Abstract

In this thesis, we establish well-posedness, regularity and exact controllability of several input-output systems described by partial differential equations (transmission Schödinger equation, fourth order Schrödinger equation) with boundary control and collocated observation. The aproach we adopt uses classical multiplier, geometric multiplier method on Riemanniann manifolds and compactness/uniqueness arguments

Key Words: Well posedness; regularity; exact controllability; transmission Schrödinger equation; fourth order Schrödinger equation, geometric multiplier method; Riemanniann manifolds

Contents

Chapter 1

Introduction

Linear finite dimensional control and observavtion systems have the following general form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

$$y(t) = Cx(t) + Du(t) \tag{1.2}$$

where $x(t) \in X = \mathbb{R}^n$, the state space, $u(t) \in U = \mathbb{R}^n$, the input space and $y(t) \in Y = \mathbb{R}^n$ is the output space, (A, B, C, D) is a quadruple of matrices of compatible dimensions, which we call the generating operators of (1.1)-(1.2). An important property of the differential equation (1.1) is that its unique continuously differentiable solution is defined by

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-\sigma)A}Bu(\sigma)d\sigma. \tag{1.3}$$

for any given initial state $x(0) = x_0 \in X$. This formula defines the state trajectories $x(.)$ also for input functions $u \in L^2([0, \infty); U)$.

For any linear systems as above, we denote by $S(t) = e^{tA}$ the semigroup on X and we introduce families of linear operators depending on $\tau \geq 0$, $\phi_\tau \in L(L^2([0, \infty); U), X)$, $L_\tau \in L(X, L^2([0, \infty); Y))$ and $\mathcal{F}_\tau \in L(L^2([0, \infty); U), L^2([0, \infty); Y))$

$$\phi_\tau u = \int_0^\tau e^{(\tau-\sigma)A}Bu(\sigma)d\sigma \tag{1.5}$$

$$(L_\tau x)(t) = \begin{cases} Ce^{tA}x, & t \in [0, \tau] \\ 0, & t > \tau \end{cases} \tag{1.6}$$

$$(\mathcal{F}_\tau u)(t) = \begin{cases} C\int_0^\tau e^{(\tau-\sigma)A}Bu(\sigma)d\sigma + Du(t), & t \in [0, \tau] \\ 0, & t \geq \tau \end{cases} \tag{1.7}$$

are the input (reachability), output (observability) and input/output maps, respectively.

$$y(t) = Ce^{tA}x_0 + C\int_0^{\tau} e^{(\tau-\sigma)A}Bu(\sigma)d\sigma + Du(t) \tag{1.8}$$

If we let $x_0 = 0$ in (1.8) and take Laplace transform, we obtain the *frequency domain* description

$$\hat{y}(s) = G(s)\hat{u}(s). \tag{1.9}$$

Where $G(s)$ is the transfer function of the system (1.1)-(1.2), given by

$$G(s) = C(sI - A)^{-1}B + D \tag{1.10}$$

Notice that the operators ϕ_τ, L_τ and G satisfies the following properties

(i) $\phi_\tau \in L(L^2([0,\infty);U), X)$ is bounded, i.e., for some and hence (for any) $t > 0$, there exists $C_t > 0$ such that

$$\|\phi_\tau(u)\|^2 = \left\| \int_0^{\tau} S(\tau - \sigma) Bu(\sigma)d\sigma \right\|^2 \leq C_t \int_0^{t} \|u(\tau)\|_U^2 \, d\tau \quad \forall u \in L^2(0,T;U). \tag{1.11}$$

(ii) $L_\tau \in L(X, L^2([0,\infty);Y))$ is bounded, i.e., for some and hence (for any) $t > 0$, there exists $C'_t > 0$ such that

$$\int_0^t \|CS(.)x\|_Y^2 \, dt \leq C'_t \|x\|_X^2 , \qquad \forall x \in D(A) \tag{1.12}$$

(iii) There is an $\alpha \in \mathbb{R}$, G is bounded on $\mathbb{C}_\alpha = \{s \in \mathbb{C} / \operatorname{Re}(s) > \alpha\}$, i.e., for some (and hence for any) $t > 0$, there exists C''_t such that

$$\int_0^{\tau} \|y(t)\|_Y^2 \, dt \leq C''_t \|u(t)\|_U^2 , \qquad \forall u \in L^2(0,t;U) \qquad \text{when } x_0 = 0 \tag{1.11}$$

Linear infinite dimensional control and observation systems can be described:

(i) directely in terms of PDEs or differential-deley equations (see Lions [**35**], Lasiecka and Triggiani [**33**], Triggiani and Yao [**56**]**,** Delfour and Mitter [**16**]**,** Hale [**27**]**,** Hale and Lunel [**28**]**,**...);

(ii) in terms of a quadruple (A, B, C, D) of abstract operators on a Banach (or Hilbert) space (see Weiss and Curtain [**61**], Curtain [**10**], Salamon [**52**],...);

(iii) as a frequency domain relationship between inputs and outputs (see Wen, Chai and Guo [**67**], [**69**], Staffans and Weiss [**53**],...).

Let us illustrate **(ii)** with the well known infinite dimensional linear systems, we choose X, U and Y, to be the state, input and output Hilbert spaces, respectively. This system is described as (1.1)-(1.2) where the (usually unbounded) A generates a C_0-semigroup $S(.)$ on X, B is a control operator from U to X, C is an observation operator from X to Y and D is a bounded operator from U to Y. If we suppose that B and C are bounded operators between compatible spaces, then the properties (1.11), (1.12) and (1.11) holds. However, many interesting infinite-dimensional systems fall outside this subclass. Usual applications include actuators and sensors supported at isolated points or on lower-dimensional hyper-surfaces, or on the boundary of, a spatial domain. As it is well known, such control and observation operators present considerable technical difficulties even at the level of state space formulations of the dynamics. Indeed, there has been and continues to be significant research devoted to formulating analogs of classical feedback control methodologies for such systems. The most general class of infinite-dimensional systems for which there is a well established theory of representation, transfer function, feedback, dynamic stabilization, controllability, observability is the class of *well-posed linear systems.* This class introduced by Salamon and Weiss in the late 1980s cover many control systems described by PDE's or differential delay equations. The aim was to provide a unifying abstract framework to formulate and solve control problems for systems described by functional and partial differential equations. Roughly speaking, a well-posed linear systems is a time invariant system such that on any finite time interval, the operator from the initial state and the input function to the final state and the output function is bounded. This means that every well-posed system has a well defined transfer function $G(.)$. An important subclass of well-posed linear systems is formed by the regular systems. A regular system ([**63**]) is a well posed system satisfying the extra requirement that $\lim\limits_{s\in\mathbb{R},s\to+\infty} G(s) = D$ exists.

There is now a rich literature on the abstract theory for regular well-posed linear systems and from a practical point of view, the construction of specific examples of infinite dimensional systems which belong to this class is of considerable importance. In recent years, a number of PDEs with boundary control and observation are proved to be well-posed and regular (see Guo and Shao [**20**], [**21**], [**19**], Guo and Zhang [**18**], [**25**], [**23**], [**24**], Allag and Rebiai [**1**], Weiss, [**64**], Byrnes, Gilliam, Shubov and Weiss [**6**]...).

Byrnes, Gilliam, Shubov and Weiss [**6**] established the well posedness and regularity of the multi dimensional heat equation with both Dirichlet and Neumann type boundary controls. Using micro local analysis, Ammari [**3**] proved that the wave equation with boundary Dirichlet input and colocated output is well-posed with state space $X = L^2(\Omega) \times H^{-1}(\Omega)$, input, output space $U = Y = L^2(\Gamma_0)$ and the regularity was proved by Guo and Zhang [**18**]**.** The well posedness and regularity of multi dimensional Euler-Bernoulli plate equation on a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$, in the state space $X = L^2(\Omega) \times H^{-2}(\Omega)$ and input output spaces $U = Y = L^2(\Gamma_0)$ was established by Guo and Shao [**21**] and they proved in [**20**] that the system composed by Schrödinger equation with Dirichlet control and colocated observation with state space $X = H^{-1}(\Omega)$ and input, output space $U = Y = L^2(\Gamma_0)$ is well-posed and regular. By using the Riemannian geometry, Guo and Zhang [**22**] showed that the wave equation with variable coefficients is well-posed and regular. Guo and Shang [**23**] established the well-posedness and regularity of an Euler-Bernoulli plate with variable coefficients and boundary control and observation by using the multiplier method on Riemannian manifold. Similarly to [**23**]**,** Wen, Chai and Guo [**69**], proved the well-posedness and regularity of Euler-Bernoulli equation with variable coefficients and Dirichlet boundary control and colocated observation. A system of transmission of Euler-Bernoulli plate equation with variable coefficients under Neumann control and colocated observation is studied by Guo and Shao **[19]**, using the mulitiplier method on a Riemannian manifold, it was shown that the system is well-posed and regular with feedthrough operator is found to be zero, then they developed under a uniqueness assumption the exact controllability by establishing the observability for the dual system.

The fourth order Schrödinger equation arises in many scientific fields such as quantum mechanics, plasma physics, nolinear optics and so on. In quantum mechanics, the solution $\varphi(x,t)$ of system (4.43) denotes the probability amplitude function and the conservation of the norms validates the Born's statistical interpretation of $\varphi(x,t)$. Furthermore, $\int_\Omega |\varphi(x,t)|^2\, d\Omega$ represents the probability of finding the particle which will not disapear in Ω. The existence and uniqueness of the solutions has been studied intensively from perspectives of mathematics (see **[29], [30]**) and the references therin.

Wen, Chai and Guo **[67],** studied the well posedness and exact controllability of a system described by the this equation on a bounded domain on $\mathbb{R}^n$ $(n \geq 2)$ with boundary control and colocated observation, with the state space $X = H^{-2}(\Omega)$ and the input and output space $U = Y = L^2(\Gamma_0)$. The Neumann boundary control problem is first discussed, it is shown that the system is well-posed. This result is then generalized to the Dirichlet boundary control problem. Then they discussed the exact controllability with the Dirichlet boundary control, which is similar to the Neumann boundary case. In addition, they proved that both systems are regular and their feedthrough operators are zero. They showed in **[68]** that the same equation with hinged boundary by either moment or Dirichlet boundary control and colocated observation are well-posed which implies that the systems are exactly controllable in some finite time interval if and only if its corresponding closed loop systems under the direct output proportional feedback are exponentially stable, so they discussed the exact controllability of the systems. In addition they showed that the systems are regular and their feedthrough operators are zero.

Wen and Chai **[70]** generalize the well-posedness and exact controllability of this equation with Neumann boundary control (**[67]**) in the case where the coefficients are spatial variable dependent. Using the multilplier method on Riemannian manifold, they showed that the system is well-posed, regular and that the feedthrough operator is zero. So in order to conclude feedback stabilization from well posedness, they studied the exact controllability under a uniquenss assumption by presenting the observability inequality for the dual system.

To facilitate the reading of the thesis, we give a brief description of the material contained in the fulfilled chapters.

Chapter 2: This chapter contains some material which will be used in this thesis such as: admissible control and observation operators, transfer functions, well-posed and regular linear systems, concepts of controllability and stability of well-posed linear systems.

Chapter 3: The aim of this chapter is to study the well posedness, regularity and exact controllability for the problem of transmission of the Schrödinger equation with Dirichlet control and colocated observation. First, we form the system into an abstract framework of a first order colocated system, this formulate enable us to show that the system is well-posed with input and output space $U = Y = L^2(\Gamma)$, state space $X = H^{-1}(\Omega)$, by using the multiplier method. The regularity of the system is also established and the feedthrough operator is found to be zero. We conclude this chapter by obtaining the exact controllability using the observability inequality of the dual system.

Chapter 4: The objectif of this chapter is to generalize the well posedness for fourth order Schrödinger equation with hinged boundary control and colocated observation **[70]** to the variable coefficients case. On the one hand, we establish the well posedness of this system in the state space V' which is the dual space of $V = \left\{\varphi \in H^3(\Omega) : \varphi|_\Gamma = \mathbf{A}\varphi| = 0\right\}$ with respect to the pivot space $L^2(\Omega)$ and the input and output space $U = Y = L^2(\Gamma_0)$ with help of multiplier method on Riemannian manifold. In addition this system is regular with zero feedthrough operator. On the other hand, we establish the exact controllability of this system by presenting the observability inequality for the dual system.

Chapter 5: We recall that in chapter 4, our system is described with hinged boundary condition. In this chapter we study the same system with impose the moment boundary control and set the Dirichlet boundary condition to be zero. We begin with showing the well posedness of this system in the state space $H_0^1(\Omega)$ and input/output space $U = Y = L^2(\Gamma_0)$, the regularity of the system is also proved with feedthrough operator is found to be zero. From the result of the well posedness, we know that this system is exactly controllable in some interval $[0, T]$ $(T > 0)$ if and only if its corresponding closed loop systems under the output proportional feedback $u = -ky$, $k > 0$ is exponentialy stable. Based on this argument, to get the feedback stabilization of this system from the well posedness, we study the exact controllability of the open-loop system.

Chapter 6: In this chapter we consider an open-loop system of a fourth order Schrödinger equation with variable coefficients, Dirichlet boundary control and colocated observation, following the approach developed in **[67]** and the multiplier method on Riemannian manifold, we show that the system is well-posed with input and output space $U = Y = L^2(\Gamma_0)$, state space V' which is the dual space of $V = \left\{\varphi \in H^3(\Omega) : \varphi|_\Gamma = \frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}\Big| = 0\right\}$ with respect to the pivot space $L^2(\Omega)$. In addition, this system is regular with zero feedthrough operator. In order to prove the feedback stabilization from well posedness, we discuss the exact controllability of this system.

Chapter 2

Basic properties of regular linear systems

In this chapter, we introduce some basic concepts concerning regular linear systems. To this aim, we define the admissible control and observation operators, transfer function and well posed linear systems. For detailed definitions, we refer to Salamon [52], Curtain [17], Prichard and Salamon [47], or to Weiss [63], [64].

Notation. Throughout this chapter, U, X and Y are Hilbert spaces which are identified with their duals. A denote the generator of a strongly continuous semigroup S. The Hilbert spaces X_1 and X_{-1} are defined as follows: X_1 is the domain of A with the norm $\|x\|_1 = \|\,(\beta I - A)\,x\|$, where $\beta \in \rho\,(A)$ is fixed and X_{-1} the completion of X with respect to the norm $\|x\|_{-1} = \|\,(\beta I - A)^{-1}\,x$. These spaces are independent of the choise of β. If $D(A^*)$ is quite with its graph norm, then X_{-1} can be identified with $D(A^*)^*$, the dual of $D(A^*)$ with respect to the scalar product of X. We have $X_1 \hookrightarrow X \hookrightarrow X_{-1}$, densely and with continuous embeddings. The semi group S can be restricted to a semigroup on X_1 and extended to a semigroup on X_{-1}. These three semigroups are isomorphic and we shall denote them by the same symbol. The generator of S on X_1 is the restriction of A to $D(A)$, and the generator of S on X_{-1} is the extension of A to X, which is bounded as an operator from X to X_{-1}. Like in the case of S, we will use the same notation for the original generator A and for its restriction and extension described above.

2.1 Admissible control and observation operators

2.1.1 Admissible control operators

The concept of an admissible control operator is motivated by the study of the solutions of the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

where $u \in L^2_{loc}([0,\infty];U)$, $x(0) \in X$ and $B \in L(U, X_{-1})$. We would like to study those operators B for which all mild solutions x of this equation are continuous X-valued functions. Such operators will be called admissible.

Let $B \in L(U, X_{-1})$ and $\tau \geq 0$. We define the input maps $\boldsymbol{\phi}_\tau \in L(L^2([0,\infty);U), X_{-1})$ by

$$\boldsymbol{\phi}_\tau(u) := \int_0^\tau S(\tau-\sigma)Bu(\sigma)d(\sigma), \qquad \forall u \in L^2([0,\infty];U) \tag{2.2}$$

Definition 2.1. The operator $B \in L(U, X_{-1})$ is called an *admissible control operator for* S with respect to X_{-1} if for some $\tau > 0$ the inputs maps $\{\boldsymbol{\phi}_\tau\}_{\tau > 0}$ are bounded from $L^2([0,\infty);U)$ to X; i. e., Rang $\boldsymbol{\phi}_\tau \subset X$.

The operator B is called bounded if $B \in L(U,X)$ and unbounded otherwise. Obeviously, every bounded B is admissible for S.

Proposition 2.1. ([57]) Suppose that $B \in L(U, X_{-1})$ is admissible; i.e., Rang $\boldsymbol{\phi}_\tau \subset X$ holds for a specific $\tau > 0$. Then for every $t \geq 0$ we have

$$\boldsymbol{\phi}_\tau \in L(L^2([0,\infty);U), X)$$

Remark 2.1. By a step function on $[0,\tau]$ (or a piecewise constant function) we mean a function that is constant on each interval of a partition of $[0,\tau]$ into finitely many intervals. We have the following equivalent characterization of admissible control operators: $B \in L(U, X_{-1})$ is admissible iff, for some $\tau > 0$, there exists a $K_t > 0$ such that for every step function $v : [0,\tau] \to U$,

$$\|\boldsymbol{\phi}_\tau(v)\| \leq K_t \|v\|_{L^2}, \tag{2.3}$$

Proposition 2.2. ([57]) Assume that $B \in L(U, X_{-1})$ is an admissible control operator for S. Then for every $x_0 \in X$ and every $u \in L^2_{loc}([0,\infty);U)$, the initial value problem

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

has a unique solution in X_{-1}. This solution is given by

$$x(t) = S(t)x_0 + \boldsymbol{\phi}_\tau u, \quad t \geq 0$$

and it satisfies

$$x \in C([0,\infty);X) \cap H^1_{loc}([0,\infty);X_{-1}).$$

2.1.2 Admissible observation operators

We now introduce the concept of an admissible observation operator, which will turn out to be the dual of the concept of an admissible control operator.

Let $C \in L(X_1, Y)$. We are interested in the output function y generated by the system

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad x(0) = x_0, & (2.4)\\ y(t) &= Cx(t) & (2.5) \end{aligned}$$

where $x_0 \in X_1$ and $t \geq 0$.

Definition 2.2. An operator $C \in L(X_1, Y)$ is called an *admissible observation operator* for $S(t)$ with respect to X_1. If for some (and hence for any) $\tau > 0$, the output map $\mathbf{L}_\tau : X_1 \to L^2([0,+\infty);Y)$, defined by

$$\mathbf{L}_\tau(x)(t) = \begin{cases} CS(t)x_0, & \text{for } t \in [0,\tau) \\ 0 & \quad t \geq \tau \end{cases} \tag{2.6}$$

has a continuous extension to X.

Equivalently, $C \in L(X_1, Y)$ is called admissible observation operator for $S(t)$ iff, for some $\tau > 0$, there exists a constants $K_\tau > 0$, such that

$$\int_0^t \|CS(t)x_0\|_Y^2 \, dt \leq K_\tau \|x_0\|_X^2, \qquad \forall x_0 \in D(A),\ \tau \geq 0, \tag{2.7}$$

The operator C is called bounded if it can be extended such that $C \in L(X, Y)$ and unbounded otherwise. Obviously, every bounded C is admissible for S.

Theorem 2.1. **([57], [52])** *Suppose that $B \in L(U, X_{-1})$. Then B is an admissible control operator for $S(t)$ if and only if B^* is an admissible observation operator for $S^*(t)$. If B is admissible, then*

$$\|\phi_\tau^* x\| = \left\| \mathbf{L}_\tau^d x \right\|$$

where L_τ^d (with $\tau \geq 0$) are the output maps corresponding to the semigroup $S^*(t)$ with the observation operator B^*.

Definition 2.3. The *Lebesgue extension* of C (with respect to $S(t)$), $C_L : D(C_L) \to Y$ is defined by

$$C_L x_0 = \lim_{\tau \to 0} C \frac{1}{\tau} \int_0^\tau S(\sigma)\, x_0 d\sigma \tag{2.8}$$

with $D(C_L) = \{x_0 \in X/ \text{ the limit in (2.8) exists}\}$, and

$$X_1 \hookrightarrow D(C_L) \hookrightarrow X$$

For every $x_0 \in X$, there holds $S(t)x_0 \in D(C_L)$ for almost every $t \geq 0$ and

$$(\mathbf{L}_\infty x_0)(t) = C_L S(t) x_0$$

A similar Λ-extension of C, denote C_Λ, is defined by

$$C_\Lambda = \lim_{\lambda \to +\infty} C\lambda (\lambda I - A)^{-1} x_0 \tag{2.9}$$

Its domain $D(C_\Lambda)$ consists of all $x_0 \in X$, for which the above limit exists. C_Λ is an extension of C_L.

2.2 Transfer functions

In this section, we use the control and observation operators to obtain a simple representation of the transfer function.

Definition 2.4. Suppose that $B \in L(U, X_{-1})$ is an admissible control operator for $S(t)$ with respect to X_{-1} and that $C \in L(X_1, Y)$ is an admissible observation operator for $S(t)$ with respect to X_1. Then we define the *transfer functions* of the triple (A, B, C) to be the solutions, $G : \rho(A) \to L(U, Y)$ of

$$\frac{G(s) - G(\beta)}{s - \beta} = -C(sI - A)^{-1} (\beta I - A)^{-1} B, \qquad \text{for } s, \beta \in \rho(A),\ s \neq \beta \tag{2.10}$$

We remark that since B is an admissible control operator for $S(t)$, $(.I-A)^{-1}B$ is an $L(U,X)$-valued analytic function and since C is an admissible observation operator for $S(t)$, $C\,(.I-A)^{-1}$ is a $L(X,Y)$-valued analytic function. Both $(.I-A)^{-1}B$ and $C\,(.I-A)^{-1}$ are analytic on some right half plane $\mathbb{C}_\alpha^+ = \{s \in \mathbb{C}/\ \mathrm{Re}(s) > \alpha\}$. Thus any transfer function is $L(U,Y)$-valued function which is analytic in some $\mathbb{C}_\alpha^+$. Moreover any two transfer functions differ only by an additive constant, $D \in L(U,Y)$. The point is that they need not necessarily be bounded on any $\mathbb{C}_\alpha^+$.

2.3 Well-posed linear system

In the previous sections, we considered the admissible control, observation operators and the transfer functions, here we consider the extra assumption on the triple (A,B,C) to be well-posed.

Definition 2.5. Under the same assumption as in Definition 2.4, we say that the triple (A,B,C) is well-posed if $B \in L(U,X_{-1})$ is an admissible control operator for $S(t)$ and $C \in L(X_1,Y)$ is an admissible observation operator for (t) and its transfer functions are bounded on some half-plane $\mathbb{C}_\alpha^+$.i.e.

$$\sup_{\mathrm{Re}(s)>\alpha>\rho} \|G(\lambda)\|_{L(U,Y)} < \infty \tag{2.11}$$

The main result in Curtain and Weiss [13] is that a triple (A,B,C) that is well-posed defines a familly of well-posed linear systems, $\Sigma = (\mathbf{T},\boldsymbol{\phi},\mathbf{L},\mathbf{F})$, where $\mathbf{T},\boldsymbol{\phi}$ and $\mathbf{L}$ are as before and the input-output map $\mathbf{F}$ is defined by

$$(\mathbf{F}_\infty u)(t) = C_\Lambda \left[\int_0^\tau S(\tau-\sigma)Bu(\sigma)d(\sigma) - (\beta I - A)^{-1}Bu(t)\right] + G(\beta)u(t) \tag{2.12}$$

for $u \in L^2_{loc}(0,\infty;U)$.

If (A,B,C) is well-posed, then the state $x(t)$ and the output $y(t)$ satisfy the following equations for almost all t

$$\left\{\begin{array}{c} x(t) = S(t)x_0 + \int_0^t S\,(t-\tau)\,Bu(\tau)d\tau \in C\,([0,\infty);X) \\ \forall x_0 \in X, u \in L^2_{loc}\,(0,\infty;U)\,, \\ y(t) = C_\Lambda\left[x(t) - (\beta I - A)^{-1}Bu(t)\right] + G(\beta)u(t) \in L^2_{loc}\,(0,\infty;Y)\,, \\ \forall u \in L^2_{loc}\,(0,\infty;U)\,, \end{array}\right. \tag{2.13}$$

A well-posed system is a system for which both the state and output depend continuously on the initial state and input function of the system. The input/output functions u and y are locally L_2 functions with values in U and in Y respectively. The boundedness property mensioned earlier means that for every $t > 0$ there is a c_t (which independent of x_0 and of u) such that

$$\|x(t)\|^2 + \int_0^t \|y(\tau)\|^2\,d\tau \le c_t^2\left[\|x(0)\|^2 + \int_0^t \|u(\tau)\|^2\,d\tau\right] \tag{2.14}$$

2.4 Regular linear systems

The paper [61] introduced an important subclass of well-posed systems, the so called *regular systems,* for which the representation (2.13) becomes much simpler.

Definition 2.6. Let Σ be a well-posed linear system, if for any $u \in U$, the following limit exists

$$\lim_{t\to 0}\frac{1}{t}\int_0^t y_u(\tau)d\tau = \mathbf{D}u : \tag{2.15}$$

where $\mathbf{D}$ is called the feedthrough operator and y_u denotes the step response corresponding to the constant input u.

In particular, if Σ is a well-posed system according to Definition 2.5, then a sufficient condition that they generates a regular linear system is:

$$\text{there exists } s \in \rho(A) \text{ such that } (sI - A)^{-1} Bu \in D(C_L), \text{ for any } u \in U \tag{2.16}$$

In this case, $C_L (sI - A)^{-1} B$ is an analytic $L(U, Y)$-valued function of s on $\rho(A)$, bounded on some right-half plane.

Theorem 2.1. **([9])** *Let $\Sigma = (S, \boldsymbol{\phi}, \mathbf{L}, \mathbf{F})$ be a regular linear system with input and output spaces U and Y, respctively. Let A be the infinitesimal generator of S, B be the admissible control operator, C be the admissible observation operator, C_L is its Lebesgue extension and $\mathbf{D}$ be the feedthrough operator of Σ. Then for any $x_0 \in X$ and any $u \in L^2_{loc}(0,\infty;U)$ the functions $x : [0,\infty) \to X$ and $y \in L^2_{loc}(0,\infty;Y)$ defined by*

$$x(t) = S(t)x_0 + \boldsymbol{\phi}_\tau u \tag{2.17}$$
$$y = \mathbf{L}_\infty x_0 + \mathbf{F}_\infty u \tag{2.18}$$

satisfy the following equations for almost all $t \geq 0$

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.19}$$
$$y(t) = C_L x(t) + \mathbf{D}u(t) \tag{2.20}$$

In particular, the function x is the unique strong continuous solution of (2.17) under the initial condition $x(0) = x_0$ and $x(t) \in D(C_L)$.

The Laplace transform of y satisfies

$$\hat{y}(s) = C(sI - A)^{-1} x_0 + G(s)\hat{u}(s) \tag{2.21}$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large.

The transfer function G is given by

$$G(s) = C_L (sI - A)^{-1} B + \mathbf{D} \tag{2.22}$$

and, for any $u \in U$

$$\lim_{\lambda\to+\infty} G(\lambda)u = \mathbf{D}u, \quad \lambda \in \mathbb{R} \tag{2.23}$$

2.5 Usefullness of regular linear systems

The motivation for introducing regular linear systems has been the simple strucure of the output equation and the simple formula for the trunsfer function, because this allow us to try to replicate classical ideas from finite-dimensional control theory in an infinite-dimensional context. Good examples of this being done are the papers ([60], [15]) on Luenberger observers, dynamic stabilization and coprime factorization. Regular systems is also used in optimal control, see [11] and the references there in, the theory of exponential stabilization by colocated feedback in [14], in the state feedback regulator theory from [42], in the PI controller theory of [38] and others. The paper [39] explores the robust stability of feedback systems with respect to small delay in the loop.

2.6 Unitary group systems with unbounded control and colocated observation

We consider the linear time invariant system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad \text{in } D(A^*)' \\ y(t) &= B^* x(t) \end{aligned} \tag{2.24}$$

where

($\mathbf{a}_1$) $A : D(A) \, (\subset X) \to X$ satisfies $A^* = -A$ so that A is the infinitesimal generator of a unitary C_0-group e^{At} on X.

($\mathbf{a}_2$) B is linear operator $U \to \left[D(A^{*\frac{1}{2}}) \right]'$, duality with respect to X as a pivot space.

The following results provide sufficient conditions for well-posedness and regularity of system (2.24)

Theorem 2.2 ([67]) *Assume assumptions (a_1) and (a_2). If for some (and hence for all) $t > 0$, the input/output map is continuous:*

$$\|y\|_{L^2(0,t;U)} \leq C_t \|u\|_{L^2(0,t;U)}, x(0) = 0, \forall u \in L^2_{loc}(0, \infty; U) \tag{2.25}$$

for some positive constant C_t indepedent of u, then B is admissible for $S(t)$ and hence system (2.24) is well posed.

Theorem 2.3 ([67]) *Under the assumptions of Theorem 2.2, the system (2.24) is regular and the feedthrough operator is zero.*

Now, we recall the concepts of controllability, stability for the system (2.24) and explore how they are related to each other.

Definition 2.8. Let $\tau > 0$.

The system (2.24) is exactly controllable on X over $[0, \tau]$ if and only if,

For any $x_0, x_1 \in X$, there exists $u \in L^2([0, \tau]; U)$ such that the solution x of (2.24) satisfies $x(\tau) = x_1$.

Equivalently ([57]):

The system (2.1) is exactly controllable on X over $[0,\tau]$ if and only if, there exists $\gamma > 0$ such that

$$\int_0^t \|B^* S^*(\tau)\varphi\|_U^2 \, d\tau \geq \gamma \|\varphi\|_X^2 \,,$$

Definition 2.9. *A* C_0-semigroup $S(t)$ on Hilbert space X is exponentially stable, if for some constants M, $\omega \geq 0$

$$\|S(t)\| \leq Me^{-\omega t}, \quad \text{for all } t \geq 0.$$

The following Theorem relates the concepts of exact controllability and uniform stabilization for system (2.24).

Theorem 2.4 ([67], [34]) *Assume assumptions* (a_1) *and* (a_2)*, then the following assertions hold true*

1. *If the open-loop system of (2.24) is well-posed with state space* X*, input/output space* $U = Y$ *and exactly controllable on* X *over* $[0,T]$ *then, the operator* $A_F = A - BB^*$ *generates an exponential stable* C_0*-semi group on* X*.*

2. *If the operator* $A_F = A - BB^*$ *generates an exponential stable* C_0*-semi group on* X*, then the system (2.24) is exactly controllable.*

Chapter 3

Well posedeness, regularity and exact controllability for the problem of transmission of the Schrödinger equation

In this chapter we shall study the system of transmission of Schrödinger equation with Dirichlet boundary control and colocated observation. Using the mutiplier method, we show that the system is well-posed with input and output space $U = Y = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$. The regularity of the system is also established and the feedthrough operator is found to be zero. Finally, the exact controllobility of the open loop system is obtained by proving the observability inequality of the dual system. This chapter was the subject of the paper [2].

3.1 System description and main result.

Let Ω be an open bounded domain of $\mathbb{R}^n (n \geq 2)$ with smooth boundary Γ and let Ω_1 be a bounded domain contained inside Ω; $\bar{\Omega}_1 \subset \Omega$ with smooth boundary Γ_1,Ω_2 is the domain $\Omega \backslash \bar{\Omega}_1$ and ν is the unit normal of Γ or Γ_1 pointing toward the exterior of Ω_2.

Let a time $T > 0$ and two distinct constants a_1, $a_2 > 0$ be given.

In this chapter, we shall concerned with the following system of transmission of the Schrödinger equation with Dirichlet control and colocated observation.

$$
\begin{aligned}
y_t(x,t) &= \mathrm{i}\,\mathrm{div}\left(a(x)\nabla y(x,t)\right), & x \in \Omega, t > 0\ , && (3.1)\\
y(x,0) &= y^0(x), & x \in \Omega, && (3.2)\\
y_2(x,0) &= u(x), & (x,t) \in \Gamma \times (0,T), && (3.3)\\
y_1(x,0) &= y_2(x,0), & (x,t) \in \Gamma_1 \times (0,T), && (3.4)\\
a_1 \frac{\partial y_1(x,t)}{\partial \nu} &= a_2 \frac{\partial y_2(x,t)}{\partial \nu}, & (x,t) \in \Gamma_1 \times (0,T), && (3.5)\\
z(x,t) &= \mathrm{i} \frac{\partial}{\partial \nu}\left(A^{-1} y_2(x,t)\right), & (x,t) \in \Gamma \times (0,T), && (3.6)
\end{aligned}
$$

where $a(x) = \begin{cases} a_1, x \in \Omega_1 \\ a_2, x \in \Omega_2 \end{cases}$,

$y(x,t) = \begin{cases} y_1(x,t), & (x,t) \in \Omega_1 \times (0,T) \\ y_2(x,t), & (x,t) \in \Omega_2 \times (0,T) \end{cases}$,

$A : H^{-1}(\Omega) \to H^{-1}(\Omega)$ is a positive self adjoint operator defined by

$$Af = -\Delta f, \qquad D(A) = H_0^1(\Omega),$$

$u(.,.)$ is the input function, and $z(.,.)$ is the output function

Equation (3.1), known as the position-dependent-mass (effective mass) Schrödinger equation, has important applications in the field of material science and condensed matter physics such as semiconductor heterostructure(see [40], [51] and the references therin).

When $a_1 = a_2$, Guo and Shao[20] ahve shown that the system (3.1)-(3.6) is well posed with input and output space $U = Y = L^2(\Gamma)$ and the state space $X = H^{-1}$ and regular with zero as the feedthrough operator. One of the aims of this chapter is to investigate the well-posedness and regularity of the system (3.1)-(3.6) in the case where

$a_1 \neq a_2$.

Indeed, we shall prove the following

Theorem 3.1. *The equations (3.1)-(3.6) determines a well-posed linear system with input and output space $U = Y = L^2(\Gamma)$ and the state space $X = H^{-1}(\Omega)$.*

Theorem 3.2. *The equations (3.1)-(3.6) is regular with zero feedthrough operator. This means that the initial state $y(.,0) = 0$ and $u(.,t) = u(t) \in U$ is a step input, then the corresponding output satisfies*

$$\underset{\sigma \to 0}{Lim} \int_{\Gamma} \left| \frac{1}{\sigma} \int_0^{\sigma} z(x,t)dt \right| d\sigma = 0 \tag{3.7}$$

The second aim is to study the exact controllability problem for the open loop system (3.1)-(3.6). Exact controllability of the Schrödinger equation with smooth coefficients in the elliptic principal part and subject to boundary control was treated in [32], [41] and [56]. To state our exact controllability result, we need the following assumptions:

(A_1) $\Gamma = \Gamma_0 \cup \Gamma_1$; Γ_0 is possibly empty while Γ_1 is nonempty and relatively open.

(A_2) $a_2 < a_1$.

(A_3) There exists a real vector field $h(.) \in \left(C^1\left(\bar{\Omega}\right)\right)^n$ such that

$$\operatorname{Re}\left(\int_{\Omega} H(x)v(x).\overline{v(x)}dx\right) \geq \rho \int_{\Omega} \|v(x)\|^2 \, dx$$

for all $v(.) \in \left(L^2(\Omega)\right)^n$ for some $\rho > 0$, where

$$H(x) = \left(\frac{\partial h_i(x)}{\partial x_j}\right), i = 1, ..., n \text{ and } j = 1, ..., n.$$

(A_{3b})

$$h(x).\nu(x) \leq 0, \qquad x \in \Gamma_1.$$

(A_{3c})

$$h(x).\nu(x) \leq 0, \qquad x \in \Gamma_0.$$

Theorem 3.3. *Let $T > 0$ be arbitrary. Assume hypotheses (A_1) and (A_3), then for any initial data $y^0 \in H^{-1}(\Omega)$, there exists a control $u \in L^2(0,T;L^2(\Gamma))$ with $u = 0$ on Γ_0 such that the corresponding solution of the system (3.1)-(3.5) satisfies $y(x,T) = 0$.*

As a consequence of Theorem 2.1, Theorem 2.3, and proposition 3.1 of [34], we have the following uniform stabilization result for the system (3.1)-(3.5) on the space $H^{-1}(\Omega)$.

Corollary 3.1. Let the hypotheses of Theorem 2.3 hold true. Then there exist positive constant M, ω such that the solution of (3.1)-(3.5) with $u = -\alpha z$ ($\alpha > 0$) satisfies

$$\|y(t)\|_X \leq M e^{-\omega t} \left\|y^0\right\|_X.$$

3.2 Abstract formulation

We define the space

$$\begin{aligned} H^2(\Omega,\Gamma_1) &= \Big\{ y \in H_0^1(\Omega) : y_i = y| \in H^2(\Omega_i); i = 1,2; \\ & \qquad a_1 \frac{\partial y_1}{\partial \nu} = a_2 \frac{\partial y_2}{\partial \nu}, \qquad \text{on } \Gamma_1 \Big\} \end{aligned}$$

with the norm

$$\|y\|^2_{H^2(\Omega,\Gamma_1)} = \|y_1\|^2_{H^2(\Omega_1)} + \|y_2\|^2_{H^2(\Omega_2)}.$$

It can be shown that $H^2(\Omega,\Gamma_1)$ is dense in $H_0^1(\Omega)$.

Let $A_1 : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the extenstion of $-\operatorname{div}(a(x)\nabla .)$ to $H_0^1(\Omega)$. This means that $A_1 f = -\operatorname{div}(a(x)\nabla f)$ whenever $f \in H^2(\Omega,\Gamma_1)$ and that $A_1^{-1} f = -\operatorname{div}(a(x)\nabla)^{-1} g$ for any $g \in L^2(\Omega)$.

Let $A_{-1} : H^{-1}(\Omega) \to (D(A))'$ be the extension of A_1 to $H^{-1}(\Omega)$. Notice that $(D(A))'$ is the dual of $D(A)$ with respect to the pivot space $H^{-1}(\Omega)$.

Define the Dirichlet map γ by

$$\gamma u = v$$

if and only if

$$\begin{aligned} \operatorname{div}(a(x)\nabla v) &= 0 \text{ in } \Omega, \\ u &= v \text{ on } \Gamma, \\ v_1 &= v_2 \text{ on } \Gamma_1, \\ a_1 \frac{\partial v_1}{\partial \nu} &= a_2 \frac{\partial v_2}{\partial \nu} \text{ on } \Gamma_1, \end{aligned}$$

then $\gamma \in L\left(L^2(\Gamma), L^2(\Omega)\right)$([34]).

