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Existence et Régularité des Solutions de Problèmes Non Linéaires d'Evolution

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THÈSE

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CHAPTER 1

General Introduction

This thesis is concerned in one hand on two problems related to the study of the blow-up in finite time of solutions of fractional differential equations, and Hamilton Jacobi equations with strong nonlinearities, and on the other hand with the decay of solutions to a heat hyperbolic system with viscoelastic term, and an abstract hyperbolic system in the presence of infinite memory.

The aim of the work in this thesis is to study the existence of solutions to some problems of evolution equations, and the related questions on the blow-up in finite time or the global existence, and in eventuality, the asymptotic behaviour of the solutions as the time t goes to infinity, namely the stability and the decay of the energy.

Throughout this general introduction, we will mainly focus on some history and on the general known results in each chapter by starting at first by motivations, and historic of each problem, and then we will give the main results of each problems that we are dealing with.

The problems we treated here are divided to four main parts, and so we will briefly mention that as follows:

The first part addresses some history and motivations to the study of fractional differential equations. In recent years, the field of fractional calculus and fractional integro-differential equations has grown considerably. Problems related to fractional derivative are interesting not only from the point of view of partial differential equations (PDE) general

theory, but also due to its applications into some phenomena from physics, finance, economics...etc. In fact it has been shown by experiments that derivatives of non-integer order can describe many phenomena better than derivatives of integer order specially hereditary phenomena and processes(see [**16, 18, 75, 167**]), and the references therein. As example fluid dynamics are well explained using fractional calculus.
Fractional derivative is a generalization of integer-order derivative and integral, the first appearance of the concept of a fractional derivative is found in the year 1695, a letter to Guillaume de L'Hôspital, Leibnitz asked, "Can we generalize ordinary derivatives to ones of arbitrary order?" L'Hôspital replied to Leibniz with another question, "What is the meaning of $d^n y/dx^n$ if $n = 1/2$; i.e, what if n is fractional?" Then Leibniz, in a letter dated September 30, 1695 [**105**] replied, "...This is an apparent paradox from which, one day, useful consequences will be drawn..."

So, a lot of contributions to the theory of fractional calculus have been done over the years, such as Laplace(1812), Fourier(1822), Abel(1823-1826), Liouville (1832-1837),Riemann(1847), Grünwald(1867-1872), Letnikov (1868-1872), Heaviside(1892-1912), Wely(1917), Erdélyi(1939-1965) and many others. For more details on the history of the subject see [**66, 127, 146**].

Then, the fractional derivative are known to be a promising tool for describing memory phenomena [**8, 19, 112, 164, 171**], and the kernel function of fractional derivative is called memory function.
In 1974, the question of the physical interpretation of fractional calculus was put forward as an open problem(see [**154**]). Only in 2002, a physical explanation was proposed by Igor Podlubny [**154**], he showed a convincing geometric and physical interpretation of fractional integration and fractional differentiation but uptill now there is still no simple answer to the open problem [**45**]. Hence, fractional calculus has emerged, over the last forty years due to its many applications as a valuable tools in the modeling of many phenomena in

various fields of science and engineering, as examples, acoustic wave propagation in inhomogeneous porous material, diffusive transport, viscoelastic materials, fluid mechanics, and many others(see[43, 62, 63, 65, 76, 87, 147]), and the references therein.

There have been significant developments in ordinary fractional and partial fractional differential equations involving both Riemann-Liouville and Caputo fractional derivatives about the questions of existence, blow-up in finite time, global existence and asymptotic properties of solutions, one can mention as examples, Kilbas et al. [87], Samko et al.[166], Agarwal et al. [3], Furati and Tatar [57, 58], Mainardi [115], Kaufmann and Miller and Ross[127], Kirane et al. [80, 89], Podlubny et al. [154, 155], and the references therein.

The existence or the nonexistence of global solutions for differential equations is as important as studying the existence of solutions. In industry, knowing the blow-up in finite time can prevent accidents and malfunctions. It helps also improve the performance of machines and extend their lifespan.
Recently, many works have been done in the literature concerning the existence, uniqueness and the blow-up of solutions for some nonlinear systems of fractional differential equations, we can cite as examples [3, 18, 44, 76, 90, 99, 110, 140] for more related results. In [89] Kirane and Malik studied the following nonlinear nonlocal fractional differential system (FDS)

$$\begin{cases} u_t(t) + D^{\alpha}_{0_+}(u - u_0)(t) = \mid v(t) \mid^q, & \forall t > 0, \\ \\ v_t(t) + D^{\beta}_{0_+}(v - v_0(t)) = \mid v(t) \mid^p, & \forall t > 0, \end{cases}$$

subject to the initial conditions

$$u(0) = u_0 > 0, \quad v(0) = v_0 > 0,$$

where $p > 1$, $q > 1$, $u(0) = u_0 > 0$, $v(0) = v_0 > 0$ are constants, $D^{\alpha}_{0_+}$ and $D^{\beta}_{0_+}$ stand for the Riemann-Liouville fractional derivatives of orders $0 < \alpha < 1$ and $0 < \beta < 1$, respectively.

They investigated the profile of the blowing-up solutions to the nonlinear nonlocal system (FDS) as well as for solutions of systems obtained by dropping either the usual derivatives or the fractional derivatives.

In [80], some results on the blow-up of the solutions and lower bounds of the maximal time have been established for the system

$$u_t(t)+\rho D^{\alpha}_{0_+}(u-u_0)(t)=e^{v(t)}, \quad t>0,\ \rho>0,$$

$$v_t(t)+\sigma D^{\beta}_{0_+}(v-v_0)(t)=e^{u(t)}, \quad t>0,\ \sigma>0,$$

$$u(0)=u_0>0, \quad v(0)=v_0>0,$$

and for the subsystem obtained by dropping the usual derivatives.

Motivated by the above researches, we study in **Chapter two**, the nonlinear fractional system (FDS)

$$\begin{cases} u_t+a_1\,{}^cD^{\alpha_1}_{0_+}u+a_2\,{}^cD^{\alpha_2}_{0_+}u+\cdots+a_n\,{}^cD^{\alpha_n}_{0_+}u=\int_0^t\frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)}f(u(s),v(s))ds, \\ \\ v_t+b_1\,{}^cD^{\beta_1}_{0_+}v+b_2\,{}^cD^{\beta_2}_{0_+}v+\cdots+b_n\,{}^cD^{\beta_n}_{0_+}v=\int_0^t\frac{(t-s)^{-\gamma_2}}{\Gamma(1-\gamma_2)}g(u(s),v(s))ds, \end{cases} \tag{0.1}$$

for $t>0$, with initial data

$$u(0)=u_0>0, \quad v(0)=v_0>0, \tag{0.2}$$

and where $0<\alpha_i<1$, $0<\beta_i<1$, $i=1,\ldots,n$, $0<\gamma_j<1$, $j=1,2$, f and g are two real continuous differentiable functions with polynomial growth defined on $\mathbb{R}\times\mathbb{R}$, a_i, b_i $i=1,\ldots,n$ are positive constants, Γ is the Euler function and ${}^cD^{\alpha_i}_{0^+}$, ${}^cD^{\beta_i}_{0^+}$, $i=1,\ldots,n$, are Caputo fractional derivatives.

The Riemann-Liouville fractional integral of order $0<\alpha<1$ with lower limit 0 is defined for a locally integrable function $\varphi:\mathbb{R}_+\to\mathbb{R}$ by

$$J^{\alpha}_{0_+}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\varphi(s)}{(t-s)^{1-\alpha}}ds,\quad t>0.$$

The left-handed and right-handed Riemann-Liouville fractional derivatives of order α with $0<\alpha<1$ of a continuous function $\psi(t)$ are defined by

$$D^{\alpha}_{0_+}\psi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{\psi(s)}{(t-s)^{\alpha}}ds,\quad t>0,$$

and

$$D^{\alpha}_{T^-}\psi(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T\frac{\psi(s)}{(s-t)^{\alpha}}ds,\quad t>0,$$

respectively. For more details about fractional integrals and fractional derivatives, the reader is referred to the books [**87**, **97**, **127**, **155**, **167**].

Our main results in this chapter can be read as follows

THEOREM 1. *Assume that the functions f and g are of class $C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$. Then system (FDS) admits a unique local classical solution on a maximal interval $(0,T_{\max})$ with the alternative: either $T_{\max}=+\infty$ and the solution is global; or*

$$T_{\max}<+\infty\quad \textit{and}\quad \lim_{t\to T_{\max}}(|u(t)|+|v(t)|)=+\infty.$$

THEOREM 2. *Assume that the assumptions of Theorem 1 hold, and that the functions f and g satisfy the growth conditions:*

$$f(\xi,\eta)\geq a|\eta|^q,\quad \textit{for all } \xi,\eta\in\mathbb{R},$$

$$g(\xi,\eta)\geq b|\xi|^p,\quad \textit{for all } \xi,\eta\in\mathbb{R},$$

for some positive constants a, b and $p,q>1$. Then for all positive initial data, the solution of the fractional differential system(FDS) blows up in a finite time.

The proofs are based on Schauder fixed point theorem, and some technical lemmas for the blow-up in finite time of the solutions.

The second part of this thesis deals with problems of stability. It was probably the first question in classical dynamical systems which was dealt with in a satisfactory way. Stability questions motivated the introduction of new mathematical (tools) in engineering, particularly in control engineering. Stability theory has been of interest to mathematicians and astronomers for a long time and had a stimulating impact on these fields. The specific problem of attempting to prove that the solar system is stable accounted for the introduction of many new methods.

The question of stability of solutions of Partial Differential Equations (PDE) has inspired a wide research, it consists in determining the asymptotic behaviour of the energy $E(t)$. The principal object is to study its limit when t tends to $+\infty$ also to determine whether this limit is zero or not, and to give in a unified way the optimal or near optimal decline rates of the energy if this limit is zero.The several type of stability are given as :

- Strong stability : $E(t) \longrightarrow 0 \quad as \;\; t \longrightarrow \infty.$
- Logarithmic stability: $\exists \lambda_1, \lambda_2 > 0, \; E(t) \leq \lambda_1 (log(t))^{-\lambda_2}.$
- Polynomial stability : $\exists \lambda_1, \lambda_2 > 0, \; E(t) \leq \lambda_1 t^{-\lambda_2}.$
- Uniform stability : $\exists \lambda_1, \lambda_2 > 0, \; E(t) \leq \lambda_1 e^{-\lambda_2 t}.$

We are interesting in viscoelasticity problems, we give at first some properties of viscoelasticity and why is it useful and interesting to study?

Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation.

The properties of viscoelastic materials are influenced by many parameters. They can include: frequency, temperature, dynamic strain rate, static pre-load, time effects such as creep and relaxation, aging, and other irreversible effects. It usually appears in fluids with complex microstructure, such as polymers. One encounters viscoelastic materials in biological science, materials sciences as well as in many industrial processes, e.g., in the chemical, food, and oil industries, [**163**].

Why it is important?

One of the characteristics is that the application of passive damping technology using viscoelastic materials is widely used in the automotive and aerospace industry in a variety of applications to reduce noise and vibration and to improve interior sound quality, Viscoelastic materials are excellent impact absorbers, It is used in automobile bumpers, on computer drives to protect from mechanical shock, in helmets (the foam padding inside), in wrestling mats, shoe insoles to reduce impact transmitted to a person's skeleton etc.

Consider the following viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\ ds = 0 & \text{in } \Omega \times (0,+\infty), \\ \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x \in \ \Omega, \end{cases} \tag{0.3}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function.

This type of problems arise in viscoelasticity. Eq.(0.3) rules the evolution of the relative displacement field u in a linearly viscoelastic solid occupying a volume Ω at rest we refer to [**50, 150, 151, 161**] for example. Eq.(0.3) can be also used to formulate a generalized Kirchhoff viscoelastic beam with memory (see [**128**]). For the thermodynamics of materials with fading memory, we refer the reader to the early works of Coleman and Noll [**35**] and Coleman and Mizel [**34**], and the references therein.

For more details concerning the physical phenomena which are modeled by differential equations with memory, as well as the problem of the modelling of materials with memory, we refer the reader to the recent and interesting paper [51].

The convolution term $\int_0^t g(t-s)\Delta u(t-s)\ ds$ reflects the memory effects of materials due to viscoelasticity.

In the absence of the memory effect ($i.e, g = 0$), the problem (0.3) reduces to the wave equation and an extensive literature on existence and asymptotic behaviour is available. We refer the reader to [52, 53], for instance, there is a large literature in the global existence and uniform stabilization of wave equations. But what would happen when a viscoelastic term occurs? or in other words, for which class of kernels g we have strong stability i.e $\lim_{t\longrightarrow\infty} \| \mathcal{U}(t) \|_{\mathcal{H}} = 0$ on some Hilbert space $\mathcal{H}$,and is it possible to get a decay estimation on $\| \mathcal{U} \|_{\mathcal{H}}$ in function of g?

In this regards with the presence of the viscoelastic term, several results concerning existence, stability and blow up of solutions have been established, different types of relaxation function have been introduced to the viscoelastic problem and several uniform and polynomial stability results have been obtained.

We start by recalling some results by the pioneer works of Dafermos [39, 40] in 1970, where the author discussed a certain one-dimensional viscoelastic problem, established some existence results, and then proved that, for smooth monotone decreasing relaxation functions, the solutions go to zero as $t \longrightarrow \infty$. However, no rate of decay has been specified.

After that a great deal of attention has been devoted to the study of the viscoelasticity problems. Let us recall as mentioned in [68, 120] the first work that dealt with uniform decay was by Dassios and Zafiropoulos [42] in which a viscoelastic problem in $\mathbb{R}^3$ was studied and a polynomial decay result was proved for exponentially decaying of g. Also, the uniform stability, for some problems in linear viscoelasticity, has been established in a book by Fabrizio and Morro [50] in 1992. After this, a very important contribution by Rivera was introduced. In fact Rivera [136] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy a bounded domain or the whole space $\mathbb{R}^n$, with zero boundary and history data and in the absence of body forces. In the bounded domain case, an exponential decay result was proved for exponentially decaying memory kernels.

For the whole space case a polynomial decay result was established and the rate of the decay was given. This result was later generalized to a situation where the kernel is decaying algebraically but not exponentially by Rivera et al.[17]. In their paper, the authors showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of the decay of the relaxation function. Also, the authors considered the case the bounded domains and the case that of a material occupying the entire space.
This result was later improved by Barreto et al. in [9], where equations related for linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same decay rate of the relaxation function. For partially viscoelastic materials, Rivera et al.[139] showed that solutions decay exponentially to zero, provided that the relaxation function decays in similar fashion, regardless to the size of the viscoelastic part of the material, we refer also to [48, 104, 162], and the references therein.

We also mention some known results in the literature related to the stabilization with finite and past history, where the relaxation function was assumed to be either of polynomial or of exponential decay see [6, 10, 14, 23, 24, 26, 27, 77, 78, 79, 134, 136, 137, 139, 141, 153, 176]. After that a series of papers have appeared for a wider class of relaxation functions based on the condition introduced by Messaoudi [120, 121]

$$g'(t) \leq -\xi(t)g(t), \tag{0.4}$$

where ξ is a differentiable nonincreasing positive function for which the usual exponential and polynomial rates are only special cases see, among others, [**72, 103, 108, 109, 153**].

For other types of relaxation function and general decay, there have been works in wich the relaxation kernels is described by the inequality $g' + \chi(g) \leq 0$ with χ convex and subject to some assumptions. Alabau-Boussouira and Cannarsa [**5**] considered (0.3) such that

$$g'(t) \leq -\chi(g(t)), \tag{0.5}$$

where χ is a non-negative function, with $\chi(0) = \chi'(0) = 0$, and χ is strictly increasing and strictly convex on $(0, k_0]$, for some $k_0 > 0$. They also required that

$$\int_0^{k0} \frac{dx}{\chi(x)} = +\infty, \quad \int_0^{k_0} \frac{x\ dx}{\chi(x)} < 1, \quad \liminf_{s\to 0^+} \frac{\chi(s)/s}{\chi'(s)/s} > \frac{1}{2}, \tag{0.6}$$

and proved a decay result for the energy of (0.3). In addition to these assumptions, if

$$\limsup_{s\to 0^+} \frac{\chi(s)/s}{\chi'(s)/s} < 1 \qquad \text{and} \qquad g'(t) = -\chi(g(t)), \tag{0.7}$$

then, an explicit rate of decay is given Messaoudi [**135**] investigated (0.3) for relaxation functions satisfying (0.5) and obtained a general relation between the decay rate of the energy and that of the relaxation function g without imposing restrictive assumptions on the behavior of g at infinity such that the usual exponential and polynomial decay rates are only special cases. We recall also the works of Lasiecka et al., where the authors discussed (0.3) with a relaxation function satisfying (0.5),where χ is a given continuous positive increasing convex function such that $\chi(0) = 0$, and developed an intrinsic method for determining optimal decay rates. We refer the reader to some works with finite and infinite memory related to condition (0.5)(see [**68**]), and the references therein.

Very recently, Messaoudi and Al-Khulaifi [125] established a general decay rate for a quasilinear viscoelastic problem

$$\begin{cases} |\, u_t \,|^\rho \, u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(x,s)\ ds = 0 & \text{in } \Omega \times (0,+\infty), \\ \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x \in \ \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$, ρ is a positive real number such that $0 < \rho \leq \frac{2}{n-2}$ if $n \geq 3$ and $\rho > 0$ if $n = 1, 2$, and the relaxation function satisfies

$$g'(t) \leq -g^p(t), t \geq 0, \qquad 1 \leq p < \frac{3}{2}. \tag{0.8}$$

They proved a general decay rate from which the exponential decay and the polynomial decay are special cases. Moreover, the optimal polynomial decay is easily and deduced without restrictive conditions. Condition (0.8) gives a better description of the growth of g at infinity and allows to obtain a precise estimate of the energy that is more general than the "stronger " one (ξ constant and $p \in [1, \frac{3}{2}[$) used in the case of past history control [123, 131] and others problems.

In this thesis, we are concerned with a general decay and optimal decay for a heat system with a viscoelastic term and for some hyperbolic systems with the presence of finite memory and past memory term. We will investigate further and generalize the main results obtained in the literature. In this thesis, our study extends and improves several earlier results.

In **chapter two**, we deal with the optimal decay of the following system

$$\begin{cases} A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\ ds = 0 & \text{in } \Omega \times (0,+\infty), \\ \\ u(x,t) = 0 & \text{in } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{0.9}$$

where $m \geq 2$, Ω is a bounded domain of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive nonincreasing function, and $A : \mathbb{R}^+ \to M_n(\mathbb{R})$ is a bounded square matrix satisfying $A \in C(\mathbb{R}^+)$, and for some positive constant c_0,

$$(A(t)v, v) \geq c_0|v|^2, \qquad \forall t \in \mathbb{R}^+,\ \forall v \in \mathbb{R}^n, \tag{0.10}$$

where $(.,.)$ and $|.|$ are the inner product and the norm, respectively, in $\mathbb{R}^n$.

For the relaxation function $g(t)$ we assume

(G_1) The function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable function and satisfies

$$g(0) > 0 \quad \text{ and } \quad 1 - \int_0^{+\infty} g(s)\ ds = l > 0.$$

(G_2) There exist a constant $p \in [1, 3/2)$ and a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \in \mathbb{R}^+.$$

We also assume that (G_3)

$$\begin{aligned} 2 \ &\leq \ m \leq \frac{2n}{n-2} \qquad \text{if } n \geq 3, \\ m \ &\geq \ 2 \qquad\qquad\quad \text{if } n = 1, 2. \end{aligned}$$

Similarly to [**159**], we give the definition of a weak solution of (0.9).

