The implication $(iii) \Rightarrow (iv)$ follows immediately from proposition 1.8$(2')$.
The implication $(i) \Rightarrow (v)$ follows immediately from A.G.M.I1.
Assume now that (iv) or (v) holds. Applying the triangular inequality in (iv) or (v), we obtain from theorem 2.2, that S is normal (with a closed range).
So that S is an EP operator satisfying (iv) or (v). Then $S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ kerS^* \end{bmatrix}$, where S_1 is invertible operator on $R(S)$. Thus we obtain the following inequality

$$\forall X \in \mathfrak{B}(R(S)), \|S_1X_1S_1^{-1} + S_1^{-1}X_1S_1\| \geq 2\|X_1\|.$$

Hence by theorem 2.4, S_1 is a selfadjoint operator in $\mathfrak{B}(R(S))$ multiplied by a nonzero scalar. Therefore $S \in \mathbb{CS}(H)$. □

2.3 Characterizations of the classes of unitary and unitary reflection operators

We start by the following proposition which gives us some preliminary characterizations of the class of unitary operators in $\mathfrak{B}(H)$.

Proposition 2.2. *[43] Let $S \in \mathfrak{I}(H)$. Then the following properties are equivalent:*
(i) $S \in \mathbb{R}^*\mathfrak{U}(H)$,
(ii) $\|S\|\|S^{-1}\| = 1$,
(iii) $\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| = \|X\|$,
(iv) $\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| \leq \|X\|$,
(v) $\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| \geq \|X\|$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is clear.
The equivalences $(ii) \Leftrightarrow (iii), (ii) \Leftrightarrow (iv)$ and $(ii) \Leftrightarrow (v)$ follow from the fact that $\sup_{\|X\|=1} \|SXS^{-1}\| = \|S\|\|S^{-1}\|$ and $\inf_{\|X\|=1} \|SXS^{-1}\| = \dfrac{1}{\|S\|\|S^{-1}\|}$. □

Lemma 2.6. *[42] Let $S \in \mathfrak{I}(H)$. If $|\langle Sx, x\rangle| \leq 1$ and $|\langle S^{-1}x, x\rangle| \leq 1$ for every unit vector x in H, then S is unitary.*

Lemma 2.7. *Let $S \in \mathfrak{I}(H)$. Then S is normal if and only if S^*S^{-1} is unitary.*

In the two next propositions, Seddik [41] define the injective norm of the operators $S \otimes S^{-1} + S^{-1} \otimes S$ and $S^* \otimes S^{-1} + S^{-1} \otimes S^*$ in the product tensor space $\mathfrak{B}(H) \otimes \mathfrak{B}(H)$.

Proposition 2.3. *Let $S \in \mathfrak{I}(H)$. Then we have*
(i) $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda \geq \sup_{\lambda,\mu \in \sigma(S)} \left|\dfrac{\lambda}{\mu} + \dfrac{\mu}{\lambda}\right|$.
If S is normal, then the two following inequalities hold:
(ii) $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = \sup_{\lambda,\mu \in \sigma(S)} \left|\dfrac{\lambda}{\mu} + \dfrac{\mu}{\lambda}\right|$,
(iii) $\|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda = \sup_{\lambda,\mu \in \sigma(S)} (\left|\dfrac{\lambda}{\mu}\right| + \left|\dfrac{\mu}{\lambda}\right|)$.

Proof. Both (i) and (ii) are immediately from proposition 1.6.
(iii). Let $S \in \mathcal{N}(H)$. From proposition 1.6 and from the fact that $\sigma(S^*) = \overline{\sigma(S)}$, we find that
$$\|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda = \sup_{\lambda,\mu \in \sigma(S)} (\left|\dfrac{\overline{\lambda}}{\mu} + \dfrac{\overline{\mu}}{\lambda}\right|) = \sup_{\lambda,\mu \in \sigma(S)} \dfrac{|\lambda|^2 + |\mu|^2}{|\lambda\mu|} = \sup_{\lambda,\mu \in \sigma(S)} (\left|\dfrac{\lambda}{\mu}\right| + \left|\dfrac{\mu}{\lambda}\right|).$$ □

Proposition 2.4. *The following inequality holds:*

$$\forall S \in \mathfrak{I}(H), \|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda = \|S\|\|S^{-1}\| + \dfrac{1}{\|S\|\|S^{-1}\|}.$$

Proof. Let $S \in \mathfrak{I}(H)$, and let $S = UP$ be the polar decomposition of S. From the fact that $\{X \in \mathfrak{B}_1(H) : rankX = 1\} = \{U^*X : X \in \mathfrak{B}_1(H), rankX = 1\}$ and

$\|S\| = \|P\|, \|S^{-1}\| = \|P^{-1}\|$, it follows that

$$\begin{aligned}\|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda &= \sup_{\|X\|=1=rankX} \|S^*XS^{-1} + S^{-1}XS^*\| \\ &= \sup_{\|X\|=1=rankX} \|PU^*XP^{-1}U^* + P^{-1}U^*XPU^*\| \\ &= \sup_{\|X\|=1=rankX} \|P(U^*X)P^{-1} + P^{-1}(U^*X)P\| \\ &= \sup_{\|X\|=1=rankX} \|PXP^{-1} + P^{-1}XP\| \\ &= \|P \otimes P^{-1} + P^{-1} \otimes P\|_\lambda.\end{aligned}$$

Since $P = |S|$ is an invertible positive operator in $\mathfrak{B}(H)$, then from proposition 2.3. It follows that

$$\begin{aligned}\|P \otimes P^{-1} + P^{-1} \otimes P\|_\lambda &= \sup_{\lambda,\mu\in\sigma(P)} (\left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|) \\ &= \sup_{\lambda,\mu\in\sigma(P)} (\frac{\lambda}{\mu} + \frac{\mu}{\lambda}) \\ &= \sup_{t\in\sigma(\mathcal{M}_P)} (t + \frac{1}{t}).\end{aligned}$$

Since $\sigma(\mathcal{M}_{P,P^{-1}}) = \sigma(P)\sigma(P^{-1})$, then $min\ \sigma(\mathcal{M}_{P,P^{-1}}) = \frac{1}{\|P\|\|P^{-1}\|} = p$ and max $\sigma(\mathcal{M}_{P,P^{-1}}) = \|P\|\|P^{-1}\| = \frac{1}{p}$. So that $\max_{t\in\sigma(\mathcal{M}_{P,P^{-1}})} \{t + \frac{1}{t} : p \leq t \leq \frac{1}{p}\} = p + \frac{1}{p}$.
Hence

$$\begin{aligned}\|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda &= \|P \otimes P^{-1} + P^{-1} \otimes P\|_\lambda \\ &= \|P\|\|P^{-1}\| + \frac{1}{\|P\|\|P^{-1}\|} \\ &= \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}.\end{aligned}$$

□

From proposition 2.3, it follows immediately that if $S \in \mathfrak{I}(H)$, then $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda \geq 2$ and $\|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda \geq 2$. It's easy to see that the two last inequalities become equalities when S is unitary. Seddik [42] characterized the class of all invertible operators in $\mathfrak{B}(H)$ for which the injective norm of $S \otimes S^{-1} + S^{-1} \otimes S$ get's it's minimal value 2.

Proposition 2.5. *Let $S \in \mathfrak{I}(H)$. The following properties are equivalent:*
(i) $\forall X \in \mathcal{F}_1(H), \|SXS^{-1} + S^{-1}XS\| \leq 2\|X\|$,
(ii) $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = 2$,
(iii) S is normal and $\sup_{\lambda,\mu\in\sigma(S)} \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = 2.$

Proof. $(i) \Rightarrow (ii)$. Assume that (i) holds. Then we have

$$2 \leq \|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = \sup_{X \in \mathcal{F}_1(H)} \|SXS^{-1} + S^{-1}XS\| \leq 2,$$

which proves (ii).
$(ii) \Rightarrow (iii)$. Firstly, we should prove that S is normal.
Using proposition 1.5, we obtain

$$\forall x, y \in (H)_1, 2 = \|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda \leq 2|\langle Sx, y\rangle\langle S^{-1}x, y\rangle|.$$

Hence the inequality $|\langle Sx, y\rangle\langle S^{-1}x, y\rangle| \geq \|x\|^2\|y\|^2$ holds for every $x, y \in H$. So we have $|\langle S^*S^{-1}x, x\rangle| \leq 1$ and $|\langle (S^*S^{-1})^{-1}x, x\rangle| \leq 1$ for every x in $(H)_1$. Then from lemma 2.6, it follows that S^*S^{-1} is unitary. Using lemma 2.5, we deduce that S is normal.
Since S is normal, then using proposition 2.3, we find that $\sup_{\lambda,\mu \in \sigma(S)} \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = 2$. Therefore (iii) holds.
$(iii) \Rightarrow (i)$. Using the fact that $\sigma(S) = \{\varphi(S) : \varphi \in \Gamma_S\}$. Then from (iii) it follows that

$$\sup_{\varphi,\psi \in \Gamma(S)} \left|\frac{\varphi(S)}{\psi(S)} + \frac{\psi(S)}{\varphi(S)}\right| = 2.$$

Since $\lambda S + \mu S^{-1}$ is normal for every complex numbers λ and μ, the following inequality holds

$$\forall \varphi \in \Gamma_S, \sup_{\psi \in \Gamma(S)} |\psi(\varphi(S)S^{-1} + \varphi(S^{-1})S)| = \|\varphi(S)S^{-1} + \varphi(S^{-1})S\| \leq 2.$$

Thus

$$\forall \varphi \in \Gamma_S, \forall f \in \mathbf{S}_1, |\varphi(f(S^{-1})S + f(S)S^{-1})| \leq 2.$$

Hence

$$\forall f \in \mathbf{S}_1, \|f(S^{-1})S + f(S)S^{-1}\| \leq 2$$

For every $x, y \in (H)_1$, define the functional $f_{x,y}$ on $\mathfrak{B}(H)$ by $f_{x,y}(T) = \langle Tx, y\rangle$. It is easy to see that $f_{x,y} \in \mathbf{S}_1$ for every $x, y \in (H)_1$.
So from the last inequality, it follows that

$$\forall u, v \in (H)_1, \|\langle S^{-1}u, y\rangle S + \langle Su, y\rangle S^{-1}\| \leq 2.$$

Thus

$$\forall u, v, x \in (H)_1, \|S(x \otimes y)S^{-1}u + S^{-1}(x \otimes y)Su\| \leq 2.$$

So, we obtain

$$\forall x, y \in (H)_1, \|S(x \otimes y)S^{-1} + S^{-1}(x \otimes y)S\| \leq 2.$$

Therefore (i) follows immediately. □

The above proposition showed that the set of all invertible operators satisfies the equality $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = 2$, does not charactrerizes the class of all unitary operators in $\mathfrak{B}(H)$; it's characterizes the class of all invertible normal operators satisfying the condition $\sup_{\lambda,\mu\in\sigma(S)} \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = 2$. Using this result Seddik [42] could find some operator inequalities that characterized the class $\mathbb{R}^*\mathfrak{U}(H)$ in the following theorem:

Theorem 2.6. *Let $S \in \mathfrak{I}(H)$. The following properties are equivalent:*
(i) $\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| = 2\|X\|$,
(ii) $\forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \leq 2\|X\|$,
(iii) $\forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \leq 2\|X\|$,
(iv) $S \in \mathbb{R}^*\mathfrak{U}(H)$.

Proof. The two implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial.
$(iii) \Rightarrow (iv)$. From (iii), it follows that $\|S \otimes S^{-1} + S^{-1} \otimes S\|_\lambda = 2$. Using proposition 2.5, we obtain that S is normal and $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \leq 2$ for every $\lambda, \mu \in \sigma(S)$.
Using the spectral measure of S, there exists a sequence (S_n) of invertible normal operators with finite spectrum such that $S_n \to S$ uniformly, and for every $\lambda \in \sigma(S)$, there exists a sequence (λ_n) such that $\lambda_n \in \sigma(S_n)$ for all n and $\lambda_n \to \lambda$.
Let $\lambda, \mu \in \sigma(S)$ and let $(\lambda_n), (\mu_n) \in \sigma(S_n)$ such that $\lambda_n \to \lambda, \mu_n \to \mu$. Let $\varepsilon > 0$. Then from (iii) and from $S_n \to S$, there exists an integer N such that

$$\forall n > N, \forall X \in \mathfrak{B}(H), \|S_n X S_n^{-1} + S_n^{-1} X S_n\| \leq (2+\epsilon)\|X\|. \qquad (*)$$

Let $n > N$. Since S_n is normal with finite spectrum, there exist p orthogonal projections$E_1, ..., E_p$ such that $E_k E_j = 0$ if $k \neq j, E_1 \oplus ...E_p = I$ and $S_n = \sum_{j=1}^p \alpha_j E_j$, where $\sigma(S_n) = \{\alpha_1, ..., \alpha_p\}$ and $\alpha_1 = \lambda_n, \alpha_2 = \mu_n$. Then using (*) by the same argument as used in theorem 2.6, we obtain

$$\forall X \in \mathfrak{B}(\mathbb{C}^2), \|C \circ X\| \leq (2+\epsilon)\|X\|, \qquad (**)$$

where $C = \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 2 \end{bmatrix}$, and $\gamma_n = \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}$.
Let $X = \begin{bmatrix} tIm\gamma_n & i \\ i & tIm\gamma_n \end{bmatrix}$ (where t is an arbitrary positive number), we obtain that

$$C \circ X = \begin{bmatrix} 2tIm\gamma_n & i\gamma_n \\ i\gamma_n & 2tIm\gamma_n \end{bmatrix}.$$

Thus

$$\|X\|^2 = \|XX^*\| = (tIm\gamma_n)^2 + 1, \|C \circ X\|^2 = \|C \circ X (C \circ X)^*\| = (2tIm\gamma_n)^2 + |\gamma_n|^2 + 4t(Im\gamma_n)^2.$$

Hence the inequality (**) becomes

$$(2tIm\gamma_n)^2 + |\gamma_n|^2 + 4t(Im\gamma_n)^2 \leq (2+\epsilon)^2((tIm\gamma_n)^2 + 1)$$

$$(2tIm\gamma_n)^2 + |\gamma_n|^2 + 4t(Im\gamma_n)^2 \leq 4((tIm\gamma_n)^2 + 1) + (4\epsilon + \epsilon^2)$$
$$(2tIm\gamma_n)^2 + |\gamma_n|^2 + 4t(Im\gamma_n)^2 \leq (2tIm\gamma_n)^2 + 4 + (4\epsilon + \epsilon^2)(t^2(Im\gamma_n)^2 + 1)$$
$$|\gamma_n|^2 + 4t(Im\gamma_n)^2 \leq 4 + (4\epsilon + \epsilon^2)(t^2(Im\gamma_n)^2 + 1).$$

Put $\gamma = \frac{\lambda}{\mu} + \frac{\mu}{\lambda}$. Letting $n \to \infty$, it follows that

$$|\gamma|^2 + 4t(Im\gamma)^2 \leq 4 + (4\epsilon + \epsilon^2)(t^2(Im\gamma)^2 + 1).$$

Letting $\epsilon \to 0$, we obtain that the inequality $4t(Im\gamma)^2 \leq 4 - |\gamma|^2$ holds for every real $t > 0$. So the inequality $4(Im\gamma)^2 \leq \frac{4 - |\gamma|^2}{t}$ holds. Letting $t \to \infty$, we obtain that $Im\gamma = 0$. Since $|\gamma| \leq 2$, thus by a simple computation, we find that $|\lambda| = |\mu|$. So that $\sigma(S)$ is included in some circle centred at the origin with radius $\|S\|$ (since S is normal). Thus $\sigma(\frac{1}{\|S\|}S)$ is included in the unit circle, and since S is normal, we obtain that $\frac{1}{\|S\|}S$ is unitary. Therefore (iv) follows immediately.
$(iv) \Rightarrow (i)$. The implication is trivial. □

In the following theorem, Seddik [41] could find three other characterizations of the class $\mathbb{R}^*\mathfrak{U}(H)$; where one of them is the set of all operators $S \in \mathfrak{I}(H)$ for which the injective norm of $S^* \otimes S^{-1} + S^{-1} \otimes S^*$ attains its minimal value 2.