Using the operators introduced above, we can rewrite (3.1), (3.3)-(3.5) on $(D(A))'$ as

$$y_t(x,t) = -\mathrm{i}A_{-1}y(t) + Bu(t)$$

where $B \in L(U, (D(A))')$ is given by

$$Bu = \mathrm{i}A_{-1}\gamma u$$

we have via, Greens second theorem,

$$\gamma^* A\psi = -\frac{\partial\psi}{\partial\nu},\ \psi \in D(A).$$

Now, we can reformulate the system (3.1)-(3.6) into an abstract form in the state space $H^{-1}(\Omega)$ as follows

$$\begin{aligned} y_t(x,t) &= -\mathrm{i}A_1 y(t) + Bu(t) && (3.8)\\ y(0) &= y^0, && (3.9)\\ z(t) &= Cu(t) && (3.10) \end{aligned}$$

3.3 Proof of Theorem 3.1

The fact that the operator $-\mathrm{i}A_1$ generates a C_0-group of unitary operators $S(t)$ on X is a consequence of a Stone's Theorem (see [46]). In order to establish the admissibility of B and C for the group $S(t)$, we need the following identity, which is a particular case of the identity (3.69) in the appendix.

Lemma 3.1. Let $m(x)$ be a real vector field on $\bar{\Omega}$ of class C^1 such that

$$m = \nu \text{ on } \Gamma\ m = 0 \text{ in } \Omega_0,$$

where Ω_0 is an open domain in $\mathbb{R}^n$ that satisfies $\bar{\Omega}_1 \subset \Omega_0 \subset \bar{\Omega}_0 \subset \Omega$.

Let $\{\xi_0, f_i\} \in H^1(\Omega_i) \times L^1(0,T,L^2(\Omega_i))$, $i = 1.2$, such that

$$\begin{aligned} \xi_1^0 &= \xi_2^0 \text{ on } \Gamma_1,\\ \xi_2^0 &= 0 \text{ on } \Gamma \end{aligned}$$

then for every week solution of

$$\begin{aligned} \xi_t(x,t) &= \mathrm{i}\,\mathrm{div}\,(a(x)\nabla\xi(x,t)) + f(x,t), && (x,t) \in \Omega \times (0,T), && (3.11)\\ \xi(x,0) &= \xi^0(x), && x \in \Omega, && (3.12)\\ \xi_2(x,0) &= 0, && (x,t) \in \Gamma \times (0,T), && (3.13)\\ \xi_1(x,t) &= \xi_2(x,t), && (x,t) \in \Gamma_1 \times (0,T), && (3.14)\\ a_1\frac{\partial\xi_1(x,t)}{\partial\nu} &= a_2\frac{\partial\xi_2(x,t)}{\partial\nu}, && (x,t) \in \Gamma_1 \times (0,T), && (3.15) \end{aligned}$$

the following identity holds true:

$$
\begin{aligned}
& a_2 \int_0^T \int_\Gamma \left| \frac{\partial \xi}{\partial \nu} \right|^2 d\Gamma dt \qquad (3.16) \\
= \; & \operatorname{Im} \left(\int_{\Omega_2} \xi m . \nabla \bar{\xi} d\Omega \right)_0^T + a_2 \operatorname{Re} \int_0^T \int_{\Omega_2} \xi \nabla \bar{\xi} . \nabla \left(\operatorname{div} m \right) d\Omega dt \\
& -2 a_2 \operatorname{Re} \int_0^T \int_{\Omega_2} \nabla \xi . m \nabla \bar{\xi} d\Omega dt + \operatorname{Re} \int_0^T \int_{\Omega_2} \bar{f} \xi \operatorname{div} m d\Omega dt - 2 \operatorname{Im} \int_0^T \int_{\Omega_2} f m . \nabla \bar{\xi} d\Omega dt.
\end{aligned}
$$

Remark 3.1. Liu and Williams [9] made use of the vector field m to establish a boundary regularity for the problem of transmission of the plate equation.

3.3.1 Admissibility of B and C for the group $S(t)$.

Since the system (3.8)-(3.10) is colocated, the dmissibility of B for the group $S(t)$ is equivalent to the admissibility of C for the group $S(t)$. But the latter means that

$$\int_0^T \int_\Gamma |CS(t)\psi|^2 \, d\Gamma dt \leq k \|\psi\|_X^2 \qquad (3.17)$$

for all $\psi \in D(A)$ and for some $T > 0$.

Here and throughout the rest of the chapter, k is a positive constant that takes different values at different occurences.

An equivalent partial differential equation characterization of the estimate (3.17) is given by

$$\int_0^T \int_\Gamma \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Gamma dt \leq k \|\varphi_0\|_{H_0^1(\Omega)}^2 \qquad (3.18)$$

where $\varphi = A^{-1}\psi$ and φ is the solution of

$$
\begin{aligned}
\varphi_t(x,t) &= \mathrm{i} \operatorname{div} \left(a(x) \nabla \varphi(x,t) \right), && (x,t) \in \Omega \times (0,T), && (3.19) \\
\varphi(x,0) &= \varphi^0(x), && x \in \Omega, && (3.20) \\
\varphi_2(x,0) &= 0, && (x,t) \in \Gamma \times (0,T), && (3.21) \\
\varphi_1(x,t) &= \varphi_2(x,t), && (x,t) \in \Gamma_1 \times (0,T), && (3.22) \\
a_1 \frac{\partial \varphi_1(x,t)}{\partial \nu} &= a_2 \frac{\partial \varphi_2(x,t)}{\partial \nu}, && (x,t) \in \Gamma_1 \times (0,T). && (3.23)
\end{aligned}
$$

Specialization of the identity (3.16) to the φ-problem (3.19)-(3.23) yields

$$\int_0^T\int_\Gamma \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Gamma dt = \operatorname{Im}\left(\int_{\Omega_2}\varphi m.\nabla\bar{\varphi}d\Omega\right)_0^T + a_2\operatorname{Re}\int_0^T\int_{\Omega_2}\varphi\nabla\bar{\varphi}.\nabla(\operatorname{div} m)\,d\Omega dt$$
$$-2a_2\operatorname{Re}\int_0^T\int_{\Omega_2}\nabla\varphi.m\nabla\bar{\varphi}d\Omega dt \tag{3.24}$$

Using Schwartz and Poincaré inequalities, we obtain from (3.24)

$$\int_0^T\int_\Gamma \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Gamma dt \leq \int_0^T\int_\Omega |\nabla\varphi|^2\,d\Omega dt + k\int_\Omega |\nabla\varphi(x,0)|^2\,d\Omega + k\int_\Omega |\nabla\varphi(x,T)|^2\,d\Omega.$$

But

$$\int_\Omega |\nabla\varphi(x,t)|^2\,d\Omega = \int_\Omega \left|\nabla\varphi^0\right|^2 d\Omega$$

Thus

$$\int_0^T\int_\Gamma \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Gamma dt \leq k\,\|\varphi_0\|^2_{H_0^1(\Omega)}$$

3.3.2 Boundedness of the input/output map.

It suffices to show that the solution of (3.1)-(3.5) with $y(x,t)=0$ satisfies

$$\int_0^T\int_\Gamma \left|\frac{\partial A^{-1}y(x,t)}{\partial\nu}\right|^2 d\Gamma dt \leq k\int_0^T\int_\Gamma |u(x,t)|^2\,d\Gamma dt \tag{3.25}$$

for all $u\in L^2(0,T;U)$.

From the admissibility of B, we have $y\in C(0,T;H^{-1}(\Omega))$ for every $y^0\in H^{-1}(\Omega)$. Let as introduce a new variable by setting

$$\begin{aligned}
\omega_t(x,t) &= \mathrm{i}\operatorname{div}(a(x)\nabla\omega(x,t)) + \mathrm{i}\gamma u(x,t), && (x,t)\in\Omega\times(0,T), && (3.26)\\
\omega(x,0) &= 0, \quad x\in\Omega, && && (3.27)\\
\omega_2(x,0) &= 0, && (x,t)\in\Gamma\times(0,T), && (3.28)\\
\omega_1(x,t) &= \omega_2(x,t), && (x,t)\in\Gamma_1\times(0,T), && (3.29)\\
a_1\frac{\partial\omega_1(x,t)}{\partial\nu} &= a_2\frac{\partial\omega_2(x,t)}{\partial\nu}, && (x,t)\in\Gamma_1\times(0,T), && (3.30)
\end{aligned}$$

The estimate (3.25) becomes

$$\int_0^T\int_\Gamma \left|\frac{\partial\omega(x,t)}{\partial\nu}\right|^2 d\Gamma dt \leq k\int_0^T\int_\Gamma |u(x,t)|^2\,d\Gamma dt \tag{3.31}$$

As for (3.18), the estimate can also be deduced from the identity (3.16). Indeed setting $f = \mathrm{i}\gamma u$ in (3.16) and using the fact that $\gamma \in L\left(L^2(\Gamma), L^2(\Omega)\right)$, we obtain

$$\begin{aligned}\int_0^T\int_\Gamma \left|\frac{\partial\omega(x,t)}{\partial\nu}\right|^2 d\Gamma dt &\leq k\int_0^T\int_\Omega |\nabla\omega|^2\, d\Omega dt + k\int_\Omega |\nabla\omega(x,T)|^2\, d\Omega + \int_0^T\int_\Gamma |u|^2\, d\Gamma dt \\ &\leq k\left(\|\omega\|^2_{C(0,T;H_0^1(\Omega))} + \|u\|^2_{L^2(0,T;L^2(\Gamma))}\right).\end{aligned}$$

this together with the admissibility of B for the C_0-group $S(t)$ yields (3.31).

3.4 Proof of Theorem 3.2

Since the system (3.1)-(3.6) is well-posed, its trunsfer function $G(s)$ is bounded on some right half-plane (see[7]). To continue, we need the following results.

The assertion of Theorem 2.2 holds if for any $u \in C_0^\infty(\Gamma)$ the solution y of

$$\begin{aligned} sy(x) &= \mathrm{i}\,\mathrm{div}\,(a(x)\nabla y(x)), & x\in\Omega, \quad (3.32)\\ y_2(x) &= u(x), & x\in\Gamma, \quad (3.33)\\ y_1(x) &= y_2(x), & x\in\Gamma_1, \quad (3.34)\\ a_1\frac{\partial y_1(x)}{\partial\nu} &= a_2\frac{\partial y_2(x)}{\partial\nu}, & x\in\Gamma_1, \quad (3.35)\end{aligned}$$

Satisfies

$$\underset{s\in\mathbb{R},s\to+\infty}{Lim}\int_\Gamma \left|\frac{1}{s}\int_0^\sigma \frac{\partial y}{\partial\nu}\right| d\Gamma = 0 \tag{3.36}$$

Proof. We know from [7] that in the frequency domain, (3.7) is equivalent to

$$\underset{s\in\mathbb{R},s\to+\infty}{Lim} G(s)u = 0 \tag{3.37}$$

In the strong topologiy of U, for any $u \in U$. Due to the boundedness of $G(s)$ and the density of $L^2(\Gamma)$ in $C_0^\infty(\Gamma)$, it suffices to establish (3.37) for all $u \in C_0^\infty(\Gamma)$. Now for $u \in C_0^\infty(\Gamma)$ and $s > 0$, let y satisfies (3.32)-(3.35) and

$$(G(s)y)(x) = \mathrm{i}\frac{\partial A^{-1}y}{\partial\nu}(x), \quad x\in\Gamma.$$

It follows from Lemma 7.1 in the appendix that there exists a function $v \in H^2(\Omega,\Gamma_1)$ satisfies the following boundary value problem:

$$\begin{aligned} \mathrm{div}\,(a(x)\nabla y(x)) &= 0, & x\in\Omega,\\ v_2(x) &= u(x), & x\in\Gamma,\\ v_1(x) &= v_2(x), & x\in\Gamma_1,\\ a_1\frac{\partial v_1(x)}{\partial\nu} &= a_2\frac{\partial v_2(x)}{\partial\nu}, & x\in\Gamma_1,\end{aligned}$$

Consequently, (3.32)-(3.35) can be written as

$$\begin{aligned}
sy(x) - \mathrm{i}\,\mathrm{div}\left(a(x)\nabla\left(y(x)-v(x)\right)\right) &= 0,\ x\in\Omega,\\
\left(y_2(x)-v_2(x)\right) &= 0,\ x\in\Gamma,\\
\left(y_1(x)-v_1(x)\right) &= y_2(x)-v_2(x),\ x\in\Gamma_1,\\
a_1\frac{\partial\left(y_1(x)-v_1(x)\right)}{\partial\nu} &= a_2\frac{\partial\left(y_2(x)-v_2(x)\right)}{\partial\nu},\ x\in\Gamma_1,
\end{aligned}$$

Hence

$$\left(G(s)y\right)(x) = \frac{a_2}{s}\frac{\partial y(x)}{\partial\nu} - \frac{a_1}{s}\frac{\partial v(x)}{\partial\nu}.$$

This gives (3.36).

Lemma 3.2. Let m be the vector field introducing in subsection 3.1. Let $u\in C_0^{\infty}(\Gamma)$. Then the solution of (3.32) satisfies

$$\begin{aligned}
a_2\int\limits_{\Gamma}\left|\frac{\partial y(x)}{\partial\nu}\right|^2 d\Gamma &= -\frac{s}{a_2}\,\mathrm{Im}\int\limits_{\Omega} ym.\nabla\bar{y}d\Omega + 2\,\mathrm{Re}\int\limits_{\Omega_2}\nabla y_2.M\nabla\bar{y}_2 d\Omega \qquad (3.38)\\
&\quad -\int\limits_{\Omega_2}\left|\nabla y_2\right|^2\mathrm{div}\,m d\Omega + \int\limits_{\Gamma}\left|\nabla_\sigma y_2\right|^2 d\Gamma.
\end{aligned}$$

Proof. We multiply both sides of (3.32) by $m.\nabla\bar{y}$ and integrate over Ω, using Green's first Theorem, we find

$$\begin{aligned}
&s\int\limits_{\Gamma}|y|^2 m.\nu d\Gamma - s\int\limits_{\Omega}\bar{y}m.\nabla y d\Omega - s\int\limits_{\Omega}|y|^2\mathrm{div}\,m d\Omega + \mathrm{i}a_1\int\limits_{\Gamma_1}\frac{\partial y_1}{\partial\nu}m.\nabla\bar{y}_1 d\Gamma \qquad (3.39)\\
&+\mathrm{i}a_1\int\limits_{\Omega_1}\nabla y_1.\nabla\left(m.\nabla\bar{y}_1\right)d\Omega - \mathrm{i}a_2\int\limits_{\Gamma_2}\frac{\partial y_2}{\partial\nu}m.\nabla\bar{y}_2 d\Gamma - \mathrm{i}a_2\int\limits_{\Gamma}\frac{\partial y_2}{\partial\nu}m.\nabla\bar{y}_2 d\Gamma\\
&+\mathrm{i}a_2\int\limits_{\Omega_2}\nabla y_2.\nabla\left(m.\nabla\bar{y}_2\right)d\Omega = 0
\end{aligned}$$

Recalling the assumptions made on the vector field m, we simplify (3.39) to

$$\begin{aligned}
&s\int\limits_{\Gamma}|y|^2 m.\nu d\Gamma - s\int\limits_{\Omega}\bar{y}m.\nabla y d\Omega - s\int\limits_{\Omega}|y|^2\mathrm{div}\,m d\Omega - \mathrm{i}a_2\int\limits_{\Gamma}\left|\tfrac{\partial y_2}{\partial\nu}\right|^2 d\Gamma\\
&\qquad + \mathrm{i}a_2\int\limits_{\Omega_2}\nabla y_2.\nabla\left(m.\nabla\bar{y}_2\right)d\Omega = 0
\end{aligned}$$

from which we obtain

$$a_2\int\limits_{\Gamma}\left|\frac{\partial y_2}{\partial\nu}\right|^2 d\Gamma = -s\int\limits_{\Omega}\bar{y}m.\nabla y d\Omega - s\int\limits_{\Omega}|y|^2\mathrm{div}\,m d\Omega + a_2\,\mathrm{Re}\int\limits_{\Omega_2}\nabla y_2.\nabla\left(m.\nabla\bar{y}_2\right)d\Omega \qquad (3.40)$$

On the other hand, we have

$$\operatorname{Re}\int_{\Omega_2}\nabla y_2.\nabla\left(m.\nabla\bar{y}_2\right)d\Omega=\operatorname{Re}\int_{\Omega_2}\nabla y_2.M\nabla\bar{y}_2 d\Omega+\frac{1}{2}\int_{\Gamma}\left|\nabla y_2\right|^2 d\Gamma-\frac{1}{2}\int_{\Omega_2}\left|\nabla y_2\right|^2\operatorname{div} m d\Omega \tag{3.41}$$

where

$$M=\left(\frac{\partial m_i}{\partial x_j}\right)_{i,j=1,\ldots,n}$$

Using the fact that

$$\left|\nabla y_2\right|^2=\left|\nabla_\sigma y_2\right|^2+\left|\frac{\partial y_2}{\partial\nu}\right|^2 \text{ on } \Gamma,$$

(3.41) becomes

$$\begin{aligned}\operatorname{Re}\int_{\Omega_2}\nabla y_2.\nabla\left(m.\nabla\bar{y}_2\right)d\Omega &= \operatorname{Re}\int_{\Omega_2}\nabla y_2.M\nabla\bar{y}_2 d\Omega-\frac{1}{2}\int_{\Omega_2}\left|\nabla y_2\right|^2\operatorname{div} m d\Omega\\ &\quad+\frac{1}{2}\int_{\Gamma}\left|\nabla_\sigma y_2\right|^2 d\Gamma+\frac{1}{2}\int_{\Gamma}\left|\frac{\partial y_2}{\partial\nu}\right|^2 d\Gamma. \end{aligned}\tag{3.42}$$

Insertion of (3.42) into (3.40) yields (3.38). ■

Lemma 3.3. Let y be a solution of (3.32)-(3.35). Then

$$s\int_{\Omega}|y|^2 d\Omega+\mathrm{i}\int_{\Omega}a(x)\left|\nabla y\right|^2 d\Omega=\mathrm{i}a_2\int_{\Gamma}\frac{\partial y_2}{\partial\nu}\bar{y}_2 d\Gamma. \tag{3.43}$$

Proof. We multiply both sides of (3.32) by $\bar{y}$ and integrate over Ω. From Green's first theorem, we have

$$\begin{aligned} s\int_{\Omega}|y|^2 d\Omega-\mathrm{i}\Bigg\{ & a_2\int_{\Gamma}\frac{\partial y_2}{\partial\nu}\bar{y}_2 d\Gamma+a_2\int_{\Gamma_1}\frac{\partial y_2}{\partial\nu}\bar{y}_2 d\Gamma-a_2\int_{\Omega_2}\left|\nabla y_2\right|^2 d\Omega \\ & -a_1\int_{\Gamma_1}\frac{\partial y_1}{\partial\nu}\bar{y}_1 d\Gamma-a_1\int_{\Omega_2}\left|\nabla y_2\right|^2 d\Omega\Bigg\}. \end{aligned}\tag{3.44}$$

Inserting the boundary condition (3.35) into (3.44), we find that this simplifies to (3.43). ■

3.4.1 Completion of the proof of Theorem 3.2

We first introduce some constants:

$a=\min\left(a_1,a_2\right)$, $\mu_1=\sup_{\bar{\Omega}}\left|m(x)\right|$, $\mu_2=\sup_{\bar{\Omega}}\left\|M(x)\right\|$, $\mu_3=\sup_{\bar{\Omega}}\left|\operatorname{div} v(x)\right|$

From (3.38), we have the estimate

$$\frac{1}{s^2}\int_{\Gamma}\left|\frac{\partial y_2}{\partial \nu}\right|^2 d\Gamma \leq \frac{\mu_1}{2a_2 s^{\frac{1}{2}}}\int_{\Omega}|y|^2\,d\Omega+\frac{\mu_2}{2a_2 s^{\frac{3}{2}}}\int_{\Omega}|\nabla y|^2\,d\Omega + \frac{\mu_3+2\mu_2}{s^2}\int_{\Omega_2}|\nabla y_2|^2\,d\Omega+\frac{1}{s^2}\int_{\Gamma}|\nabla_\sigma y_2|^2\,d\Gamma \tag{3.45}$$

On the other hand, (3.43) implies

$$\frac{1}{s^{\frac{1}{2}}}\int_{\Omega}|y|^2\,d\Omega \leq \frac{a_2}{2s^{\frac{1}{2}}}\int_{\Gamma}|y_2|^2\,d\Gamma+\frac{a_2}{2s^{\frac{5}{2}}}\int_{\Gamma}\left|\frac{\partial y_2}{\partial \nu}\right|^2 d\Gamma, \tag{3.46}$$

$$\frac{1}{s^{\frac{3}{2}}}\int_{\Omega}\int_{\Omega}|\nabla y|^2\,d\Omega \leq \frac{a_2}{2as^{\frac{1}{2}}}\int_{\Gamma}|y_2|^2\,d\Gamma+\frac{a_2}{2as^{\frac{5}{2}}}\int_{\Gamma}\left|\frac{\partial y_2}{\partial \nu}\right|^2 d\Gamma. \tag{3.47}$$

substituting (3.46),(3.47) into (3.45), we get

$$\frac{1}{s^2}\int_{\Gamma}\left|\frac{\partial y_2}{\partial \nu}\right|^2 d\Gamma \leq \left(\frac{\mu_1}{4s^{\frac{1}{2}}}+\frac{\mu_1}{4as^{\frac{1}{2}}}+\frac{a_2\left(\mu_3+2\mu_2\right)}{2as}\right)\int_{\Gamma}|y_2|^2\,d\Gamma + \left(\frac{\mu_1}{4s^{\frac{1}{2}}}+\frac{\mu_1}{4as^{\frac{1}{2}}}+\frac{a_2\left(\mu_3+2\mu_2\right)}{2as}\right)\frac{1}{s^2}\int_{\Gamma}\left|\frac{\partial y_2}{\partial \nu}\right|^2 d\Gamma+\frac{1}{s^2}\int_{\Gamma}|\nabla_\sigma y_2|^2\,d\Gamma. \tag{3.48}$$

since

$$y_2 = u \quad \text{on } \Gamma\times(0,T)$$

and

$$\|y\|^2_{H^1(\Gamma)} = \|y\|^2_{L^2(\Gamma)}+\||\nabla_\sigma y|\|^2_{L^2(\Gamma)},$$

we rewrite (3.48) as follows:

$$\frac{1}{s^2}\int_{\Gamma}\left|\frac{\partial y_2}{\partial \nu}\right|^2 d\Gamma \leq \left(\frac{\mu_1}{4s^{\frac{1}{2}}}+\frac{\mu_1}{4as^{\frac{1}{2}}}+\frac{a_2\left(\mu_3+2\mu_2\right)}{2as}\right)\|u\|^2_{H^1(\Gamma)} + \left(\frac{\mu_1}{4s^{\frac{1}{2}}}+\frac{\mu_1}{4as^{\frac{1}{2}}}+\frac{a_2\left(\mu_3+2\mu_2\right)}{2as}\right)\frac{1}{s^2}\int_{\Gamma}\left|\frac{\partial y_2}{\partial \nu}\right|^2 d\Gamma.$$

This last estimate shows that

$$\underset{s\in\mathbb{R},s\to+\infty}{Lim}\int_{\Gamma}\left|\frac{1}{s}\int_0^{\sigma}\frac{\partial y}{\partial \nu}\right| d\Gamma = 0.$$

Remark 3.1. The result can also be proved by applying Theorem 5.2 of [67] which appeared shortly after the publication of their paper.

3.4.2 Proof of Theorem 3.3.

Let

$$E(t)=\int_{\Omega}a(x)\left|\nabla\varphi\right|^{2}d\Omega$$

bet the energy corresponding to the solution of the system (3.19)-(3.23). Then

$$E(t)=E(0)\text{ for all }t>0$$

By classical duality theory, to prove Theorem 3.2 it is enough to establish the associated observability inequalitiy

$$\int_{0}^{T}\int_{\Gamma}\left|\frac{\partial\varphi}{\partial\nu}\right|^{2}d\Gamma dt\geq k\left\|\varphi_{0}\right\|_{H_{0}^{1}(\Omega)}^{2} \tag{3.50}$$

where φ is the solution of the homogenuous system (3.19)-(3.23)

To this end, we apply the identity (3.69) to the $\varphi-$problem (3.19)-(3.23), to obtain

$$\begin{aligned}
&a_{2}\int_{0}^{T}\int_{\Gamma}\left|\frac{\partial\varphi_{2}}{\partial\nu}\right|^{2}h.\nu d\Gamma dt && (3.51)\\
=\;&2a_{1}\left(1-\frac{a_{1}}{a_{2}}\right)\int_{0}^{T}\int_{\Gamma_{1}}\left|\frac{\partial\varphi_{1}}{\partial\nu}\right|^{2}h.\nu d\Gamma dt+a_{2}\int_{0}^{T}\int_{\Gamma_{1}}\left|\nabla\varphi_{2}\right|^{2}h.\nu d\Gamma dt\\
&-a_{1}\int_{0}^{T}\int_{\Gamma_{1}}\left|\nabla\varphi_{1}\right|^{2}h.\nu d\Gamma dt+\operatorname{Im}\left[\int_{\Omega}\varphi h.\nabla\bar{\varphi}d\Omega\right]_{0}^{T}+2\operatorname{Re}\int_{0}^{T}\int_{\Omega}a(x)\nabla\varphi.H\nabla\bar{\varphi}d\Omega dt\\
&+\operatorname{Re}\int_{0}^{T}\int_{\Omega}a(x)\varphi\nabla\bar{\varphi}.\nabla(\operatorname{div}h)d\Omega dt. && (2.52)
\end{aligned}$$

But

$$\begin{aligned}
\left|\nabla\varphi_{i}\right|^{2} &= \left|\frac{\partial\varphi_{i}}{\partial\nu}\right|^{2}+\left|\nabla_{\sigma}\varphi_{i}\right|^{2}\text{ on }\Gamma_{1}\times(0,T),\ i=1,2\\
\left|\nabla_{\sigma}\varphi_{1}\right|^{2} &= \left|\nabla_{\sigma}\varphi_{2}\right|^{2}\text{ on }\Gamma_{1}\times(0,T),
\end{aligned}$$

then (A_2) and (A_{3b}) imply that

$$\begin{aligned}
&2a_{1}\left(1-\frac{a_{1}}{a_{2}}\right)\int_{0}^{T}\int_{\Gamma_{1}}\left|\frac{\partial\varphi_{1}}{\partial\nu}\right|^{2}h.\nu d\Gamma dt+a_{2}\int_{0}^{T}\int_{\Gamma_{1}}\left|\nabla\varphi_{2}\right|^{2}h.\nu d\Gamma dt-a_{1}\int_{0}^{T}\int_{\Gamma_{1}}\left|\nabla\varphi_{1}\right|^{2}h.\nu d\Gamma dt\\
=\;&2a_{1}\left(1-\frac{a_{1}}{a_{2}}\right)\int_{0}^{T}\int_{\Gamma_{1}}\left|\frac{\partial\varphi_{1}}{\partial\nu}\right|^{2}h.\nu d\Gamma dt-(a_{1}-a_{2})\int_{0}^{T}\int_{\Gamma_{1}}\left|\nabla_{\sigma}\varphi_{1}\right|^{2}h.\nu d\Gamma dt\geq0. && (3.53)
\end{aligned}$$

from (2.52) and (3.53), we deduce that

$$
\begin{aligned}
2\rho T\int\limits_{\Omega} a(x)\left|\nabla\varphi^0\right|^2 d\Omega \;\leq\; & a_2\int\limits_0^T\int\limits_{\Gamma_1}\left|\frac{\partial\varphi_2}{\partial\nu}\right|^2 h.\nu d\Gamma dt-\operatorname{Im}\left[\int\limits_{\Omega}\varphi h.\nabla\bar{\varphi}d\Omega\right]_0^T \\
& +\operatorname{Re}\int\limits_0^T\int\limits_{\Omega} a(x)\varphi\nabla\bar{\varphi}.\nabla(\operatorname{div} h)d\Omega dt. \qquad (3.54)
\end{aligned}
$$

Application of Schwartz and Poincaré inequalities to the $\int\limits_{\Omega}$-terms on the right hand side (3.54) yields

$$
\begin{aligned}
\left(2\rho T-\frac{c_1\varepsilon}{a_2}\right)\int\limits_{\Omega} a(x)\left|\nabla\varphi^0\right|^2 d\Omega \;\leq\; & a_2c_1\int\limits_0^T\int\limits_{\Gamma_1}\left|\frac{\partial\varphi_2}{\partial\nu}\right|^2 h.\nu d\Gamma dt \qquad (3.55)\\
& +\frac{1}{2}\left(\left(c_2c_p+c_2\right)T+\frac{2c_1c_p}{a_2\varepsilon}\right)\|\varphi\|^2_{C(0,T;H_0^1(\Omega)}
\end{aligned}
$$

where $c_1=\sup\limits_{\bar{\Omega}}|h(x)|$, $c_2=\sup\limits_{\bar{\Omega}}|\nabla(\operatorname{div} h)|$,

c_p is the Poincaré constant: $\int\limits_{\Omega}|\varphi|^2 d\Omega\leq c_p\int\limits_{\Omega}|\nabla\varphi|^2 d\Omega$, and

ε is an arbitary positive small constant

The sought-after estimate follows from (3.55) by a compactness/uniqueness argument.

3.5 Appendix

Lemma 3.4. Let f be a solution to the following elliptic problem:

$$
\begin{aligned}
\operatorname{div}(a(x)\nabla f(x)) &= g(x) \quad ,x\in\Omega, && (3.56)\\
f_2(x) &= u(x), \qquad x\in\Gamma, && (3.57)\\
f_1(x) &= f_2(x), \qquad x\in\Gamma_1, && (3.58)\\
a_1\frac{\partial f_1(x)}{\partial\nu} &= a_2\frac{\partial f_2(x)}{\partial\nu}, \qquad x\in\Gamma_1, && (3.59)
\end{aligned}
$$

for $g\in L^2(\Omega)$ and $u\in H^{\frac{3}{2}}(\Omega)$. Then there exists a constant k independent of f, g and u such that

$$\|f\|_{H^2(\Omega,\Gamma_1)}=k\{\|g\|_{L^2(\Omega)}+\|u\|_{H^{\frac{3}{2}}(\Gamma)}\}.$$

Proof. Let f be a solution to (3.56)-(3.59). Then f can be written as

$$f(x)=\begin{cases} f_1(x), & x\in\Omega_1\\ f_2(x), & x\in\Omega_2\end{cases},$$

where f_2 and f_1 are respectively the solution of

$$\begin{aligned} a_2\Delta f_2(x) &= g(x), \quad x \in \Omega, \\ f_2(x) &= u(x), \quad x \in \Gamma, \end{aligned}$$

and

$$\begin{aligned} a_1\Delta f_1(x) &= g(x), \quad x \in \Omega_1, \\ f_1(x) &= f_2(x), \ x \in \Gamma_1, \\ a_1\frac{\partial f_1(x)}{\partial \nu} &= a_2\frac{\partial f_2(x)}{\partial \nu}, \ x \in \Gamma_1, \end{aligned} \tag{3.60}$$

From elliptic regularity theory (see[37]), we have

$$\|f_2\|_{H^2(\Omega)} \leq k\{\|g\|_{L^2(\Omega)} + \|u\|_{H^{\frac{3}{2}}(\Gamma)}\}. \tag{3.61}$$

It follows from the trace theorem that $f_2|_{\Gamma_1} \in H^{\frac{3}{2}}(\Gamma_1)$ and

$$\|f_2\|_{H^{\frac{3}{2}}(\Gamma_1)} \leq k\,\|f_2\|_{H^2(\Omega)}\,. \tag{3.62}$$

(3.60) together with (3.62) implies again via the elliptic regularity that $f_1 \in H^2(\Omega_1)$ and

$$\|f_1\|_{H^2(\Omega_1)} \leq k\{\|g\|_{L^2(\Omega)} + \|f_2\|_{H^{\frac{3}{2}}(\Gamma_1)}\}. \tag{3.63}$$

combining (3.61),(3.62) and (3.63), we obtain

$$\|f_1\|_{H^2(\Omega_1)} + \|f_2\|_{H^2(\Omega)} \leq k\{\|g\|_{L^2(\Omega)} + \|u\|_{H^{\frac{3}{2}}(\Gamma)}\}.$$

from which follows the desired estimate, since

$$\|f\|_{H^2(\Omega,\Gamma_1)} = \|f_1\|^2_{H^2(\Omega_1)} + \|f_2\|^2_{H^2(\Omega)}\,. \qquad \blacksquare$$

Lemma 3.5. Let h be a real vector field of class C^1 on $\bar{\Omega}$. Then for every solution of the problem

$$\begin{aligned} \xi_t(x,t) &= \mathrm{i}\,\mathrm{div}\,(a(x)\nabla\xi(x,t)) + g(x,t), \quad (x,t) \in \Omega \times (0,T), && (3.64) \\ \xi(x,0) &= \xi^0(x), \qquad x \in \Omega, && (3.65) \\ \xi_2(x,0) &= 0, \qquad (x,t) \in \Gamma \times (0,T), && (3.66) \\ \xi_1(x,t) &= \xi_2(x,t), \qquad (x,t) \in \Gamma_1 \times (0,T), && (3.67) \\ a_1\frac{\partial \xi_1(x,t)}{\partial \nu} &= a_2\frac{\partial \xi_2(x,t)}{\partial \nu}, \qquad (x,t) \in \Gamma_1 \times (0,T), && (3.68) \end{aligned}$$

we have

$$
\begin{aligned}
&a_2\int_0^T\int_\Gamma\left|\frac{\partial\xi_2}{\partial\nu}\right|^2 h.\nu d\Gamma dt+2a_1\left(\frac{a_1}{a_2}-1\right)\int_0^T\int_{\Gamma_1}\left|\frac{\partial\xi_1}{\partial\nu}\right|^2 h.\nu d\Gamma dt \qquad (3.69)\\
&-a_2\int_0^T\int_{\Gamma_1}\left|\nabla\xi_2\right|^2 h.\nu d\Gamma dt+a_1\int_0^T\int_{\Gamma_1}\left|\nabla\xi_1\right|^2 h.\nu d\Gamma dt\\
=\ &\operatorname{Im}\left[\int_\Omega\varphi h.\nabla\bar\varphi d\Omega\right]_0^T+2\operatorname{Re}\int_0^T\int_\Omega a(x)\nabla\xi.H\nabla\bar\xi d\Omega dt\\
&+\operatorname{Re}\int_0^T\int_\Omega a(x)\xi\nabla\bar\xi.\nabla(\operatorname{div}h)d\Omega dt+\operatorname{Im}\int_0^T\int_\Omega\bar g\xi\operatorname{div}hd\Omega dt-2\operatorname{Im}\int_0^T\int_\Omega gh.\nabla\bar\xi d\Omega dt
\end{aligned}
$$

Proof. The identity (3.69) will be established for strong solutions and the general case will follow by a standard density argument. to this end, let $\left\{\xi_i^0,f_i\right\}\in H^2(\Omega_i)\times H^1(\Omega_i)\times L^1(0,T,H^1(\Omega_i))$, $i=1.2$, such that

$$
\begin{aligned}
\xi_1^0&=\xi_2^0 \text{ on } \Gamma_1,\\
g_1&=g_2 \text{ on } \Gamma_1\times(0,T),\\
\xi_2^0&=0 \text{ on } \Gamma\\
g_2&=0 \text{ on } \Gamma\times(0,T),\\
a_1\frac{\partial\xi_1^0}{\partial\nu}&=a_2\frac{\partial\xi_2^0}{\partial\nu},\ \text{ on } \Gamma_1,
\end{aligned}
$$

we multiply both sides of (3.64) by $h.\nabla\bar\xi$ and integrate over $\Omega\times(0,T)$ to obtain

$$
\int_0^T\int_\Omega\xi_t h.\nabla\bar\xi d\Omega dt=\mathrm{i}\int_0^T\int_\Omega\operatorname{div}\left(a(x)\nabla\xi\right)h.\nabla\bar\xi d\Omega dt+\int_0^T\int_\Omega gh.\nabla\bar\xi d\Omega dt. \qquad (3.70)
$$

we have

$$
\begin{aligned}
\int_0^T\int_\Omega\xi_t h.\nabla\bar\xi d\Omega dt&=\left[\int_\Omega\xi h.\nabla\bar\xi d\Omega\right]_0^T-\int_0^T\int_\Gamma\xi\bar\xi_t h.\nu d\Gamma dt \qquad (3.71)\\
&+\int_0^T\int_\Omega\left(-\mathrm{i}\operatorname{div}\left(a(x)\nabla\bar\xi\right)+\bar g\right)h.\nabla\bar\xi d\Omega dt+\int_0^T\int_\Omega\xi\bar\xi_t\operatorname{div}hd\Omega dt.
\end{aligned}
$$

substituting (3.71) into (3.70), we get

$$
\begin{aligned}
&\left[\int_\Omega\xi h.\nabla\bar\xi d\Omega\right]_0^T-\int_0^T\int_\Gamma\xi\bar\xi_t h.\nu d\Gamma dt+\int_0^T\int_\Omega\left(-\mathrm{i}\operatorname{div}\left(a(x)\nabla\bar\xi\right)+\bar g\right)h.\nabla\bar\xi d\Omega dt\\
&+\int_0^T\int_\Omega\xi\bar\xi_t\operatorname{div}hd\Omega dt=\mathrm{i}\int_0^T\int_\Omega\left(\operatorname{div}\left(a(x)\nabla\bar\xi\right)\right)h.\nabla\bar\xi d\Omega dt+\int_0^T\int_\Omega gh.\nabla\bar\xi d\Omega dt.
\end{aligned}
$$

hence

$$2\operatorname{Re}\int_0^T\int_\Omega \left(\operatorname{div}\left(a(x)\nabla\bar{\xi}\right)\right)h.\nabla\bar{\xi}d\Omega dt = \operatorname{Im}\left[\int_\Omega \xi h.\nabla\bar{\xi}d\Omega\right]_0^T - \operatorname{Im}\int_0^T\int_\Gamma \xi\bar{\xi}_t h.\nu d\Gamma dt$$
$$+\operatorname{Im}\int_0^T\int_\Omega \xi\bar{\xi}_t \operatorname{div} h d\Omega dt - 2\operatorname{Im}\int_0^T\int_\Omega g h.\nabla\bar{\xi}d\Omega dt. \tag{3.72}$$

Using Green's first theorem along with identity

$$2\operatorname{Re}\int_\Omega \nabla\omega.\nabla(h.\nabla\bar{\omega})d\Omega = 2\operatorname{Re}\int_\Omega \nabla\omega.H\nabla\bar{\omega}d\Omega + \int_\Omega h.\nabla\left(|\nabla\omega|^2\right)d\Omega,$$

we rewrite the left hand side of (3.72) as

$$2\operatorname{Re}\int_0^T\int_\Omega \left(\operatorname{div}\left(a(x)\nabla\bar{\xi}\right)\right)h.\nabla\bar{\xi}d\Omega dt \tag{3.73}$$
$$= \quad 2a_2\operatorname{Re}\int_0^T\int_{\Gamma_1}\frac{\partial\xi_2}{\partial\nu}h.\nabla\bar{\xi}_2 d\Gamma dt + 2a_2\operatorname{Re}\int_0^T\int_\Gamma \frac{\partial\xi_2}{\partial\nu}h.\nabla\bar{\xi}_2 d\Gamma dt$$
$$-2a_1\operatorname{Re}\int_0^T\int_{\Gamma_1}\frac{\partial\xi_1}{\partial\nu}h.\nabla\bar{\xi}_1 d\Gamma dt - a_2\int_0^T\int_\Gamma |\nabla\xi_2|^2 h.\nu d\Gamma dt - a_2\int_0^T\int_{\Gamma_1}|\nabla\xi_2|^2 h.\nu d\Gamma dt$$
$$+a_1\int_0^T\int_{\Gamma_1}|\nabla\xi_1|^2 h.\nu d\Gamma dt - 2a_2\int_0^T\int_{\Omega_2}\nabla\xi_2.H\nabla\bar{\xi}_2 d\Omega dt + a_2\int_0^T\int_{\Omega_2}|\nabla\xi_2|^2\operatorname{div} h d\Omega dt$$
$$-2a_1\operatorname{Re}\int_0^T\int_{\Omega_1}\nabla\xi_1.H\nabla\bar{\xi}_1 d\Omega dt + a_1\int_0^T\int_{\Omega_2}|\nabla\xi_1|^2\operatorname{div} h d\Omega dt$$

recalling the boundary conditions(3.66)-(3.68), we have

$$h.\nabla\xi_2 \quad = \quad \frac{\partial\xi_2}{\partial\nu}h.\nu \text{ on } \Gamma\times(0,T), \tag{3.74}$$
$$h.\nabla\left(\xi_1-\xi_2\right) \quad = \quad \frac{\partial\left(\xi_1-\xi_2\right)}{\partial\nu}h.\nu$$
$$= \quad \left(1-\frac{a_1}{a_2}\right)\frac{\partial\xi_1}{\partial\nu} \text{ on } \Gamma_1\times(0,T). \tag{3.75}$$

Inserting (3.74) and (3.75), we find that this simplifies to

$$
\begin{aligned}
& 2\operatorname{Re}\int_0^T\int_\Omega \left(\operatorname{div}\left(a(x)\nabla\bar{\xi}\right)\right) h.\nabla\bar{\xi}\,d\Omega dt && (3.76)\\
= \quad & -2a_1\left(1-\frac{a_1}{a_2}\right)\int_0^T\int_{\Gamma_1}\left|\frac{\partial\xi_1}{\partial\nu}\right|^2 h.\nu d\Gamma dt + a_2\int_0^T\int_\Gamma\left|\frac{\partial\xi_2}{\partial\nu}\right|^2 h.\nu d\Gamma dt - a_2\int_0^T\int_{\Gamma_1}\left|\nabla\xi_2\right|^2 h.\nu d\Gamma dt \\
& +a_1\int_0^T\int_{\Gamma_1}\left|\nabla\xi_1\right|^2 h.\nu d\Gamma dt - 2\operatorname{Re}\int_0^T\int_\Omega a(x)\nabla\xi.H\nabla\bar{\xi}\,d\Omega dt + \int_0^T\int_\Omega a(x)\left|\nabla\xi\right|^2 \operatorname{div} h\,d\Omega dt
\end{aligned}
$$

Now, we consider the third integral on the right hand side of (3.72). Applying Green's first theorem and taking into consideration the boundary condition (3.66), we obtain

$$
\begin{aligned}
& \operatorname{Im}\int_0^T\int_\Omega \xi\bar{\xi}_t \operatorname{div} h\,d\Omega dt && (3.77)\\
= \quad & \operatorname{Re}\int_0^T\int_\Omega a(x)\left|\nabla\xi\right|^2 \operatorname{div} h\,d\Omega dt + \operatorname{Re}\int_0^T\int_\Omega a(x)\xi\nabla\bar{\xi}.\nabla\left(\operatorname{div} h\right) d\Omega dt + \operatorname{Im}\int_0^T\int_\Omega \bar{g}\xi \operatorname{div} h\,d\Omega dt.
\end{aligned}
$$

Substituting (3.76) and (3.77) into (3.72) and using the boundary condition (3.66), we obtain (3.69). ■

Chapter 4

Well posedness and exact controllability of fourth order Schrödinger equation with variable coefficients, hinged boundary control and colocated observation

The objectif of this chapter is to generalize the well-posedness for fourth order Schrödinger equation with hinged boundary control and colocated observation [**70**] to the variable coefficients case. On the one hand, we establish the well-posedness of this system in the state space V' which is the dual space of $V = \left\{\varphi \in H^3(\Omega) : \varphi|_\Gamma = \mathbf{A}\varphi| = 0\right\}$ with respect to the pivot space $L^2(\Omega)$ and the input/output space $U = Y = L^2(\Gamma_0)$ with help of multiplier method on Riemannian manifold. In addition this system is regular with feedthrough operator is found to be zero. On the other hand, we establish the exact controllability of this system by presenting the observability inequality for the dual system.