Definition 3. *A weak solution of* (0.9) *on* $[0, T]$ *is a function*

$$u \in C\left([0,T); (H_0^1(\Omega))^n\right) \cap C^1\left((0,T); (L^m(\Omega))^n\right)$$

which satisfies

$$\begin{aligned} \int_0^t \int_\Omega \ & (\nabla u(x,s) - \textstyle\int_0^s \nabla u(x,\tau)d\tau) \cdot \nabla\phi(x,s)\ dx\ ds \\ & + \textstyle\int_0^t A(s)|u_t|^{m-2}u_t(x,s) \cdot \phi(x,s)\ dx\ ds = 0, \end{aligned} \tag{0.11}$$

for all t in $[0,T)$ and all ϕ in $C\Big([0,T);(H_0^1(\Omega))^n\Big)$.

Similarly to [159], we assume the existence of a solution. For the linear case ($m=2$), one can easily establish the existence of a weak solution by the Galerkin method. In the one-dimensional case ($n=1$), the existence is established in a more general setting by Yin [177].

The classical energy associated with problem (0.9) is given by

$$E(t) = \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)\ ds\right)||\nabla u(t)||_2^2, \quad \forall t \in \mathbb{R}^+, \tag{0.12}$$

where $||.||_q = ||.||_{(L^q(\Omega))^n}$, for $1 \le q < +\infty$, and

$$(g \circ \nabla u)(t) = \int_0^t g(t-\tau)||\nabla u(.,t) - \nabla u(.,\tau)||_2^2\ d\tau, \quad \forall t \in \mathbb{R}^+. \tag{0.13}$$

We state the damping produced by the memory term forces solutions to go to rest in an exponential or polynomial way depending on p as mentioned by our main result.

THEOREM 4. Let u be solution of (0.9) Then, there exist strictly two positive constants λ_0 *and* λ_1 such that the energy satisfies, for all $t \in \mathbb{R}^+$,

$$E(t) \le \lambda_0 e^{-\lambda_1 \int_0^t \xi(s)\ ds} \qquad \textit{if } p = 1, \tag{0.14}$$

$$E(t) \le \lambda_0 \left(1 + \int_0^t \xi^{2p-1}(s)\ ds\right)^{\frac{-1}{2p-2}} \qquad \textit{if } p > 1. \tag{0.15}$$

Moreover, if ξ and p in (G_2) satisfy

$$\int_0^{+\infty} \left(1 + \int_0^t \xi^{2p-1}(s)\ ds\right)^{\frac{-1}{2p-2}} dt < +\infty, \tag{0.16}$$

then, for all $t \in \mathbb{R}^+$,

$$E(t) \le \lambda_0 \left(1 + \int_0^t \xi^{p}(s)\ ds\right)^{\frac{-1}{p-1}} \qquad \textit{if } p > 1. \tag{0.17}$$

The proof is essentially based on some particular case of the well-known Jensen inequality which will be of essential use in obtaining our result and also on some preliminary Lemmas (see Chapter 2 for details).

Lemma 5. Assume that g satisfies $(G1)$ and $(G2)$ and u is the solution of (0.9) then there exists a positive constant k_0 such that

$$\xi(t)(g\circ\nabla u)(t) \leq k_0\left(-E'(t)\right)^{\frac{1}{2p-1}}, \quad \forall t \in \mathbb{R}^+. \tag{0.18}$$

Lemma 6. Let u be a solution of problem (0.9). Then, for any $\delta > 0$, we have

$$\|\nabla u(t)\|_2^2 \leq c_4\delta E(t) - \frac{C_\delta}{c_0}E'(t) + c_5(g\circ\nabla u)(t), \quad \forall t \in \mathbb{R}^+, \tag{0.19}$$

where c_0 is introduced in (0.10), c_4 and c_5 are two positive constants, and C_δ is a positive constant depending on δ.

The following examples illustrate our result and show the optimal decay rate in the polynomial case:

Example 7. *Let $g(t) = a(1+t)^{-\nu}$, where $\nu > 2$, and $a > 0$ so that*

$$\int_0^{+\infty} g(t)\ dt < 1. \tag{0.20}$$

We have

$$g'(t) = -a\nu(1+t)^{-\nu-1} = -b\left(a(1+t)^{-\nu}\right)^{\frac{\nu+1}{\nu}},$$

where $b = \nu a^{-\frac{1}{\nu}}$. Then (G_2) holds with $\xi(t) = b$ and $p = \frac{\nu+1}{\nu} \in (1, \frac{3}{2})$. Therefore (0.16) yields

$$\int_0^{+\infty}\left(b^{2p-1}t+1\right)^{\frac{-1}{2p-2}}\ dt < +\infty,$$

and hence, by (0.17), we get

$$E(t) \leq C(1+t)^{\frac{-1}{p-1}} = C(1+t)^{-\nu},$$

which is the optimal decay.

Example 8. . *Let* $g(t) = ae^{-(1+t)^{\nu}}$, *where* $0 < \nu \leq 1$, *and* $a > 0$ *is chosen so that* (0.20) *holds. Then*

$$g'(t) = -a\nu(1+t)^{\nu-1}e^{-(1+t)^{\nu}}.$$

Therefore (G_2) *holds with* $p = 1$ *and* $\xi(t) = \nu(1+t)^{\nu-1}$. *Consequently, we can use* (0.14) *to deduce*

$$E(t) \leq Ce^{-\lambda(1+t)^{\nu}}.$$

The third part of this thesis is reserved to another subject which treats the stability of an abstract system in the presence of infinite memory.

In **chapter three**, we aims in this part at investigating the asymptotic behaviour of the following initial boundary value problem :

$$u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)\ ds = 0, \qquad \forall t > 0, \tag{0.21}$$

with initial conditions

$$\begin{cases} u(-t) = u_0(t), \qquad \forall t \in \mathbb{R}_+, \\ \\ u_t(0) = u_1, \\ \end{cases} \tag{0.22}$$

where Ω is a bounded domain with sufficiently smooth boundary $\partial\Omega$, u_0 and u_1 are given history and initial data, g is a positive and nonincreasing function called the relaxation function. This type of viscoelasticity problems has been widely studied in the literature and several stability results have been established (see [**7**, **46**, **49**, **51**, **111**, **128**, **138**, **150**, **152**]), and the references therein. In the particular case $A = B = -\Delta$ on $L^2(\Omega)$ with Dirichlet boundary conditions, Eq.(0.21)- (0.22) describes the dynamics of linear viscoelastic solids(see [**150**] for example) and it can also used to formulate a generalized Kirchhoff viscoelastic beam with memory(see [**128**]), and the references therein.

In the case $A = \alpha B$ with $\alpha > 0$, Dafermos [**39**] showed that the energy tends asymptotically to zero, but no decay rate was given. Under the condition that g decays exponentially, the exponential decay of solutions of this system was obtained by Fabrizio and Lazzari [49], Giorgi et al. [64], Liu and Zheng [111], and Rivera and Naso [138], and Guesmia, Messaoudi [69] (in different contexts and using different approaches).

For problems with finite history (viscoelasticity), we mention some results related to

$$u_{tt} + \Delta^2 u - \int_0^t g(s)\Delta^2 u(t-s)\ ds = 0, \qquad \forall t > 0,$$

we refer to Lagnese [101] and Rivera et al.[9], where it was proved that the energy decays exponentially if the relaxation function g decays exponentially and polynomially if g decays polynomially. The same results were obtained by Alabau-Boussouira et al.[6] for a more general abstract equation. In [129, 130], Rivera et al. investigated a class of abstract viscoelastic systems of the form

$$u_{tt} + Au - \left(g * A^{\beta} u\right)(t) = 0, \qquad \forall t > 0, \tag{0.23}$$

where A is a strictly positive, self-adjoint operator with domain $\mathcal{D}(A)$ a subset of a Hilbert space $H$and $*$ denotes the convolution product in the variable t. The authors showed that solutions for (0.23), when $0 < \beta < 1$, decay polynomially even if the kernel g decays exponentially, while in the case $\beta = 1$, the solution energy decays at the same decay rate as the relaxation function.

For a more general decay to problem (0.23), Han and Wang [71] showed that the rate of the decay of the energy is exactly the rate of decay of g, which is not necessarily of polynomial or exponential decay type by considering relaxation function satisfying

$$g'(t) \le -\xi(t) g(t), \quad \forall t \ge 0; \tag{0.24}$$

where $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is nonincreasing differentiable function such that

$$\exists k > 0, \quad \mid \frac{\xi'(t)}{\xi(t)} \mid \leq k, \quad \forall t \geq 0. \tag{0.25}$$

Problems related to (0.21)-(0.22) have been studied by many authors and several stability results have been established; see [**36, 67, 150, 151**]. The exponential and polynomial decay of the solutions of equation (0.21)-(0.22) have been studied in [**68**], where it was assumed that $(\mathbb{H}_1)$ holds wich will be cited later and

- There exists an increasing strictly convex function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \qquad \textit{and} \qquad \lim_{t \longrightarrow +\infty} G'(t) = +\infty,$$

  such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))}\, ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty.$$

The author established a general decay estimate given in term of the convex function G. His result generalizes the usual exponential and polynomial decay results found in the literature. He considered two cases corresponding to the following two conditions on A and B:

$$\exists a_2 > 0 : \qquad \|A^{\frac{1}{2}} v\|^2 \leq a_2 \|B^{\frac{1}{2}} v\|^2, \quad \forall v \in \mathcal{D}(A^{\frac{1}{2}}). \tag{0.26}$$

or

$$\exists a_2 > 0 : \qquad \|A^{\frac{1}{2}} v\|^2 \leq a_2 \|A^{\frac{1}{2}} B^{\frac{1}{2}} v\|^2, \quad \forall v \in \mathcal{D}(A^{\frac{1}{2}} B^{\frac{1}{2}}). \tag{0.27}$$

The main question and difficulty of our study is that : can us get an optimal decay with infinite memory for the polynomial case for the problem (0.21)-(0.22)? To overcome this difficulty, we try to adopt the method introduced in [**125**] for finite history, with some modifications imposed by the nature of our problem.

We shall present some necessary assumptions and prove some important inequalities that will become useful in later stages.

Let us assume that

$(\mathbb{H}_1)$ There exist positive constants a_0 and a_1 such that

$$a_1\|v\|^2 \leq \|B^{\frac{1}{2}}v\|^2 \leq a_0\|A^{\frac{1}{2}}v\|^2, \quad \forall v \in \mathcal{D}(A^{\frac{1}{2}}).$$

$(\mathbb{H}_2)$ $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable nonincreasing function satisfying

$$0 < g_0 < \frac{1}{a_0}.$$

$(\mathbb{H}_3)$ There exists a nonincreasing differentiable function $\xi: \mathbb{R}_+ \to \mathbb{R}_+$ and $1 \leq p < \frac{3}{2}$ satisfying (1.6).

Equation (0.21)-(0.22) can be rewritten as an abstract linear first-order system of the form

$$\begin{cases} \mathcal{U}_t + \mathcal{A}\mathcal{U}(t) = 0, \qquad \forall t > 0, \\ \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{0.28}$$

where $\mathcal{U}_0 = (u_0(0), u_1, \eta_0)^T \in \mathcal{H} = \mathcal{D}(A^{\frac{1}{2}}) \times H \times L^2_g(\mathbb{R}_+, \mathcal{D}(B^{\frac{1}{2}}))$, $\mathcal{U} = (u, u_t, \eta^t)^T$ and $L^2_g(\mathbb{R}_+, \mathcal{D}(B^{\frac{1}{2}}))$ is the weighted space with respect to the measure $g(s)ds$ defined by

$$L^2_g(\mathbb{R}_+, \mathcal{D}(B^{\frac{1}{2}})) = \left\{ z : \mathbb{R}_+ \longrightarrow \mathcal{D}(B^{\frac{1}{2}}),\ \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}z(s)\|^2\ ds < +\infty \right\}$$

endowed with the inner product

$$\langle z_1, z_2 \rangle_{L^2_g(\mathbb{R}_+, \mathcal{D}(B^{\frac{1}{2}}))} = \int_0^{+\infty} g(s)\langle B^{\frac{1}{2}}z_1(s), B^{\frac{1}{2}}z_2(s)\rangle\ ds.$$

The operator $\mathcal{A}$ is defined by

$$\mathcal{A}(v,w,z)^T = \left(-w, Av - g_0 Bv + \int_0^{+\infty} g(s)Bz(s)\ ds, \frac{\partial z}{\partial s} - w\right)^T,$$

where $g_0 = \int_0^{+\infty} g(s)\ ds$,

$$\mathcal{D}(\mathcal{A}) = \left\{(v,w,z)^T \in \mathcal{H},\ v \in \mathcal{D}(A),\ w \in \mathcal{D}(A^{\frac{1}{2}})),\ z \in \mathcal{L}_g, \int_0^{+\infty} g(s)z(s)\ ds \in \mathcal{D}(B)\right\},$$

and $\mathcal{L}_g = \left\{z \in L_g^2(\mathbb{R}_+, \mathcal{D}(B^{\frac{1}{2}})), \partial_s z \in L_g^2(\mathbb{R}_+, \mathcal{D}(B^{\frac{1}{2}})), z(0) = 0\right\}$.

the space $\mathcal{H}$ endowed with the inner product

$$\langle (v_1,w_1,z_1)^T, (v_2,w_2,z_2)^T\rangle_{\mathcal{H}} = \langle A^{\frac{1}{2}}v_1, A^{\frac{1}{2}}v_2\rangle - g_0\left\langle B^{\frac{1}{2}}v_1, B^{\frac{1}{2}}v_2\right\rangle + \langle w_1,w_2\rangle + \langle z_1, z_2\rangle_{L_g^2(\mathbb{R}_+,\mathcal{D}(B^{\frac{1}{2}}))}$$

is a Hilbert space, Therefore, the classical semigroup theory implies that (see [**149**]), for any $\mathcal{U}_0 \subset \mathcal{H}$, the system (0.28) has a unique weak solution

$$\mathcal{U} \in \mathcal{C}(\mathbb{R}_+, \mathcal{H}).$$

Moreover, if $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A})$, then the solution of (0.28) is classical; that is

$$\mathcal{U} \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C(\mathbb{R}_+, \mathcal{D}(\mathcal{A})).$$

We recall that the energy related with problem (0.21)-(0.22) is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \\ &= \frac{1}{2}\left(\|A^{\frac{1}{2}}u(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2 + \|u_t(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\ ds\right). \end{aligned}$$

and satisfies

$$E'(t) = \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\ ds, \quad \forall t \in \mathbb{R}_+, \tag{0.29}$$

Our main results in this part can be state as follows

Theorem 9. *Assume that* $(\mathbb{H}_1)$, $(\mathbb{H}_2)$ *and* $(\mathbb{H}_3)$ *hold.*

(1) *Let* $\mathcal{U}_0 \in H$ *and* $\mathcal{U}$ *be the solution of* (0.21) − (0.22).
If (0.26) *holds, and if,*

$$\exists m_0 > 0 : \quad \|B^{\frac{1}{2}} u_0(s)\| \leq m_0, \qquad \forall s > 0,$$

then there exists a positive constant C *such that, for all* $t \in \mathbb{R}_+$,

$$E(t) \leq C(1+t)^{-\frac{1}{2p-2}} \xi^{-\frac{2p-1}{2p-2}}(t) \left[1 + \int_0^t (s+1)^{\frac{1}{2p-2}} \xi^{\frac{2p-1}{2p-2}}(s) h^{2p-1}(s)\ ds\right],$$

where $h(t) = \xi(t) \int_t^{+\infty} g(s)\ ds$.

Moreover, if

$$\int_0^{+\infty} (1+t)^{-\frac{1}{2p-2}} \xi^{-\frac{2p-1}{2p-2}}(t) \left[1 + \int_0^t (s+1)^{\frac{1}{2p-2}} \xi^{\frac{2p-1}{2p-2}}(s) h^{2p-1}(s)\ ds\right]\ dt < +\infty,$$

then, for all $t \in \mathbb{R}_+$,

$$E(t) \leq C(1+t)^{-\frac{1}{p-1}} \xi^{-\frac{p}{p-1}}(t) \left[1 + \int_0^t (s+1)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s) h^{p}(s)\ ds\right].$$

(2) *Let* $\mathcal{U}_O \in D(A) \times D(A^{\frac{1}{2}}) \times L^2_g(\mathbb{R}_+, D(A^{\frac{1}{2}} B^{\frac{1}{2}})$ *and* $\mathcal{U}$ *be the solution of* (0.21) − (0.22).
If (0.27) *holds, and if,*

$$\exists m_0 > 0 : \quad \|A^{\frac{1}{2}} B^{\frac{1}{2}} u_0(s)\| \leq m_0, \quad \forall s > 0,$$

then there exists a positive constant C *such that, for all* $t \in \mathbb{R}_+$,

$$E(t) \leq C \left(\frac{E_2(0) + E^{2p-1}(0) + \int_0^t h^{2p-1}(s)\ ds}{\int_0^t \xi^{2p-1}(s)\ ds} \right)^{\frac{1}{2p-1}},$$

where

$$E_2(t) \;=\; \frac{1}{2}\left(\|Au(t)\|^2 - g_0\|A^{\frac{1}{2}}B^{\frac{1}{2}}u(t)\|^2 + \|A^{\frac{1}{2}}u'(t)\|^2\right) + \frac{1}{2}\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\ ds.$$

Moreover, if

$$\int_0^{+\infty}\left(\frac{E_2(0)+E^{2p-1}(0)+\int_0^t h^{2p-1}(s)\ ds}{\int_0^t \xi^{2p-1}(s)\ ds}\right)^{\frac{1}{2p-1}} < +\infty,$$

then for all $t \in \mathbb{R}_+$,

$$E(t) \le C\left(\frac{E_2(0)+E^{p}(0)+\int_0^t h^{p}(s)\ ds}{\int_0^t \xi^{p}(s)\ ds}\right)^{\frac{1}{p}}.$$

The method of proof of Theorem 5 is based on multiplier method and makes use of the general Young's inequality and Jensen's inequality, and some lemmas and corollary.

Finally, We illustrate the energy decay rate given by Thoerem 3 through an example and we compare our results with the one of [**68, 69**].

The fourth part is devoted to some Hamilton-Jacobi equations.

We shall consider some systems of Hamilton-Jacobi equations as the following form

$$\begin{cases} u_t + \Delta^2 u = |\nabla v|^{\alpha_1} + |v|^{\beta_1} & \text{in } \Omega\times(0,\infty), \\ v_t + \Delta^2 v = |\nabla u|^{\alpha_2} + |u|^{\beta_2} & \text{in } \Omega\times(0,\infty), \\ u = v = \Delta u = \Delta v = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{0.30}$$

where $\alpha_i, \beta_i,\ i = 1,2$ are positive constants, such that $\alpha_i, \beta_i > 1$.
with exponents in the nonlinearities.

Our aim is to investigate the existence of weak solutions for these systems , and the blowing-up in finite time of the solutions, namely sufficient assumptions on the exponents and the initial data which ensure these results.
This study is motivated by some recent works on parabolic equations with gradients nonlinearities (see[**15, 102, 169**]), and the references therein.

Our main results can be stated as follows

THEOREM 10. *(Existence of weak solutions)*
Assume that

$$1 < \alpha_i < \frac{n+8}{n+2}, \qquad 1 < \beta_i < \frac{n+8}{n}, \quad i = 1,2. \tag{0.31}$$

Then for all $u_0, v_0 \in L^2(\Omega)$, *there exists at least a maximal weak solution of problem* (0.30)

THEOREM 11. *(Blow-up in finite time of solutions)*
Suppose that

$$1 < \alpha_i < \frac{n+8}{n+2}, \qquad 1 < \beta_i < \frac{n+8}{n}, \quad i = 1,2.$$

Then for $u_0, v_0 \in L^2(\Omega)$ *and* u_0 *or* v_0 *sufficiently large, the problem* (0.30) *cannot admit a globally defined weak solution.*

The proofs are based on the Galerkin method and interpolation inequalities and Kaplan method for the blowing-up in finite of the solutions.

