République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université de Batna 2, Mostefa Ben Boulaid
Faculté des Mathématiques et de l'Informatique
Département de Mathématiques
Laboratoire des Techniques Mathématiques, LTM

# THÈSE

**Présentée en vue d'obtenir du diplôme de doctorat en mathématiques**

**Discipline:**
***Équations aux Dérivées Partielles et Applications***

*Présentée et soutenue publiquement par*


**Nedjem-Eddine RAMDANI**


# Sur les méthodes de projection et applications aux équations intégrales et intégro-différentielles

*Soutenue le : 14 Novembre 2018.*

*Devant le jury composé de:*

| | | | |
|---|---|---|---|
| Ameur SEDDIK | Professeur | Université de Batna 2 | Président |
| Abdelaziz MENNOUNI | Professeur | Université de Batna 2 | Rapporteur |
| Amar YOUKANA | Professeur | Université de Batna 2 | Examinateur |
| Ali HAKEM | Professeur | Université de Sidi-Bel-Abbès | Examinateur |
| Lakhdar CHITER | Professeur | Université de Sétif 1 | Examinateur |
| Fadila BENTALHA | Professeure | Université de Batna 2 | Invitée |

THÈSE

Democratic and Popular Republic oF Algeria
Ministry of Higher Education and Scientific Research
University of Batna 2, Mostefa Ben Boulaid
Faculty of Mathematics and Computer Science
Department of Mathematics
Laboratory of Mathematical Techniques, LMT

# THESIS

**Submitted for the degree of doctorate in mathematics**

**Discipline:**
***Partial Differential Equations and Applications***

*Presented by*

**Nedjem-Eddine Ramdani**

# On projection methods and applications to integral and integro-differential equations

*Defended on : Novembre,* $14^{th}$ *2018.*

*Jury Members:*

| | | | |
|---|---|---|---|
| Ameur Seddik | Professor | University of Batna 2 | President |
| Abdelaziz Mennouni | Professor | University of Batna 2 | Supervisor |
| Amar Youkana | Professor | University of Batna 2 | Examiner |
| Ali Hakem | Professor | University of Sidi-Bel-Abbes | Examiner |
| Lakhdar Chiter | Professor | University of Setif 1 | Examiner |
| Fadila Bentalha | Professor | University of Batna 2 | Invited |

THESIS

# Acknowledgments

I would like to warmly acknowledge and express my deep sense of gratitude and indebtedness to everyone who has supported me during the accomplishment of my thesis.

Firstly, a special thanks goes to my supervisor Professor Abdelaziz Mennouni for his help, support, guidance, encouragement and suggestions over all the years of my study and preparation of the final manuscript of this thesis.

My sincere thanks also go to the members of my thesis committee: Pr. Ameur Seddik, Pr. Amar Youkana, Pr. Ali Hakem and Pr. Lakhdar Chiter for reviewing and evaluating this thesis.

I am sincerely grateful to Pr. Fadila Bentalha for being part of the examination committee.

Pursuing a doctorate would have never been an option without the full support that I got from my family, my friends and my colleagues.

Moreover, I would like to thank my teachers of the mathematics department at university of Batna 2, every one with his/her name. Especially, Professor Salah-Eddine Rebiai, Said Guedjiba, Ahmed Zerrouk Mokrane and all the other teachers, without forget the Head of Department: Mister Mahmoud Brahimi.

I am thankful to all my friends for their help and support over the years and a special thanks goes to: : Abderrahmane Youkana, Salah-Eddine Boutiah, Halim Bouhanef, Moussa Berghout...

Least and not last, I am grateful to my parents, my wife all my family for all their love, encouragement and supported me along the way.

The journey to a doctorate is a long ride and having my family and my friends along that ride made it more appreciated. So, to all these amazing people who went with me along this journey, I sincerely thank you.

# Dedication

***Dedicated to my wife***

**To my dear parents**

***To my brothers and sisters***

# Contents

# Introduction

We have the habit of working on Differential Equations, but, as we know, they are not always easy to handle. Another type of equations characterized with easiness of solving and a better relevance to model phenomenas has emerged; they are called: *Integral equations*.
An integral equation is an equation where the unknown function is written under the integral sign. Their typical forms are:

$$\int k(s,t)u(t)dt = f(s)$$
$$u(s) + \int k(s,t)u(t)dt = f(s),$$

where $u$ is the unknown function, $f$ is the known function called the right hand side and $k(.,.)$ is called the kernel.

The first application of integral equations was done by Danial Bernoulli in around 1730 to study oscillations of stretched cord. However, the term integral equation was used for the first time by Paul du Bois-Reymond in 1888.
Nowadays, integral equations occur in many problems in a lot of fields, especially Fredholm integral equations. Their importance leads us to study this kind of equations. There are numerous researchers who studied such problems and they proposed several methods like expansion method [68], wavelet moment method [16], a discrete collocation method [44], a local radial basis function method [8], and other methods [63, 97, 87, 81, 79, 80].
Integral equations intervene in a lot of areas, including: Physics [76], mechanics [58], dynamics [108], electrostatics and electromagnetic [107], oscillating magnetics [9],$\cdots$ etc.
The aim of this thesis is to develop new methods to numerically solve some classes of Fredholm integral and integro-differential equations.
The thesis is organized as follows: Firstly, in the introductory chapter, we start with the classification of integral equations and the projection method. Then, we recall briefly

some basic concepts, notions and fundamental theorems such as: the Riesz theorem, the Fredholm alternative and the application to Fredholm integral equations. To stress the importance of integral equations, some real life examples are given.

In the second chapter, an interesting method has been developed a lot after the work of Galerkin to solve integral equations called modified projection method. One of the papers published that caught our attention was Kulkarni's[56] work, which used a projection method based on orthogonal projection and piecewise polynomials of degree $\leq r-1$ to prove that the convergence is of order $4r$. However, the other works were restricted to $[0,1]$. One of the contributions of this thesis is the extension of the domain from $[0,1]$ to the whole real line. Also, we used the Hermite polynomials instead of piecewise polynomials. Finally, we gave the convergence order which is different from the previous work.

The purpose of the third chapter is is devoted to the study of integro-differential equations which appear in many scientific fields such as: physics, biology and engineering. In [79], the author introduced a projection method based on the Legendre polynomials to solve an integro-differential equations of the first order with Cauchy kernel. In the same perspective, our contribution in this chapter consists of a presentation of a modified collocation method for solving integro-differential equations of second order. To do so, we use a sequence of orthogonal finite rank projections called airfoil polynomials of the first kind. We concluded this chapter by giving numerical examples.

The Fourth chapter to study another type of integral equations which has widely occurred in several problems in many fields called singular integral equations. There are many classes of singular integral equations with logarithmic kernel. In this work, we were interested in a more general logarithmic kernel and we showed the existence and uniqueness of the solution. In addition, we discussed the projection method to solve the generalized logarithmic integral equation. After that, we presented our method which is based on shifted-Legendre polynomials. Then, we proved the existence of the solution for the approximate equation. In the end, we gave a new error estimate for the numerical solution and we illustrated our approach with numerical examples.

In the last chapter, we were interested in the work of Xiao [51] which introduced the application of the wavelet method in the resolution of integral equations. One of the features provided by such an application is that symmetric systems can be solved in $O(n \log n)$ operations, where $n$ is the size of the systems. Our contribution here is a generalization of this method to the second kind. Furthermore, we extended it to non-symmetric kernel. this latter occurs in many problems and fields. In the end, we proved that the new system can be solved in $O(2n \log n)$ operations.

# Chapter 1

# Preliminary

## 1.1 Classification of integral equations

The classification of integral equations based on many basic characteristics

1. The integral bounds

- ▶ Both of them are constants: Fredholm equation.
- ▶ One of them is variable: Volterra equation.

2. The placement of the unknown function $u$:

- ▶ Just inside the integral: First kind.
- ▶ Both inside and outside the integral: Second kind. The Fredholm integral equation of the first kind is given by:

$$\int_a^b k(s,t)u(t)dt = f(s), \quad a \leq s \leq b.$$

The Volterra integral equation of the second kind is given by:

$$u(s) - \int_a^s k(s,t)u(t)dt = f(s), \quad a \leq s \leq b.$$

3. The nature of the known function $f$:

- ▶ Equal zero: homogeneous equation.

- Different from zero: non-homogeneous equation. For example:

$$u(s) - \int_a^b k(s,t)u(t)dt = 0, \quad a \le s \le b$$

is a homogeneous Fredholm integral equation of the second kind.

The following equation

$$\int_a^s k(s,t)u(t)dt = f(s), \quad a \le s \le b$$

is called non-homogeneous Volterra integral equation of the first kind.

4. The linearity: with respect to the unknown function $u$:

- Linear integral equations.
- Nonlinear integral equations. For example, the equation

$$u(s) - \int_a^b k(s,u(t))dt = f(s), \quad a \le s \le b,$$

is a nonlinear Fredholm integral equation of the second kind.

5. The nature of the kernel:

- Regular integral equations.
- Singular integral equations.

For example, the following equation

$$u(s) - \int_0^1 \frac{u(t)}{(s-t)^\alpha}dt = f(s), \quad 0 < \alpha < 1, \quad 0 \le s \le 1$$

is called weakly singular integral equation of the second.

Moreover,

$$u(s) - \int_0^1 \frac{u(t)}{\sqrt{s-t}}dt = f(s), \quad 0 \le s \le 1$$

is the Abel's integral equation of the second.

The integral equation

$$u(s) - \int_0^1 \frac{u(t)}{s-t}dt = f(s), \quad 0 < \alpha < 1, \quad 0 \le s \le 1$$

is the Cauchy integral equation of the second.

**Remark 1.1** *The kernel $k(.,.)$ could play the role of several functions, exclusively:*

1. *The power-law functions (linear,quadratic,polynomial...);*
2. *The exponential functions;*
3. *The hyperbolic functions ($\cosh, \sinh, \tanh, \coth$);*
4. *The trigonometric functions ($\cos, \sin, \tan, \cot$);*
5. *The logarithmic functions.*

An other type of integral equations occur with derivative on the unknown function $u$, called integro-differential equations.
For example:

$$u''(s) - \int_{-1}^{1} k(s,t)u(t)dt = f(s), \quad -1 \leq s \leq 1$$

is an integro-differential equation of the second order.

## 1.2 Compact operator

**Definition 1.1** *Let $X$ and $Y$ be normed spaces and $T : X \to Y$ be a linear operator. Recall that $T$ is a bounded operator if and only if for every bounded subset $M$ of $X$ the set $T(M)$ is bounded in $Y$ so that the closure of $T(M)$, is closed and bounded. If the closure of $T(M)$ is compact for every bounded subset $M$ of $X$, then the operator $T$ is said to be a compact operator.*

**Theorem 1.1** *A linear operator $T : X \to Y$ from a normed space $X$ into a normed space $Y$ is compact if and only if, for each bounded sequence $(\varphi_n)$ in $X$ the sequence $(A\varphi_n)$ contains a convergent subsequence in $Y$.*

*Proof* : See [55]. □

**Definition 1.2** *Let $H$ be a Hilbert space and $T \in B(H)$. The adjoint of $T$ is the operator $T^* \in B(H)$ with the property that, for all $\phi, \psi \in H, (T\phi, \psi) = (\phi, T^*\psi)$.*

**Lemma 1.1** *Let $H$ be a Hilbert space. If $T \in B(H)$ is compact then $T^*$ is also compact.*

*Proof* : See [94]. □

## 1.3 Integral operator

Let us consider an integral equation of the form

$$u(s) - \int_D k(s,t)u(t)dt = f(s),$$

where $k$ is a suitable function on $D \times D$. Define the following integral operator

$$(Tu)(s) = \int_D k(s,t)u(t)dt.$$

The above equation reads as:

$$u - Tu = f.$$

### 1.3.1 Compact integral operator

**Theorem 1.2** *Let $X := C(G)$ be the space of real continuous functions on a compact subset $G \subset \mathbb{R}$. The integral operator with continuous kernel is a compact operator on $X$.*

*Proof* : See [46]. □

**Theorem 1.3** *Let $K$ be a compact integral operator on $X$ and $T$ a bounded integral operator. Then both $KT$ and $TK$ are compact integral operators.*

*Proof* : See [46]. □

## 1.4 Projection and Riesz representation theorem

### 1.4.1 Approximation based on projection

Let $X$ be a Banach space and $T : X \longrightarrow X$ be a linear operator. Let us denote by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ the null space ( or the kernel space) and the range of $T$, respectively. Let $(\pi_n)$ be a sequence of nonzero bounded projections defined on $X$, that is each $\pi_n$ is bounded operator and $\pi_n^2 = \pi_n$, hence $\|\pi_n\| \geq 1$.

The following three conditions are equivalent to each other if $\pi_n$ is a bounded projection defined on Hilbert space $X$:

$$\pi_n^* = \pi_n, \quad \|\pi_n\| \leq 1, \quad \mathcal{N}(\pi_n) = \mathcal{R}(\pi_n)^\perp.$$

If one of these conditions is satisfied, then $\pi_n$ is called an orthogonal projection. (For more details see [84]).

Now, we would like to obtain an approximate solution for (1.3) using a sequence of bounded projections $\pi_n$ each one of finite rank,

In the beginning, we approach (1.3), we get the approximate problem:

$$(I - \pi_n T)u_n = \pi_n f, \quad u_n \in X_n.$$

**Lemma 1.2** *Let $X$ be a Banach space and $(\pi_n)_{n\geq 0}$ a sequence of bounded projections on $X$, with*

$$\pi_n u \to u, \quad as \quad n \to \infty, \quad for\ all \quad u \in X.$$

*Let $T : X \to X$ be compact. Then*

$$\|T - \pi_n T\| \to 0, \quad n \to \infty.$$

*Proof* : See [11]. □

### 1.4.2 Riesz-Fredholm theory

**Lemma 1.3 (Riesz Lemma)** *Let $X$ be a normed space and let $M \subset X$ a closed subspace, such that $M \neq X$, then:*

$$\forall \epsilon \geq 0,\ \exists u \in X,\ such\ that\ \|u\| = 1\ and\ dist(u, M) \geq 1 - \epsilon.$$

*Proof* : See [22]. □

**Theorem 1.4** *Let $X$ be a Banach space, and let $T : X \to X$ be compact. Then:*

- $\mathcal{N}(I-T)$ *is a subspace of finite dimension.*
- $Im(I-T)$ *is a closed linear subspace.*
- *There exists* $r \in \mathbb{N}$*, called Riesz number of the operator* $T$ *such that:*

$$\{0\} = Ker((I-T)^0) \subsetneqq \mathcal{N}((I-T)^1) \subsetneqq \cdots \subsetneqq \mathcal{N}((I-T)^r) = \mathcal{N}((I-T)^{r+1}) = \cdots$$
$$X = \mathcal{R}((I-T)^0) \supsetneqq \mathcal{R}((I-T)^1) \supsetneqq \cdots \supsetneqq \mathcal{R}((I-T)^r) = \mathcal{R}((I-T)^{r+1}) = \cdots$$

*Moreover, we have the direct sum :*

$$X = \mathcal{N}((I-T)^r) \oplus \mathcal{R}((I-T)^r).$$

*Proof* : See [45]. □

**Theorem 1.5** *Let* $X$ *be a Banach space, and let* $T : X \to X$ *be compact operator. Then* $(I-T)$ *is injective if and only if it is surjective. Furthermore, the inverse operator* $(I-T)^{-1}$ *exists and is bounded.*

*Proof* : See [33] □

### 1.4.3 The Fredholm alternative theorem

**Theorem 1.6** *Let* $X$ *be a Hilbert space with scalars the complex numbers,* $T : X \to X$ *be a compact operator, and let* $\lambda$ *be a nonzero eigenvalue of* $T$*. Then:*

- $\overline{\lambda}$ *is an eigenvalue of the adjoint operator* $T^*$*. In addition,* $\mathcal{N}(\lambda I - T)$ *and* $\mathcal{N}(\overline{\lambda} I - T^*)$ *have the same dimension.*
- *The equation* $(\lambda I - T)u = f$ *is solvable if and only if* $(f, z) = 0 \quad z \in \mathcal{N}(\overline{\lambda} I - T^*)$.