4.1 System description and statement of main results.

The system what are concerned with in this paper is described by the following PDEs

$$\begin{cases} iw_t(x,t) + \mathbf{A}^2 w(x,t) = 0 & x \in \Omega, t > 0 \\ w(x,t) = u(x,t) & x \in \Gamma_0, t \geq 0 \\ w(x,t) = 0 & x \in \Gamma_1, t \geq 0 \\ \mathbf{A}w(x,t) = 0 & x \in \partial\Omega, t \geq 0 \\ y(x,t) = -i\frac{\partial(A_1^{-2})w(x,t)}{\partial\nu_{\mathbf{A}}} & x \in \Gamma_0, t \geq 0 \end{cases} \tag{4.1}$$

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is an open bounded region with C^3-boundary $\partial\Omega = \Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$ and assume that Γ_0 $(int\Gamma_0 \neq \varnothing)$ and Γ_1 are relatively open in $\partial\Omega$ and $\Gamma_0 \cap \Gamma_1 = \varnothing$.

The operator A_1 is defined in (4.4) later and $\mathbf{A}$ is a second order partial differential operator

$$\mathbf{A} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) \tag{4.2}$$

which for some constants $a, b > 0$, satisfies

$$\begin{aligned} a\sum_{i=1}^{n} |\xi_i|^2 &\leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\bar{\xi}_j \leq b\sum_{i=1}^{n} |\xi_i|^2 , \\ a_{ij} &= a_{ji} \in C^{\infty}(\mathbb{R}^n), \quad \forall i,j = 1,2,...,n. \\ \forall x &\in \bar{\Omega},\ \xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{C}^n, \quad , \end{aligned} \tag{4.3}$$

we define the operator A_1 as follows

$$\begin{aligned} A_1 f &= -\mathbf{A} f \\ \forall f &\in D(A_1) = H^2(\Omega) \cap H_0^1(\Omega) \end{aligned} \tag{4.4}$$

and define

$$\begin{aligned} \nu_{\mathbf{A}} &= \left(\sum_{k=1}^{n} \nu_k a_{k1}(x), \sum_{k=1}^{n} \nu_k a_{k2}(x), \sum_{k=1}^{n} \nu_k a_{k3}(x), ..., \sum_{k=1}^{n} \nu_k a_{kn}(x) \right) \\ \frac{\partial}{\partial \nu_{\mathbf{A}}} &= \sum_{i,j=1}^{n} a_{ij}(x)\nu_i \frac{\partial}{\partial x_j} \end{aligned} \tag{4.5}$$

where ν is the unit normal vector of $\partial\Omega$ pointing outwards of Ω, u and y are the boundary control and the boundary observation of the system (4.1) respectively.

Now, let A the positive self-adjoint operator in $L^2(\Omega)$ defined by

$$\begin{aligned} A\varphi &= \mathbf{A}^2\varphi, \\ D(A) &= \left\{\varphi \in L^2(\Omega), \mathbf{A}^2\varphi \in L^2(\Omega), \varphi|_{\Gamma} = \mathbf{A}\varphi|_{\Gamma} = 0\right\} \\ &= \left\{\varphi \in H^4(\Omega); \varphi|_{\Gamma} = \mathbf{A}\varphi|_{\Gamma} = 0\right\} \end{aligned} \tag{4.6}$$

One can show that $A^{\frac{1}{2}} = A_1$ where A_1 is given by (4.4)

Let $V = D(A^{\frac{3}{4}}) = \left\{\varphi \in H^3(\Omega); \varphi|_{\Gamma} = \mathbf{A}\varphi|_{\Gamma} = 0\right\}$ and V' its dual space with respect to the pivot space $L^2(\Omega)$, in the sence of Gelfand's triple inclusions

$$V \hookrightarrow L^2(\Omega) \hookrightarrow V'.$$

The following Theorem shows that the system (4.1) is well-posed with state space V' and input/output space $U = Y = L^2(\Gamma_0)$.

Theorem 4.1 *The system (4.1) is well-posed. More precisely, for any $T > 0$, initial value $w_0 \in V'$ and control input $u \in L^2(0,T;U)$ there exists a constant C_T that is independent of w_0* and u *such that*

$$\|w(.,T)\|_{V'}^2 + \|y\|_{L^2(0,T;U)}^2 \leqslant C_T(\|w_0\|_{V'}^2 + \|u\|_{L^2(0,T;U)}) \tag{4.7}$$

It is proved in [24, Theorem 5.8] (see also [43, Theorem 5.2]) that if the abstract system (4.1) introduced later is well-posed, it must be regular in the sense of Weiss with the zero feedthrough operator.

The following result is hence a consequence of Theorem 4.1.

Corollary 4.1 *The system (4.1) is regular with zero feedthrough operator. This means that if the initial state* $w(.,0) = 0$ *and* $u(.,t) = u(t) \in U$ *is a step input, then the correspending output satisfies*

$$\underset{\sigma \to 0}{Lim} \int_{\Gamma} \left| \frac{1}{\sigma} \int_{0}^{\sigma} y(x,t)dt \right|^2 d\Gamma = 0 \tag{4.8}$$

The second aim is to study the exact controllability problem for the open loop system (4.1), this is the result of Theorem 4.1 under a certain geomtric condition on Ω.

(H1) There is a vector field N on $(\mathbb{R}^n, g)$ such that

$$DN(X,X) = b(x)\,|X|_g^2\,, \quad \forall X \in T_x\mathbb{R}^n,\ x \in \Omega. \tag{H1}$$

where $b(x)$ is a function defined on Ω so that

$$b_0 = \inf_{x\in\Omega} b(x) > 0. \tag{H2}$$

(H2)

$$\Gamma \text{ satisfies } N(x).\nu > 0 \text{ on } \Gamma_0 \tag{H3}$$

Theorem 4.2 *Under assumptions (H1)-(H3), system (4.1) is exactly controllable on some* $[0,T]$, $T > 0$. *That is, given initial data* $w(.,0) = w_0 \in V'$ *and time* $T > 0$, *there exists a boundary control* $u \in L(0,T; L^2(\Gamma_0))$, *such that the unique solution to the system (4.1) satisfies* $w(T) = 0$.

The following result is a direct consequence Theorems 4.1 and 4.2.

Corollary 4.2

Let the Hypotheses of Theorem 4.2 hold true. Then system (4.1) is exponentially stable under the proportional output feedback $u = -ky$ for any $k > 0$.

4.2 Abstarct formulation

In this section we cast the system (4.1) into an abstract framework of a first order collocated system in the state space V' and input/output space $U = Y = L^2(\Gamma_0)$. Extend the operator $\tilde{A}$ of A to the domain V as follows:

$$\langle \tilde{A}\varphi, \psi\rangle_{V'} = \langle A^{1/2}\varphi, A^{1/2}\psi\rangle_{V'}, \quad \forall \varphi, \psi \in V. \tag{4.9}$$

Then $\tilde{A}$ is a positive self-adjoint operato in V'. In fact,

$$\begin{aligned} \langle \tilde{A}\varphi, \varphi\rangle_{V'} &= \langle A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\varphi\rangle_{V'} = \langle A^{-\frac{1}{4}}\varphi, A^{-\frac{1}{4}}\varphi\rangle_{L^2(\Omega)} \\ &\geq C\,\|\varphi\|^2_{L^2(\Omega)} \geq C'\left\|A^{-\frac{3}{4}}\varphi\right\|^2_{L^2(\Omega)} \\ &= C'\,\|\varphi\|^2_{V'}\,, \quad \forall \varphi \in V. \end{aligned} \tag{4.10}$$

where C and $C^{'}$ are constants. We identify $H = V^{'}$ with it's dual $H^{'}$. Then the following Gelfand triple continuous inclusions positive hold true:

$$D(\tilde{A}^{\frac{1}{2}}) \hookrightarrow H = H^{'} \hookrightarrow D(\tilde{A}^{\frac{1}{2}})^{'}. \tag{4.11}$$

Define an extension $\hat{A} \in L(D(\tilde{A}^{1/2}), D(\tilde{A}^{1/2})')$ of $\tilde{A}$:

$$\langle \hat{A}f, g\rangle_{D(\tilde{A}^{\frac{1}{2}})', D(\tilde{A}^{\frac{1}{2}})} = \langle \tilde{A}^{\frac{1}{2}}f, \tilde{A}^{\frac{1}{2}}g\rangle_{V'}, \quad \forall f, g \in D(\tilde{A}^{\frac{1}{2}}). \tag{4.12}$$

Let γ be the Dirichlet map: $\gamma \in L(L^2(\Gamma_0), H^{1/2}(\Omega))$ [34, p. 188-189] so that $\gamma u = \phi$ if and only if

$$\begin{cases} \mathbf{A}^2\phi = 0, \quad x \in \Omega, \\ \phi(x)|_{\Gamma_0} = u(x), \quad \phi(x)|_{\Gamma_1} = 0\ , \\ \mathbf{A}\phi(x)|_{\Gamma} = 0 \end{cases} \tag{4.13}$$

By virtue of the above map, one can write (4.1) in $D(\tilde{A}^{\frac{1}{2}})^{'}$ as

$$\dot{w} = \mathrm{i}\hat{A}w + Bu. \tag{4.14}$$

where $B \in L(U, D(\tilde{A}^{\frac{1}{2}})^{'})$ is given by

$$Bu = -\mathrm{i}\hat{A}\gamma u, \quad \forall u \in U. \tag{4.15}$$

Define $B^* \in L(D(\tilde{A}^{\frac{1}{2}}), U)$ by

$$\langle B^*f, u\rangle_U = \langle f, Bu\rangle_{D(\tilde{A}^{\frac{1}{2}}), D(\tilde{A}^{\frac{1}{2}})'}, \quad \forall f \in D(\tilde{A}^{\frac{1}{2}}),\ u \in U.$$

Then for any $f \in D(\tilde{A}^{\frac{1}{2}})$ and $u \in C_0^\infty(\Gamma_0)$, we have

$$\begin{aligned} \langle f, Bu\rangle_{D(\tilde{A}^{\frac{1}{2}}), D(\tilde{A}^{\frac{1}{2}})'} &= \langle f, -\mathrm{i}\hat{A}\gamma u\rangle_{D(\hat{A}^{1/2}), D(\hat{A}^{1/2})'} = -\mathrm{i}\langle \hat{A}^{1/2}f, \hat{A}^{1/2}\gamma u\rangle_{V'} \\ &= -\mathrm{i}\langle A_1^2(A_1^{-2}f), \tilde{A}\gamma u\rangle_{V''} \\ &= -\mathrm{i}\langle A^{-3/4}A_1^2(A_1^{-2}f), A^{-3/4}A\gamma u\rangle_{L^2(\Omega)}, \text{ with } A^{\frac{1}{2}} = A_1 \\ &= -\mathrm{i}\langle A_1(A_1^{-2}f), \gamma u\rangle_{L^2(\Omega).} = \langle u, -\mathrm{i}\frac{\partial(A_1^{-2}f))}{\partial\nu_{\mathbf{A}}}, u\rangle_{L^2(\Gamma_0).}. \end{aligned}$$

We have used in the last step Green's second theorem.

Since $C_0^\infty(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, we obtain

$$B^*f = -\mathrm{i}\left.\frac{\partial(A_1^{-2}f)}{\partial\nu_{\mathbf{A}}}\right|_{\Gamma_0}, \quad \forall f \in D(\tilde{A}^{1/2}). \tag{4.16}$$

we have thus formulated the open loop system (4.1) into an abstract first-order form in the state space V':

$$\dot{w} = \mathrm{i}\hat{A}w + Bu \tag{4.17}$$
$$y = B^*w \tag{4.18}$$

where $\hat{A}$, B and B^* are defined by (4.12), (4.15) and (4.16) respectively.

The operator A defined in (4.6) is a positive self adjoint in $L^2(\Omega)$, then the operator $\tilde{A}$ defined in (4.10) is a positive self adjoint in V', this follows that the operator $\mathrm{i}\hat{A}$ is a skew-adjoint $\left(\mathrm{i}\hat{A}\right)^* = -\mathrm{i}\hat{A}$ and from the Stone's Theorem, the operator $\mathrm{i}\hat{A}$ is the infinitesimal generator of a C_0-unitary group on V'.

4.3 Proof of Theorem 4.1

To prove Theorem 4.1, we need the following Lemma which comes from Theorem 8.4 of [24].

Lemma 4.1 If there exist constants $T > 0$ and $C_T > 0$ such that the input and the output of system (4.1) satisfy

$$\int_0^T \|y(t)\|_U^2 \, dt \leq C_T \int_0^T \|u(t)\|_U^2 \, dt, \quad \forall u \in L^2(0,T;L^2(\Gamma_0)) \tag{4.19}$$

with $w(.,0) = 0$, the system (4.1) is well-posed.

By lemma 4.1, Theorem 4.1 amounts to saying that the solution to system (4.1) with zero initial data satisfies

$$\|y(t)\|^2_{L^2(0,T;L^2(\Gamma_0))} \leq C_T \|u(t)\|^2_{L^2(0,T;L^2(\Gamma_0))}, \quad \forall u \in L^2(0,T;L^2(\Gamma_0)).$$

Make a transformation $z = A_1^{-3} w(t) \in C(0,T;V)$. Instead with (4.1), we consider the following system in V :

$$\begin{cases} z_t(x,t) = \mathrm{i}\mathbf{A}^2 z(x,t) + \mathrm{i}\left(G_1 u(.,t))\right)(x,t), & (x,t) \in \Omega \times (0,T] =: Q, \\ z(x,0) = z_0(x), & x \in \Omega, \\ z(x,t) = \mathbf{A} z(x,t) = 0, & ,(x,t) \in \partial\Omega \times [0,T] =: \Sigma, \\ y(x,t) = \mathrm{i}\frac{\partial(\mathbf{A}z(x,t))}{\partial \nu_{\mathbf{A}}}, & ,(x,t) \in \Gamma_0 \times [0,T] =: \Sigma_0, \end{cases} \tag{4.20}$$

where we used the following fact in the first equation of (4.20):

$$A^{-1}\gamma u = -G_1 u, \ \forall u \in L^2(\Gamma_0)$$

so Theorem 4.1 holds true if and only if for some (and hence for all) $T > 0$, there exists a $C_T > 0$ such that the solution to (4.1) satisfies (consider smooth u if necessary)

$$\int_0^T \int_{\Gamma_0} |\frac{\partial(\mathbf{A}z(x,t))}{\partial \nu_{\mathbf{A}}}|^2 d\Gamma dt \leqslant C_T \int_0^T \int_{\Gamma_0} |u(x,t)|^2 d\Gamma dt. \tag{4.21}$$

Proof of Theorem 4.1 The proof will be spit into three steps

Step 1. As indicated in the beginning of the proof of Theorem.4.1, since $\partial\Omega$ is of class C^3, it follows from Lemma 2.1 [25, Lemma 4.1], that there exists a C^2 vector field h on $\overline{\Omega}$ such that

$$h(x) = \mu(x), \ x \in \Gamma; \quad |N(x)|_g \leqslant 1, \ x \in \Omega.$$

Multiply both sides of the first equation in (4.20) by $h(\mathbf{A}\overline{z})$ and integrate over Q to obtain

$$\int_Q z_t h(\mathbf{A}\overline{z}) dQ - \mathrm{i} \int_Q \mathbf{A}^2 z h(\mathbf{A}\overline{z}) dQ + \mathrm{i} \int_Q G_1 u h(\mathbf{A}\overline{z}) dQ = 0. \tag{4.22}$$

Computing the second term on the left-hand side of (4.22) gives

$$
\begin{aligned}
& \mathrm{i}\int_Q \mathbf{A}^2 z h(\overline{z})dQ \qquad (4.23)\\
= \;& \mathrm{i}\int_Q \Delta_g\left(\mathbf{A}z\right)h(\overline{\mathbf{A}}z)dQ+\mathrm{i}\int_Q Dp\left(\mathbf{A}z\right)h(\mathbf{A}\overline{z})dQ\\
= \;& \mathrm{i}\left[\int_\Sigma \frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}h(\mathbf{A}\overline{z})d\Sigma-\int_Q\langle\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(\mathbf{A}\overline{z}\right)\rangle_g dQ\right]+\mathrm{i}\int_Q Dp\left(\mathbf{A}z\right)h(\mathbf{A}\overline{z})dQ\\
= \;& \mathrm{i}\left[\int_\Sigma \frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}h(\mathbf{A}\overline{z})d\Sigma-\int_Q Dh\left(\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(\mathbf{A}\overline{z}\right)\right)dQ\right]\\
& +\mathrm{i}\left[-\frac{1}{2}\int_Q div_g\left(\left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 h\right)dQ+\frac{1}{2}\int_Q\left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 div_g h dQ\right]\\
& +\mathrm{i}\int_Q Dp\left(\mathbf{A}z\right)h(\mathbf{A}\overline{z})dQ
\end{aligned}
$$

and by virtue of the divergence formula, we have

$$
\begin{aligned}
\mathrm{i}\int_Q \mathbf{A}^2 z h(\mathbf{A}\overline{z})dQ \;=\;& \mathrm{i}\left[\int_\Sigma \frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}h(\mathbf{A}\overline{z})d\Sigma-\int_Q Dh\left(\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(\mathbf{A}\overline{z}\right)\right)dQ\right]\\
& +\mathrm{i}\left[-\frac{1}{2}\int_\Sigma\left\langle\left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 h,\mu\right\rangle_g d\Sigma+\frac{1}{2}\int_Q\left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 div_g h dQ\right]\\
& +\mathrm{i}\int_Q Dp\left(\mathbf{A}z\right)h(\mathbf{A}\overline{z})dQ.
\end{aligned}
$$

Then

$$
\begin{aligned}
& \mathrm{Im}\left(\mathrm{i}\int_Q \mathbf{A}^2 z h(\mathbf{A}\overline{z})dQ\right) \qquad (4.24)\\
= \;& \mathrm{Re}\int_\Sigma \frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}h(\mathbf{A}\overline{z})d\Sigma-\mathrm{Re}\int_Q Dh\left(\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(\mathbf{A}\overline{z}\right)\right)dQ\\
& -\frac{1}{2}\int_\Sigma\left\langle\left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 h,\mu\right\rangle_g d\Sigma+\frac{1}{2}\int_Q\left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 div_g h dQ\\
& +\mathrm{Re}\int_Q Dp\left(\mathbf{A}z\right)h(\mathbf{A}\overline{z})dQ.
\end{aligned}
$$

Moreover, by Lemma 2.1 [18, Lemma 4.1] we have

$$
\begin{aligned}
& \mathrm{Im}\left(\mathrm{i}\int_Q h(\mathbf{A}\overline{z})dQ\right) \qquad (4.25)\\
= \;& \frac{1}{2}\int_\Sigma\left|\frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}\right|^2 d\Sigma-\mathrm{Re}\int_Q Dh\left(\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(\mathbf{A}\overline{z}\right)\right)dQ\\
& +\frac{1}{2}\int_Q\left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 div_g h dQ+\int_Q Dp\left(\mathbf{A}z\right)h(\mathbf{A}\overline{z})dQ
\end{aligned}
$$

Next, we compute the first term on the left hand side of (4.22), by virtue of the divergence formula, we have

$$\begin{aligned} div_g\left(z_t\mathbf{A}\bar{z}h\right) &= z_t div_g\left(\mathbf{A}\bar{z}h\right)+\mathbf{A}\bar{z}h\left(z_t\right) \\ &= z_t\left[\mathbf{A}\bar{z}div_g\left(\mathbf{A}\bar{z}h\right)+h\left(\mathbf{A}\bar{z}\right)\right]+\mathbf{A}\bar{z}h\left(z_t\right) \\ &= z_t\mathbf{A}\bar{z}div_g\left(h\right)+z_t h\left(\mathbf{A}\bar{z}\right)+\mathbf{A}\bar{z}h\left(z_t\right) \\ &= z_t\mathbf{A}\bar{z}div_g\left(N\right)+z_t N\left(\mathbf{A}\bar{z}\right)+\frac{d}{dt}\left[\mathbf{A}\bar{z}N\left(z\right)\right]-\mathbf{A}\bar{z}_t N\left(z\right) \\ &= \left(\mathrm{i}\mathbf{A}^2 z+\mathrm{i}G_1 u\right)\mathbf{A}\bar{z}div_g\left(h\right)+z_t N\left(\mathbf{A}\bar{z}\right) \\ &\quad +\frac{d}{dt}\left[\mathbf{A}\bar{z}N\left(z\right)\right]-\mathbf{A}\bar{z}_t N\left(z\right) \end{aligned} \tag{4.26}$$

In which,

$$\begin{aligned} &\int_Q \mathbf{A}\bar{z}_t h\left(z\right)dQ \\ =&\int_Q \Delta_g\left(\bar{z}_t\right)h(z)dQ+\int_Q Dp\left(\bar{z}_t\right)h(z)dQ \\ =&\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu}h(z)d\Sigma+\int_Q \bar{z}_t\Delta_g\left(h\left(z\right)\right)dQ+\int_Q Dp\left(\bar{z}_t\right)h\left(z\right)dQ \\ =&\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu}h(z)d\Sigma-\int_Q\langle\nabla_g\left(\bar{z}_t\right),\nabla_g\left(h(z)\right)\rangle_g dQ+\int_Q Dp\left(\bar{z}_t\right)h(z)dQ \\ =&\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu}h(z)d\Sigma+\int_Q Dp\left(\bar{z}_t\right)h\left(z\right)dQ \\ &+\int_Q \bar{z}_t\left[\left(\Delta h\right)\left(z\right)+2\langle Dh,D^2 z\rangle_{(T_x\mathbb{R}^n)^2}+h\left(\mathbf{A}z\right)\right]dQ \\ &+\int_Q \bar{z}_t\left[Ricc\left(h,Dz\right)-D^2 p\left(h,Dz\right)-D^2 z\left(h,Dp\right)\right]dQ \end{aligned} \tag{4.27}$$

Integrating the equality (4.25) over Q by taking (4.27) into account yields

$$\begin{aligned} \int_Q div_g\left(z_t\mathbf{A}\bar{z}h\right)dQ &= \int_Q \bar{z}_t\mathbf{A}\bar{z}div_g\left(h\right)dQ+\int_Q z_t h\left(\mathbf{A}\bar{z}\right)dQ \\ &\quad +\int_Q\frac{d}{dt}\left[\mathbf{A}\bar{z}h\left(z\right)\right]dQ-\int_Q\mathbf{A}\bar{z}_t h\left(z\right)dQ \end{aligned}$$

Then

$$\begin{aligned} \int_Q z_t h\left(\mathbf{A}\bar{z}\right)dQ &= \int_Q\mathbf{A}\bar{z}_t h\left(z\right)dQ-\int_Q\bar{z}_t\mathbf{A}\bar{z}div_g\left(h\right)dQ \\ &\quad -\int_Q\frac{d}{dt}\left[\mathbf{A}\bar{z}h\left(z\right)\right]dQ \end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Im}\left[\int_Q z_t h\left(\mathbf{A}\bar{z}\right) dQ\right] \qquad (4.28)\\
= \;& -\frac{1}{2}\int_Q G_1 u \mathbf{A}\bar{z} div_g\left(h\right) dQ - \frac{1}{2}\int_Q \mathbf{A}^2 z \mathbf{A}\bar{z} div_g\left(h\right) dQ + \frac{\mathrm{i}}{2}\int_\Omega \mathbf{A}\bar{z} h\left(z\right) d\Omega\Big]_0^T\\
& -\frac{\mathrm{i}}{2}\int_Q \bar{z}_t \Delta\left(h\left(z\right)\right) dQ - \mathrm{i}\int_Q \bar{z}_t \langle Dh, D^2 z\rangle_{(T_x\mathbb{R}^n)^2} dQ + \frac{\mathrm{i}}{2}\int_Q \bar{z}_t D^2 p\left(h, Dz\right) dQ\\
& -\frac{\mathrm{i}}{2}\int_Q \bar{z}_t Ricc\left(h, Dz\right) dQ + \frac{\mathrm{i}}{2}\int_Q \bar{z}_t D^2 z\left(h, Dp\right) dQ\\
& -\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu} h(z) d\Sigma - \frac{\mathrm{i}}{2}\int_Q Dp\left(\bar{z}_t\right) h\left(z\right) dQ
\end{aligned}$$

while

$$\int_Q Dp\left(\bar{z}_t\right) h\left(z\right) dQ = -\int_Q \bar{z}_t Dp\left(h\left(z\right)\right) dQ - \int_Q \bar{z}_t h\left(z\right) div_g\left(Dp\right) dQ \qquad (4.29)$$

Combining (4.29), (4.28) and (4.27) to obtain

$$\begin{aligned}
&\operatorname{Im}\left(\int_Q z_t h\left(\mathbf{A}\bar{z}\right) dQ\right) \qquad (4.30)\\
= \;& -\frac{1}{2}\int_Q G_1 u \mathbf{A}\bar{z} div_g\left(h\right) dQ - \frac{1}{2}\int_\Sigma \frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}\mathbf{A}\bar{z} div_g\left(h\right) d\Sigma + \frac{1}{2}\int_Q \left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 div_g h dQ\\
& +\frac{1}{2}\int_Q \mathbf{A}\bar{z}\langle\nabla_g\left(\mathbf{A}z\right), \nabla_g\left(div_g\left(h\right)\right)\rangle_g dQ + \frac{\mathrm{i}}{2}\int_\Omega\left(\mathbf{A}\bar{z} h\left(z\right)\right) d\Omega\Big]_0^T - \frac{\mathrm{i}}{2}\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu} h\left(z\right) d\Sigma\\
& +\frac{1}{2}\int_Q\left(\mathbf{A}^2\bar{z} + \overline{G_1 u}\right)\left(\Delta h\right)\left(z\right) dQ - \frac{1}{2}\int_Q\left(\mathbf{A}^2\bar{z} + \overline{G_1 u}\right)\langle Dh, D^2 z\rangle_{(T_x\mathbb{R}^n)^2} dQ\\
& -\frac{1}{2}\int_Q\left(\mathbf{A}^2\bar{z} + \overline{G_1 u}\right) D^2 z\left(h, Dp\right) dQ - \frac{1}{2}\int_Q\left(\mathbf{A}^2\bar{z} + \overline{G_1 u}\right) D^2 p\left(h, Dz\right) dQ\\
& +\frac{1}{2}\int_Q\left(\mathbf{A}^2\bar{z} + \overline{G_1 u}\right) Ricc\left(h, Dz\right) dQ
\end{aligned}$$

then

$$\begin{aligned}
& \operatorname{Im}\left(\int_Q z_t h\left(\mathbf{A}\bar{z}\right)dQ\right) \qquad (4.31)\\
= & -\frac{1}{2}\int_Q G_1 u\mathbf{A}\bar{z}div_g\left(h\right)dQ+\frac{1}{2}\int_Q\left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 div_g\left(h\right)dQ\\
& +\int_Q \mathbf{A}\bar{z}\langle\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(div_g\left(h\right)\right)\rangle_g dQ+\frac{\mathrm{i}}{2}\int_\Omega\left(\mathbf{A}\bar{z}h\left(z\right)\right)d\Omega\Big]_0^T-\frac{1}{2}\int_\Sigma\frac{\partial\left(\bar{z}_t\right)}{\partial\mu}\frac{\partial z}{\partial\mu}d\Sigma\\
& -\frac{1}{2}\int_\Sigma\frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}\left[\mathbf{A}\bar{z}div_g\left(h\right)+\left(\Delta h\right)\left(z\right)+\langle Dh,D^2z\rangle_{(T_x\mathbb{R}^n)^2}-D^2p\left(h,Dz\right)\right]d\Sigma\\
& +\frac{1}{2}\int_\Sigma\frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}\left[Ricc\left(h,Dz\right)+D^2z\left(h,Dp\right)+h\left(z\right)div_g\left(Dp\right)\right]d\Sigma\\
& +\frac{1}{2}\int_Q\langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(\Delta h\right)\left(z\right)\rangle_g dQ+\frac{1}{2}\int_Q\langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(\langle Dh,D^2z\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ\\
& -\frac{1}{2}\int_Q\langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(D^2p\left(h,Dz\right)\right)\rangle_g dQ-\frac{1}{2}\int_Q\langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(Ricc\left(h,Dz\right)\right)\rangle_g dQ\\
& -\frac{1}{2}\int_Q\langle\nabla_g\left(\mathbf{A}\bar{z}\right),\left(\nabla_g\left(D^2z\left(h,Dp\right)\right)\right)\rangle_g dQ-\frac{1}{2}\int_Q\langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(Dp\left(h\left(z\right)\right)\right)\rangle_g dQ\\
& -\frac{1}{2}\int_Q\langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(h\left(z\right)div_g\left(Dp\right)\right)\rangle_g dQ\\
& +\int_Q Dp\left(\mathbf{A}z\right)\mathbf{A}\bar{z}div_g\left(h\right)dQ+\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)\left(\Delta h\right)\left(z\right)dQ\\
& -\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)\left(D^2p\left(h,Dz\right)\right)dQ-\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)Ricc\left(h,Dz\right)dQ\\
& +\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)\left(\langle Dh,D^2z\rangle_{(T_x\mathbb{R}^n)^2}\right)dQ-\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)D^2z\left(h,Dp\right)dQ\\
& -\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)Dp\left(h\left(z\right)\right)dQ-\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)h\left(z\right)div_g\left(Dp\right)dQ\\
& +\frac{1}{2}\int_Q\overline{G_1u}\left[\left(\Delta h\right)\left(z\right)+\langle Dh,D^2z\rangle_{(T_x\mathbb{R}^n)^2}-D^2p\left(h,Dz\right)+Ricc\left(h,Dz\right)\right]dQ\\
& -\frac{1}{2}\int_Q\overline{G_1u}\left[D^2z\left(h,Dp\right)+Dp\left(h\left(z\right)\right)+h\left(z\right)div_g\left(Dp\right)\right]dQ
\end{aligned}$$

Used the properties of the vector field h, we have

$$
\begin{aligned}
& \operatorname{Im}\left(\mathrm{i}\int_Q \mathbf{A}^2 z N(\mathbf{A}\bar{z})dQ\right) \qquad (4.32)\\
= \quad & \frac{1}{2}\int_\Sigma \left|\frac{\partial(\mathbf{A}z)}{\partial\mu}\right|^2 d\Sigma - \operatorname{Re}\left[\int_Q Dh\left(\left(\nabla_g(\mathbf{A}z), \nabla_g(\mathbf{A}\bar{z})\right)\right)dQ\right]\\
& -\frac{1}{2}\int_\Sigma \left\langle |\nabla_g(\mathbf{A}z)|_g^2 h, \mu\right\rangle_g d\Sigma + \frac{1}{2}\int_Q |\nabla_g(\mathbf{A}z)|_g^2 div_g(h)\,dQ\\
& + \operatorname{Re}\int_Q Dp(\mathbf{A}z)\,h(\mathbf{A}\bar{z})\,dQ\\
= \quad & \frac{1}{2}\int_\Sigma \left|\frac{\partial(\mathbf{A}z)}{\partial\mu}\right|^2 d\Sigma - \operatorname{Re}\left[\int_Q Dh\left(\left(\nabla_g(\mathbf{A}z), \nabla_g(\mathbf{A}\bar{z})\right)\right)dQ\right]\\
& +\frac{1}{2}\int_Q |\nabla_g(\mathbf{A}z)|_g^2 div_g(h)\,dQ + \operatorname{Re}\int_Q Dp(\mathbf{A}z)\,h(\mathbf{A}\bar{z})\,dQ.
\end{aligned}
$$

In which

$$
\begin{aligned}
& -\operatorname{Re}\left(\frac{\mathrm{i}}{2}\int_\Sigma \frac{\partial(\bar{z}_t)}{\partial\mu}\frac{\partial z}{\partial\mu}d\Sigma\right) - \operatorname{Re}\left(\frac{1}{2}\int_\Sigma \frac{\partial(\mathbf{A}z)}{\partial\mu}\mathbf{A}\bar{z}\,div_g(h)\,d\Sigma\right) \qquad (4.35)\\
& -\frac{1}{2}\operatorname{Re}\left(\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu}(\Delta h)(z)\,d\Sigma\right) - \operatorname{Re}\left(\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu}\langle Dh, D^2 z\rangle_{(T_x\mathbb{R}^n)^2}d\Sigma\right)\\
& -\frac{1}{2}\operatorname{Re}\left(\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu}D^2 p(h, Dz)\,d\Sigma\right) + \frac{1}{2}\operatorname{Re}\left(\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu}Ricc(h, Dz)\,d\Sigma\right)\\
& +\frac{1}{2}\operatorname{Re}\left(\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu}D^2 z(h, Dp)\,d\Sigma\right) - \frac{1}{2}\operatorname{Re}\left(\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu}Dp(h(z))\,d\Sigma\right)\\
& +\frac{1}{2}\operatorname{Re}\left(\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu}Dp(h(z))\,d\Sigma\right) + \frac{1}{2}\operatorname{Re}\left(\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu}h(z)\,div_g(Dp)\,d\Sigma\right)\\
\leq \quad & \frac{9}{32}\int_\Sigma \left|\frac{\partial(\mathbf{A}z)}{\partial\mu}\right|^2 d\Sigma + C\|z\|^2_{L^2(0,T;H^3(\Omega))} + \left.\frac{1}{4}\int_\Gamma \left|\frac{\partial z}{\partial\mu}\right|^2 d\Gamma\right]_0^T
\end{aligned}
$$

In the last step where we have used the Sobolev trace theorem with constant $C > 0$. Combining (4.22), (4.31), (4.35) and (4.36) gives

$$
\frac{7}{32}\int_\Sigma \left|\frac{\partial(\mathbf{A}z)}{\partial\mu}\right|^2 d\Sigma \leq R_1 + R_2 + b_{0,T} \qquad (4.36)
$$

where

$$
\begin{aligned}
R_1 \;=\;& \frac{1}{2}\operatorname{Re}\left(\int_Q |\nabla_g(\mathbf{A}z)|_g^2 div_g(h)\,dQ\right) + \operatorname{Re}\left(\int_Q \langle \nabla_g(\mathbf{A}z), \nabla_g(div_g(h))\rangle_g\right) \\
&+\frac{1}{2}\operatorname{Re}\left(\int_Q \langle \nabla_g(\mathbf{A}\bar{z}), \nabla_g(\Delta h)(z)\rangle_g dQ + \int_Q \langle \nabla_g(\mathbf{A}\bar{z}), \nabla_g\left(\langle Dh, D^2 z\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ\right) \\
&+\frac{1}{2}\operatorname{Re}\left(\int_Q \langle \nabla_g(\mathbf{A}\bar{z}), \nabla_g\left(\langle Dh, D^2 z\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ - \int_Q \langle \nabla_g(\mathbf{A}\bar{z}), \nabla_g\left(D^2 p(h, Dz)\right)\rangle_g dQ\right) \\
&-\frac{1}{2}\operatorname{Re}\left(\int_Q \langle \nabla_g(\mathbf{A}\bar{z}), \nabla_g(Ricc(h, Dz))\rangle_g dQ + \int_Q \langle \nabla_g(\mathbf{A}\bar{z}), \left(\nabla_g D^2 z(h, Dp)\right)\rangle_g dQ\right) \\
&-\frac{1}{2}\operatorname{Re}\left(\int_Q \langle \nabla_g(\mathbf{A}\bar{z}), \nabla_g(Dp(h(z)))\rangle_g dQ + \int_Q \langle \nabla_g(\mathbf{A}\bar{z}), \nabla_g(h(z)\,div_g(Dp))\rangle_g dQ\right) \\
&+\operatorname{Re}\int_Q \mathbf{A}z Dp(h(\mathbf{A}\bar{z}))\,dQ - \operatorname{Re}\int_Q h(\mathbf{A}\bar{z})\,Dp(\mathbf{A}z)\,div_g(Dp)\,dQ + C\|z\|^2_{L^2(0,T;H^3(\Omega))}
\end{aligned}
$$

and

$$
\begin{aligned}
R_2 \;=\;& -\frac{1}{2}\int_Q G_1 u \mathbf{A}\bar{z} div_g(h)\,dQ - \operatorname{Re}\int_Q G_1 u h(\mathbf{A}\bar{z})\,dQ \qquad (4.37) \\
&-\frac{1}{2}\operatorname{Re}\int_Q \overline{G_1 u}\left[(\Delta h)(z) + \langle Dh, D^2 z\rangle_{(T_x\mathbb{R}^n)^2} - D^2 p(h, Dz) - D^2 z(h, Dp)\right] dQ \\
&-\frac{1}{2}\int_Q \overline{G_1 u}\left[Dp(h(z)) - Ricc(h, Dz) - h(z)\,div_g(Dp)\right] dQ \\
b_{0,T} \;=\;& -\frac{1}{2}\int_\Omega (\mathbf{A}\bar{z}h(z))\,d\Omega\Big]_0^T + \frac{1}{4}\int_\Gamma \left|\frac{\partial z}{\partial \mu}\right|^2 d\Gamma\Big]_0^T
\end{aligned}
$$

Step 2. Evaluation of R_1.