CHAPTER 2

Finite Time Blow-up of Solutions for a Nonlinear System of Fractional Differential Equations

This chapter is the subject of the following publication:

Abdelaziz Mennouni; Abderrahmane Youkana.

Finite time blow-up of solutions for a nonlinear system of fractional differential equations .

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Abstract. In this work we study the blow-up in finite time of solutions for the Cauchy problem of fractional ordinary equations

$$u_t + a_1\, {}^cD^{\alpha_1}_{0_+} u + a_2\, {}^cD^{\alpha_2}_{0_+} u + \cdots + a_n\, {}^cD^{\alpha_n}_{0_+} u = \int_0^t \frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)} f(u(s), v(s))ds,$$
$$v_t + b_1\, {}^cD^{\beta_1}_{0_+} v + b_2\, {}^cD^{\beta_2}_{0_+} v + \cdots + b_n\, {}^cD^{\beta_n}_{0_+} v = \int_0^t \frac{(t-s)^{-\gamma_2}}{\Gamma(1-\gamma_2)} g(u(s), v(s))ds,$$

for $t > 0$, where the derivatives are Caputo fractional derivatives of order α_i, β_i, and f and g are two continuously differentiable functions with polynomial growth. First, we prove the existence and uniqueness of local solutions for the above system supplemented with initial conditions, then we establish that they blow-up in finite time.

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Keywords and phrases. Fractional differential equation; Caputo fractional derivative; blow-up in finite time.

1. Introduction

In this work, we study the system of ordinary fractional differential equations

$$\begin{cases} u_t + a_1\ {}^cD^{\alpha_1}_{0_+}u + a_2\ {}^cD^{\alpha_2}_{0_+}u + .. + a_n\ {}^cD^{\alpha_n}_{0_+}u = \frac{1}{\Gamma(1-\gamma_1)}\int_0^t (t-s)^{-\gamma_1} f(u(s),v(s))ds, \\ \\ v_t + b_1\ {}^cD^{\beta_1}_{0_+}v + b_2\ {}^cD^{\beta_2}_{0_+}v + .. + b_n\ {}^cD^{\beta_n}_{0_+}v = \frac{1}{\Gamma(1-\gamma_2)}\int_0^t (t-s)^{-\gamma_2} g(u(s),v(s))ds, \end{cases} \tag{1.1}$$

for $t > 0$, with initial data

$$u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \tag{1.2}$$

and where $0 < \alpha_i < 1,\ 0 < \beta_i < 1,\ i = 1, \dots, n,\ 0 < \gamma_j < 1,\ j = 1, 2,\ f$ and g are two real continuous differentiable functions defined on $\mathbb{R} \times \mathbb{R}$, a_i, b_i $i = 1, \dots, n$ are positive constants, Γ is the Euler function and ${}^cD^{\alpha_i}_{0^+}$, ${}^cD^{\beta_i}_{0^+}$, $i = 1, \dots, n$, are Caputo fractional derivatives.

In recent years, fractional differential equations have played an important role in the study of models for many phenomena in various fields of physics, biology and engineering, such as aerodynamics, viscoelasticity, control of dynamic systems, electrochemistry, porous media, etc (see [**16**, **18**, **75**, **167**]), and the references therein; their study attracted the attention of many researchers (see for instance [**88**, **90**, **110**, **140**]), and the references therein. In addition, a particular attention was given for the study of the local existence and uniqueness of solutions for these systems and their properties like the blow-up in finite time, the global existence, the asymptotic behavior, etc. (see [**18**, **90**, **110**, **140**]).

In [**89**], the profile of the blowing-up solutions has been investigated for the following nonlinear nonlocal system:

$$u_t(t) + D^{\alpha}_{0_+}(u - u_0)(t) = |v(t)|^q, \quad t > 0,\ q > 1,$$

$$v_t(t) + D^{\beta}_{0_+}(v - v_0)(t) = |u(t)|^p, \quad t > 0,\ p > 1,$$

$$u(0) = u_0 > 0, \quad v(0) = v_0 > 0,$$

as well as for solutions of systems obtained by dropping either the usual derivatives or the fractional derivatives.

In [80], some results on the blow-up of the solutions and lower bounds of the maximal time have been established for the system

$$u_t(t)+\rho D^{\alpha}_{0_+}(u-u_0)(t)=e^{v(t)}, \quad t>0,\ \rho>0,$$

$$v_t(t)+\sigma D^{\beta}_{0_+}(v-v_0)(t)=e^{u(t)}, \quad t>0,\ \sigma>0,$$

$$u(0)=u_0>0, \quad v(0)=v_0>0,$$

and the subsystem obtained by dropping the usual derivatives.

In the spirit of the interesting works [56, 80, 89], we prove that the non global existence of solutions to (1.1)-(1.2) holds for polynomial nonlinearities. For the existence of solutions for the system (1.1)-(1.2), we will use the Schauder theorem.

Our paper is organized as follows: In Section 2, we give some preliminary results for fractional derivatives. In Section 3, we will prove the local existence and uniqueness of the solutions. In Section 4, we will state and prove our main result on the blow- up in finite time of solutions for system (1.1)-(1.2).

2. Preliminaries and mathematical background

For the convenience of the reader, we recall basic facts from fractional calculus,for more details on fractional calculus see [166, 87]

The Riemann-Liouville fractional integral of order $0<\alpha<1$ with lower limit 0 is defined for a locally integrable function $\varphi:\mathbb{R}_+\to\mathbb{R}$ by

$$J^{\alpha}_{0_+}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\varphi(s)}{(t-s)^{1-\alpha}}ds, \quad t>0,$$

where Γ is the Euler Gamma function.

The left-handed and right-handed Riemann-Liouville fractional derivatives of order α with $0<\alpha<1$ of a continuous function $\psi(t)$ are defined by

$$D^{\alpha}_{0_+}\psi(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\psi(s)}{(t-s)^{\alpha}}ds, \quad t>0,$$

and

$$D^{\alpha}_{T^-}\psi(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{\psi(s)}{(s-t)^{\alpha}}ds, \quad t>0,$$

respectively. One can see that

$$\frac{d}{dt}J^{1-\alpha}_{0^+}\psi(t) = D^{\alpha}_{0+}\psi(t), \quad t>0.$$

The integration by parts formula (see [**167**]) in $[0,T]$ reads

$$\int_0^T h(t)D^{\alpha}_{0_+}k(t)dt = \int_0^T (D^{\alpha}_{T\_}h(t))k(t)dt,$$

for functions h,k in $C([0,T])$ such that $D^{\alpha}_{0^+}k$ and $D^{\alpha}_{T^-}h$ are continuous.

The Caputo fractional derivative of order $0<\alpha<1$ of an absolutely continuous function $\phi(t)$ of order $0<\alpha<1$ is defined by

$${}^cD^{\alpha}_{0_+}\phi(t) = J^{1-\alpha}_{0_+}\frac{d}{dt}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\phi'(s)ds.$$

The relation between the Riemann-Liouville and the Caputo fractional derivatives for an absolutely continuous function $\phi(t)$ is given by

$${}^cD^{\alpha}_{0_+}\phi(t) = D^{\alpha}_{0_+}(\phi(t)-\phi(0)), \quad 0<\alpha<1.$$

3. Existence and uniqueness of solutions

In this section, we deal with the existence and uniqueness of local solutions for problem (1.1)-(1.2). We say that (u,v) is a local classical solution if it satisfies equations (1.1)-(1.2) on some interval $(0,T^*)$. Our main result in this section reads as follows.

THEOREM 12. *Assume that the functions f and g are of class $C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$. Then system (1.1)-(1.2) admits a unique local classical solution on a maximal interval $(0,T_{\max})$ with the alternative: either $T_{\max}=+\infty$ and the solution is global; or*

$$T_{\max}<+\infty \quad \textit{and} \quad \lim_{t\to T_{\max}}(|u(t)|+|v(t)|)=+\infty.$$

PROOF. For the sake of completeness, we give the proof of the existence of solutions of (1.1)-(1.2). Let $k>0$ be a positive constant and

$$h:=\min\{\sigma_1,\ \sigma_2\}>0, \tag{3.1}$$

where

$$\sigma_1:=\min\Big\{\min_{1\le i\le n}\Big(\frac{1}{2n^2\bar{a}\max_{1\le i\le n}(\frac{1}{\Gamma(2-\alpha_i)})}\Big)^{\frac{1}{1-\alpha_i}},\Big(\frac{k\Gamma(2-\gamma_1)}{2M}\Big)^{\frac{1}{1-\gamma_1}}\Big\},$$

$$\sigma_2:=\min\Big\{\min_{1\le i\le n}\Big(\frac{1}{2n^2\bar{b}\max_{1\le i\le n}\frac{1}{\Gamma(2-\beta_i)})}\Big)^{\frac{1}{1-\beta_i}},\Big(\frac{k\Gamma(2-\gamma_2)}{2M}\Big)^{\frac{1}{1-\gamma_2}}\Big\},$$

$$\bar{a}=\max_{1\le i\le n}\{a_i\},\quad \bar{b}=\max_{1\le i\le n}\{b_i\},$$

and M is a positive constant which will be defined later.
Let $C([0,h])\times C([0,h])$ be the space of all continuous functions (χ,ψ) on $[0,h]$ equipped with the norm

$$\|(\chi,\psi)\|_\infty=\max(\|\chi\|_\infty,\ \|\psi\|_\infty),$$

where

$$\|\chi\|_\infty=\max_{0\le t\le h}|\chi(t)|,\quad \|\psi\|_\infty=\max_{0\le t\le h}|\psi(t)|.$$

For simplicity, we assume $\alpha_1\le\alpha_2\le\cdots\le\alpha_n$ and $\beta_1\le\beta_2\le\cdots\le\beta_n$.

Now, in order to prove the existence of solutions for problem (1.1)-(1.2), we rewrite it as a system of integral equations in $C([0,h])\times C([0,h])$,

$$
\begin{aligned}
x(t) &= -a_1 J_{0_+}^{1-\alpha_1}x(t) - a_2 J_{0_+}^{1-\alpha_2}x(t) - \cdots - a_n J_{0_+}^{1-\alpha_n}x(t) + J_{0^+}^{1-\gamma_1} f(u_0\\
&\quad + \int_0^t x(s)ds, v_0 + \int_0^t y(s)ds),\\
y(t) &= -b_1 J_{0_+}^{1-\beta_1}y(t) - b_2 J_{0_+}^{1-\beta_2}y(t) - \cdots - b_n J_{0_+}^{1-\beta_n}y(t) + J_{0^+}^{1-\gamma_2} g(u_0\\
&\quad + \int_0^t x(s)ds, v_0 + \int_0^t y(s)ds),
\end{aligned}
\tag{3.2}
$$

via the transformation

$$u(t) = u_0 + \int_0^t x(s)ds, \quad v(t) = v_0 + \int_0^t y(s)ds,$$

and the relation ${}^cD_{0+}^{\alpha}\psi(t) = J_{0^+}^{1-\alpha}\frac{d}{dt}\psi(t)$, and we shall prove the existence of local solutions for (3.2).

Let us define the operator $A : C([0,h])\times C([0,h]) \to C([0,h])\times C([0,h])$ by

$$A(x,y) = (A_1(x,y),\ A_2(x,y)),$$

where

$$
\begin{aligned}
A_1(x(t),y(t)) &= -\sum_{i=1}^{n} a_i J_{0_+}^{1-\alpha_i}x(t)\\
&\quad + J_{0^+}^{1-\gamma_1} f\Big(u_0 + \int_0^t x(s)ds, v_0 + \int_0^t y(s)ds\Big),\\
A_2(x(t),y(t)) &= -\sum_{i=1}^{n} b_i J_{0_+}^{1-\beta_i}y(t)\\
&\quad + J_{0^+}^{1-\gamma_2} g\Big(u_0 + \int_0^t x(s)ds, v_0 + \int_0^t y(s)ds\Big).
\end{aligned}
\tag{3.3}
$$

Let us define the set

$$D := \Big\{(x,y) \in C([0,h])\times C([0,h]),\ \|(x,y)\|_\infty = \sup(\|x\|_\infty,\ \|y\|_\infty) \le k\},$$

as a domain of the operator A, which is a convex, bounded, and closed subset of the Banach space $C([0,h])\times C([0,h])$. Since f and g are continuously differentiable on $[u_0 - kh, u_0 +$

$kh] \times [v_0 - kh, v_0 + kh]$, there exists a positive constant M such that, for any t in $[0,h]$ and any (x,y) in D,

$$\Big|f(u_0 + \int_0^t x(s)ds,\ v_0 + \int_0^t y(s))ds\Big| \leq M, \tag{3.4}$$

$$\Big|g((u_0 + \int_0^t x(s)ds,\ v_0 + \int_0^t y(s))ds\Big| \leq M, \tag{3.5}$$

and for any (u_j, v_j) in $[u_0 - kh, u_0 + kh] \times [v_0 - kh, v_0 + kh]$, $j = 1,2$, and any t in $[0,h]$, there exist two positive constants L_1 and L_2 depending on u_0, v_0, k, h and on f and g respectively such that

$$|f(u_1(t), v_1(t)) - f(u_2(t), v_2(t))| \leq L_1 \|(u_1(t) - u_2(t), v_1(t) - v_2(t))\|, \tag{3.6}$$

$$|g(u_1(t), v_1(t)) - g(u_2(t), v_2(t))| \leq L_2 \|(u_1(t) - u_2(t), u_1(t) - u_2(t))\|, \tag{3.7}$$

where $\|(u_1(t) - u_2(t), v_1(t) - v_2(t))\| = |u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|$.

Now, by using (3.1) and (3.6) and (3.7), for all $z_1 = (x_1, y_1) \in D$ and $z_2 = (x_2, y_2) \in D$ satisfying $\|z_1 - z_2\|_\infty < \delta$, where δ is a positive constant which will be defined later, we obtain

$$
\begin{aligned}
&\|A_1(z_1)-A_1(z_2)\|_\infty\\
&=\sup_{0\le t\le h}|-\sum_{i=1}^{n}a_iJ_{0_+}^{1-\alpha_i}x_1(t)+J_{0^+}^{1-\gamma_1}f(u_0+\int_0^t x_1(s)ds,\ v_0+\int_0^t y_1(s)ds)\\
&\quad+\sum_{i=1}^{n}a_iJ_{0_+}^{1-\alpha_i}x_2(t)-J_{0^+}^{1-\gamma_1}f(u_0+\int_0^t x_2(s)ds,\ v_0+\int_0^t y_2(s)\,ds)\ |\\
&\le\sup_{0\le t\le h}|-\sum_{i=1}^{n}a_iJ_{0_+}^{1-\alpha_i}(x_1(t)-x_2(t))\\
&\quad+J_{0^+}^{1-\gamma_1}\{f(u_0+\int_0^t x_1(s)ds,v_0+\int_0^t y_1(s)ds)\\
&\quad-f(u_0+\int_0^t x_2(s)ds,v_0+\int_0^t y_2(s)ds)\}|\\
&\le\sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\alpha_i)}\int_0^h(t-s)^{-\alpha_i}\|z_1-z_2\|_\infty ds+\frac{L_1}{\Gamma(2-\gamma_1)}h^{2-\gamma_1}\|z_1-z_2\|_\infty\\
&\le\Big(n\bar{a}\max_{1\le\ i\le n}\{\frac{1}{\Gamma(2-\alpha_i)}\}\sum_{i=1}^{n}h^{1-\alpha_i}+\frac{L_1}{\Gamma(2-\gamma_1)}\ h^{2-\gamma_1}\Big)\delta,
\end{aligned}
\tag{3.8}
$$

and in the same way, we obtain

$$
\|A_2(z_1)-A_2(z_2)\|_\infty\le\Big(n\bar{b}\max_{1\le\ i\le n}\{\frac{1}{\Gamma(2-\beta_i)}\}\sum_{i=1}^{n}h^{1-\beta_i}+\frac{L_2}{\Gamma(2-\gamma_2)}h^{2-\gamma_2}\Big)\delta. \tag{3.9}
$$

Now, given an $\varepsilon>0$, pick $\delta=\min\left\{\frac{\varepsilon}{\omega_1},\frac{\varepsilon}{\omega_2}\right\}$, where

$$
\begin{aligned}
\omega_1&:=n\bar{a}\max_{1\le\ i\le n}\left\{\frac{1}{\Gamma(2-\alpha_i)}\right\}\sum_{i=1}^{n}h_i^{1-\alpha_i}+\frac{L_1}{\Gamma(2-\gamma_1)}h^{2-\gamma_1}\,,\\
\omega_2&:=n\bar{b}\max_{1\le\ i\le n}\left\{\frac{1}{\Gamma(2-\beta_i)}\right\}\sum_{i=1}^{n}h_i^{1-\beta_i}+\frac{L_2}{\Gamma(2-\gamma_2)}h^{2-\gamma_2}\,.
\end{aligned}
$$

One can see that $\|A(z_1)-A(z_2)\|_\infty < \varepsilon$, consequently, A is a continuous operator on D. Next, from (3.3), (3.4), (3.5) and (3.1), for all $z=(x,y)\in D$ we have

$$\begin{aligned}
\|A_1(z)\|_\infty &\leq \sup_{0\leq t\leq h}\Big|\sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\alpha_i)}\int_0^t (t-s)^{-\alpha_i}x(s)\,ds \\
&\quad +\frac{1}{\Gamma(1-\gamma_1)}\int_0^t (t-s)^{-\gamma_1} f(u_0+\int_0^t x(s)ds, v_0+\int_0^t y(s))ds\Big| \\
&\leq \sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\alpha_i)}\|z\|_\infty \int_0^h (t-s)^{-\alpha_i}ds + \frac{1}{\Gamma(1-\gamma_1)}\int_0^t (t-s)^{-\gamma_1} M\,ds \\
&\leq nk\bar{a}\max_{1\leq i\leq n}\Big\{\frac{1}{\Gamma(2-\alpha_i)}\Big\}\sum_{i=1}^{n} h^{1-\alpha_i} + \frac{1}{\Gamma(2-\gamma_1)} M h^{1-\gamma_1} \leq k. \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
\|A_2(z)\|_\infty &\leq \sup_{0\leq t\leq h}\Big|\sum_{i=1}^{n}\frac{b_i}{\Gamma(1-\beta_i)}\int_0^t (t-s)^{-\beta_i}x(s)\,ds + \frac{1}{\Gamma(2-\gamma_2)} M h^{1-\gamma_1}\Big| \\
&\leq \sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\beta_i)}\|z\|_\infty \int_0^h (t-s)^{-\beta_i}ds + \frac{1}{\Gamma(2-\gamma_2)} M h^{1-\gamma_2} \\
&\leq nk\bar{b}\max_{1\leq i\leq n}\Big\{\frac{1}{\Gamma(2-\beta_i)}\Big\}\sum_{i=1}^{n} h^{1-\beta_i} + \frac{1}{\Gamma(2-\gamma_2)} M h^{1-\gamma_2} \leq k. \qquad (3.11)
\end{aligned}$$

Inequalities (3.10) and (3.11) assert that $A(D)\subset D$. Thus, the set $A(D)$ is uniformly bounded. Now, for all $0\leq t_1\leq t_2\leq h$ with $|t_1-t_2|<\eta$, and all $z=(x,y)\in C([0,h])\times C([0,h])$, from (3.6) we have

$$
\begin{aligned}
|A_1(\ z(t_1)) - A_1(\ z(t_2)\ )| = && \Big| -\textstyle\sum_{i=1}^{n} \frac{a_i}{\Gamma(1-\alpha_i)} \int_0^{t_1} (t_1-s)^{-\alpha_i} x(s)ds \\
&+& \tfrac{1}{\Gamma(1-\gamma_1)} \int_0^{t_1} (t_1-s)^{-\gamma_1} f(u_0+\int_0^s x(\tau)d\tau, v_0+\int_0^s y(\tau)d\tau)ds \\
&+& \textstyle\sum_{i=1}^{n} \frac{a_i}{\Gamma(1-\alpha_i)} \int_0^{t2} (t_2-s)^{-\alpha_i} x(s)ds \\
&-& \tfrac{1}{\Gamma(1-\gamma_1)} \int_0^{t_2} (t_2-s)^{-\gamma_1} f(u_0+\int_0^s x(\tau)d\tau,\ v_0+\int_0^s y(\tau)d\tau)ds \Big| \\
&\le& \textstyle\sum_{i=1}^{n} \frac{a_i}{\Gamma(1-\alpha_i)} \int_0^{t_1} \Big((t_1-s)^{-\alpha_i} - (t_2-s)^{-\alpha_i}\Big) |x(s)|ds \\
&+& \textstyle\sum_{i=1}^{n} \frac{a_i}{\Gamma(1-\alpha_i)} \int_{t_1}^{t_2} (t_2-s)^{-\alpha_i} |x(s)|ds \\
&+& \tfrac{1}{\Gamma(1-\gamma_1)} \int_0^{t_1} \Big((t_1-s)^{-\gamma_1} - (t_2-s)^{-\gamma_1}\Big) \\
&\times& \Big| f(u_0+\int_0^s x(\tau)d\tau, v_0+\int_0^s y(\tau)d\tau) \Big| ds \\
&+& \tfrac{1}{\Gamma(1-\gamma_1)} \int_{t_1}^{t_2} (t_2-s)^{-\gamma_1} \Big| f(u_0+\int_0^s x(\tau)d\tau, v_0+\int_0^s y(\tau)d\tau) \Big| ds \\
&\le& k\bar{a} \textstyle\sum_{i=1}^{n} \frac{1}{\Gamma(2-\alpha_i)} (t_2-t_1)^{1-\alpha_i} + \frac{2M}{\Gamma(2-\gamma_1)} (t_2-t_1)^{1-\gamma_1}
\end{aligned}
\tag{3.12}
$$

Similarly, we obtain

$$
\begin{aligned}
&|A_2(z(t_1)) - A_2(z(t_2))| \\
&\le k\bar{b} \sum_{i=1}^{n} \frac{1}{\Gamma(2-\beta_i)} (t_2-t_1)^{1-\beta_i} + \frac{2M}{\Gamma(2-\gamma_2)} (t_2-t_1)^{1-\gamma_2}.
\end{aligned}
\tag{3.13}
$$

From (3.12) and (3.13) it yields that $A(D)$ is equicontinuous, and so by using Arzela-Ascoli theorem, we find that $A(D)$ is relatively compact in $C([0,h]) \times C([0,h])$.