**Remark 1.2** *Let* $X$ *be a Hilbert space with scalars the complex numbers,* $T : X \to X$ *be a compact operator. It is known that*

$$\mathcal{R}(\lambda I - T) = \mathcal{N}(\overline{\lambda} I - T^*)^{\perp};$$

*so that*

$$X = \mathcal{N}(\overline{\lambda} I - T^*) \oplus \mathcal{R}(\lambda I - T).$$

**Theorem 1.7 (Fredholm alternative)** *Let $X$ be a Banach space, and let $T : X \to X$ be compact operator. Then the equation $(I - T)u = f$ has a unique solution $u \in X$ for all $f \in X$ if and only if the homogeneous equation $(I - T)u = 0$ has only the trivial solution $u = 0$. In such a case, the operator $(I - T) : X \to X$ has a bounded inverse $(I - T)^{-1}$.*

*Proof* : See [11]. □

## 1.5 Application of integral equations for the investigation of differential equations

As we mentioned above, integral equations play an important role in the theory of ordinary and partial differential equations. Moreover, many results of the theory of differential equations have been obtained by the investigation of the corresponding integral equations.

### 1.5.1 Cauchy problem for first-order ODEs to integral equations

The Cauchy problem: find a solution of the equation

$$u'(s) = H(s, u(s)), \quad 0 \leq s \leq 1,$$

that satisfies the initial condition

$$u(s_0) = u_0,$$

for a given $s_0 \in [0, 1]$ and some know function $u_0$.
By integration, the above Cauchy problem reduced to the equivalent integral equation:

$$u(s) = u_0 - \int_{s_0}^{s} H(t, u(t))dt.$$

Thus, we can solve the Cauchy problem by solving its equivalent form given by the integral equation.

### 1.5.2 Cauchy problem for second-order ODEs to integral equations

Consider the determination of $u$ from

$$\begin{cases} u''(s) &= H(s,u(s)), \quad 0 \le s \le 1, \\ u(0) &= u_0, \quad u(1) = u_1, \end{cases} \tag{1.1}$$

where $u_0' = C$.
One integration gives

$$u'(s) = \int_0^s H(t,u(t))dt + C, \quad 0 \le s \le 1,$$

satisfying $u'(0) = u_0' = C$ and a second integration produces

$$u(s) = \int_0^s d\tau \int_0^t H(t,u(t))dt + Cs + u_0.$$

If we assume that $H$ is a continuous function of both variables, then

$$\int_0^s d\tau \int_0^t H(t,u(t))dt = \int_0^s (s-t)H(t,u(t))dt.$$

Finally, the integral equation corresponding to (1.1) is giving by

$$u(s) = \int_0^s (s-t)H(s,u(t))dt + Cs + u_0, \quad 0 \le s \le 1.$$

Now, we need to determine $C$. Since $u(1) = u_1$, we get

$$C = u_1 - u_0 - \int_0^1 (1-t)H(t,u(t))ds.$$

Thus, the corresponding integral equation is giving as follows

$$u(s) = -\int_0^1 k(s,t)H(t,u(t))dt + (u_1 - u_0)s + u_0, \quad 0 \le s \le 1,$$

such that

$$k(s,t) = \begin{cases} t(1-s) & if(t \le s), \\ s(1-t) & if(s \le t). \end{cases}$$

### 1.5.3 Examples:

**Example 1.1** *We have the following boundary problem*

$$u''(s) = -\lambda u(s), \quad 0 \le s \le 1,$$
$$u(0) = u_0, \quad u(1) = u_1.$$

*the above problem becomes a Fredholm integral equation of second kind*

$$u(s) = \lambda \int_0^1 k(s,t)u(t)dt + f(s), \quad 0 \le s \le 1,$$

*where*

$$f(s) = (u_1 - u_0)s + u_0.$$

**Example 1.2 (Airy's equation for second-order ODEs)** *Let $u$ satisfies the following Airy's equation:*

$$u''(s) = su(s), \quad 0 \le s \le 1,$$

*and*

$$u(0) = 1, \quad u'(0) = 0.$$

*One puts in* (1.1)

$$H(s,t) = st, \ u_0 = 1, \ u_0' = 0.$$

*Then, we obtain the integral equation*

$$u(s) = \int_0^s (s-t)tu(t)dt + 1, \quad 0 \le s \le 1. \tag{1.2}$$

*We solve the Airy equation by solving the latter equation.*

*We can proceed by the same way to the second kind Volterra equation, using* (1.2) *for illustration. We introduce the operator $H$ by*

$$(Hu)(s) = \int_0^s (s-t)tu(t)dt, \quad 0 \le s \le 1,$$

*and let $f(s) = 1$. Thus, the abbreviate form is given as follows*

$$u = f + Hu.$$

*Then,*

$$\begin{aligned} u &= f + H(f + Hu) \\ &= f + Hf + H(Hu) \\ &= f + Hf + H^2 u, \end{aligned}$$

*where*

$$\begin{aligned} (Hf)(s) &= \int_0^s (s-t)t\,dt = \frac{1}{6}s^3, \quad 0 \le s \le 1 \\ (H^2 f)(s) &= \int_0^s (s-t)t(Kf)(t)dt \\ &= \int_0^s (s-t)t(\frac{1}{6}t^3)dt = \frac{1}{180}s^6, \quad 0 \le s \le 1. \end{aligned}$$

*Hence, by the same way we obtain*

$$u = f + Hf + H^2 f + H^3 u,$$

*In the end, the solution reads as*

$$u(s) = 1 + \frac{1}{6}s^3 + \frac{1}{180}s^6 + (H^3 u)(s), \quad 0 \le s \le 1.$$

***Remark 1.3*** *The Airy equation appears for modeling numerous physics phenomena such us diffraction phenomena and aerodynamic.*

**Example 1.3 (Boundary value problems)** *Let $u$ satisfy*

$$u''(s) + \lambda v(s)u(s) = 0, \quad 0 \le s \le 1,$$

$$u(0) = 0, \quad u(1) = 0.$$

*Thus, its equivalent integral equation is giving as follows*

$$u(s) = \lambda \int_0^s t(1-s)v(t)u(t)dt + \lambda \int_s^1 s(1-t)v(t)u(t)dt. \quad 0 \le s \le 1,$$

# Chapter 2

# Convergence analysis of projection methods for solving singular integral equations on the real line

This chapter is the subject of an article submitted to the Electronic Journal of Differential Equations.

**Abstract**

In this work, we extend the application of projection methods to approach the solution of singular integral equations on the real line using Hermite functions. We obtain the orders of convergence for the methods of Galerkin, Sloan, Kulkarni and Kulkarni's iterated method, respectively. We prove the following convergence orders: Galerkin's order $n^{-\frac{r}{2}}$, Sloan's order $n^{-r}$, Kulkarni's order $n^{-\frac{3r}{2}}$ and Kulkarni's iterated order $n^{-2r}$.

## 2.1 Introduction

In the last decades, projection methods have been developed starting from the work of Galerkin. This development founded by Sloan enhancing the Galerkin method adopting the iteration techniques, which has been used to solve integral equations. In addition, the Kantorovich method and its iterated version have been included in those projection methods along with the Sloan method.

Among the interesting papers published after Sloan, we distinguish Kulkarni's work, where she based on the orthogonal projection and interpolatory projection to approach solutions of compact operator equation with smooth kernel. She used piecewise polynomials of the degree $\leq r-1$ to prove that the convergence order is $4r$. Therefore, her results represent an improvement to the works presented in the literature, precisely, Galerkin's order $r$ and Sloan's order $2r$ (see [56]).

Several problems of engineering, quantum mechanics, probability theory, and statistical mechanics are described in terms of integral equations on the half real line and real line (cf. [59]). As a solution to these problems, the Hermite functions play a crucial role in the approximation theory (cf. [52]). According to Anselone and Baker, there exist computable bounds for the errors occuring in certain classes of Wiener-Hopf compact operators (see [2]).

Unlike the previous works where the kernel is defined on $[0,1] \times [0,1]$, the present paper considers the kernel as defined on $\mathbb{R}$. The main idea of this work is to solve the Fredholm integral equation of the second kind over the real line. Furthermore, we use the Hermite polynomials instead of piecewise polynomials.

Anselone and Sloan [3], have applied a numerical approximation schemes of quadrature type to solve the Wiener-Hopf integral equation. In [12], the author has developed several numerical methods to solve integral equations on the half line. Moreover, Sloan and Spence [100] have proved convergence results using projection methods to solve integral equations on the half line; the authors have investigated superconvergence of the iterated solution obtained from the collocation method.

More recently, many authors have used the Hermite functions to solve different types of equatios (cf. [90]). Among these, the authors of [4] have applied some addition formulae similar to the well-known expansion of the Hermite polynomials. The decomposition of powers in terms of the Hermite polynomials offers a different form of the formal moment generating function. A similar method without a CA system has been presented in [49]. Moreover, the elements of the matrix for the Gaussian-type potential have been calculated using the properties of the generating function of the Hermite polynomials (see [96]). In [110], the authors have found that the quantum state for the two-variable Hermite polynomial-Gaussian laser modes of the electromagnetic field are the Hermite-polynomial excitation on the two-mode squeezed vacuum state. In [17], the authors have shown how the combined use of the generating function method and of the theory of multivariable Hermite polynomials is naturally suited to evaluate integrals of Gaussian functions and of multiple products of Hermite polynomials. A more general class of multiple Hermite polynomials has been studied in cf. [53].

In [81] the author presents a projection method to solve operator equations with bounded operator in Hilbert spaces, and applies that method to solve the Cauchy integral equations in two cases: Galerkin projections and Kulkarni methods, respectively, using a sequence of orthogonal finite rank projections. In [78], the author introduces a modified method which is based on the trapezoidal and Simpson's rules, to solve a Volterra integral equations of the second kind. In [82], the authors have studied projection approximations to solve Cauchy integro-differential equations using airfoil polynomials of the first kind. In [79], the author introduces a projection method based on the Legendre polynomials, to solve integro-differential equations with a Cauchy kernel. In the same context, the author studies a collocation method, for approximate solution of an integro-differential equations with logarithmic kernel, using airfoil polynomials (see [80]).

The main results of our work are illustrated through the convergence order for a set of methods: for the Galerkin method, $n^{-\frac{r}{2}}$; for the Sloan method, $n^{-r}$ (with a better constant than the former); for the Kulkarni method, $n^{-\frac{3r}{2}}$; and finally for the iterated Kulkarni method, $n^{-2r}$.

This chapter is organized as follows: Section 2 is devoted to derive some basic definitions and preliminary results concerning the Hermite functions. In Section 3, we introduce a singular integral equation on the real line, we discuss the compactness of integral operator. Section 4 applies the improved Galerkin method to solve a Fredholm integral equation of the second kind over the real line, using the Hermite projection. Section 5 presents the Kulkarni method. In section 6, we prove the convergence of our methods, Galerkin, Sloan, Kulkarni and iterated Kulkrani respectively, and we give new error estimates.

## 2.2 Hermite functions

In this section, we briefly recall and discuss some basic formulae about the Hermite functions.

The Hermite polynomials $H_n$, are the eigenfunctions of the singular Sturm-Liouville problem in $\mathbb{R}$

$$\left(e^{-x^2}H_n'(x)\right)' + 2ne^{-x^2}H_n(x) = 0.$$

The analogue of the Rodriguez formula is

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

The Hermite polynomials have been introduced in [26].

Hermite polynomials $H_n$ are defined by the generating function [89]

$$\begin{aligned} H_n(t) &\sim \frac{\Gamma(n+1)}{\Gamma(n/2+1)} e^{t^2/2} \cos\left(\sqrt{2n+1}t - \frac{n\pi}{2}\right), \qquad t \in \mathbb{R}, \\ &\sim n^2 e^{t^2/2} \cos\left(\sqrt{2n+1}t - \frac{n\pi}{2}\right), \qquad t \in \mathbb{R}. \end{aligned}$$

We note that $H_n$ are unnormalized system in $L^2(\mathbb{R})$, but, as indicated in [89, 105], they are orthogonal in $\mathbb{R}$ with respect to the function $e^{-x^2}$, they satisfy

$$\int_{-\infty}^{+\infty} H_k(t) H_j(t) e^{-t^2} dt = 2^k k! \sqrt{\pi} \delta_{kj}.$$

Let us consider the normalized Hermite functions $\psi_n$ of the degree $n$

$$\psi_n(t) \;:=\; \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(t).$$

Hence, $\psi_n$ are unnormalized system in $L^2(\mathbb{R})$, which means,

$$\int_{-\infty}^{+\infty} \psi_k(t) \psi_j(t) dt \neq \delta_{kj}.$$

where $\delta_{kj}$ is the Kronecker delta function.

Clearly, Hermite functions $\psi_n$ verify the orthonormality relation with respect to the weight $\omega(t) := e^{-t^2}$.