Let $G_1 u = 0$ in the first identity of (4.19) and note that $z = A_1^{-3} w \in V$. It is known that (4.19) associates with a C_0-group solution in V. That is to say, For any $z_0 \in V$, there exists aunique solution $z \in V$ the solution to (4.20), which depends continuously on z_0.This fact together with (4.37), (4.36) implies that

$$
\int_\Sigma \left|\frac{\partial(\mathbf{A}z)}{\partial \nu_{\mathbf{A}}}\right|^2 d\Sigma \le C_T \|z_0\|_V^2 \qquad (4.38)
$$

Hence the operator B^* is admissible, and so is B [9]. Therfore,

$$
u \longrightarrow w \text{ is continuous from } L^2(0,T;L^2(\Gamma_0)) \text{ to } C(0,T;V'). \qquad (4.39)
$$

Moreover

$$
z = A_1^{-3} w \in C(0,T;V) \text{ depends continously on } u \in L^2(0,T;L^2(\Gamma_0)). \qquad (4.40)
$$

Therefore,

$$
R_1 \le C_T \|u\|^2_{L^2(0,T;L^2(\Gamma_0))} \qquad \forall u \in L^2(0,T;L^2(\Gamma_0)). \qquad (4.41)
$$

Step 3. Evaluation of R_2 and $b_{0,T}$

Since $G_1 u \in L^2(0,T; H^{\frac{1}{2}}(\Omega))$, the terms in R_2 and $b_{0,T}$ depend continuously on $u \in L^2(0,T; L^2(\Gamma_0))$, from this facts and (4.41), we obtain

$$R_2 + b_{0,T} \leq C_T \|u\|^2_{L^2(0,T;L^2(\Gamma_0))} \qquad \forall u \in L^2(0,T; L^2(\Gamma_0)). \tag{4.42}$$

From (4.42), (4.41), and (4.36), it is seen that (4.20) holds true. ■

4.4 Proof of Theorem 4.2

We show the exact controllability by means of the Hilbert Uniqueness Method **(HUM)**, which is stated as Theorem 11.2.1 in [57] for well-posed systems as the duality principe between exact controllability and observability. Since by Theorem 4.1, system (4.1) is well-posed, which is formulated into the abstract form (4.14) and $(i\hat{A})^* = -i\hat{A}$ in V'. It follows from theorem 11.2.1 of [57] that $\dot{w} = \mathrm{i}\hat{A}w + Bu$ is exactly controllable if and only if $\dot{w} = \mathrm{i}\hat{A}w$, $y = B^*w$ is exactly observable. More precisely, the exact controllability of system (4.1) is equivalent of the exact observability of the following dual problem of (4.1):

$$\begin{cases} i\varphi_t(x,t) + \mathbf{A}^2\varphi(x,t) = 0, & \text{in } \Omega \times (0,T) = Q, \\ \varphi(x,t) = 0, \ \mathbf{A}\varphi = 0 & \text{on } \partial\Omega \times (0,T) = \Sigma, \\ \varphi(x,0) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{4.43}$$

with the output $y = B^* A^3 \varphi$. That is to say, the "observability inequality" holds true for system (4.43) in the sence of (see (4.20), (4.21)):

$$\int_{\Sigma_0} |\frac{\partial\left(\mathbf{A}\varphi\left(x,t\right)\right)}{\partial\nu_{\mathbf{A}}}|^2 d\Sigma \geq C_T \left\|\varphi^0\right\|^2_V ., \qquad \forall \varphi^0 \in V, \tag{4.44}$$

for some (and hence for all) positive $T > 0$.

To prove (5.39), we let A defined by (4.6) and let φ be a solution to (5.38). Then $i\mathrm{A}$ generates a strongly continuous unitary group on the space $V = D(\mathrm{A}^{\frac{3}{4}})$ and hence

$$\begin{aligned} \|\varphi(t)\|_V &= \left\|(\mathrm{A}^{\frac{3}{4}}\varphi(t)\right\|_V = \|\mathrm{e}^{i\mathrm{A}t}\varphi^0\|_V \\ &= \left\|\varphi^0\right\|_V = \left\|\mathrm{A}^{\frac{3}{4}}\varphi^0\right\|_{L^2(\Omega)}. \end{aligned} \tag{4.45}$$

Next, we claim that for $f \in D(\mathrm{A}^{\frac{3}{4}})$, the norms

$$\|f\|_{D(\mathrm{A}^{\frac{3}{4}})} = \left\|\mathrm{A}^{\frac{3}{4}}f\right\|_{L^2(\Omega)} \text{ and } \left\{\int_\Omega |\nabla_g\left(\mathbf{A}f\right)|^2_g\, dx\right\}^{\frac{1}{2}} \text{ are equivalent.} \tag{4.46}$$

Actually, $\left\{\int_\Omega |\nabla_g\left(\mathbf{A}f\right)|^2_g\, dx\right\}^{\frac{1}{2}}$ being a norm is a trivial fact, since the norms $\|f\|_{D(\mathrm{A}^{\frac{1}{4}})} = \left\|\mathrm{A}^{\frac{1}{4}}f\right\|_{L^2(\Omega)}$ and $\left\{\int_\Omega |\nabla_g\left(f\right)|^2_g\, dx\right\}^{\frac{1}{2}}$ are equivalent by the Poincaré inequality, the norms $\|f\|_{D(\mathrm{A}^{\frac{3}{4}})} = \left\|\mathrm{A}^{\frac{3}{4}}f\right\|_{L^2(\Omega)} = \left\|\mathrm{A}^{\frac{1}{4}}\left(\mathbf{A}f\right)\right\|_{L^2(\Omega)}$ and $\left\{\int_\Omega \left|\nabla_g\left(\mathbf{A}f\right)\right|_g^2 dx\right\}^{\frac{1}{2}}$ are equivalent (see [67])

Proof of Theorem 4.2 We split the proof into three steps.

Step 1. Multiply the both sides of the first equation of (4.43) by $N(\mathbf{A}\bar{\varphi})$ and integrate on Q to obtain

$$\int_Q \varphi_t N(\mathbf{A}\overline{\varphi})dQ - \mathrm{i}\int_Q \mathbf{A}^2\varphi N(\mathbf{A}\overline{\varphi})dQ = 0. \tag{4.47}$$

By making use of the computation procedure form by setting $N = h$ (4.24)-(4.32), we get

$$\begin{aligned}
&\mathrm{Im}\left(\int_Q \varphi_t N\left(\mathbf{A}\bar{\varphi}\right)dQ\right) \qquad\qquad (4.48)\\
=\;& \frac{1}{2}\int_Q \left|\nabla_g\left(\mathbf{A}\varphi\right)\right|_g^2 div_g\left(N\right)dQ + \int_Q \mathbf{A}\bar{\varphi}\langle\nabla_g\left(\mathbf{A}\varphi\right), \nabla_g\left(div_g\left(N\right)\right)\rangle_g dQ\\
&+\frac{\mathrm{i}}{2}\int_\Omega \left(\mathbf{A}\bar{\varphi}N\left(\varphi\right)\right)d\Omega\Big]_0^T - \frac{1}{2}\int_\Sigma \frac{\partial\left(\bar{\varphi}_t\right)}{\partial\nu_{\mathbf{A}}}\frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}d\Sigma\\
&-\frac{1}{2}\int_\Sigma \frac{\partial\left(\mathbf{A}\varphi\right)}{\partial\nu_{\mathbf{A}}}\left[\mathbf{A}\bar{\varphi}div_g\left(N\right) + \left(\Delta N\right)\left(\varphi\right) + \langle DN, D^2 z\rangle_{(T_x\mathbb{R}^n)^2} - D^2 p\left(N, D\varphi\right)\right]d\Sigma\\
&+\frac{1}{2}\int_\Sigma \frac{\partial\left(\mathbf{A}z\right)}{\partial\nu_{\mathbf{A}}}\left[Ricc\left(N, D\varphi\right) + D^2\varphi\left(N, Dp\right) + N\left(\varphi\right)div_g\left(Dp\right)\right]d\Sigma\\
&+\frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{\varphi}\right), \nabla_g\left(\Delta N\right)\left(\varphi\right)\rangle_g dQ + \frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{\varphi}\right), \nabla_g\left(\langle DN, D^2\varphi\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ\\
&-\frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{\varphi}\right), \nabla_g\left(D^2 p\left(N, D\varphi\right)\right)\rangle_g dQ - \frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{\varphi}\right), \nabla_g\left(Ricc\left(N, D\varphi\right)\right)\rangle_g dQ\\
&-\frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{\varphi}\right), \left(\nabla_g\left(D^2\varphi\left(N, Dp\right)\right)\right)\rangle_g dQ.
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Im}\left(\mathrm{i}\int_Q \mathbf{A}^2\varphi N(\mathbf{A}\overline{\varphi})dQ\right) \qquad\qquad (4.49)\\
=\;& \frac{1}{2}\mathrm{Re}\int_\Sigma \left(\frac{\partial\left(\mathbf{A}\varphi\right)}{\partial\nu_{\mathbf{A}}}\right)N\left(\mathbf{A}\bar{\varphi}\right)d\Sigma - \mathrm{Re}\left[\int_Q DN\left(\left(\nabla_g\left(\mathbf{A}\varphi\right), \nabla_g\left(\mathbf{A}\bar{\varphi}\right)\right)\right)dQ\right]\\
&-\frac{1}{2}\int_\Sigma \left|\nabla_g\left(\mathbf{A}\varphi\right)\right|_g^2 N.\nu d\Sigma + \frac{1}{2}\int_Q \left|\nabla_g\left(\mathbf{A}\varphi\right)\right|_g^2 div_g\left(N\right)dQ.
\end{aligned}$$

By (4.47), (4.48) and (4.49), it follows that

$$
\begin{aligned}
&\sum_{i=1}^{4} L_i \qquad (4.50)\\
= \;& \frac{1}{2}\operatorname{Re}\int_\Sigma \left(\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right) N(\mathbf{A}\bar{\varphi})\,d\Sigma - \frac{1}{2}\int_\Sigma |\nabla_g(\mathbf{A}\varphi)|_g^2\, N.\nu d\Sigma\\
&+\frac{1}{2}\int_\Sigma \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\left[D^2\varphi(N,Dp) - D^2p(N,D\varphi) + N(\varphi)\,div_g(Dp)\right]d\Sigma\\
&-\frac{1}{2}\int_\Sigma \frac{\partial(\mathbf{A}z)}{\partial\nu_{\mathbf{A}}}\left[Ricc(N,D\varphi) + \langle DN, D^2 z\rangle_{(T_x\mathbb{R}^n)^2}\right]d\Sigma\\
= \;& \operatorname{Re}\left[\int_Q DN\left((\nabla_g(\mathbf{A}\varphi), \nabla_g(\mathbf{A}\bar{\varphi}))\right)dQ\right] + \frac{\mathrm{i}}{2}\int_\Omega (\mathbf{A}\bar{\varphi}N(\varphi))\,d\Omega\Bigg]_0^T\\
&-\frac{1}{2}\int_\Sigma \frac{\partial(\bar{\varphi}_t)}{\partial\nu_{\mathbf{A}}}\frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}d\Sigma + \int_Q \mathbf{A}\bar{\varphi}\langle\nabla_g(\mathbf{A}\varphi), \nabla_g(div_g(N))\rangle_g dQ\\
&+\Bigg[\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g(\Delta N)(\varphi)\rangle_g dQ + \frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g\left(\langle DN, D^2\varphi\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ\\
&-\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g\left(D^2p(N,D\varphi)\right)\rangle_g dQ - \frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g(Ricc(N,D\varphi))\rangle_g dQ\\
&-\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{\varphi}), \left(\nabla_g\left(D^2\varphi(N,Dp)\right)\right)\rangle_g dQ\Bigg]\\
= \;& \sum_{i=1}^{4} M_i
\end{aligned}
$$

We first compute the four ters in the LHS of (4.50). For $\varepsilon > 0$,

$$L_1 \le \mu_1 \int_\Sigma \left[\frac{1}{\varepsilon}\left|\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right|^2 + \varepsilon|\nabla_g(\mathbf{A}\bar{\varphi})|_g^2\right]d\Sigma \qquad (4.51)$$

By (H2), we have

$$L_2 = -\frac{1}{2}\int_\Sigma |\nabla_g(\mathbf{A}\varphi)|_g^2\, N.\nu d\Sigma \le 0 \qquad (4.52)$$

and by (A.40), (A.41) (in Appendix A), we have

$$L_3 \le \frac{1}{4\varepsilon}\int_\Sigma \left|\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma + C\varepsilon\,\|\varphi\|^2_{L^2(0,T;D(\mathrm{A}^{\frac{3}{4}}))} \qquad (4.53)$$

$$L_4 \le \frac{1}{4\varepsilon}\int_\Sigma \left|\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma + C\varepsilon\,\|\varphi\|^2_{L^2(0,T;D(\mathrm{A}^{\frac{3}{4}}))} \qquad (4.54)$$

where in the last step we used the trace Theorem and the Poincaré inequality.

Adding (4.51), (4.52), (4.53), and (4.54), we get

$$\begin{aligned} \text{LHS of (4.50)} \quad &\leq \quad \left(\frac{1}{2\varepsilon}+\frac{\mu_1}{\varepsilon}\right)\int_\Sigma \left|\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma \\ &+\mu_1\varepsilon\int_\Sigma |\nabla_g(\mathbf{A}\bar{\varphi})|_g^2 d\Sigma - \frac{1}{2}\int_\Sigma |\nabla_g(\mathbf{A}\varphi)|_g^2 N.\nu d\Sigma \\ &+2C\varepsilon\|\varphi\|^2_{L^2(0,T;D(\mathrm{A}^{\frac{3}{4}}))}. \end{aligned} \tag{4.55}$$

Choosing $\varepsilon > 0$ sufficiently small so that $\mu_1\varepsilon\int_\Sigma |\nabla_g(\mathbf{A}\bar{\varphi})|_g^2 d\Sigma - \frac{1}{2}\int_\Sigma |\nabla_g(\mathbf{A}\varphi)|_g^2 N.\nu d\Sigma <$ 0 and making use of (4.45), we obtain

$$\begin{aligned} \text{LHS of (4.50)} \quad &\leq \quad \left(\frac{1}{2\varepsilon}+\frac{\mu_1}{\varepsilon}\right)\int_\Sigma \left|\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma \\ &+2C\varepsilon\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})}. \end{aligned} \tag{4.56}$$

Next, we estimate the RHS of (4.50). First by (H1) and (4.45)

$$\begin{aligned} M_1 \quad &\geq \quad b_0T\|\varphi\|^2_{L^2(0,T;D(\mathrm{A}^{\frac{3}{4}}))} \\ &= \quad b_0T\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} \end{aligned} \tag{4.57}$$

$$\begin{aligned} |M_2| \quad &= \quad \left|\frac{\mathrm{i}}{2}\int_\Omega (\mathbf{A}\bar{\varphi}N(\varphi))\,d\Omega\right]_0^T\right| \\ &\leq \quad 2\mu_1\left(\|\mathbf{A}\overline{\varphi}(T)\|_{L^2(\Omega)}\|\nabla_g\varphi(T)\|_{L^2(\Omega)} + \left\|\mathbf{A}\overline{\varphi^0}\right\|_{L^2(\Omega)}\left\|\nabla_g\varphi^0\right\|_{L^2(\Omega)}\right) \\ &\leq \quad \mu_1\left(\varepsilon\|\mathbf{A}\overline{\varphi}(T)\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|\nabla_g\varphi(T)\|^2_{L^2(\Omega)} + \varepsilon\left\|\mathbf{A}\overline{\varphi^0}\right\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\left\|\nabla_g\varphi^0\right\|^2_{L^2(\Omega)}\right) \\ &\leq \quad 2\mu_1\varepsilon\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} + \frac{2\mu_1}{\varepsilon}\|\nabla_g\varphi(T)\|^2_{C(0,T;L^2(\Omega))}. \end{aligned} \tag{4.58}$$

$$\begin{aligned} |M_3| \quad &= \quad \left|-\frac{1}{2}\int_\Sigma \frac{\partial(\bar{\varphi}_t)}{\partial\nu_{\mathbf{A}}}\frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}d\Sigma\right| \leq \int_\Gamma \left|\frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}\right|^2 d\Gamma\Bigg|_0^T \\ &\leq \quad \frac{\mu_3}{4}\left(\|\mathbf{A}\overline{\varphi}(T)\|^2_{L^2(\Omega)} + \|\mathbf{A}\varphi(T)\|^2_{L^2(\Omega)} + \left\|\mathbf{A}\overline{\varphi^0}\right\|^2_{L^2(\Omega)} + \left\|\mathbf{A}\varphi^0\right\|^2_{L^2(\Omega)}\right) \\ &\leq \quad \frac{\mu_3}{2}\left(\frac{1}{\varepsilon}\|\mathbf{A}\varphi(t)\|^2_{C(0,T;L^2(\Omega))} + \varepsilon\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})}\right) \end{aligned} \tag{4.59}$$

by (A.40) and (A.41) (in Appendix A)

$$|M_4| \leq \frac{\mu_3}{\varepsilon}C\|\mathbf{A}\varphi(t)\|^2_{C((0,T);L^2(\Omega))} + C\mu_3\varepsilon\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} \tag{4.60}$$

Combining (4.57), (4.58), (4.59) and (4.60) gives

$$\begin{aligned} \text{RHS of (4.50)} \quad &\geq \quad b_0T\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} - (C\mu_3 + 2\mu_1\mu_3 + 4\mu_1)\,\varepsilon\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} \\ &-\frac{(\mu_3C + 2\mu_1\mu_3 + 4\mu_1)}{\varepsilon}\|\mathbf{A}\varphi(t)\|^2_{C(0,T;L^2(\Omega))} \end{aligned} \tag{4.61}$$

Therefore, from (4.55), (4.61) and (H1), we have

$$\begin{aligned} & \left(\frac{1+2\mu_1}{2\varepsilon}\right)\int_{\Sigma}\left|\frac{\partial\left(\mathbf{A}\varphi\right)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma+\frac{\left(\mu_3 C+2\mu_1\mu_3+4\mu_1\right)}{\varepsilon}\left\|\mathbf{A}\varphi\left(t\right)\right\|^2_{C(0,T;L^2(\Omega))} \\ \geq\ & b_0\left(T-\frac{\left(C\mu_3+2\mu_1\mu_3+4\mu_1\right)}{b_0}\varepsilon\right)\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})}. \end{aligned} \tag{4.62}$$

Step 2. We claim that the inequality (4.62) implies that any $T>0$, there exists a $C_T>0$ such that for all $\varphi^0\in D(\mathrm{A}^{\frac{3}{4}})$,

$$\left\|\mathbf{A}\varphi\left(t\right)\right\|^2_{C((0,T);L^2(\Omega))}\leq C_T\int_{\Sigma_0}\left|\frac{\partial\left(\mathbf{A}\varphi\right)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma \tag{4.63}$$

and for any sequence $\{T_q\}$ with $T_q\to\infty$ as $q\to\infty$,

$$\underline{\lim}_{q\to\infty}C_{T_q}=0. \tag{4.64}$$

we first assume (4.63) is invalid to obtain a contradiction. To this purpose, let $\{\varphi_n\}$ be the solutions of the following system over $[0,T]$:

$$\left\{\begin{array}{c} i\varphi_n'(x,t)+\mathbf{A}^2\varphi_n(x,t)=0,\quad \text{in }\Omega\times(0,T)=Q,\\ \varphi_n(x,t)=0,\ \ \mathbf{A}\varphi_n(x,t)=0\qquad \text{on }\partial\Omega\times(0,T)=\Sigma,\\ \varphi_n(x,0)=\varphi_n^0(x)\quad \text{in }\Omega. \end{array}\right. \tag{4.65}$$

such that

$$\left\|\mathbf{A}\varphi\left(t\right)\right\|^2_{C((0,T);L^2(\Omega))}=1 \tag{4.66}$$

and

$$\int_{\Sigma_0}\left|\frac{\partial\left(\mathbf{A}\varphi\right)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma\to 0\text{ as }n\to\infty. \tag{4.67}$$

by (4.67), we have

$$\left\|\varphi_n^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})}\leq C\text{ uniformly for }n \tag{4.68}$$

with some constant $C>0$. Hence, there exists a subsequence of $\left\{\varphi_n^0\right\}$, still denote by itself without confusion, and a function $\varphi^0\in D(\mathrm{A}^{\frac{3}{4}})$ such that

$$\varphi_n^0\to\varphi^0\text{ weakly in }D(\mathrm{A}^{\frac{3}{4}}). \tag{4.69}$$

Let $\tilde{\varphi}$ be the solution to (4.65) associated with the initial data φ^0. Then we can claim that there exists a $\tilde{\varphi}\in L^{\infty}\left(0,T;V\right)$ such that

$$\varphi_n\to\tilde{\varphi}\text{ weak}^*\text{ in }L^{\infty}\left(0,T;D(\mathrm{A}^{\frac{3}{4}})\right). \tag{4.70}$$

In fact, since

$$\varphi_n(t)=U(t)\varphi_n^0,\quad \tilde{\varphi}(t)=U(t)\varphi^0, \tag{4.71}$$

where $U(t)$ is the unitary group generated by iA in $D(\mathrm{A}^{\frac{3}{4}})$. For any $\psi \in L^{\infty}\left(0,T;D(\mathrm{A}^{\frac{3}{4}})'\right)$, it follows that

$$\begin{aligned} & \int_0^T \left(\mathrm{A}^{\frac{3}{4}}\left(\varphi_n(t)-\tilde{\varphi}(t)\right), \mathrm{A}^{-\frac{3}{4}}\psi(t)\right) dt \\ = & \int_0^T \left(\mathrm{A}^{\frac{3}{4}}U(t)\left(\varphi_n^0-\varphi^0\right), \mathrm{A}^{-\frac{3}{4}}\psi(t)\right)_{L^2(\Omega)} dt \\ = & \int_0^T \left(\mathrm{A}^{\frac{3}{4}}\left(\varphi_n^0-\varphi^0\right), \mathrm{A}^{-\frac{3}{4}}\left(U(t)\right)^{-1}\psi(t)\right)_{L^2(\Omega)} dt \to 0 \text{as } n\to\infty. \end{aligned} \tag{4.72}$$

where in the last step we used the Lebesgue dominated Theorem, (4.69) and the property that $\|U(t)\|$ is uniformly bounded over $t\in[0,T]$. Equation (4.70) then follows from (4.72). Since (4.69) implies that $\{\varphi_n\}$ is uniformly bounded in $L^{\infty}\left(0,T;D(\mathrm{A}^{\frac{3}{4}})\right)$, this together with the compact imbeding: $D(\mathrm{A}^{\frac{3}{4}})\hookrightarrow D(\mathrm{A}^{\frac{2}{4}})=H^2(\Omega)\cap H_0^1(\Omega)$ implies that there exists a subsequence of $\{\varphi_n\}$, still denoted by itself without confusion, such that

$$\varphi_n \to \tilde{\varphi} \text{ strongly in } L^{\infty}\left(0,T;D(\mathrm{A}^{\frac{2}{4}})\right). \tag{4.73}$$

From (4.66) and (4.73), we obtain

$$1=\|\mathbf{A}\varphi_n\|^2_{C((0,T);L^2(\Omega))}\to\|\mathbf{A}\tilde{\varphi}\|^2_{C((0,T);L^2(\Omega))}=1, \tag{4.74}$$

Moreover, by (4.67), it follows that

$$\frac{\partial\left(\mathbf{A}\tilde{\varphi}\right)}{\partial\nu_{\mathbf{A}}}=0 \text{ on } \Gamma_{0.} \tag{4.75}$$

Thus, $\tilde{\varphi}$ satisfies

$$\begin{cases} i\tilde{\varphi}_t+\mathbf{A}^2\tilde{\varphi}=0, & \text{in } \Omega\times(0,T)=Q, \\ \tilde{\varphi}=\mathbf{A}\tilde{\varphi}=0 & \text{on } \partial\Omega\times(0,T)=\Sigma, \\ \frac{\partial(\mathbf{A}\tilde{\varphi})}{\partial\nu_{\mathbf{A}}}=0 \text{ on } \Sigma_{0.} \end{cases} \tag{4.76}$$

Now, setting $\phi=\mathbf{A}\tilde{\varphi}=-\mathrm{A}^{\frac{2}{4}}\tilde{\varphi}$, we obtain the system

$$\begin{cases} i\phi_t+\mathbf{A}^2\phi=0, & \text{in } \Omega\times(0,T)=Q, \\ \phi=\mathbf{A}\phi=0 & \text{on } \partial\Omega\times(0,T)=\Sigma, \\ \frac{\partial\phi}{\partial\nu_{\mathbf{A}}}=0 \text{ on } \Sigma_{0.} \end{cases} \tag{4.77}$$

where the boundary condition $\mathbf{A}\phi=0$ follows from $\mathbf{A}\phi=\mathbf{A}^2\tilde{\varphi}=-i\tilde{\varphi}_t$ in Q.

Its restriction in Σ vanishes by (4.76). Therfore, (4.77) implies that $\phi\equiv0$ in Q [36] or $\tilde{\varphi}\equiv0$. But this contradics (4.74). Then (4.63) follows.

Next, we prove (4.64). For notation convnience, we set for any $T>0$ and $\varphi^0\in D(\mathrm{A}^{\frac{3}{4}})$ that

$$N_T(\varphi^0)=\|\mathbf{A}\varphi\|^2_{C((0,T);L^2(\Omega))}, \quad D_T(\varphi^0)=\int_{\Sigma_0}\left|\frac{\partial\left(\mathbf{A}\varphi\right)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma, \tag{4.78}$$

Then we can take C_T in (4.63) as

$$C_T = \sup_{\varphi^0 \in V, \varphi^0 \neq 0} \frac{N_T(\varphi^0)}{D_T(\varphi^0)} < \infty \tag{4.79}$$

Suppose on the contrary that there exists a sequence $T_q \to \infty$ such that $C_{Tq} \geq \alpha > 0$ for all q. Then from (4.79), for every sufficiently large q, there is an initial value $\varphi_q^0 \in D(\mathrm{A}^{\frac{3}{4}})$ such that

$$\frac{N_T(\varphi_q^0)}{D_T(\varphi_q^0)} \geq C_{Tq} - \frac{1}{q} \geq \alpha - \frac{1}{q}, \tag{4.80}$$

we may suppose without loss of generality that the $\varphi_q^0 \in D(\mathrm{A}^{\frac{3}{4}})$ satisfies

$$N_{T_q}(\varphi_q^0) = \|\mathbf{A}\varphi_q\|^2_{C((0,T);L^2(\Omega))} = 1 \tag{4.81}$$

and

$$D_T(\varphi_q^0) = \int_{\Sigma_0} \left| \frac{\partial\left(\mathbf{A}\varphi\right)}{\partial \nu_{\mathbf{A}}} \right|^2 d\Sigma \leq \left(\alpha - \frac{1}{q}\right)^{-1} < C \text{ for all } q \tag{4.82}$$

with some $C > 0$. Let $\varphi(t, \varphi_q^0)$ be the solution of (4.43) corresponding to the initial value φ_q^0. By (4.62), (4.81), (4.82) we have

$$\left(T_q - \varepsilon'\right) \|\varphi_q^0\|^2_{D(\mathrm{A}^{\frac{3}{4}})} < C \text{ uniformly in } q, \tag{4.83}$$

where $\varepsilon' = \frac{(C\mu_3 + 2\mu_1\mu_3 + 4\mu_1)}{b_0}\varepsilon$. This gives further

$$\varphi_q^0 \to 0 \text{ in } D(A^{\frac{3}{4}}) \text{ as } q \to \infty. \tag{4.84}$$

Since $\varphi(t, \varphi_q^0) = U(t)\varphi_q^0$ where $U(t)$ is a unitary group, we have

$$\varphi(t, \varphi_q^0) \to 0 \text{ in } C\left(0, \infty; D(A^{\frac{2}{4}})\right) \text{ as } q \to \infty. \tag{4.85}$$

Therefore, (4.85) implies $\lim_{q\to\infty} N_T(\varphi_q^0) = 0$ which contradicts (4.81). (4.64) is thus proved.

Step 3. From (4.62)-(4.64), we finally get

$$\int_{\Sigma_0} \left| \frac{\partial \varphi}{\partial \nu_{\mathbf{A}}} \right|^2 d\Sigma \geq C'' \left(T - \frac{(C\mu_3 + 2\mu_1\mu_3 + 4\mu_1)}{b_0}\varepsilon \right) \|\varphi^0\|^2_{D(\mathrm{A}^{\frac{3}{4}})}$$

where $C'' = \frac{2b_0\varepsilon}{(1+2\mu_1)+2(C\mu_3+2\mu_1\mu_3+4\mu_1)C_T} > 0$. So (4.44) holds for all $T > 0$. The proof is complete. ■

Remark 4.1 Using the inequality of admissibility (4.38) we obtain that the solution of (4.43) satisfies the following inequality

$$\int_{\Sigma_0} |\frac{\partial\left(\mathbf{A}\varphi\right)}{\partial \nu_{\mathbf{A}}}|^2 d\Sigma \geq C_T \|\varphi^0\|^2_{D(A^{\frac{3}{4}})}, \tag{4.86}$$

for any $T > 0$ with some constant $C_T > 0$. By (4.44) and (4.86), we see that for any $T > 0$, the norm $\int_{\Sigma_0} |\frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}}|^2 d\Sigma$ on the space $D(A^{\frac{3}{4}})$ is equivalent to the norm $\|\varphi^0\|^2_{D(A^{\frac{3}{4}})} = \left\|A^{\frac{3}{4}}\varphi^0\right\|^2_{L^2(\Omega)}$.

Chapter 5

Well posedness and exact controllability of fourth order Schrödinger equation with variable coefficients, moment boundary control and colocated observation

We recall that in chapter 4 our system is described with hinged boundary condition. In this chapter we study the same system with impose the moment boundary control and set the Dirichlet boundary condition to be zero. We begin with showing the well-posedness of this system in the state space $H_0^1(\Omega)$ and input/output space $U = Y = L^2(\Gamma_0)$, the regularity of the system is also proved with feedthrough operator is found to be zero. From the result of the well-posedness, we know that this system is exactly controllable in some interval $[0, T]$, $(T > 0)$ if and only if its corresponding closed loop systems under the output proportional feedback $u = -ky$, $k > 0$ is exponentialy stable. Based on this argument, to get the feedback stabilization of this system from the well-posedness, we study the exact controllability of the open-loop system.

5.1 System description and statement of main results.

The system what we concerned within is discribed by the following PDE's

$$\begin{cases} iw_t(x,t) + \mathbf{A}^2 w(x,t) = 0, & x \in \Omega, t > 0 \\ \mathbf{A}w(x,t) = u(x,t) & x \in \Gamma_0, t \geq 0 \\ \mathbf{A}w(x,t) = 0 & x \in \Gamma_1, t \geq 0 \\ w(x,t) = 0 & x \in \partial\Omega, t \geq 0 \\ y(x,t) = i\frac{\partial(A_1^{-1})w(x,t)}{\partial \nu_{\mathbf{A}}} & x \in \Gamma_0, t \geq 0 \end{cases} \tag{5.1}$$

Where, u is also standing for the boundary control input and y is the output.

Here we still use Ω defined in chapter 4, system (4.1), A_1 and A defined by (4.4) and (4.6), respectively.

Let $H_1 = H_0^1(\Omega)$, $\mathbf{H} = H^{-1}(\Omega)$. The following Theorem shows that the system (5.1) is well posed with the state space H_1, and the input and out put space $U = Y = L^2(\Gamma_0)$.

Theorem 5.1 *The system (5.1) is well-posed. More precisely, for any $T > 0$, initial value $w_0 \in H_1$ and control input $u \in L^2(0,T;U)$ there exists a unique solution $w \in C(0,T;H_1)$ to (5.1) such that*

$$\|w(.,T)\|_{H_1}^2 + \|y\|_{L^2(0,T;U)}^2 \leqslant C_T(\|w_0\|_{H_1}^2 + \|u\|_{L^2(0,T;U)}^2) \tag{5.2}$$

where C_T is used to represent the constant that depends only on T.

It is proved in [26, Theorem 5.8] (see also [68, Theorem 5.2]) that if the abstract system (5.1) introduced later is well-posed, it must be regular in the sense of Weiss with the zero feedthrough operator.

The following result is hence a consequence of Theorem 5.1.

Corollary 5.1 The system (5.1) is regular and the feedthrough operator is zero.

The second aim is to study the exact controllability problem for the open loop system (5.1), this is the result of Theorem 5.1 under a certain geomtric condition on Ω.

(H1) There is a vector field N on $(\mathbb{R}^n, g)$ such that

$$DN(X,X) = b(x)\,|X|_g^2, \quad \forall X \in T_x\mathbb{R}^n,\ x \in \Omega. \tag{H1}$$

where $b(x)$ is a function defined on Ω so that

$$b_0 = \inf_{x\in\Omega} b(x) > 0. \tag{H2}$$

(H2)

$$\Gamma \text{ satisfies } N(x).\nu > 0 \text{ on } \Gamma_0 \tag{H3}$$

Theorem 5.2 *Under assumptions (H1) and (H2), system (5.1) is exactly controllable on some $[0,T]$, $T > 0$. That is, given initial data $w(.,0) = w_0 \in H_0^1(\Omega)$ and time $T > 0$, there exists a boundary control $u \in L(0,T;L^2(\Gamma_0))$, such that the unique solution to the system (5.1) satisfies $w(T) = 0$.*

The following result is a direct consequence of Theorems 5.1 and 5.2.

Corollary 5.2 Let the Hypotheses of Theorem 5.2 hold true. Then system (5.1) is exponentially stable under the proportional output feedback $u = -ky$ for any $k > 0$.

5.2 Abstract formulation

We formulate system (5.1) as an abstract framework of a first order colocated system in the state space $H_1 = H_0^1(\Omega)$ and control and output space $U = Y = L^2(\Gamma_0)$.