Finally, by Schauder theorem, we conclude that the operator A has at least one fixed point, this means that the system of integral equations (3.2) has at least one local continuous solution (x,y) defined on $[0,h]$. Now, since for all $t \in [0,h]$,

$$
u(t) = u_0 + \int_0^t x(s)ds, \quad v(t) = v_0 + \int_0^t y(s)ds, \tag{3.14}
$$

where x and y are solutions of system (3.2) of integral equations, it follows that $u'(t) = x(t)$, $v'(t) = y(t)$ for any t in $(0, h)$.

Using the definition of Caputo fractional derivative, we find for all t in $(0, h)$,

$$\begin{aligned} {}^{c}D_{0_+}^{\alpha_i} u(t) &= J_{0_+}^{1-\alpha_i} x(t) = \frac{1}{\Gamma(1-\alpha_i)} \int_0^T (t-s)^{-\alpha_i} x(s)\, ds, \quad i = 1, \dots, n, \\ {}^{c}D_{0_+}^{\beta_i} v(t) \ &= J_{0_+}^{1-\beta_i} y(t) = \frac{1}{\Gamma(1-\beta_i)} \int_0^T (t-s)^{-\beta_i} y(s)\, ds, \quad i = 1, \dots, n. \end{aligned} \tag{3.15}$$

Combining (3.14), (3.15) and (3.2), for all t in $(0, h)$ we obtain

$$\begin{aligned} u'(t) + \sum_{i=1}^{n} a_i J_{0_+}^{1-\alpha_i} \frac{du(t)}{dt} &= J_{0_+}^{1-\gamma_1} f(u(s), v(s)) \\ v'(t) + \sum_{i=1}^{n} b_i J_{0_+}^{1-\beta_i} \frac{dv(t)}{dt} &= J_{0_+}^{1-\gamma_2} g(u(s), v(s)). \end{aligned} \tag{3.16}$$

Since $(u(0), v(0)) = (u_0, v_0)$, we conclude that (u, v) is a classical solution for (1.1)-(1.2) on $(0, h)$, and this solution may be extended (see [**30**]) to a maximal interval $(0, T_{\max})$ with the alternative: either $T_{\max} = +\infty$ and the solution is global; or

$$T_{\max} < +\infty \quad \text{and} \quad \lim_{t \to T_{\max}} (|u(t)| + |v(t)|) = +\infty.$$

Next, we shall prove uniqueness. Assume that the Cauchy problem (1.1)-(1.2) admits two classical solutions (u_1, v_1) and (u_2, v_2) with the same initial data (u_0, v_0) on $(0, T_{\max})$. Observe that for all $t \in (0, \rho)$ with $\rho < T_{\max}$, these solutions satisfy the following equalities:

$$\begin{aligned} (u_1 - u_2)_t + \sum_{i=1}^{n} a_i D_{0+}^{\alpha_i} (u_1 - u_2) &= J_{0^+}^{1-\gamma_1} (f(u_1, v_1) - f(u_2, v_2)), \\ (v_1 - v_2)_t + \sum_{i=1}^{n} b_i D_{0+}^{\beta_i} (v_1 - v_2) &= J_{0^+}^{1-\gamma_2} (g(u_1, v_1) - g(u_2, v_2)). \end{aligned} \tag{3.17}$$

Integrating (3.17) over $(0,t)$ yields

$$\begin{aligned}
&(u_1-u_2)(t)+\int_0^t \sum_{i=1}^{n} a_i D_{0_+}^{\alpha_i}(u_1-u_2)(s))\,ds \\
&=\int_0^t J_{0^+}^{1-\gamma_1}(f(u_1(s),v_1(s))-f(u_2(s),v_2(s)))ds \\
&(v_1-v_2)(t)+\int_0^t \sum_{i=1}^{n} b_i\ D_{0_+}^{\beta_i}(u_1-u_2)(s))\,ds \\
&=\int_0^t J_{0^+}^{1-\gamma_2}(g(u_1(s),v_1(s))-g(u_2(s),v_2(s)))ds.
\end{aligned} \tag{3.18}$$

Let $\theta := \max\{\alpha_1,\alpha_2,\dots,\alpha_n,\beta_1,\beta_2,\dots,\beta_n,\gamma_1,\gamma_2\}$. Using (3.18) and the fact that f and g are locally Lipshitz on $[0,h]$, thanks to (3.6) and (3.7), for all $t\in(0,\rho)$, we have

$$\begin{aligned}
|u_1(t)-u_2(t) \leq\ & \textstyle\int_0^t \left(\sum_{i=1}^n \frac{a_i}{\Gamma(1-\alpha_i)}(t-s)^{-\alpha_i}+L_1\frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)}\right)\|u_1(s)-u_2(s),v_1(s)-v_2(s))\|ds \\
\leq\ & \textstyle\int_0^t \Big\{ \sum_{i=1}^n \frac{a_i}{\Gamma(1-\alpha_i)}(t-s)^{\theta-\alpha_i} \\
+\ & \textstyle\frac{L_1}{\Gamma(1-\gamma_1)}(t-s)^{\theta-\gamma_1}\}(t-s)^{-\theta}\|u_1(s)-u_2(s),v_1(s)-v_2(s))\|ds \\
\leq\ & \textstyle d_1\int_0^t (t-s)^{-\theta}\|u_1(s)-u_2(s),v_1(s)-v_2(s))\|ds
\end{aligned} \tag{3.19}$$

where

$$d_1 := n\bar{a}\max_{1\leq i\leq n}\Big\{\frac{1}{\Gamma(1-\alpha_i)}\rho^{\theta-\alpha_i}\Big\}+\frac{L_1}{\Gamma(1-\gamma_1)}\rho^{\theta-\gamma_1},$$

and

$$\|u_1(t)-u_2(t),\ v_1(t)-v_2(t))\| = |u_1(t)-u_2(t)|+|v_1(t)-v_2(t)|.$$

Similarly,

$$
\begin{aligned}
|v_1(t)-v_2(t)| &\le \int_0^t \Big(\sum_{i=1}^n \frac{b_i}{\Gamma(1-\beta_i)}(t-s)^{-\beta_i} \\
&\qquad + L_2\frac{(t-s)^{-\gamma_2}}{\Gamma(1-\gamma_2)}\Big)\|u_1(s)-u_2(s),v_1(s)-v_2(s))\|ds \\
&\le \int_0^t \Big\{\sum_{i=1}^n \frac{b_i}{\Gamma(1-\beta_i)}(t-s)^{\theta-\beta_i} + \frac{L_2}{\Gamma(1-\gamma_2)}(t-s)^{\theta-\gamma_2}\Big\} \\
&\qquad \times (t-s)^{-\theta}\|u_1(s)-u_2(s),v_1(s)-v_2(s))\|ds \\
&\le d_2\int_0^t (t-s)^{-\theta}\|u_1(s)-u_2(s),v_1(s)-v_2(s)\|ds, \qquad (3.20)
\end{aligned}
$$

where

$$
d_2 := n\bar{b}\max_{1\le i\le n}\Big\{\frac{1}{\Gamma(1-\beta_i)}\rho^{\theta-\beta_i}\Big\} + \frac{L_2}{\Gamma(1-\gamma_2)}\rho^{\theta-\gamma_2}.
$$

Then from (3.19) and (3.20), we find

$$
\begin{aligned}
&\|(u_1(t)-u_2(t),v_1(t)-v_2(t)\| \\
&\le (d_1+d_2)\int_0^t (t-s)^{-\theta}\|u_1(s)-u_2(s),v_1(s)-v_2(s)\|\,ds \quad \forall t\in(0,\rho).
\end{aligned} \tag{3.21}
$$

Finally using Gronwall's inequality (see [**74**, p. 6]), we deduce the uniqueness and this completes the proof. □

4. Blow up results

Our main result of this study concerns the blow up of solutions of the system (1.1)-(1.2) whenever the nonlinear terms satisfy certain growth conditions. Our main result reads as follows.

THEOREM 13. *Assume that the assumptions of Theorem 12 hold, and that the functions f and g satisfy the growth conditions:*

$$
f(\xi,\eta) \ge a|\eta|^q, \quad \text{for all } \xi,\eta\in\mathbb{R}, \tag{4.1}
$$

$$
g(\xi,\eta) \ge b|\xi|^p, \quad \text{for all } \xi,\eta\in\mathbb{R}, \tag{4.2}
$$

for some positive constants a, b. Then for all positive initial data, the solution of (1.1)-(1.2) *blows up in a finite time.*

Proof. We proceed by contradiction. We assume that $T_{\max} = +\infty$ and we consider the function used in [**56**]

$$\phi(t) = \begin{cases} T^{-\lambda}(T-t)^{\lambda} & \text{for } t \in [0,T],\ \lambda \gg 1, \\ 0 & \text{for } t > T. \end{cases} \tag{4.3}$$

Then by multiplying the first equation in (1.1) by ϕ and integrating over $(0,T)$, we obtain

$$\begin{aligned} &\int_0^T u_t(t)\phi(t)dt + \int_0^T \sum_{i=1}^{n} a_i (D_{0_+}^{\alpha_i}(u(t)-u_0))\phi(t)dt \\ &= \int_0^T (J_{0_+}^{1-\gamma_1} f(u(t),v(t)))\phi(t)dt. \end{aligned} \tag{4.4}$$

Let

$$\psi(t) := \int_0^t \phi(s)ds = -\frac{1}{\lambda+1}T^{-\lambda}(T-t)^{\lambda+1} \quad t \in [0,T].$$

Integrating by parts, and since $\psi(T) = 0$, yields

$$\begin{aligned} \int_0^T (J_{0_+}^{1-\gamma_1} f(u(t),v(t)))\phi(t)dt &= -\int_0^T \frac{d}{dt}(J_{0_+}^{1-\gamma_1} f(u(t),v(t)))\psi(t)dt \\ &= -\int_0^T (D_{0_+}^{\gamma_1} f(u(t),v(t)))\psi(t)dt \\ &= -\int_0^T (D_{T_-}^{\gamma_1}\psi(t)) f((u(t),v(t))dt. \end{aligned} \tag{4.5}$$

Recall (see [**56**]) the formulas

$$D_{T^-}^{\gamma_j}\phi(t) = C_{\lambda,\gamma_j} T^{-\lambda}(T-t)^{\lambda-\gamma_j}, \quad \text{where} C_{\lambda,\gamma_j} = \frac{\lambda\Gamma(\lambda-\gamma_j)}{\Gamma(\lambda-2\gamma_j+1)},$$

and

$$D_{T^-}^{\gamma_j}\psi(t) = -\frac{1}{\lambda+1}C_{\lambda+1,\gamma_j}T^{-\lambda}(T-t)^{\lambda+1-\gamma_j} = -C'_{\lambda,\gamma_j}\phi(t)(T-t)^{1-\gamma_j}, \tag{4.6}$$

for $j=1,2$, where $C'_{\lambda,\gamma_j}=\frac{1}{\lambda+1}C_{\lambda+1,\gamma_j}$, $j=1,2$. Then

$$-\int_0^T (D^{\gamma_1}_{0_+}\psi(t))f(u(t),v(t))dt=\int_0^T C'_{\lambda,\gamma_1}\phi(t)(T-t)^{1-\gamma_1}f(u(t),v(t))dt. \tag{4.7}$$

From (4.4), (4.5) and (4.7) and since u_0 is positive and ϕ is in $C^1([0,T])$, thanks to (4.3), an integration by parts yields

$$\begin{aligned}&C'_{\lambda,\gamma_1}\int_0^T \phi(t)(T-t)^{1-\gamma_1}f(u(t),v(t))dt\\ &\leq -\int_0^T u(t)\phi'(t)dt+\sum_{i=1}^n\int_0^T u(t)D^{\alpha_i}_{T_-}(a_i\phi(t))dt.\end{aligned} \tag{4.8}$$

Observe that if p' is the conjugate of p, then

$$\begin{aligned}\int_0^T u(t)(-\phi'(t))dt &= \int_0^T u(t)(\phi(t))^{\frac{1}{p}}(\phi(t))^{-1/p}(T-t)^{\frac{1-\gamma_2}{p}}(T-t)^{\frac{-(1-\gamma_2)}{p}}(-\phi'(t))dt\\ &\leq C'_{\lambda,\gamma_2}\frac{b}{4}\int_0^T |u(t)|^p\phi(t)(T-t)^{1-\gamma_2}\,dt\\ &+ \left(\frac{4}{bC'_{\lambda,\gamma_2}}\right)^{p'/p}\int_0^T(\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|(\phi'(t))|^{p'}dt\\ &\leq C'_{\lambda,\gamma_2}\frac{1}{4}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}dt\\ &+ \left(\frac{4}{bC'_{\lambda,\gamma_2}}\right)^{p'/p}\int_0^T(\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|\phi'(t)|^{p'}dt,\end{aligned} \tag{4.9}$$

and for all $1 \leq i \leq n$,

$$
\begin{aligned}
&\int_0^T u(t)(D_{T_-}^{\alpha_i}(a_i\phi(t))dt \\
&= \int_0^T u(t)(\phi(t))^{\frac{1}{p}}(\phi(t))^{-\frac{1}{p}}(T-t)^{\frac{1-\gamma_2}{p}}(T-t)^{\frac{-(1-\gamma_2)}{p}} D_{T_-}^{\alpha_i}(a_i\phi(t))dt \\
&\leq C'_{\lambda,\gamma_2}\frac{b}{4n}\int_0^T |u(t)|^p\phi(t)(T-t)^{1-\gamma_2}dt \\
&\quad + \Big(\frac{4n}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p} a_i^{p'}\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|(D_{T_-}^{\alpha_i}\phi(t))|^{p'}dt \\
&\leq C'_{\lambda,\gamma_2}\frac{1}{4n}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}dt \\
&\quad + \Big(\frac{4n}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}\bar{a}^{p'}\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)p'/p}|(D_{T_-}^{\alpha_i}\phi(t))|^{p'}dt.
\end{aligned}
\tag{4.10}
$$

Furthermore,

$$
\begin{aligned}
&C'_{\lambda,\gamma_1}\int_0^T f(u(t),v(t))\phi(t)(T-t)^{1-\gamma_1}dt \\
&\leq \frac{1}{2}C'_{\lambda,\gamma_2}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}\,dt \\
&\quad + \Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|\phi'(t)|^{p'}dt \\
&\quad + \bar{a}^{p'}\Big(\frac{4n}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}\sum_{i=1}^n\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}\;|D_{T_-}^{\alpha_i}\phi(t))|^{p'}dt.
\end{aligned}
\tag{4.11}
$$

Analogously, if q' is the conjugate of q, we obtain

$$
\begin{aligned}
&C'_{\lambda,\gamma_2}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}dt \\
&\leq -\int_0^T v(t)\phi'(t)dt + \sum_{i=1}^n\int_0^T v(t)D_{T_-}^{\beta_i}(\;b_i(t)\phi(t))dt \\
&\leq \frac{1}{2}C'_{\lambda,\gamma_1}\int_0^T f(u(t),v(t))\phi(t)(T-t)^{1-\gamma_1}dt \\
&\quad + \Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}\int_0^T (\phi(t))^{-q'/q}(T-t)^{-(1-\gamma_1)q'/q}|\phi'(t)|^{q'}dt \\
&\quad + \bar{b}^{q'}\Big(\frac{4n}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}\sum_{i=1}^n\int_0^T (\phi(t))^{-q'/q}(T-t)^{-(1-\gamma_1)q'/q}|D_{T_-}^{\beta_i}\phi(t)|^{q'}dt.
\end{aligned}
\tag{4.12}
$$

Denote

$$
\begin{aligned}
A &:= C'_{\lambda,\gamma_1}\int_0^T f(u(t),v(t))\phi(t)(T-t)^{1-\gamma_1}dt,\\
B &:= C'_{\lambda,\gamma_2}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}dt,\\
C &:= \int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|\phi'(t)\ |^{p'}dt,\\
D &:= \int_0^T (\phi(t))^{-q'/q}(T-t)^{-(1-\gamma_1)q'/q}|\phi'(t)|^{q'}dt,\\
E &:= \int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}\sum_{i=1}^n |D^{\alpha_i}_{T-}\phi(t)|^{p'}dt,\\
F &:= \int_0^T (\phi(t))^{-q'/q}(T-t)^{-(1-\gamma_1)q'/q}\sum_{i=1}^n |D^{\beta_i}_{T-}\phi(t)|^{q'}dt.
\end{aligned}
$$

From (4.11) and (4.12) we have

$$
\begin{aligned}
A &\leq \frac{1}{2}B+\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C+n^{p'/p}\bar{a}^{p'}E),\\
B &\leq \frac{1}{2}A+\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F),
\end{aligned}
$$

then

$$
\begin{aligned}
A &\leq \frac{1}{2}\Big(\frac{1}{2}A+\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F)\Big)+\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C+n^{\frac{p'}{p}}\bar{a}^{p'}E)\\
&= \frac{1}{4}A+\frac{1}{2}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F)+\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C+n^{\frac{p'}{p}}\bar{a}^{p'}E);
\end{aligned}
$$

thus

$$
A \leq \frac{2}{3}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F)+\frac{4}{3}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C+n^{\frac{p'}{p}}\bar{a}^{p'}E)
$$

and

$$B \le \frac{1}{2}\Big(\frac{2}{3}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F)+\frac{4}{3}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C+n^{\frac{p'}{p}}\bar{a}^{p'}E)\Big)$$
$$+\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F)$$
$$\le \frac{4}{3}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F)+\frac{2}{3}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C+n^{\frac{p'}{p}}\bar{a}^{p'}E).$$