Theorem 2.7. *Let $S \in \mathfrak{I}(H)$. The following properties are equivalent:*
(i) $S \in \mathbb{R}^*\mathfrak{U}(H)$,
(ii) $\forall X \in \mathfrak{B}(H), \|S^*XS^{-1} + S^{-1}XS^*\| \leq 2\|X\|$,
(iii) $\forall X \in \mathfrak{B}(H), \|S^*XS^{-1} + S^{-1}XS^*\| = 2\|X\|$,
(iv) $\|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda = 2$.

Proof. $(i) \Rightarrow (ii)$. The implication is trivial.
$(ii) \Rightarrow (iii)$. Assume that (ii) holds. Using proposition 1.8(2′), then (iii) follows immediately.
$(iii) \Rightarrow (iv)$. The implication is trivial.
$(iv) \Rightarrow (i)$. Using proposition 2.4, it follows that

$$2 = \|S^* \otimes S^{-1} + S^{-1} \otimes S^*\|_\lambda = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}.$$

Hence $\|S\|\|S^{-1}\| = 1$. Using proposition 2.2, then (i) follows immediately. □

Remark 2.4. *From the two above theorems, it follows that the class $\mathbb{R}^*\mathfrak{U}(H)$ is given by*

$$\begin{aligned}
\mathfrak{U}(H) &= \{S \in \mathfrak{I}_1(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| = 2\|X\|\} \\
&= \{S \in \mathfrak{I}_1(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1}\| + \|S^{-1}XS\| \leq 2\|X\|\} \\
&= \{S \in \mathfrak{I}_1(H) : \forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \leq 2\|X\|\} \\
&= \{S \in \mathfrak{I}_1(H) : \forall X \in \mathfrak{B}(H), \|S^*XS^{-1} + S^{-1}XS^*\| \leq 2\|X\|\} \\
&= \{S \in \mathfrak{I}_1(H) : \forall X \in \mathfrak{B}(H), \|S^*XS^{-1} + S^{-1}XS^*\| = 2\|X\|\}.
\end{aligned}$$

From theorems 2.4 and 2.6, Seddik [42] deduced the following corollary that gives a complete characterization of the class $\mathbb{C}^*\mathfrak{U}_r(H)$ of all unitary reflection operators in $\mathfrak{B}(H)$ multiplied with a nonzero complex.

Corollary 2.1. *Let $S \in \mathfrak{B}(H)$. Then the equality $\|SXS^{-1} + S^{-1}XS\| = 2\|X\|$, holds for all X in $\mathfrak{B}(H)$ if and only if λS is a unitary reflection for some nonzero complex number λ.*

2.4 Characterizations of the class of partial isometries

Definition 2.1. *An operator $S \in \mathfrak{B}(H)$ is said to be a partial isometry if and only if $\|Sx\| = \|x\|$, for all $x \in (kerS)^{\perp}$.*

Proposition 2.6. *Let $S \in \mathfrak{B}(H)$. Then the following statements are mutually equivalent:*
(i) S is a partial isometry,
(ii) S^* is a partial isometry,
(iii) $SS^*S = S$,
(iv) $S^*SS^* = S^*$,
(v) SS^* is an orthogonal projection,
(vi) S^*S is an orthogonal projection.

By the definition of Moore-Penrose inverse and the above proposition, we find that $S \in \mathfrak{R}(H)$ is a partial isometry if and only if $S^+ = S^*$.

In 2004, Mbekhta [28] has given the following characterization of partial isometries in Hilbert spaces.

Theorem 2.8. *Let $S \in \mathfrak{R}(H)$. Then S is a nonzero partial isometry if and only if $\|S\| = \|S^+\| = 1$.*

Proof. $\Rightarrow$ The implication is trivial.
Conversely, Assume that $\|S\| = \|S^+\| = 1$. Now, let $x \in H$, then $\|S^+x\| = \|S^+SS^+x\| \leq \|SS^+x\| \leq \|S^+x\|$. Therefore, for every $x \in H$, we have

$$\|S^+x\|^2 = \|SS^+x\|^2 \Rightarrow \langle (I - S^*S)S^+x, S^+x\rangle = 0.$$

Since $\|S\| = 1$, then the operator $(I - S^*S))$ is positive. Let $(I - S^*S)^{\frac{1}{2}}$ be its positive square root, then, for every $x \in H, \|(I - S^*S)^{\frac{1}{2}}S^+x\|^2 = \langle (I - S^*S)S^+x, S^+x\rangle = 0$. Consequently, $(I - S^*S)^{\frac{1}{2}}S^+ = 0$. Thus $(I - S^*S)S^+ = 0$. Hence $S^+ = S^*SS^+$. Finally we obtain that $S = SS^+S = SS^*SS^+S = SS^*S$, which proves that S is a partial isometry. □

Proposition 2.7. *Let Let $S \in \mathfrak{R}(H)$ be a nonzero operator. Then $\|S\|\|S^+\| = 1$ if and only if $\dfrac{S}{\|S\|}$ is a partial isometry.*

Proof. Let $T = \dfrac{S}{\|S\|}$, it's obvious that $\|T\| = 1$. From the other side we have $\|T^+\| = \|S\|\|S^+\|$. Using theorem 2.8, we find that $\dfrac{S}{\|S\|}$ is a partial isometry if and only if $\|S\|\|S^+\| = 1$. □

Based on the theorem of Mbekhta, Menkad [30] could find a complete characterization of the class of partial isometries in terms of operator inequalities. The following theorem is an extension of theorem 2.7 from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{R}(H)$.

Theorem 2.9. *Let $S \in \mathfrak{R}(H)$ be a nonzero operator. Then the following properties are equivalent:*
(i) $\dfrac{S}{\|S\|}$ *is a partial isometry,*
(ii) $\forall X \in \mathfrak{B}(H), \|S^*XS^+\| + \|S^+XS^*\| \leq 2\|SS^+XS^+S\|$,
(iii) $\forall X \in \mathfrak{B}(H), \|S^*XS^+ + S^+XS^*\| = 2\|SS^+XS^+S\|$,
(iv) $\|S^* \otimes S^+ + S^+ \otimes S^*\|_\lambda = 2$.

Proof. Without loss of generality, we can take $\|S\| = 1$.
$(i) \Rightarrow (ii)$. Since S is partial isometry, then $S^* = S^+$. Therefore

$$\begin{aligned}\forall X \in \mathfrak{B}(H), \|S^*XS^+\| + \|S^+XS^*\| =& 2\|S^*XS^*\| = 2\|S^*SS^*XS^*SS^*\| \\ &\leq 2\|SS^*XS^*S\| = \|SS^+XS^+S\|.\end{aligned}$$

Thus

$$\forall X \in \mathfrak{B}(H), \|S^*XS^+\| + \|S^+XS^*\| \leq 2\|SS^+XS^+S\|.$$

$(ii) \Rightarrow (iii)$. Assume that (ii) holds. Then

$$\forall X \in \mathfrak{B}(H), \|S^*XS^+ + S^+XS^*\| \leq 2\|SS^+XS^+S\|.$$

Using the above inequality and the inequality $(2')$ from proposition 1.8, we obtain the equality (iii).
$(iii) \Rightarrow (iv)$. From (iii) and proposition 1.5, it follows that

$$\begin{aligned}\|S^* \otimes S^+ + S^+ \otimes S^*\|_\lambda &= \sup_{\|X\|=1=rankX} \|S^*XS^+ + S^+XS^*\| \\ &= \sup_{\|X\|=1=rankX} 2\|SS^+XS^+S\| = 2\|SS^+\|\|S^+S\| = 2.\end{aligned}$$

$(iv) \Rightarrow (i)$. Assume that (iv) holds. Then we obtain

$$\forall X \in \mathcal{F}_1(H), \|S^*XS^+ + S^+XS^*\| \leq 2\|X\|.$$

If we replace X by SS^+XS^+S, the above inequality becomes

$$\forall X \in \mathcal{F}_1(H), \|S^*XS^+ + S^+XS^*\| \leq 2\|SS^+XS^+S\|.$$

Let $S = UP$ be the polar decomposition of S. Then the above inequality becomes

$$\forall X \in \mathcal{F}_1(H), \|PU^*XP^+U^* + P^+U^*XPU^*\| \leq 2\|PP^+U^*XP^+P\|.$$

Thus

$$\forall X \in \mathcal{F}_1(H), \|PU^*XP^+ + P^+U^*XP\| \leq 2\|PP^+U^*XP^+P\|.$$

By replacing again X by UX, we get

$$\forall X \in \mathcal{F}_1(H), \|PXP^+ + P^+XP\| \leq 2\|PP^+XP^+P\|.$$

Thus

$$\|P \otimes P^+ + P^+ \otimes P\|_\lambda \leq 2.$$

P is selfadjoint operator with a closed range, then P is an EP operator has the following matrix representation $P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$ (where P_1 invertible) with respect to the orthogonal direct sum $H = R(P) \oplus kerP$. Choose $X = X_1 \oplus 0$ (where $X_1 \in \mathfrak{B}(R(P))$ and $\|X_1\| = rankX_1 = 1$), then the above inequality becomes

$$\|P_1 \otimes P_1^{-1} + P_1^{-1} \otimes P_1\|_\lambda \leq 2.$$

Using proposition 2.4, we obtain that

$$\|P_1\|\|P_1^{-1}\| + \frac{1}{\|P_1\|\|P_1^{-1}\|} \leq 2.$$

Hence $\|P_1\|\|P_1^{-1}\| = 1 = \|P\|\|P^+\| = \|S\|\|S^+\|$. So (i) follows immediately from Proposition 2.7. □

3. NEW CHARACTERIZATIONS OF SOME SUBCLASSES OF OPEARTORS

This chapter cosists of our contribution in the characterizations of some distinguished subclasses of operators in terms of operator inequalities.
We start by giving some new forms of operator inequalities as follows:

$$\forall X \in \mathfrak{B}(H),\ \|S^*SX + XSS^*\| \geq \|S^*XS\| + \|SXS^*\|, \quad (S \in \mathfrak{I}(H)), \tag{N10}$$
$$\forall X \in \mathfrak{B}(H),\ \|S^2X\| + \|XS^2\| = \|S^*SX\| + \|XSS^*\|, \quad (S \in \mathfrak{R}(H)), \tag{N11}$$
$$\forall X \in \mathfrak{B}(H),\ \|S^2X\| + \|XS^2\| \geq \|S^*SX\| + \|XSS^*\|, \quad (S \in \mathfrak{R}(H)), \tag{N12}$$
$$\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| \leq 2\|SXS\|, \quad (S \in \mathfrak{R}(H)), \tag{N13}$$
$$\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| = 2\|SXS\|, \quad (S \in \mathfrak{B}(H)), \tag{S8}$$
$$\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| \geq 2\|SXS\|, \quad (S \in \mathfrak{B}(H)). \tag{S9}$$

Also, we shall interest to the following extensions of the three properties (N1-N3) from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{I}_1(H)$:

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{\#}\right\| + \left\|S^{\#}XS\right\| = \left\|S^*XS^{\#}\right\| + \left\|S^{\#}XS^*\right\|, \quad (S \in \mathfrak{I}_1(H)), \tag{N14}$$
$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{\#}\right\| + \left\|S^{\#}XS\right\| \leq \left\|S^*XS^{\#}\right\| + \left\|S^{\#}XS^*\right\|, \quad (S \in \mathfrak{I}_1(H)), \tag{N15}$$
$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{\#}\right\| + \left\|S^{\#}XS\right\| \geq \left\|S^*XS^{\#}\right\| + \left\|S^{\#}XS^*\right\|, \quad (S \in \mathfrak{I}_1(H)). \tag{N16}$$

Finally, we give characterizations of subclasses of nonnormal operators, precisely the subclass $\mathcal{V}(H)$ of all isometry operators in $\mathfrak{B}(H)$ and subclasses of partial isometry operators in $\mathfrak{B}(H)$.
Note that this work was published in [7].

3.1 *Subclasses of normal operators*

In this section, we prove that the property $(N10)$ characterizes the class $\mathcal{N}_0(H)$, the properties $(N11-N13)$ characterize the class $\mathcal{N}_{cr}(H)$, and the two properties $(S8)$, $(S9)$ characterize the class $\mathbb{CS}(H)$.
We also prove that the properties $(N14-N16)$ characterize the class $\mathcal{N}_{cr}(H)$.
At the end of this section, we deduce some characterizations of the class $\mathfrak{U}(H)$.

Proposition 3.1. *Let $S \in \mathfrak{I}(H)$. Then the following properties are equivalent:*
(*i*) $S \in \mathcal{N}_0(H)$,
(*ii*) $\forall X \in \mathfrak{B}(H),\ \|S^*SX + XSS^*\| \ \geq\ \|S^*XS\| + \|SXS^*\|$.

Proof. $(i) \Rightarrow (ii)$. This implication follows immediately using McIntosh inequality and proposition 2.1.
$(ii) \Rightarrow (i)$. Assume (ii) holds.
Applying (ii) for $X = S^{-2}$, we obtain

$$\|S^*S^{-1} + S^{-1}S^*\| \geq \|S^*S^{-1}\| + \|S^{-1}S^*\|.$$

Hence

$$\|S^*S^{-1} + S^{-1}S^*\| = \|S^*S^{-1}\| + \|S^{-1}S^*\|.$$

Using [3, Theorem 2], we obtain

$$1 = w((S^*S^{-1})(S^{-1}S^*)^*) = \|S^*S^{-1}\|\|S^{-1}S^*\| = \|S^*S^{-1}\|\|(S^*S^{-1})^{-1}\|.$$

So, From lemma 2.6.(i), $S^*S^{-1} = \|S^*S^{-1}\|U$, for some unitary operator U in $\mathfrak{B}(H)$. Hence $\|S^*S^{-1}\| = 1$. Then S^*S^{-1} is unitary. Using lemma 2.6(ii), we obtain that S is normal. □

From the above proposition, it's natural to ask the following problem:

Problem 3.1. *Is it true that the class of normal operators is characterized by*

$$\forall X \in \mathfrak{B}(H), \|S^*SX + XSS^*\| \geq \|S^*XS\| + \|SXS^*\|, (S \in \mathfrak{B}(H))?$$

We shall extend the properties $(N1-N3)$ in proposition 2.2 and proposition 2.3 from the domain $\mathfrak{I}(H)$ to the domain $\mathfrak{I}_1(H)$ (where the usual inverse is replaced by the group inverse).