Extend the operator $\tilde{A}$ of A into the space V as

Define an extension operator $\tilde{A}$ of A to the domain $\mathbf{H}$ as follows:

$$\langle \tilde{A}\varphi, \psi\rangle_{\mathbf{H}} = \langle A^{1/2}\varphi, A^{1/2}\psi\rangle_{\mathbf{H}}, \quad \forall \varphi, \psi \in V. \tag{5.3}$$

Then $\tilde{A}$ is a positive self-adjoint operator in $\mathbf{H}$:

Then $\tilde{A}$ is a positive self-adjoint operator in $\mathbf{H}$ as follows,

$$\begin{aligned} \langle \tilde{A}\varphi, \varphi\rangle_{\mathbf{H}} &= \langle A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\varphi\rangle_{\mathbf{H}} = \langle A^{\frac{1}{4}}\varphi, A^{\frac{1}{4}}\varphi\rangle_{L^2(\Omega)} \\ &\geq C\left\|\varphi\right\|^2_{L^2(\Omega)} \geq C^{'}\left\|A^{-\frac{1}{4}}\varphi\right\|^2_{L^2(\Omega)} = C^{'}\left\|\varphi\right\|^2_{\mathbf{H}}, \quad \forall\varphi\in V, \end{aligned} \tag{5.4}$$

Where C and $C^{'}$ are constants. We identify $\mathbf{H}$ with it's dual $\mathbf{H}^{'}$, then the following Gelfand inclusions hold true:

$$D(\tilde{A}^{-\frac{1}{2}}) \hookrightarrow \mathbf{H} = \mathbf{H}^{'} \hookrightarrow D(\tilde{A}^{-\frac{1}{2}})^{'}. \tag{5.5}$$

Define an extension $\hat{A} \in L(D(\tilde{A}^{1/2}), D(\tilde{A}^{1/2})')$ of $\tilde{A}$:

$$\langle \hat{A}f, g\rangle_{D(\tilde{A}^{\frac{1}{2}})', D(\tilde{A}^{\frac{1}{2}})} = \langle \tilde{A}^{\frac{1}{2}}f, \tilde{A}^{\frac{1}{2}}g\rangle_{\mathbf{H}}, \quad \forall f, g \in D(\tilde{A}^{\frac{1}{2}}). \tag{5.6}$$

Let G_1 be the Dirichlet map: $G_1 \in L(L^2(\Gamma_0), H^{5/2}(\Omega))$ by $G_1 u = \phi$ if and only if

$$\begin{cases} \mathbf{A}^2\phi = 0, \quad x\in\Omega, \\ \phi(x)|_{\Gamma} = 0, \\ \mathbf{A}\phi(x)|_{\Gamma_0} = u(x), \quad \mathbf{A}\phi(x)|_{\Gamma_1} = 0. \end{cases} \tag{5.7}$$

By virtue of the operators $\hat{A}$ and G_1, system (5.1) can be written in $D(\tilde{A}^{\frac{1}{2}})^{'}$ as

$$\dot{w} = \mathrm{i}\hat{A}w + Bu. \tag{5.8}$$

where $B \in L(U, D(\tilde{A}^{\frac{1}{2}})^{'})$ is given by

$$Bu = -\mathrm{i}\hat{A}G_1 u, \quad \forall u \in U. \tag{5.9}$$

Define $B^* \in L(D(\tilde{A}^{\frac{1}{2}}), U)$ by

$$\langle B^* f, u\rangle_U = \langle f, Bu\rangle_{D(\tilde{A}^{\frac{1}{2}}), D(\tilde{A}^{\frac{1}{2}})'}, \quad \forall f \in D(\tilde{A}^{\frac{1}{2}}) = H^1_0(\Omega),\ u\in U. \tag{5.10}$$

Then for any $f \in D(\tilde{A}^{\frac{1}{2}})^{'}$ and $u \in C_0^\infty(\Gamma_0)$, we have

$$\begin{aligned} \langle f, Bu\rangle_{D(\tilde{A}^{\frac{1}{2}}), D(\tilde{A}^{\frac{1}{2}})'} &= \langle f, -\mathrm{i}\hat{A}G_1 u\rangle_{D(\tilde{A}^{\frac{1}{2}}), D(\tilde{A}^{\frac{1}{2}})'} = \langle \tilde{A}^{1/2}f, -\mathrm{i}\tilde{A}^{1/2}G_1 u\rangle_{\mathbf{H}} \\ &= -\mathrm{i}\langle A_1(A_1^{-1}f), \tilde{A}G_1 u\rangle_{\mathbf{H}} \\ &= -\mathrm{i}\langle \mathbf{A}^2(\mathbf{A}^{-2}f), \tilde{A}G_1 u\rangle_{\mathbf{V}'} \\ &= -\mathrm{i}\langle A^{-1/4}A_1(A_1^{-1}f), A^{-1/4}AG_1 u\rangle_{L^2(\Omega).} \\ &= -\mathrm{i}\langle A_1(A_1^{-1}f), A_1 G_1 u\rangle_{L^2(\Omega).} \\ &= -\mathrm{i}\langle A_1^2(A_1^{-1}f), G_1 u\rangle_{L^2(\Omega).} \\ &= \langle u, -\mathrm{i}\frac{\partial(A_1{}^{-1}f))}{\partial\nu_{\mathbf{A}}}, u\rangle_{L^2(\Gamma_0)}. \end{aligned}$$

In the last step, we have used Greens formula and the definitions G_1 and A_1.

Since $C_0^\infty(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, we obtain

$$B^* f = -\mathrm{i}\frac{\partial(A_1^{-1} f))}{\partial \nu_{\mathbf{A}}}\Big|_{\Gamma_0}, \quad \forall f \in D(\tilde{A}^{1/2}) = H_0^1(\Omega). \tag{5.11}$$

Combing the above operators, we cast the open loop system (5.1) into an abstract first-order form in H_0^1:

$$\begin{aligned} \dot{w} &= \mathrm{i}\hat{A}w + Bu \\ y &= B^* w \end{aligned} \tag{5.12}$$

where $\hat{A}$, B and B^* are defined by (5.6), (5.9) and (5.10) respectively.

5.3 Proof of Theorem 5.1

Make a transformation $z = A_1^{-1} w$, then $z \in C\left(0, T; V\right)$, Then z satisfies

$$\begin{cases} z_t(x,t) = \mathrm{i}\mathbf{A}^2 z(x,t) + \mathrm{i}\left(\gamma u(.,t))\right)(x,t), & (x,t) \in \Omega \times (0,T] =: Q, \\ z(x,0) = z_0(x), & x \in \Omega, \\ z(x,t) = \mathbf{A} z\left(x,t\right) = 0 & ,(x,t) \in \partial\Omega \times [0,T], \end{cases} \tag{5.13}$$

and from (5.11) the output of (5.1) is changed into the form

$$y(x,t) = B^* w(x,t) = B^* A_1 A_1^{-1} w(x,t) = B^* A_1 z(x,t) = \mathrm{i}\frac{\partial z(x,t)}{\partial \nu_{\mathbf{A}}} \quad x \in \Gamma_0,\ t > 0 \tag{5.14}$$

and we used the fact in the first equation of (5.13)

$$A_1^{-1}\hat{A}G_1 u = \hat{A}^{\frac{1}{2}} G_1 u = -\gamma u, \qquad \forall u \in L^2(\Gamma_0)$$

where $\gamma \in L(L^2(\Gamma_0), H^{1/2}(\Omega))$ is defined by $\gamma u = \varphi$ if and only if

$$\begin{gathered} \mathbf{A}\phi = 0, \quad x \in \Omega, \\ \phi(x)|_{\Gamma_0} = u(x), \quad \phi(x)|_{\Gamma_1} = 0. \end{gathered}$$

Therefore, to prove theorem (5.1), we need only to prove that

$$\int_0^T \int_{\Gamma_0} |\frac{\partial z\left(x,t\right)}{\partial \nu_{\mathbf{A}}}|^2 d\Gamma dt \leqslant C_T \int_0^T \int_{\Gamma_0} |u(x,t)|^2 d\Gamma dt. \tag{5.16}$$

For the system (5.13) with the output (5.14).

Since $\partial\Omega$ is of class C^3, it follows from [25, Lemma 4.1] that there exists a C^2-vector field q on $\overline{\Omega}$ such that

$$q(x) = \mu(x),\ x \in \Gamma; \quad |N(x)|_g \leqslant 1,\ x \in \Omega. \tag{5.17}$$

Now, multiply both sides of the first equation in (5.13) by $q(\overline{z})$ and integrate over Q to obtain

$$\int_Q z_t q(\overline{z}) dQ - \mathrm{i}\int_Q \mathbf{A}^2 z q(\overline{z}) dQ = \mathrm{i}\int_Q \gamma u q(\overline{z}) dQ \tag{5.18}$$

Computing the first term on the left hand side of (5.18) and integrating by parts we obtain

$$\begin{aligned}\int_Q z_t q(\overline{z})dQ &= \int_\Omega zq(\overline{z})d\Omega\Big|_0^T - \int_Q zq(\bar{z}_t)dQ \qquad (5.19)\\ &= \left(\int_\Omega div_g(|z|^2 q)d\Omega - \int_\Omega \overline{z}q(z)\,d\Omega - \int_\Omega |z|^2 div_g(q)d\Omega\right)\Big|_0^T \\ &\quad -\left(\int_Q div_g(z\bar{z}_t q)\,dQ - \int_Q \bar{z}_t q(z)\,dQ - \int_Q z\bar{z}_t div_g(q)dQ\right)\end{aligned}$$

and hence

$$\begin{aligned}&2\mathrm{i}\,\mathrm{Im}\int_Q z_t q(\overline{z})dQ \qquad (5.20)\\ &= \int_Q z\bar{z}_t div_g(q)\,dQ - \left(\int_\Omega \overline{z}q(z)\,d\Omega + \int_\Omega |z|^2 div_g(q)d\Omega\right)\Big|_0^T \\ &= -\mathrm{i}\int_Q z\overline{\gamma u}\,div_g(q)\,dQ - \mathrm{i}\int_Q z\mathbf{A}^2\overline{z}div_g(q)\,dQ \\ &\quad -\left(\int_\Omega \overline{z}q(z)\,d\Omega + \int_\Omega |z|^2 div_g(q)d\Omega\right)\Big|_0^T\end{aligned}$$

A straight computation shows that

$$\begin{aligned}&\int_Q z\mathbf{A}^2\overline{z}div_g(q)\,dQ \qquad (5.21)\\ &= \int_\Sigma \frac{\partial(\mathbf{A}\overline{z})}{\partial\nu_{\mathbf{A}}} z\,div_g(q)\,d\Sigma - \int_\Sigma \frac{\partial(z\,div_g(q))}{\partial\nu_{\mathbf{A}}}\mathbf{A}\overline{z}d\Sigma + \int_Q \mathbf{A}(z\,div_g(q))\,\mathbf{A}\overline{z}dQ \\ &= \int_Q \mathbf{A}\overline{z}\left(div_g(q)\,\mathbf{A}z + z\mathbf{A}div_g(q) + 2\langle\nabla_g z, \nabla_g(div_g(q))\rangle_g\right)dQ \\ &= \int_Q |\mathbf{A}z|^2 div_g(q)\,dQ + \int_Q z\mathbf{A}\overline{z}\mathbf{A}div_g(q)\,dQ + 2\int_Q \mathbf{A}\overline{z}\langle\nabla_g z, \nabla_g(div_g(q))\rangle_g\,dQ\end{aligned}$$

Where we have used the fact that $\mathbf{A}(\varphi\psi) = \psi\mathbf{A}\varphi + \varphi\mathbf{A}\psi + 2\langle\nabla_g\varphi, \nabla_g\psi\rangle_g$, substituting (5.21) in (5.20) to yield

$$\begin{aligned}&\mathrm{Im}\int_Q z_t q(\overline{z})dQ \qquad (5.22)\\ &= -\frac{1}{2}\int_Q z\overline{\gamma u}\,div_g(q)\,dQ - \frac{1}{2}\int_Q |\mathbf{A}z|^2 div_g(q)\,dQ - \frac{1}{2}\int_Q z\mathbf{A}\overline{z}\mathbf{A}div_g(q)\,dQ \\ &\quad - \int_Q \mathbf{A}\overline{z}\langle\nabla_g z, \nabla_g(div_g(q))\rangle_g\,dQ + \frac{\mathrm{i}}{2}\left(\int_\Omega \overline{z}q(z)\,d\Omega + \int_\Omega |z|^2 div_g(q)d\Omega\right)\Big|_0^T\end{aligned}$$

Next, compute the second term of the left-hand side of (5.18) to yield

$$
\begin{aligned}
& \operatorname{Im} \mathrm{i}\left(\int_Q \mathbf{A}^2 z q(\bar{z}) dQ\right) && (5.23)\\
= \ & \operatorname{Re}\left(\int_Q \mathbf{A}^2 z q(\bar{z}) dQ\right)\\
= \ & \operatorname{Re}\int_Q \Delta_g(\mathbf{A}z)\, q(\bar{z}) dQ + \operatorname{Re}\int_Q (Dp)(\mathbf{A}z)\, q(\bar{z}) dQ\\
= \ & \operatorname{Re}\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu} q(\bar{z}) d\Sigma + \operatorname{Re}\int_Q \mathbf{A}z\Delta_g(q(\bar{z})) + \operatorname{Re}\int_Q (Dp)(\mathbf{A}z)\, q(\bar{z}) dQ\\
= \ & \operatorname{Re}\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu} q(\bar{z}) d\Sigma + \operatorname{Re}\int_Q \mathbf{A}z(\Delta q)(\bar{z}) dQ + \operatorname{Re}\int_Q \mathbf{A}z q(\mathbf{A}\bar{z}) dQ\\
& + 2\operatorname{Re}\int_Q \mathbf{A}z\langle Dq, D^2\bar{z}\rangle_{(T_x\mathbb{R}^n)^2} dQ - \operatorname{Re}\int_Q \mathbf{A}z D^2 p(q, D\bar{z})\, dQ\\
& - \operatorname{Re}\int_Q \mathbf{A}z D^2\bar{z}(q, Dp)\, dQ + \operatorname{Re}\int_Q \mathbf{A}z Ricc(q, D\bar{z})\, dQ + \operatorname{Re}\int_Q (Dp)(\mathbf{A}z)\, q(\bar{z}) dQ
\end{aligned}
$$

Furthermore, the following inequalities hold true:

$$
\left\|\frac{\partial z}{\partial\nu_{\mathbf{A}}}\right\|_{L^2(\Gamma)} \le C\|z\|_{H^2(\Omega)} \le C'\|\mathbf{A}z\|_{L^2(\Omega)} \qquad (5.24)
$$

Where C,$C' > 0$ are constants. The first inequality of (5.24) comes from the trace theorem, and the second one is attributed to the equivalence of the norms $\|z\|_{H^2(\Omega)}$ and $\|\mathbf{A}z\|_{L^2(\Omega)}$ in the space $H^2(\Omega)$.

Combining (5.18), (5.22), (5.23), we get

$$
\begin{aligned}
& C\int_\Sigma \left|\frac{\partial z}{\partial\mu}\right|^2 d\Sigma && (5.25)\\
\le \ & \inf_{\bar{\Omega}} |div_g(q)|_g \int_Q |\mathbf{A}z|_g^2\, dQ\\
\le \ & -\int_Q \overline{\gamma u}\, div_g(q)\, dQ - \int_Q \mathbf{A}\bar{z}\mathbf{A}(div_g(q))\, dQ\\
& -2\int_Q \mathbf{A}\bar{z}\langle\nabla_g z, \nabla_g(div_g(q))\rangle_g\rangle\, dQ + \mathrm{i}\left(\int_\Omega \bar{z}q(z)\, d\Omega + \int_\Omega |z|^2 div_g(q) d\Omega\right)\Bigg|_0^T\\
& +2\operatorname{Re}\int_\Sigma \frac{\partial(\mathbf{A}\bar{z})}{\partial\mu} q(\bar{z}) d\Sigma - 2\operatorname{Re}\int_Q \mathbf{A}z(\Delta q)(\bar{z}) dQ - 2\operatorname{Re}\int_Q \mathbf{A}z q(\mathbf{A}\bar{z}) dQ\\
& -4\operatorname{Re}\int_Q \mathbf{A}z\langle Dq, D^2\bar{z}\rangle_{(T_x\mathbb{R}^n)^2} dQ + 2\operatorname{Re}\int_Q \mathbf{A}z D^2 p(q, D\bar{z})\, dQ\\
& +2\operatorname{Re}\int_Q \mathbf{A}z D^2\bar{z}(q, Dp)\, dQ - 2\operatorname{Re}\int_Q \mathbf{A}z Ricc(q, D\bar{z})\, dQ - 2\operatorname{Re}\int_Q (Dp)(\mathbf{A}z)\, q(\bar{z}) dQ\\
& -2\operatorname{Re}\left(\mathrm{i}\int_Q \gamma u q(\bar{z}) dQ\right)
\end{aligned}
$$

In which

$$\mathrm{Re}\int_{\Sigma}\frac{\partial(\mathbf{A}\overline{z})}{\partial\mu}q(\overline{z})d\Sigma \le \frac{7}{32}\int_{\Sigma}\left|\frac{\partial(\mathbf{A}\overline{z})}{\partial\mu}\right|^2 d\Sigma + C^{'}\left\|z\right\|^2_{L^2(0,T;H^2(\Omega))} \tag{5.26}$$

where C,$C^{'} > 0$ are constants.

Moreover, we can use (5.32) by making $h = q$, to have

$$\frac{7}{32}\int_{\Sigma}\left|\frac{\partial(\mathbf{A}z)}{\partial\mu}\right|^2 d\Sigma \le R_1^{'} + R_2^{'} + b_{0,T}^{'} \tag{5.27}$$

where $R_1^{'}$ and $b_{0,T}^{'}$ are the same as the representations of R_1 and $b_{0,T}$ in (5.33), respectively, and the representation of $R_2^{'}$ is as follows:

$$\begin{aligned} R_2^{'} &= -\frac{1}{2}\mathrm{Re}\int_Q \gamma u\mathbf{A}\overline{z}div\left(q\right)dQ \\ &\quad -\frac{1}{2}\mathrm{Re}\int_Q \overline{\gamma u}\left[(\Delta h)(z) + \langle Dh, D^2 z\rangle_{(T_x\mathbb{R}^n)^2} - D^2 p\left(h, Dz\right) + Ricc\left(h, Dz\right) - D^2 z\left(h, Dp\right)\right] \\ &\quad -\frac{1}{2}\mathrm{Re}\int_Q \overline{\gamma u}\left[Dp\left(h\left(z\right)\right) - h\left(z\right)div_g\left(Dp\right)dQ\right] - \mathrm{Re}\int_Q \gamma uq\left(\mathbf{A}\overline{z}\right)dQ \end{aligned} \tag{5.28}$$

Combining (5.25), (5.26), and (5.27) to get

$$C\int_{\Sigma}\left|\frac{\partial z}{\partial\mu}\right|^2 d\Sigma \le R_1^{''} + R_2^{''} + b_{0,T}^{''} \tag{5.29}$$

Where

$$\begin{aligned} & R_1^{''} \\ =\; & R_1^{'} - \int_Q \mathbf{A}\overline{z}\mathbf{A}\left(div_g\left(q\right)\right)dQ - 2\int_Q \mathbf{A}\overline{z}\left\langle\nabla_g z, \nabla_g\left(div_g\left(q\right)\right)\rangle_g\right\rangle dQ - 2\,\mathrm{Re}\int_Q \mathbf{A}z\left(\Delta q\right)(\overline{z})dQ \\ & -4\,\mathrm{Re}\int_Q \mathbf{A}z\langle Dq, D^2\overline{z}\rangle_{(T_x\mathbb{R}^n)^2}dQ + 2\,\mathrm{Re}\int_Q \mathbf{A}zD^2 p\left(q, D\overline{z}\right)dQ + 2\,\mathrm{Re}\int_Q \mathbf{A}zD^2\overline{z}\left(q, Dp\right)dQ \\ & +2\,\mathrm{Re}\int_Q \mathbf{A}zD^2\overline{z}\left(q, Dp\right)dQ - 2\,\mathrm{Re}\int_Q \mathbf{A}zRicc\left(q, D\overline{z}\right)dQ - 2\,\mathrm{Re}\int_Q \left(Dp\right)\left(\mathbf{A}z\right)q(\overline{z})dQ \\ & -\frac{1}{2}\int_Q \left|\mathbf{A}z\right|_g^2 div_g\left(q\right)dQ + C^{'}\left\|z\right\|^2_{L^2(0,T;H^2(\Omega))} \end{aligned} \tag{5.30}$$

$$R_2^{''} = R_2^{'} - \int_Q \overline{\gamma u}div_g\left(q\right)dQ - 2\,\mathrm{Re}\left(\mathrm{i}\int_Q \gamma uq(\overline{z})dQ\right)$$

$$b_{0,T}^{''} = b_{0,T}^{'} + \mathrm{i}\left(\int_{\Omega}\overline{z}q\left(z\right)d\Omega + \int_{\Omega}\left|z\right|^2 div_g(q)d\Omega\right)\Big|_0^T$$

Now, we estimate $R_1^{''}$, $R_2^{''}$, $b_{0,T}^{''}$ in the following steps.

Step 1. Evaluation of $R_1^{''}$. It is found that the dual system of (5.13) is

$$\begin{cases} z_t(x,t) = \mathrm{i}\tilde{A}z(x,t), \\ z(0) = z_0, \\ y = B^* A_1 z, \end{cases} \tag{5.31}$$

Where $\tilde{A}$ and A_1 are given by (5.4), (4.4) respectively. It is well known that system (5.13) associates with a C_0-group solution in the space V. That is to say, for any $z_0 \in V$ to (5.31) depends continuously on z_0. By this facts and letting $\gamma u = 0$ in (5.18), we obtain

$$\int_{\Sigma} \left| \frac{\partial z}{\partial \nu_{\mathbf{A}}} \right|^2 d\Sigma \leq C_T \|z_0\|_V^2 \tag{5.32}$$

Which is equivalent to saying that for any initial $w_0 \in V^{'}$, the solution to system (5.13) with u satisfies

$$\int_{\Sigma} \left| \frac{\partial (A_1^{-1} w))}{\partial \nu_{\mathbf{A}}} \right|^2 d\Sigma \leq C_T \|w_0\|_{H_0^1(\Omega)}^2 \tag{5.33}$$

Hence the operator B^* is admissible, and so is B [9]. In other words,

$$u \longrightarrow w \text{ is continuous from } L^2(0,T;L^2(\Gamma_0)) \text{ to } C(0,T;V'). \tag{5.34}$$

Moreover

$$z \in C(0,T;V) \text{ depends continously on } u \ \in L^2(0,T;L^2(\Gamma_0)). \tag{5.35}$$

Therefore

$$R_1^{''} \leq \|u\|_{L^2(0,T;L^2(\Gamma_0))}^2 \qquad\qquad \forall u \in L^2(0,T;L^2(\Gamma_0)). \tag{5.36}$$

Step 2. Evaluation of $\mathbf{R}_2^{''}$ and $\mathbf{b}_{0,T}^{''}$. This can be easy done from the representation of $R_2^{'}$ and $b_{0,T}^{'}$ in (5.30),and by virtue of (5.35). we can readly obtain

$$R_2^{''} + b_{0,T}^{''} \leq \|u\|_{L^2(0,T;L^2(\Gamma_0))}^2, \qquad\qquad \forall u \in L^2(0,T;L^2(\Gamma_0)). \tag{5.37}$$

From (5.29), (5.36), and (5.37), it is seen that (5.16) holds true.■

5.4 Proof of Theorem 5.2

Since by Theorem 5.1, the operator B is admissible in system (5.12), the exact controllability of system (5.1) is equivalent to the exact observability to the dual problem of (5.1):

$$\begin{cases} i\varphi_t + \mathbf{A}^2\varphi(x,t) = 0, & \text{in } \Omega \times (0,T) = Q, \\ \varphi = 0, \ \mathbf{A}\varphi = 0 & \text{on } \partial\Omega \times (0,T) = \Sigma, \\ \varphi(x,0) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{5.38}$$

with the output $y = B^* A\varphi$. That is to say, the "observability inequality" holds true for system (5.38) in the sence of (5.13), (5.16):

$$\int_{\Sigma_0} |\frac{\partial \left(\varphi\left(x,t\right)\right)}{\partial \nu_{\mathbf{A}}}|^2 d\Sigma \geq C_T \left\|\varphi^0\right\|_V^2 ., \qquad \forall \varphi^0 \in V, \tag{5.39}$$

for some (and hence for all) positive $T > 0$.

To prove (5.39), we let A defined by (4.6) and let φ be a solution to (5.38). Then iA generates a strongly continuous unitary group on the space $V = D(A^{\frac{3}{4}})$ and hence

$$\begin{aligned} \|\varphi(t)\|_V &= \left\|(A^{\frac{3}{4}}\varphi(t)\right\|_V = \left\|e^{iAt}\varphi^0\right\|_V \\ &= \left\|\varphi^0\right\|_V = \left\|A^{\frac{3}{4}}\varphi^0\right\|_{L^2(\Omega)}. \end{aligned} \tag{5.40}$$

Next, we claim that for $f \in D(A^{\frac{3}{4}})$, the norms

$$\|f\|_{D(\mathrm{A}^{\frac{3}{4}})} = \left\|A^{\frac{3}{4}}f\right\|_{L^2(\Omega)} \text{ and } \left\{\int_\Omega |\nabla_g(\mathbf{A}f)|_g^2\,dx\right\}^{\frac{1}{2}} \text{ are equivalent.} \tag{5.41}$$

Actually, $\left\{\int_\Omega |\nabla_g(\mathbf{A}f)|_g^2\,dx\right\}^{\frac{1}{2}}$ being a norm is a trivial fact, since the norms $\|f\|_{D(\mathrm{A}^{\frac{1}{4}})} = \left\|A^{\frac{1}{4}}f\right\|_{L^2(\Omega)}$ and $\left\{\int_\Omega |\nabla_g(f)|_g^2\,dx\right\}^{\frac{1}{2}}$ are equivalent by the Poincaré inequality, the norms $\|f\|_{D(\mathrm{A}^{\frac{3}{4}})} = \left\|\mathrm{A}^{\frac{3}{4}}f\right\|_{L^2(\Omega)} = \left\|\mathrm{A}^{\frac{1}{4}}(\mathbf{A}f)\right\|_{L^2(\Omega)}$ and $\left\{\int_\Omega |\nabla_g(\mathbf{A}f)|_g^2\,dx\right\}^{\frac{1}{2}}$ are equivalent.

Proof of Theorem 5.2

Multiplying both sides of the first equation of (5.38) by $N(\bar{\varphi})$ and integrating over Q, applying the same computation procedure from (5.18)-(5.23) in the proof of Theorem 5.1, we obtain

$$\begin{aligned} & -\mathrm{Re}\int_\Sigma \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}N(\overline{\varphi})d\Sigma \\ = \ & \mathrm{Re}\int_Q \mathbf{A}zq(\mathbf{A}\overline{z})dQ + \frac{1}{2}\int_Q |\mathbf{A}\varphi|^2\,div_g(N)\,dQ - \frac{1}{2}\int_Q \varphi\mathbf{A}\overline{\varphi}\mathbf{A}div_g(N)\,dQ \\ & + \int_Q \mathbf{A}\overline{\varphi}\left\langle\nabla_g\varphi, \nabla_g(div_g(N))\rangle_g\right\rangle dQ - \frac{\mathrm{i}}{2}\left(\int_\Omega \overline{\varphi}N(\varphi)\,d\Omega + \int_\Omega |\varphi|^2\,div_g(N)d\Omega\right)\Big|_0^T \\ & + \mathrm{Re}\int_Q \mathbf{A}\varphi\left[(\Delta N)(\overline{\varphi}) + N(\mathbf{A}\overline{\varphi}) + 2\langle DN, D^2\bar{\varphi}\rangle_{(T_x\mathbb{R}^n)^2}\right]dQ \\ & - \mathrm{Re}\int_Q \mathbf{A}\varphi\left[D^2p(N, D\bar{\varphi}) + D^2\bar{\varphi}(N, Dp) - Ricc(N, D\bar{\varphi})\right]dQ \end{aligned} \tag{5.42}$$

To obtain the observability inequality, we define $T \in (T_x\mathbb{R}^n)^2$ for any $x \in \bar{\Omega}$ as follows:

$$T(X,Y) = DN(X,Y) + DN(Y,X), \quad \forall X,Y \in T_x\mathbb{R}^n. \tag{5.43}$$

It is clear that $T(.,.)$ is symmetric, and from $(H1)$, we have

$$DN(X,Y) + DN(Y,X) = 2b(x)\left\langle X,Y\right\rangle_g, \ \forall X,Y \in T_x\mathbb{R}^n, \ x \in \bar{\Omega} \tag{5.44}$$

Fix $x \in \bar{\Omega}$, and let $\{e_i\}_{i=1}^n$ be an orthogonormal basis of $(T_x\mathbb{R}^n, g)$. By (5.44), we have 2

$$\begin{aligned} \left\langle DN, D^2\varphi\right\rangle_{T_x\mathbb{R}^n} &= \sum_{i,j=1}^n DN(e_i,e_j)D^2\varphi(e_i,e_j) \\ &= b(x)\Delta_g\varphi = b(x)\left(\mathbf{A}\varphi - Dp(\varphi)\right). \end{aligned} \tag{5.45}$$

Combining (5.44), (5.45) and (5.42) we obtain

$$\begin{aligned}
& -\operatorname{Re}\int_{\Sigma}\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}N(\overline{\varphi})d\Sigma \qquad (5.46)\\
= & \operatorname{Re}\int_{Q}\mathbf{A}zq(\mathbf{A}\overline{z})dQ+\frac{1}{2}\int_{Q}|\mathbf{A}\varphi|^{2}\,div_{g}(N)\,dQ\\
& +\int_{Q}\mathbf{A}\overline{\varphi}\,\langle\nabla_{g}\varphi,\nabla_{g}(div_{g}(N))\rangle_{g}\rangle\,dQ-\frac{1}{2}\int_{Q}\varphi\mathbf{A}\overline{\varphi}\mathbf{A}div_{g}(N)\,dQ\\
& +\operatorname{Re}\int_{Q}\mathbf{A}\varphi\left[(\Delta N)(\overline{\varphi})-2\int_{Q}b(x)\mathbf{A}\varphi Dp(\bar{\varphi})\right]dQ\\
& -\operatorname{Re}\int_{Q}\mathbf{A}\varphi\left[DN(D\overline{\varphi},Dp)-D^{2}p(N,D\overline{\varphi})+Ricc(N,D\overline{\varphi})\right]dQ\\
& -\frac{\mathrm{i}}{2}\left(\int_{\Omega}\overline{\varphi}N(\varphi)\,d\Omega+\int_{\Omega}|\varphi|^{2}\,div_{g}(N)d\Omega\right)\Bigg|_{0}^{T}\\
= & \sum_{i=1}^{6}M_{i}
\end{aligned}$$

We compute the LHS of (5.46), to get

$$-\operatorname{Re}\int_{\Sigma}\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}N(\overline{\varphi})d\Sigma\leq\frac{1}{\varepsilon}\int_{\Sigma}\left|\frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}\right|^{2}N.\nu d\Sigma+\varepsilon\mu_{1}\int_{\Sigma}\left|\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right|^{2}d\Sigma. \qquad (5.47)$$

A direct computation shows that

$$\int_{Q}\mathbf{A}zq(\mathbf{A}\overline{z})dQ=\int_{Q}\left(\operatorname{div}_{g}\left(|\mathbf{A}\varphi|^{2}N\right)-\mathbf{A}\bar{\varphi}N(\mathbf{A}\varphi)-|\mathbf{A}\varphi|_{g}^{2}\,div_{g}(N)\right)dQ$$

This implies that

$$\operatorname{Re}\int_{Q}\mathbf{A}zq(\mathbf{A}\overline{z})dQ=\frac{1}{2}\int_{Q}|\mathbf{A}\varphi|_{g}^{2}\,div_{g}(N)\,dQ$$

Multiply both sides of the first equation of (5.38) by $\bar{\varphi}$ and $\mathbf{A}\bar{\varphi}$, respectively and integrate over Q to obtain

$$\begin{aligned}
i\int_{Q}\varphi_{t}\bar{\varphi}dQ+\int_{Q}|\mathbf{A}\varphi|_{g}^{2}\,dQ &= 0\\
i\int_{Q}\nabla_{g}\varphi_{t}.\nabla_{g}\bar{\varphi}dQ+\int_{Q}|\nabla_{g}(\mathbf{A}\varphi)|_{g}^{2}\,dQ &= 0
\end{aligned}$$

Then

$$\int_{Q}|\mathbf{A}\varphi|_{g}^{2}\,dQ=\int_{Q}|\nabla_{g}(\mathbf{A}\varphi)|_{g}^{2}\,dQ+i\int_{Q}\nabla_{g}\varphi_{t}.\nabla_{g}\bar{\varphi}dQ-i\int_{Q}\varphi_{t}\bar{\varphi}dQ. \qquad (5.48)$$

Next, we evaluate the terms of the RHS of (5.46) and using (5.48), we obtain

$$\begin{aligned} M_1 &= \int_Q |\mathbf{A}\varphi|_g^2 \, div_g(N)\, dQ \qquad (5.49)\\ &\geq \mu_2 \left(\int_Q |\nabla_g(\mathbf{A}\varphi)|_g^2 \, dQ + i\int_Q \nabla_g\varphi_t . \nabla_g \bar{\varphi} dQ - i\int_Q \varphi_t \bar{\varphi} dQ \right) \\ &\geq \mu_2 \left\| (A^{\frac{3}{4}}\varphi^0 \right\|^2_{L^2((0,T)\times\Omega)} + \mu_2 \left(i\int_Q \nabla_g\varphi_t . \nabla_g \bar{\varphi} dQ - i\int_Q \varphi_t \bar{\varphi} dQ \right) \\ &= \mu_2 T \left\| (A^{\frac{3}{4}}\varphi^0 \right\|^2_{L^2((0,T)\times\Omega)} + \mu_2 i\int_Q \nabla_g\varphi_t . \nabla_g \bar{\varphi} dQ - \mu_2 i\int_Q \varphi_t \bar{\varphi} dQ \\ &= \mu_2 T \left\| \varphi^0 \right\|^2_{D(A^{\frac{3}{4}})} + \mu_2 i\int_Q \nabla_g\varphi_t . \nabla_g \bar{\varphi} dQ - \mu_2 i\int_Q \varphi_t \bar{\varphi} dQ \end{aligned}$$

by (5.41), the Poincaré inequality, we obtain

$$\begin{aligned} |M_2| &= \left| \int_Q \mathbf{A}\overline{\varphi} \left\langle \nabla_g \varphi, \nabla_g (div_g(N)) \rangle_g \right\rangle dQ - \frac{1}{2}\int_Q \varphi \mathbf{A}\overline{\varphi}\mathbf{A} div_g(N)\, dQ \right. \qquad (5.50)\\ &\quad + \mathrm{Re}\int_Q \mathbf{A}\varphi \left[(\Delta N)(\overline{\varphi}) - 2\int_Q b(x)\mathbf{A}\varphi Dp(\bar{\varphi}) \right] dQ \\ &\quad \left. - \mathrm{Re}\int_Q \mathbf{A}\varphi \left[DN(D\bar{\varphi}, Dp) - D^2 p(N, D\bar{\varphi}) + Ricc(N, D\bar{\varphi}) \right] dQ \right| \\ &\leq \frac{\mu_3'}{\varepsilon} \left\| \mathbf{A}\varphi(t) \right\|^2_{C((0,T);L^2(\Omega))} + \mu_3 \varepsilon \left\| \varphi^0 \right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} \end{aligned}$$

$$\begin{aligned} |M_3| &= \left| -\frac{\mathrm{i}}{2} \left(\int_\Omega \overline{\varphi} N(\varphi)\, d\Omega \right) \Big|_0^T \right| \leq \frac{\mu_1}{2} \left(\left\| \overline{\varphi(T)} \right\| \left\| \nabla_g \varphi(T) \right\| + \left\| \overline{\varphi^0} \right\| \left\| \nabla_g \varphi^0 \right\| \right) \qquad (5.51)\\ &\leq \frac{\mu_1}{4} \left(\frac{1}{\varepsilon} \left\| \nabla_g \varphi(T) \right\|^2 + \varepsilon \left\| \overline{\varphi(T)} \right\|^2 + \frac{1}{\varepsilon} \left\| \nabla_g \varphi^0 \right\|^2 + \varepsilon \left\| \overline{\varphi^0} \right\|^2 \right) \\ &\leq \frac{\mu_1}{2} \left(\frac{1}{\varepsilon} \mu_3 \left\| \mathbf{A}\varphi(t) \right\|^2_{C((0,T);L^2(\Omega))} + \varepsilon \left\| \varphi^0 \right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} \right), \end{aligned}$$

$$\begin{aligned} |M_4| &= \left| -\frac{\mathrm{i}}{2} \left(\int_\Omega |\varphi|^2 \, div_g(N) d\Omega \right) \Big|_0^T \right| \qquad (5.52)\\ &= \left| -\frac{\mathrm{i}}{2} \left(\int_\Omega \varphi\bar{\varphi}\, \mathrm{div}_g(N) d\Omega \right) \Big|_0^T \right| \\ &\leq \frac{\mu_1}{2} \left(\left\| \varphi(T) \right\| \left\| \overline{\varphi(T)} \right\| + \left\| \varphi^0 \right\| \left\| \overline{\varphi^0} \right\| \right) \\ &\leq \frac{\mu_1}{4} \left(\frac{1}{\varepsilon} \left\| \varphi(T) \right\|^2 + \varepsilon \left\| \overline{\varphi(T)} \right\| + \frac{1}{\varepsilon} \left\| \varphi^0 \right\|^2 + \varepsilon \left\| \overline{\varphi^0} \right\|^2 \right) \\ &\leq \frac{\mu_1}{2} \left(\frac{1}{\varepsilon} \mu_3' \left\| \mathbf{A}\varphi(t) \right\|^2_{C((0,T);L^2(\Omega))} + \varepsilon \left\| \varphi^0 \right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} \right), \end{aligned}$$

$$\begin{aligned} |M_5| &= \left|-\mu_2 i\int_Q \nabla_g\varphi_t.\nabla_g\bar{\varphi}dQ\right| \leq \int_Q |\nabla_g\varphi_t|\,|\nabla_g\bar{\varphi}|\,dQ \leq \frac{\mu_2}{2}\int_\Omega |\nabla_g\varphi|_g^2\,d\Omega\Big|_0^T \qquad (5.53)\\ &\leq \frac{\mu_2}{2}\left(\|\nabla_g\varphi(T)\|\left\|\overline{\nabla_g\varphi(T)}\right\| + \|\nabla_g\varphi^0\|\left\|\overline{\nabla_g\varphi^0}\right\|\right)\\ &\leq \frac{1}{\varepsilon}\frac{\mu_3\mu_2}{2}\|\mathbf{A}\varphi(t)\|^2_{C((0,T);L^2(\Omega))} + \frac{\mu_2}{2}\varepsilon\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})}\end{aligned}$$

$$\begin{aligned} |M_6| &= \left|\mu_2 i\int_Q \varphi_t\bar{\varphi}dQ\right| \leq \mu_2\int_Q |\varphi_t|\,|\varphi|\,dQ \leq \mu_2\int_\Omega |\varphi|^2\Big|_0^T \qquad (5.54)\\ &\leq \mu_2\left(\|\varphi(T)\|\left\|\overline{\varphi(T)}\right\| + \|\varphi^0\|\left\|\overline{\varphi^0}\right\|\right)\\ &\leq \frac{\mu_3'}{\varepsilon}\|\mathbf{A}\varphi(t)\|^2_{C((0,T);L^2(\Omega))} + \varepsilon\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})},\end{aligned}$$

where μ_i and μ_i' $i = 1,2,3$ are constants. Combining (5.49)-(5.54) gives

$$\begin{aligned}\text{RHS of (5.46)} &\geq \mu_2 T\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} - \left(2+\frac{\mu_2}{2}+\mu_3\right)\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} \qquad (5.55)\\ &\quad -\frac{\left(\frac{\mu_2}{2}+1\right)\mu_3' + \left(\frac{\mu_2}{2}+1\right)\mu_3}{\varepsilon}\|\mathbf{A}\varphi(t)\|^2_{C((0,T);L^2(\Omega))}\end{aligned}$$

Therefore, by the estimates of the LHS and RHS of (5.46) in (5.47) and (5.55) respectively, we have by taking (H2), (5.24) and (4.86) into account, that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\Sigma_0}\left|\frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma + \frac{\mu_1 C'^2 + \left(\frac{\mu_2}{2}+1\right)\mu_3' + \left(\frac{\mu_2}{2}+1\right)\mu_3}{\varepsilon}\|\mathbf{A}\varphi(t)\|^2_{C((0,T);L^2(\Omega))} \qquad (5.56)\\ &\geq \mu_2\left(T - \frac{2+\frac{\mu_2}{2}+\mu_3+\mu_1 C_T''}{\mu_2}\varepsilon\right)\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})}.\end{aligned}$$

We claim that the inequality (5.56) implies that for any $T > 0$, there exists $\alpha > 0$, such that for all $\varphi^0 \in D(\mathrm{A}^{\frac{3}{4}})$,

$$\|\mathbf{A}\varphi(t)\|^2_{C((0,T);L^2(\Omega))} \leq \alpha\int_{\Sigma_0}\left|\frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}\right|^2 N.\nu d\Sigma, \qquad (5.57)$$

and for any sequence T_n with $T_n \to \infty$,

$$\lim_{n\to\infty}\alpha_{T_n} = 0. \qquad (5.58)$$

The proofs for aforementioned claims are similar to step to in the proof of Theorem 4.2. Actually, we will assume that they are not true to obtain a contradiction , to this purpose, let be the solution sequence $\{\varphi_n\}$ to the system of (5.38) such that

$$\begin{aligned}\|\mathbf{A}\varphi_n\|^2_{C((0,T);L^2(\Omega))} &= 1, \qquad (5.59)\\ \int_{\Sigma_0}\left|\frac{\partial\varphi_n}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma &\to 0, \quad n\to\infty \qquad (5.60)\end{aligned}$$

By (5.56), we can obtain (5.54) and hence (5.55), (5.56) and

$$1 = \|\mathbf{A}\varphi_n\|^2_{C((0,T);L^2(\Omega))} \to \|\mathbf{A}\tilde{\varphi}\|^2_{C((0,T);L^2(\Omega))} = 1. \tag{5.61}$$

In addition, owing to (5.60), we have

$$\frac{\partial \tilde{\varphi}}{\partial \nu_{\mathbf{A}}} = 0 \text{ on } \Gamma_{0.}$$

Thus, $\tilde{\varphi}$ satisfies

$$\begin{cases} i\tilde{\varphi}_t(x,t) + \mathbf{A}^2\tilde{\varphi}(x,t) = 0, & \text{in } \Omega \times (0,T) = Q, \\ \tilde{\varphi}(x,t) = 0,\ \mathbf{A}\tilde{\varphi}(x,t) = 0 & \text{on } \partial\Omega \times (0,T) = \Sigma, \\ \frac{\partial \tilde{\varphi}}{\partial \nu_{\mathbf{A}}} = 0 & \text{on } \Sigma_{0.} \end{cases}$$

This leads to $\tilde{\varphi} \equiv 0$ in Q, contradicting (5.59). (5.57) is thus proved.