We introduce the following inner product

$$\langle f, g \rangle_\omega := \int_{-\infty}^{+\infty} f(t) \overline{g(t)} \omega(t) dt.$$

Let us introduce the following weighted space

$$L^2_\omega(\mathbb{R}, \mathbb{C}) := \left\{ \varphi : \mathbb{R} \to \mathbb{C}, \quad \int_{-\infty}^{+\infty} \omega(t) \left|\varphi(t)\right|^2 dt < \infty \right\}.$$

The norm in $L^2_\omega(\mathbb{R}, \mathbb{C})$ is defined by

$$\|\varphi\|_w := \left( \int_{-\infty}^{+\infty} \omega(t) \left|\varphi(t)\right|^2 dt \right)^{\frac{1}{2}}.$$

Any $\varphi \in L^2_\omega(\mathbb{R})$ can be developed in a Fourier-Hermite series of the form

$$\varphi(s) := \sum_{k=0}^{\infty} \hat{\varphi}_k \psi_k(s),$$

with

$$\hat{\varphi}_k := \int_{-\infty}^{+\infty} \omega(t)\varphi(t)\psi_k(t)dt.$$

The theory of the Hermite polynomials are illustrated in [64].

## 2.3 Integral equations on the real line

Let $\mathcal{H} := L^2_\omega(\mathbb{R}, \mathbb{C})$ be the universe of our discourse. Consider the following integral equation on the real line

$$\varphi(s) - \int_{-\infty}^{+\infty} \Gamma(t,s)\varphi(t)\omega(t)dt = f(s), \quad s \in \mathbb{R}, \tag{2.1}$$

$$\Gamma(t,s) := k(t,s) + \int_{-\infty}^{+\infty} \frac{k(t,\tau)}{\tau - s} d\tau,$$

where $f$ and $k(.,.)$ are given functions. Let

$$k_t(s) := k(t,s) = k_s(t).$$

We assume that

$$\sup_t \|k_t\|_2 + \left\| k_t^{(r')} \right\|_2 < \infty, \quad \text{for some} \quad r' \geq 0.$$

Hence, following [36], the operator

$$T\varphi(s) := \int_{-\infty}^{+\infty} \Gamma(s,t)\varphi(t)\omega(t)dt, \qquad s \in \mathbb{R},$$

is compact from $\mathcal{H}$ into itself. In [36], the authors propose a simple numerical method to approximate the solution of (2.1), using Lagrange polynomials. In this paper, we use a set of methods to approximate the solution of (2.1), using a sequence of orthogonal finite rank projections based on Hermite polynomials.

Let $X_n$ denote the space spanned by the first $n+1$ of Hermite functions. Let us consider $(\Pi_n)_{n\geq 1}$, the sequence of bounded projections of finite rank defined by

$$\Pi_n x := \sum_{j=0}^{n} \langle x, \psi_j \rangle_\omega \psi_j.$$

Hence,

$$\lim_{n\to\infty} \|\Pi_n \psi - \psi\|_\omega = 0, \qquad \text{for all } \psi \in \mathcal{H}.$$

As proved in [7], $\Pi_n$ is the best approximation associated with the inner product of the space $\mathcal{H}$, (see, also, [43, 20]).

## 2.4 Galerkin projection

Let $T_n^G := \Pi_n T \Pi_n$ denote the Galerkin projection with the corresponding approximate equation

$$\varphi_n^G - T_n^G \varphi_n^G = \Pi_n f, \tag{2.2}$$

the approximate solution $\varphi_n^G$ is given by

$$\varphi_n^G = \sum_{j=0}^{n} x_n(j)\psi_j(s)$$

for some scalars $x_n(j)$. Equation (2.2) reads as

$$\sum_{j=0}^{n} x_n(j) \left[\psi_j - \Pi_n T \psi_j\right] = \Pi_n f.$$

We obtain the linear system

$$\sum_{j=0}^{n} x_n(j) \left[\psi_j - \sum_{i=0}^{n} \langle T\psi_j, \psi_i \rangle_\omega \psi_i\right] = \sum_{i=0}^{n} \langle f, \psi_i \rangle_\omega \psi_i.$$

That is

$$(I - A_n)x_n = b_n,$$

where

$$\begin{aligned} A_n(k,j) &:= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Gamma(s,t)\psi_j(t)\omega(t)\psi_k(s)\omega(s)dtds, \\ b_n(k) &:= \int_{-\infty}^{+\infty} f(s)\psi_k(s)\omega(s)ds. \end{aligned}$$

## 2.5 Kulkarni method

Consider the approximate operator

$$T_n^K := \Pi_n T + T\Pi_n - \Pi_n T \Pi_n,$$

Let $\varphi_n^K$ the approximate solution of equation (2.1) using $T_n^K$.

The approximate equation is

$$\varphi_n^K - T_n^K \varphi_n^K = f. \tag{2.3}$$

Let

$$u_n := \Pi_n \varphi_n^K.$$

Since $\Pi_n u_n = u_n$, there exist scalars $x_n(j)$ such that

$$u_n = \sum_{j=0}^{n} x_n(j)\psi_j,$$

Equation (2.3) reads as

$$u_n - \left[\Pi_n T \Pi_n + \Pi_n T\left(I - \Pi_n\right) T \Pi_n\right] u_n = \Pi_n f + \Pi_n T\left(I - \Pi_n\right) f,$$

so that

$$\sum_{j=0}^{n} x_n(j)\left[\psi_j - (\Pi_n T\psi_j + \Pi_n T(I - \Pi_n) T\psi_j)\right] = \Pi_n f + \Pi_n T(I - \Pi_n) f,$$

and hence

$$\sum_{j=0}^{n} x_n(j)\left[\psi_j - \sum_{k=1}^{n}\left(\langle T\psi_j,\psi_k\rangle_\omega + \langle T(I-\Pi_n)T\psi_j,\psi_k\rangle_\omega\right)\psi_k\right]$$
$$= \sum_{k=1}^{n}\left[\langle f,\psi_k\rangle_\omega + \langle T(I-\Pi_n)f,\psi_k\rangle_\omega\right]\psi_k.$$

Performing the inner product with $\psi_i$, we obtain

$$x_n(i) - \sum_{j=0}^{n} x_n(j)\left[\langle T\psi_j,\psi_i\rangle_\omega + \langle T(I-\Pi_n)T\psi_j,\psi_i\rangle_\omega\right] = \langle f,\psi_i\rangle_\omega + \langle T(I-\Pi_n)f,\psi_i\rangle_\omega,$$

which becomes

$$x_n(i) - \sum_{j=0}^{n}\left[\langle T\psi_j,\psi_i\rangle_\omega + \langle T^2\psi_j,\psi_i\rangle_\omega - \sum_{k=1}^{n}\langle T\psi_j,\psi_k\rangle_\omega\langle T\psi_k,\psi_i\rangle_\omega\right]x_n(j)$$

$$= \langle f,\psi_i\rangle_\omega + \langle Tf,\psi_i\rangle_\omega - \sum_{k=1}^{n}\langle f,\psi_k\rangle_\omega\langle T\psi_k,\psi_i\rangle_\omega, \qquad i=1,\dots,n. \tag{2.4}$$

After solving system (2.4), the solution $\varphi_n^K$ is recovered as

$$\varphi_n^K = u_n + (I-\Pi_n)Tu_n + (I-\Pi_n)f.$$

Therefore

$$\varphi_n^K = u_n + Tu_n - \sum_{k=1}^{n}\langle Tu_n,\psi_k\rangle_\omega\psi_k + f - \sum_{k=1}^{n}\langle f,\psi_k\rangle_\omega\psi_k.$$

## 2.6 Convergence analysis

Since $T$ is compact from $\mathcal{H}$ into itself, the approximate equation (2.3) has a unique solution $\varphi_n$ (see [1, 11]).

Let us consider the weighted Sobolev space

$$H_\omega^m := \left\{\varphi\in\mathcal{H} \quad \|\varphi\|_{H_\omega^m} < +\infty\right\}, \qquad m\geq 0,$$

where

$$\|\varphi\|_{H_\omega^m} := \sum_{j=0}^{m}\|\varphi^{(j)}\|_{L_\omega^2}.$$

In [39], Freud proved the following estimate

$$\|\varphi - \Pi_n\varphi\|_\omega \le cn^{-\frac{r}{2}}\|\varphi\|_{r,\omega} \quad \text{for any} \quad \varphi \in H^r_\omega \quad \text{and} \quad r \ge 0. \tag{2.5}$$

The convergence order of Galerkin projection method is given in the following

**Theorem 2.1** *Assume that $f \in H^r_\omega$. The following estimate holds:*

$$\|\varphi_n^G - \varphi\|_\omega \le \alpha[\|f\|_{r,\omega} + 2\,\|T\|\,\|\varphi\|_{r,\omega}]n^{-\frac{r}{2}}, \quad r \ge 0,$$

*for some positive constant $\alpha$.*

*Proof* : In fact

$$\varphi_n^G - \varphi = (I - T_n^G)^{-1}[(\Pi_n - I)f + (T_n^G - T)\varphi].$$

Since

$$(T_n^G - T)\varphi = \Pi_n T(\Pi_n\varphi - \varphi) + (\Pi_n - I)T\varphi.$$

Following [1], $I - T_n^G$ is invertible and $\left\|(I - T_n^G)^{-1}\right\| < C_0$, we get

$$\varphi_n^G - \varphi = (I - T_n^G)^{-1}[(\Pi_n - I)f + \Pi_n T(\Pi_n\varphi - \varphi) + (\Pi_n - I)T\varphi].$$

Hence

$$\begin{aligned}\|\varphi_n^G - \varphi\|_\omega &= \|(I - T_n^G)^{-1}[(\Pi_n - I)f + \Pi_n T(\Pi_n\varphi - \varphi) + (\Pi_n - I)T\varphi]\|_\omega \\ &\le \|(I - T_n^G)^{-1}\|\ [\|(\Pi_n - I)f\|_\omega + \|\Pi_n\|\ \|T\|\|(\Pi_n\varphi - \varphi)\|_\omega + \|(\Pi_n - I)T\varphi\|_\omega] \\ &\le \|(I - T_n^G)^{-1}\|\ [C_1 n^{-\frac{r}{2}}\|f\|_{r,\omega} + \|T\|C_2 n^{-\frac{r}{2}}\|\varphi\|_{r,\omega} + C_3 n^{-\frac{r}{2}}\|T\|\|\varphi\|_{r,\omega}]\end{aligned}$$

for some positive constants $C_1$, $C_2$, $C_3$.

Letting $\alpha := C_0 \min\{C_1, C_2, C_3\}$ the proof is complete. □

Denote by $\overline{\Gamma_s}$ the conjugate of $\Gamma_s(.,)$. From the same perspective, the convergence order of Sloan projection method is given in the following theorem.

**Theorem 2.2** *Let $r \ge 0$, assume that $f \in H^r_\omega$. The following estimate holds:*

$$\|\varphi - \varphi_n^S\|_\omega \le \beta n^{-r}\|\varphi\|_{r,\omega}\|\overline{\Gamma_s}\|_{r,\omega},$$

*for some positive constant* $\beta$.

*Proof* : Indeed

$$\begin{aligned}\varphi_n^S - \varphi &= (I - T_n^S)^{-1} f - (I-T)^{-1} f \\ &= (I - T_n^S)^{-1}[(I-T) - (I - T_n^S)](I-T)^{-1} f \\ &= (I - T_n^S)^{-1}(T_n^S - T)\varphi.\end{aligned}$$

We note that

$$\sup_n \|(I - T_n^S)^{-1}\| < \infty.$$

Since

$$(T_n^S - T)\varphi = T(\Pi_n - I)\varphi,$$

we get

$$\varphi_n^S - \varphi = (I - T_n^S)^{-1}[T(\Pi_n - I)\varphi].$$

Hence

$$\|\varphi_n^S - \varphi\|_\omega \le \|(I - T_n^S)^{-1}\| \; \|T(\Pi_n - I)\varphi\|_\omega.$$

This leads to

$$\|\varphi - \varphi_n^S\|_\omega \le \beta n^{-r} \|\varphi\|_{r,\omega} \|\overline{\Gamma_s}\|_{r,\omega},$$

where $\beta$ is a finite constant and $\Gamma_s(t) = \Gamma(s,t)$. □

Similarly, the convergence order of Kulkarni method is given in the following theorem.

**Theorem 2.3** *Let* $r \ge 0$*, assume that* $f \in H_\omega^r$*. The following estimate holds:*

$$\|\varphi - \varphi_n^K\|_\omega \le \gamma \, n^{-\frac{3r}{2}} \|\varphi\|_{r,\omega} \|\overline{\Gamma_s}\|_{r,\omega},$$

*for some positive constant* $\gamma$.

*Proof* : Following [56, 81],

$$\varphi_n^K - \varphi = (I - T_n^K)^{-1}(T_n^K - T)\varphi,$$

According to the compactness of $T$, the theory put forward in [11] shows that the inverse operator $(I - T_n^K)^{-1}$ exists and is uniformly bounded for $n$ large enough, which means that

$$\sup_n \|(I - T_n^K)^{-1}\| < \infty.$$

Since

$$\begin{aligned}(T_n^K - T)\varphi &= [\Pi_n T(I - \Pi_n) - T(I - \Pi_n)]\,\varphi \\ &= \Pi_n T(I - \Pi_n)\varphi - T(I - \Pi_n)\varphi \\ &= -(I - \Pi_n)T(I - \Pi_n)\varphi,\end{aligned}$$

we get

$$\varphi_n^K - \varphi = -(I - T_n^K)^{-1}[(I - \Pi_n)T(I - \Pi_n)\varphi],$$

$$\begin{aligned}\|\varphi_n^K - \varphi\|_\omega &= \|(I - T_n^K)^{-1}[(I - \Pi_n)T(I - \Pi_n)\varphi]\|_\omega \\ &\le \|(I - T_n^K)^{-1}\| \, \|[(I - \Pi_n)T(I - \Pi_n)\varphi]\|_\omega.\end{aligned}$$

Using (2.5),

$$\|\varphi_n^K - \varphi\|_\omega \le \|(I - T_n^K)^{-1}\| \; C_4 n^{-\frac{r}{2}} \; \|T(I - \Pi_n)\varphi\|_{r,\omega}$$

for some positive constant $C_4$.

Consequently,

$$\|\varphi - \varphi_n^K\|_\omega \le \gamma\, n^{-\frac{3r}{2}} \|\varphi\|_{r,\omega} \|\overline{\Gamma_s}\|_{r,\omega},$$

which completes the proof. □

In the following, we provide the convergence order of iterated Kulkarni approximation:

$$\tilde{\varphi}_n^K = T\varphi_n^K + f.$$

**Theorem 2.4** *Let* $r \ge 0$*, assume that* $f \in H_\omega^r$*. The following estimate holds:*

$$\|\tilde{\varphi}_n^K - \varphi\|_\omega \le \delta \|(I - T)^{-1}\| \; \left[\|\overline{\Gamma_s}\|_{r,\omega} + \|T\|^2\right] \|\varphi\|_{r,\omega} \|\overline{\Gamma_s}\|_{r,\omega} n^{-2r}$$

*for some positive constant* $\delta$.