Proposition 3.2. *Let $S \in \mathfrak{I}_1(H)$. Then the following properties are equivalent:*

(i) $S \in \mathcal{N}_{cr}(H)$,

(ii) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^\#\right\| + \left\|S^\#XS\right\| = \left\|S^*XS^\#\right\| + \left\|S^\#XS^*\right\|,$

(iii) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^\#\right\| + \left\|S^\#XS\right\| \leq \left\|S^*XS^\#\right\| + \left\|S^\#XS^*\right\|,$

(iv) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^\#\right\| + \left\|S^\#XS\right\| \geq \left\|S^*XS^\#\right\| + \left\|S^\#XS^*\right\|.$

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds. Since $S^\# = S^+$, then (ii) holds using theorem 2.4.

$(ii) \Rightarrow (iii)$. The implication is trivial.

$(iii) \Rightarrow (i)$. Assume (iii) holds.

Let $X = S(S^+)^2S - S^+S^2S^+$. Since $S^*SS^+ = S^+SS^* = S^*$ and $SS^+S^\# = S^\#S^+S = S^\#$, we obtain that

$$\begin{aligned}
S^*XS^\# &= S^*S(S^+)^2SS^\# - S^*S^+S^2S^+S^\# = S^*S^+SS^\# - S^*S^+SS^\# = 0,\\
S^\#XS^* &= S^\#S(S^+)^2SS^* - S^\#S^+S^2S^+S^* = S^\#SS^+S^* - S^\#SS^+S^* = 0,\\
S^\#XS &= S^\#S(S^+)^2SS - S^\#S^+S^2S^+S = S^\#S(S^+)^2S^2 - S^\#S,\\
SXS^\# &= SS(S^+)^2SS^\# - SS^+S^2S^+S^\# = S^2(S^+)^2SS^\# - SS^\#.
\end{aligned}$$

Applying (iii) for $X = S(S^+)^2S - S^+S^2S^+$, we get that

$$\left\|S^2(S^+)^2SS^\# - SS^\#\right\| + \left\|S^\#S(S^+)^2S^2 - S^\#S\right\| = 0.$$

Then $\left\|S^\#S(S^+)^2S^2 - S^\#S\right\| = 0$, so $S^\#SS^{+2}S^2 - S^\#S = 0$. Using the matrices representation with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$ of the operators S, S^+, and $S^\#$, we obtain that

$$\begin{aligned} S^\#SS^{+2}S^2 &= \begin{bmatrix} S_1^*K^{-1}S_1^*K^{-1}S_1^2 + S_1^{-1}S_2S_2^*K^{-1}S_1^*K^{-1}S_1^2 & * \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I_1 & S_1^{-1}S_2 \\ 0 & 0 \end{bmatrix} \\ &= S^\#S, \end{aligned}$$

(where I_1 is the identity on $R(S)$).
Hence

$$S_1^*K^{-1}S_1^*K^{-1}S_1^2 + S_1^{-1}S_2S_2^*K^{-1}S_1^*K^{-1}S_1^2 = I_1.$$

It follows that

$$S_1^{-1}(S_1S_1^* + S_2S_2^*)K^{-1}S_1^*K^{-1}S_1^2 = I_1.$$

Thus $S_1^{-1}S_1^*K^{-1}S_1^2 = I_1$, which implies that $S_1S_1^*K^{-1} = I_1$.
So $K = S_1S_1^*$. Consequently $S_2 = 0$. Since S_1 is invertible , then S is an EP operator.

Now applying again (iii) for $X = X_1 \oplus 0$ (where $X_1 \in \mathfrak{B}(R(S))$), we obtain

$$\forall X_1 \in \mathfrak{B}(R(S)), \left\|S_1X_1S_1^{-1}\right\| + \left\|S_1^{-1}X_1S_1\right\| \leq \left\|S_1^*X_1S_1^{-1}\right\| + \left\|S_1^{-1}X_1S_1\right\|.$$

Using proposition 2.4 with the Hilbert space $R(S)$, we find that S_1 is normal. Hence S is normal.

$(i) \Rightarrow (iv)$. This implication is trivial.

$(iv) \Rightarrow (i)$. Assume (iv) holds. If we put $X = x \otimes y$ (where $x, y \in H$) in (iv), then we obtain

$$\forall x, y \in H, \|Sx\| \left\|(S^\#)^*y\right\| + \left\|S^\#x\right\| \|S^*y\| \geq \|S^*x\| \left\|(S^\#)^*y\right\| + \left\|S^\#x\right\| \|Sy\|.$$

From this last inequality, it follows that $\ker S = \ker S^*$. Then S is an EP operator. Hence $S = S_1 \oplus 0$ with respect to the orthogonal direct sum $H = R(S) \oplus \ker S$, and where S_1 is an invertible operator in $\mathfrak{B}(R(S))$. Applying (iv) for $X = X_1 \oplus 0$ (where $X_1 \in \mathfrak{B}(R(S))$), we obtain the following inequality

$$\forall X_1 \in \mathfrak{B}(R(S)),\ \left\|S_1XS_1^{-1}\right\| + \left\|S_1^{-1}XS_1\right\| \geq \left\|S^*XS_1^{-1}\right\| + \left\|S_1^{-1}XS^*\right\|.$$

Then from this last inequality, and using proposition 2.3 with the Hilbert space $R(S)$, we obtain that S_1 is normal. Therefore S is normal. $\square$

In the two next propositions, we shall give another characterizations of the class $\mathcal{N}_{cr}(H)$ of all normal operators with closed ranges in $\mathfrak{B}(H)$.

Proposition 3.3. *Let $S \in \mathfrak{R}(H)$. Then the following properties are equivalent:*
(i) $S \in \mathcal{N}_{cr}(H)$,
(ii) $\forall x \in H, \|S^2x\| = \|S^*Sx\|; \|(S^*)^2x\| = \|SS^*x\|$,
(iii) $\forall x \in H, \|S^2x\| \geq \|S^*Sx\|; \|(S^*)^2x\| \geq \|SS^*x\|$,
(iv) $\forall X \in \mathfrak{B}(H), \mathfrak{B}(H), \|S^2X\| + \|XS^2\| = \|S^*SX\| + \|XSS^*\|$,
(v) $\forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| \geq \|S^*SX\| + \|XSS^*\|$,
(vi) $S^*D_SS = 0 = SD_SS^*$,
(vii) $S^*D_SS \leq 0 \leq SD_SS^*$.

Proof. $(i) \Rightarrow (ii)$. The implication follows immediately from proposition 2.1.
$(ii) \Rightarrow (iv)$. Asume (ii) holds. Let $X \in \mathfrak{B}(H), x \in (H)_1$, Using (ii), we obtain that

$$\forall X \in \mathfrak{B}(H), \forall x \in (H)_1, \|S^2Xx\| = \|S^*SXx\|; \|(S^*)^2Xx\| = \|SS^*Xx\|,$$

by taking the supremum over $x \in (H)_1$, we find

$$\forall X \in \mathfrak{B}(H), \|S^2X\| = \|S^*SX\|; \|(S^*)^2X\| = \|SS^*X\|,$$

Thus

$$\forall X \in \mathfrak{B}(H), \|S^2X\| = \|S^*SX\|; \|XS^2\| = \|XSS^*\|,$$

Therefore (iv) holds.
$(iv) \Rightarrow (v)$. This implication is trivial.
$(v) \Rightarrow (i)$. Using the A.G.M.I in (v) and using theorem 2.4, the implication follows immediately.
The implication $(ii) \Rightarrow (iii)$ is trivial.
Hence the properties (i), (ii), (iv), (v), and (iii) are equivalent.
$(ii) \Leftrightarrow (vi)$. The proof follows immediately from the two following equivalences:

$$\begin{cases} (S^*D_SS = 0) \Leftrightarrow (\forall x \in (H), \|S^2x\| = \|S^*Sx\|). \\ (SD_SS^* = 0) \Leftrightarrow (\forall x \in (H), \|(S^*)^2x\| = \|SS^*x\|). \end{cases}$$

$(iii) \Leftrightarrow (vii)$. By the same argument as used above.
Therefore the properties $(i - vii)$ are equivalent. □

Proposition 3.4. *Let $S \in \mathfrak{R}(H)$. Then the two following properties are equivalent:*
(i) $S \in \mathcal{N}_{cr}(H)$,
(ii) $\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| \leq 2\,\|SXS\|$.

Proof. The proof is trivial if $S = 0$. Assume now that $S \neq 0$.

$(i) \Rightarrow (ii)$. The implication follows immediately using the triangular inequality and the proposition 2.1.

$(ii) \Rightarrow (i)$. Assume (ii) holds. We shall prove (i) in two steps.

Step 1. Suppose that S is invertible.

Then applying (ii) for $X = S^{-1}(S^{-1})^*$, we obtain

$$\begin{aligned} 1 + \left\|S^*S^{-1}\right\|^2 &= \left\|S^*S^{-1}(S^*S^{-1})^* + I\right\| \\ &\leq 2\left\|S^*S^{-1}\right\|. \end{aligned}$$

Hence $\|S^*S^{-1}\| = 1$. On the other hand, we remark that (ii) holds if we replace S by S^*. It follows that, $\|S^*S^{-1}\| = 1 = \|(S^*S^{-1})^{-1}\|$. So, we have S^*S^{-1} is unitary. Thus S is normal.

Step 2. General case $S \in \mathfrak{R}(H)$.

If we put $X = x \otimes y$ (where $x, y \in H$) in (ii), we obtain

$$\forall x, y \in H, \|S^*x \otimes S^*y + Sx \otimes Sy\| \leq 2\|Sx\|\|S^*y\|.$$

From this last inequality, it follows immediately that $\ker S^* = \ker S$. Then S is an EP operator. Hence $S = S_1 \oplus 0$ with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$, and where S_1 is invertible. Applying (ii) for $X = X_1 \oplus 0$ (where $X_1 \in \mathfrak{B}(R(S))$, we obtain

$$\forall X_1 \in \mathfrak{B}(R(S)), \|S_1^*X_1S_1 + S_1X_1S_1^*\| \leq 2\|S_1X_1S_1\|.$$

From step 1, S_1 normal. Hence S is normal. □

As an immediate consequence of the previous proposition we have

Corollary 3.1. *The following property characterizes the class $\mathcal{N}_0(H)$*

$$\forall X \in \mathfrak{B}(H), \ \left\|S^*S^{-1}X + XS^{-1}S^*\right\| \leq 2\|X\|, (S \in \mathfrak{I}(H)).$$

Remark 3.1. *Let $S \in \mathfrak{B}(H)$. The three following properties characterize the class $\mathcal{N}(H)$ of all normal operators in $\mathfrak{B}(H)$*

$$\forall X \in \mathfrak{B}(H), \|S^*XS\| + \|SXS^*\| = 2\|SXS\| \qquad (*)$$

$$\forall X \in \mathfrak{B}(H), \|S^*XS\| + \|SXS^*\| \geq 2\|SXS\| \qquad (**)$$

$$\forall X \in \mathfrak{B}(H), \|S^*XS\| + \|SXS^*\| \leq 2\|SXS\| \qquad (***)$$

It's easy to see that every normal operator satisfies the three above properties. Conversely , for (*) and (**) it's enough to replace X by $x \otimes y$ (for $x, y \in H$), then take the supremum for $\|y\| = 1$. For the last property we replace replace X by $x \otimes x$ (for $x \in H$).

Proposition 3.5. *Let $S \in \mathfrak{B}(H)$. Then the following properties are equivalent:*
$(i) S \in \mathcal{N}(H)$,
$(ii) \forall X \in \mathfrak{B}(H), \ \sqrt{\|S^2X\|\|XS^2\|} \geq \|SXS\|$.

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds. Then for $X \in \mathfrak{B}(H)$, we have

$$\begin{aligned}
\sqrt{\|S^2X\|\,\|XS^2\|} &= \sqrt{\|S^*SX\|\,\|XSS^*\|} \\
&\geq \sqrt{\|SS^*X^*S^*SX\|} \\
&\geq \sqrt{r(SS^*X^*S^*SX)} \\
&= \sqrt{r\left(S^*X^*S^*SXS\right)} \\
&= \sqrt{r\left((SXS)^*SXS\right)} \\
&= \|SXS\|
\end{aligned}$$

Therefore (ii) holds.
$(ii) \Rightarrow (i)$. Assume (ii) holds.
By taking $X = x \otimes y$ in (ii), where $x, y \in (H)_1$, we find the following

$$\forall x, y \in (H)_1,\ \left\|S^2x\right\|\left\|S^{*2}y\right\| \geq \|Sx\|^2\,\|S^*y\|^2 .$$

Then

$$\forall x, y \in (H)_1,\ \|S\|^2\left\|S^{*2}y\right\| \geq \left\|S^2x\right\|\left\|S^{*2}y\right\| \geq \|Sx\|^2\,\|S^*y\|^2 .$$

Thus

$$\forall x, y \in (H)_1,\ \|S\|^2\left\|S^{*2}y\right\| \geq \sup_{\|x\|=1}\left(\left\|S^2x\right\|\left\|S^{*2}y\right\|\right) \geq \|S\|^2\,\|S^*y\|^2 .$$

Hence $\|S^{*2}y\| \geq \|S^*y\|^2$, for every $y \in (H)_1$. Then S^* is paranormal.
By the same argument, we obtain that S is also paranormal.
Using [47], we deduce that S is normal. □

Remark 3.2. *In [44], Seddik proved that the class $\mathcal{N}_{cr}(H)$ is characterized by the following inequality*

$$\forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| \geq 2\|SXS\|, (S \in \mathfrak{R}(H)),$$

So it is natural to ask: does the following property characterizes the class $\mathcal{N}(H)$

$$\forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| \geq 2\|SXS\|, (S \in \mathfrak{B}(H))? \qquad \text{(PN4)}$$

The above question is still unsolved, but we may remark that the following property

$$\forall X \in \mathfrak{B}(H), \sqrt{\|S^2X\|\|XS^2\|} \geq \|SXS\|, (S \in \mathfrak{B}(H)), \qquad \text{(PN5)}$$

implies the property (PN4). And the above proposition shows that $(PN5)$ characterizes exactely the class $\mathcal{N}(H)$.

Proposition 3.6. *Let $S \in \mathfrak{B}(H)$. Then the three following properties are equivalent:*

(i) $S \in \mathbb{CS}(H)$,

(ii) $\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| = 2\,\|SXS\|$,

(iii) $\forall X \in \mathfrak{B}(H),\ \|S^*XS + SXS^*\| \geq 2\,\|SXS\|$.