To prove (5.58), we set

$$N_T(\varphi^0) = \|\mathbf{A}\varphi\|^2_{C((0,T);L^2(\Omega))}\,, \quad D_T(\varphi^0) = \int_{\Sigma_0} \left|\frac{\partial \varphi}{\partial \nu_{\mathbf{A}}}\right|^2 d\Sigma,$$

for each $T > 0$ and each $\varphi^0 \in D(\mathrm{A}^{\frac{3}{4}})$. The remaining proof for (5.58) is similar to the proof of (**??**), then

$$\int_{\Sigma_0} \left|\frac{\partial \varphi}{\partial \nu_{\mathbf{A}}}\right|^2 d\Sigma \geq C\left(T - \frac{2 + \frac{\mu_2}{2} + \mu_3 + \mu_1 C''_T}{\mu_2}\varepsilon\right) \|\varphi^0\|^2_{D(\mathrm{A}^{\frac{3}{4}})}$$

where $C = \frac{\mu_2\varepsilon}{1+\left[\mu_1 C'^2 + \left(\frac{\mu_2}{2}+1\right)\mu_3' + \left(\frac{\mu_2}{2}+1\right)\mu_3\right]\alpha} > 0$. The proof is complete. ■

Chapter 6

Well posedness, regularity and exact controllability of fourth order Schrödinger equation with variable coefficients, Dirichlet boundary control and colocated observation

In this chapter we consider an open-loop system of fourth order Schrödinger equation with variable coefficients, Dirichlet boundary control and collocated observation, following the approach developed in **[67]** and the multiplier method on Riemannian manifold, we show that the system is well-posed with input/output space $U = Y = L^2(\Gamma_0)$, state space V' which is the dual space of $V = \left\{\varphi \in H^3(\Omega) : \varphi|_\Gamma = \frac{\partial \varphi}{\partial \nu_{\mathbf{A}}}\Big| = 0\right\}$ with respect to the pivot space $L^2(\Omega)$. In addition, this system is regular with feedthrough operator is zero. In order to prove the feedback stabilization from well posedness, we discuss the exact controllability of this control system.

6.1 System description and statement of main results

The system that we are concerned with in this chapter is described as follows

$$\begin{cases} iw_t(x,t) + \mathbf{A}^2 w(x,t) = 0 & x \in \Omega, t > 0 \\ w(x,t) = 0 & x \in \partial\Omega, t \geq 0 \\ \frac{\partial w(x,t)}{\partial \nu_{\mathbf{A}}} = 0 & x \in \Gamma_1, t \geq 0 \\ \frac{\partial w(x,t)}{\partial \nu_{\mathbf{A}}} = u(x,t) & x \in \Gamma_0, t \geq 0 \\ z(x,t) = i\frac{\partial \mathbf{A}(A)^{-\frac{3}{2}} w(x,t)}{\partial \nu_{\mathbf{A}}} & x \in \Gamma_0, t \geq 0 \end{cases} \tag{6.1}$$

Where, u is also standing for the boundary control input and y is the output.

Here we still use Ω defined in chapter 4 and 5. Now, let A be the positive self-adjoint operator in $L^2(\Omega)$ defined by

$$Af = \mathbf{A}^2 f, \quad D(A) = H^4(\Omega) \cap H_0^2(\Omega). \tag{6.5}$$

one can show that $A^{1/2} = -\mathbf{A}$. Moreover,

$$V = D(A^{\frac{3}{4}}) = \left\{ \varphi \in H^3(\Omega) : \varphi|_\Gamma = \frac{\partial \varphi}{\partial \nu_{\mathbf{A}}} \bigg| = 0 \right\},$$

Then the following theorem states that the system is well-posed with the state space V' which is the dual of the space V with respect to the pivot space $L^2(\Omega)$ in the sence of Gelfand's triple inclusions $V \to L^2(\Omega) \to V'$ and the input and output space $U = Y = L^2(\Gamma_0)$.

Theorem 6.1 *The system (6.1) is well-posed. More precisely, for any $T > 0$, initial value $w_0 \in V'$ and control input $u \in L^2(0, \infty; U)$ there exists a unique solution $w \in C(0, T; V')$ to (6.1) such that*

$$\|w(.,T)\|_{V'}^2 + \|z\|_{L^2(0,T;U)}^2 \leqslant C_T(\|w_0\|_{V'}^2 + \|u\|_{L^2(0,T;U)}^2)$$

where C_T is used to represent the constant that depends only on Ω, Γ_0 and T.

Theorem 6.2 *The system (6.1) is regular with zero feedthrough operator. This means that if the initial state $w(.,0) = 0$ and $u(.,t) = u(t) \in U$ is a step input, then the corresponding output satisfies*

$$\underset{\sigma \to 0}{Lim} \int_\Gamma \left| \frac{1}{\sigma} \int_0^\sigma z(x,t)dt \right|^2 d\sigma = 0 \tag{6.6}$$

The second aim is to study the exact controllability problem for the open loop system (6.1), this is the result of Theorem 6.1 under a certain geomtric condition on Ω.

(H'1) There is a vector field N on $(\mathbb{R}^n, g)$ such that

$$DN(X,X) = b(x)\,|X|_g^2\,, \quad \forall X \in T_x\mathbb{R}^n,\ x \in \Omega. \tag{H'1}$$

where $b(x)$ is a function defined on Ω, so that

$$b_0 = \inf_{x \in \Omega} b(x) > 0. \tag{H'2}$$

(H'2)

$$\Gamma \text{ satisfies } N(x).\nu > 0 \text{ on } \Gamma_0 \tag{H3}$$

Theorem 6.3 *Under assumptions (H'1) and (H'2), system (6.1) is exactly controllable on some $[0,T]$, $T > 0$. That is, given initial data $w(.,0) = w_0 \in V'$ and time $T > 0$, there exists a boundary control $u \in L(0,T; L^2(\Gamma_0))$, such that the unique solution to the system (6.1) satisfies $w(T) = 0$.*

The following result is a direct consequence Theorems 4.1 and 4.2.

Corollary 6.1 Let the hypotheses of Thorem 6.3 hold true. Then system (6.1) is exponentially stable under the proportional output feedback $u = -ky$ for any $k > 0$.

6.2 Abstract formulation

In this section we cast the system (6.1) into an abstract first order system in the state space V' and control and output space $U = Y = L^2(\Gamma_0)$.

Define an extension operator $\tilde{A}$ of A defined as in (4.6) to the space V' by

$$\langle \tilde{A}\varphi, \psi\rangle_{V'} = \langle A^{1/2}\varphi, A^{1/2}\psi\rangle_{V'}, \quad \forall \varphi, \psi \in V.$$

$\tilde{A}$ is a positive self-adjoint operator in V'. In fact,

$$\begin{aligned}
\langle \tilde{A}\varphi, \varphi\rangle_{V'} &= \langle A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\varphi\rangle_{V'} = \langle A^{-\frac{1}{4}}\varphi, A^{-\frac{1}{4}}\varphi\rangle_{L^2(\Omega)} \\
&\geq C\|\varphi\|^2_{L^2(\Omega)} \geq C'\left\|A^{-\frac{3}{4}}\varphi\right\|^2_{L^2(\Omega)} \\
&= C'\|\varphi\|^2_{V'}, \quad \forall \varphi \in V,
\end{aligned}$$

where C and C' are constants. We identify $H = V'$ with it's dual H'. Then the following Gelfand triple continuous inclusions positive hold true:

$$D(\tilde{A}^{\frac{1}{2}}) \hookrightarrow H = H' \hookrightarrow D(\tilde{A}^{\frac{1}{2}})'.$$

Define an extension $\hat{A} \in L(D(\tilde{A}^{1/2}), D(\tilde{A}^{1/2})')$ of $\tilde{A}$:

$$\langle \hat{A}f, g\rangle_{D(\tilde{A}^{\frac{1}{2}})', D(\tilde{A}^{\frac{1}{2}})} = \langle \tilde{A}^{\frac{1}{2}}f, \tilde{A}^{\frac{1}{2}}g\rangle_{V'}, \quad \forall f, g \in D(\tilde{A}^{\frac{1}{2}}). \tag{6.7}$$

Define the map $\gamma \in (L^2(\Gamma_0), H^{1/2}(\Omega))$ [37, pp. 188-189] so that $\gamma u = \phi$ if and only if

$$\begin{cases} \mathbf{A}^2\phi = 0, \quad x \in \Omega, \\ \left.\frac{\partial\phi(x)}{\partial\nu_{\mathbf{A}}}\right| = 0,\ \phi(x)|_{\Gamma_1} = 0,\ \phi(x)|_{\Gamma_0} = u \end{cases} \tag{6.8}$$

By virtue of $\hat{A}$ and γ system (6.1) can be written in $D(\tilde{A})'$ as

$$\dot{w} = \mathrm{i}\hat{A}w + Bu. \tag{6.9}$$

where $B \in L(U, D(\tilde{A}^{1/2})')$ is given by

$$Bu = -\mathrm{i}\hat{A}\gamma u, \quad \forall u \in U. \tag{6.10}$$

Define $B^* \in L(D(\tilde{A}^{1/2})', U)$ by

$$\langle B^*f, u\rangle_U = \langle f, Bu\rangle_{D(\tilde{A}^{\frac{1}{2}}), D(\tilde{A}^{\frac{1}{2}})'}, \quad \forall f \in D(\tilde{A}^{\frac{1}{2}}) = H^1_0(\Omega),\ u \in U. \tag{6.13}$$

Then for any $f \in D(\tilde{A}^{\frac{1}{2}})$ and $u \in C_0^\infty(\Gamma_0)$, we have

$$\begin{aligned}
\langle f, Bu\rangle_{D(\tilde{A}^{\frac{1}{2}}), D(\tilde{A}^{\frac{1}{2}})'} &= \langle f, -\mathrm{i}\hat{A}\gamma u\rangle_{D(\tilde{A}^{1/2}), D(\tilde{A}^{1/2})'} \qquad (6.14) \\
&= -\mathrm{i}\langle \tilde{A}^{1/2}f, \tilde{A}^{1/2}\gamma u\rangle_{V'} \\
&= -\mathrm{i}\langle \tilde{A}^{3/2}\left(\tilde{A}^{-3/2}f\right), \tilde{A}\gamma u\rangle_{V'} \\
&= -\mathrm{i}\langle A^{-3/4}A^{3/2}(A^{-3/2}f), A^{-3/4}A\gamma u\rangle_{L^2(\Omega)}, \text{ with } A^{\frac{1}{2}} = A_1 \\
&= \langle \mathbf{A}^2(A^{-3/2}f), -\mathrm{i}\gamma\mathbf{A}u\rangle_{L^2(\Omega).} \\
&= \langle \mathrm{i}\frac{\partial(\mathbf{A}(A^{-3/2}f))}{\partial\nu_{\mathbf{A}}}, u\rangle_{U.}
\end{aligned}$$

We have used in the last step Green's second theorem.

Since $C_0^\infty(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, we obtain

$$B^* f = -\mathrm{i}\frac{\partial(\mathbf{A}(A^{-3/2}f))}{\partial\nu_{\mathbf{A}}}\Bigg|_{\Gamma_0}, \quad \forall f \in D(A^{1/2}) = H_0^1(\Omega). \tag{6.15}$$

Thus, we have formulated the open loop system (6.1) into an abstract first-order form in H:

$$\begin{aligned} \dot{w} &= \mathrm{i}\hat{A}w + Bu \\ z &= B^* w \end{aligned} \tag{6.16}$$

where $\hat{A}$, B and B^* are defined by (6.7), (6.10) and (6.13) respectively.

6.3 Proof of Theorem 6.1

In order to prove Theorem 6.1, we need the following Lemma which comes from [8, Theorem A.1].

Lemma 6.1. If there exist constants $T > 0$, $C_T > 0$, such that the input and output of system (6.1) satisfy

$$\int_0^T \|z(t)\|_U^2\,dt \leq C_T \int_0^T \|u(t)\|_U^2\,dt, \quad \forall u \in L^2(0,T;L^2(\Gamma_0)) \tag{6.17}$$

with $y(.,0) = 0$, the system (6.1) is well-posed

make a transformation $z = A^{-3/2}w \in H^1\left(0,T;H^4(\Omega)\right)$, Then z satisfies

$$\begin{cases} z_t(x,t) = \mathrm{i}\mathbf{A}^2 z(x,t) - \mathrm{i}\left(A^{-1/2}\gamma u(.,t))\right)(x,t)\,, & (x,t) \in \Omega\times(0,T] =: Q, \\ z(x,0) = 0, & x \in \Omega, \\ z(x,t) = \frac{\partial z(x,t)}{\partial\nu_{\mathbf{A}}} = 0, & ,(x,t) \in \partial\Omega\times[0,T] =: \Sigma, \\ y(x,t) = \mathrm{i}\frac{\partial(\mathbf{A}z(x,t))}{\partial\nu_{\mathbf{A}}}, & ,(x,t) \in \partial\Omega\times[0,T]\,, \end{cases} \tag{6.18}$$

and from (6.16) the output of (6.18) is changed into the form

$$\begin{aligned} y(x,t) &= B^* w(x,t) = B^* A_1 A_1^{-1} w(x,t) = B^* A_1 z(x,t) \\ &= -\mathrm{i}\frac{\partial(\mathbf{A}(A^{-3/2}f))}{\partial\nu_{\mathbf{A}}}(x,t) \quad x \in \Gamma_0,\ t > 0 \end{aligned} \tag{6.19}$$

So Theorem 6.1, holds true if and only if for some (and hence for all) $T > 0$, there exists a $C_T > 0$ such that the solution to (6.18) satisfies (consider smooth u if necessary)

$$\int_0^T \int_{\Gamma_0} |\frac{\partial\left(\mathbf{A}z\left(x,t\right)\right)}{\partial\nu_{\mathbf{A}}}|^2 d\Gamma dt \leqslant C_T \int_0^T \int_{\Gamma_0} |u(x,t)|^2 d\Gamma dt. \tag{6.20}$$

We proceed the proof in three steps.

Step 1. Since $\partial\Omega$ is of class C^3, it follows from [25, Lemma 4.1] that there exists a C^2 vector field N on $\overline{\Omega}$ such that

$$N(x) = \mu(x),\ x \in \Gamma; \quad |N(x)| \leqslant 1,\ x \in \Omega. \tag{6.21}$$

Let $P(x,T) = \left(A^{-1/2}\gamma u\,(.,t)\right)(x,t)$, then by definition of A and γ we have

$$\int_0^T \|P(x,t)\|^2_{L^2(\Omega)}\,dt \le C_T \int_0^T \|u(.,t)\|^2_{L^2(\Gamma_0)}\,dt, \tag{6.22}$$

Now, multiply both sides of the first equation in (6.18) by $N(\mathbf{A}\overline{z})$ and integrate over Q to obtain

$$\int_Q z_t N(\mathbf{A}\overline{z})dQ - \mathrm{i}\int_Q \mathbf{A}zN(\mathbf{A}\overline{z})dQ + \mathrm{i}\int_Q PN(\mathbf{A}\overline{z})dQ = 0. \tag{6.23}$$

Compute the second term on the left-hand side of (6.23) to yield

$$\begin{aligned}
& \mathrm{i}\int_Q \mathbf{A}^2 zN(\mathbf{A}\overline{z})dQ \qquad (6.24)\\
= \; & \mathrm{i}\int_Q \Delta_g\left(\mathbf{A}z\right)N(\mathbf{A}\overline{z})dQ + \mathrm{i}\int_Q Dp\left(\mathbf{A}z\right)N(\mathbf{A}\overline{z})dQ\\
= \; & \mathrm{i}\left[\int_\Sigma \frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}N(\mathbf{A}\overline{z})d\Sigma - \int_Q \langle\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(\mathbf{A}\overline{z}\right)\rangle_g dQ\right]\\
& +\mathrm{i}\int_Q Dp\left(\mathbf{A}z\right)N(\mathbf{A}\overline{z})dQ\\
= \; & \mathrm{i}\left[\int_\Sigma \left|\frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}\right|^2 d\Sigma - \int_Q DN\left(\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(\mathbf{A}\overline{z}\right)\right)dQ\right]\\
& +\mathrm{i}\left[-\frac{1}{2}\int_Q div_g\left(\left|\nabla_g\left(\mathbf{A}z\right)\right|^2_g N\right)dQ + \int_Q \left|\nabla_g\left(\mathbf{A}z\right)\right|^2_g div_g N dQ\right]\\
& +\mathrm{i}\int_Q Dp\left(\mathbf{A}z\right)N(\overline{\mathbf{A}z})dQ
\end{aligned}$$

Then

$$\begin{aligned}
& \mathrm{Im}\left(\mathrm{i}\int_Q \mathbf{A}^2 zN(\mathbf{A}\overline{z})dQ\right)\\
= \; & \mathrm{Re}\left(\int_Q \mathbf{A}^2 zN(\mathbf{A}\overline{z})dQ\right)\\
= \; & \int_\Sigma \left|\frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}\right|^2 d\Sigma - \mathrm{Re}\left(\int_Q DN\left(\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(\mathbf{A}\overline{z}\right)\right)dQ\right)\\
& -\frac{1}{2}\int_Q div_g\left(\left|\nabla_g\left(\mathbf{A}z\right)\right|^2_g N\right)dQ + \mathrm{Re}\left(\int_Q Dp\left(\mathbf{A}z\right)N(\mathbf{A}\overline{z})dQ\right)
\end{aligned}$$

Next, we compute the first term of LHS of (6.23) and by virtue of the divergence formula, we have

$$\begin{aligned}
div_g\left(z_t\bar{z}N\right) &= z_t div_g\left(\mathbf{A}\bar{z}N\right) + \mathbf{A}\bar{z}N\left(z_t\right) \qquad (6.25)\\
&= z_t\left[\mathbf{A}\bar{z}div_g\left(N\right) + N\left(\mathbf{A}\bar{z}\right)\right] + \mathbf{A}\bar{z}N\left(z_t\right)\\
&= z_t\mathbf{A}\bar{z}div_g\left(N\right) + z_tN\left(\mathbf{A}\bar{z}\right) + \mathbf{A}\bar{z}N\left(z_t\right)\\
&= \left(\mathrm{i}\mathbf{A}^2 z - \mathrm{i}P\right)\left(\mathbf{A}\bar{z}div_g\left(N\right)\right) + z_tN\left(\mathbf{A}\bar{z}\right) + \frac{d}{dt}\left[\mathbf{A}\bar{z}N\left(z\right)\right] - \mathbf{A}\bar{z}_tN\left(z\right)
\end{aligned}$$

in which

$$\begin{aligned}
& \int_Q \mathbf{A}\bar{z}_t N(z)\, dQ \qquad (6.26)\\
= & \int_Q \left[\Delta_g \bar{z}_t + Dp(\bar{z}_t)\right] N(z)\, dQ\\
= & \int_Q \Delta_g \bar{z}_t N(z)\, dQ + \int_Q Dp(\bar{z}_t) N(z)\, dQ\\
= & \int_\Sigma \frac{\partial(\bar{z}_t)}{\partial\mu} N(z)\, d\Sigma + \int_Q \bar{z}_t \Delta_g (N(z))\, dQ + \int_Q Dp(\bar{z}_t) N(z)\, dQ\\
= & \int_\Sigma \frac{\partial(\bar{z}_t)}{\partial\mu} N(z)\, d\Sigma + \int_Q Dp(\bar{z}_t) N(z)\, dQ\\
& + \int_Q \bar{z}_t \left[\Delta(N(z)) + 2\langle DN, D^2 z\rangle_{(T_x\mathbb{R}^n)^2} + N(\mathbf{A}z)\right] dQ\\
& + \int_Q \bar{z}_t \left[Ricc(N, Dz) - D^2 p(N, Dz) - D^2 z(N, Dp)\right] dQ
\end{aligned}$$

where we have used (A.33). Then

$$\begin{aligned}
& \int_Q \mathbf{A}\bar{z}_t N(z)\, dQ \qquad (6.27)\\
= & \int_\Sigma \frac{\partial(\bar{z}_t)}{\partial\mu} N(z)\, d\Sigma + \int_Q \bar{z}_t (\Delta N)(z)\, dQ + 2\int_Q \bar{z}_t \langle DN, D^2 z\rangle_{(T_x\mathbb{R}^n)^2}\\
& + \int_Q \bar{z}_t N(\mathbf{A}z)\, dQ - \int_Q \bar{z}_t D^2 p(N, Dz)\, dQ + \int_Q \bar{z}_t Ricc(N, Dz)\, dQ\\
& - \int_Q \bar{z}_t D^2 z(N, Dp)\, dQ + \int_Q Dp(\bar{z}_t) N(z)\, dQ
\end{aligned}$$

integrating the equality (6.26) over Q by taking (6.27) into account, yields

$$\begin{aligned}
\int_Q z_t N \mathbf{A}\bar{z}\, dQ = & \;\mathrm{i}\int_Q P\mathbf{A}\bar{z}\, div_g(N)\, dQ - \mathrm{i}\int_Q \mathbf{A}^2 z \mathbf{A}\bar{z}\, div_g(N)\, dQ - \int_Q \frac{d}{dt}(\mathbf{A}\bar{z}N(z))\, dQ\\
& + \int_\Sigma \frac{\partial(\bar{z}_t)}{\partial\mu} N(z)\, d\Sigma + \int_Q \bar{z}_t (\Delta N)(z)\, dQ + 2\int_Q \bar{z}_t \langle DN, D^2 z\rangle_{(T_x\mathbb{R}^n)^2}\\
& + \int_Q \bar{z}_t N(\mathbf{A}z)\, dQ - \int_Q \bar{z}_t D^2 p(N, Dz)\, dQ + \int_Q \bar{z}_t Ricc(N, Dz)\, dQ\\
& - \int_Q \bar{z}_t D^2 z(N, Dp)\, dQ + \int_Q Dp(\bar{z}_t) N(z)\, dQ
\end{aligned}$$

this implies that

$$\begin{aligned}
&2\mathrm{i}\,\mathrm{Im}\left(\int_Q z_t N\left(\mathbf{A}\bar{z}\right)dQ\right)\\
=\;& \mathrm{i}\int_Q P\mathbf{A}\bar{z}div_g\left(N\right)dQ - \mathrm{i}\int_Q \mathbf{A}^2 z\mathbf{A}\bar{z}div_g\left(N\right)dQ - \int_Q \frac{d}{dt}\left(\mathbf{A}\bar{z}N\left(z\right)\right)dQ\\
&+\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu}N\left(z\right)d\Sigma + \int_Q \bar{z}_t\left(\Delta N\right)\left(z\right)dQ + 2\int_Q \bar{z}_t\langle DN, D^2 z\rangle_{\left(T_x\mathbb{R}^n\right)^2}\\
&+\int_Q \bar{z}_t N\left(\mathbf{A}z\right)dQ - \int_Q \bar{z}_t D^2 p\left(N, Dz\right)dQ + \int_Q \bar{z}_t Ricc\left(N, Dz\right)dQ\\
&-\int_Q \bar{z}_t D^2 z\left(N, Dp\right)dQ + \int_Q Dp\left(\bar{z}_t\right)N\left(z\right)dQ
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathrm{Im}\left(\int_Q z_t N\left(\mathbf{A}\bar{z}\right)dQ\right) \qquad (6.28)\\
=\;& \frac{1}{2}\int_Q P\mathbf{A}\bar{z}div_g\left(N\right)dQ - \frac{1}{2}\int_Q \mathbf{A}^2 z\mathbf{A}\bar{z}div_g\left(N\right)dQ - \frac{\mathrm{i}}{2}\int_Q \frac{d}{dt}\left(\mathbf{A}\bar{z}N\left(z\right)\right)dQ\\
&+\frac{\mathrm{i}}{2}\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu}N\left(z\right)d\Sigma + \frac{\mathrm{i}}{2}\int_Q \bar{z}_t\left(\Delta N\right)\left(z\right)dQ + \mathrm{i}\int_Q \bar{z}_t\langle DN, D^2 z\rangle_{\left(T_x\mathbb{R}^n\right)^2}\\
&+\frac{\mathrm{i}}{2}\int_Q \bar{z}_t N\left(\mathbf{A}z\right)dQ - \frac{\mathrm{i}}{2}\int_Q \bar{z}_t D^2 p\left(N, Dz\right)dQ + \frac{\mathrm{i}}{2}\int_Q \bar{z}_t Ricc\left(N, Dz\right)dQ\\
&-\frac{\mathrm{i}}{2}\int_Q \bar{z}_t D^2 z\left(N, Dp\right)dQ + \frac{\mathrm{i}}{2}\int_Q Dp\left(\bar{z}_t\right)N\left(z\right)dQ
\end{aligned}$$

while

$$\int_Q Dp\left(\bar{z}_t\right)N\left(z\right)dQ = -\int_Q \bar{z}_t Dp\left(N\left(z\right)\right)dQ - \int_Q \bar{z}_t N\left(z\right)div_g\left(Dp\right)dQ \qquad (6.29)$$

$$\begin{aligned}
&\mathrm{Im}\left(\int_Q z_t N\left(\mathbf{A}\bar{z}\right)dQ\right)\\
=\;& \frac{1}{2}\int_Q P\mathbf{A}\bar{z}div_g\left(N\right)dQ - \frac{1}{2}\int_Q \mathbf{A}^2 z\mathbf{A}\bar{z}div_g\left(N\right)dQ + \frac{\mathrm{i}}{2}\int_\Omega \frac{d}{dt}\left(\mathbf{A}\bar{z}N\left(z\right)\right)d\Omega\Big]_0^T\\
&+\frac{\mathrm{i}}{2}\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu}N\left(z\right)d\Sigma - \frac{\mathrm{i}}{2}\int_Q \bar{z}_t\left(\Delta N\right)\left(z\right)dQ - \mathrm{i}\int_Q \bar{z}_t\langle DN, D^2 z\rangle_{\left(T_x\mathbb{R}^n\right)^2}dQ\\
&+\frac{\mathrm{i}}{2}\int_Q \bar{z}_t D^2 p\left(N, Dz\right)dQ - \frac{\mathrm{i}}{2}\int_Q \bar{z}_t Ricc\left(N, Dz\right)dQ + \frac{\mathrm{i}}{2}\int_Q \bar{z}_t D^2 z\left(N, Dp\right)dQ\\
&+\frac{\mathrm{i}}{2}\int_Q \bar{z}_t Dp\left(N\left(z\right)\right)dQ + \frac{\mathrm{i}}{2}\int_Q \bar{z}_t N\left(z\right)div_g\left(Dp\right)dQ
\end{aligned}$$

Then

$$\begin{aligned}
& \operatorname{Im}\left(\int_Q z_t N\left(\mathbf{A}\bar{z}\right) dQ\right) \qquad (6.30)\\
= \; & \frac{1}{2}\int_Q P\mathbf{A}\bar{z}div_g\left(N\right) dQ - \frac{1}{2}\int_Q \mathbf{A}^2 z\mathbf{A}\bar{z}div_g\left(N\right) dQ + \frac{\mathrm{i}}{2}\int_\Omega \left(\mathbf{A}\bar{z}N\left(z\right)\right) d\Omega\Big]_0^T \\
& -\frac{\mathrm{i}}{2}\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu} N\left(z\right) d\Sigma - \frac{1}{2}\int_Q \left(\mathbf{A}^2\bar{z} - \bar{P}\right)\left(\Delta N\right)\left(z\right) dQ \\
& -\int_Q \left(\mathbf{A}^2\bar{z} - \bar{P}\right)\left\langle DN, D^2 z\right\rangle_{\left(T_x\mathbb{R}^n\right)^2} dQ + \frac{1}{2}\int_Q \left(\mathbf{A}^2\bar{z} - \bar{P}\right) D^2 p\left(N, Dz\right) dQ \\
& -\frac{1}{2}\int_Q \left(\mathbf{A}^2\bar{z} - \bar{P}\right) Ricc\left(N, Dz\right) dQ + \frac{1}{2}\int_Q \left(\mathbf{A}^2\bar{z} - \bar{P}\right) D^2 z\left(N, Dp\right) dQ \\
& +\frac{1}{2}\int_Q \left(\mathbf{A}^2\bar{z} - \bar{P}\right) N\left(z\right) div_g\left(Dp\right) dQ + \frac{1}{2}\int_Q \left(\mathbf{A}^2\bar{z} - \bar{P}\right) Dp\left(N\left(z\right)\right) dQ
\end{aligned}$$

and hence

$$\begin{aligned}
& \operatorname{Im}\left(\int_Q z_t N\left(\mathbf{A}\bar{z}\right) dQ\right) \\
= \; & \frac{1}{2}\int_Q P\mathbf{A}\bar{z}div_g\left(N\right) dQ - \frac{1}{2}\int_\Sigma \frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}\mathbf{A}\bar{z}div_g\left(N\right) d\Sigma + \frac{1}{2}\int_Q \left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 div_g\left(N\right) dQ \\
& + \int_Q \mathbf{A}\bar{z}\langle\nabla_g\left(\mathbf{A}z\right), \nabla_g\left(div_g\left(N\right)\right)\rangle_g dQ + \int_Q Dp\left(\mathbf{A}z\right)\mathbf{A}\bar{z}div_g\left(N\right) dQ \\
& +\frac{\mathrm{i}}{2}\int_\Omega \left(\mathbf{A}\bar{z}N\left(z\right)\right) d\Omega\Big]_0^T - \frac{\mathrm{i}}{2}\int_\Sigma \frac{\partial\left(\bar{z}_t\right)}{\partial\mu} N\left(z\right) d\Sigma - \frac{1}{2}\int_Q \mathbf{A}^2\bar{z}\left(\Delta N\right)\left(z\right) dQ \\
& -\int_Q \mathbf{A}^2\bar{z}\langle DN, D^2 z\rangle_{\left(T_x\mathbb{R}^n\right)^2} dQ + \frac{1}{2}\int_Q \mathbf{A}^2\bar{z}D^2 p\left(N, Dz\right) dQ \\
& -\frac{1}{2}\int_Q \mathbf{A}^2\bar{z}Ricc\left(N, Dz\right) dQ + \frac{1}{2}\int_Q \mathbf{A}^2\bar{z}D^2 z\left(N, Dp\right) dQ \\
& +\frac{1}{2}\int_Q \mathbf{A}^2\bar{z}Dp\left(N\left(z\right)\right) dQ + \frac{1}{2}\int_Q \mathbf{A}^2\bar{z}N\left(z\right) div_g\left(Dp\right) dQ + \frac{1}{2}\int_Q \bar{P}\left(\Delta N\right)\left(z\right) dQ \\
& + \int_Q \bar{P}\langle DN, D^2 z\rangle_{\left(T_x\mathbb{R}^n\right)^2} dQ - \frac{1}{2}\int_Q \bar{P}D^2 p\left(N, Dz\right) dQ \\
& +\frac{1}{2}\int_Q \bar{P}Ricc\left(N, Dz\right) dQ - \frac{1}{2}\int_Q \bar{P}D^2 z\left(N, Dp\right) dQ \\
& -\frac{1}{2}\int_Q \bar{P}Dp\left(N\left(z\right)\right) dQ - \frac{1}{2}\int_Q \bar{P}N\left(z\right) div_g\left(Dp\right) dQ
\end{aligned}$$

Then

$$
\begin{aligned}
&\operatorname{Im}\left(\int_Q z_t N\left(\mathbf{A}\bar{z}\right) dQ\right) \qquad (6.31)\\
= \;&\frac{1}{2}\int_Q P\mathbf{A}\bar{z}div_g\left(N\right)dQ - \frac{1}{2}\int_\Sigma \frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}\bar{z}div_g\left(N\right)d\Sigma\\
&+\frac{1}{2}\int_Q \left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 div_g\left(N\right)dQ + \frac{\mathrm{i}}{2}\int_\Omega \left(\mathbf{A}\bar{z}N\left(z\right)\right)d\Omega\Big]_0^T\\
&+\int_Q \mathbf{A}\bar{z}\langle\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(div_g\left(N\right)\right)\rangle_g dQ + \int_Q Dp\left(\mathbf{A}z\right)\mathbf{A}\bar{z}div_g\left(N\right)dQ\\
&+\frac{1}{2}\int_\Sigma \frac{\partial\left(\mathbf{A}\bar{z}\right)}{\partial\mu}\left[D^2p\left(N,Dz\right) - \left(\Delta N\right)\left(z\right) - \langle DN, D^2z\rangle_{(T_x\mathbb{R}^n)^2} + Ricc\left(N,Dz\right)\right]d\Sigma\\
&+\frac{1}{2}\int_\Sigma \frac{\partial\left(\mathbf{A}\bar{z}\right)}{\partial\mu}\left[D^2z\left(N,Dp\right) + Dp\left(N\left(z\right)\right) + N\left(z\right)div_g\left(Dp\right)\right]d\Sigma\\
&+\frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(\Delta N\right)\left(z\right)\rangle_g dQ + \frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(\langle DN, D^2z\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ\\
&-\frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(D^2p\left(N,Dz\right)\right)\rangle_g dQ - \frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(Ricc\left(N,Dz\right)\right)\rangle_g dQ\\
&-\frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{z}\right),\left(\nabla_g D^2z\left(N,Dp\right)\right)\rangle_g dQ - \frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(Dp\left(N\left(z\right)\right)\right)\rangle_g dQ\\
&-\frac{1}{2}\int_Q \langle\nabla_g\left(\mathbf{A}\bar{z}\right),\nabla_g\left(N\left(z\right)div_g\left(Dp\right)\right)\rangle_g dQ\\
&+\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)\left(\Delta N\right)\left(z\right)dQ + \frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)\left(\langle DN, D^2z\rangle_{(T_x\mathbb{R}^n)^2}\right)dQ\\
&-\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)Ricc\left(N,Dz\right)dQ - \frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)D^2z\left(N,Dp\right)dQ\\
&-\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)Dp\left(N\left(z\right)\right)dQ - \frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)N\left(z\right)div_g\left(Dp\right)dQ\\
&-\frac{1}{2}\int_Q Dp\left(\mathbf{A}\bar{z}\right)\left(D^2p\left(N,Dz\right)\right)dQ\\
&+\frac{1}{2}\int_Q \bar{P}\left[\left(\Delta N\right)\left(z\right) + \langle DN, D^2z\rangle_{(T_x\mathbb{R}^n)^2} - D^2p\left(N,Dz\right)\right]dQ\\
&+\frac{1}{2}\int_Q \bar{P}\left[Ricc\left(N,Dz\right) - D^2z\left(N,Dp\right) - Dp\left(N\left(z\right)\right) - N\left(z\right)div_g\left(Dp\right)\right]dQ
\end{aligned}
$$

compute the second term of the LHS of (6.23) to obtain

$$
\begin{aligned}
&\operatorname{Im}\left(\mathrm{i}\int_Q \mathbf{A}^2 zN(\mathbf{A}\bar{z})dQ\right) \qquad (6.32)\\
= \;&\frac{1}{2}\int_\Sigma \left|\frac{\partial\left(\mathbf{A}z\right)}{\partial\mu}\right|^2 d\Sigma - \frac{1}{2}\int_\Sigma \left|\frac{\partial\left(\mathbf{A}z\right)}{\partial\tau}\right|^2 d\Sigma - \operatorname{Re}\left[\int_Q DN\left(\left(\nabla_g\left(\mathbf{A}z\right),\nabla_g\left(\mathbf{A}\bar{z}\right)\right)\right)dQ\right]\\
&+\operatorname{Re}\left[\int_Q Dp\left(\mathbf{A}z\right)N\left(\mathbf{A}\bar{z}\right)dQ\right] + \frac{1}{2}\int_Q \left|\nabla_g\left(\mathbf{A}z\right)\right|_g^2 div_g\left(N\right)dQ
\end{aligned}
$$

Where $\tau = \tau(x)$ is the tangential vector at $x \in \Gamma$ and we have used the divergence formula.

$$
\begin{aligned}
& \frac{1}{2}\int_{\Sigma}\frac{\partial(\mathbf{A}z)}{\partial\mu}\left[N(z)\,div_g(Dp)-\mathbf{A}\bar{z}div_g(N)-\langle DN,D^2z\rangle_{(T_x\mathbb{R}^n)^2}\right]d\Sigma \qquad (6.33)\\
& +\frac{1}{2}\int_{\Sigma}\frac{\partial(\mathbf{A}z)}{\partial\mu}\left[D^2p(N,Dz)+Ricc(N,Dz)+D^2z(N,Dp)-(\Delta N)(z)\right]d\Sigma\\
\leq\; & \int_{\Sigma}\left|\frac{\partial(\mathbf{A}z)}{\partial\mu}\right|^2d\Sigma+\int_{\Sigma}|\mathbf{A}\bar{z}div_g(N)|^2d\Sigma+\int_{\Sigma}|(\Delta N)(z)|^2d\Sigma+\int_{\Sigma}\left|\langle DN,D^2z\rangle_{(T_x\mathbb{R}^n)^2}\right|^2d\Sigma\\
& +\int_{\Sigma}|Ricc(N,Dz)|^2d\Sigma+\int_{\Sigma}\left|D^2z(N,Dp)\right|^2d\Sigma+\int_{\Sigma}|N(z)\,div_g(Dp)|^2d\Sigma\\
\leq\; & \int_{\Sigma}\left|\frac{\partial(\mathbf{A}z)}{\partial\mu}\right|^2d\Sigma+C\|z\|^2_{L^2(0,T;H^3(\Omega))}
\end{aligned}
$$

In the last step where we have used the Sobolev trace theorem with constant $C > 0$. Combining (6.23), (6.31), (6.32) and (6.33) gives

$$
\int_{\Sigma}\left|\frac{\partial(z)}{\partial\mu}\right|^2d\Sigma \leq R_0+R_1+R_2+b_{0,T} \qquad (6.34)
$$

wher

$$
\begin{aligned}
R_0 &= \frac{1}{2}\int_{\Sigma}\left|\frac{\partial(\mathbf{A}z)}{\partial\tau}\right|^2d\Sigma \qquad (6.35)\\
R_1 &= \mathrm{Re}\left(\int_Q DN\left(\nabla_g(\mathbf{A}z),\nabla_g(\mathbf{A}\bar{z})\right)dQ\right)+\int_Q\mathbf{A}\bar{z}\langle\nabla_g(\mathbf{A}z),\nabla_g(div_g(N))\rangle_g dQ\\
&\quad+\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{z}),\nabla_g(\Delta N)(z)\rangle_g dQ\\
&\quad+\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{z}),\nabla_g\left(\langle DN,D^2z\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ-\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{z}),\nabla_g\left(D^2p(N,Dz)\right)\rangle_g dQ\\
&\quad+\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{z}),\nabla_g\left(\langle DN,D^2z\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ\\
&\quad-\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{z}),\nabla_g(Ricc(N,Dz))\rangle_g dQ-\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{z}),\left(\nabla_g D^2z(N,Dp)\right)\rangle_g dQ\\
&\quad-\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{z}),\nabla_g(Dp(N(z)))\rangle_g dQ\\
&\quad-\frac{1}{2}\int_Q\langle\nabla_g(\mathbf{A}\bar{z}),\nabla_g(N(z)\,div_g(Dp))\rangle_g dQ\\
R_2 &= \frac{1}{2}\int_Q\bar{P}\left[(\Delta N)(z)+\langle DN,D^2z\rangle_{(T_x\mathbb{R}^n)^2}-D^2p(N,Dz)\right]dQ\\
&\quad+\frac{1}{2}\int_Q\bar{P}\left[Ricc(N,Dz)-D^2z(N,Dp)-Dp(N(z))-N(z)\,div_g(Dp)\right]dQ\\
b_{0,T} &= \frac{\mathrm{i}}{2}\int_{\Omega}(\mathbf{A}\bar{z}N(z))\,d\Omega\Big]_0^T
\end{aligned}
$$

We now estimate R_0. To this purpose, we first introduce the following operator: $\mathbf{B}$ = first order differential operator on Ω, tangential to Γ (i.e., wihtout transversal derivatives to Γ when it is expressed in local coordinates) and with smooth coefficients on $\bar{\Omega}$.
Next, we define a new variable

$$\eta = \mathbf{B}z \in C\left(0, T; H^2(\Omega)\right) \tag{6.36}$$

and apply $\mathbf{B}$ to system (6.18) to obtain

$$\begin{cases} \boldsymbol{\eta}_t(x,t) - \mathrm{i}\mathbf{A}^2\boldsymbol{\eta} = P & (x,t) \in \Omega \times (0,T] =: Q, \\ \boldsymbol{\eta}(x,0) = 0, & x \in \Omega, \\ \boldsymbol{\eta}(x,t) = \frac{\partial \boldsymbol{\eta}(x,t)}{\partial \nu_{\mathbf{A}}} = 0, & , (x,t) \in \partial\Omega \times [0,T] =: \Sigma, \end{cases} \tag{6.37}$$

where

$$\begin{aligned} S &= -\mathrm{i}\left[\mathbf{A}^2, \mathbf{B}\right] z - \mathrm{i}\mathbf{B}A^{-1/2}\gamma u, \\ Kz &= \left[\mathbf{A}^2, \mathbf{B}\right] z \in C\left(0, T; H^{-1}(\Omega)\right) \text{ is the interior commutator.} \end{aligned} \tag{6.38}$$

Since $\left.\frac{\partial \boldsymbol{\eta}}{\partial \nu_{\mathbf{A}}}\right|_{\partial\Omega \times [0,T]} = \left[\frac{\partial}{\partial \nu_{\mathbf{A}}}, \mathbf{B}\right] z$ is smooth, we can replace it with the homogenuous boundary value without loss of generality. In this way, we get

$$\begin{aligned} &\int_\Gamma \left|\frac{\partial(\mathbf{A}z)}{\partial \tau}\right|^2 d\Gamma \\ =& \int_\Gamma |\mathbf{B}(\mathbf{A}z)|^2 d\Gamma = \int_\Gamma |\mathbf{A}(\mathbf{B}z)|^2 d\Gamma + l_1 = \int_\Gamma |\mathbf{A}\boldsymbol{\eta}|^2 d\Gamma + l_1 \end{aligned} \tag{6.39}$$

Where l_1 denotes the lower order terms of z, so we need to evaluate only $\int_\Gamma |\mathbf{A}\boldsymbol{\eta}|^2 d\Gamma$ for the system (6.36) in order to evaluate R_0.