*Proof* : In fact

$$\tilde{\varphi}_n^K - \varphi = T(\varphi_n^K - \varphi) = (I-T)^{-1}T[(I-\Pi_n)T(I-\Pi_n)(\varphi_n^K - \varphi + \varphi)].$$

We get

$$\|\tilde{\varphi}_n^K - \varphi\|_\omega \leq \|(I-T)^{-1}\| \, \|T(I-\Pi_n)T(I-\Pi_n)\varphi\|_\omega + \|T(I-\Pi_n)T(I-\Pi_n)(\varphi_n^K - \varphi)\|_\omega].$$

On one hand

$$\|T(I-\Pi_n)T(I-\Pi_n)\varphi\|_\omega \leq \beta^2 n^{-2r}\|\varphi\|_{r,\omega}\|\overline{\Gamma_s}\|_{r,\omega}^2.$$

On the other hand,

$$\begin{aligned}\|T(I-\Pi_n)T(I-\Pi_n)(\varphi_n^K - \varphi)\|_\omega &\leq \|T\|^2\, Cn^{-\frac{r}{2}}\|\varphi_n^K - \varphi\|_\omega \\ &\leq \|T\|^2\, C\gamma\, n^{-2r}\|\varphi\|_{r,\omega}\|\overline{\Gamma_s}\|_{r,\omega},\end{aligned}$$

which leads to

$$\|T(I-\Pi_n)T(I-\Pi_n)(\varphi_n^K - \varphi)\|_\omega \leq C_4\, \|T\|^2\, n^{-2r}\|\varphi\|_{r,\omega}\|\overline{\Gamma_s}\|_{r,\omega}, \quad \text{with} \quad C_4 := C\gamma.$$

Hence

$$\|\tilde{\varphi}_n^K - \varphi\|_\omega \leq \|(I-T)^{-1}\| \, \left[\beta^2 n^{-2r}\|\varphi\|_{r,\omega}\|\overline{\Gamma_s}\|_{r,\omega}^2 + \|T\|^2\, C_4 n^{-2r}\|\varphi\|_{r,\omega}\|\overline{\Gamma_s}\|_{r,\omega}\right],$$

and

$$\|\tilde{\varphi}_n^K - \varphi\|_\omega \leq \|(I-T)^{-1}\| \, \left[\beta^2\|\overline{\Gamma_s}\|_{r,\omega} + \|T\|^2\, C_4\right] \|\varphi\|_{r,\omega}\|\overline{\Gamma_s}\|_{r,\omega} n^{-2r}.$$

Letting $\delta := \min\{\beta^2, C_4\}$ we get the desired result. □

## 2.7 Conclusion

A convergence analysis of Hermite projection methods have been presented to solve integral equations over the real line. New error estimates have been obtained. Our main results of the convergence order have been given for the methods of Galerkin, Sloan, Kulkarni and its iterated version, respectively:

$$\begin{aligned}
\|\varphi_n^G - \varphi\|_\omega &= O(n^{-\frac{r}{2}}),\\
\|\varphi_n^S - \varphi\|_\omega &= O(n^{-r}),\\
\|\varphi_n^K - \varphi\|_\omega &= O(n^{-\frac{3r}{2}}),\\
\|\tilde{\varphi}_n^K - \varphi\|_\omega &= O(n^{-2r}).
\end{aligned}$$

# Chapter 3

# Collocation method to solve second order Cauchy integro-differential equations

This chapter is the subject of the following paper: A. Mennouni, N.E. Ramdani, *Collocation method to solve second order Cauchy integro-differential equations*, in *Differential and Difference Equations with Applications, ICDDEA 2017 proceedings, Editors: Pinelas, S., Caraballo, T., Kloeden, P., Graef, J.R. (Eds.), springer 2018, ISBN 978-3-319-75647-9.*


**abstract**

In this chapter, we present a collocation method for solving the following second-order Cauchy integro-differential equation

$$x''(s) + \oint_{-1}^{1} \frac{\omega(t)x(t)}{s-t}dt = f(s), \quad -1 < s < 1,$$

$$x'(-1) = x(1) = 0,$$

in the space $\mathcal{X} := \mathcal{C}^0([-1,1], \mathbb{C})$, with domain

$$\mathcal{D} := \{x \in \mathcal{X} \ : \ x'' \in \mathcal{X}, \quad x'(-1) = x(1) = 0\}.$$

The integral is a Cauchy principal value, and

$$\omega(s) := \sqrt{\frac{1+s}{1-s}}$$

is the weight function.
We come up with a modified collocation method to build an approximate solution $x_n$ using the airfoil polynomials of the first kind.
Finally, we establish a numerical example to exhibit the theoretical results.

## 3.1 Introduction and mathematical background

Integro-differential equations appear in many applications in scientific fields such as biological, physical, and engineering problems. In [24], the authors have presented a high-order methods for the numerical solution of Volterra integro-differential equations. In [25], the authors have derived m-stage Runge-Kutta-Nystrom methods for the numerical solution of general second-order Volterra integro-differential equations. These implicit methods are based on collocation techniques in certain polynomial spline spaces. The modified trapezoidal method adapted for general second order initial value problems has been being given in [29]. In [40], the authors have presented a direct methods for a class of second order Volterra integro-differential equations which explicitly contain a first order derivative. In [81], the author has studied and presented a projection method for solving operator equations with bounded operator in Hilbert spaces. In [79], the author has introduced a projection method based on the Legendre polynomials, for solving integro-differential equations with Cauchy kernel. In [80], the author has studied a collocation method, for approximate solution of an integro-differential equations with logarithmic kernel, using airfoil polynomials. The goal of this study is to present a collocation method for solving second order integro-differential equations, using airfoil polynomials.

Let $L^2([-1,1],\mathbb{C})$, be the space of complex-valued Lebesgue square integrable (classes of) functions on $[-1,1]$.

We recall that the so-called airfoil polynomials are used as expansion functions to compute the pressure on an airfoil in steady or unsteady subsonic flow.

The airfoil polynomial $t_n$ of the first kind is defined by

$$t_n(x) = \frac{\cos[(n+\frac{1}{2})\arccos x]}{\cos(\frac{1}{2}\arccos x)}.$$

The airfoil polynomial $u_n$ of the second kind is defined by

$$u_n(x) = \frac{\sin[(n+\frac{1}{2})\arccos x]}{\sin(\frac{1}{2}\arccos x)}.$$

## 3.2 Approximate solution

Consider the following second order Fredholm integro-differential equation with Cauchy kernel:

$$\varphi''(s) + \oint_{-1}^{1} \frac{\omega(t)\varphi(t)}{t-s}dt = f(s), \quad -1 < s < 1. \tag{3.1}$$

$$\varphi'(-1) = \varphi(1) = 0,$$

with the domain

$$\mathcal{D} := \{\varphi \in \mathcal{X} \ : \ \varphi'' \in \mathcal{X}, \quad \varphi'(-1) = \varphi(1) = 0\}.$$

Letting

$$S\varphi(s) := \varphi''(s), \quad T\varphi(s) := \oint_{-1}^{1} \frac{\omega(t)\varphi(t)}{t-s}dt.$$

The following two formulas (cf. [37])

$$(1+s)t_i'(s) = (i+\frac{1}{2})u_i(s) - \frac{1}{2}t_i(s),$$

$$(1-s^2)t_i''(s) + (1-2s)t_i'(s) + n(n-1)t_i(s) = 0$$

give

$$t_i''(s) = \frac{(2s-1)(n+1/2)}{(1-s^2)(1+s)}u_i(s) - \frac{(2s-1)+2n(n-1)}{2(1-s^2)}. \tag{3.2}$$

We recall that (cf. [37]),

$$\oint_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\frac{t_i(t)}{t-s}dt = \pi u_i(s). \tag{3.3}$$

Our goal is to approximate the solution of equation (3.1) via the airfoil polynomials of the first kind $t_n$ as

$$\varphi_n(s) = \sum_{i=0}^{n} a_i t_i(s).$$

Consider the set of $n+1$ collocation points $s_j$, which are the zeros of $u_{n+1}$:

$$s_j = -\cos\frac{2j-1}{2n+3}\pi, \quad j = 0, 1, \ldots, n.$$

Letting

$$(V_1 y)(s) = \int_{-1}^{s} y(t)dt,$$

$$(V_2 \psi)(s) = \int_{s}^{1} \psi(t)dt.$$

We recall that $V_1, V_2 : \mathcal{H} \to \mathcal{D}$ are compact.

Moreover,

For all $\varphi \in \mathcal{D}$,

$$(V_2 V_1 S)\varphi = -\varphi.$$

Consider the space $C^{0,\lambda}[-1,1]$ of all functions $\varphi$ defined on $[-1,1]$ satisfying the following Hölder condition: $\exists M \geq 0$ such that

$$\forall s_1, s_2 \in [-1,1], \quad |\varphi(s_1) - \varphi(s_2)| \leq M\, |s_1 - s_2|^{\lambda},$$

where $0 < \lambda \leq 1$.

Let

$$\mathcal{H} := \left\{ \varphi \in L^2[-1,1] \; : \; \varphi'' \in L^2([-1,1]), \quad \varphi'(-1) = \varphi(1) = 0 \right\}.$$

Note that the operator $T$ is bounded from $L^2[-1,1]$ into itself and also from $C^{0,\lambda}[-1,1]$ into itself.

Consider hat functions $e_0, e_1, e_2, \ldots, e_n$ in $C^0[-1,1]$ such that

$$e_j(x_k) = \delta_{j,k}.$$

Define the projection operators $\pi_n$ from $C^0[-1,1]$ into the space of continuous functions by

$$\pi_n g(x) := \sum_{j=0}^{n} g(x_j) e_j(x).$$

Let us define the operators

$$V_n := V_2 V_1 \pi_n T, \quad V := V_2 V_1 T.$$

Consider the following approximate equation in the unknown $\varphi_n$:

$$-\varphi_n + V_n \varphi_n = V_2 V_1 f.$$

**Theorem 3.1** *Assume that $f \in C^0[-1,1]$. There exists a positive constant $\alpha$, such that*

$$\|\varphi - \varphi_n\|_\infty \leq \alpha\| (V_n - V)\, \varphi\|_\infty$$

*for $n$ large enough.*

*Proof* : It is well-known that $\|\pi_n x - x\|_\infty \to 0$, for all $x \in C^0[-1,1]$. Since $V_2 V_1$ is compact, it is clear that $V$ is compact. In (cf. [11] and [55]) it is shown that the inverse operator $(-I + V_n)^{-1}$ exists and is uniformly bounded for $n$ large enough. On the other hand,

$$\varphi_n - \varphi = [V_n \varphi_n - V_2 V_1 f] - [V\varphi - V_2 V_1 f]\,,$$

hence

$$\varphi_n - \varphi = [V_n \varphi_n - V\varphi]\,.$$

This leads to

$$\varphi_n - \varphi = [(V_n - V)\varphi + V_n(\varphi_n - \varphi)]\,.$$

Thus

$$(-I + V_n)(\varphi - \varphi_n) = (V_n - V)\varphi.$$

Consequently

$$\begin{aligned} \varphi - \varphi_n &= (-I + V_n)^{-1} [(V_n - V)\varphi]\,, \\ \|\varphi_n - \varphi\|_\infty &\leq \alpha\|(V_n - V)\varphi\|_\infty, \end{aligned}$$

where

$$\alpha := \sup_{n \geq N} \left\|(-I + V_n)^{-1}\right\|,$$

which is finite. $\square$

Thus, we obtain the following system:

$$S\varphi_n(s_j) + T\varphi_n(s_j) = f(s_j), \quad j = 0, 1, \ldots, n.$$

By (3.2) and (3.3), we get for $j = 0, 1, \ldots, n$

$$\sum_{i=0}^{n} a_i \left\{ \left[ \frac{(2s_j - 1)(i + 1/2)}{(1 - s_j^2)(1 + s_j)} - \pi \right] u_i(s_j) - \frac{(2s_j - 1) + 2i(i-1)}{2(1 - s_j^2)} t_i(s_j) \right\} = f(s_j).$$

## 3.3 Numerical results

Let us consider the integro-differential equation (3.1), with the following exact solution

$$\varphi(x) = x^3 - 3x + 2.$$

Table (3.1) gives the numerical results for Example 1.

| $x$ | n=6 | n=22 | n=120 |
|---|---|---|---|
| -0.8 | 0.133e-1 | 0.154e-2 | 0.142e-3 |
| -0.6 | 0.172e-1 | 0.179e-2 | 0.147e-3 |
| -0.4 | 0.124e-1 | 0.163e-2 | 0.241e-4 |
| -0.2 | 0.321e-1 | 0.165e-2 | 0.134e-3 |
| 0.0 | 0.156e-1 | 0.187e-2 | 0.201e-4 |
| 0.2 | 0.179e-1 | 0.177e-2 | 0.443e-4 |
| 0.4 | 0.195e-1 | 0.165e-2 | 0.781e-4 |
| 0.6 | 0.541e-1 | 0.167e-2 | 0.795e-4 |
| 0.8 | 0.325e-1 | 0.167e-2 | 0.807e-4 |

Table 3.1: *Example 1*

# Chapter 4

# Numerical solution of generalized logarithmic integral equations of the second kind by projections

This chapter is the subject of an article submitted to the Malaysian Journal of Mathematical Sciences (MJMS).

**Abstract**

In this work, we present new techniques to solve integral equations of the second kind with logarithmic kernel. First, we show the existence and uniqueness of the solution for the problem in Hilbert spaces. Next, we discuss a projection method for solving integral equations with logarithmic kernel of the second kind; the present method based on shifted Legendre polynomials. We examine the existence of the solution for the approximate equation, and we provide a new error estimate for the numerical solutions. At the end, numerical examples are provided to illustrate the theoretical results.