Proof. The proof is trivial if $S = 0$. Assume now that $S \neq 0$.

The two implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial.

$(iii) \Rightarrow (i)$. Assume (iii) holds.

By taking $X = x \otimes y$ (where $x, y \in (H)_1$) in (iii), we obtain

$$\forall x, y \in (H)_1, \ \|S^*x\| \|S^*y\| + \|Sx\| \|Sy\| \geq 2 \|Sx\| \|S^*y\| . \qquad (A)$$

From (A), we deduce that

$$\forall x, y \in (H)_1, \ \|S\| (\|S^*x\| + \|Sx\|) \geq 2 \|Sx\| \|S^*y\| .$$

By taking the supremum in the above inequality over $y \in (H)_1$, we deduce that

$$\forall x \in H, \ \|S^*x\| \geq \|Sx\| . \qquad (B)$$

Using the same argument as used before, we deduce from (A) that

$$\forall x \in H, \ \|Sx\| \geq \|S^*x\| . \qquad (C)$$

Them from (B) and (C), S is normal. So from (iii) and proposition 2.1, it follows that

$$\forall X \in \mathfrak{B}(H), \ \|S^*XS + SXS^*\| = \|S^*XS\| + \|SXS^*\| . \qquad (D)$$

Let now $X \in \mathfrak{B}(H)$ and put $A = SXS^*$, $B = S^*XS$. Then from (D), we deduce that

$$\begin{aligned} \|A\|^2 + \|B\|^2 + 2\|A\|\|B\| &= \|(A+B)^*(A+B)\| \\ &= \|AA + BB + 2Re(B^*A)\| \\ &\leq \|A\|^2 + \|B\|^2 + 2\|B^*A\| \\ &\leq \|A\|^2 + \|B\|^2 + 2\|A\|\|B\|. \end{aligned}$$

Hence, $\|B^*A\| = \|A\| \|B\|$. So, we have

$$\forall X \in \mathfrak{B}(H), \ \|S^*XS\| \|SXS^*\| = \|(S^*XS)^* (SXS^*)\| .$$

Since $S \neq 0$ and S is normal, then we may choose a vector $y \in H$ such that $Sy \neq 0$ and $S^*y \neq 0$. By taking $X = x \otimes y$ (where x arbitrary in H) in the last inequality, we obtain the following

$$\forall x \in H, \ \|S^*x\| \|Sx\| = |\langle S^*x, Sx \rangle| .$$

Thus S^*x and Sx are linearly dependent, for every $x \in H$. Since S is normal, then for every $x \in H$ such that $Sx \neq 0$, there exists a complex numbers $\lambda(x)$ (depending in x) of modulus one such that $S^*x = \lambda(x)Sx$. By taking $X = x \otimes y$ in (D), for $x, y \in H$ such that $Sx \neq 0$, $Sy \neq 0$, we deduce that $\left|1 + \lambda(x)\overline{\lambda(y)}\right| = 2$ and so $\lambda(x) = \lambda(y)$. Hence there exits a constant real number θ such that $S^*x = e^{i\theta}Sx$, for every $x \in H$. Thus $S^* = e^{i\theta}S$. Put $M = e^{i\frac{\theta}{2}}S$. Then M is selfadjoint in $\mathfrak{B}(H)$ and $S = e^{-i\frac{\theta}{2}}M$. Therefore (i) holds. □

From the preceding proposition, we can obtain the following corollary:

Corollary 3.2. *The class $\mathbb{C}^*\mathbb{S}_0(H)$ is characterized by each of the two following properties*

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*S^{-1}X + XS^{-1}S^*\right\| = 2\left\|X\right\|, (S \in \mathfrak{I}(H)),$$
$$\forall X \in \mathfrak{B}(H),\ \left\|S^*S^{-1}X + XS^{-1}S^*\right\| \geq 2\left\|X\right\|, (S \in \mathfrak{I}(H)).$$

Proposition 3.7. *Let $S \in \mathfrak{B}(H)$, such that $S^2 \neq 0$. Then the following properties are equivalent:*

(i) $S \in \mathbb{R}^*\mathfrak{U}(H)$,

(ii) $\forall X \in \mathfrak{B}(H),\ \|S^2X\| + \|XS^2\| \leq 2\|SXS\|$,

(iii) $\forall X \in \mathfrak{B}(H),\ \|S^2X + XS^2\| \leq 2\|SXS\|$.

Proof. The two implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial.
$(iii) \Rightarrow (i)$. Assume that $S^2 \neq 0$. If we put $X = x \otimes y$ (for $x,\ y \in H$) in (iii), we obtain

$$\forall x, y \in H,\ \|S^2x \otimes y + x \otimes S^{*2}y\| \leq 2\|Sx\|\|S^*y\|. \qquad (**)$$

We shall prove (i) in three steps.
Step 1. Prove that S is injective.
Let $x \in kerS$, then (**) becomes

$$\forall y \in H,\ \|x\|\|S^{*2}y\| = 0.$$

Since $S^2 \neq 0$, we may choose y in H such that $S^{*2}y \neq 0$, it follows that $x = 0$. Hence S injective.
Step 2. Prove that S^* is injective.
By the same argument as above, we find that S^* injective.
Step 3. Prove that $R(S)$ closed.
Let $(x_n)_{n\geq 1}$ be a sequence in H, and let y in H such that $Sx_n \to y$. Applying $(**)$ for $x = x_n - x_m$ (where $n, m \geq 1$), we obtain

$$\forall y \in H, \forall n, m \geq 1;\ \|x_n - x_m\|\|S^{*2}y\| \leq 2\|Sx_n - Sx_m\|\|S^*y\| + \|S(Sx_n - Sx_m)\|\|y\|.$$

We may choose y such that $S^{*2}y \neq 0$, thus by passing to the limit, we obtain that $(x_n)_{n\geq 1}$ is a Cauchy sequence, which implies that $(x_n)_{n\geq 1}$ converges to some vector $x \in H$. By the continuity of S we find $y = Sx$. Thus $R(S)$ is closed.
By the three above steps, we find that S is invertible. If we replace X by $S^{-1}XS^{-1}$ in (iii), we obtain

$$\forall X \in \mathfrak{B}(H), \|SXS^{-1} + S^{-1}XS\| \leq 2\|X\|.$$

Using theorem 2.8, we get $S \in \mathbb{R}^*\mathfrak{U}(H)$. □

Remark 3.3. *By the same argument as used in the above theorem, we can see that a nonzero operator which satisfies each of the property (*),(**) or (***) is an invertible operator*

$$\forall X \in \mathfrak{B}(H), \|S^2X\| + \|XS^2\| = 2\|SXS\|,\ (S \in \mathfrak{B}(H)) \qquad (*)$$

$$\forall X \in \mathfrak{B}(H), \|S^*SX + XSS^*\| = 2\|SXS\|, \ (S \in \mathfrak{B}(H)) \tag{**}$$

$$\forall X \in \mathfrak{B}(H), \|S^2X + XS^2\| = 2\|SXS\|, (S \in \mathfrak{B}(H)) \tag{***}$$

Replace X by $S^{-1}XS^{-1}$ and using theorem 2.8, we can see that property (*) and (**) characterize the class $\mathbb{R}\mathfrak{U}(H)$ and the property (***) characterizes the class $\mathbb{R}\mathfrak{U}_r(H)$.

The following proposition characterizes the class $\mathbb{R}^*\mathfrak{U}(H)$ using the notion of parallelism.

Proposition 3.8. *Let $S \in \mathfrak{I}(H)$ such that $0 \notin W(S)$, then the following properties are equivalent:*
$(i) S\|S^{-1}$,
$(ii) S \in \mathbb{R}^*\mathfrak{U}(H)$.

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds. Then using [3], we obtain that

$$w(S^*S^{-1}) = w(SS^{*-1}) = \left\|S^*S^{-1}\right\| = \left\|SS^{*-1}\right\| = \|S\| \left\|S^{-1}\right\|.$$

So both S^*S^{-1} and SS^{*-1} are normaloid operators.
We also have $(S^*S^{-1})^{-1}S = S(S^*S^{-1})^*$ with $0 \notin W(S)$.
Applying [10, Theorem 1], we get that S^*S^{-1} is unitary. Hence

$$\left\|S^*S^{-1}\right\| = 1 = \|S\| \left\|S^{-1}\right\|.$$

Therefore $S \in \mathbb{R}^*\mathfrak{U}(H)$.
$(ii) \Rightarrow (i)$. The implication is trivial. □

Remark 3.4. *The above proposition doesn't holds without the condition $0 \notin W(S)$. In $\mathfrak{B}(\mathbb{C}^2)$, $S = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = S^{-1}$. We notice that $S\|S^{-1}$, but S is not unitary.*

3.2 Isometry operators and subclasses of partial isometry operators

The purpose of this section is to establish the following propositions that provide a useful characterizations of subclasses of nonnormal operators brecisely the subclass $\mathcal{V}(H)$ of all isometry operators in $\mathfrak{B}(H)$ and the subclass of all quasinormal partial isomerty operators in $\mathfrak{B}(H)$ (that contain $\mathfrak{U}(H)$). Those properties that give the characterizations have new forms that missed the symmetry form. Finally, we deduce another characterizations of the class $\mathfrak{U}(H)$.

Proposition 3.9. *Let $S \in (\mathfrak{R}(H))_1$. Then the two following properties are equivalent:*

(i) $S \in \mathcal{V}(H)$,

(ii) $\forall X \in \mathfrak{B}(H), \|X\| + \|S^+XS\| \leq 2\|SXS^+\|$.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (i)$. Assume (ii) holds. We prove (i) in three steps:

Step 1. Prove that S is injective.

If we replace X by $x \otimes y$ (for $x, y \in H$) in (ii), we obtain

$$\forall x, y \in H, \|x\| \|y\| + \left\|S^+ x\right\| \|S^* y\| \leq 2 \|Sx\| \left\|(S^+)^* y\right\|.$$

By taking $x \in \ker S$ and choosing $y \neq 0$ in the above inequality, we obtain that $x = 0$. Hence S is injective.

Step 2. Prove that $(S^2)^+ S = S^+$.

Since S is injective with a closed range, then S^2 is also injective with a closed range. Thus $S^+ S = (S^2)^+ S^2 = I$.

It's known that S^+ is the unique solution of the four following equations: $SXS = S$, $XSX = X$, $(XS)^* = XS$, $(SX)^* = SX$. It is easy to see that $(S^2)^+ S$ satisfies the first three equations. Now, prove that $(S^2)^+ S$ satisfies the last equation. Since the operator $S(S^2)^+ S$ is a projection, it suffices to prove that its norm is less than or equal to one. By taking $X = S(S^2)^+ S$ in (ii), we obtain

$$\begin{aligned} \left\|S(S^2)^+ S\right\| + \left\|S^+ S(S^2)^+ SS\right\| &\leq 2 \left\|SS(S^2)^+ SS^+\right\| \\ &\leq 2 \left\|S^2 (S^2)^+\right\| \left\|SS^+\right\| \\ &\leq 2. \end{aligned}$$

Thus $\|S(S^2)^+ S\| \leq 1$. Hence $S((S^2)^+ S)$ is a projection of norm less or equal to one. Thus $S(S^2)^+ S$ is a selfadjoint projection. So $(S^2)^+ S$ satisfies the above four equations. Therefore $(S^2)^+ S = S^+$.

Step 3. Prove that $S \in \mathcal{V}(H)$.

Since $S^2 (S^2)^+ = SS^+ S^2 (S^2)^+$ and that SS^+, $S^2 (S^2)^+$ are selfadjoint, then $S^2 (S^2)^+ = S^2 (S^2)^+ SS^+$. So from step 2, we obtain $S^2 (S^2)^+ = S^2 (S^+)^2$. Since S^2 is injective, we have $(S^+)^2 = (S^2)^+$.

Return to (ii) and replace X by SXS^+, we obtain

$$\forall X \in \mathfrak{B}(H), \left\|SXS^+\right\| + \|X\| \leq 2 \left\|S^2 X (S^2)^+\right\|.$$

Put $P = |S|$, $R = |S^2|$. Since S and S^2 are both injective with closed ranges, then P and R are invertible. By taking the polar decomposition of each of the two operators S and S^2 in the last inequality, we obtain

$$\forall X \in \mathfrak{B}(H), \left\|PXP^{-1}\right\| + \|X\| \leq 2 \left\|RXR^{-1}\right\|. \tag{*}$$

If we replace X by $R^{-1} XR$ in the above inequality, we obtain

$$\forall X \in \mathfrak{B}(H), \left\|PR^{-1} XRP^{-1} \right\| + \left\| R^{-1} XR\right\| \leq 2 \|X\|.$$

So from proposition 1.8(5′), we get that

$$\forall X \in \mathfrak{B}(H), \left\|PR^{-1}XRP^{-1}\right\| + \left\|R^{-1}XR\right\| \leq \left\|RXR^{-1}\right\| + \left\|R^{-1}XR\right\|.$$

Thus

$$\forall X \in \mathfrak{B}(H), \left\|PR^{-1}XRP^{-1}\right\| \leq \left\|RXR^{-1}\right\|.$$

Hence

$$\forall X \in \mathfrak{B}(H), \left\|PR^{-2}X\left(PR^{-2}\right)^{-1}\right\| \leq \|X\|.$$

So, we obtain that

$$\forall X \in \mathfrak{B}(H), \left\|PR^{-2}XR^{2}P^{-1}\right\| = \|X\|.$$

Using again $(*)$ and replace X by $R^{-2}XR^{2}$, we find

$$\forall X \in \mathfrak{B}(H), \left\|PR^{-2}XR^{2}P^{-1}\right\| + \left\|R^{-2}XR^{2}\right\| \leq 2\left\|R^{-1}XR\right\|.$$

So, we have

$$\forall X \in \mathfrak{B}(H), \left\|X\right\| + \left\|R^{-2}XR^{2}\right\| \leq 2\left\|R^{-1}XR\right\|.$$

Thus,

$$\forall X \in \mathfrak{B}(H), \left\|RXR^{-1}\right\| + \left\|R^{-1}XR\right\| \leq 2\|X\|.$$

Using theorem 2.8 and since R is positive, we obtain $R = \|R\| I$. Then $(*)$ becomes

$$\forall X \in \mathfrak{B}(H), \left\|PXP^{-1}\right\| \leq \|X\|.$$

Thus $1 \leq \|P\| \left\|P^{-1}\right\| = \sup_{\|X\|=1} \left\|PXP^{-1}\right\| \leq 1$. Hence $\|P\| \left\|P^{-1}\right\| = 1$. So we have $P = \|P\| I = I$. Hence $S^*S = P^2 = I$. Therefore $S \in \mathcal{V}(H)$. □

From the preceding proposition, we can obtain the following corollary:

Corollary 3.3. *The class $\mathfrak{U}(H)$ is characterized by each of the two following properties:*

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|SXS^{-1}\right\| \leq 2\left\|S^{-1}XS\right\|, \left(S \in (\mathfrak{I}(H))_1\right),$$

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|S^{-1}XS\right\| \leq 2\left\|SXS^{-1}\right\|, \left(S \in (\mathfrak{I}(H))_1\right).$$

In the three next propositions, we shall give some characterizations of some subclasses of partial isometry operators in $\mathfrak{B}(H)$.