Now, multiply both sides of the first equation of (6.36) by $N(\bar{\boldsymbol{\eta}})$ and integrate over Q. We obtain

$$\frac{1}{2}\int_\Sigma |\mathbf{A}\boldsymbol{\eta}|^2 d\Sigma = R_0' + R_1' + R_2' + b_{0,T}' \tag{6.40}$$

with

$$\begin{aligned} R_0' &= \mathrm{Re}\int_Q \bar{\boldsymbol{\eta}}\langle\nabla_g \eta, \nabla_g(div_g(N))\rangle_g dQ + \frac{1}{2}\mathrm{Re}\,\eta\mathbf{A}\bar{\boldsymbol{\eta}}\mathbf{A}(div_g(N))\,dQ \\ &+ \mathrm{Re}\int_Q \mathbf{A}\boldsymbol{\eta}\left[(\Delta N)(\boldsymbol{\eta}) + 2\langle DN, D^2\boldsymbol{\eta}\rangle_{(T_x\mathbb{R}^n)^2}\right] dQ, \\ &+ \mathrm{Re}\int_Q \mathbf{A}\boldsymbol{\eta}\left[Ricc(N, D\boldsymbol{\eta}) - D^2 p(N, D\boldsymbol{\eta}) - D^2\boldsymbol{\eta}(N, Dp)\right] dQ \\ R_1' &= -\frac{1}{2}\mathrm{Re}\int_Q \mathbf{A}\bar{\boldsymbol{\eta}} Dp(\boldsymbol{\eta} div_g(N))\,dQ + \frac{1}{2}\mathrm{Re}\int_Q \boldsymbol{\eta} div_g(N)\,Dp(\mathbf{A}\bar{\boldsymbol{\eta}})\,dQ \\ &+ \mathrm{Re}\int_Q N(\bar{\boldsymbol{\eta}})\,Dp(\mathbf{A}\boldsymbol{\eta})\,dQ, \\ R_2' &= -\frac{1}{2}\mathrm{Re}\int_Q \bar{S}\boldsymbol{\eta} div_g(N)\,dQ + \mathrm{Im}\int_Q \bar{S}N(\boldsymbol{\eta})\,dQ, \\ b_{0,T}' &= \frac{1}{2}\mathrm{Im}\int_\Omega \bar{\boldsymbol{\eta}}N(\boldsymbol{\eta})\,d\Omega\Big|_0^T + \int_\Omega |\boldsymbol{\eta}|^2 div_g(N)\,d\Omega\Big|_0^T \end{aligned} \tag{6.41}$$

compute the two terms of $R_1^{'}$ respectively to obtain

$$\begin{aligned} & -\frac{1}{2}\operatorname{Re}\int_Q \bar{S}\boldsymbol{\eta} div_g(N)\,dQ \qquad (6.42) \\ = & -\frac{1}{2}\operatorname{Re}\int_Q \boldsymbol{\eta}\left[\mathbf{A}^2,\mathbf{B}\right]\boldsymbol{\eta} div_g(N)\,dQ - \frac{1}{2}\operatorname{Re}\int_Q \boldsymbol{\eta}\mathbf{B}\left(A^{-1/2}\gamma\bar{u}\right)div_g(N)\,dQ \end{aligned}$$

and

$$\begin{aligned} & \operatorname{Im}\int_Q \bar{S}N(\boldsymbol{\eta})\,dQ \qquad (6.43) \\ = & -\operatorname{Re}\int_Q \boldsymbol{\eta}\left[\mathbf{A}^2,\mathbf{B}\right]\boldsymbol{\eta}N(\bar{\boldsymbol{\eta}})\,dQ - \operatorname{Re}\int_Q \mathbf{B}\left(A^{-1/2}\gamma\bar{u}\right)N(\bar{\boldsymbol{\eta}})\,dQ \end{aligned}$$

Then, from (6.34) and (6.39)-(6.43), we have

$$\frac{5}{16}\int_\Sigma \left|\frac{\partial(\mathbf{A}z)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma \le R_1'' + R_2'' + b_{0,T}'', \qquad (6.44)$$

Where

$$\begin{aligned} R_1'' &= R_1 + R_0^{'} + R_1^{'} - \frac{1}{2}\operatorname{Re}\int_Q \boldsymbol{\eta}\left[\mathbf{A}^2,\mathbf{B}\right]\boldsymbol{\eta} div_g(N)\,dQ \qquad (6.45) \\ & \quad -\operatorname{Re}\int_Q \boldsymbol{\eta}\left[\mathbf{A}^2,\mathbf{B}\right]\boldsymbol{\eta}N(\bar{\boldsymbol{\eta}})\,dQ + \frac{1}{2}l_1 \\ R_2'' &= R_2 - \operatorname{Re}\int_Q \mathbf{B}\left(A^{-1/2}\gamma\bar{u}\right)N(\bar{\boldsymbol{\eta}})\,dQ - \frac{1}{2}\operatorname{Re}\int_Q \boldsymbol{\eta}\mathbf{B}\left(A^{-1/2}\gamma\bar{u}\right)div_g(N)\,dQ \\ b_{0,T}'' &= b_{0,T} + b_{0,T}^{'} \end{aligned}$$

In the following steps, we estimate $R_1^{''}$, $R_2^{''}$, $b_{0,T}^{''}$ separately

Step 2. Evaluation of $\mathbf{R}_1^{''}$. It is found that the dual system of (6.16) is

$$\left\{\begin{array}{c} z_t(x,t) - \mathrm{i}A_1 z(x,t) = 0, \\ z(0) = z_0, \\ y = B^* A^{3/2} z, \end{array}\right. \qquad (6.46)$$

Where A_1, B^* are given by (4.4), (6.15) respectively, It is well known that system (6.46) associates with a C_0-group solution in the space V that is to say, for any $z_0 \in V$ to (6.46) depends continusly on z_0. By this facts and letting $\gamma u = 0$ in (6.45), we obtain

$$\int_\Sigma \left|\frac{\partial(\mathbf{A}z)}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma \le C_T\|z_0\|_V^2, \qquad (6.47)$$

which is equivalent to saying that for any initial $w_0 \in V^{'}$, the solution to system (6.18) with u satisfies

$$\int_\Sigma \left|\frac{\partial(\mathbf{A}(A^{-3/2}w))}{\partial\nu_{\mathbf{A}}}\right|^2 d\Sigma \le C_T\|z_0\|_{V'}^2. \qquad (6.48)$$

Hence the operator B^* is admissible, and so is B. Therfore,

$$g \longrightarrow w \text{ is continuous from } L^2(0,T;L^2(\Gamma_0)) \text{ to } C(0,T;V'). \tag{6.49}$$

By virtue of 6.49, we have

$$z \in C(0,T;V) \text{ depends continously on } g \in L^2(0,T;L^2(\Gamma_0)). \tag{6.50}$$

Therefore

$$R_1'' \leq \|u\|^2_{L^2(0,T;L^2(\Gamma_0))} \qquad \forall u \in L^2(0,T;L^2(\Gamma_0)). \tag{6.51}$$

Step 3. Evaluation of $\mathbf{R}_2''$ and $\mathbf{b}_{0,T}''$. This can be easy done from the representation of R_2'' and $b_{0,T}''$ in (6.22),and by virtue of (6.50). we can readly obtain

$$R_2'' + b_{0,T}'' \leq \|u\|^2_{L^2(0,T;L^2(\Gamma_0))} \qquad \forall u \in L^2(0,T;L^2(\Gamma_0)). \tag{6.52}$$

From (6.44), (6.51), and (6.52), it is seen that (6.20) holds true. ■

6.4 Proof of Theorem 6.2

From the appendix of [53], the transfer function of (6.16) is

$$H(\lambda) = \lambda B^* \left(\lambda^2 + \tilde{A}\right)^{-1} B \tag{6.53}$$

Where $\tilde{A}$, B and B^* are difined by (6.9), (6.10), (6.15) respectively. Moreover, the well-posedness claimed by Theorem 6.1 implies that there exist a positive constants M, $\alpha > 0$, such that

$$\underset{\mathrm{Re}\,\lambda \geq \alpha}{Sup} \|H(\lambda)\|_{\mathbf{L}(U)} = M < +\infty \tag{6.54}$$

Proposition 6.1 The Theorem 6.2 is valid, if for any $u \in C_0^\infty(\Gamma_0)$ and $\varepsilon > 0$, the solution $v_\varepsilon$to the following sysytem

$$\begin{cases} \mathrm{i}v_\varepsilon(x) - \varepsilon \mathbf{A}^2 v(x) = 0 & x \in \Omega, \\ v_\varepsilon(x) = u(x) & x \in \Gamma, \\ \frac{\partial(v_\varepsilon(x))}{\partial \nu_{\mathbf{A}}} = 0 & x \in \Gamma, \end{cases} \tag{6.55}$$

Satisfies

$$\underset{\varepsilon \to 0^+}{Lim} \int_\Gamma \left| \varepsilon \frac{\partial v_\varepsilon}{\partial \nu_{\mathbf{A}}} \right|^2 d\Gamma = 0 \tag{6.56}$$

Proof. We need only to show that $H(\lambda)u$ converges to zero in the strong topology of U along the positive real axis ([53]), that is,

$$\underset{\lambda \to +\infty}{Lim} H(\lambda)u = 0, \tag{6.57}$$

for any $u \in L^2(\Gamma)$. By density argument and (6.54), it suffices to show that (6.57) holds for all $u \in C_0^\infty(\Gamma_0)$. To this purpose, let

$$v_\lambda(x) = \left(\left(\lambda^2 - \mathrm{i}\tilde{A}\right)^{-1} Bu \right)(x)\,. \tag{6.58}$$

Then v_λ satisfies

$$\begin{cases} \lambda v_\lambda(x) - \mathrm{i}\mathbf{A}^2 v_\lambda(x) = 0, & x \in \Omega, \\ v_\lambda(x) = u(x) & x \in \Gamma, \\ \frac{\partial v_\lambda(x)}{\partial \nu_{\mathbf{A}}} = 0 & x \in \Gamma, \end{cases} \tag{6.59}$$

and

$$(H(\lambda)u)(x) = \lambda \frac{\partial \left(\mathbf{A}((A_1)^{-3/2} v_\lambda(x)\right)}{\partial \nu_{\mathbf{A}}}, \qquad \forall x \in \Gamma \tag{6.60}$$

Because $u \in C_0^\infty(\Gamma)$, there exists a unique solution to (6.59) Let $\tilde{v} \in H^4(\Omega)$ be the unique solution to the following boundary value problem

$$\begin{cases} \mathbf{A}^2 \tilde{v}(x) = 0, & x \in \Omega, \\ \tilde{v}(x) = u(x) & x \in \Gamma, \\ \frac{\partial \tilde{v}(x)}{\partial \nu_{\mathbf{A}}} = 0 & x \in \Gamma, \end{cases} \tag{6.61}$$

Then (6.59) becomes

$$\begin{cases} \lambda v_\lambda(x) - \mathrm{i}\mathbf{A}^2 \left(v_\lambda(x) - \tilde{v}(x)\right) = 0, & x \in \Omega, \\ v_\lambda(x) - \tilde{v}(x) = 0 & x \in \Gamma, \\ \frac{\partial (v_\lambda(x) - \tilde{v}(x))}{\partial \nu_{\mathbf{A}}} = 0 & x \in \Gamma, \end{cases} \tag{6.62}$$

$$\lambda \mathrm{i}\mathbf{A}((A)^{-3/2} v_\lambda(x) = v_\lambda(x) - \tilde{v}(x) \tag{6.63}$$

Therefore, (6.60) is found to be

$$(H(\lambda)u)(x) = \frac{1}{\lambda} \frac{\partial v_\lambda(x)}{\partial \nu_{\mathbf{A}}} - \frac{1}{\lambda} \frac{\partial \tilde{v}(x)}{\partial \nu_{\mathbf{A}}} \tag{6.64}$$

If we set $v_\varepsilon(x) = v_\lambda(x)$ with $\varepsilon = \frac{1}{\lambda}$, we obtained the required result immediately.

From the boundary condition of (6.59), it is easy to know that (6.56) holds. This completes the proof of Theorem 6.2. ■

6.5 Proof of Theorem 6.3

In this section we establish the exact controllability of system (6.1), by means of the Hilbert Uniqueness Method (the proof is similar to that in theorem 4.3 in[67]). Since by Theotem 6.1, the operator B is admissible in system (6.16), the exact controllability of system (6.1) is equivalent of the exact observability of the following dual problem of (6.1):

$$\begin{cases} i\varphi_t(x,t) + \mathbf{A}^2 \varphi(x,t) = 0, & \text{in } \Omega \times (0,T) = Q, \\ \varphi(x,t) = \frac{\partial \varphi}{\partial \nu_{\mathbf{A}}}(x,t) = 0 & \text{on } \partial\Omega \times (0,T) = \Sigma, \\ \varphi(x,0) = \varphi_0(x) & \text{in } \Omega. \end{cases} \tag{6.65}$$

That is, the "observability inequality" holds for system (6.65) in the sence of (see (6.18) and (6.20))

$$\int_{\Sigma_0} |\frac{\partial \left(\mathbf{A}\varphi\left(x,t\right)\right)}{\partial \nu_{\mathbf{A}}}|^2 d\Sigma \geq C_T \left\|\varphi^0\right\|_V^2 ., \qquad \forall \varphi^0 \in V,\ T > T_0 \tag{6.66}$$

for some $T_0 > 0$.

In order to prove (6.66), we let A defined by (6.65) and let φ be a solution to (6.66). Then iA generates a strongly continuous unitary group on the space $V = D(A^{\frac{3}{4}})$ and hence

$$\begin{aligned} \|\varphi(t)\|_V &= \left\|(A^{\frac{3}{4}}\varphi(t)\right\|_V = \left\|\mathrm{e}^{iAt}\varphi^0\right\|_V \\ &= \left\|\varphi^0\right\|_V = \left\|A^{\frac{3}{4}}\varphi^0\right\|_{L^2(\Omega)}. \end{aligned} \tag{6.67}$$

Next, we claim that for $f \in D(A^{\frac{3}{4}})$, the norms

$$\|f\|_{D(\mathrm{A}^{\frac{3}{4}})} = \left\|A^{\frac{3}{4}}f\right\|_{L^2(\Omega)} \text{ and } \left\{\int_\Omega |\nabla_g(\mathbf{A}f)|_g^2\,dx\right\}^{\frac{1}{2}} \text{ are equivalent.} \tag{6.68}$$

Actually, $\left\{\int_\Omega |\nabla_g(\mathbf{A}f)|_g^2\,dx\right\}^{\frac{1}{2}}$ being a norm is a trivial fact, since the norms $\|f\|_{D(\mathrm{A}^{\frac{1}{4}})} = \left\|\mathrm{A}^{\frac{1}{4}}f\right\|_{L^2(\Omega)}$.If $|\nabla_g(f)|_g^2 \in L^2(\Omega)$ with $f|_\Gamma = \frac{\partial f}{\partial \nu_{\mathbf{A}}} = 0$, then it follows that $\frac{\partial(\mathbf{A}f)}{\partial x_j} = \mathbf{A}f_{x_j} = l_j \in L^2(\Omega)$ and $f_{x_j}\big|_\Gamma = 0$ and hence $f_{x_j} \in H^2(\Omega)$ by the elliptic regularity theory. THis together with $f|_\Gamma = 0$ yields $f \in L^2(\Omega)$, by the Poincaré inequality. Thus $f \in H^3(\Omega)$ and so (6.68) follows (see [67]).

Lemma 6.1. Suppose that (H'1) and (H'2), the following inequality holds true:

$$\begin{aligned} &\left(\frac{\mu_1 + C_N}{2\varepsilon}\right)\int_\Sigma \left(\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right)^2 d\Sigma + \frac{2\mu_2}{\varepsilon}\|\nabla_g\varphi(T)\|^2_{C(0,T;L^2(\Omega))} \\ \geq\ &\left[\left[b_0 - \left(\frac{\mu_1}{2}C + C'\right)\varepsilon\right]T - 2(\mu_2 + 3\mu_3)\varepsilon\right]\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} \end{aligned} \tag{6.69}$$

where $\varepsilon > 0$ is sufficiently small and $\mu_1 = \max\limits_{x\in\partial\Omega}|N(x)|, \mu_2 = \frac{1}{4}\max\limits_{x\in\Omega}|N(x)|,$

Proof. Multiply the both sides of the first equation of (6.65) by $N(\mathbf{A}\bar{\varphi})$ and integrate on Q to obtain

$$\int_Q \varphi_t N(\mathbf{A}\overline{\varphi})dQ - \mathrm{i}\int_Q \mathbf{A}^2\varphi N(\mathbf{A}\overline{\varphi})dQ = 0. \tag{6.70}$$

By making use of the computation procedure from (6.24)-(6.31) by setting $h(x)$ to $N(x)$, we get respectively

$$\begin{aligned} &\mathrm{i}\int_Q \mathbf{A}^2\varphi N(\mathbf{A}\bar{\varphi})dQ \\ =\ &\mathrm{Re}\int_\Sigma \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}N(\mathbf{A}\overline{\varphi})d\Sigma - \mathrm{Re} - \int_Q \langle\nabla_g(\mathbf{A}\varphi), \nabla_g(\mathbf{A}\bar{\varphi})\rangle_g dQ \\ =\ &\mathrm{Re}\int_\Sigma \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}N(\mathbf{A}\bar{\varphi})d\Sigma - \mathrm{Re}\int_Q DN(\nabla_g(\mathbf{A}\varphi), \nabla_g(\mathbf{A}\bar{\varphi}))\,dQ \\ &-\frac{1}{2}\int_\Sigma |\nabla_g(\mathbf{A}\varphi)|_g^2 N.\nu d\Sigma + \frac{1}{2}\int_Q |\nabla_g(\mathbf{A}\varphi)|_g^2\, div_g N dQ \end{aligned} \tag{6.71}$$

and

$$\begin{aligned}
& \operatorname{Im}\left(\int_Q \varphi_t N(\mathbf{A}\bar{\varphi})\, dQ\right) && (6.72)\\
= \; & \frac{\mathrm{i}}{2}\int_\Omega (\mathbf{A}\bar{\varphi}N(\varphi))\, d\Omega\Big|_0^T + \frac{1}{2}\int_Q |\nabla_g(\mathbf{A}\varphi)|_g^2 div_g(N)\, dQ \\
& + \int_Q \mathbf{A}\bar{\varphi}\langle\nabla_g(\mathbf{A}\varphi), \nabla_g(div_g(N))\rangle_g dQ \\
& + \frac{1}{2}\int_\Sigma \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\left[D^2 p(N, D\varphi) - (\Delta N)(\varphi) - \langle DN, D^2\varphi\rangle_{(T_x\mathbb{R}^n)^2} + Ricc(N, D\varphi)\right] d\Sigma \\
& + \frac{1}{2}\int_\Sigma \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\left[D^2\varphi(N, Dp) + N(\varphi)\, div_g(Dp)\right] d\Sigma \\
& + \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g(\Delta N)(\varphi)\rangle_g dQ + \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g\left(\langle DN, D^2\varphi\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ \\
& - \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g\left(D^2 p(N, D\varphi)\right)\rangle_g dQ - \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g(Ricc(N, D\varphi))\rangle_g dQ \\
& - \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g\left(D^2 z(N, Dp)\right)\rangle_g dQ - \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g(Dp(N(\varphi)))\rangle_g dQ
\end{aligned}$$

By (6.70), (6.71) and (6.72), we get

$$\begin{aligned}
\sum_{i=1}^3 L_i \; = \; & \operatorname{Re}\int_\Sigma \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}} N(\mathbf{A}\bar{\varphi}) d\Sigma - \frac{1}{2}\int_\Sigma |\nabla_g(\mathbf{A}\varphi)|_g^2\, N.\nu d\Sigma && (6.73)\\
& + \left[\frac{1}{2}\int_\Sigma \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\left[(\Delta N)(\varphi) - D^2 p(N, D\varphi) + D^2\varphi(N, Dp) - Ricc(N, D\varphi)\right] d\Sigma\right. \\
& \left. + \frac{1}{2}\int_\Sigma \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\left[\langle DN, D^2\varphi\rangle_{(T_x\mathbb{R}^n)^2} - N(\varphi)\, div_g(Dp)\right] d\Sigma\right] \\
= \; & \operatorname{Re}\int_Q DN(\nabla_g(\mathbf{A}\varphi), \nabla_g(\mathbf{A}\bar{\varphi}))\, dQ - \frac{1}{2}\operatorname{Im}\int_\Omega (\mathbf{A}\bar{\varphi}N(\varphi))\, d\Omega\Big|_0^T \\
& + \left[\frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g(\Delta N)(\varphi)\rangle_g dQ + \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g\left(\langle DN, D^2\bar{\varphi}\rangle_{(T_x\mathbb{R}^n)^2}\right)\rangle_g dQ\right. \\
& - \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g\left(D^2 p(N, D\bar{\varphi})\right)\rangle_g dQ - \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g(Ricc(N, D\varphi))\rangle_g dQ \\
& \left. - \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g\left(D^2\varphi(N, Dp)\right)\rangle_g dQ - \frac{1}{2}\int_Q \langle\nabla_g(\mathbf{A}\bar{\varphi}), \nabla_g(Dp(N(\varphi)))\rangle_g dQ\right] \\
= \; & \sum_{i=1}^3 M_i
\end{aligned}$$

We first compute the three terms in the LHS of (6.73).

For any $\varepsilon > 0$,

$$\begin{aligned} L_1 &= \operatorname{Re} \int_{\Sigma} \frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}} N(\mathbf{A}\bar{\varphi}) d\Sigma \qquad (6.74) \\ &\leq \frac{\mu_1}{2} \int_{\Sigma} \left[\frac{1}{\varepsilon} \left(\frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}} \right)^2 + \varepsilon \left| \mathbf{A}\bar{\varphi} \right|^2 \right] d\Sigma \\ &\leq \frac{\mu_1}{2\varepsilon} \int_{\Sigma} \left(\frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}} \right)^2 d\Sigma + \frac{\mu_1}{2} \varepsilon \int_{\Sigma} \left| \mathbf{A}\bar{\varphi} \right|^2 d\Sigma \\ &\leq \frac{\mu_1}{2\varepsilon} \int_{\Sigma} \left(\frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}} \right)^2 d\Sigma + \frac{\mu_1}{2} C\varepsilon \left\| \bar{\varphi} \right\|^2_{L^2(0,T;D(A^{\frac{3}{4}}))}, \end{aligned}$$

$$\begin{aligned} L_3 = & \left[\frac{1}{2} \int_{\Sigma} \frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}} \left[(\Delta N)(\varphi) - D^2 p(N, D\varphi) + D^2 \varphi(N, Dp) \right] d\Sigma \right. \qquad (6.75) \\ & \left. + \frac{1}{2} \int_{\Sigma} \frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}} \left[\langle DN, D^2 \varphi \rangle_{(T_x \mathbb{R}^n)^2} - N(\varphi)\, div_g(Dp) \right] d\Sigma - Ricc(N, D\varphi) \right] \\ \leq & \frac{C_N}{2\varepsilon} \int_{\Sigma} \left(\frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}} \right)^2 d\Sigma + C' \varepsilon \left\| \varphi \right\|^2_{L^2(0,T;D(A^{\frac{3}{4}}))} \end{aligned}$$

where in L_1 and L_3 we used the trace theorem and the Poincaré inequality:

$$\begin{aligned} \left\| \mathbf{A}\bar{\varphi} \right\|_{L^2(\Gamma)} &\leq C \left\| \mathbf{A}\bar{\varphi} \right\|_{L^2(\Gamma)} \leq C \left\| \bar{\varphi} \right\|_{H^3(\Omega)} \leq \left\| \bar{\varphi} \right\|_{D(A^{\frac{3}{4}})}. \\ \left\| D\bar{\varphi} \right\|_{L^2(\Gamma)} &\leq C \left\| D\bar{\varphi} \right\|_{L^2(\Gamma)} \leq C \left\| \bar{\varphi} \right\|_{H^3(\Omega)} \leq \left\| \bar{\varphi} \right\|_{D(A^{\frac{3}{4}})} \end{aligned}$$

and by (H2)

$$L_2 = -\frac{1}{2} \int_{\Sigma} \left| \nabla_g (\mathbf{A}\varphi) \right|_g^2 N.\nu d\Sigma \leq 0 \qquad (6.76)$$

Adding (6.74), (6.75) and (6.76), we get

$$\text{LHS of (6.73)} \leq \left(\frac{\mu_1 + C_N}{2\varepsilon} \right) \int_{\Sigma} \left(\frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}} \right)^2 d\Sigma + \left(\frac{\mu_1}{2} C + C' \right) \varepsilon \left\| \varphi \right\|^2_{L^2(0,T;D(A^{\frac{3}{4}}))}. \qquad (6.77)$$

Making use of (6.67), we obtain

$$\text{LHS of (6.73)} \leq \left(\frac{\mu_1 + C_N}{2\varepsilon} \right) \int_{\Sigma} \left(\frac{\partial(\mathbf{A}\varphi)}{\partial \nu_{\mathbf{A}}} \right)^2 d\Sigma + \left(\frac{\mu_1}{2} C + C' \right) \varepsilon T \left\| \varphi^0 \right\|^2_{D(A^{\frac{3}{4}})}.$$

Next, we estimate the three terms in the right hand side (RHS). Using (H'1) we get at first

$$\begin{aligned} M_1 &= \operatorname{Re} \int_Q DN\left(\nabla_g (\mathbf{A}\varphi), \nabla_g (\mathbf{A}\bar{\varphi}) \right) dQ \geq \int_Q b(x) \left| \nabla_g (\mathbf{A}\varphi) \right|_g^2 dQ \qquad (6.78) \\ &\geq b_0 \int_Q \left| \nabla_g (\mathbf{A}\varphi) \right|_g^2 dQ = b_0 \left\| \varphi \right\|^2_{L^2(0,T;D(A^{\frac{3}{4}}))} \\ &\geq b_0 T \left\| \varphi^0 \right\|^2_{D(A^{\frac{3}{4}})}. \end{aligned}$$

$$
\begin{aligned}
|M_2| &= \left|-\frac{1}{2}\operatorname{Im}\int_\Omega (\mathbf{A}\bar{\varphi}N(\varphi))\,d\Omega\Big|_0^T\right| \qquad (6.79)\\
&\leq \mu_2\left(\|\mathbf{A}\overline{\varphi}(T)\|_{L^2(\Omega)}\|\nabla_g\varphi(T)\|_{L^2(\Omega)}+\left\|\mathbf{A}\overline{\varphi^0}\right\|_{L^2(\Omega)}\left\|\nabla_g\varphi^0\right\|_{L^2(\Omega)}\right)\\
&\leq \mu_2\left(\varepsilon\|\mathbf{A}\overline{\varphi}(T)\|^2_{L^2(\Omega)}+\frac{1}{\varepsilon}\|\nabla_g\varphi(T)\|^2_{L^2(\Omega)}+\varepsilon\left\|\mathbf{A}\overline{\varphi^0}\right\|^2_{L^2(\Omega)}+\frac{1}{\varepsilon}\left\|\nabla_g\varphi^0\right\|^2_{L^2(\Omega)}\right)\\
&\leq \frac{2\mu_2}{\varepsilon}\|\nabla_g\varphi(T)\|^2_{C(0,T;L^2(\Omega))}+2\mu_2\varepsilon\left\|\varphi^0\right\|^2_{D(A^{\frac{3}{4}})}.
\end{aligned}
$$

and

$$
|M_3|\leq\frac{6\mu_3}{\varepsilon}\|\nabla_g\varphi(T)\|^2_{C(0,T;L^2(\Omega))}+6\mu_3\varepsilon\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})} \qquad (6.80)
$$

By (6.69), (6.78), (6.79) and (6.80)

$$
\text{RHS of (6.73)}\geq b_0T\left\|\varphi^0\right\|^2_{D(A^{\frac{3}{4}})}-2(\mu_2+3\mu_3)\varepsilon\left\|\varphi^0\right\|^2_{D(\mathrm{A}^{\frac{3}{4}})}-\frac{2\mu_2}{\varepsilon}\|\nabla_g\varphi(T)\|^2_{C(0,T;L^2(\Omega))} \qquad (6.81)
$$

Inequality (6.69) then follows from (6.73), (6.77) and (6.81). ■

Remark 6.1. We reestimate the second term of the RHS of (6.73) to obtain

$$
\begin{aligned}
M_2 &= -\frac{1}{2}\operatorname{Im}\int_\Omega(\mathbf{A}\bar{\varphi}N(\varphi))\,d\Omega\Big|_0^T \qquad (6.82)\\
&\leq 2\mu_2\left(\|\mathbf{A}\overline{\varphi}(T)\|_{L^2(\Omega)}\|\nabla_g\varphi(T)\|_{L^2(\Omega)}+\left\|\mathbf{A}\overline{\varphi^0}\right\|_{L^2(\Omega)}\left\|\nabla_g\varphi^0\right\|_{L^2(\Omega)}\right)\\
&\leq \mu_2\left(\|\mathbf{A}\overline{\varphi}(T)\|^2_{L^2(\Omega)}+\|\nabla_g\varphi(T)\|^2_{L^2(\Omega)}+\left\|\mathbf{A}\overline{\varphi^0}\right\|^2_{L^2(\Omega)}+\left\|\nabla_g\varphi^0\right\|^2_{L^2(\Omega)}\right)\\
&\leq 2\mu_2\left(\|\varphi(T)\|^2_{D(A^{\frac{3}{4}})}+\left\|\varphi^0\right\|^2_{D(A^{\frac{3}{4}})}\right)\\
&\leq 4\mu_2\left\|\varphi^0\right\|^2_{D(A^{\frac{3}{4}})}.
\end{aligned}
$$

This improve (6.81) as

$$
\text{RHS of (6.73)}\geq b_0T\left\|\varphi^0\right\|^2_{D(A^{\frac{3}{4}})}-(4\mu_2+6\mu_3)\left\|\varphi^0\right\|^2_{D(A^{\frac{3}{4}})}. \qquad (6.83)
$$

It then follows from (6.73), (6.77) and (6.83) that

$$
\int_\Sigma\left(\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right)^2 d\Sigma\geq\frac{\left[b_0-\left(\frac{\mu_1}{2}C+C'\right)\varepsilon\right]2\varepsilon}{\mu_1+C_N}\left(T-\frac{(4\mu_2+6\mu_3)}{b_0-\left(\frac{\mu_1}{2}C+C'\right)\varepsilon}\right)\left\|\varphi^0\right\|^2_{D(A^{\frac{3}{4}})} \qquad (6.84)
$$

Choosing $\varepsilon>0$ small enough so that $C''=\frac{\left[b_0-\left(\frac{\mu_1}{2}C+C'\right)\varepsilon\right]2\varepsilon}{\mu_1+C_N}>0$, (6.66) then follows for $C_T=C''(T-T_0)$, $T_0=\frac{(4\mu_2+6\mu_3)}{b_0-\left(\frac{\mu_1}{2}C+C'\right)\varepsilon}$ and $T>T_0$.

In what follows, we prove that the requirement T_0 in (6.66) can be taken as $T_0=0$.