Taking into account (4.4), (4.9) and (4.10), we deduce that

$$\begin{aligned} u_0\int_0^T D^{\alpha_1}_{T^-}\phi(t)dt &= \frac{u_0}{a_1}\int_0^T D^{\alpha_1}_{T^-}(a_1\phi(t))dt \\ &\le \frac{1}{a_1}\Big(-\int_0^T u(t)\ \phi'(t)dt+\int_0^T\sum_{i=1}^n u(t)\ D^{\alpha_i}_{T^-}(a_i\phi(t))\,dt\Big) \\ &\le \frac{1}{a_1}\Big(\frac{1}{2}B+\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C+n^{\frac{p'}{p}}\bar{a}^{p'}E)\Big) \\ &\le \frac{1}{a_1}\Big(\frac{2}{3}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F)+\frac{4}{3}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C+n^{\frac{p'}{p}}\bar{a}^{p'}E)\Big). \end{aligned}$$

For $\lambda>\max\{\frac{p'}{p}+p'-1,\frac{q'}{q}+q'-1\}$, it holds

$$\int_0^T D^{\alpha_i}_{T^-}\phi(t)dt = C_{\alpha_i,\lambda}T^{1-\alpha_i}, \tag{4.13}$$

where

$$C_{\alpha_i,\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha_i+2)},\quad \forall 1\le i\le n.$$

Also there exists a positive constant K such that

$$C\le KT^{(\gamma_2-1)\frac{p'}{p}+1-p'},\quad D\le KT^{(\gamma_1-1)\frac{q'}{q}+1-q'},$$
$$E\le K\sum_{i=1}^n T^{(\gamma_2-1)\frac{p'}{p}+1-p'\alpha_i},\quad F\le K\sum_{i=1}^n T^{(\gamma_1-1)\frac{q'}{q}+1-q'\beta_i},\quad \forall 1\le i\le n. \tag{4.14}$$

Consequently,

$$
\begin{aligned}
&u_0\int_0^T D_{T_-}^{\alpha_1}\phi(t)dt\\
&\leq \frac{2}{3a_1}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}K\Big(T^{(\gamma_1-1)\frac{q'}{q}+1-q'}+n^{q'/q}\bar{b}^{q'}\sum_{i=1}^{n}T^{(\gamma_1-1)\frac{q'}{q}+1-q'\beta_i}\Big)\\
&\quad+\frac{4}{3a_1}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}K\Big(T^{(\gamma_2-1)\frac{p'}{p}+1-p'}+n^{\frac{p'}{p}}\bar{a}^{p'}\sum_{i=1}^{n}T^{(\gamma_2-1)\frac{p'}{p}+1-p'\alpha_i}\Big).
\end{aligned}
\tag{4.15}
$$

Using (4.13) and (4.15), we obtain

$$
\begin{aligned}
u_0\leq{}& C_{\alpha_1,\lambda}^{-1}K\Big\{\frac{2}{3a_1}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}\Big(T^{(\gamma_1-1)\frac{q'}{q}+\alpha_1-q'}\\
&+n^{q'/q}\bar{b}^{q'}\sum_{i=1}^{n}T^{(\gamma_1-1)\frac{q'}{q}+\alpha_1-q'\beta_i}\Big)\Big\}\\
&+C_{\alpha_1,\lambda}^{-1}K\Big\{\frac{4}{3a_1}\Big(\frac{4}{bC'_{\lambda,\gamma_1}}\Big)^{p'/p}\Big(T^{(\gamma_2-1)\frac{p'}{p}+\alpha_1-p'}\\
&+n^{\frac{p'}{p}}\bar{a}^{p'}\sum_{i=1}^{n}T^{(\gamma_2-1)\frac{p'}{p}+\alpha_1-p'\alpha_i}\Big)\Big\}.
\end{aligned}
\tag{4.16}
$$

Similarly we obtain

$$
\begin{aligned}
v_0\int_0^T D_{T_-}^{\beta_1}\phi(t)dt &\leq \frac{1}{b_1}\Big(\frac{1}{2}A+\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F)\Big)\\
&\leq C_{\beta,\lambda}^{-1}\Big(\frac{4}{3b_1}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D+n^{q'/q}\bar{b}^{q'}F)\\
&+\frac{2}{3b_1}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C+n^{\frac{p'}{p}}\bar{a}^{p'}E)\Big),
\end{aligned}
\tag{4.17}
$$

which yields

$$
\begin{aligned}
v_0 \leq\, & C_{\beta_1,\lambda}^{-1} K\Big\{\frac{4}{3b_1}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}\Big(T^{(\gamma_1-1)\frac{q'}{q}+\beta_1-q'} \\
& + n^{q'/q}\bar{b}^{q'}\sum_{i=1}^{n} T^{(\gamma_1-1)\frac{q'}{q}+\beta_1-q'\beta_i}\Big)\Big\} \\
& + C_{\beta_1,\lambda}^{-1} K\Big\{\frac{2}{3b_1}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}\Big(T^{(\gamma_2-1)\frac{p'}{p}+\beta_1-p'} \\
& + n^{\frac{p'}{p}}\bar{a}^{p'}\sum_{i=1}^{n} T^{(\gamma_2-1)\frac{p'}{p}+\beta_1-p'\alpha_i}\Big)\Big\}.
\end{aligned}
\tag{4.18}
$$

One can observe that

$$
\begin{gathered}
(\gamma_1-1)\frac{q'}{q}+\alpha_1-q'<0, \quad (\gamma_2-1)\frac{p'}{p}+\alpha_1-p'<0, \\
(\gamma_1-1)\frac{q'}{q}+\beta_1-q'<0, \quad (\gamma_2-1)\frac{p'}{p}+\beta_1-p'<0, \\
(\gamma_2-1)\frac{p'}{p}+\alpha_1-p'\alpha_i<0, \quad (\gamma_1-1)\frac{q'}{q}+\beta_1-q'\beta_i<0, \quad \forall 1\leq i\leq n, \\
(\gamma_2-1)\frac{p'}{p}+\beta_1-p'\alpha_i<\beta_1-\alpha_1, \quad (\gamma_1-1)\frac{q'}{q}+\alpha_1-q'\beta_i<\alpha_1-\beta_1, \\
\forall 1\leq i\leq n.
\end{gathered}
\tag{4.19}
$$

Inequalities (4.19) reduce to

$$
(\gamma_2-1)\frac{p'}{p}+\beta_1-p'\alpha_i<0, \quad \forall 1\leq i\leq n,
$$

or

$$
(\gamma_1-1)\frac{q'}{q}+\alpha_1-q'\beta_i<0, \quad \forall 1\leq i\leq n.
$$

Taking the limit when T approaches infinity in (4.16) and (4.18), we find

$$
0<u_0\leq 0 \quad \text{or} \quad 0<v_0\leq 0. \tag{4.20}
$$

This leads to a contradiction and consequently the maximal time of existence for the solution to (1.1)-(1.2) is finite and this completes the proof. □

CHAPTER 3

A General Decay and Optimal Decay Result in a Heat System with a Viscoelastic Term

This chapter is the subject of a submitted paper :

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A General Decay and Optimal Decay Result in a Heat System with a Viscoelastic Term.

Abstract. We consider a quasilinear heat system in the presence of an integral term and establish a general and optimal decay result which improves and generalizes several stability results in the literature.

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Keywords and phrases. General decay, Optimal decay, Relaxation function, Viscoelastic quasilinear heat equation

1. Introduction

In this work, we consider the following problem:

$$\begin{cases} A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\, ds = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{in } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $m \geq 2$, Ω is a bounded domain of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive nonincreasing function, and $A : \mathbb{R}^+ \to M_n(\mathbb{R})$ is a bounded square matrix satisfying $A \in C(\mathbb{R}^+)$ and, for some positive constant c_0,

$$(A(t)v, v) \geq c_0|v|^2, \qquad \forall t \in \mathbb{R}^+,\ \forall v \in \mathbb{R}^n, \tag{1.2}$$

where $(.,.)$ and $|.|$ are the inner product and the norm, respectively, in $\mathbb{R}^n$. The equation in consideration arises from various mathematical models in engineering and physics. For instance, in the study of heat conduction in materials with memory, the classical Fourier law is replaced by the following form (cf. [**145**]):

$$q = -d\nabla u - \int_{-\infty}^{t} \nabla \left(k(x,t)u(x,\tau)\right)\, d\tau,$$

where u is the temperature, d the diffusion coefficient and the integral term represents the memory effect in the material. This type of problems has considered by a number of researchers; see [**145, 158, 177**] and the references therein. From a mathematical point of view, we expect that the integral term would be dominated by the leading term in the equation, so that the theory of parabolic equation can be applied. In fact, this has been confirmed by the work of Yin [**177**], in which he considered a general equation of the form

$$u_t = divA(x,t,u,u_x) + a(x,t,u,u_x) + \int_0^t divB(x,t,\tau,u,u_x)\, d\tau$$

and proved the existence of a unique weak solution under suitable conditions on A, B and a. See more results concerning global existence and asymptotic behavior in Nakao and Ohara [**142**], Nakao and Chen [**143**], and Engler *et al.* [**47**]. Pucci and Serrin [**159**] discussed the following system:

$$A(t)|u_t|^{m-2}u_t = \Delta u - f(x,u),$$

for $m > 1$ and f satisfying

$$(f(x,u),u) \geq 0$$

and showed that strong solutions tend to the rest state as $t \to +\infty$, however, no rate of decay has been given. Berrimi and Messaoudi [1] showed that, if A satisfies (1.2), then solutions with small initial energy decay exponentially for $m = 2$ and polynomially if $m > 2$. Messaoudi and Tellab [**124**] considered (1.1), under condition (1.2) and for relaxation function g satisfying a general decay condition of the form

$$g'(t) \leq -\xi(t)g(t), \quad \forall t \in \mathbb{R}^+,$$

for some nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$, and established a general decay result, from which the exponential and polynomial decay rates of [**13**] are only special cases. Recently, Liu and Chen [**107**] investigated (1.1), with a nonlinear source term, and established a general decay result under suitable conditions on g and the nonlinear source term. They also proved a blow-up result for the solution with both positive and negative initial energy.

In this work, we discuss (1.1) when g is of a more general decay, and establish a general and optimal decay result, which improves those of Berrimi and Messaoudi [**13**], Liu and Chen [**107**], and Messaoudi and Tellab [**124**].

2. Preliminaries

In this section, we present some material needed in the proof of our result. For the relaxation function g we assume that

(G_1) The function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function satisfying

$$g(0) > 0 \quad \text{and} \quad 1 - \int_0^{+\infty} g(s)\, ds = l > 0.$$

(G_2) There exist a constant $p \in [1, 3/2)$ and a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \in \mathbb{R}^+.$$

(G_3) We also assume that

$$\begin{aligned} 2 &\leq m \leq \frac{2n}{n-2} \qquad \text{if } n \geq 3, \\ m &\geq 2 \qquad\qquad\quad \text{if } n = 1,2. \end{aligned}$$

REMARK 14. *There are many functions satisfying (G1) and (G2). Examples of such functions are, for $b>0$, $\alpha>0$, $\nu>1$ and $a>0$ small enough,*

$$g_1(t) = ae^{-b(t+1)^{\alpha}} \qquad \textit{and} \qquad g_2(t) = \frac{a}{(1+t)^{\nu}}.$$

We will also be using the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, $L^r(\Omega) \hookrightarrow L^q(\Omega)$, for $2 \leq q \leq r < +\infty$, and Poincaré's inequality. The same embedding constant C_* will be used, and C denotes a generic positive constant.

We introduce the following:

$$E(t) = \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right) ||\nabla u(t)||_2^2, \quad \forall t \in \mathbb{R}^+, \tag{2.1}$$

where $||.||_q = ||.||_{(L^q(\Omega))^n}$, for $1 \leq q < +\infty$, and

$$(g \circ \nabla u)(t) = \int_0^t g(t-\tau)||\nabla u(.,t) - \nabla u(.,\tau)||_2^2\, d\tau, \quad \forall t \in \mathbb{R}^+. \tag{2.2}$$

Similarly to [**159**], we give the definition of a weak solution of (1.1).

DEFINITION 15. *A weak solution of* (1.1) *on* $[0,T]$ *is a function*

$$u \in C\left([0,T);(H_0^1(\Omega))^n\right) \cap C^1\left((0,T);(L^m(\Omega))^n\right)$$

which satisfies

$$\int_0^t \int_\Omega \left(\nabla u(x,s) - \int_0^s \nabla u(x,\tau) d\tau\right) \cdot \nabla\phi(x,s)\, dx\, ds$$
$$+ \int_0^t A(s)|u_t|^{m-2}u_t(x,s) \cdot \phi(x,s)\, dx\, ds = 0,$$

for all t in $[0,T)$ and all ϕ in $C\left([0,T);(H_0^1(\Omega))^n\right)$.

REMARK 16. *Similarly to [**159**], we assume the existence of a solution. For the linear case ($m=2$), one can easily establish the existence of a weak solution by the Galerkin method. In the one-dimensional case ($n=1$), the existence is established in a more general setting by Yin [**177**].*

We state an important lemma [**125**].

LEMMA 17. Assume that g satisfies $(G1)$ and $(G2)$ and u is the solution of (1.1), then there exists a positive constant k_0 such that

$$\xi(t)(g\circ\nabla u)(t)\le k_0\left(-E'(t)\right)^{\frac{1}{2p-1}},\quad \forall t\in\mathbb{R}^+. \tag{2.3}$$

We also recall the following particular case of the well-known Jensen inequality which will be of essential use in obtaining our result: let $f:\Omega\to\mathbb{R}^+$ and $h:\Omega\to\mathbb{R}^+$ be integrable functions on Ω such that

$$\int_\Omega h(x)\ dx = k>0.$$

Then, for any $p>1$, we have

$$\frac{1}{k}\int_\Omega (f(x))^{\frac{1}{p}}h(x)\ dx\le\left(\frac{1}{k}\int_\Omega f(x)h(x)\ dx\right)^{\frac{1}{p}}. \tag{2.4}$$

3. Decay results

In this section, we state and prove our main result. We start with these lemma.

LEMMA 18. Let u be the solution of (1.1). Then the energy satisfies

$$E'(t)=-\int_\Omega A(t)|u_t|^m\ dx-\frac{1}{2}g(t)||\nabla u(t)||_2^2+\frac{1}{2}(g^{'}\circ\nabla u)(t)\le 0,\quad \forall t\in\mathbb{R}^+. \tag{3.1}$$

Proof. By multiplying the first equation in (1.1) by u_t and integrating over Ω, we get (3.1), after routine manipulations. $\square$

Lemma 19. . Let u be a solution of problem (1.1). Then, for any $\delta > 0$, we have

$$\|\nabla u(t)\|_2^2 \le c_4 \delta E(t) - \frac{C_\delta}{c_0} E'(t) + c_5 (g \circ \nabla u)(t), \quad \forall t \in \mathbb{R}^+, \tag{3.2}$$

where c_0 is introduced in (1.2), c_4 and c_5 are two positive constants, and C_δ is a positive constant depending on δ.

Proof. Multiplying the first equation in (1.1) by u and integrating over Ω, we get

$$\|\nabla u(t)\|_2^2 = -\int_\Omega A(t)|u_t|^{m-2} u_t u(x,t)\, dx + \int_\Omega \int_0^t g(t-s)\nabla u(x,s)\cdot \nabla u(x,t)\, ds\, dx. \tag{3.3}$$

Now, we estimate the right-hand side of (3.3). By using Young's and Poincaré's inequalities, the boundedness of A, conditions (G_1) and (G_3), and the fact that

$$E(t) \le E(0),$$

we find, for any $\delta > 0$,

$$\begin{aligned} -\int_\Omega A(t)|u_t|^{m-2} u_t u\, dx &\le \delta \|u(.,t)\|_m^m + C_\delta \|u_t(.,t)\|_m^m \\ &\le \delta C_*^m \|\nabla u(.,t)\|_2^m + C_\delta \|u_t(.,t)\|_m^m \\ &\le \delta C_*^m \left(\frac{2E(0)}{l}\right)^{\frac{m-2}{2}} \left(\frac{2}{l}E(t)\right) + C_\delta \|u_t(.,t)\|_m^m \\ &\le c_1 \delta E(t) - \frac{C_\delta}{c_0} E'(t). \end{aligned} \tag{3.4}$$

Next, we estimate the second term of the right-hand side of (3.3) carefully. By Young's inequality, we easily see that

$$\int_\Omega \nabla u(x,t)\ .\int_0^t g(t-s)\nabla u(x,s)\ ds\ dx \leq \frac{1}{2}\|\nabla u(.,t)\|_2^2$$
$$+\ \ \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(x,s)-\nabla u(x,t)|\ +|\nabla u(x,t)|\right)\ ds\ \right)^2\ dx. \tag{3.5}$$

Using the fact that

$$\int_0^t g(s)\ ds \leq 1-l$$

and Young's and Hölder's inequalities, we obtain, for any $\eta>0$,

$$\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(x,s)-\nabla u(x,t)|\ +|\nabla u(x,t)|\right)\ ds\ \right)^2\ dx$$

$$\begin{aligned}
= & \int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(s)-\nabla u(t)|\right)\right)^2\ dx+\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)|\ ds\right)^2\ dx\\
& +2\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)|)\ ds\right)\ \ \left(\int_0^t g(t-s)|\nabla u(t)|\ ds\right)\ dx
\end{aligned}$$

$$\begin{aligned}
\leq\ & (1+\eta)\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)|\ ds\right)^2\ dx+\left(1+\frac{1}{\eta}\right)\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|\ ds\right)^2\ dx\\
\leq\ & (1+\eta)(1-l)^2\|\nabla u(.,t)\|_2^2+\left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t).
\end{aligned} \tag{3.6}$$

Substuting (3.6) in (3.5), we get

$$\begin{aligned}
\int_\Omega \nabla u(x,t)\ .\int_0^t g(t-s)\nabla u(x,s)\ ds\ dx\ \ \leq\ & \frac{1}{2}\left(1+(1+\eta)(1-l)^2\right)\|\nabla u(.,t)\|_2^2\\
+\ & \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t).
\end{aligned} \tag{3.7}$$

Combining (2.4), (3.4) and (3.7), we find

$$\begin{aligned} \|\nabla u(.,t)\|_2^2 &\leq c_1\delta E(t) - \frac{C_\delta}{c_0}E'(t) \\ &+ \frac{1}{2}\left(1+(1+\eta)(1-l)^2\right)\|\nabla u(.,t)\|_2^2 \\ &+ \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t). \end{aligned} \tag{3.8}$$

We then choose $0<\eta<l(2-l)/(1-l)^2$, which makes $c_2=\frac{1}{2}\left(1+(1+\eta)(1-l)^2\right)<1$, and, therefore, (3.8) takes the form

$$\|\nabla u(.,t)\|_2^2 \leq c_1\delta E(t) - \frac{C_\delta}{c_0}E'(t) + c_2\|\nabla u(.,t)\|_2^2 + c_3(g\circ\nabla u)(t),$$

where $c_3=\frac{1}{2}(1+\frac{1}{\eta})(1-l)$. This yields (3.2) with $c_4=\frac{c_1}{1-c_2}$ and $c_5=\frac{c_3}{1-c_2}$. □