Proposition 3.10. *Let $S \in (\mathfrak{R}(H))_1$. Then the following properties are equivalent:*
$(i) S^* \| S^+$,
$(ii) S$ is a partial isometry.

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds. Then using [3] we find that

$$w(SS^+) = \|SS^+\| = \|S\| \, \|S^+\| = 1,$$

Therefore S is a partial isometry.
$(ii) \Rightarrow (i)$ Trivial. □

Proposition 3.11. *Let $S \in \mathfrak{I}_1(H)$. Then the following properties are equivalent:*
$(i) S^* \| S^{\#}$,
$(ii) S \in \mathbb{R}^*(\mathfrak{U}(H) \oplus 0)$.

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds. Then using [3], we find that

$$w(SS^{\#}) = \|SS^{\#}\| = \|S\| \, \|S^{\#}\| ,$$

thus $SS^{\#}$ is normaloid and idempotent. Hence $SS^{\#}$ is orthogonal projection which give us that S is EP operator.
this yield that

$$w(SS^{\#}) = 1 = \|S\| \, \|S^{\#}\| = \|S_1\| \, \|S_1^{-1}\| .$$

Therefore $S \in \mathbb{R}^*(\mathfrak{U}(H) \oplus 0)$.
The implication $(ii) \Rightarrow (i)$ is trivial. □

Proposition 3.12. *Let $S \in (\mathfrak{R}(H))_1$. Then the following properties are equivalent:*

(i) S is a direct sum of an isometry and zero,

(ii) S is a partial isometry and quasinormal,

(iii) $\forall X \in \mathfrak{B}(H), \|X\| + \|SXS^+\| \geq 2 \|S^+XS\|$.

Proof. $(i) \Rightarrow (ii)$. Since $S = (S_1 \oplus 0)$, where S_1 is an isometry, then $S^* = S_1^* \oplus 0 = S_1^*(S_1S_1^*)^{-1} \oplus 0 = S^+$. Thus from proposition 2.6, S is a partial isometry. Hence $SS^*S = S = S_1 \oplus 0 = (S_1^*S_1)S_1 \oplus 0 = S^*SS$.
$(ii) \Rightarrow (i)$. Assume (ii) holds. Then $SS^*S = S = S^*SS$, thus $R(S) \subset R(S^*)$. Hence $S = S_1 \oplus 0$ with respect to the orthogonal decomposition $H = R(S^*) \oplus kerS$, where S_1 is injective.
On the other hand, we have $S^* = S_1^* \oplus 0$ and $S^+ = S_1^*(S_1S_1^*)^{-1} \oplus 0$. From the fact that $S^* = S^+$, we obtain $(S_1^*S_1)^{-1}S_1^* = S_1^*$, since S_1 is injective, then $(S_1^*S_1)^{-1} = I_1$.
So that $S_1^*S_1 = I_1$. Hence (i) holds.
Therefore $(i) \Leftrightarrow (ii)$.
$(ii) \Rightarrow (iii)$. Assume (ii) holds. Then $S = S^*S^2$, so we have $R(S) \subset R(S^*)$. Using Douglas Theorem, we obtain $\|Sx\| \geq \|S^*x\|$, for every $x \in H$. Hence, $\|SX\| \geq \|S^*X\|$, for every $X \in \mathfrak{B}(H)$.

Now let $X \in \mathfrak{B}(H)$. Then we have $\|X\| \geq \|S^*XS\| = \|S^+XS\|$ (since $\|S\| = 1$ and $S^* = S^+$), and

$$\|SXS^+\| = \|SXS^*\| \geq \|S^*XS^*\| = \|SX^*S\| \geq \|S^*X^*S\| = \|S^*XS\| = \|S^+XS\|$$

Hence $\|X\| + \|SXS^{+}\| \geq 2\|S^{+}XS\|$. This proves (iii).

$(iii) \Rightarrow (ii)$. Assume (iii) holds. It follows immediately that

$$1 + \left\|S^{+}\right\| \|S\| \geq 2\left\|S^{+}\right\| \|S\|.$$

Hence

$$\left\|S^{+}\right\| \|S\| \leq 1.$$

Thus,

$$\left\|S^{+}\right\| \|S\| = 1.$$

Since $\|S\| = 1$, then using theorem 2.8, S is a partial isometry.

It remains to prove that S is quasinormal. By taking $X = Sx \otimes Sx$ (where $x \in H$) in (iii), we obtain

$$\forall x \in H, \|Sx\|^2 + \left\|S^2x\right\|^2 \geq 2\|S^*Sx\|^2. \qquad (*)$$

Since S^* is also a partial isometry, then

$$\forall x \in H, \|S^*Sx\| = \|Sx\|. \qquad (**)$$

From (*) and (**), we obtain the following inequality

$$\forall x \in H, \left\|S^2x\right\| \geq \|Sx\|.$$

Since $\|S\| = 1$, we have

$$\forall x \in H, \left\|S^2x\right\| = \|Sx\|.$$

Hence $S^*(I - S^*S)S = 0$ (where $I - S^*S \geq 0$). Then $(I - S^*S)S = 0$. So $S = (S^*S)\,S = S\,(S^*S)$. Therefore S is quasinormal. □

As an immediate consequence of the above proposition, we have the following corollary:

Corollary 3.4. *The class $\mathfrak{U}(H)$ is characterized by each of the the two following properties:*

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|SXS^{-1}\right\| \geq 2\left\|S^{-1}XS\right\|, (S \in (\mathfrak{I}(H))_1),$$
$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|S^{-1}XS\right\| \geq 2\left\|SXS^{-1}\right\|, (S \in (\mathfrak{I}(H))_1).$$

As an immediate consequence of both Corollary 3.3 and Corollary 3.4, we deduce the following characterization:

Corollary 3.5. *The class $\mathfrak{U}(H)$ is characterized by the following property:*

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|SXS^{-1}\right\| = 2\left\|S^{-1}XS\right\|, (S \in (\mathfrak{I}(H))_1).$$

4. CHARACTERIZATIONS OF THE EXTENSIONS OF THE IDEMPOTENTS AND THE ORTHOGONAL PROJECTIONS

This chapter contains two sections, in the first section we are interesting by the study of two classes of operators in $\mathfrak{B}(H)$, the generalized projections and hypergeneralized projections, which are extension of the orthogonal projections.
In the second section, we define a new class of operator in $\mathfrak{B}(H)$ which is extension of idempotents.

4.1 Characterizations of generalized and hypergeneralized projections

An operator $A \in \mathfrak{B}(H)$ is said to be an idempotent (projection) if $A^2 = A$, n-idempotent if $A^n = A$, and an orthogonal projection if $A^2 = A = A^*$. The notion of orthogonal projection has been generalized in variant directions. For example, the generalized projections and the hypergeneralized projections whose concepts was introduced in 1997 by GroB and Trenkler [17] in finite dimensional Hilbert space, those operators extend the idea of orthogonal projections by removing the idempotency requirement. The variety properties of generalized projections and hypergeneralized projections in finite dimensional Hilbert space have been studied by many authors (see [1],[2],[17]).
Later many authors extended the concept of generalized projections and hypergeneralized projections on an infinite dimensional Hilbert space (see [9],[27],[36]).
In [9], Deng and Li have given the following characterization of n-idempotent operators in Hilbert space which is important in the characterizations of generalized and hypergeneralized projections.

Lemma 4.1. *Let $A \in \mathfrak{B}(H)$. Then $A^n = A$ if and only if*
(i) $\sigma(A) \subset \{0, e^{\frac{i2k\pi}{n-1}} : 0 \leq k \leq n-2\}$.
(ii) there exists a resolution $\{E(\lambda), \lambda \in \sigma(A)\}$ of the identity I and an invertible operator S such that

$$SAS^{-1} = \sum_{\lambda \in \sigma(A)} \oplus \lambda E(\lambda) .$$

where $E(\lambda), \lambda \in \sigma(A)$ are orthogonal projections adding up to unity, $\sum\limits_{\lambda \in \sigma(A)} E(\lambda) = I$ and $E(\lambda)E(\mu) = E(\mu)E(\lambda) = 0$ if $\lambda, \mu \in \sigma(A)$ and $\lambda \neq \mu$.

Definition 4.1. *1. An operator $A \in \mathfrak{B}(H)$ is called generalized projection if $A^2 = A^*$,* *2. An operator $A \in \mathfrak{R}(H)$ is called hypergeneralized projection if $A^2 = A^+$.*

Notation 4.1. *We denote by $\mathcal{P}(H)$ the set of all idempotent operators, $\mathcal{OP}(H)$ the set of all orthogonal projections on H.*
The set of all generalized projections on H is denoted by $\mathcal{GP}(H)$, and the set of all hypergeneralized projections on H is denoted by $\mathcal{HGP}(H)$.
The set of all EP oprators on H is denoted by $\mathcal{EP}(H)$.

In [36], Sonja Radosavljevic and Dragan gived the following theorems that show some characterisations of the two both classes generalized and hypergeneralized projections.

Theorem 4.1. *Let $A \in \mathfrak{B}(H)$. Then the following properties are equivalent:*
(i) A is a generalized projection,
(ii) A is a normal operator and $A^4 = A$,
(iii) A is a partial isometry and $A^4 = A$.

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds. Then, we have

$$AA^* = AA^2 = A^3 = A^2A = A^*A,$$

$$A^4 = (A^2)^2 = (A^*)^2 = (A^2)^* = (A^*)^* = A.$$

Hence (ii) holds.
$(ii) \Rightarrow (i)$. Since A is normal, then using the spectral measure, A has the following spectral representation

$$A = \int \lambda dE_\lambda,$$

where E_λ is the spectral projection associated with $\lambda \in \sigma(A)$.
From $A^4 = A$, and using lemma 4.1, we obtain that $\sigma(A) \subset \{0, 1, e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}\}$.
Since A is normal and $\sigma(A) \subset \{0, 1, e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}\}$, then A has the form

$$A = 0E(0) \oplus E(1) \oplus e^{\frac{2\pi i}{3}} E(e^{\frac{2\pi i}{3}}) \oplus e^{\frac{-2\pi i}{3}} E(e^{\frac{-2\pi i}{3}}),$$

such that $E(\lambda) = 0$ if $\lambda \in \{0, 1, e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}\} \setminus \sigma(A)$, $E(\lambda) \neq 0$ if $\lambda \in \sigma(A)$, and $E(0) \oplus E(1) \oplus E(\frac{2\pi i}{3}) \oplus E(\frac{-2\pi i}{3}) = I$.
We may remark that $\lambda^2 = \bar{\lambda}$, for every $\lambda \in \{0, 1, e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}\}$. Hence

$$\begin{aligned} A^2 &= 0E(0) \oplus E(1) \oplus e^{\frac{4\pi i}{3}} E(e^{\frac{2\pi i}{3}}) \oplus e^{\frac{-4\pi i}{3}} E(e^{\frac{-2\pi i}{3}}) \\ &= 0E(0) \oplus E(1) \oplus e^{\frac{-2\pi i}{3}} E(e^{\frac{2\pi i}{3}}) \oplus e^{\frac{2\pi i}{3}} E(e^{\frac{-2\pi i}{3}}) \\ &= A^*. \end{aligned}$$

Therefore A is a generalized projection.
$(i) \Rightarrow (iii)$. Assume (i) holds. Then we know $A = A^4 = AA^2A = AA^*A$. Multiplying the left side (or the right side) by A^*, we get $A^*A = A^*AA^*A$ (or $AA^* = AA^*AA^*A$), which proves that A^*A (or AA^*) is the orhtogonal projection onto $R(A^*A) = R(A^*) = ker(A)^\perp$

(or $R(AA^*) = R(A) = ker(A^*)^\perp$). Then, from proposition 2.6(v), it follows that A is a partial isometry.
$(iii) \Rightarrow (i)$. Assume (iii) holds. Since A is a partial isometry, then $A^* = A^+$ and AA^* is the orthogonal projection onto $R(AA^*) = R(A)$. Thus, $AA^*A = P_{R(A)}A = A$.
From other side, since $A^4 = A$ then $A^4 = AA^2A = A$. The uniqueness of A^+ implies $A^2 = A^*$. □

The following theorem gives matrix representation of generalized projections based on the previous characterization.

Theorem 4.2. *Let $A \in \mathfrak{B}(H)$ be a generalized projection. Then A is a closed range operator and A^3 is an orthogonal projection on $R(A)$. Moreover, H has decomposition $H = R(A) \oplus ker(A)^\perp = R(A) \oplus ker(A)$ and A has the following matrix representaton*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(A) \\ ker A \end{bmatrix} \to \begin{bmatrix} R(A) \\ ker A \end{bmatrix},$$

where the restriction $A_1 = A\,|_{R(A)}$ is unitary on $R(A)$.

Proof. Since A is a generalized projection, then A is a partial isometry implying that

$$A^3 = AA^* = P_{R(A)},$$

$$A^3 = A^*A = P_{ker(A)^\perp}.$$

Thus, A^3 is an orthogonal projection onto $R(A) = ker(A)^\perp = R(A^*)$. Consequently, $R(A)$ is a closed subset in H as a range of an orthogonal projection on a Hilbert space. So H has the following decomposition $H = R(A^*) \oplus ker(A) = R(A) \oplus ker(A)$. Now, A has the following matrix representation in accordance with this decomposition:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(A) \\ ker A \end{bmatrix} \to \begin{bmatrix} R(A) \\ ker A \end{bmatrix},$$

where $A_1^2 = A_1^*, A_1^4 = A_1$ and $A_1A_1^* = A_1^*A_1 = A_1^3 = I_{R(A)}$. □

Theorem 4.3. *Let $A \in \mathfrak{R}(H)$. Then the following properties are equivalent:*
(i) A is a hypergeneralized projection,
(ii) A^3 is an orthogonal projection onto $R(A)$,
(iii) A is an EP and $A^4 = A$.

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds. Then $A^3 = AA^+ = P_{R(A)}$ conclusion follows.
$(ii) \Rightarrow (i)$. Assume (ii) holds. A direct verification of the Moore-Penrose equations shows that $A^2 = A^+$.
$(i) \Rightarrow (iii)$. Assume (i) holds. Thus

$$AA^+ = AA^2 = A^3 = A^2A = A^+A,$$

we conclude that A is an EP operator. Then $A^+ = A^\#$. Thus $(A^+)^n = (A^n)^+$, for all $n \geq 1$. Hence

$$A^4 = (A^2)^2 = (A^+)^2 = (A^2)^+ = (A^+)^+ = A.$$

$(iii) \Rightarrow (i)$. Assume (iii) holds. Since A is an EP operator, then $A^+ = A^\#$. Hence $A^2A^+ = A$. Since $A^4 = A^2A^2 = A$, then from uniquencess of A^+, it follows that $A^2 = A^+$. □

Theorem 4.4. *Let $A \in \mathfrak{R}(H)$ be an hypergeneralized projection. Then A^3 is an orthogonal projection on $R(A)$. Moreover, H has decomposition $H = R(A) \oplus ker(A)^\perp = R(A) \oplus ker(A)$ and A has the following matrix representation*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(A) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA \end{bmatrix},$$

where the restriction $A_1 = A\mid_{R(A)}$ satisfies $A_1^3 = I_{R(A)}, A_1^2 = A_1^+$.