## 4.1 Introduction

In past years, several problems of mathematics, engineering, physics and related sciences are described in terms of integral equations, specifically, singular integral equations ( cf. [59]). An important class of these equations is the singular integral equations with logarithmic kernel. Projection approximation methods play an important role in numerical analysis, especially the projection method is an effective means of numerical solution of integral equations ( cf.[11, 55]). [81] presented a projection method for solving operator

equations with bounded operator in Hilbert spaces, and he applied the method for solving the Cauchy integral equations for two cases: Galerkin projections and Kulkarni projections respectively, using a sequence of orthogonal finite rank projections. [78] introduced a modified method which is based on the trapezoidal and Simpson's rules, for solving a Volterra integral equations of the second kind. [82] discussed a projection method for solving Cauchy integro-differential equations via airfoil polynomials of the first kind. [99] discussed the solutions of boundary integral equations and systems of logarithmic integral equations of the first kind. [19] solved a logarithmic singular integral equation in two disjoint finite intervals by using function theoretic method, under some conditions. [27] considered a direct method to solve a singular integral equations of the first kind, involving the combination of a logarithmic and a Cauchy type singularity. [23] solved an integral equation of the first kind with logarithmic kernel as an improperly posed problem. [86] presented a closed form solutions for an important class of difference singular integral equations of the first kind. He considered the kernel as the sum of a polynomials, the second one is multiplied by a logarithm. [66] introduced a method to solve a singular integral equations with logarithmic or Cauchy kernels. [106] investigated the analytical properties of a logarithmic singular integral equation. He developed two product integration methods to solve this class of integral equation, the first one based on Euler's method, but in the second one the author used a product trapezoidal method. [88] considered the Galerkin method for singular Fredholm integral equations of the second kind with weakly kernel and its corresponding eigenvalue problem on $[-1, 1]$. More recently, [10] described a collocation method using the radial basis function to numerically solve a boundary singular integral equations of the second kind with logarithmic kernels. These class of integral equations obtained from boundary value problems of Laplace equations with linear Robin boundary conditions. Recently many researchers developed numerical methods that solve integral equations of the first kind with logarithmic kernel via collocation and Galerkin methods, see [30, 31], [47, 48], [101, 102] and references therein. Based on the above, integral equations with logarithmic kernel are an important type of singular integral equations. This class of singular integral equations has an important applications in several problems of economics, fluid dynamics, electrodynamics, elasticity, fracture mechanics, biology and other scientific fields and the latest high technology.
The goal of this chapter is to introduce a shifted Legendre projection method to solve generalized integral equations with logarithmic kernel of the second kind. We use new techniques to show the existence and uniqueness of the solution for the present problem in Hilbert spaces and the solution of its corresponding approximate problem.

## 4.2 Existence and uniqueness of solutions

Let $H$ be a Hilbert space. $BL(H)$ will denote the space of bounded linear operators from $H$ into itself, and $sp$ denotes the spectrum.
Let $T \in BL(H)$, and let $T^*$ be the adjoint of $T$. We recall that $T$ is selfadjoint if $T^* = T$, and that $T$ is skew-Hermitian if $T^* = -T$.

**Lemma 4.1** *Let $T \in BL(H)$.*

1. *If $T$ is self-adjoint, then $sp\ (T) \subseteq \mathbb{R}$.*
2. *If $T$ is skew-Hermitian, then $sp\ (T) \subseteq \mathrm{i}\mathbb{R}$.*

*Proof* :

1. See [94].
2. It is clear that

$$(\mathrm{i}T)^* = -\mathrm{i}T^* = \mathrm{i}T,$$

so that the operator $\mathrm{i}T$ is self-adjoint, say $sp\ (\mathrm{i}T) \subseteq \mathbb{R}$, hence $sp\ (T) \subseteq \mathrm{i}\mathbb{R}$.

$\square$

Throughout our paper, denote by $\mathcal{H} := L^2([0,1], \mathbb{C})$. Let us consider the generalized integral equation with logarithmic kernel

$$\int_0^1 h(s,\varsigma) \ln|\varsigma - s|\, \varphi(\varsigma) d\varsigma = \lambda\varphi(s) + f(s), \quad 0 \leq s \leq 1. \tag{4.1}$$

We examine the numerical solution of this equation. Our discussion will be in two important cases.

### 4.2.1 Case $\overline{h(s,\varsigma)} = -h(\varsigma, s)$

In the first case, we assume that $\lambda$ is real and non-zero, and $h(.,.)$ is continuous function, moreover,

$$\overline{h(s,\varsigma)} = -h(\varsigma, s).$$

Letting

$$Su(s) \quad := \quad \int_0^1 h(s,\varsigma) \ln|\varsigma - s|\, u(\varsigma) d\varsigma. \quad u \in \mathcal{H}, \quad 0 \leq s \leq 1.$$

We recall that $S \in BL(\mathcal{H})$, further $S^* = -S$.

Equation (4.1) is equivalent to

$$(S - \lambda I)\varphi = f.$$

**Theorem 4.1** *For all $f \in \mathcal{H}$ , the logarithmic integral equation (4.1) has a unique solution $\varphi \in \mathcal{H}$.*

*Proof* : Since

$$\overline{h(s,\varsigma)} = -h(\varsigma, s),$$

it follows that $S$ is skew-Hermitian operator, and by Lemma 4.1, we get $sp\ (S) \subseteq \mathrm{i}\mathbb{R}$.

This shows that $\lambda \notin sp\ (S)$, consequently the operator $S - \lambda I$ is invertible. □

**Theorem 4.2** *The following estimate holds:*

$$\|(S - \lambda I)^{-1}\| \leq \frac{1}{|\lambda|}.$$

*Proof* : As in [94], for all $u \in \mathcal{H}$,

$$\begin{aligned} Re\,\langle (S - \lambda I)\, u, u\rangle &= \frac{1}{2}\left[\langle (S - \lambda I)\, u, u\rangle + \overline{\langle (S - \lambda I)\, u, u\rangle}\right] \\ &= \frac{1}{2}\left[-2\lambda\,\langle u, u\rangle + \langle Su, u\rangle + \langle u, Su\rangle\right] \\ &= -\lambda\,\langle u, u\rangle\,, \end{aligned}$$

so that

$$|\lambda|\,\|u\|^2 = |\mathrm{Re}\,\langle (S - \lambda I)u, u\rangle| \leq |\langle (S - \lambda I)u, u\rangle| \leq \|(S - \lambda I)u\|\,\|u\|\,,$$

which yields

$$\|(S-\lambda I)^{-1}\| \leq \frac{1}{|\lambda|}.$$

$\square$

### 4.2.2 Case $\overline{h(s,\varsigma)} = h(\varsigma, s)$

In this case, we assume that $\lambda \in \mathrm{i}\mathbb{R}^*$ and $h(.,.)$ satisfies

$$\overline{h(s,\varsigma)} = h(\varsigma, s),$$

hence

$$S^* = S.$$

**Theorem 4.3** *For all $f \in \mathcal{H}$ , the logarithmic integral equation (4.1) has a unique solution $\varphi \in \mathcal{H}$.*

*Proof* : Since

$$\overline{h(s,\varsigma)} = h(\varsigma, s),$$

we deduce that $S$ is selfadjoint operator. Hence

$$sp\ (S) \subseteq \mathbb{R},$$

and hence $\lambda \notin sp\ (S)$, which proves that the operator $S - \lambda I$ is invertible. $\square$

**Theorem 4.4** *The following estimate holds:*

$$\|(S-\lambda I)^{-1}\| \leq \frac{1}{|\mathrm{Im}(\lambda)|}.$$

*Proof* : As in [94], for all $u \in \mathcal{H}$, we have

$$\begin{aligned}
|\langle (S-\lambda I)\, u, u\rangle|^2 &= \left|\langle Su, u\rangle - \lambda \|u\|^2\right|^2 \\
&= \left|\langle S, u\rangle - \mathrm{Re}(\lambda)\, \|u\|^2\right|^2 + |\mathrm{Im}(\lambda)|^2 \|u\|^4 \\
&\geq |\mathrm{Im}(\lambda)|^2 \|u\|^4,
\end{aligned}$$

so that

$$|\mathrm{Im}(\lambda)|\,\|u\|^2 \leq |\langle (S-\lambda I)\,u,u\rangle|\,,$$

which yields

$$\|(S-\lambda I)^{-1}\| \leq \frac{1}{|\mathrm{Im}(\lambda)|}.$$

□

## 4.3 Bounded finite rank orthogonal projections

Let $(L_n)_{n\geq 0}$ denote the sequence of Legendre polynomials. We recall that the Legendre polynomials $L_n(.)$ can be defined by

$$L_n(s) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}^2 \left(\frac{1+s}{2}\right)^{n-k} \left(\frac{1-s}{2}\right)^k.$$

Their generating function is given by

$$\frac{1}{\sqrt{1-2st+t^2}} = \sum_{n=0}^{\infty} L_n(s)t^n.$$

Note that the first few Legendre polynomials are

$$\begin{aligned}
L_0(s) &:= 1;\\
L_1(s) &:= s;\\
L_2(s) &:= \frac{1}{2}\left(3s^2-1\right);\\
L_3(s) &:= \frac{1}{2}\left(5s^3-3s\right);\\
L_4(s) &:= \frac{1}{8}\left(35s^4-30s^2+3\right).
\end{aligned}$$

The Legendre polynomials of higher degrees are often found by employing a three-term recurence relation (see [91]).
Let us consider the following shifted Legendre polynomials

$$\ell_n(s) := L_n(2s-1), \quad 0 \leq s \leq 1.$$

Recall that the shifted Legendre polynomials are orthogonal on $[0,1]$.

An explicit formula for the shifted Legendre polynomials is given by

$$\ell_n(s) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-s)^k.$$

The first few shifted Legendre polynomials are:

$$\begin{aligned}
\ell_0(s) &:= 1; \\
\ell_1(s) &:= 2s - 1; \\
\ell_2(s) &:= 6s^2 - 6s + 1; \\
\ell_3(s) &:= 20s^3 - 30s^2 + 12s - 1; \\
\ell_4(s) &:= 70s^4 - 140s^3 + 90s^2 - 20s + 1.
\end{aligned}$$

Letting

$$e_j(t) := \frac{1}{\sqrt{2j+1}} \ell_j(t), \quad 0 \leq t \leq 1.$$

We define the space $\mathcal{H}_n$ spanned by $\{e_j, \quad j = 0 \ldots n\}$. We associate to $\mathcal{H}_n$ the sequence $(\pi_n)_{n\geq 0}$ of bounded finite rank orthogonal projections onto $\mathcal{H}_n$ given by

$$\pi_n u := \sum_{j=0}^{n} \langle u, e_j \rangle \, e_j.$$

We recall that, for all $\psi \in \mathcal{H}$,

$$\lim_{n\to\infty} \|\pi_n \psi - \psi\| = 0.$$

We consider the following approximate operator $S_n := \pi_n S \pi_n$.

### 4.3.1 Case $\overline{h(s,\varsigma)} = -h(\varsigma, s)$

Since $S$ is skew-Hermitian operator, it follows that $(S_n)_{n\geq 1}$ is a sequence of skew-Hermitian operator from $\mathcal{H}$ into itself.

**Theorem 4.5** *We assume that $\lambda \in \mathbb{R}^*$. For all $n$, the operator $S_n - \lambda I$ is invertible, and*

$$\| \left(S_n - \lambda I\right)^{-1} \| \leq \frac{1}{|\lambda|}.$$

*Proof* : The proof is similar to the proof of Theorem 4.2. Obviously that $\lambda \notin sp\ (S_n)$ (cf. Lemma 4.1), and hence the operator $S_n - \lambda I$ is invertible. We easily obtain

$$Re \langle (S_n - \lambda I)\, u, u\rangle \quad = \quad -\lambda \langle u, u\rangle\,,$$

so that

$$|\lambda| \langle u, u\rangle \leq |\langle (S_n - \lambda I)\, u, u\rangle| \leq \|(S_n - \lambda I)\, u\| \, \|u\|\,,$$

that is to say

$$\|(S_n - \lambda I)\, u\| \geq |\lambda| \, \|u\|\,.$$

Clearly, we have the bound

$$\|\, (S_n - \lambda I)^{-1} \,\| \leq \frac{1}{|\lambda|}.$$

□

### 4.3.2 Case $\overline{h(s,\varsigma)} = h(\varsigma, s)$

Since $S$ is selfadjoint operator, it follows that $(S_n)_{n\geq 1}$ is a sequences of selfadjoint operators from $\mathcal{H}$ into itself.

**Theorem 4.6** *We assume that $\lambda \in \mathrm{i}\mathbb{R}^*$. For all $n$, the operator $S_n - \lambda I$ is invertible, moreover*

$$\|\, (S_n - \lambda I)^{-1} \,\| \leq \frac{1}{|\mathrm{Im}(\lambda)|}.$$

*Proof* : The proof is similar to the proof of Theorem 4.4. As before, for all $u \in \mathcal{H}$, we have

$$\begin{aligned} |\langle (S_n - \lambda I)\, u, u\rangle|^2 &= \left|\langle S_n u, u\rangle - \lambda \|u\|^2\right|^2 \\ &= \left|\langle S_n u, u\rangle - \mathrm{Re}(\lambda) \|u\|^2\right|^2 + |\mathrm{Im}(\lambda)|^2 \|u\|^4 \\ &\geq |\mathrm{Im}(\lambda)|^2 \|u\|^4\,, \end{aligned}$$

hence

$$|\mathrm{Im}(\lambda)| \, \|u\|^2 \leq |\langle (S_n - \lambda I)\, u, u\rangle|\,.$$

This leads to

$$\|(S_n - \lambda I)^{-1}\| \leq \frac{1}{|\mathrm{Im}(\lambda)|}.$$

□

## 4.4 Approximate solutions

The approximate problem is the following equation for $\varphi_n$:

$$S_n\varphi_n - \lambda\varphi_n = \pi_n f, \tag{4.2}$$

For all $n$, the approximate equation (4.2) has a unique solution $\varphi_n$, given by

$$\varphi_n = \sum_{j=0}^{n} c_j e_j,$$

for some scalars $c_j$. Equation (4.2) reads as

$$\sum_{j=0}^{n} c_j \left[\pi_n S e_j - \lambda e_j\right] = \pi_n f,$$

so that

$$\sum_{j=0}^{n} c_j \left[\sum_{i=0}^{n} \langle S e_j, e_i\rangle e_i - \lambda e_j\right] = \sum_{i=0}^{n} \langle f, e_i\rangle e_i,$$

that is to say, the coefficients $c_j$ are obtained by solving the following linear system

$$(A_n - \lambda I)x_n = b_n, \tag{4.3}$$

where

$$\begin{aligned} A_n(k,j) &:= \int_0^1\int_0^1 h(s,\varsigma)\ln|\varsigma - s|\, e_j(\varsigma)e_k(s)d\varsigma ds,\\ b_n(k) &:= \int_0^1 e_k(s)f(s)ds. \end{aligned}$$

As in [11, 38], we introduce a quadrature rule for computing $A_n(k,j)$.

Let us consider the following change of variables

$$\tau := \varsigma - s, \quad \upsilon := \varsigma + s,$$

that is

$$\varsigma = \frac{\tau + \upsilon}{2}, \quad s = \frac{-\tau + \upsilon}{2}.$$

Letting

$$V(\tau, \upsilon) := \frac{1}{2} h(\frac{-\tau + \upsilon}{2}, \frac{\tau + \upsilon}{2}) \ln |\tau| \, e_j(\frac{\tau + \upsilon}{2}) e_k(\frac{-\tau + \upsilon}{2}),$$

$$\alpha(\tau) := \max(-\tau, \tau), \quad \beta(\tau) := \min(2 - \tau, 2 + \tau),$$

and

$$\vartheta(\tau) := \int_{\alpha(\tau)}^{\beta(\tau)} V(\tau, \upsilon) d\upsilon,$$

we get

$$A_n(k, j) := \int_{-1}^{1} \vartheta(\tau) d\tau.$$

Consider the following partition points of the interval $[-1, 1]$:

$$-1 + s_j, \quad -s_j, \quad s_j, \quad 1 - s_j, \quad j = 0, 1, \ldots, m.$$

with

$$s_j = \frac{1}{2} (\frac{j}{m})^q, \quad j = 0, 1, \ldots, m.$$

Hence

$$A_n(k, j) = \sum_{i=1}^{4m} \int_{\tau_{i-1}}^{\tau_i} \vartheta(\tau) d\tau,$$

where $\tau_0, \tau_1, \ldots \tau_{4m}$ are partition points in ascending order.