THEOREM 20. Let u be the solution of (1.1). Then, there exist strictly two positive constants λ_0 *and* λ_1 such that the energy satisfies, for all $t\in \mathbb{R}^+$,

$$E(t)\leq \lambda_0 e^{-\lambda_1\int_0^t \xi(s)\ ds} \qquad \textit{if } p=1, \tag{3.9}$$

$$E(t)\leq \lambda_0\left(1+\int_0^t \xi^{2p-1}(s)\ ds\right)^{\frac{-1}{2p-2}} \qquad \textit{if } p>1. \tag{3.10}$$

Moreover, if ξ and p in (G_2) satisfy

$$\int_0^{+\infty}\left(1+\int_0^t \xi^{2p-1}(s)\ ds\right)^{\frac{-1}{2p-2}} dt<+\infty, \tag{3.11}$$

then, for all $t\in\mathbb{R}^+$,

$$E(t)\leq \lambda_0\left(1+\int_0^t \xi^{p}(s)\ ds\right)^{\frac{-1}{p-1}} \qquad \textit{if } p>1. \tag{3.12}$$

REMARK 21. *Estimates* (3.10) *and* (3.11) *yield*

$$\int_0^{+\infty} E(t)\ dt<+\infty. \tag{3.13}$$

Proof. From (3.1) and for any $\kappa > 0$, we have

$$\begin{aligned}
E'(t) &\leq 0 = -\kappa E(t) + \kappa E(t) \\
&\leq -\kappa E(t) + \kappa \left(\frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right) \|\nabla u(.,t)\|_2^2 \right) \\
&\leq -\kappa E(t) + \frac{\kappa}{2}(g \circ \nabla u)(t) + \frac{\kappa}{2}\|\nabla u(.,t)\|_2^2.
\end{aligned}$$

Recalling Lemma (19), we get

$$\begin{aligned}
E'(t) &\leq -\kappa E(t) + \frac{\kappa}{2}(g \circ \nabla u)(t) \\
&\quad + \frac{\kappa}{2}\left(c_4 \delta E(t) - \frac{C_\delta}{c_0}E'(t) + c_5 (g \circ \nabla u)(t)\right) \\
&\leq -\kappa \left(1 - \frac{c_4}{2}\delta\right) E(t) - \frac{\kappa C_\delta}{2c_0}E'(t) + \frac{\kappa(1+c_5)}{2}(g \circ \nabla u)(t).
\end{aligned}$$

Then we have

$$\left(1 + \frac{\kappa C_\delta}{2c_0}\right) E'(t) \leq -\kappa \left(1 - \frac{c_4}{2}\delta\right) E(t) + \frac{\kappa(1+c_5)}{2}(g \circ \nabla u)(t).$$

By choosing δ small enough, we obtain, for two positive constants λ and γ,

$$E'(t) \leq -\lambda E(t) + \gamma (g \circ \nabla u)(t). \tag{3.14}$$

Case of $p = 1$. Multiplying (3.14) by $\xi(t)$ and exploiting (G_2), we get

$$\begin{aligned}
\xi(t)E'(t) &\leq -\lambda \xi(t) E(t) + \gamma (\xi g \circ \nabla u)(t) \\
&\leq -\lambda \xi(t) E(t) - \gamma (g' \circ \nabla u)(t) \\
&\leq -\lambda \xi(t) E(t) - \gamma E'(t).
\end{aligned} \tag{3.15}$$

We then set $L = (\xi + \gamma)E \sim E$ to obtain, from (3.15) and the fact that $\xi' \leq 0$,

$$L'(t) \leq -\lambda \xi(t) E(t) \leq -\lambda_1 \xi(t) L(t). \tag{3.16}$$

A simple integration of (3.16) leads to

$$L(t) \leq Ce^{-\lambda_1 \int_0^t \xi(s)\ ds}.$$

This gives (3.9), by virtue of $L \sim E$.

<u>**Case of** $p > 1$</u>. To establish (3.10), we again consider (3.14) and use Lemma 2.1 to get

$$\xi(t)E'(t) \leq -\lambda\xi(t)E(t) + C\left(-E'(t)\right)^{\frac{1}{2p-1}}.$$

Multiplication of the last inequality by $\xi^\alpha E^\alpha(t)$, where $\alpha = 2p-2 > 0$, gives

$$\frac{1}{\alpha+1}\xi^{\alpha+1}\frac{d}{dt}E^{\alpha+1}(t) \leq -\lambda\xi^{\alpha+1}(t)E^{\alpha+1}(t) + c\left(\xi E\right)^\alpha(t)\left(-E'(t)\right)^{\frac{1}{\alpha+1}}.$$

Use of Young's inequality, with $q = \alpha+1$ and $q^* = \frac{\alpha+1}{\alpha}$, yields, for any $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{\alpha+1}\xi^{\alpha+1}\frac{d}{dt}E^{\alpha+1}(t) &\leq -\lambda\xi^{\alpha+1}(t)E^{\alpha+1}(t) + C\left(\varepsilon\xi^{\alpha+1}(t)E^{\alpha+1}(t) - C_\varepsilon E'(t)\right)\\ &= -(\lambda-\varepsilon C)\xi^{\alpha+1}(t)E^{\alpha+1}(t) - C_\varepsilon E'(t).\end{aligned}$$

We then choose $0 < \varepsilon < \frac{\lambda}{C}$ and recall that $\xi' \leq 0$, to obtain, for $c_6 > 0$,

$$\left(\xi^{\alpha+1}E^{\alpha+1}(t)\right)'(t) \leq \xi^{\alpha+1}\frac{d}{dt}E^{\alpha+1}(t) \leq -c_6\xi^{\alpha+1}(t)E^{\alpha+1}(t) - CE'(t);$$

which implies

$$\left(\xi^{\alpha+1}E^{\alpha+1} + CE\right)'(t) \leq -c_6\xi^{\alpha+1}(t)E^{\alpha+1}(t).$$

Let $W = \xi^{\alpha+1}E^{\alpha+1} + CE \sim E$. Then

$$W'(t) \leq -C\xi^{\alpha+1}(t)W^{\alpha+1}(t) = -C\xi^{2p-1}(t)W^{2p-1}(t).$$

Integrating over $(0,t)$ and using the fact that $W \sim E$, we obtain, for some $\lambda_0 > 0$,

$$E(t) \leq \lambda_0\left(\int_0^t \xi^{2p-1}(s)\ ds + 1\right)^{\frac{-1}{2p-2}};$$

so (3.10) holds.

To establish (3.12), we follow the approach of [**125**]

$$\eta(t) = \int_0^t \|\nabla u(t) - \nabla u(t-s)\|_2^2 \, ds.$$

Using Remark (21), we have

$$\begin{aligned}
\eta(t) &\leq 2\int_0^t \left(\|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2\right) \, ds \\
&\leq \frac{4}{1-l}\int_0^t (E(t)+E(t-s)) \, ds \\
&= \frac{8}{1-l}\int_0^t E(s) \, ds < \frac{8}{1-l}\int_0^{+\infty} E(s) \, ds < +\infty.
\end{aligned}$$

This implies that

$$\sup_{t\in\mathbb{R}^+} \eta^{1-\frac{1}{p}}(t) < +\infty. \tag{3.17}$$

Assume that $\eta(t) > 0$. Then, from (3.14), we find

$$\begin{aligned}
\xi(t)E'(t) &\leq -\lambda\xi(t)E(t) + \gamma\xi(t)(g\circ\nabla u)(t) \qquad (3.18)\\
&= -\lambda\xi(t)E(t) + \gamma\frac{\eta(t)}{\eta(t)}\int_0^t (\xi^p(s)g^p(s))^{\frac{1}{p}} \|\nabla u(t) - \nabla u(t-s)\|_2^2 \, ds.
\end{aligned}$$

Applying Jensen's inequality (2.4) for the second term of the right-hand side of (3.18) with

$$\omega = [0,t], \quad f(s) = \xi^p(s)g^p(s) \quad \text{and} \quad h(s) = \|\nabla u(t) - \nabla u(t-s)\|_2^2$$

to get

$$\xi(t)E'(t) \leq -\lambda\xi(t)E(t) + \gamma\eta(t)\left(\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 \, ds\right)^{\frac{1}{p}}.$$

Therefore, using (3.17) we obtain

$$\begin{aligned}\xi(t)E'(t) &\leq -\lambda\xi(t)E(t)+\gamma\eta^{1-\frac{1}{p}}(t)\left(\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\nabla u(t)-\nabla u(t-s)\|_2^2\ ds\right)^{\frac{1}{p}} \\ &\leq -\lambda\xi(t)E(t)+C(-g'\circ\nabla u)^{\frac{1}{p}}(t)\end{aligned}$$

and then

$$\xi(t)E'(t)\leq -\lambda\xi(t)E(t)+C(-E'(t))^{\frac{1}{p}}. \tag{3.19}$$

If $\eta(t)=0$, then $s\to\nabla u(s)$ is a constant function on $[0,t]$. Therefore

$$(g\circ\nabla u)(t)=0,$$

and hence we have, from (3.14),

$$E'(t)\leq -\lambda E(t),$$

which implies (3.19).

Now, multiplying (3.19) by $\xi^{\alpha}(t)E^{\alpha}(t)$, for $\alpha=p-1$, and repeating the same computations as in above, we arrive at, for some $\lambda_0>0$,

$$E(t)\leq \lambda_0\left(\int_0^t \xi^p(s)\ ds+1\right)^{\frac{-1}{p-1}}.$$

This completes the proof of our main result. □

The following examples illustrate our result and show the optimal decay rate in the some polynomial case:

Example 22. *Let $g(t)=a(1+t)^{-\nu}$, where $\nu>2$, and $a>0$ so that*

$$\int_0^{+\infty} g(t)\ dt<1. \tag{3.20}$$

We have

$$g'(t) = -a\nu(1+t)^{-\nu-1} = -b\left(a(1+t)^{-\nu}\right)^{\frac{\nu+1}{\nu}},$$

where $b = \nu a^{-\frac{1}{\nu}}$. *Then* (G_2) *holds with* $\xi(t) = b$ *and* $p = \frac{\nu+1}{\nu} \in (1, \frac{3}{2})$. *Therefore* (3.11) *yields*

$$\int_0^{+\infty} \left(b^{2p-1}t+1\right)^{\frac{-1}{2p-2}}\ dt < +\infty,$$

and hence, by (3.12), *we get*

$$E(t) \leq C(1+t)^{\frac{-1}{p-1}} = C(1+t)^{-\nu},$$

which is the optimal decay.

EXAMPLE 23. *Let* $g(t) = ae^{-(1+t)^{\nu}}$, *where* $0 < \nu \leq 1$, *and* $a > 0$ *is chosen so that* (3.20) *holds. Then*

$$g'(t) = -a\nu(1+t)^{\nu-1}e^{-(1+t)^{\nu}}.$$

Therefore (G_2) *holds with* $p = 1$ *and* $\xi(t) = \nu(1+t)^{\nu-1}$. *Consequently, we can use* (3.9) *to deduce*

$$E(t) \leq Ce^{-\lambda(1+t)^{\nu}}.$$

CHAPTER 4

Stability of an Abstract System with Infinite History

This chapter is the subject of a submitted paper:

Stability of an Abstract System with Infinite History.

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Abstract. This work is concerned with stabilization of an abstract linear dissipative integrodiffrential equation with infinite memory modeling linear viscoelasticity where the relaxation function satisfies $g'(t) \leq -\xi(t)g^p(t),\ \forall t \geq 0,\ 1 \leq p < \frac{3}{2}$ and $\xi : \mathbb{R}_+ \to \mathbb{R}_+$. Our result improves earlier results in the literature.

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1. Introduction

Let us denote by $\mathcal{H}$ a Hilbert space with inner product and related norm denoted by $\langle , \rangle$ and $\|.\|$ respectively. Let $A : \mathcal{D}(A) \longrightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \longrightarrow \mathcal{H}$ be self-adjoint linear positive definite operators with domains $\mathcal{D}(A) \subset \mathcal{D}(B) \subset \mathcal{H}$ such that the embeddings are dense and compact.

We are interested in energy decay of the solution u to the following initial boundary value problem

$$u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)\ ds = 0, \qquad \forall t > 0, \tag{1.1}$$

with initial conditions

$$\begin{cases} u(-t) = u_0(t), & \forall t \in \mathbb{R}_+, \\ \\ u_t(0) = u_1, \\ \end{cases} \tag{1.2}$$

where u_0 and u_1 are given history and initial data, g is a positive and nonincreasing function called the relaxation function.

1.0.1. *Well Posedness.* By following the brilliant intuition of Dafermos [**40, 38**], we introduce the relative history of u defined as

$$\begin{cases} \eta^t(s) = u(t) - u(t-s), & \forall t, s \in \mathbb{R}_+, \\ \\ \eta_0(s) = \eta^0(s) = u_0(0) - u_0(s), & \forall s \in \mathbb{R}_+. \\ \end{cases}$$

Equation (1.1)-(1.2) can be rewritten as an abstract linear first-order system of the form

$$\begin{cases} \mathcal{U}_t + \mathcal{A}\mathcal{U}(t) = 0, & \forall t > 0, \\ \\ \mathcal{U}(0) = \mathcal{U}_0, \\ \end{cases} \tag{1.3}$$

where $\mathcal{U}_0 = (u_0(0), u_1, \eta_0)^T \in \mathcal{H} = \mathcal{D}(A^{\frac{1}{2}}) \times H \times L^2_g(\mathbb{R}_+, \mathcal{D}(B^{\frac{1}{2}}))$, $\mathcal{U} = (u, u_t, \eta^t)^T$ and $L^2_g(\mathbb{R}_+, \mathcal{D}(B^{\frac{1}{2}}))$ is the weighted space with respect to the measure $g(s)ds$ defined by

$$L^2_g(\mathbb{R}_+, \mathcal{D}(B^{\frac{1}{2}})) = \left\{ z : \mathbb{R}_+ \longrightarrow \mathcal{D}(B^{\frac{1}{2}}),\ \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} z(s)\|^2\ ds < +\infty \right\}$$

endowed with the inner product

$$\langle z_1, z_2\rangle_{L^2_g(\mathbb{R}_+,\mathcal{D}(B^{\frac{1}{2}}))} = \int_0^{+\infty} g(s)\langle B^{\frac{1}{2}} z_1(s), B^{\frac{1}{2}} z_2(s)\rangle \; ds.$$

The operator $\mathcal{A}$ is defined by

$$\mathcal{A}(v,w,z)^T = \left(-w, Av - g_0 Bv + \int_0^{+\infty} g(s)Bz(s)\; ds, \frac{\partial z}{\partial s} - w\right)^T,$$

where $g_0 = \int_0^{+\infty} g(s)\; ds$,

$$\mathcal{D}(\mathcal{A}) = \left\{(v,w,z)^T \in \mathcal{H},\; v \in \mathcal{D}(A),\; w \in \mathcal{D}(A^{\frac{1}{2}})),\; z \in \mathcal{L}_g,\; \int_0^{+\infty} g(s)Bz(s)\; ds \in H\right\},$$

and $\mathcal{L}_g = \left\{z \in L^2_g(\mathbb{R}_+,\mathcal{D}(B^{\frac{1}{2}})), \partial_s z \in L^2_g(\mathbb{R}_+,\mathcal{D}(B^{\frac{1}{2}})), z(0)=0\right\}$.

As shown in [**132**] for example, under the assumptions $(\mathbb{H}_1)$ and $(\mathbb{H}_2)$ below, the space $\mathcal{H}$ endowed with the inner product

$$\langle (v_1,w_1,z_1)^T, (v_2,w_2,z_2)^T\rangle_{\mathcal{H}} = \langle A^{\frac{1}{2}}v_1, A^{\frac{1}{2}}v_2\rangle - g_0\left\langle B^{\frac{1}{2}}v_1, B^{\frac{1}{2}}v_2\right\rangle + \langle w_1, w_2\rangle + \langle z_1, z_2\rangle_{L^2_g(\mathbb{R}_+,\mathcal{D}(B^{\frac{1}{2}}))}$$

is a Hilbert space, $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$ with dense embedding, and $\mathcal{A}$ is the infinitesimal generator of a linear contraction $\mathcal{C}_0$ semigroup on $\mathcal{H}$. Therefore, the classical semigroup theory implies that (see [**149**]), for any $\mathcal{U}_0 \subset \mathcal{H}$, the system (1.3) has a unique weak solution

$$\mathcal{U} \in \mathcal{C}(\mathbb{R}_+, \mathcal{H}).$$

Moreover, if $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A})$, then the solution of (1.3) is classical; that is

$$\mathcal{U} \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C(\mathbb{R}_+, \mathcal{D}(\mathcal{A})).$$

1.0.2. *Stability.* Problems related to (1.1)-(1.2) have been studied by many authors and several stability results have been established; see [**36, 51, 67, 150, 151**]. The exponential and polynomial decay of the solutions of equation (1.1)-(1.2) have been studied in [68], where it was assumed that $(\mathbb{H}_1)$ below holds and

- (A_1) There exists an increasing strictly convex function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \qquad and \qquad \lim_{t \longrightarrow +\infty} G'(t) = +\infty,$$

  such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))}\, ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty$$

The author established a general decay estimate given in term of the convex function G. His result generalizes the usual exponential and polynomial decay results found in the literature. He considered two cases corresponding to the following two conditions on A and B:

$$\exists a_2 > 0 : \qquad \|A^{\frac{1}{2}} v\|^2 \leq a_2 \|B^{\frac{1}{2}} v\|^2, \quad \forall v \in \mathcal{D}(A^{\frac{1}{2}}) \tag{1.4}$$

or

$$\exists a_2 > 0 : \qquad \|A^{\frac{1}{2}} v\|^2 \leq a_2 \|A^{\frac{1}{2}} B^{\frac{1}{2}} v\|^2, \quad \forall v \in \mathcal{D}(A^{\frac{1}{2}} B^{\frac{1}{2}}). \tag{1.5}$$

The study of viscoelastic problem (1.1)-(1.2) in the particular case $A = B$ was considered by Guesmia and Messaoudi [69]. The authors considered (1.6) below with $p = 1$ and extended the decay result known for problems with finite history to those with infinite history. In addition, they improved, in some cases, some decay results obtained earlier in [68].

Very recently, the authors of [125] considered the condition

$$g'(t) \leq -\xi(t) g^p(t), \qquad \forall t \in \mathbb{R}_+, \tag{1.6}$$

where ξ is a positive and nonincreasing function and $1 \leq p < \frac{3}{2}$, with the objective of improving the decay rate for problems with finite memory.

Condition (1.6) gives a better description of the growth of g at infinity and allows to obtain a precise estimate of the energy that is more general than the "stronger" one (ξ constant and $p \in [1, \frac{3}{2}[$) used in the case of past history control [**123, 131**]. We also refer the reader to some recent researches under the condition (1.6) with finite history and viscolelastic term [**125**] for related results. The authors proved a general decay rate from which the exponential decay is only a special case. Moreover, the optimal polynomial decay is easily and directly obtained without restrictive conditions.

With the above motivations and inspired by the approach of [**125**], in this paper, we intend to study the general decay result to problem (1.1)-(1.2) under suitable assumptions on the initial data and the relaxation function g. Our main contribution is an enhancement to the results of [**68, 69**] in a way that our result gives a better rate of decay in the polynomial case.

The plan of this paper is as follows. In Section 2, we present some assumptions, preliminaries and some technical lemmas needed to establish the proofs of our results. Section 3 is devoted to the statement and the proof of our main result.

2. Preliminaries

In this section, we shall present some necessary assumptions and prove some important inequalities that will become useful in later stages.
Let us assume that
($\mathbb{H}_1$) There exist positive constants a_0 and a_1 such that

$$a_1\|v\|^2 \leq \|B^{\frac{1}{2}}v\|^2 \leq a_0\|A^{\frac{1}{2}}v\|^2, \quad \forall v \in \mathcal{D}(A^{\frac{1}{2}}).$$

($\mathbb{H}_2$) $g:\mathbb{R}_+\to\mathbb{R}_+$ is a differentiable nonincreasing function satisfying

$$0<g_0<\frac{1}{a_0}.$$

($\mathbb{H}_3$) There exists a nonincreasing differentiable function $\xi:\ \mathbb{R}_+\to\mathbb{R}_+$ and $1<p<\frac{3}{2}$ satisfying (1.6).