Proof. Since A is hypergeneralized projection, then A is an EP. So we have the following decomposition of the space $H = R(A) \oplus ker(A)$ and A has the required representation. □

Remark 4.1. *From the above theorems, we have if $A \in \mathfrak{B}(H)$ is a generalized projection, then $A^2 = A^* = A^+ = A^\#$. So A is an hypergeneralized projection. Besides, Both of generalized and hypergeneralized projections are EP operators.*
Every orthogonal projection is a generalized projection.
So, we have the inclusions

$$\mathcal{OP}(H) \subseteq \mathcal{GP}(H) \subseteq \mathcal{HGP}(H) \subseteq \mathcal{EP}(H).$$

Theorem 4.5. *Let $A \in \mathfrak{B}(H)$. Then the following holds:*
(i) $A \in \mathcal{GP}(H)$ if and only if $A^* \in \mathcal{GP}(H)$.
(ii) If $A \in \mathfrak{R}(H)$, then $A \in \mathcal{GP}(H)$ if and only if $A^+ \in \mathcal{GP}(H)$.
(iii) If $ind(A) \leq 1$, then $A \in \mathcal{GP}(H)$ if and only if $A^\# \in \mathcal{GP}(H)$.

Proof. (i) If $A \in \mathcal{GP}(H)$, then

$$(A^*)^* = A = A^4 = (A^2)^2 = (A^*)^2;$$

meaning that $A^* \in \mathcal{GP}(H)$. Conversely, if $A^* \in \mathcal{GP}(H)$, then $A^2 = ((A^*)^*)^2 = ((A^*)^2)^* = A^*$. Hence $A \in \mathcal{GP}(H)$.
(ii) If $A \in \mathcal{GP}(H)$, then $A^+ = A^2 = A^*$. Thus $(A^+)^2 = (A^2)^2 = A = (A^*)^* = (A^+)^*$ implying $A^+ \in \mathcal{GP}(H)$.
If $A^+ \in \mathcal{GP}(H)$, then $(A^+)^2 = (A^+)^* = (A^+)^+ = A$ and $(A^+)^4 = A^+$. Thus

$$A^2 = (A^+)^4 = A^+,$$

$$A^* = ((A^+)^*)^* = A^+.$$

Hence $A \in \mathcal{GP}(H)$.
(*iii*) If $A \in \mathcal{GP}(H)$, then $A^+ = A^\#$. So part (*ii*) of this theorem implies that $A^\# \in \mathcal{GP}(H)$.
To prove the converse, it is enough to see that $A^\# \in \mathcal{GP}(H)$ implies $(A^\#)^2 = (A^\#)^* = (A^\#)^+ = (A^\#)^\# = A$ and $(A^\#)^4 = A^\#$. Hence

$$A^2 = (A^\#)^4 = A^\# = ((A^\#)^*)^* = A^*.$$

Therefore $A \in \mathcal{GP}(H)$. □

Remark 4.2. *Let us mention an alternative proof for the previous theorem. If $A^+ \in \mathcal{GP}(H)$, then A and A^+ have representations*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(A^*) \\ kerA^* \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA^* \end{bmatrix}, A^+ = \begin{bmatrix} A_1^*K^{-1} & 0 \\ A_2^*K^{-1} & 0 \end{bmatrix},$$

where $K = (A_1A_1^* + A_2A_2^*)$. From $(A^+)^2 = (A^+)^*$, we get

$$\begin{bmatrix} A_1^*K^{-1}A_1^*K^{-1} & 0 \\ A_2^*K^{-1}A_1^*K^{-1} & 0 \end{bmatrix} = \begin{bmatrix} K^{-1}A_1 & K^{-1}A_2 \\ 0 & 0 \end{bmatrix},$$

which implies $A_2 = 0$, and $K = A_1A_1^*$. So

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, A^+ = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $(A_1^{-1})^2 = (A_1^{-1})^*$, then the same equality is also satisfied for A_1. Hence $A \in \mathcal{GP}(H)$. Similarly, to prove that $A^\# \in \mathcal{GP}(H)$ implies $A \in \mathcal{GP}(H)$, assume that $H = R(A) \oplus ker(A^*)$. Then, $A, A^\#$ have the following representation

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, A^\# = \begin{bmatrix} A_1^{-1} & (A_1^{-1})^2A_2 \\ 0 & 0 \end{bmatrix}.$$

Since $(A^\#)^2 = (A^\#)^*$, we get $A_2 = 0$ and $(A_1^{-1})^2 = (A^{-1}1)^*$.
Consequently, $(A_1)^2 = A_1^*$ which proves that $A \in \mathcal{GP}(H)$.

Theorem 4.6. *Let $A \in \mathfrak{R}(H)$. Then the following holds:*
(i) $A \in \mathcal{HGP}(H)$ if and only if $A^* \in \mathcal{HGP}(H)$.
(ii) $A \in \mathcal{HGP}(H)$ if and only if $A^+ \in \mathcal{HGP}(H)$.
(iii) If $ind(A) \leq 1$, then $A \in \mathcal{HGP}(H)$ if and only if $A^\# \in \mathcal{HGP}(H)$.

Proof. Proofs of (*i*) and (*ii*) are similar to proofs of theorem 4.5(*i*) and (*ii*).
For the proof of (*iii*), we should only prove that $A^\# \in \mathcal{HGP}(H)$ implies $A \in \mathcal{HGP}(H)$, since the implication ($\Rightarrow$) is analogous to the same part of theorem 4.5.
Let $H = R(A) \oplus ker(A^*)$ and $ind(A) \leq 1$. Then

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, A^\# = \begin{bmatrix} A_1^{-1} & (A_1^{-1})^2A_2 \\ 0 & 0 \end{bmatrix}, (A^\#)^+ = \begin{bmatrix} (A_1^{-1})^*K^{-1} & 0 \\ (A_2^{-1})^*K^{-1} & 0 \end{bmatrix},$$

where $K = A_1^{-1}(A_1^{-1})^* + A_2^{-1}(A_2^{-1})^*$. From $(A^\#)^+ = (A^\#)^2$, we get $A_2 = 0$ and $A_1 = A_1^{-2}$. Multiplying with A_1^2, the last equation becomes $A_1^3 = I_{R(A)}$. Hence $A \in \mathcal{HGP}(H)$. □

They also examined under what conditions product, sum and difference of generalized (hypergeneralized) projections is still a generalized (hypergeneralized) projection. The following theorem gives very useful matrix representation of generalized projections.

Theorem 4.7. *Let $A, B \in \mathcal{GP}(H)$ and $H = R(A) \oplus kerA$. Then A and B have the following representations with respect to the decomposition of the space:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(A) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} R(A) \\ kerA \end{bmatrix} \to \begin{bmatrix} R(A) \\ kerA \end{bmatrix},$$

where
$B_1^* = B_1^2 + B_2B_3$,
$B_2^* = B_3B_1 + B_4B_3$,
$B_3^* = B_1B_2 + B_2B_4$
$B_4^* = B_3B_2 + B_4^2$.
Furtheremore, $B_2 = 0$ if and only if $B_3 = 0$.

Proof. Let $H = R(A) \oplus ker(A)$. The representation of A follows from theorem 4.2 and let B has the followig representation

$$A = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then, from

$$B^2 = \begin{bmatrix} B_1^2 B_2 B_3 & B_1B_2 + B_2B_4 \\ B_3B_1 + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix} = \begin{bmatrix} B_1^* & B_3^* \\ B_2^* & B_4^* \end{bmatrix} = B^*,$$

the conclusion follows directly.
If $B_2 = 0$, then $B_3^* = B_1B_2 + B_2B_4 = 0$ and $B_3 = 0$. Analogously, $B_3 = 0$ implies $B_2 = 0$. $\square$

Theorem 4.8. *Let $A \in \mathfrak{B}(H)$ be a generalized projection. Then $I - A$ is a normal operator. Moreover, $I - A$ is a generalized projection if and only if A is an orthogonal projection.*
If $I - A$ is a generalized projection, then A is a normal operator and A is a generalized projection if and only if $I - A$ is an orthogonal projecton.

Proof. If A is a generalized projection, then A is a normal operator and $A^4 = A$, which implies that A has the form

$$A = 0E(0) \oplus 1E(1) \oplus e^{\frac{2\pi i}{3}} E(e^{\frac{2\pi i}{3}}) \oplus e^{\frac{-2\pi i}{3}} E(e^{\frac{-2\pi i}{3}}),$$

where $E(\lambda)$ is the orthogonal projection such that $E(\lambda) \neq 0$ if $\lambda \in \sigma(A), E(\lambda) = 0$, if $\lambda \in \{0, 1, e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}\} \setminus \sigma(A)$ and $E(0) \oplus E(1) \oplus E(e^{\frac{2\pi i}{3}}) \oplus E(e^{\frac{-2\pi i}{3}}) = I$.
Thus

$$I - A = (1-0)E(0) \oplus (1-1)E(1) \oplus (1 - e^{\frac{2\pi i}{3}})E(e^{\frac{2\pi i}{3}}) \oplus (1 - e^{\frac{-2\pi i}{3}})E(e^{\frac{-2\pi i}{3}}),$$

and

$$(I - A)^2 = E(0) \oplus (1 - e^{\frac{2\pi i}{3}})^2 E(e^{\frac{2\pi i}{3}}) \oplus (1 - e^{\frac{-2\pi i}{3}})^2 E(e^{\frac{-2\pi i}{3}}),$$
$$(I - A)^* = E(0) \oplus (1 - e^{\frac{2\pi i}{3}})^* E(e^{\frac{2\pi i}{3}}) \oplus (1 - e^{\frac{2\pi i}{3}})^* E(e^{\frac{2\pi i}{3}}),$$

Hence, $(I-A)^2 = (I-A)^*$ if and only if $(1 - e^{\frac{2\pi i}{3}})^2 E(e^{\frac{2\pi i}{3}}) = (1 - e^{\frac{2\pi i}{3}})^* E(e^{\frac{2\pi i}{3}})$ and $(1 - e^{\frac{-2\pi i}{3}})^2 E(e^{\frac{-2\pi i}{3}}) = (1 - e^{\frac{-2\pi i}{3}})^* E(e^{\frac{-2\pi i}{3}})$.
This is true if and only if $E(e^{\frac{2\pi i}{3}}) = 0$ and $E(e^{\frac{-2\pi i}{3}}) = 0$, which is equivalent to $\sigma(A) = \{0, 1\}$ and A is an orthogonal projection. □

Theorem 4.9. *Let $A, B \in \mathcal{GP}(H)$. If $AB = BA$, then $AB \in \mathcal{GP}(H)$*

Proof. If

$$AB = \begin{bmatrix} A_1 B_1 & A_1 B_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1 A_1 & 0 \\ B_3 A_1 & 0 \end{bmatrix} = BA,$$

it is clear that $A_1 B_1 = B_1 A_1, B_2 = 0$ and $B_3 = 0$. Form theorem 4.7 we conclude that $B_1^* = B_1^2, B_4^* = B_4^2$, and

$$(AB)^2 = \begin{bmatrix} (A_1 B_1)^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (A_1 B_1)^* & 0 \\ 0 & 0 \end{bmatrix} = (AB)^*$$

□

Theorem 4.10. *Let $A, B \in \mathcal{GP}(H)$. Then $A + B \in \mathcal{GP}(H)$ if and only if $AB = BA = 0$.*

Proof. If

$$(A+B)^2 = \begin{bmatrix} (A_1 + B_1)^2 + B_2 B_3 & (A_1 + B_1) B_2 B_4 \\ B_3(A_1 + B_1) + B_4 B_3 & B_3 B_2 + B_4^2 \end{bmatrix} = \begin{bmatrix} (A_1 + B_1)^* & B_3^* \\ B_2^* & B_4^* \end{bmatrix} = (A+B)^*,$$

it is clear that $(A_1 + B_1)^2 = A_1^2 + A_1 B_1 + B_1 A_1 + B_1^2 = (A_1 + B_1)^*$. Since $B_1^* = B_1^2 + B_2 B_3$, we get $A_1 B_1 + B_1 A_1 = 0$. Thus, $B_1 = 0$ which implies $B_2 = B_3 = 0, B_4^2 = B_4^*$. In this case we obtain $AB = BA = 0$.
Conversely, if $AB = BA = 0$, then $A_1 B_1 = B_1 A_1 = 0$, implying $B_1 = B_2 = B_3 = 0, B_4^2 = B_4^*$, and obviously, $(A+B)^2 = (A+B)^*$. □

Theorem 4.11. *Let $A, B \in \mathcal{GP}(H)$. Then $A - B \in \mathcal{GP}(H)$ if and only if $AB = BA = B^*$.*

Proof. If

$$(A-B)^2 = \begin{bmatrix} (A_1 - B_1)^2 + B_2B_3 & -(A_1 - B_1) + B_2B_4 \\ -B_3(A_1 + B_1) + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix} = \begin{bmatrix} (A_1 - B_1)^* & -B_3^* \\ -B_2^* & -B_4^* \end{bmatrix} = (A-B)^*,$$

we get $B_2 = B_3 = 0, B_4^2 = -B_4^*$. Since $B_4^2 = B_4^*$, then $B_4 = 0$.
We also get that

$$(A_1 - B_1)^2 = A_1^2 - A_1B_1 - B_1A_1 + B_1^2 = A_1^* - B_1^*,$$

This is true if and only if $A_1B_1 = B_1A_1 = B_1^*$, and in that case $AB = BA = B^*$. □

For hypergeneralized projections they didn't establish equivalency like the one they established for the generalized projections because they needed additional conditions to ensure that $(A + B)^+ = A^+ + B^+$.