Letting

$$G(t) := \sum_{i=1}^{4m} \frac{\tau_i - \tau_{i-1}}{2} \vartheta(\frac{\tau_i - \tau_{i-1}}{2} t + \frac{\tau_i - \tau_{i-1}}{2}),$$

we obtain

$$A_n(k, j) = \int_0^1 G(t) dt.$$

In order to compute $b_n(k)$ accurately and effectively, we use the Legendre Gauss Lobatto quadrature (cf. [26], pp. 331).

### 4.4.1 Convergence Analysis

Let $\rho > 0$ and $H^\rho(0, 1)$ be the classical Sobolev space, and let $\|.\|_\rho$ denote its norm. (Cf. [13], pp. 119.)

As in [13] there exists $c > 0$ such that, for all $x \in H^\rho([0,1],\mathbb{C})$,

$$\|(I-\pi_n)x\| \leq cn^{-\rho}\|x\|_\rho. \tag{4.4}$$

**Theorem 4.7** *Assume that $f \in H^\rho([0,1],\mathbb{C})$, and*

$$\overline{h(s,\varsigma)} = -h(\varsigma,s), \quad \lambda \in \mathrm{i}\mathbb{R}^*.$$

*There exists a positive constant $c$, such that:*

$$\|\varphi_n - \varphi\| \leq \frac{cn^{-\rho}}{|\lambda|}\left[(1+2\|S\|)\|\varphi\|_\rho + \|f\|_\rho\right],$$

*for $n$ large enough.*

*Proof* : We have

$$\begin{aligned}
\varphi_n - \varphi &= (S_n - \lambda I)^{-1}\pi_n f - (S-\lambda I)^{-1} f \\
&= (S_n - \lambda I)^{-1}\pi_n f - (S_n-\lambda I)^{-1} f + (S_n - \lambda I)^{-1} f - (S-\lambda I)^{-1} f \\
&= (S_n - \lambda I)^{-1}(\pi_n - I) f + (S_n - \lambda I)^{-1}\left[(S-\lambda I) - (S_n - \lambda I)\right](S-\lambda I)^{-1} f \\
&= (S_n - \lambda I)^{-1}\left[(\pi_n - I)f + (S-S_n)\varphi\right].
\end{aligned}$$

But

$$\left\|(S_n - \lambda I)^{-1}\right\| \leq \frac{1}{|\lambda|},$$

and

$$(S-S_n)\varphi = (I-\pi_n)S\varphi + \pi_n S(\pi_n - I)\varphi,$$

and since $\|\pi_n\| = 1$, then using (4.4), we get the desired result. □

**Theorem 4.8** *Assume that $f \in H^\rho([0,1],\mathbb{C})$, and*

$$\overline{h(s,\varsigma)} = h(\varsigma,s).$$

*There exists a positive constant $c$, such that:*

$$\|\varphi_n - \varphi\| \leq \frac{cn^{-\rho}}{|\mathrm{Im}(\lambda)|}\left[(1+2\|S\|)\|\varphi\|_\rho + \|f\|_\rho\right],$$

*for $n$ large enough.*

*Proof* : Proceed in the similar manner as above, and using

$$\left\|(S_n - \lambda I)^{-1}\right\| \leq \frac{1}{|\mathrm{Im}(\lambda)|},$$

we get the desired result. □

### 4.4.2 Classical Fredholm integral equation with logarithmic kernel of the second kind

In this section, we turn our attention to the following classical Fredholm integral equation with logarithmic kernel

$$\int_0^1 \ln|\varsigma - s|\,\varphi(\varsigma)d\varsigma = \lambda\varphi(s) + f(s), \quad 0 \leq s \leq 1.$$

We assume that this equation has unique solution in $\mathcal{H}$.

Letting

$$Ku(s) \quad := \quad \int_0^1 \ln|\varsigma - s|\,u(\varsigma)d\varsigma, \quad u \in \mathcal{H}, \quad 0 \leq s \leq 1,$$

$$K_n := \pi_n K \pi_n.$$

We recall that $K$ is compact from $\mathcal{H}$ into itself, (see [11] pp. 8), further, $\|K\| = 1 + \ln 2$, (see [57], pp. 228). It is shown that the inverse operator $(I - K_n)^{-1}$ exists and is uniformly bounded for $n$ large enough, (see [11] pp. 55).

As above, there exists a positive constant $c$, such that:

$$\|\varphi_n - \varphi\| \leq cn^{-\rho}\left[(3 + 2\ln 2)\|\varphi\|_\rho + \|f\|_\rho\right],$$

for $n$ large enough.

## 4.5 Numerical Computations

In this section, we present some numerical examples to illustrate the theoretical results obtained in the previous sections. For computations we use Gauss quadrature rule to solve the linear system. The errors of the projection method are presented for different kernels, and for several values of $n$. In examples 4 and 5, we compare the present results with previous results presented in many works related logarithmic singular integral equations. We use the suggested method to solve the particular integral equations in all examples. We evaluate $A_n(k, j)$ and $b_n(k)$. Once the above matrix equation (4.3) is solved, we find $x_n := [c_j, \quad j = 0, \cdots, n]$, hence the solution $\varphi_n$ is built through

$$\varphi_n(t) = \sum_{j=0}^{n} \frac{c_j}{\sqrt{2j+1}} \ell_j(t), \quad 0 \leq t \leq 1.$$

The main advantages of the present method are that we give a new theoretical framework for the logarithmic singular Fredholm integral equations by projections, the method is applicable even for the particular cases mentioned above with a better accuracy and we obtain new error estimate, which is small in comparison with other methods. However, it may be difficult to use the present method for solving nonlinear integral equations.

## Example 1

Let us first consider the generalized integral equation with logarithmic kernel (4.1) where $f$ is chosen such that the exact solution is

$$\varphi(s) = s(s-1),$$

and

$$h(s, \varsigma) = s^2 - \varsigma^2, \quad \lambda = 2.$$

It is clear that

$$h(\varsigma, s) = -h(s, \varsigma).$$

Table 4.1 shows the rate of convergence of the method.

| $n$ | $\|\varphi - \varphi_n\|$ |
|---|---|
| 3 | $4.266 \times 10^{-3}$ |
| 4 | $8.575 \times 10^{-4}$ |
| 5 | $1.125 \times 10^{-4}$ |
| 6 | $4.751 \times 10^{-6}$ |

Table 4.1: *Absolute errors for Example 1*

## Example 2

Let us consider the generalized integral equation with logarithmic kernel (4.1) where $f$ is chosen such that the exact solution is

$$\varphi(s) = s^2 - s + 1,$$

and

$$h(s, \varsigma) = (s-1)(\varsigma - 1), \quad \lambda = \mathrm{i}.$$

It is clear that $\overline{h(s, \varsigma)} = h(\varsigma, s)$.

| $n$ | $\|\varphi - \varphi_n\|$ |
|---|---|
| 3 | $1.373 \times 10^{-5}$ |
| 4 | $3.735 \times 10^{-6}$ |
| 5 | $5.211 \times 10^{-7}$ |
| 5 | $6.283 \times 10^{-7}$ |
| 6 | $8.305 \times 10^{-8}$ |

Table 4.2: *Absolute errors for Example 2*

We present in Table 4.2 the corresponding absolute errors for the example 2.

## Example 3

Here, we consider the following integral equation with logarithmic kernel

$$u(s) - \int_0^1 \ln|s-t|\, u(\varsigma) d\varsigma = (1 - \ln(s)e^s + \mathrm{Ei}\ (s) + e^{s-1}\ln(1-s) - \mathrm{Ei}\ (x-1))e^{-s}.$$

with the exact solution

$$u(s) = e^{-s}.$$

We note that $\mathrm{Ei}\ (.)$ is the exponential integral function, which is defined as:

$$\mathrm{Ei}\ (s) := \int_{-s}^{\infty} \frac{e^{-t}}{t} dt.$$

| $n$ | $\|\varphi - \varphi_n\|_2$ |
|---|---|
| 3 | $1.94539 \times 10^{-4}$ |
| 4 | $1.22031 \times 10^{-4}$ |
| 5 | $6.32020 \times 10^{-5}$ |
| 6 | $1.47377 \times 10^{-5}$ |
| 7 | $8.57667 \times 10^{-6}$ |

Table 4.3: *Absolute errors for Example 3*

The corresponding absolute errors for the example 3 are presented in Table 4.3.

## Example 4: (Cf. [18] pp. 536)

In [18], the author has used the modified quadrature method and the repeated Simpson's rule with step $h := \frac{1}{n}$ to approximate the solution of the following Fredholm integral equation with logarithmic kernel of the second kind

$$u(s) - \int_0^1 \ln|s-t|\, u(\varsigma) d\varsigma = \frac{3}{2}s - \frac{1}{2}\ln(s)s^2 + \frac{1}{2}\ln(-s+1)s^2 + \frac{1}{4} - \frac{1}{2}\ln(-s+1).$$

with the exact solution

$$u(s) = s.$$

As in [18], we use a computer working to 10 decimal digits. We compare his results with

| $s$ | $\lvert\varphi(s) - \varphi_n(s)\rvert$ in [18] | $\lvert\varphi(s) - \varphi_n(s)\rvert$ for the present method |
|---|---|---|
| 0 | $3.6 \times 10^{-3}$ | $4.9 \times 10^{-10}$ |
| 0.25 | $7.5 \times 10^{-5}$ | $4. \times 10^{-10}$ |
| 0.5 | $3.6 \times 10^{-12}$ | $3.1 \times 10^{-13}$ |
| 0.75 | $7.5 \times 10^{-5}$ | $0.$ |
| 1 | $3.6 \times 10^{-3}$ | $1.4 \times 10^{-10}$ |

Table 4.4: *Absolute errors for Example 4*

our results for $n = 4$ ( see Table (4.4).

## Example 5: (Cf. [11] pp. 117)

The author of [11] has introduced the Nyström method to numerically solve the following integral equation with logarithmic kernel of the second kind

$$u(s) - \int_0^1 \ln|s-t|\, u(\varsigma)d\varsigma = e^s + \ln(s) - e^s\mathrm{Ei}\,(-s) - e^1\ln(1-s) + e^s\mathrm{Ei}\,(1-s),$$

with the exact solution

$$u(s) = e^s.$$

Numerical results of the present method are given in Table 4.5. In [11], the corresponding uniform norm of the error is $1.16 \times 10^{-3}$, for $n = 10$. However, only at $n = 6$, the corresponding uniform norm of error by the present method is $2.1 \times 10^{-5}$.

| $n$ | $\|\varphi - \varphi_n\|_2$ |
|---|---|
| 2 | $6.29377 \times 10^{-3}$ |
| 3 | $5.28887 \times 10^{-3}$ |
| 4 | $3.31887 \times 10^{-4}$ |
| 5 | $1.68863 \times 10^{-5}$ |
| 6 | $5.00531 \times 10^{-6}$ |

Table 4.5: *Absolute errors for Example 5*

## 4.6 Conclusion

In this paper, we have proposed a projection method to numerically solve generalized Fredholm integral equation with logarithmic kernel of the second kind. For our analysis we used new techniques via spectral theory. The proposed method is based on the shifted Legendre polynomials. We feel that the present method can be used to solve other classes of integral and integro-differential equations.

# Chapter 5

# A new class of Fredholm integral equations of the second kind with non symmetric kernel: Solving by wavelets method

This chapter is the subject of the following publication: A. Mennouni, N.E. Ramdani and Kh. Zennir *A New Class of Fredholm Integral Equations of the Second Kind with Non Symmetric Kernel: Solving by Wavelets Method*, Boletim da Sociedade Paranaense de Matemática, 2018.

**Abstract**

In this chapter, we introduce an efficient modification of the wavelets method to solve a new class of Fredholm integral equations of the second kind with non symmetric kernel. This method based on orthonormal wavelet basis, as a consequence three systems are obtained, a Toeplitz system and two systems with condition number close to $1$. Since the preconditioned conjugate gradient normal equation residual (CGNR) and preconditioned conjugate gradient normal equation error (CGNE) methods are applicable, we can solve the systems in $O(2n\log(n))$ operations, by using the fast wavelet transform and the fast Fourier transform.

## 5.1 Introduction

Integral equation perform role effectively in many fields of science and engineering. Recently, there are a lot of orthonormal basis function that have been used to find an approx-

imate solution, mention Fourier functions [6], Legendre polynomials [79] and wavelets [50, 54, 61, 69, 70, 73, 74, 103]. Although, the wavelet bases is one of the most interesting basis, especially for large scale problems, in which the kernel can be constituted as sparse matrix.
We reminder that usually it is difficult to construct the exact solution of linear and non-linear Fredholm integral equation via the well-known methods. A lot of different useful methods have been developed to approximate the solutions of these equations, for instance collocation methods are studied in [67, 95], spectral methods are given in [62, 72], transform methods are introduced in [5, 15, 92], and homotopy perturbation method is presented in [41] and others.
More recently, the multiresolution analysis has been considered by many researchers (see [51, 54, 70, 73, 74, 109]). For instance, wavelets method play a key role to find the unique solution for some Fredholm integral equations.
In the present chapter, we present wavelet basis to find the approximate solution of the following Fredholm integral equation of second kind:

$$u(t) - 2^{\beta} \int_{0}^{+\infty} k(2^{\alpha}s - 2^{\alpha}t)u(t)dt = f(t), \quad s \in [0, +\infty[, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \tag{5.1}$$

where $u(.)$ is the unknown function, $f(.)$ is the right hand side and $k(s-t)$ is a non symmetric kernel.

A considerable part of this proposal is based on a study by [Jin and Yuan,1998], in which the authors focused on the first kind with symmetric kernel. In contrast to their work, we focused on the second kind with non symmetric kernel and as we know that the symmetric property is necessary condition to apply Conjugate Gradient method and in our case we don't have this property so we dealt with the equivalent two systems that have the symmetric property.

The outline of this chapter is as follows: In section 2, we describe the basic formulation of wavelets and preliminary which are necessary for our development. Section 3, is devoted to the discretization of the integral equation. In section 4, we study the condition number of the matrix operator and we give the operation cost to solve the systems.