Lemma 6.2. Suppose that (H1) and (H2) holds. Then for any $T>0$ and nonegative integer k, there exists a $C_T>0$ such that

$$
\|\nabla_g\varphi(T)\|^2_{C(0,T;L^2(\Omega))}\leq C_T\left\{\int_\Sigma\left(\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right)^2 d\Sigma+\|\mathbf{A}\varphi\|^2_{H^{-k}(\Sigma)}\right\}. \qquad (6.85)
$$

Proof. We suppose that (6.85) is not true and obtain a contradiction. Let $\{\varphi_n\}$ be the solutions of the following system over $[0, T]$:

$$\begin{cases} i\varphi_n' + \mathbf{A}^2\varphi_n(x,t) = 0, \quad \text{in } \Omega \times (0,T) = Q, \\ \varphi_n = \frac{\partial \varphi}{\partial \nu_{\mathbf{A}}} = 0 \quad \text{on } \partial\Omega \times (0,T) = \Sigma, \\ \varphi_n(x,0) = \varphi_n^0(x) \quad \text{in } \Omega. \end{cases} \tag{6.86}$$

such that

$$\|\nabla_g \varphi_n (T)\|^2_{C(0,T;L^2(\Omega))} = 1 \tag{6.87}$$

and

$$\int_{\Sigma_0} \left| \frac{\partial (\mathbf{A}\varphi_n)}{\partial \nu_{\mathbf{A}}} \right|^2 d\Sigma + \|\mathbf{A}\varphi_n\|^2_{H^{-k}(\Omega)} \to 0 \text{ as } n \to \infty. \tag{6.88}$$

By (6.69), we have

$$\left\|\varphi_n^0\right\|_V^2 \le C \text{ uniformly for } n \tag{6.89}$$

for some constant $C > 0$. Hence, there exists a subsequence of $\{\varphi_n^0\}$, still denote by itself without confusion, and a function $\varphi^0 \in V$ such that

$$\varphi_n^0 \to \varphi^0 \text{ weakly in } V. \tag{6.90}$$

Let $\tilde{\varphi}$ be the solution to (6.86) associated with the initial data φ^0. Then we can claim that there exists a $\tilde{\varphi} \in L^\infty(0,T;V)$ such that

$$\varphi_n \to \tilde{\varphi} \text{ weak}^* \text{ in } L^\infty(0,T;V). \tag{6.91}$$

In fact, since

$$\varphi_n(t) = U(t)\varphi_n^0, \quad \tilde{\varphi}(t) = U(t)\varphi^0, \tag{6.92}$$

where $U(t)$ is the unitary group generated by iA in $D(A^{\frac{3}{4}})$. For any $\psi \in L^\infty\left(0,T;D(A^{\frac{3}{4}})'\right)$, it follows that

$$\begin{aligned} &\int_0^T \left(A^{\frac{3}{4}}(\varphi_n(t) - \tilde{\varphi}(t)), A^{-\frac{3}{4}}\psi(t)\right) dt \qquad (6.93) \\ = &\int_0^T \left(A^{\frac{3}{4}}U(t)\left(\varphi_n^0 - \varphi^0\right), A^{-\frac{3}{4}}\psi(t)\right)_{L^2(\Omega)} dt \\ = &\int_0^T \left(A^{\frac{3}{4}}\left(\varphi_n^0 - \varphi^0\right), A^{-\frac{3}{4}}(U(t))^{-1}\psi(t)\right)_{L^2(\Omega)} dt \to 0 \text{as } n \to \infty. \end{aligned}$$

where in the last step we used the Lebesgue dominated Theorem, (6.90) and the property that $\|U(t)\|$ is uniformly bounded over $t \ge 0$. Equation (6.91) then follows from (6.93). Since (6.91) implies that $\{\varphi_n\}$ is uniformly bounded in $L^\infty(0,T;V)$, this together with the compact imbeding: $D(A^{\frac{3}{4}}) \hookrightarrow D(A^{\frac{1}{4}}) = H_0^1(\Omega)$ implies that there exists a subsequence of $\{\varphi_n\}$, still denoted by itself without confusion, such that

$$\varphi_n \to \tilde{\varphi} \text{ strongly in } L^\infty\left(0,T;H_0^1(\Omega)\right). \tag{6.94}$$

From (6.87) and (6.94), we obtain

$$1 = \|\nabla_g \varphi_n\|^2_{C(0,T;L^2(\Omega))} \to \|\nabla_g \tilde{\varphi}\|^2_{C(0,T;L^2(\Omega))} = 1, \tag{6.95}$$

Moreover, by (6.88), it follows that

$$\frac{\partial\left(\mathbf{A}\tilde{\varphi}\right)}{\partial\nu_{\mathbf{A}}}=\mathbf{A}\tilde{\varphi}=0 \text{ on } \partial\Omega, \tag{6.96}$$

Thus, $\tilde{\varphi}$ satisfies

$$\begin{cases} i\tilde{\varphi}_t+\mathbf{A}^2\tilde{\varphi}=0, \quad \text{in } \Omega\times(0,T)=Q,\\ \tilde{\varphi}=\frac{\partial\tilde{\varphi}}{\partial\nu_{\mathbf{A}}}=0 \qquad \text{on } \partial\Omega\times(0,T)=\Sigma,\\ \qquad\qquad \frac{\partial(\mathbf{A}\tilde{\varphi})}{\partial\nu_{\mathbf{A}}}=\mathbf{A}\tilde{\varphi}=0 \text{ on } \Sigma. \end{cases} \tag{6.97}$$

in $t\in[0,T]$. By the Holmogren classical uniqueness Theorem (see [31, Theorem 5.33, p. 129]), we conclude that $\tilde{\varphi}\equiv 0$ in Q.

This contradicts (6.95. Lemma 6.2 is proved. ■

It is noted by (6.69) and (6.85), there exists a C_T' such that

$$\left\|\varphi^0\right\|_V^2\leq C_T'\left(\int_\Sigma\left(\frac{\partial\left(\mathbf{A}\varphi\right)}{\partial\nu_{\mathbf{A}}}\right)^2 d\Sigma+\|\mathbf{A}\varphi\|_{H^{-k}(\Sigma)}^2\right) \tag{6.98}$$

for any $T>0$.

Lemma 6.3 Suppose that φ satisfies

$$\begin{cases} i\varphi_t+\mathbf{A}^2\varphi=0, \quad \text{in } \Omega\times(0,T)=Q,\\ \varphi=\frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}=0 \qquad \text{on } \partial\Omega\times(0,T)=\Sigma,\\ \qquad\qquad \frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}=0 \text{ on } \Sigma. \end{cases}$$

Then for any $T>0$, $\tilde{\varphi}=0$ in Q.

Proof. Introduce $F=L^\infty\left(0,T;D(A^{\frac{3}{4}})\right)$, which contains the solution φ to system (6.65) with the initial value $\varphi^0\in D(A^{\frac{3}{4}})$. In addition, let E be the space composed of all solutions of system (6.65) in F that satisfy the boundary condition

$$\left.\frac{\partial\left(\mathbf{A}\tilde{\varphi}\right)}{\partial\nu_{\mathbf{A}}}\right|_\Sigma=0. \tag{6.99}$$

Now, we show that E is finite-dimentional. To this end, we only need to show that $B_F\cap E$ is sequentially compact, where B_F is the unit ball of F. Let $\varphi\in B_F\cap E$.

Then

$$\begin{cases} i\left(\varphi_t\right)_t+\mathbf{A}^2\varphi_t=0, \quad \text{in } \Omega\times(0,T)=Q,\\ \varphi_t=\frac{\partial\varphi_t}{\partial\nu_{\mathbf{A}}}=\frac{\partial(\mathbf{A}\varphi_t)}{\partial\nu_{\mathbf{A}}}=0 \qquad \text{on } \partial\Omega\times(0,T)=\Sigma. \end{cases} \tag{6.100}$$

Furthermore, by virtue of the interior regularity and the Sobolev trace Theorem, it follows that $\mathbf{A}\varphi|_\Sigma\in H^{\frac{1}{2}}(\Sigma)$ and $\mathbf{A}\varphi_t|_\Sigma\in H^{-1}\left(0,T;L^2\left(\Gamma\right)\right)$. This together with (6.98) and boundary conditions of (6.100) gives

$$\begin{aligned} \|\varphi_t(0)\|_V^2 &\leq C_T'\left(\int_\Sigma\left(\frac{\partial\left(\mathbf{A}\varphi_t\right)}{\partial\nu_{\mathbf{A}}}\right)^2 d\Sigma+\|\mathbf{A}\varphi_t\|_{H^{-k}(\Sigma)}^2\right) \\ &\leq C_T\|\mathbf{A}\varphi_t\|_{H^{-1}(0,T;L^2(\Gamma))}^2\leq C_T\left\|\varphi^0\right\|_V^2 \qquad \forall T>0. \end{aligned} \tag{6.101}$$

Hence $\varphi_t(0) \in V$ and

$$\varphi_t = U(t)\varphi_t(0) \in C(0,T;V). \tag{6.102}$$

This shows that

$$\varphi_t = i\mathbf{A}^2\varphi \in C(0,T;V). \tag{6.103}$$

Therefore,

$$\varphi \in C(0,T;H^7(\Omega)).$$

We then have

$$B_F \cap E \hookrightarrow C(0,T;H^7(\Omega)) \hookrightarrow C(0,T;V), \tag{6.104}$$

where each inclusion is compact embedding. This shows that E must be finite-dimensional. By the arguments of [4], the elements of E are solutions of an equivalent finite-dimensional ordinary differential equation with constant coefficients. Since such a solution must vanish for all $T > T_0 > 0$ by remark 6.1, for all $T > 0$. ■

Lemma 6.4. Suppose that (H1), (H2), (H3) holds, Then for any $T > 0$, there exists a $C_T > 0$ such that

$$\|\nabla_g\varphi\|^2_{C(0,T;L^2(\Omega))} \le C_T \int_\Sigma \left(\frac{\partial(\mathbf{A}\varphi)}{\partial\nu_{\mathbf{A}}}\right)^2 d\Sigma \tag{6.105}$$

Proof. The proof is similar to that of Lemma 6.2. We suppose that (6.105) is not true and obtain a contradiction. Let $\{\varphi_n\}$ be the solutions of system (6.86) over $[0,T]$, which satisfies such that

$$\|\nabla_g\varphi_n(T)\|^2_{C(0,T;L^2(\Omega))} = 1, \tag{6.106}$$

$$\int_\Sigma \left(\frac{\partial(\mathbf{A}\varphi_n)}{\partial\nu_{\mathbf{A}}}\right)^2 d\Sigma \to 0 \text{ as } n \to \infty. \tag{6.107}$$

By (6.69), we have (6.89) and hence (6.90)-(6.95). By (6.107), it follows that (6.107), it follows that

$$\frac{\partial(\mathbf{A}\tilde{\varphi})}{\partial\nu_{\mathbf{A}}} = 0 \text{ on } \Sigma. \tag{6.108}$$

Thus $\tilde{\varphi}$ satisfies

$$\begin{cases} i\tilde{\varphi}_t + \mathbf{A}^2\tilde{\varphi} = 0, \quad \text{in } \Omega\times(0,T) = Q, \\ \tilde{\varphi} = \frac{\partial\tilde{\varphi}}{\partial\nu} = 0 \qquad \text{on } \Sigma, \\ \frac{\partial(\mathbf{A}\tilde{\varphi})}{\partial\nu_{\mathbf{A}}} = 0 \text{ on } \Sigma. \end{cases} \tag{6.109}$$

for all $t \in [0,T]$. By Lemma 6.3, we conclude that

$$\tilde{\varphi} = 0 \text{ in } Q, \tag{6.110}$$

which contradics (6.95) so that (6.105) follows.■

Proof of Theorem 6.3. Combining (H1), (H2), (H3) and (6.105), we see that (6.66) is true for any $T > 0$. This complites the proof of Theorem 6.3. ■

Chapter 7

Conclusion and open problems

Regular linear systems form a very general class of infinite-dimensional systems whose basic properties are rich enough to develop a parallel of the theory of control for finite-dimensional systems. In the literature, several examples of regular linear systems described by partial differential equations has been given ([19], [20], [21], [1],...) and the aim of this thesis is to provide further regular PDE's systems. Indeed, we have established the well-posedness and regularity of a several input/output systems, namely:

1. Problem of transmission of the Schrödinger equation, in the state space $X = H^{-1}(\Omega)$, input/output space $U = Y = L^2(\Gamma)$.
2. Fourth order Schrödinger equation with variable coefficients, hinged boundary control and colocated observation, in the state space V' which is the dual space of $V = \left\{\varphi \in H^3(\Omega) : \varphi|_\Gamma = \mathbf{A}\varphi| = 0\right\}$, input/output space $U = Y = L^2(\Gamma_0)$.
3. Fourth order Schrödinger equation with variable coefficients, moment boundary control, zero Dirichlet boundary condition and colocated observation, in the state space $H_0^1(\Omega)$, input/output space $U = Y = L^2(\Gamma_0)$.
4. Fourth order Schrödinger equation with variable coefficients, Dirichlet boundary control and colocated observation, in the state space V' which is the dual space of $V = \left\{\varphi \in H^3(\Omega) : \varphi|_\Gamma = \frac{\partial\varphi}{\partial\nu_{\mathbf{A}}}\Big| = 0\right\}$, input/output space $U = Y = L^2(\Gamma_0)$.

The following open problems can be made regarding the material presented in this thesis.

1. The exact controllability problem for the transmission wave equation with Dirichlet boundary control has been considered in [36]. It would be interesting to study the well-posedness and the regularity of this control system with the corresponding colocated observation.

2. In [58], the authors have considered the system

$$\begin{array}{ll} w_{tt}(x,t) = \Delta w(x,t) & \text{on } \Omega \times [0,\infty)\,, \\ w(x,t) = 0 & \text{on } \Gamma_0 \times [0,\infty)\,, \\ \frac{\partial}{\partial \nu} w(x,t) + |b(x)|^2\, w_t(x,t) = \sqrt{2} b(x) u(x,t) & \text{on } \Gamma_1 \times [0,\infty)\,, \\ \frac{\partial}{\partial \nu} w(x,t) - |b(x)|^2\, w_t(x,t) = \sqrt{2} b(x) y(x,t) & \text{on } \Gamma_1 \times [0,\infty)\,, \\ w(x,0) = w_0(x),\ w_t(x,0) = z_0(x) \text{ on } \Omega, & \end{array}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$ the function $b \in L^\infty(\Gamma_1)$, such that $b(x) \neq 0$, for almost every $x \in \Gamma_1$, $w_0(x)$ and $z_0(x)$ are the initial state of the system. They showed that this system is a conservative linear system with input, output space $U = L^2(\Gamma_0)$ and state space $\mathbf{H}_{\frac{1}{2}} \times \mathbf{H}$, with $\mathbf{H}_{\frac{1}{2}} = D(A_0^{\frac{1}{2}}) = H^1_{\Gamma_0}(\Omega)$ and

$$\begin{array}{rcl} A_0 w & = & -\Delta w, \\ D(A_0) & = & \left\{ w \in H^1_{\Gamma_0}(\Omega) \,/\, \Delta w \in \mathbf{H},\ \gamma_1 w = 0 \right\}. \end{array}$$

with γ_1 is the Neumann trace operator and $\mathbf{H} = L^2(\Omega)$. By conservative system, we mean

$$\|w(.,T), w_t(.,T)\|^2_{\mathbf{H}_{\frac{1}{2}} \times \mathbf{H}} + \int_0^T \|y(t)\|^2_{L^2(\Gamma_0)}\, dt = \|w_0, z_0\|^2_{\mathbf{H}_{\frac{1}{2}} \times \mathbf{H}} + \int_0^T \|u(t)\|^2_{L^2(\Gamma_0)}\, dt.$$

A study of regularity of this system is desirable.

1. In [5], the authors established sufficient conditions for the admissibility and the observability of observation operators for semilinear systems of the form

$$\begin{array}{rcl} y_t(t) & = & Ay(t) + F(y(t)), \quad y(0) = x, \quad t \geq 0, \ \ x \in X, \\ z(t) & = & Cy(t) \end{array}$$

where A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ in a Banach space X, F is a nonlinear continuous function on X and z is the output function. Motivated by this paper, one may consider for future investigation, the well-posedness problems for semilinear control system of the form

$$\begin{array}{rcl} y_t(t) & = & Ay(t) + Bu(t) + F(y(t)), \quad y(0) = x, \quad t \geq 0, \ \ x \in X, \\ z(t) & = & Cy(t) \end{array}$$

with $B \in L(U, X_{-1})$ and $C \in L(X_1, Y)$.

2. Although the class of well-posed linear systems includs many input/state/output systems described by PDE or delay differential equations, there are important systems that do not belong to this class, for example the heat equation with Dirichlet control and Neumann observation (see ([34]) for other examples). Therefore a new abstract framework for linear infinite-dimensional systems that covers these examples is needed, some works in this direction, has been done in ([45], [44]).

Appendix A

Riemannian Metric generated by the Principal part A

In this appendix, we recall some well known results from the Riemannian Metric.

Recalling the coefficients $a_{ij}(x) = a_{ji}(x)$ of $\mathbf{A}$, let $A(x)$ and $G(x)$ be respectively, the coefficients matrix and its inverse, and the determinant of $G(x)$ by $\rho(x)$

$$\begin{aligned} A(x) &= a_{ij}(x); \quad G(x) = [A(x)]^{-1} = (g_{ij}(x)); \\ \rho(x) &= \det [g_{ij}(x)], \quad i,j = 1,...,n; \ x \in \mathbb{R}^n. \end{aligned} \tag{A.1}$$

Both $A(x)$ and $G(x)$ are $n \times n$ matrices.$A(x)$ is positive definite for any $x \in \mathbb{R}^n$ by assumption

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\bar{\xi}_j > 0, \qquad \forall x \in \bar{\Omega}, \xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n, \tag{A.2}$$

A.1 Riemannian metric.

Let $\mathbb{R}^n$ be the usual topology. Define the inner product and the norm on the tangent space $T_x\mathbb{R}^n$ by

$$g(X,Y) = \langle X, Y \rangle_g = \sum_{i,j=1}^{n} g_{ij}(x)\alpha_i\beta_j \tag{A.3}$$

$$|X|_g = \langle X, X \rangle_g^{\frac{1}{2}}, \qquad \forall X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in T_x\mathbb{R}^n \tag{A.4}$$

It is easy checked that $(\mathbb{R}^n, g)$ is a Riemannian manifold with the Riemannian metric g.

A.2 Euclidean metric.

For each $x \in \mathbb{R}^n$, denote by

$$X \cdot Y = \sum_{i=1}^{n} \alpha_i\beta_i, \ \ |X|_0 = \langle X, X \rangle^{\frac{1}{2}} \tag{A.5}$$

The Euclidian metric on $\mathbb{R}^n$,for $x \in \mathbb{R}^n$ and with reference to (A.1), set

$$A(x)X(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(x)\alpha_j \right) \frac{\partial}{\partial x_i}, \quad \forall X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i} \in T_x\mathbb{R}^n \tag{A.6}$$

Thus recalling the co-normal derivative defined in (6.20), we have

$$\frac{\partial w(x)}{\partial \nu_{\mathbf{A}}} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(x) \frac{\partial w}{\partial x_i} \right) \nu_i = (A(x)\nabla_0 w) \cdot \nu$$

A.3 Covariant derivative and covriant differential

Denote the Levi Civita connection in the Riemannian metric g by D. Let

$$H = \sum_{k=1}^{n} h_k \frac{\partial}{\partial x_k}; \quad X = \sum_{i=1}^{n} \xi_k \frac{\partial}{\partial x_k} \tag{A.7}$$

be the vector fields on $(\mathbb{R}^n, g)$. The covariantdifferential DH of H determines a bilinear form on $T_x\mathbb{R}^n \times T_x\mathbb{R}^n$, for each $x \in \mathbb{R}^n$, defined by

$$DH(X,Y) = \langle D_X H, Y \rangle_g, \quad \forall X, Y \in T_x\mathbb{R}^n \tag{A.8}$$

Where $D_X H$ is the covariant derivative of H with respect to X, This computed as follows, in the notation of (A.7), (A.8), by using the axioms of a connection,

$$D_X H = \sum_{l=1}^{n} \left(X(h_l) + \sum_{i,k=1}^{n} h_k \alpha_i \Gamma_{ik}^l \right) \frac{\partial}{\partial x_l} \tag{A.9}$$

$$X(h_l) = \sum_{i=1}^{n} \alpha_i \frac{\partial h_l}{\partial x_i} \tag{A.10}$$

Γ_{ik}^l being the connection coefficients (Cristoffel symbols) of the connection D.

$$\Gamma_{ik}^l = \frac{1}{2} \sum_{l=1}^{n} a_{lp} \left(\frac{\partial g_{kp}}{\partial x_i} + \frac{\partial g_{ip}}{\partial x_k} - \frac{\partial g_{ik}}{\partial x_p} \right), \quad g_{ij} = (a_{ij})^{-1} \tag{A.11}$$

Inserting (A.11) into (A.9) and then (A.9) into (A.8) yields

$$DH(X,Y) = \langle D_X H, Y \rangle_g = \sum_{l,j=1}^{n} \left[X(h_l) + \sum_{i,k=1}^{n} h_k \alpha_i \Gamma_{ik}^l \right] \alpha_j g_{lj} \tag{A.12}$$

$$\text{by } (A.10) \quad = \sum_{l,j=1}^{n} \left[\frac{\partial h_l}{\partial x_i} g_{lj} + \sum_{i,k=1}^{n} h_k g_{lj} \Gamma_{ik}^l \right] \alpha_i \alpha_j \tag{A.13}$$

Thus , in $T_x\mathbb{R}^n \times T_x\mathbb{R}^n$, $DH(\cdot,\cdot)$ is equivalent to the $n \times n$ matrix

$$\left(m_{ij} = \frac{\partial h_l}{\partial x_i} g_{lj} + \sum_{i,k=1}^{n} h_k g_{lj} \Gamma_{ik}^l \right) \tag{A.14}$$

A.4 Hessian in the Riemannian metric g.

Let $f \in C^2(\mathbb{R}^n)$. By definition, the Hessian of f with respect to the metric g is

$$\begin{aligned} D^2 f(X,X) &\equiv \langle D_X(\nabla_g f), X\rangle_g & \text{(A.15)}\\ &= \sum_{i,j=1}^{n} \alpha_i \left(\frac{\partial f_l}{\partial x_i} g_{lj} + \sum_{i,k=1}^{n} f_k g_{lj}\Gamma_{ik}^l\right)\alpha_j & \text{(A.16)}\end{aligned}$$

Where $H = \nabla_g f$, $h_l = (\nabla_g f)_l = f_l$ is the l- coordinate of $\nabla_g f$:

$$(\nabla_g f)_l = f_l = \sum_{p=1}^{n} a_{lp}\frac{\partial f}{\partial x_p} \quad \text{(A.17)}$$

Put

$$m_{ij} = \frac{\partial f_l}{\partial x_i} g_{lj} + \sum_{i,k=1}^{n} f_k g_{lj}\Gamma_{ik}^l, \quad i,j = 1,...,n. \quad \text{(A.18)}$$

Then

$$\begin{cases} D^2 f \text{ is positive on } T_x\mathbb{R}^n \times T_x\mathbb{R}^n \text{ if and only if the} \\ n\times n \text{ matrix } \left(m_{ij} = \frac{\partial h_l}{\partial x_i} g_{lj} + \sum_{i,k=1}^{n} h_k g_{lj}\Gamma_{ik}^l\right) \\ i,j = 1,...,n \text{ is positive, with } f_l \text{ given by (A.17)} \end{cases}$$

The following lemma provide further relationships[56, lemma 2.1]

Lemma A.1. Let f, $h \in C^1(\bar{\Omega})$ and let H, X be vector fields. Then with reference to the above notation, we have

(*a*)

$$\langle H(x), A(x)X(x)\rangle_g = H(x)\cdot X(x), \qquad x \in \mathbb{R}^n; \quad \text{(A.19)}$$

(*b*)

$$X(f) = X\cdot\nabla_0 f = \langle X, \nabla_g f\rangle_g \quad \text{(A.20)}$$

Where

$$\begin{aligned} Df(x) &= \nabla_g f(x) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\frac{\partial f}{\partial x_j}\right)\frac{\partial}{\partial x_i}, \ \operatorname{div}_0(H) = \sum_{i=1}^{n}\frac{\partial h_i(x)}{\partial x_j} & \text{(A.21)}\\ \operatorname{div}_g(H) &= \sum_{i=1}^{n}\frac{1}{\sqrt{\rho(x)}}\frac{\partial}{\partial x_i}\left(\sqrt{\rho(x)}h_i(x)\right) & \text{(A.22)}\\ \Delta_g\varphi &= \sum_{i,j=1}^{n}\frac{1}{\sqrt{\rho(x)}}\frac{\partial}{\partial x_i}\left(\sqrt{\rho(x)}a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\right) = \mathbf{A}\varphi - (Dp)\,\varphi, \text{ and} & \text{(A23)}\\ Dp &= \sum_{i,j=1}^{n}\frac{a_{ij}}{\sqrt{\rho(x)}}\frac{\partial\sqrt{\rho(x)}}{\partial x_i}\frac{\partial}{\partial x_j} & \\ p &= \frac{1}{2}\ln\left(\det\left[a_{ij}(x)\right]\right) & \text{(A.24)}\end{aligned}$$

Where div_0 is the divergence operator in Euclidean space $\mathbb{R}^n$, and ∇_g, div_g and Δ_g are the gradiant operator, the divergence operator and the Beltrami- Laplace operator in $(\mathbb{R}^n, g)$ respectively.

(c) The normal derivative $\frac{\partial y}{\partial \nu_{\mathbf{A}}}$ defined by

$$\frac{\partial y}{\partial \nu_{\mathbf{A}}} = (A(x)\nabla_0 y) \cdot \nu = \nabla_g y \cdot \nu \tag{A.25}$$

(d)

$$\langle \nabla_g f, \nabla_g h \rangle_g = \nabla_g f\,(h) = \langle A(x) \cdot \nabla_0 f, \nabla_g h \rangle_g = \nabla_0 f \cdot A(x) \nabla_0 h \tag{A.26}$$

(e)

$$\mathbf{A} y = \operatorname{div}_0 (\nabla_g y), \qquad y \in C^2(\bar{\Omega}) \tag{A.27}$$

(f)

$$\begin{aligned} \langle \nabla_g f, \nabla_g (H(f)) \rangle_g (x) = DH\,(\nabla_g f, \nabla_g f)\,(x) + \tfrac{1}{2} \operatorname{div}_0 \left(|\nabla_g f|_g^2 H \right)(x) \\ - \tfrac{1}{2} |\nabla_g f|_g^2 (x) \operatorname{div}_0 (H)\,(x), \quad x \in \mathbb{R}^n; \end{aligned} \tag{A.28}$$

Let $\mu = \frac{\nu_{\mathbf{A}}}{|\nu_{\mathbf{A}}|_g}$ be the unit outward-pointing normal to $\partial\Omega$ in terms of the Riemmannian metric g. The following Lemma [54, p. 128,138] provides some useful identities.

Lemma A.2. Let $\varphi, \psi \in C^2(\bar{\Omega})$ and let N be a vector field on $(\mathbb{R}^n, g)$. Then we have

(1). **Divergence formulae**

$$\operatorname{div}_0(\varphi N) = \varphi \operatorname{div}_0(N) + N(\varphi), \operatorname{div}_g(\varphi N) = \varphi \operatorname{div}_g(N) + N(\varphi), \tag{A.29}$$

$$\int_\Omega \operatorname{div}_0(N) d\Omega = \int_\Gamma N \cdot \nu d\Gamma, \int_\Omega \operatorname{div}_g(N) d\Omega = \int_\Gamma \langle N, \mu \rangle_g d\Gamma \tag{A.30}$$

(2) **Green's formulae**

$$\langle \mathbf{A}\varphi, \psi \rangle_{L^2(\Omega)} = \int_\Omega \mathbf{A}\varphi \bar{\psi} d\Omega = \int_\Gamma \bar{\psi} \frac{\partial \varphi}{\partial \nu_{\mathbf{A}}} d\Gamma - \int_\Omega \langle \nabla_g \varphi, \nabla_g \psi \rangle_g d\Omega, \tag{A.31}$$

$$\langle \Delta_g \varphi, \psi \rangle_{L^2(\Omega)} = \int_\Omega \Delta_g \varphi \bar{\psi} d\Omega = \int_\Gamma \bar{\psi} \frac{\partial \varphi}{\partial \mu} d\Gamma - \int_\Omega \langle \nabla_g \varphi, \nabla_g \psi \rangle_g d\Omega, \tag{A.32}$$

$$\mathbf{A}(\varphi\psi) = \psi \mathbf{A}\varphi + 2 \langle \nabla_g \varphi, \nabla_g \psi \rangle_g + \varphi \mathbf{A} \boldsymbol{\psi}$$

where we have

$$2 \langle \nabla_g \varphi, \nabla_g \psi \rangle_g = \left(\frac{\partial \varphi}{\partial x_1}, ..., \frac{\partial \varphi}{\partial x_n} \right) .A(x). \left(\frac{\partial \psi}{\partial x_1}, ..., \frac{\partial \psi}{\partial x_n} \right)^\tau$$

Lemma A.3. We denote by $\mathbf{X}(\mathbb{R}^n)$ the set of all vector fields on $\mathbb{R}^n$. Let $\Delta : \mathbf{X}(\mathbb{R}^n) \to \mathbf{X}(\mathbb{R}^n)$ the hodge Laplace operator. Then [56, (2.2.7),(2.2.14)]:

$$\begin{aligned} \Delta_g (N(\varphi)) &= (\Delta N)(\varphi) + 2 \langle DN, D^2 \varphi \rangle_{T_x \mathbb{R}^n} + N(\Delta_g \varphi) + Ricc\,(N, D\varphi) \qquad \text{(A.33)} \\ N(\Delta_g \varphi) &= N(\mathbf{A}\varphi) - D^2 p\,(N, D\varphi) - D^2 \varphi\,(N, Dp), \quad \forall \varphi \in C^2(\mathbb{R}^n), \end{aligned}$$

Where $Ricc$ is the Ricci curvature tensor of the Riemannian metric g,$D^2\varphi$, D^2p are the Hessian of φ and p, respectively, in terms of the Riemannian metric g.

For a fixed $x \in \mathbb{R}^n$. Let E_1, E_2,..., E_n be a frame field normal at x on $(\mathbb{R}^n, g)$, which means that $\langle E_i, E_j\rangle = \delta_{ij}$ in some neighborhood of x and $(D_{E_i}E_j)(x) = 0$ for $i, j = 1, ..., n$.

Set $N = \sum\limits_{i=1}^{n} \gamma_i E_i$, then $N(\varphi) = \sum\limits_{i=1}^{n} \gamma_i E_i(\varphi)$, where $E_i(\varphi)$ is the covariant derivative of φ with respect to E_i under the Riemannian metric g. Then

$$\begin{aligned} \langle Dp, D(N(\varphi))\rangle_g &= E_i(p)\, E_i(N(\varphi)) \\ &= E_i(p)\left[E_i(\gamma_j)\, E_j(\varphi) + \gamma_j E_i E_j(\varphi)\right] \\ &= DN(D\varphi, Dp) + D^2\varphi(N, Dp) \end{aligned} \tag{A.34}$$

From (A.33) and (A.34), we obtain

$$\begin{aligned} \mathbf{A}(N(\varphi)) &= (\Delta_g + Dp)(N(\varphi)) \\ &= \Delta_g(N(\varphi)) + \langle Dp, D(N(\varphi))\rangle_g \\ &= (\Delta N)(\varphi) + 2\langle DN, D^2\varphi\rangle_{T_x\mathbb{R}^n} + N(\mathbf{A}\varphi) - D^2p(N, D\varphi) \\ &\quad + Ricc(N, D\varphi) - DN(D\varphi, Dp), \end{aligned} \tag{A.35}$$

Lemma A.4. [19, Lemma 4.1] Let ψ a smooth function on $\bar{\Omega}$ satisfy $\psi|_\Gamma = 0$.Then there exists a continuous function $q(x)$ on Γ which is independent of ψ such that

$$\begin{aligned} \Delta_g\psi(x) &= \frac{\partial^2\psi}{\partial\mu^2} + q(x)\frac{\partial\psi}{\partial\mu}, \\ \frac{\partial\psi}{\partial\mu} &= \frac{1}{|\nu_{\mathbf{A}}|_g}\frac{\partial\psi}{\partial\nu_{\mathbf{A}}}\ , \quad \forall x \in \Gamma \end{aligned} \tag{A.36}$$

Morever, if ψ satisfies $\left.\frac{\partial\psi}{\partial\nu_{\mathbf{A}}}\right|_\Gamma = 0$, then

$$N(\psi)|_\Gamma = 0 \ on\ \bar{\Omega} \qquad \text{for any vector field } N \tag{A.37}$$

So,

$$\mathbf{A}(\psi) = \Delta_g\psi + (Df)(\psi) = \Delta_g\psi = \frac{\partial^2\psi}{\partial\mu^2} = \frac{1}{|\nu_{\mathbf{A}}|_g^2}\frac{\partial^2\psi}{\partial\nu_{\mathbf{A}}^2} \quad \text{on } \Gamma, \tag{A.38}$$

and

$$\begin{aligned} \frac{\partial N(\psi)}{\partial\nu_{\mathbf{A}}} &= N\left(\frac{\partial\psi}{\partial\nu_{\mathbf{A}}}\right) = \left\langle N, \frac{\nu_{\mathbf{A}}}{|\nu_{\mathbf{A}}|_g}\right\rangle_g \frac{\nu_{\mathbf{A}}}{|\nu_{\mathbf{A}}|_g}\left(\frac{\partial\psi}{\partial\nu_{\mathbf{A}}}\right) \\ &= N\cdot\nu\frac{1}{|\nu_{\mathbf{A}}|_g^2}\frac{\partial^2\psi}{\partial\nu_{\mathbf{A}}^2} = \mathbf{A}\psi N\cdot\nu \quad \text{on } \Sigma \end{aligned} \tag{A.39}$$

Lemma A.5. Let φ a complex function defined on $\bar{\Omega}$ with suitable regularity. Then there exist some constants C, possibly depending on g, N, and Ω, such that:

(1).

$$\begin{aligned}
\sup_{x\in\bar{\Omega}} |N|_g &\leq C, \quad \sup_{x\in\bar{\Omega}} |DN|_g \leq C, \quad \sup_{x\in\bar{\Omega}} |\mathrm{div}_g(N)| \leq C, \\
\sup_{x\in\bar{\Omega}} |Dp|_g &\leq C, \quad \sup_{x\in\bar{\Omega}} |\nabla_g (\mathrm{div}_g(N))| \leq C, \\
\sup_{x\in\bar{\Omega}} |\mathbf{A} (\mathrm{div}_g N)|_g &\leq C, \quad \sup_{x\in\bar{\Omega}} |Df (\mathrm{div}_g(N))| \leq C, \\
\sup_{x\in\bar{\Omega}} |\mathbf{A} (\mathrm{div}_g N)|_0 &\leq C, \quad \sup_{x\in\bar{\Omega}} |Df (\mathrm{div}_0(N))| \leq C, \\
\sup_{x\in\bar{\Omega}} |\mathrm{div}_g (N)|_0 &\leq C, \sup_{x\in\partial\Omega} \left| \frac{1}{|\nu_{\mathbf{A}}|_g} \right| \leq C, \sup_{x\in\partial\Omega} |Df.\nu| \leq C.
\end{aligned} \tag{A.40}$$

(2).

$$\begin{gathered}
|N(\varphi)| \leq C |\nabla_g \varphi|_g , \quad |Dp(\varphi)| \leq C |\nabla_g \varphi|_g , \quad |Dp(\varphi)| \leq C |\nabla_g \varphi|_g \\
\left| \langle \nabla_g \varphi, \nabla_g (\mathrm{div}_g(N)) \rangle_g \right|_g \leq C |\nabla_g \varphi|_g , \ |(\Delta N)(\varphi)|_g \leq C |\Delta N|_g |\nabla_g \varphi|_g \leq C |\nabla_g \varphi|_g \\
\left| \langle DN, D^2\varphi \rangle_{T_x\mathbb{R}^n} \right| \leq C |DN|_g \left|D^2\varphi\right|_g \leq C \left|D^2\varphi\right|_g , \\
\left|D^2 p (N, D\varphi)\right| \leq C \left|D^2 p\right|_g |N|_g |D\varphi|_g \leq C |D\varphi|_g , \\
\left|D^2 \varphi (N, Dp)\right| \leq C \left|D^2 \varphi\right|_g |N|_g |Dp|_g \leq C \left|D^2\varphi\right|_g , \\
|Ric (N, D\varphi)| \leq C |Ric|_g |N|_g |D\varphi|_g \leq C |D\varphi|_g ,
\end{gathered} \tag{A.41}$$

(3).

$$\int_\Omega |\varphi|^2 \, d\Omega \leq C \|\varphi\|^2_{H^2(\Omega)} , \quad \int_\Omega |D\varphi|^2_g \, d\Omega \leq C \|\varphi\|^2_{H^2(\Omega)} , \quad \int_\Omega \left|D^2\varphi\right|^2_g d\Omega \leq C \|\varphi\|^2_{H^2(\Omega)} \tag{A.42}$$

Appendix B

Linear semigroup theory

In this appendix we recall some basic properties of the theory of semigroups

B.1 Strongly continuous semigroups

Definition B.1. A strongly continuous semigroup is an operator- valued function $S(t)$ that satisfies the following properties:

1. $S(t+s) = S(t)S(s)$ for any $s,\, t \geq 0$
2. $S(0) = I_X$
3. $\|S(t)y - y\| \to 0$ when $t \to 0^+$, for any $y \in X$.

Some elementary properties of semigroups are given in the following theorem

Theorem B.1. *[12] A strongly continuous semi group on a Hilbert space H. $S(t)$ has the following properties*

1. $\|S(.)\|$ *is bounded on every finite subinterval of* $[0, \infty)$;
2. $S(.)$ *is strongly continuous for all* $t \in [0, \infty)$;
3. *If* $\omega_0 = \inf_{t>0} \frac{1}{t} (\log \|S(t)\|)$, *then* $\lim_{t \to +\infty} \frac{1}{t} (\log \|S(t)\|) < +\infty$;
4. *For all* $y \in H$ *we have that* $\frac{1}{t} \int_0^t S(s) y ds \to y$ *for* $t \to 0^+$
5. *For any* $\omega > \omega_0$, *there exists a constant* $M_\omega \geq 1$ *such that* $\|S(t)\| \leq Me^{\omega t}$ *for all* $t \geq 0$;

This constant ω_0 is called the grouwth bound of the semigroup.

Let $D(A)$ denote the subspace of all elements such that $\frac{(S(t)y - y)}{t}$ converges in H for $t \to 0^+$.Define the operator on $D(A)$:

$$Ay = \lim_{t \to 0} \frac{(S(t)y - y)}{t} \tag{A43}$$

Definition B.2. The operator given by (A43) is the infinitesimal generator of the semi group $S(t)$.

B.2 The Hille-Yosida Theorem

In the sequel we shall denote by $G(M,w)$ the set of all strongly continuous semigrous S such that $\|S(t)\| \leq Me^{\omega t}$, $t \geq 0$.

Lemma B.4. We have

$$\{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda > w\} \subset \rho(A)$$
$$R(\lambda, A)y = \int_0^\infty e^{-\lambda t} S(t)ydt,\ y \in X, \operatorname{Re}\lambda > w.$$

Definition B.3. (**Groups of bounded operators**) A parameter family $T(t), -\infty < t < \infty$, of bounded linear operators on banach space X is a C_0 group of bounded operators if it satisfies

1. $S(0) = I_X$
2. $S(t+s) = S(t)S(s)$ for $-\infty < t, s < \infty$,
3. $\lim\limits_{t\to 0} S(t)x = x$ for $x \in X$.

Definition B.4. (*Self adjoint operators*). Let H be a Hilbert space with the scalar product $(.,.)$. An operator A in H is symmetric if $\overline{D(A)} = H$ and $A \subset A^*$, that is, $(Ax, y) = (x, Ay)$ for all $x, y \in D(A)$. A self adjoint if $A = A^*$.

Theorem B.2.*(Stone Theorem)* *([46]). A is the infinitesimal generator of a C_0 group of unitary on a Hilbert space H if and only if iA is self adjoint $(iA) = (iA)^*$.*

The result below which is known as the Hille- Yosida Theorem provide a complete characterization of infinitasimal generators [12]

Theorem B.3. ***(Hille- Yoshida Theorem)****([46]) Let $A : D(A) \subset X \to X$ be a closed operators. Then A is the infinitesimal generator of a strongly semigroup belonging to $G(M,w)$ if and only if*

(i)- $\{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda > w\} \subset \rho(A)$

(ii)- $\|R^r(\lambda, A)\| \leq \frac{M}{(\lambda-\omega)^r}$, *for all* $r \geq 1$, $\forall \lambda > w$

(iii)- $D(A)$ is dence in X.

Remark B.1. To use the Hille-Yosida Theorem requires to check infinite conditions. However if $M = 1$ it is enough to ask *(ii)* only for $r = 1$, in such a case $S \in G(M,w)$. If $w \leq 0$ we say that $S(.)$ is a *contraction semigroup.*

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