Theorem 4.12. *Let $A, B \in \mathcal{HGP}(H)$. If $AB = BA$, then $AB \in \mathcal{HGP}(H)$*

Proof. Let $H = R(A) \oplus ker A$ and $A, B \in \mathcal{HGP}(H)$ have representations

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{bmatrix}, (AB)^2 = \begin{bmatrix} A_1B_1A_1B_1 & A_1B_1A_1B_2 \\ 0 & 0 \end{bmatrix}.$$

it is clear to see that

$$(AB)^+ = \begin{bmatrix} (A_1B_1)^*D^{-1} & 0 \\ (A_1B_2)^*D^{-1} & 0 \end{bmatrix},$$

where $D = A_1B_1(A_1B_1)^* + A_1B_2(A_1B_2)^* > 0$ is invertible.
Assume that the two hypergeneralized projections A, B commute, i.e., that

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1A_1 & 0 \\ B_3A_1 & 0 \end{bmatrix} = BA,$$

This implies $B_2 = 0$, $B_3 = 0$, $A_1B_1 = B_1A_1$ and it is easy to see that $(AB)^2 = (AB)^+$. □

Theorem 4.13. *Let $A, B \in \mathcal{HGP}(H)$. If $AB = BA = 0$, then $A + B \in \mathcal{HGP}(H)$*

Proof. From the matrix representations it is easy to see that $AB = BA = 0$ implies $B_1 = B_2 = B_3 = 0$ and $B_4^2 = B_4^+$. Now,

$$(A + B)^2 = A^2 + B^2 = A^+ + B^+ = (A + B)^+.$$

□

4.2 Characterizations of a new class of operators

Based on the two classes of generalized projections and hypergeneralized projections, we shall introduce a new class of operators which is define by $\Omega_{\#} = \{A \in \mathfrak{I}_1(H) : \; A^2 = A^{\#}\}$, this class of operators extend the idea of idempotent operators unlike generalized projections and hypergeneralized projections which extend the idea of orthogonal projections by removing the idempotency requirement.

Theorem 4.14. *Let $A \in \mathfrak{B}(H)$. Then the following properties are equivalent:*
(i) $A \in \Omega_{\#}$,
(ii) $A^4 = A$,
(iii) there exists a resolution $\{E(\lambda), \lambda \in \sigma(A)\}$ of the identity I and an invertible operator S such that

$$SAS^{-1} = 0E(0) \oplus 1E(1) \oplus e^{\frac{2\pi i}{3}} E(e^{\frac{2\pi i}{3}}) \oplus e^{\frac{-2\pi i}{3}} E(e^{\frac{2\pi i}{3}}),$$

where $\sigma(A) \subset \{0, 1, e^{\frac{2\pi i}{3}}, e^{\frac{-2\pi i}{3}}\}$, $E(\lambda_i)$, where $\lambda_i \in \sigma(A)$ are orthogonal projections adding up to unity ($\sum_{i=1}^{4} E(\lambda_i) = I$) and $E(\lambda_i)E(\lambda_j) = E(\lambda_j)E(\lambda_i) = 0$ if $\lambda_i, \lambda_j \in \sigma(A)$ and $\lambda_i \neq \lambda_j$.

Proof. $(i) \Rightarrow (ii)$. Since $A^2 = A^{\#}$, then $A^3 = AA^{\#} = A^{\#}A = P_{R(A)}$. Thus $A^4 = A$ conclusion follows.
$(ii) \Rightarrow (iii)$. This implication follows immediately by using lemma 4.1.
$(iii) \Rightarrow (i)$. Assume (iii) holds. We may remark that for every $\forall \lambda \in \sigma(A) \setminus 0 : \lambda^2 = \lambda^{-1}, 0^{\#} = 0$. Thus

$$\begin{aligned}
SA^2S^{-1} &= (SAS^{-1})^2 \\
&= 0E(0) \oplus 1E(1) \oplus (e^{\frac{i2\pi}{3}})^2 E(e^{\frac{i2\pi}{3}}) \oplus (e^{\frac{i4\pi}{3}})^2 E(e^{\frac{i4\pi}{3}}) \\
&= 0E(0) \oplus 1E(1) \oplus (e^{\frac{i2\pi}{3}})^{-1} E(e^{\frac{i2\pi}{3}}) \oplus (e^{\frac{i4\pi}{3}})^{-1} E(e^{\frac{i4\pi}{3}}) \\
&= (SAS^{-1})^{\#} \\
&= SA^{\#}S^{-1}.
\end{aligned}$$

Therefore $A^2 = A^{\#}$. ☐

Remark 4.3. *The class $\Omega_{\#}$ contains both idempotents and orthogonal projections unlike the generalized and hypergeneralized projections which contains only orthogonal projections.*

$$\mathcal{OP}(H) \subset \mathcal{GP}(H) \subset \mathcal{HGP}(H) \subset \Omega_{\#}, \; \mathcal{P}(H) \subset \Omega_{\#}.$$

Proposition 4.1. *Let $A \in \mathfrak{B}(H)$. Then*
(i)$A \in \Omega_{\#}$ and $R(A) \perp N(A)$ then $A \in \mathcal{HGP}(H)$,
(ii)$A \in \Omega_{\#}$ and A is normal, then $A \in \mathcal{GP}(H)$,
(iii)$A \in \Omega_{\#}$ and A is partial isometry, then $A \in \mathcal{GP}(H)$,
(iv)$A \in \Omega_{\#}$ and A selfadjoint then $A \in \mathcal{OP}(H)$,
(v)$A \in \Omega_{\#}$ and A positive then $A \in \mathcal{OP}(H)$.

Proof. $(i) A \in \Omega_{\#}$, then $H = R(A) \oplus N(A)$. Since $R(A) \perp N(A)$, then $H = R(A) \oplus^{\perp} N(A)$. Thus A is an EP operator, hence $A^+ = A^{\#}$. Therefore $A^2 = A^+$.
$(ii) A \in \Omega_{\#}$ and A is normal, then from the above theorem $A^4 = A$ and A is normal. Using theorem 4.1, we find that $A \in \mathcal{GP}(H)$.
$(iii) A \in \Omega_{\#}$, and A is a partial isometry, then $A^4 = A$ and A is a partial isometry. Using theorem 4.1, we find that $A \in \mathcal{GP}(H)$.
$(iv) A \in \Omega_{\#}$ and A selfadjoint then, using (ii) we find that $A^2 = A^*$. Since A selfadjoint, then $A^2 = A^* = A$.
$(v) A \in \Omega_{\#}$ and A positive , then A is selfadjoint and using (iv), we find that $A \in \mathcal{OP}(H)$. □

Proposition 4.2. *Let $A \in \mathfrak{B}(H)$. Then the following holds:*
(i) $A \in \Omega_{\#}$ if and only if $A^* \in \Omega_{\#}$,
(ii) $A \in \Omega_{\#}$ if and only if $A^{\#} \in \Omega_{\#}$,
(iii) If $A \in \Omega_{\#}$ and $C \in \mathfrak{B}(H)$ is unitary equivalent to A, then $C \in \Omega_{\#}$.

Proof. (i) If $A \in \Omega_{\#}$, then $(A^*)^2 = (A^2)^* = (A^{\#})^* = (A^*)^{\#}$ implying $A^* \in \Omega_{\#}$.
Conversely, if $A^* \in \Omega_{\#}$, then $A^2 = ((A^*)^*)^2 = ((A^*)^2)^* = ((A^*)^{\#})^* = ((A^*)^*)^{\#} = A^{\#}$ meaning that $A \in \Omega_{\#}$.
(ii) If $A \in \Omega_{\#}$, then $A^4 = A$ and $(A^{\#})^2 = A^{\#}A^{\#} = A^2A^2 = A^4 = A = (A^{\#})^{\#}$. Therefore $A^{\#} \in \Omega_{\#}$.
Conversely, if $A^{\#} \in \Omega_{\#}$, then $(A^2)^{\#} = (A^{\#})^2 = (A^{\#})^{\#} = A$. So $A^{\#} \in \Omega_{\#}$.
(iii) C is unitary equivalent of A, then there exists a unitary operator U such that $C = UAU^*$.
Hence $C^2 = UAU^*UAU^* = UA^2U^* = UA^{\#}U^* = (UAU^*)^{\#} = C^{\#}$. Therefore $C \in \Omega_{\#}$. □

Proposition 4.3. *Let $T_1, ..., T_n \in \Omega_{\#}$. Then $T_1 \oplus ... \oplus T_n$ and $T_1 \otimes ... \otimes T_n$ are two operators in $\Omega_{\#}$.*

Proof. Since $(T_1 \oplus ... \oplus T_n)^2 = T_1^2 \oplus ... \oplus T_n^2 = T_1^{\#} \oplus ... \oplus T_n^{\#} = (T_1 \oplus ... \oplus T_n)^{\#}$. Then $(T_1 \oplus ... \oplus T_n) \in \Omega_{\#}$.
For $x_1, ..., x_n \in H, (T_1 \otimes ... \otimes T_n)^2(x_1, ..., x_n) = (T_1^2 \otimes ... \otimes T_n^2)(x_1, ..., x_n) = T_1^2x_1 \otimes ... \otimes T_n^2x_n = T_1^{\#}x_1 \otimes ... \otimes T_n^{\#}x_n = (T_1^{\#} \otimes ... \otimes T_n^{\#})(x_1, ..., x_n) = (T_1 \otimes ... \otimes T_n)^{\#}(x_1, ..., x_n)$.
So $(T_1 \otimes ... \otimes T_n)^2 = (T_1 \otimes ... \otimes T_n)^{\#}$. Thus $(T_1 \otimes ... \otimes T_n) \in \Omega_{\#}$. □

Theorem 4.15. *Let $A \in \mathfrak{R}(H)$. Then the following statements are equivalent:*
(1) $A \in \Omega_{\#}$,
(2) $ind(A) \leq 1$ and $A^{n+2} = A^{\#}A^n = A^nA^{\#}, \ \forall n \in \mathbb{N}$,
(3) $ind(A) \leq 1$ and $(A^{\#})^{n-2} = (A^{\#})^{n+1};, \ \forall n \in \mathbb{N}$,
(4)A^3 is an idempotent and $asc(A) \leq 1$,
(5)A^3 is an idempotent and $dsc(A) \leq 1$,
(6) $ind(A) \leq 1$ and $A^4A^+ + A^+A^4 = A^+A + AA^+$,
(7) $dsc(A) \leq 1$ and $A^*A^4A^+ = A^*$,
(8) $dsc(A) \leq 1$ and $A^+A^4A^* = A^*$,

(9) $asc(A) < \infty$ and there exists some $X \in \mathfrak{B}(H)$ such that $A^4A^+X = A$ and $AA^+X = A$,
(10) $ind(A) \leq 1$ and there exists some $X \in \mathfrak{B}(H)$ such that $A^{\#}(A^2)^+X = A^2$ and $AA^+X = A^2$,
(11) $ind(A) \leq 1$ and there exists some $X \in \mathfrak{B}(H)$ such that $A(A^{2^{\#}})^+X = A^{2^{\#}}$ and $AA^+X = A^{2^{\#}}$.

Proof. Property (1) implies conditions (2) − (11) is trivial.
Conversely, we prove that each of statements (2)-(11) implies that $A \in \Omega_{\#}$.
The two implications (2) ⇒ (1) and (3) ⇒ (1) are trivial.
(4) ⇒ (1). A^3 is an idempotent, then $A^6 = A^3$. Thus $A^2(A^4 - A) = 0$. Since $asc(A) \leq 1$, then $A(A^4 - A) = 0$. So $A^2(A^3 - I) = 0$, thus $A(A^3 - I) = 0$. Hence $A^4 = A$. Using theorem 4.14, we obtain that $A \in \Omega_{\#}$.
(5) ⇒ (1). Same method as the above implication.
(6) ⇒ (1). Since $ind(A) \leq 1$, then $A^{\#}$ exists. Multiplying the equality $A^4A^+ + A^+A^4 = A^+A + AA^+$ by $A^{\#}$ from both left and right sides, we get that $A = A^{\#^2}$. Hence $A \in \Omega_{\#}$.
(7) ⇒ (1). Multiplying the equality $A^*A^4A^+ = A^*$ by A from the left side, we obtain $A^*A^4 = A^*A$. Then $(A^*A)^+A^*A^4 = (A^*A)^+A^*A$. Since $dsc(A) \leq 1$, it follows that $A^3 = (A^*A)^+A^*A$. Hence $A^4 = A$. Using theorem 4.14 we obtain that $A \in \Omega_{\#}$.
(8) ⇒ (1). This implication can be proved in the same way as the previous one.
(9) ⇒ (1). The condition $A^4A^+X = A$ is equivalent to $R(A) = R(A^2)$, then $dsc(A) \leq 1$ and since $asc(A) < \infty$, then $ind(A) \leq .1$ Hence $A^{\#}$ exists. Multiplying the equality $A^4A^+X = A$ by $(A^{\#})^2$ from the left side, we get $AA^+X = (A^{\#})^3$ and since $AA^+X = A$, then $(A^{\#})^2 = A$. Consequently $A \in \Omega_{\#}$.
(10) ⇒ (1). Multiplying the equality $A^{\#}(A^2)^+X = A^2$ by AA^+ from the left side, we find $AA^+X = A^5$ and since $AA^+X = A^2$, then $A^2 = A^5$. Multiplying this last equality by $(A^{\#})^3$, we obtain that $A \in \Omega_{\#}$.
(11) ⇒ (1). This implication can be proved in the same way as the previous one. □

Now, we study the properties of this class by examine the conditions which imply that the product, sum and difference of those operators belongs to the same class of operators.

Theorem 4.16. *Let $A, B \in \Omega_{\#}$. If $AB = BA$, then $AB \in \Omega_{\#}$.*

Proof. Since $(AB)^2 = A^2B^2 = A^{\#}B^{\#} = (AB)^{\#}$, then $AB \in \Omega_{\#}$. □

Lemma 4.2. *[4] Let $\mathfrak{A}$ be an algebra with unity. If $a, b \in \mathfrak{A}^{\#}$ satisfy $ab = ba = 0$, then $a + b \in \mathfrak{A}^{\#}$ and*

$$(a + b)^{\#} = a^{\#} + b^{\#}.$$

Theorem 4.17. *Let $A, B \in \Omega_{\#}$. If $AB = BA = 0$, then $A + B \in \Omega_{\#}$.*

Proof. Since $AB = BA = 0$, then

$$(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + B^2 = A^\# + B^\#.$$

Using the above lemma, we find that

$$(A+B)^2 = A^\# + B^\# = (A+B)^\#.$$

Therefore $A + B \in \Omega_\#$. □

Corollary 4.1. *Let $A, B \in \Omega_\#$. If $AB = BA = 0$, then $A - B \in \Omega_\#$.*

Finally, we try to reduce the above conditions using the matrix representation. For the group inverse of upper triangular operator matrix.
For that, we need the following lemma that gives the group inverse of upper triangular operator matrix (see [21, Theorem 1]).

Lemma 4.3. *Let H, K be Hilbert spaces, and $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ be an operator on $H \oplus K$. Then the following assertions hold.*
(i) If $D^\#$ exists, then $M^\#$ exists if and only if $A^\#$ exists and $A^\pi B D^\pi = 0$.
(ii) If $A^\#$ and $D^\#$ exist, then $M^\#$ exists if and only if $A^\pi B D^\pi = 0$. In this case,

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^\# = \begin{bmatrix} A^\# & Y \\ 0 & D^\# \end{bmatrix},$$

where $Y = (A^\#)^2 B D^\pi + A^\pi B (D^\#)^2 A^\# B D^\# (A^\pi = I - A^\# A)$.