## 5.2 Preliminaries

### 5.2.1 Wavelet Bases

The basic tool for our method to approximate the solution of (5.1) is wavelet bases. For the convenience of the reader, we recall here some basic concepts and well-known results concerning the multiresolution analysis (MRA for short). As in [35, 51], let us consider a function $\varphi \in L^2(\mathbb{R})$, called the father wavelet (or scaling function), with a compact support $[0, a], a > 0$. We assume that

$$\varphi(. - k), \quad k \in \mathbb{Z} \tag{5.2}$$

form an orthonormal sequence in $L^2(\mathbb{R})$. Let $V_0$ be the closed linear subspace of $L^2(\mathbb{R})$ generated by (5.2). The multiresolution analysis (MRA), depending on the $\varphi(.)$ consists of:

(i)
$$f(.) \in V_0 \quad \text{if and only if} \quad f(2^j.) \in V_{j+1} \quad \text{for all} \quad j \in \mathbb{Z};$$

(ii)
$$\cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots ;$$

(iii)
$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}) \text{ and } \bigcap_{j\in\mathbb{Z}} V_j = 0;$$

(iv) The sequence (5.2) forms a Riesz basis of $V_0$.

Let $W_j$ be the orthogonal complement of $V_j$ in $V_{j+1}$, i.e.,

$$V_{j+1} = V_j \oplus W_j.$$

According to the above definition, we have

$$\bigoplus_{-\infty}^{+\infty} W_j = L^2(\mathbb{R}).$$

Following [32, 51, 85], there exists at least one function $\psi \in W_0$ such that

$$\psi(. - k), \quad k \in \mathbb{Z}$$

is an orthonormal basis of $W_0$. The function $\psi$ is called the mother wavelet.

A wavelet $\phi \in L^2(\mathbb{R})$ is called orthonormal if the family of functions generated from $\phi$ by

$$\phi_{j,k}(s) = 2^{j/2}\phi(2^j s - k), \quad j, k \in \mathbb{Z},$$

is orthonormal, that is,

$$\langle \phi_{j,k}, \phi_{m,n} \rangle = \delta_{j,m}\delta_{k,n}.$$

Let us introduce the following two wavelet sequences:

$$\varphi_{j,k}(s) = 2^{j/2}\varphi(2^j s - k), \quad j, k \in \mathbb{Z},$$

and

$$\psi_{j,k}(s) = 2^{j/2}\psi(2^j s - k), \quad j, k \in \mathbb{Z}.$$

We recall that

$$\langle \psi_{m,k}, \varphi_{m,l} \rangle = \langle \psi_{n,k}, \varphi_{n,l} \rangle, \quad \text{for all} \quad m, n, k, l \in \mathbb{Z}.$$

Therefore, the wavelet sequence $\{\psi_{j,k}\}$ forms a Riesz basis of $H^{\mathfrak{s}}(\mathbb{R})$ for $\mathfrak{s} \geq 0$.

Assume that $\mathbb{B}_1$ and $\mathbb{B}_2$ two bases in $V_n$ with:

$$\mathbb{B}_1 = (\varphi_{n,k}(.))_k, \quad k \in \mathbb{Z},$$

and

$$\mathbb{B}_2 = \bigcup_{-\infty < j \leq n-1} (\psi_{j,k}(.))_k, k \in \mathbb{Z}.$$

We note that $\mathbb{B}_1$ and $\mathbb{B}_2$ follow from the father wavelet $\varphi$ and the mother wavelet $\psi$, respectively.

### 5.2.2 Wavelet Transform

**Definition 5.1 (Continuous wavelet transform)** *The continuous wavelet transform of the mother wavelet $\varphi$ is defined by*

$$(S_\varphi f)(j,k) = \int_{-\infty}^{+\infty} f(t)\overline{\varphi_{j,k}(t)}dt = \langle f, \varphi_{a,b} \rangle.$$

**Definition 5.2 (Discrete wavelet transform)** *The discrete wavelet transform of the fa-*

*ther wavelet $\psi$ is defined by*

$$(S_\psi f)(j,k) = \int_{-\infty}^{+\infty} f(t)\overline{\psi_{j,k}(t)}dt = \langle f, \psi_{j,k}\rangle.$$

### 5.2.3 Condition number

Condition number of a matrix gives the information about the singularity of the corresponding matrix.

**Definition 5.3 (Condition number)** *Let $A$ be an $n \times n$ invertible matrix. Define $\kappa(A)$, the condition number of $A$, by*

$$\kappa(A) = \|A\|\|A^{-1}\|.$$

The condition number of an $n \times n$ invertible matrix $A$ is defined as the ratio of its maximum singular value to its minimum singular value, that is, for

$$\lambda_M := \max\{|\lambda|, \lambda \quad \text{is an eigenvalue of} \quad A\},$$

and

$$\lambda_m := \min\{|\lambda|, \lambda \quad \text{is an eigenvalue of} \quad A\},$$

we have

$$\kappa(A) = \frac{\lambda_M}{\lambda_m}.$$

### 5.2.4 Preconditioning and diagonal scaling

A preconditioner $P$ of a matrix $A$ is given by $P^{-1}A$ which its condition number smaller than the original matrix. In order to solve linear systems of the form $Ax = b$, preconditioners are used for numerous iterative methods. Then, while the condition number of the matrix $A$ decreases, for a lot of iterative linear solvers the rate of convergence increases. Hence, preconditioning is a very effective tool uses to reduce the condition number of the matrix $A$ small.

Diagonal scaling (DS) is a special case of preconditioning and it is an efficient tool used to reduce the condition number of matrix $A$ for ensuring the convergence and the accuracy of the first method. In our case, in order to reduce the condition number of the matrix $A$ we apply the diagonal matrix $D$, in a way to speed up the method.

### 5.2.5 Conjugate Gradient Method

Conjugate Gradient (CG) method used to solve linear system of the form $Ax = b$ to obtain a quick convergence when $\kappa(A)$ is smaller.

Generally, Conjugate Gradient method used for large problems in order to attain a modest accuracy in a reasonable number of iterations.

#### Conjugate gradient normal equation residual and error

The conjugate gradient method can be applied on the normal equations. The CGNE and CGNR methods are variants of this approach that are the simplest methods for non symmetric or indefinite systems. Since other methods for such systems are in general rather more complicated than the conjugate gradient method, transforming the system to a symmetric definite one and then applying the conjugate gradient method is attractive for its coding simplicity.
CGNR solves the system

$$(A^T A)x = A^T b.$$

CGNE solves the system

$$(AA^T)y = d.$$

## 5.3 Discretization of integral equation

Let $H^{\mathfrak{s}}(\mathbb{R})$ and $H^{\mathfrak{s}'}(\mathbb{R})$ be two Sobolev spaces, with $\mathfrak{s} \geq \mathfrak{s}' \geq 0$. Letting

$$(Ku)(s) := 2^{\beta} \int_0^{+\infty} k(2^{\alpha}s - 2^{\alpha}t)u(t)dt, \tag{5.3}$$

we assume that $k(2^a. - 2^a.) \in H^{\mathfrak{s}}(\mathbb{R})$ is continuous non symmetric kernel.

The integral operator $K$ from $H^{\mathfrak{s}}(\mathbb{R})$ into $H^{\mathfrak{s}'}(\mathbb{R})$ is compact. For a given function $f \in H^{\mathfrak{s}'}(\mathbb{R})$, we try to find $u \in H^{\mathfrak{s}}(\mathbb{R})$ such that Equation (5.1) can be rewritten in operator form as follows:

$$(I - K)u = f$$

and we assume that $1$ is not an eigenvalue of $K$. Hence, the equivalent variational form follows:

$$\begin{cases} \text{find } u \in H^{\mathfrak{s}}(\mathbb{R}) \text{ such that} \\ B(u, v) = F(v), \text{ for all } v \in H^{\mathfrak{s}}(\mathbb{R}), \end{cases} \tag{5.4}$$

where

$$B(u,v) = \langle u,v\rangle - \langle Ku,v\rangle = \int_0^{+\infty} u(s)v(s)ds - \int_0^{+\infty}\int_0^{+\infty} k(s-t)u(t)v(s)dsdt,$$

and

$$F(v) = \int_0^{+\infty} f(s)v(s)ds.$$

Since

$$(Ku,v) \leq \beta \|Ku\|_{H^{\mathfrak{s}'}} \|v\|_{H^{\mathfrak{s}'}}$$

it follows that $(Ku,v)$ is a continuous bilinear form on $H^{\mathfrak{s}'}(\mathbb{R}) \times H^{\mathfrak{s}}(\mathbb{R})$.
We assume that

$$(Ku,u) \geq \rho \|u\|^2_{H^{\mathfrak{s}}}, \quad \text{for some constant} \quad \rho > 0.$$

Hence, $(Ku,v)$ is coercive form on $H^{\mathfrak{s}'}(\mathbb{R}) \times H^{\mathfrak{s}}(\mathbb{R})$.

### 5.3.1 Projection of $(I-A)$ with respect to $\mathbb{B}_1$ and $\mathbb{B}_2$

• Let the matrix $(I-A_n)$ relative to the basis $B_1$, which is is the projection of the matrix $(I-A)$ on the subspace $V_n$.

The elements of the matrix $(I-A_n)$ are given as follows

$$\begin{aligned} t_{p,q} &= \langle \varphi_{n,p}, \varphi_{n,q}\rangle - \langle K\varphi_{n,p}, \varphi_{n,q}\rangle \\ &= \int_0^{+\infty} \varphi_{n,p}(s)\varphi_{n,q}(s)ds - 2^{\beta}\int_0^{+\infty}\int_0^{+\infty} k(2^{\alpha}s - 2^{\alpha}t)\varphi_{n,p}(t)\varphi_{n,q}(s)dtds. \end{aligned} \tag{5.5}$$

For all $u, v \in H^{\mathfrak{s}}(\mathbb{R})$, we suppose that $u_n, v_n$ are the projections of $u, v$ on $V_n$ respectively. Which implies that (5.4) becomes

$$\int_0^{+\infty} u_n(s)v_n(s)ds - \int_0^{+\infty}\int_0^{+\infty} k(s-t)u_n(t)v_n(s)dtds = \int_0^{+\infty} f(s)v_n(s)ds \tag{5.6}$$

Let

$$u_n = \sum_p x_p\varphi_{n,p} \text{ and } v_n = \varphi_{n,q}, \qquad \text{for all} \quad q \in \mathbb{Z}. \tag{5.7}$$

By substituting (5.7) into (5.6), we get a linear system given as follows

$$(I - T_\infty)x = b, \tag{5.8}$$

where $(I - T_\infty)_{p,q} = t_{p,q}$ is given by (5.5), and

$$(x)_p = x_p, \qquad (b)_q = \int_0^{+\infty} f(s)\varphi_{n,q}(s)ds.$$

We mention that $\varphi$ has the compact support $[0, a]$, which leads us to $t_{p,q} = t_{p-q}$.

$$\begin{aligned}
t_{p,q} &= \int_0^{+\infty} \varphi_{n,p}(s)\varphi_{n,q}(s)ds - \int_0^{+\infty}\int_0^{+\infty} 2^\beta k(2^\alpha s - 2^\alpha t)\varphi_{n,p}(t)\varphi_{n,q}(s)dtds \\
&= \delta_{p,q} - 2^{\beta+n}\int_0^{+\infty}\int_0^{+\infty} k(2^\alpha s - 2^\alpha t)\varphi(2^n t - p)\varphi(2^n s - q)dtds \\
&= \delta_{p,q} - 2^{\beta+n}\int_{2^{-n}p}^{2^{-n}(a+p)}\int_{2^{-n}q}^{2^{-n}(a+q)} k\left(2^\alpha s - 2^\alpha t\right)\varphi(2^n t - p)\varphi(2^n s - q)dtds \\
&= \delta_{p,q} - 2^\beta \times 2^{-n}\int_0^a\int_0^a k\left[2^{-n} \times 2^\alpha(s - t + p - q)\right]\varphi(t)\varphi(s)dtds \\
&= \delta_{p,q} - 2^{-n+\beta}\int_0^a\int_0^a k\left[2^{-n+\alpha}(s - t + p - q)\right]\varphi(t)\varphi(s)dtds \\
&= t_{p-q}.
\end{aligned}$$

Hence $(I - T_\infty)$ is a Toeplitz matrix.

• The matrix representation of $(I - A_n)$ relative to the basis $\mathbb{B}_2$ has the elements given as follows

$$\begin{aligned}
a_{p,q,i,j} &= \langle \psi_{p,q}\psi_{i,j}\rangle - \langle K\psi_{p,q}, \psi_{i,j}\rangle \\
&= \int_0^{+\infty} \psi_{p,q}(s), \psi_{i,j}(s)ds - 2^\beta\int_0^{+\infty}\int_0^{+\infty} k(2^\alpha s - 2^\alpha t)\psi_{p,q}(t)\psi_{i,j}(s)dtds,
\end{aligned} \tag{5.9}$$

for $-\infty < p, i < n$ and $-\infty < q, j < +\infty$.

Writing

$$u_n = \sum_{p,q} x_{p,q}\psi_{p,q}, \text{ and } v_n = \psi_{p,q}, \quad -\infty < p < n, \quad \text{for all} \quad q \in \mathbb{Z}. \tag{5.10}$$

We substitute (5.10) into (5.6), we obtain the linear system

$$(I - A_\infty)x = d, \tag{5.11}$$

where $(I - A_\infty)_{p,q,i,j} = a_{p,q,i,j}$ is unsymmetric given by (5.9), $x = (x_{p,q})^T$ and $d = (d_{p,q})^T$

are vectors with $d_{p,q} = \displaystyle\int_0^{+\infty} f(s)\psi_{p,q}(s)ds.$

## 5.4 Solving the linear systems

### 5.4.1 Condition number

From the previous section we obtained two different linear systems. One of them is the Toeplitz system (5.8) (relative to $\mathbb{B}_1$) and the other one is the systems (5.11) (relative to $\mathbb{B}_2$).

Let us focus on studying the condition number of the last linear system. Actually, we will develop the idea of Zhang [109] and in order to do that. Firstly, we present the following Lemma which plays an important role for reducing the condition number of the matrix.