Throughout the sequel, we denote by C a generic positive constant which may vary from line to line. We start with the follwing lemma.

Lemma 24. *Let F,h be two non-negative functions such that F is derivable and h is continuous on $\mathbb{R}_+$, and α, c_1 and c_2 be three positive constants such that*

$$F'(t)\leq -c_1\xi^{\alpha+1}(t)F^{\alpha+1}(t)+c_2h^{\alpha+1}(t),\quad \forall t\in\mathbb{R}_+. \tag{2.1}$$

Then, for some constant $C>0$, we have

$$F(t)\leq C(1+t)^{-\frac{1}{\alpha}}\xi^{-\frac{\alpha+1}{\alpha}}(t)\left[1+\int_0^t(s+1)^{\frac{1}{\alpha}}\xi^{\frac{\alpha+1}{\alpha}}(s)h^{\alpha+1}(s)\ ds\right],\qquad \forall t\in\mathbb{R}_+. \tag{2.2}$$

Proof. Multiplying (2.1) by ξ^{β}, where $\beta>1$ that will be defined later, we find

$$\xi^{\beta}(t)F'(t)\leq -c_1\xi^{\alpha+1+\beta}(t)F^{\alpha+1}(t)+c_2\xi^{\beta}(t)h^{\alpha+1}(t). \tag{2.3}$$

By taking advantage of the fact that ξ is a nonincreasing function, we get

$$\left(\xi^{\beta}(t)F(t)\right)'\leq -c_1\xi^{\alpha+1+\beta}(t)F^{\alpha+1}(t)+c_2\xi^{\beta}(t)h^{\alpha+1}(t). \tag{2.4}$$

Also by noting $\varphi(t)=\xi^{\beta}(t)F(t)$, and taking $\beta=\frac{\alpha+1}{\alpha}$, we obtain

$$\varphi'(t)\leq -c_1\varphi^{\alpha+1}(t)+c_2\xi^{\beta}(t)h^{\alpha+1}(t). \tag{2.5}$$

Following the same steps as in [28], we then find that

$$\varphi(t) \leq C(1+t)^{-\frac{1}{\alpha}} \left[1 + \int_0^t \xi^{\frac{\alpha+1}{\alpha}}(s) h^{\alpha+1}(s)(s+1)^{\frac{1}{\alpha}} \, ds\right]. \tag{2.6}$$

Therefore (2.2) is established. □

LEMMA 25. *Assuming that g satisfies $(\mathbb{H}_2)$ and $(\mathbb{H}_3)$ then*

$$\int_0^{+\infty} \xi(t) g^{1-\sigma}(t) \, dt < +\infty, \qquad \forall \sigma < 2-p. \tag{2.7}$$

PROOF. See [125]. □

3. Decay of Solutions

In this section, we aim to investigate the asymptotic behavior of solutions for problem (1.1)-(1.2).

Note that, for any regular solution u of the problem (1.1)-(1.2), it is straightforward to see that

$$E'(t) = \frac{1}{2} \int_0^{+\infty} g'(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2 \, ds, \quad \forall t \in \mathbb{R}_+, \tag{3.1}$$

where

$$\begin{aligned} E(t) &= \frac{1}{2} \|\mathcal{U}(t)\|_{\mathcal{H}}^2 \\ &= \frac{1}{2} \left(\|A^{\frac{1}{2}} u(t)\|^2 - g_0 \|B^{\frac{1}{2}} u(t)\|^2 + \|u_t(t)\|^2 + \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2 \, ds \right). \end{aligned} \tag{3.2}$$

THEOREM 26. *Assume that $(\mathbb{H}_1)$, $(\mathbb{H}_2)$ and $(\mathbb{H}_3)$ hold.*

(1) *Let $\mathcal{U}_0 \in H$ and $\mathcal{U}$ be the solution of (1.3).*

If (1.4) holds and

$$\exists m_0 > 0: \quad \|B^{\frac{1}{2}} u_0(s)\| \leq m_0, \qquad \forall s > 0, \tag{3.3}$$

then there exists a positive constant C such that, for all $t \in \mathbb{R}_+$,

$$E(t) \le C(1+t)^{-\frac{1}{2p-2}} \xi^{-\frac{2p-1}{2p-2}}(t) \left[1 + \int_0^t (s+1)^{\frac{1}{2p-2}} \xi^{\frac{2p-1}{2p-2}}(s) h^{2p-1}(s)\ ds\right], \tag{3.4}$$

where $h(t) = \xi(t) \int_t^{+\infty} g(s)\ ds$.

Moreover, if

$$\int_0^{+\infty} (1+t)^{-\frac{1}{2p-2}} \xi^{-\frac{2p-1}{2p-2}}(t) \left[1 + \int_0^t (s+1)^{\frac{1}{2p-2}} \xi^{\frac{2p-1}{2p-2}}(s) h^{2p-1}(s)\ ds\right]\ dt < +\infty, \tag{3.5}$$

then, for all $t \in \mathbb{R}_+$,

$$E(t) \le C(1+t)^{-\frac{1}{p-1}} \xi^{-\frac{p}{p-1}}(t) \left[1 + \int_0^t (s+1)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s) h^{p}(s)\ ds\right]. \tag{3.6}$$

(2) *Let $\mathcal{U}_0 \in D(A) \times D(A^{\frac{1}{2}}) \times L_g^2(\mathbb{R}_+, D(A^{\frac{1}{2}} B^{\frac{1}{2}})$ and $\mathcal{U}$ be the solution of (1.3). If (1.5) holds and*

$$\exists m_0 > 0 : \quad \|A^{\frac{1}{2}} B^{\frac{1}{2}} u_0(s)\| \le m_0, \quad \forall s > 0, \tag{3.7}$$

then there exists a positive constant C such that, for all $t \in \mathbb{R}_+$,

$$E(t) \le C \left(\frac{E_2(0) + E^{2p-1}(0) + \int_0^t h^{2p-1}(s)\ ds}{\int_0^t \xi^{2p-1}(s)\ ds} \right)^{\frac{1}{2p-1}}, \tag{3.8}$$

where

$$E_2(t) = \frac{1}{2}\left(\|Au(t)\|^2 - g_0\|A^{\frac{1}{2}} B^{\frac{1}{2}} u(t)\|^2 + \|A^{\frac{1}{2}} u'(t)\|^2\right) + \frac{1}{2}\int_0^{+\infty} g(s) \|A^{\frac{1}{2}} B^{\frac{1}{2}} \eta^t(s)\|^2\ ds. \tag{3.9}$$

Moreover, if

$$\int_0^{+\infty} \left(\frac{E_2(0) + E^{2p-1}(0) + \int_0^t h^{2p-1}(s)\ ds}{\int_0^t \xi^{2p-1}(s)\ ds} \right)^{\frac{1}{2p-1}} < +\infty, \tag{3.10}$$

then for all $t \in \mathbb{R}_+$,

$$E(t) \leq C \left(\frac{E_2(0) + E^p(0) + \int_0^t h^p(s)\ ds}{\int_0^t \xi^p(s)\ ds} \right)^{\frac{1}{p}}. \tag{3.11}$$

Lemma 27. *Assume that g satisfies $(\mathbb{H}_2)$ and $(\mathbb{H}_3)$, and u is the solution of (1.1)-(1.2). Then, for any $0 < \sigma < 1$ and any $t > 0$, we have*

$$\int_0^t g(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2\ ds \leq C \left[E(0) \int_0^t g^{1-\sigma}(s)\ ds \right]^{\frac{p-1}{p-1+\sigma}} \left[\int_0^t g^p(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2\ ds \right]^{\frac{\sigma}{p-1+\sigma}}, \tag{3.12}$$

and for $\sigma = \frac{1}{2}$, we have

$$\int_0^t g(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2\ ds \leq C \left[E(0) \int_0^t g^{\frac{1}{2}}(s)\ ds \right]^{\frac{2p-2}{p-1}} \left[\int_0^t g^p(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2\ ds \right]^{\frac{1}{2p-1}}. \tag{3.13}$$

Proof. By making use of Hölder's inequality, we obtain

$$\begin{aligned} \int_0^t g(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2\ ds = & \quad \int_0^t g^{1-\frac{\sigma p}{p-1+\sigma}}(s) \|B^{\frac{1}{2}} \eta^t(s)\|^{2-\frac{2\sigma}{p-1+\sigma}}\ g(s)^{\frac{p\sigma}{p-1+\sigma}} \|B^{\frac{1}{2}} \eta^t(s)\|^{\frac{2\sigma}{p-1+\sigma}}\ ds \\ \leq & \quad \left(\int_0^t g^{1-\sigma}(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2\ ds \right)^{\frac{p-1}{p-1+\sigma}} \left(\int_0^t g^p(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2\ ds \right)^{\frac{\sigma}{p-1+\sigma}}. \end{aligned} \tag{3.14}$$

Using $(\mathbb{H}_1)$, $(\mathbb{H}_2)$ and the the definition of E, we then find that

$$E(t) \geq \frac{1 - a_0 g_0}{2} \left(\|A^{\frac{1}{2}} u(t)\|^2 + \|u'(t)\|^2 + \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta^t(s)\ ds \right), \qquad \forall t \in \mathbb{R}_+. \tag{3.15}$$

Therefore, from (3.14) and (3.15), we deduce that

$$\begin{aligned}\int_0^t g(s)^{1-\sigma}\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds &\leq \int_0^t g^{1-\sigma}(s)\|B^{\frac{1}{2}}u(t)+B^{\frac{1}{2}}u(t-s)\|^2\,ds\\ &\leq 2\|B^{\frac{1}{2}}u(t)\|^2\int_0^{+\infty} g^{1-\sigma}(s)\,ds+2\int_0^t g^{1-\sigma}(s)\|B^{\frac{1}{2}}u(t-s)\|^2\,ds\\ &\leq 2a_0\|A^{\frac{1}{2}}u(t)\|^2\int_0^{+\infty} g^{1-\sigma}(s)\,ds+2a_0\int_0^t g^{1-\sigma}(s)\|A^{\frac{1}{2}}u(t-s)\|^2\,ds\\ &\leq CE(0)\int_0^{+\infty} g^{1-\sigma}(s)ds. \qquad (3.16)\end{aligned}$$

Finally, by inserting (3.16) in (3.14), we get (3.12). Inequality (3.13) is simply a particular case of (3.12). □

COROLLARY 28. *Assume that g satisfies $(\mathbb{H}_2)$ and $(\mathbb{H}_3)$, and u is the solution of (1.1)-(1.2). Then, for all $t\in\mathbb{R}_+$,*

$$\xi(t)\int_0^t g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds \leq C\,[-E'(t)]^{\frac{1}{2p-1}}. \qquad (3.17)$$

PROOF. Using (3.13), Lemma (25) (for $\sigma=\frac{1}{2}$) and Young's inequality, we obtain

$$\begin{aligned}\xi(t)\int_0^t g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds &\leq C\xi(t)^{\frac{2p-2}{2p-1}}(t)\left[\int_0^t g^{\frac{1}{2}}(s)\,ds\right]^{\frac{2p-2}{2p-1}}\xi^{\frac{1}{2p-1}}(t)\left[\int_0^t g^p(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right]^{\frac{1}{2p-1}}\\ &\leq C\left[\int_0^t \xi(s)g^{\frac{1}{2}}(s)\,ds\right]^{\frac{2p-2}{2p-1}}\left[\int_0^t \xi(s)g^p(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right]^{\frac{1}{2p-1}}\\ &\leq C\left[-\int_0^t g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right]^{\frac{1}{2p-1}}\\ &\leq C[-E'(t)]^{\frac{1}{2p-1}}. \qquad (3.18)\end{aligned}$$

□

For the following Lemma, we adopt the result from [**68**] without proof.

LEMMA 29. *There exist positive constants $M,\alpha_0,\alpha_1,\alpha_2$ such that the functional*

$$I_3(t) \;:=\; ME+I_1+\frac{g_0}{2}I_2+\alpha_0 E \qquad (3.19)$$

is equivalent to E and satisfies, for all $t \in \mathbb{R}_+$,

$$I_3'(t) \le -\alpha_1 E(t) + \alpha_2 \left(\int_0^{+\infty} g(s) \|A^{\frac{1}{2}} \eta^t(s)\|^2 \; ds + \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2 \; ds \right), \tag{3.20}$$

where the functionals I_1, I_2 are given by

$$I_1(t) := \langle u_t(t), u(t) \rangle$$

and

$$I_2(t) := -\langle u_t(t), \int_0^{+\infty} g(s) \eta^t(s) \; ds \rangle.$$

Proof. (**Theorem3**)

Case1: (1.4) **holds**.

We have

$$\int_0^{+\infty} g(s) \|A^{\frac{1}{2}} \eta^t(s)\|^2 \; ds \le a_2 \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2 \; ds. \tag{3.21}$$

It then follows from (3.21) and (3.20), that, for some positive constant C,

$$I_3'(t) \quad \le \quad -\alpha_1 E(t) + C \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2 \; ds. \tag{3.22}$$

Using Corollary (28), we multiply the estimate (3.22) by $\xi(t)$ to arrive at

$$\xi(t) I_3'(t) \quad \le \quad -\alpha_1 \xi(t) E(t) + C \xi(t) \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2 \; ds. \tag{3.23}$$

Now, from $(\mathbb{H}_1)$ and (3.15) one can see that for all $s > t$,

$$\begin{aligned} \|B^{\frac{1}{2}} \eta^t(s)\|^2 \quad &\le \quad 2a_0 \|A^{\frac{1}{2}} u(t)\|^2 + 2 \|B^{\frac{1}{2}} u(t-s)\|^2 \\ &\le \quad \frac{4a_0}{1 - a_0 g_0} E(0) + 2 \sup_{\tau < 0} \|B^{\frac{1}{2}} u(\tau)\|^2 \\ &\le \quad \frac{4a_0}{1 - a_0 g_0} E(0) + 2m_0^2. \end{aligned} \tag{3.24}$$

This leads to

$$\xi(t)\int_t^{+\infty} g(s)\|B^{\frac{1}{2}}\eta^t(s)\ ds\|^2\ ds \leq \left(\frac{4}{1-g_0}E(0)+2m_0^2\right)\xi(t)\int_t^{+\infty} g(s)\ ds := Ch(t). \quad (3.25)$$

From Corollary (28) and (3.25), the inequality (3.23) takes the form

$$\xi(t)I_3'(t) \ \ \leq \ \ -\alpha_1\xi(t)E(t)+C[-E'(t)]^{\frac{1}{2p-1}}+Ch(t). \quad (3.26)$$

Multiplying the last inequality by $\xi^\alpha E^\alpha$, where $\alpha = 2p-2>0$, we find

$$\xi(t)^{\alpha+1}E^\alpha(t)I_3'(t) \leq -\alpha_1\xi^{\alpha+1}(t)E^{\alpha+1}(t)+C\xi^\alpha(t)E^\alpha(t)[-E'(t)]^{\frac{1}{2p-1}}+Ch(t)\xi^\alpha(t)E^\alpha(t). \quad (3.27)$$

Exploiting Young's inequality, we get for any $\epsilon>0$,

$$\xi^{\alpha+1}(t)E^\alpha(t)I_3'(t) \ \ \leq \ \ -\alpha_1\xi^{\alpha+1}E^{\alpha+1}(t)-C_\epsilon E'(t)+2\epsilon\xi^{\alpha+1}(t)E^{\alpha+1}(t)+C_\epsilon h^{\alpha+1}(t). \quad (3.28)$$

Next, let $F(t) := \xi^{\alpha+1}(t)E^\alpha(t)I_3(t)+C_\epsilon E(t)\sim E(t)$. Then, for ϵ small enough, there exists a positive constant $\tilde{\alpha}_1$ such that

$$F'(t) \leq -\tilde{\alpha}_1\xi^{\alpha+1}(t)F^{\alpha+1}(t)+Ch^{\alpha+1}(t). \quad (3.29)$$

In view of Lemma (24) and taking into account that $F\sim E$, we get

$$E(t) \leq C(1+t)^{-\frac{1}{2p-2}}\xi^{-\frac{2p-1}{2p-2}}(t)\left[1+\int_0^t (s+1)^{\frac{1}{2p-2}}\xi^{\frac{2p-1}{2p-2}}(s)h^{2p-1}(s)\ ds\right]. \quad (3.30)$$

To get (3.6), again we use estimate (3.22)

$$\xi(t)I_3'(t) \ \ \leq \ \ -\alpha_1\xi(t)E(t)+\alpha_2\xi(t)\int_0^t g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\ ds+\alpha_2\xi(t)\int_t^{+\infty} g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\ ds.$$

Applying Jensen's inequality, the estimate (3.5) and the fact that ξ is non-increasing, we find

$$
\begin{aligned}
\xi(t)\int_0^t g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds &\le \frac{\nu(t)}{\nu(t)}\int_0^t [\xi^p(s)g^p(s)]^{\frac{1}{p}}\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds \\
&\le C\nu(t)\left[\frac{1}{\nu(t)}\int_0^t \xi^p(s)g^p(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right]^{\frac{1}{p}} \\
&= c\nu^{1-\frac{1}{p}}(t)\left[\int_0^t \xi^p(s)g^p(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right]^{\frac{1}{p}}, \qquad (3.31)
\end{aligned}
$$

where

$$
\begin{aligned}
\nu(t) &:= \int_0^t \|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds \le C\int_0^t \left(\|A^{\frac{1}{2}}u(t)\|^2 + \|A^{\frac{1}{2}}u(t-s)\|^2\right)\,ds \\
&\le C\int_0^t [E(t)+E(t-s)]\,ds \\
&\le 2C\int_0^t E(t-s))\,ds \\
&\le 2C\int_0^t E(s)\,ds < 2C\int_0^\infty E(s)\,ds < \infty, \qquad (3.32)
\end{aligned}
$$

and with the assumption that $\nu(t) > 0$.

Using (3.31) and (1.4), the assumption $(\mathbb{H}_3)$ and the fact that ξ is non-increasing, we get

$$
\begin{aligned}
\xi(t)\int_0^t g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds &\le C\nu^{\frac{p-1}{p}}(t)\xi^{p-1}(0)\left[\int_0^t \xi(s)g^p(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right]^{\frac{1}{p}} \\
&\le C\left[\int_0^t -g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right]^{\frac{1}{p}} \\
&\le C[-E'(t)]^{\frac{1}{p}}. \qquad (3.33)
\end{aligned}
$$

Thus, from (3.25) and (3.33), it yields that

$$
\xi(t)I_3'(t) \le -\alpha_1\xi(t)E(t) + C[-E'(t)]^{\frac{1}{p}} + Ch(t). \qquad (3.34)
$$

We multiply (3.34) by $\xi^{\alpha}(t)E^{\alpha}(t)$, for $\alpha = p-1$. This yields

$$\xi^{\alpha+1}(t)E^{\alpha}(t)I_3'(t) + CE'(t) \leq -\beta_1 \xi^{\alpha+1}(t)E^{\alpha+1}(t) + \beta_2 h^{\alpha+1}(t). \tag{3.35}$$

Now, let $F_0(t) := \xi^{\alpha+1}(t)E^{\alpha}(t)I_3(t) + C_{\epsilon}E(t) \sim E(t)$, then we have

$$F_0'(t) \leq -\beta_1 \xi^{\alpha+1}(t)F_0^{\alpha+1}(t) + \beta_2 h^{\alpha+1}(t). \tag{3.36}$$

Then Lemma (24) implies that

$$F_0(t) \leq C(1+t)^{-\frac{1}{p-1}}\ \xi^{-\frac{p}{p-1}}(t)\left[1 + \int_0^t (s+1)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s)\ h^p(s)\ ds\right]. \tag{3.37}$$

Hence, we infer that

$$E(t) \leq C(1+t)^{-\frac{1}{p-1}}\ \xi^{-\frac{p}{p-1}}(t)\left[1 + \int_0^t (s+1)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s)\ h^p(s)\ ds\right], \quad \forall t > 0. \tag{3.38}$$

Case 2: (1.5) **holds**.