Theorem 4.18. *Let $P \in \mathfrak{B}(H)$ be an orthogonal projection and let $A \in \Omega_\#$. If $(I - P)AP = 0$, then $AP \in \Omega_\#$. Similarly, If $PA(I-P) = 0$, then $PA \in \Omega_\#$.*

Proof. Let $H = R(P) \oplus (R(P))^\perp$. Then

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, AP = \begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix}$$

$(I - P)AP = 0$ i.e. $A_3 = 0$, then A has matrix form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}, A^2 = \begin{bmatrix} A_1^2 & A_1 A_2 + A_2 A_4 \\ 0 & A_4^2 \end{bmatrix},$$

using the above lemma, we find that

$$A^\# = \begin{bmatrix} A_1^\# & (A_1^\#)^2 A_2 (I - A_4 A_4^\#) - A_1^\# A_2 A_4^\# \\ 0 & A_4^\# \end{bmatrix}.$$

$A \in \Omega_\#$ gives that $A_1^\# = A_1^2$ and consequently

$$(AP)^2 = \begin{bmatrix} A_1^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{\#} & 0 \\ 0 & 0 \end{bmatrix} = (AP)^{\#}.$$

□

Theorem 4.19. *Let $P \in \mathfrak{B}(H)$ be an orthogonal projection and let $A \in \Omega_{\#}$. If $(I-P)AP = 0$ or $PA(I-P) = 0$, then $A(I-P) \in \Omega_{\#}$. Similarly, if $(I-P)AP = 0$ or $PA(I-P) = 0$, then $(I-P)A \in \Omega_{\#}$.*

Proof. Let $H = R(P) \oplus (R(P))^{\perp}$. Then

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, AP = \begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix}.$$

If $(I-P)AP = 0$ i.e. $A_3 = 0$, then A has matrix form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}, A^2 = \begin{bmatrix} A_1^2 & A_1A_2 + A_2A_4 \\ 0 & A_4^2 \end{bmatrix},$$

and it is easy to see that

$$A^{\#} = \begin{bmatrix} A_1^{\#} & (A_1^{\#})^2A_2(I - A_4A_4^{\#}) - A_1^{\#}A_2A_4^{\#} \\ 0 & A_4^{\#} \end{bmatrix}.$$

Since $A \in \Omega_{\#}$, it is clear that $A_4^{\#} = A_4^2$ and consequently

$$(A(I-P))^2 = \begin{bmatrix} 0 & A_2A_4 \\ 0 & A_4^2 \end{bmatrix} = \begin{bmatrix} 0 & A_2(A_4^{\#})^2 \\ 0 & A_4^{\#} \end{bmatrix} = (A(I-P))^{\#}.$$

□

Theorem 4.20. *Let $Q \in \mathfrak{B}(H)$ be an idempotent and let $A \in \Omega_{\#}$. If $(I-Q)AQ = 0$, then $AQ, QA, A(I-Q)$, and $(I-Q)A$ are operators in $\Omega_{\#}$.*

Proof. Let $H = R(Q) \oplus (R(Q))^{\perp}$. Then

$$Q = \begin{bmatrix} I & Q_1 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

$(I-Q)AQ = 0$ i.e. $A_3 = 0$. Then $A \in \Omega_{\#}$ gives $A_1^2 = A_1^{\#}$ and $A_4^2 = A_4^{\#}$. So consequently

$$(AQ)^2 = \begin{bmatrix} A_1^2 & A_1^2Q_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{\#} & A_1^{\#}Q_1 \\ 0 & 0 \end{bmatrix} = (AQ)^{\#}.$$

$$(QA)^2 = \begin{bmatrix} A_1^2 & A_1(A_2 + Q_1A_4) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^{\#} & A_1^{\#2}(A_2 + Q_1A_4) \\ 0 & 0 \end{bmatrix} = (QA)^{\#}.$$

$$(A(I-Q))^2 = \begin{bmatrix} 0 & (A_1Q_1 + A_2)A_4 \\ 0 & A_4^2 \end{bmatrix} = \begin{bmatrix} 0 & (A_1Q_1 + A_2)A_4^{\#2} \\ 0 & A_4^{\#} \end{bmatrix} = (A(I-Q))^{\#}.$$

$$((I-Q)A)^2 = \begin{bmatrix} 0 & Q_1 A_4^2 \\ 0 & A_4^2 \end{bmatrix} = \begin{bmatrix} 0 & Q_1 A_4^{\#} \\ 0 & A_4^{\#} \end{bmatrix} = ((I-Q)A)^{\#}.$$

□

Theorem 4.21. *Let $Q \in \mathfrak{B}(H)$ be an idempotent and let $P \in \mathfrak{B}(H)$ be an orthogonal projection. If $(I-P)QP = 0$, then $PQ, QP, P(I-Q)$, and $(I-Q)P$ are operators in $\Omega_{\#}$.*

Proof. Let $H = R(Q) \oplus (R(Q))^{\perp}$. Then

$$Q = \begin{bmatrix} I & Q_1 \\ 0 & 0 \end{bmatrix}, P = \begin{bmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{bmatrix}.$$

$(I-P)QP = 0$ i.e. $P_2 = 0$. Then $P \in \Omega_{\#}$ gives $P_1^2 = P_1^{\#}$ and $P_3^2 = P_3^{\#}$. So consequently

$$(QP)^2 = \begin{bmatrix} P_1^2 & P_1 Q_1 P_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1^{\#} & P_1^{\#2} Q_1 P_3 \\ 0 & 0 \end{bmatrix} = (QP)^{\#}.$$

$$(PQ)^2 = \begin{bmatrix} P_1^2 & P_1^2 Q_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1^{\#} & P_1^{\#} Q_1 \\ 0 & 0 \end{bmatrix} = (PQ)^{\#}.$$

$$(P(I-Q))^2 = \begin{bmatrix} 0 & P_1 Q_1 P_3 \\ 0 & P_3^2 \end{bmatrix} = \begin{bmatrix} 0 & P_1 Q_1 P_3^{\#2} \\ 0 & P_3^{\#} \end{bmatrix} = (p(I-Q))^{\#}.$$

$$((I-Q)P)^2 = \begin{bmatrix} 0 & Q_1 P_3^2 \\ 0 & P_3^2 \end{bmatrix} = \begin{bmatrix} 0 & Q_1 P_3^{\#} \\ 0 & P_3^{\#} \end{bmatrix} = ((I-Q)P)^{\#}.$$

□

BIBLIOGRAPHY

[1] J.K. Baksalary, O.M. Baksalary, and X. Liu, Further properties of generalized and hypergeneralized projectors. Linear Algebra and its Applications 389 (2004) 295-303.

[2] J.K. Baksalary and X. Liu, An alternative characterization of generalized projectors. Linear Algebra and its Applications 388 (2004) 61-65.

[3] M. Barra and M. Boumazgour, Inner derivations and norm equality, Proc. Amer. Math. Soc. 130(2001), 471-476.

[4] J. Benítez, X. Liuy, and T. Zhu, Additive results for the group inverse in an algebra with applications to block operators. Linear and Multilinear Algebra Vol. 59, No. 3, March 2011, 279-289.

[5] A. Ben-Israel, T.N.E. Greville, Generalized Inverses, Theory and Applications, second ed., Springer, 2003.

[6] R.H. Bouldin, The pseudo-inverse of a product, SIAM J. Appl. Math. 25 (1973) 489–495.

[7] C. Bouraya and A. Seddik, On the characterizations of some distinguished suclasses of Hilbert space operators, Acta Sci. Math. (Szgred) 84(2018), 611-627.

[8] G. Corach, H. Porta, and L. Recht, An operator inequality, Linear Algebra Appl., 142(1990), 153-158.

[9] C. Deng, Q. Li, and H. Du, Generalized n-idempotents and hyper-generalized n-idempotents, Northeast.Math. J. 22 (2006) 387–394.

[10] C.R. Deprima, Remarks on operators with inverses similar to thier adjoints, Proc. Amer. Math. Soc. 43(1974) 478-480.

[11] R.G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, Proc. Amer. Math. Soc. 17 (1966), 413 – 415.

[12] H. Du and Y. Li, The spectral characterization of generalized projections, Linear Algebra Appl.410(2005) 150-159.

[13] D.S. Djordjević and J. J. Koliha, Characterizing hermitian, normal and EP operators. Filomat 21:1 (2007) 39-54.

[14] D.S. Djordjević and N. Dincić, Reverse order law for the Moore–Penrose inverse. J. Math. Anal. Appl. 361 (2010) 252–261.

[15] M. Drazin, Pseudo - inverse in assosiative rings and semi groups. Amer. math. monthly 1958.

[16] T.N.E. Greville, Note on the generalized inverse of a matrix product, SIAM Rev. 8 (1966) 518–521.

[17] J. GroB and G. Trenkler, Generalized and hypergeneralized projectors, Linear Algebra Appl., 1997, 264, 463–474

[18] P.A. Fillmore and J.P. William, On Operator Ranges, Advances in mathematics 7 (1971), 254-281.

[19] J. Fujii, M. Fujii, T. Furuta, and R. Nakamoto, Norm inequalities equivalent to Heinz inequality, Proc. Amer. Math. Soc., 118(1993), 827 – 830.

[20] R. Hart and M. Mbekhta, On generalized inverse in $\mathbb{C}^*$ algebra, Studia mathematica,103(1992), 71-77.

[21] R.E. Hartwig and J.M. Shoaf, Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices, Austral. J. Math. 24(A) (1977) 10–34.

[22] E. Heinz, Beitrage zur storungstheory der spctralzerlegung, Math. Ann 123 (1951) 415-438.

[23] S. Izumino, The product of operators with closed range and an extension of the reverse order law, Tühoku Math. Journ. 34(1982), 43-52.

[24] I. Kaplansky, Rings with a polynomial identity, Bull. Amer. Math. Soc, 54 (1948), 575-580.

[25] M. Khosravi, Corach-Porta-Recht inequality for closed range operators, Math. Ineq. Appl., 16(2013), 477-481.

[26] M. Khosravi, A characterization of the class of partial isometries. Linear Algebra Appl. 437 (2012), 1300–1304.

[27] L. Lebtahi and Néstor Thome, A note on k-generalized projections. Linear algebra and its applications 420 (2007) 572-575.

[28] M. Mbekhta, Partial isometry and generalized inverses, Acta Sci. Math (Szeged) 70 (2004), 767-781.

[29] A. McIntosh, Heinz inequalities and perturbation of spectral families, Macquarie Mathematical Reports, Macquarie Univ., 1979.

[30] S. Menkad and A. Seddik, Operator inequalities and normal operators, Banach J. Math. Anal., 6(2012), 187-193.

[31] S. Menkad, Partial isometries and norm inequalities for operators, Matematiqki nesvik. 67, 4 (2015), 269–276.

[32] E. H. Moore, On the reciprocal of the general algebraic matrix , Bull. Amer. Math. Soc, 26 (1920), 394-395.

[33] D. Mosić and D.S. Djordjević, Moore-Penrose-invertible normal and Hermitian elements in rings. Linear Algebra and its Applications 431 (2009) 732-745.

[34] R. Penrose, A generalized inverse for matrices, Procedings of the Combridge Philosophical, Society, 51 (1955), 406-413.

[35] R. Rado, Note on generalized inverse of matrices, Proc. Combridge philos. Soc. 52(1965) 600-601.

[36] S. Radosavljevic and D.S.Djordjevic, On pairs of generalized and hypergeneralized projections in Hilbert space. Functional Analysis, Approximation and Computation 5:2 (2013),67-75.

[37] P. B. Ramanujan and S. M. Patel, Operators whose ascent is 0 or 1. Proc. Amer. Math. Soc. 29(1971), 557-560.

[38] A. Seddik, Some results related to Corach-Porta-Recht inequality, Proc. Amer. Math. Soc. 129(2001), 3009-3015.

[39] A. Seddik, Rank one operators and norm of elementary operators, Linear algebra Appl., 424(2007), 177-183.

[40] A. Seddik, On the injective norm of $\sum_{i=1}^{n} A_i \otimes B_i$ of and characterization of normaloid operators. Oper. Matrices 2 (2008) 67–77.

[41] A. Seddik, On the injective norm and characterization of some subclasses of normal operators by inequalities or equalities, J. Math. Anal. Appl. 351(2009), 277-284.

[42] A. Seddik, Characterization of the class of unitary operators by operator inequalities, Linear and Multilinear algebra, 59(2011), 1069-1074.

[43] A. Seddik, Closed operator inequalities and open problems, Math. Ineq. Appl. 14(2011), 17-157.

[44] A. Seddik, Moore-Penrose inverse and operator inequalities, Extracta mathematicae Vol. 30, Núm. 1,29-39 (2015) .

[45] J.G. Stampli and J.P. Williams, Growth conditions and the numerical range in Banach algebra, Tohoku Math. J.,20(1968), 417-442.

[46] J. Von Neumann, On regular rings, Proc. Natl. Acad. Sci USA 1936, 22 (12), 707-713.

[47] T. Yamazaka and M. Yanagida, Relations between two operator inequalities and their applications to paranormal operators, Acta Sci. Math. (Szeged), 69(2003), 377-389.

Abstract

The first chapter summarizes the background that will be required for the thesis which is represented by some basic notions and theorems of operator theory and also some famous inequalities in the theory of operator and the relation between them.
Based on these inequalities Seddik could find a complete characterisations of some distinguished subclasses of operator. In the second chapter, we give a detailed study of all the characterisations found of some subclasses by operator inequalities.
The third chapter consists of our contribution in the characterisations of some subclasses by operator inequalities.
The last chapter contains two sections, in the first we study the variety properties of the two classes generalized projections and hypergeneralized projections which extend the idea of orthogonal projections by removing the idempotency requirement.
Based on the two classes, we define in the second section a new class of operators that extend the idea of idempotency and give a complete characterisation of this class.

Keywords

Closed range operator, Moore-Penrose inverse, group inverse, selfadjoint operator, unitary operator, normal operator, partial isometry operator, isometry operator, operator inequality.

Résumé

Le premier chapitre résume le contexte requis pour la thèse qui est représentée par quelques notions de base et théorèmes de la théorie de l'opérateur et aussi quelques inégalités célèbres dans la théorie de l'opérateur et la relation entre eux.
En se basant sur ces inégalités, Seddik pourrait trouver une caractérisation complète de certaines sous-classes distinguées. Le deuxième chapitre, nous étudions toutes les caractérisations trouvées de certaines sous-classes par les inégalités de l'opérateur.
Le troisième chapitre consiste en notre contribution à la caractérisation de certaines sous-classes par les inégalités de l'opérateur.
Le dernier chapitre contient deux sections, dans la première section on étudie les différentes Propriétés des deux classes de projections généralisées et les projections hypergénéralisées qui étendent l'idée de projections orthogonales en supprimant l'exigence d'idempotence.
Basé sur les deux classes, nous définissons dans la deuxième section une nouvelle classe d'opérateurs qui étend l'idée d'idempotence et donnons une caractérisation complète de cette classe.

Mots clés

Operateur à image fermé, Moore-Penrose inverse, group inverse, operateur autoadjoint, operateur unitaire, opérateur normal, opérateur isométrie partielle, opérateur isométrique, inegalité d'opérateur.