**Lemma 5.1** *([51, 85, 109]) Let*

$$f = \sum_{j,k} \langle f, \psi_{j,k}\rangle \psi_{j,k}.$$

*Then $f \in H^{\mathfrak{s}}(\mathbb{R})$ if and only if*

$$\sum_{j,k} |\langle f, \psi_{j,k}\rangle|^2 (1 + 4^{j\mathfrak{s}}) < +\infty, \quad -r < \mathfrak{s} < r,$$

*where $r$ is the regularity of the MRA. Moreover, since $\{\psi_{j,k}\}$ is a Riesz basis of $H^{\mathfrak{s}}(\mathbb{R})$, we also have*

$$C_1 \sum_{j,k} |\langle f, \psi_{j,k}\rangle|^2 (1 + 4^{j\mathfrak{s}}) \leq \|f\|^2_{H^{\mathfrak{s}}} \leq C_2 \sum_{j,k} |\langle f, \psi_{j,k}\rangle|^2 (1 + 4^{j\mathfrak{s}}), \tag{5.12}$$

*where $C_2 \geq C_1 > 0$ are constants.*

Secondly, we know that $(I - A_\infty)$ in system (5.11) is unsymmetric. Then, system (5.11) becomes

$$(I - A_\infty)^T (I - A_\infty)x = (I - A_\infty)^T d, \tag{5.13}$$

$$(I - A_\infty)(I - A_\infty)^T y = d, \ x = (I - A_\infty)^T y. \tag{5.14}$$

Now, let $\phi \in V_n$ with $\phi = \sum_{j,k} w_{j,k}\psi_{j,k}$. We have

$$\begin{aligned} B_1(\phi,\phi) &:= \sum_{j,k}\sum_{i,\ell} w_{j,k}w_{i,\ell}\left[\langle (I-A_\infty)^T(I-A_\infty)\psi_{j,k},\psi_{i,\ell}\rangle\right] \\ &= w^T(I-A_\infty)^T(I-A_\infty)w, \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} B_2(\phi,\phi) &:= \sum_{j,k}\sum_{i,\ell} w_{j,k}w_{i,\ell}\left[\langle (I-A_\infty)(I-A_\infty)^T\psi_{j,k},\psi_{i,\ell}\rangle\right] \\ &= w^T(I-A_\infty)(I-A_\infty)^T w, \end{aligned} \tag{5.16}$$

where $w := (w_{j,k})^T$ is a vector. By the assumption that $B(u,v) \in \{B_1(u,v), B_2(u,v)\}$ is a continuous elliptic bilinear from on the space $H^{\mathfrak{s}}(\mathbb{R}) \times H^{\mathfrak{s}}(\mathbb{R})$, i.e.,

$$\begin{aligned} &B(u,v) \le \beta\|u\|_{H^{\mathfrak{s}}}\cdot\|v\|_{H^{\mathfrak{s}}}, \\ &B(u,u) \ge \alpha\|u\|^2_{H^{\mathfrak{s}}}. \end{aligned}$$

Since $\phi \in V_j$, we get $\phi \in H^{\mathfrak{s}}$.

Consequently,

$$C_3\|\phi\|^2_{H^{\mathfrak{s}}} \le B(\phi,\phi) \le C_4\|\phi\|^2_{H^{\mathfrak{s}}}, \quad \text{for some constants} \quad C_4 \ge C_3 > 0. \tag{5.17}$$

**Condition number of system** (5.15)

From (5.15) and (5.17), we get

$$C_3\|\phi\|^2_{H^{\mathfrak{s}}} \le w^T(I-A_\infty)^T(I-A_\infty)w \le C_4\|\phi\|^2_{H^{\mathfrak{s}}}.$$

By using (5.12), we obtain

$$C_1\sum_{j,k}|\langle w,\psi_{j,k}\rangle|^2(1+4^{j\mathfrak{s}}) \le \|\phi\|^2_{H^{\mathfrak{s}}} \le C_2\sum_{j,k}|\langle w,\psi_{j,k}\rangle|^2(1+4^{j\mathfrak{s}}),$$

then

$$C_1\sum_{j,k}|w_{j,k}|^2 2^{2j\mathfrak{s}} \le \|\phi\|^2_{H^{\mathfrak{s}}} \le C_2\sum_{j,k}|w_{j,k}|^2 + C_2\sum_{j,k}|w_{j,k}|^2 2^{2j\mathfrak{s}}.$$

Thus,

$$C_1\sum_{j,k}|2^{j\mathfrak{s}}w_{j,k}|^2 \le \|\phi\|^2_{H^{\mathfrak{s}}} \le C_0\sum_{j,k}|2^{j\mathfrak{s}}w_{j,k}|^2,$$

so that

$$C_3C_1\sum_{j,k}|2^{js}w_{j,k}|^2 \leq C_3\|\phi\|^2_{H^s} \leq w^T(I-A_\infty)^T(I-A_\infty)w \leq C_4\|\phi\|^2_{H^s} \leq C_4C_0\sum_{j,k}|2^{js}w_{j,k}|^2.$$

Consequently,

$$C_5\sum_{j,k}|2^{js}w_{j,k}|^2 \leq w^T(I-A_\infty)^T(I-A_\infty)w \leq C_6\sum_{j,k}|2^{js}w_{j,k}|^2, \quad \text{for some constants} \quad C_5 \geq C_6 > 0.$$

By using diagonal scaling $D$, we get

$$C_5\|w\|^2 \leq w^TD^{-1/2}(I-A_\infty)^T(I-A_\infty)D^{-1/2}w \leq C_6\|w\|^2,$$

where $\|\cdot\|$ is the $L^2$-norm. In the end, the condition number of $(I-A_\infty)^T(I-A_\infty)$ is close to 1, that is,

$$k(D^{-1/2}(I-A_\infty)^T(I-A_\infty)D^{-1/2}) = O(1).$$

**Condition number of system** (5.16)

From (5.16) and (5.17), we get

$$C_3\|\phi\|^2_{H^s} \leq w^T(I-A_\infty)(I-A_\infty)^Tw \leq C_4\|\phi\|^2_{H^s}.$$

By following the same steps of the previous system we obtain that the condition number of $(I-A_\infty)(I-A_\infty)^T$ after a diagonal scaling is

$$k(D^{-1/2}(I-A_\infty)(I-A_\infty)^TD^{-1/2}) = O(1).$$

### 5.4.2 Operation cost of the corresponding systems

In order to numerically solve the system (5.8), we use a finite interval. For this reason, let us consider the finite section $T_n$ of $T_\infty$. Thus, the Toeplitz system (5.8) becomes an $n-by-n$ system

$$(I-T_n)x = b. \tag{5.18}$$

Now, we introduce the relation between $(I-T_n)$ and $(I-A_n)$, which is similar to the one given by the authors of [51] as follows

$$(I-A_n) = W_n(I-T_n)W_n^{-1},$$

where $(I - A_n)$ is the finite section of $(I - A_\infty)$ and $W_n$ is a finite section of $W$ which is the wavelet transform matrix between two orthonormal wavelet bases $\mathbb{B}_1$ and $\mathbb{B}_2$.
Hence, we solve the Toeplitz system (5.8) by solving its equivalent form

$$\left(W_n(I - T_n)W_n^{-1}\right) W_n x = W_n b,$$

i.e.,

$$(I - A_n)\tilde{x} = \tilde{b}, \tag{5.19}$$

where $\tilde{x} := W_n x$ and $\tilde{b} := W_n b$.

Now, we are going to solve the system (5.19). However, the matrix $(I - A_n)$ does not have a small condition number. Then we would like to apply PCG method with diagonal preconditioner $D_n$ in order to obtain a new matrix with a smaller condition number. Unfortunately, $(I - A_n)$ does not have the symmetric property. That means the PCG method will not work. Thus, two systems are obtained with symmetric property.

$$(I - A_n)_n^T(I - A_n)\tilde{x} = (I - A_n)T\tilde{b}, \tag{5.20}$$

$$(I - A_n)(I - A_n)^T\tilde{y} = \tilde{b},\ \tilde{x} = (I - A_n)^T\tilde{y}. \tag{5.21}$$

with $(I - A_n)^T(I - A_n)$ and $(I - A_n)(I - A_n)^T$ are symmetric.

Now, in order to solve the system (5.19), we solve its two equivalent systems (5.20) and (5.21). We know that the matrices $(I - A_n)^T(I - A_n)$ and $(I - A_n)(I - A_n)^T$ do not have a small condition number. Thus we apply conjugate gradient normal equation residual CGNR method to (5.20) and conjugate gradient normal equation error CGNE method to (5.21) with diagonal preconditioner $D_n$ in order to obtain a new matrices with a smaller condition number.

More precisely, by applying the diagonal preconditioner to (5.20), we have then the following preconditioned system

$$D_n^{-1}(I - A_n)^T(I - A_n)\tilde{x} = D_n^{-1}(I - A_n)^T\tilde{b}, \tag{5.22}$$

with the condition number

$$k\left(D_n^{-1}(I - A_n)^T(I - A_n)\right) = k\left(D_n^{-1/2}(I - A_n)^T(I - A_n)D_n^{-1/2}\right) = O(1).$$

We apply again the diagonal preconditioner to (5.21), we get the following preconditioned system

$$D_n^{-1}(I-A_n)^T(I-A_n)\tilde{y} = D_n^{-1}\tilde{b},\ \tilde{x} = (I-A_n)^T\tilde{y}, \tag{5.23}$$

with the condition number

$$k\left(D_n^{-1}(I-A_n)(I-A_n)^T\right) = k\left(D_n^{-1/2}(I-A_n)(I-A_n)^T D_n^{-1/2}\right) = O(1).$$

Hence, we can solve the system (5.22) by applying the conjugate gradient normal equation residual CGNR method and (5.23) by applying the conjugate gradient normal equation error CGNE method which give as a linear convergence rate (see[42]).

Thus, the equivalent form of (5.22) is

$$\tilde{A}_n y_1 = z_1, \tag{5.24}$$

where

$$y_1 := D_n\tilde{x},\ z_1 := D_n^{-1}(I-A_n)^T\tilde{b},$$

and

$$\tilde{A}_n := D_n^{-1}(I-A_n)^T(I-A_n)D_n^{-1}.$$

The equivalent form of (5.23) is

$$\tilde{A'}_n y_2 = z_2, \tag{5.25}$$

where

$$y_2 = D_n\tilde{y}, \quad z_2 = D_n^{-1}\tilde{b},$$

and

$$\tilde{A'_n} = D_n^{-1}(I-A_n)(I-A_n)^T D_n^{-1}.$$

In each iteration of CGNR and CGNE methods, requires computing $(I-A_n)^T v_1$ and $(I-A_n)v_2$ for some vectors $v_1$ and $v_2$ respectively, and then solving (5.24) and (5.25) (see[98]).

Well, after some updates to CG method, we can solve the systems $D_n\tilde{x} = y_1$ and $D_n\tilde{y} = y_2$ respectively.
For solving the above systems, we use the algorithm presented in [42].

• For the case $(I - A_n)^T v_1$, since

$$(I - A_n) = W_n(I - T_n)W_n^{-1},$$

we get

$$(I - A_n)^T v_1 = (W_n^{-1})^T (I - T_n^T) u_1,$$

where $u_1 = W_n^T v_1$, and by using FWT we could then compute $u_1$ in $O(n)$ operations ([21, 98]).

In addition, by using FFT we could then compute $(I - T_n)u_1$ in $O(n \log n)$ operations ([28, 104]).

In the end, to solve $(I - A_n)v_1 = (W_n^{-1})^T (I - T_n)u_1$ we use FWT and Strang's algorithm given in [104]. Therefore, the operation cost decreased to $O(n \log n)$. Regarding the system $D_n \tilde{x} = y_1$ we just need $O(n)$ operations.

Hence, the cost per iteration for (5.20) is $O(n \log n)$.

• For the case $(I - A_n)v_2$, by similar way as above, we get the cost per iteration for (5.21) is $O(n \log n)$.

Consequently, the total cost per iteration is $O(2n \log n)$.
Finally, we can solve the systems (5.18), (5.19) in $O(2n \log n)$, as a result of the independence of the iterations and $n$.

# Conclusions and perspectives

In this thesis, we are concerned with the resolution of some classes of Fredholm integral and integro-differential equations, where we presented modified projection methods in order to solve them and we illustrated our results by given some numerical examples.

The aim of our work is to construct approximate solution of linear integral and integro-differential equations using projection methods based on several orthogonal polynomials. This work may be extended to nonlinear integral and integro-differential equations and other classes of singular integral equations.
As a future work, under which conditions the previous methods could be applied for Volterra integral equations. These methods can be also applied to nonlinear integral and integro-differential equations, but some modifications are required.

Specifically, we hope to use the present methods for approximate the solution of the integral equations of the form

$$a(s)u(s) + b(s)\sum_{k=1}^{m}\int_a^s H_k(s,t,\psi(t))u(t)dt = f(s), \qquad m \in \mathbb{N}^*, \quad a \leq s \leq b,$$

$$a(s)u(s) + b(s)\sum_{k=1}^{m}\int_a^s H_k(s,t,\psi(t))\ln|s-t|\, h(s,t)u(t)dt = f(s), \qquad m \in \mathbb{N}^*, \quad a \leq s \leq b,$$

$$a(s)u(s) + \frac{b(s)}{\pi}\int_{-1}^{1}\frac{h(s,t)k(s,t,\psi(t))}{s-t}u(t)dt = f(s), \quad , \quad 0 \leq s \leq 1.$$

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## Abstract

The main objective of this thesis is to study some classes of integral and integro-differential equations with regular and singular kernels. We apply projection methods based on different orthogonal polynomials to solve Fredholm integral equation of the second kind; we also apply collocation method based on the airfoil polynomial to numerically solve an integro-differential equation of second order. We introduce a wavelets method to solve a new class of Fredholm integral equations of the second kind with non symmetric kernel.



**Key words:** Integral equations, integro-differential equations, projection methods, collocation methods.



## Résumé

L'objectif principal de cette thèse est d'étudier quelques classes des équations intégrales et intégro-différentielles avec des noyaux singuliers et réguliers. Pour résoudre les équations intégrales de Fredholm de la deuxième espèce, nous appliquons les méthodes de projections basées sur différents polynômes orthogonaux. Nous appliquons aussi des méthodes de collocations basées sur les polynômes Tchebychev pour résoudre numériquement des équations intégro-différentielles de second ordre. Nous introduisons une méthode des ondelettes pour résoudre une nouvelle classe d'équations intégrales de Fredholm du second type avec un noyau non symétrique.



**Mots clés:** Equations intégrales, équations intégro-différentielles, méthodes de projection, méthodes de collocation.



## ملخص:

الهدف الرئيسي لهذه الأطروحة هو دراسة بعض اصناف المعادلات التكاملية و التكامل-تفاضلية بنواة مفردة و نواة منتظمة. لحل المعادلات التكاملية لفريدهولم من النوع الثاني نطبق طرق الإسقاط المعتمدة على مختلف كثيرات الحدود المتعامدة. نطبق ايضا طرق التجميع المعتمدة على كثيرات الحدود شيبيشيف لحل المعادلات التكامل-تفاضلية من الدرجة الثانية عدديا. نولد طريقة المويجات من اجل حل صنف جديد من المعادلات التكاملية لفريدهولم من النوع الثاني بنواة غير متناظرة.



**الكلمات المفتاحية:** المعادلات التكاملية، المعادلات التكامل-تفاضلية، طرق الاسقاط، طرق التجميع.