As in [**68**], and similar to the approach of [**132**], we recall that the energy E_2 related with problem (1.1)-(1.2) and associated with $\mathcal{A}^{\frac{1}{2}}\mathcal{U}$ corresponding to $\mathcal{U}_0 \in \left(\mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times L_g^2(\mathbb{R}^+, \mathcal{D}(A^{\frac{1}{2}}B^{\frac{1}{2}}))\right)$ defined on $\mathbb{R}^+$ by

$$\begin{aligned} E_2(t) &= \|\mathcal{A}^{\frac{1}{2}}\mathcal{U}\|_{\mathcal{H}}^2 \\ &= \frac{1}{2}\left(\|Au(t)\|^2 - g_0\|A^{\frac{1}{2}}B^{\frac{1}{2}}u(t)\|^2 + \|A^{\frac{1}{2}}u'(t)\|^2\right) + \frac{1}{2}\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\ ds \end{aligned}$$

with

$$E_2'(t) = \frac{1}{2}\int_0^{+\infty} g'(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\ ds \leq 0, \qquad \forall t \in \mathbb{R}_+. \tag{3.39}$$

We observe that in view of assumption $(\mathbb{H}_1)$, we have

$$\|A^{\frac{1}{2}}B^{\frac{1}{2}}v\|^2 \leq a_0\|Av\|^2, \qquad \forall v \in \mathcal{D}(A^{\frac{1}{2}}B^{\frac{1}{2}}). \tag{3.40}$$

Multiplying (3.20) by $\xi(t)$, using (1.5) and (3.40), we get

$$\xi(t)I_3'(t) \leq -\alpha_1\xi(t)E(t)+\alpha_2a_2(1+a_0)\xi(t)\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\ ds. \tag{3.41}$$

Thanks to $(\mathbb{H}_2)$, we get

$$E_2(t) \geq \frac{1-g_0a_0}{2}\left(\|Au(t)\|^2+\|A^{\frac{1}{2}}u'(t)\|^2+\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\ ds\right) > 0. \tag{3.42}$$

From (3.42), we have

$$\|Au(s)\|^2 \leq \frac{2}{1-g_0a_0}E_2(s) \leq \frac{2}{1-g_0a_0}E_2(0), \quad \forall s \in \mathbb{R}_+. \tag{3.43}$$

Therefore, for all $s > t$

$$\|A^{\frac{1}{2}}u(t-s)\| \leq 2\sup_{\tau>0}\|A^{\frac{1}{2}}B^{\frac{1}{2}}u_0(\tau)\|^2 \leq 2m_0^2. \tag{3.44}$$

Combining (3.43), (3.44) and (3.40), we find

$$\begin{aligned}
\xi(t)\int_t^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\ ds &\leq 2\xi(t)\int_t^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}u(t)\|^2\ ds \\
&+ 2\xi(t)\int_t^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}u(t-s)\|^2\ ds \\
&\leq \left(\frac{4a_0}{1-g_0a_0}E_2(0)+4a_0m_0^2\right)\xi(t)\int_t^{+\infty} g(s)\ ds \\
&\leq Ch(t).
\end{aligned} \tag{3.45}$$

If we repeat the same steps as in Lemma (27), and replace $A^{\frac{1}{2}}B^{\frac{1}{2}}$ by A in estimate (3.16), we end up with

$$\begin{aligned}\xi(t)\int_0^t g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\, ds &\leq C\left[E_2(0)\int_0^t \xi(s)g^{\frac{1}{2}}(s)\, ds\right]^{\frac{2p-2}{p-1}}\left[\int_0^t \xi(s)g^p(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\, ds\right]^{\frac{1}{2p-1}}\\ &\leq C\left[\int_0^t \xi(s)g^p(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\, ds\right]^{\frac{1}{2p-1}}. \end{aligned} \tag{3.46}$$

Using condition (1.5), $(\mathbb{H}_3)$ and combining (3.45) and (3.46), we obtain

$$\begin{aligned}\xi(t)\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\, ds &\leq a_2\xi(t)\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\, ds\\ &\leq C[-E_2'(t)]^{\frac{1}{2p-1}}+C\ h(t).\end{aligned} \tag{3.47}$$

Hence, estimate (3.41) will take the form

$$\xi(t)I_3'(t)\leq -\alpha_1\xi(t)E(t)+C[-E_2'(t)]^{\frac{1}{2p-1}}+C\ h(t). \tag{3.48}$$

Next, multiplying (3.48) by $\xi^\alpha E^\alpha$ with $\alpha=2p-2$, and using Hölder's inequality, we get, for some $\tilde{\alpha}_2>0$,

$$\xi^{\alpha+1}(t)E^\alpha(t)I_3'(t)\leq -\tilde{\alpha}_2\xi^{\alpha+1}(t)E^{\alpha+1}(t)-CE_2'(t)+Ch^{\alpha+1}(t). \tag{3.49}$$

Integrating over $(0,t)$, we find

$$\int_0^t \xi^{\alpha+1}(s)E^{\alpha+1}(s)\, ds \leq -\frac{1}{\tilde{\alpha}_2}\int_0^t \xi^{\alpha+1}(s)E^\alpha(s)I_3'(s)\, ds+\frac{C_\epsilon}{\tilde{\alpha}_2}E_2(0)+\frac{C_\epsilon}{\tilde{\alpha}_2}\int_0^t h^{\alpha+1}(s)\, ds. \tag{3.50}$$

Now, taking the advantage of the fact that $I_3\sim E$ and that the energy E and the function ξ are nonincreasing, we get

$$\begin{aligned}E^{\alpha+1}(t)\int_0^t \xi^{\alpha+1}(s)\, ds &\leq -\frac{1}{\tilde{\alpha}_2}\xi^{\alpha+1}(t)E^{\alpha+1}(t)+\frac{1}{\tilde{\alpha}_2}\xi^{\alpha+1}(0)E^{\alpha+1}(0)+\frac{C}{\tilde{\alpha}_2}E_2(0)+\frac{C}{\tilde{\alpha}_2}\int_0^t h^{\alpha+1}(s)\, ds\\ &\leq \frac{1}{\tilde{\alpha}_2}\xi^{\alpha+1}(0)E^{\alpha+1}(0)+\frac{C}{\tilde{\alpha}_2}E_2(0)+\frac{C}{\tilde{\alpha}_2}\int_0^t h^{\alpha+1}(s)\, ds.\end{aligned} \tag{3.51}$$

This yields

$$E(t) \leq C\left(\frac{E_2(0)+E^{\alpha+1}(0)+\int_0^t h^{\alpha+1}(s)\ ds}{\int_0^t \xi^{\alpha+1}(s)\ ds}\right)^{\frac{1}{\alpha+1}}. \tag{3.52}$$

Therefore,

$$E(t) \leq C\left(\frac{E_2(0)+E^{2p-1}(0)+\int_0^t h^{2p-1}(s)\ ds}{\int_0^t \xi^{2p-1}(s)\ ds}\right)^{\frac{1}{2p-1}}. \tag{3.53}$$

In order to get (3.11), we use estimate (3.42) and follow the same procedure as in estimate (3.33). We take the operator $A^{\frac{1}{2}}B^{\frac{1}{2}}$ instead of $A^{\frac{1}{2}}$ to find

$$\xi(t)\int_0^t g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\ ds \leq c[-E'(t)]^{\frac{-1}{p}}. \tag{3.54}$$

Consequently, from (3.45) and (3.54), estimate (3.41) will be reduced to

$$\xi(t)I_4'(t) \quad \leq \quad -\alpha_1\xi(t)E(t)+C[-E'(t)]^{\frac{-1}{p}}+Ch(t). \tag{3.55}$$

Finally, by repeating the same steps from (3.49) to (3.53) with $\alpha = p-1$, we find the estimate (3.11), and the proof is now complete. □

Example: We illustrate the energy decay rate given by Thoerem 3 through the following example

Let $g(t) = a(1+t)^{-q}, \quad q > 2$, where $a > 0$ is a constant so that $\int_0^{+\infty} g(t)\ dt < \frac{1}{a}$, then we have

$$g'(t) = -a_0q(1+t)^{-q-1} = -b(a_0(1+t)^{-q})^{\frac{q+1}{q}} = -bg^p(t), \quad p = \frac{q+1}{q},\ b > 0.$$

Therefore, for the Case (1.4), estimate (3.6) with $\xi(t) = b$ yields

$$E(t) \leq C(1+t)^{-\frac{1}{p-1}}\xi^{-\frac{p}{p-1}}(t)\left(1+\int_0^t (s+1)^{\frac{1}{p-1}}\xi^{\frac{p}{p-1}}(s)h^p(s)\ ds\right).$$

Let us compute

$$h(t)=\xi(t)\int_t^{+\infty} g(s)\ ds \;=\; b\int_t^{+\infty} a_0(1+s)^{-q}\ ds$$
$$= \frac{a_0 b}{q-1}(1+t)^{1-q}, \quad q=\frac{1}{p-1}. \tag{3.56}$$

Observe that, for some positive constant C, it yields

$$\int_0^t (s+1)^{\frac{1}{p-1}}\xi^{\frac{p}{p-1}}(s)h^p(s)\ ds = b\frac{ab}{q-1}\int_0^t (1+s)^{p(1-q)+\frac{1}{p-1}}\ ds = C(1+t)^{p(1-q)+\frac{1}{p-1}+1}-C.$$

Then, it yields

$$\begin{aligned} E(t) &\leq C(1+t)^{-\frac{1}{p-1}}\xi^{-\frac{p}{p-1}}(t)\left(1+C(1+t)^{p(1-q)+\frac{1}{p-1}+1}\right) \\ &= C(1+t)^{-\frac{1}{p-1}}+C(1+t)^{p(1-q)+1} \\ &= C(1+t)^{-q}+C(1+t)^{\frac{-q^2+q+1}{q}} \leq C(1+t)^{\frac{-q^2+q+1}{q}}. \end{aligned} \tag{3.57}$$

For the Case (1.5), estimate (3.11) gives

$$\begin{aligned} E(t) &\leq \left(\frac{E_2(0)+E^p(0)+\int_0^t h^p(s)\ ds}{\int_0^t \xi^p(s)\ ds}\right)^{\frac{q}{q+1}} \\ &\leq bt^{-\frac{q}{q+1}}\left(E_2(0)+E^p(0)+\int_0^t h^p(s)\ ds\right)^{\frac{q}{q+1}} \\ &\leq C\ t^{-\frac{q}{q+1}}\left(1-(1+t)^{\frac{(1-q)(q+1)}{q}+1}\right)^{\frac{q}{q+1}} \leq C\ t^{-\frac{q}{q+1}}. \end{aligned} \tag{3.58}$$

Let us compare our result with the one of [**68, 69**]. In this way, let us recall the approach of [**68**] with $B=A$, there exists a positive constant c_1 such that

$$E(t)\leq c_1(1+t)^{-c_2}, \qquad \forall t\in\mathbb{R}_+, \tag{3.59}$$

where c_2 is generated by the calculations and it is generally small.

Furthermore the approach of [**68**] in polynomial case under (A_2) and with $G(t)=t^{\frac{1}{p}+1}$, for any $p \in \left]0, \frac{q-1}{2}\right[$ gives

If (1.4) holds,

$$E(t) \leq C(1+t)^{-p}, \quad \forall p \in \left]0, \frac{q-1}{2}\right[, \tag{3.60}$$

and if (1.5) holds,

$$E(t) \leq C(1+t)^{-\frac{p}{p+1}}, \quad \forall p \in \left]0, \frac{q-1}{2}\right[. \tag{3.61}$$

Now, since $\frac{q-1}{2} < \frac{q^2-q-1}{2}$ for $q > 2$. Then from (3.57), (3.59) and (3.60), we conclude that our estimate (3.57) gives a better decay than (3.59) and (3.60).

For the case (1.5), we see that $\frac{q}{q+1} > \frac{p}{p+1}$, for any $p \in]0, \frac{q-1}{2}[$. Then estimate (3.58) has better decay than estimate (3.61).

As a conclusion our approach improves and has a better decay rate than the one of [**68, 69**].

CHAPTER 5

Finite Time of Solutions for Some Hamilton Jacobi-Equations

This chapter is a part of a subject proposed by Pr. M. Ziane.

Abstract. In this work, we investigate the existence of weak solutions and the blow-up in finite time of solutions for some Hamiltion-Jacobi with exponents in the nonlineatities.

1. Introduction

In the final part of this thesis, we consider some initial boundary value problems for nonlinear parabolic systems whith source terms depending on the gradient of the solutions, known as " Viscous Hamilton-Jacobi " equations, and which are given by

$$\begin{cases} u_t + \Delta^2 u = f(u, v, |\nabla u|, |\nabla v|) & \text{in } \Omega \times (0, \infty), \\ v_t + \Delta^2 v = g(u, v, |\nabla u|, |\nabla v|) & \text{in } \Omega \times (0, \infty), \\ u = v = \Delta u = \Delta v = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded domain Ω with regular boundary $\partial\Omega$, and $\Delta^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i^2} \frac{\partial^2}{\partial x_j^2}$ is the bi-Laplacien operator, $|\nabla u| = \left(\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{1}{2}}$, (u_0, v_0) is the initial data, and f and g are functions of class C^1 with polynomial growth.

and with appropriate assumptions.

Without loss of generality, and for the sake of simplicity, we study the following system and all the results obtained here carry out to the general system (1.1) under the appropriate conditions.

$$\begin{cases} u_t + \Delta^2 u = |\nabla v|^{\alpha_1} + |v|^{\beta_1} & \text{in } \Omega \times (0,\infty), \\ \\ v_t + \Delta^2 v = |\nabla u|^{\alpha_2} + |u|^{\beta_2} & \text{in } \Omega \times (0,\infty), \\ \\ u = v = \Delta u = \Delta v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{1.2}$$

where $\alpha_i, \beta_i,\ i = 1,2$, are positive constants such that $\alpha_i, \beta_i > 1$.

It is well known that the Kuramoto-Sivashinsky Equation is given by

$$\phi_t + \Delta^2\phi + \Delta\phi + \frac{1}{2} \mid \nabla\phi \mid^2 = 0, \tag{1.3}$$

subject to appropriate initial data and boundary conditions. This equation has attracted a great deal of interest as a model for complex spatio-temporal dynamics in spatially extended systems, and as a paradigm for finite-dimensional dynamics in a partial differential equation.

Precisely, this equation appears in hydrodynamics as a model for the flow of thin soap films flowing down an inclined surface, and in combustion theory as a model for the propagation of flame fronts ([**98**], [**168**]). In the one-dimensional case, (KSE) was considered by several authors both analytically and computationally (see, e.g. [**113, 126, 172, 144**]), and the references therein. It has been shown that the long term dynamics of this equation are finite-dimensional. In particular, it possesses a globally invariant, finite-dimensional exponentially attracting inertial manifold.

The question of global existence of solutions in two dimensional or higher to (1.3) under the physical Dirichlet-like boundary conditions

$$u = \Delta u = 0 \qquad on\ \partial\Omega, \tag{1.4}$$

is sill an open problem in nonlinear analysis of partial differential equations [102].

To avoid dealing with the average of the solution to this equation, most authors consider, instead, the system of evolution equations for $u = \nabla\phi$

$$u_t + \Delta^2 u + \Delta u + \frac{1}{2}\nabla \mid u \mid^2 = 0, \tag{1.5}$$

or in this way

$$u_t + \Delta^2 u + \Delta u + (u \cdot \nabla)u = 0, \tag{1.6}$$

wich is also called the Kuramoto Sivashinsky equation (KSE).

For partial results, in case $n \geq 2$, we quote some recent results obtained in [60, 61, 157, 156].

The motivation of our work is due to the following problem:

$$\begin{cases} u_t + \Delta^2 u = |\nabla u|^p, & \text{in } \Omega \times (0, \infty), \\ u = \Delta u = 0, & \text{on } \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

Bellout, Benachour and S Titi [15] introduced this problem of hyper-viscous Hamilton-Jacobi-like equations parametrized by the exponent p in the nonlinear term, where in the case of the usual Hamilton-Jacobi nonlinearity, $p = 2$. They established some results concerning the local existence of weak solutions, strong solutions and uniqueness of strong solutions for this family of equations, and established also some results on the blow-up in finite time and the global existence of the solutions.

The most important is that they showed that the blow-up in finite time of the weak solutions holds when $p > 2$ and u_0, sufficiently large which is the same reult obtained by Souplet [169] to the following generalization of the viscous Hamilton-Jacobi equation:

$$\begin{cases} u_t - \Delta u = |\nabla u|^p, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

This study is also motivated by some parabolic systems considered by much authors with gradient terms as nonlinearites, one can cite as examples ([**4, 114**]), and the references therein.

Numerous papers are devoted to the study of nonlinear parabolic equations of the form

$$\frac{\partial u}{\partial t} = \Delta u + \mu \mid \nabla u \mid^p + f(x,u), \tag{1.7}$$

and related form for which blow-up results and global (in time) solvability (see [**31, 54, 73, 85, 165, 170**] and the reference therein) were obtained.
In this work, we investigate the questions of the short time and the existence of weak solutions for the system (1.1), and give some generalisation for some results in [**15**].

In order to study this problem, we give some preliminaries which are useful in the next.

2. Preliminaries

We first give some notations and definitions.
Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and

$$Q_t = \Omega \times (0,t), \qquad \Gamma_t = \partial\Omega \times (0,t), \qquad \Omega_t = \Omega \times \{t\}.$$

Let us introduce the space $E = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. By classical results of elliptic regularity, the dot product $\langle u,v \rangle = \int_\Omega \Delta u \Delta v \; dx$ makes E a Hilbert space. Here $\mid \nabla u \mid = (\nabla u, \nabla u)^{\frac{1}{2}}$ and $(.,.)$ is the usual Euclidean dot product in $\mathbb{R}^n$. We will denote the dual space of E by E', and by $\|w\|_{k,p}$ the norm in $\mathbb{W}^{k,p}(\Omega)$.

The rest of this work is organized as follows. In Section 3, we will establish the local existence of the weak solution of (1.2) under some conditions on the power nonlinearities

$\alpha_i, \beta_i,\ i=1,2$. Section 4 is devoted to the blow-up results .

From [**15**], our basic definition of weak solution to (1.2) is the following

Definition 30. *A weak solution (u,v) to the problem (1.2) on the interval $(0,T)$ with $(u_0,v_0)\in(L^2(\Omega))^2$ is a measurable function such that*

$$\begin{array}{lcl}(u,v) & \in & (L^2((0,T),E)\cap L^\infty\Big((0,T),L^2(\Omega)\Big))^2,\\ & \cap & (L^2_{loc}((0,T),W^{4,2}(\Omega)))^2,\\ (\frac{\partial u}{\partial t},\frac{\partial v}{\partial t}) & \in & (L^2((0,T),W^{-n-2,2}(\Omega))^2,\\ (|\nabla u|^{\alpha_2},|\nabla v|^{\alpha_1}) & \in & (L^1(\Omega\times(0,T)))^2,\\ (|u|^{\beta_2},|v|^{\beta_1}) & \in & (L^1(\Omega\times(0,T)))^2.\end{array}$$

The initial data and the boundary conditions are satisfied in the sens of distributions and trace successively, and for all test functions $\phi(x,t)\in C^\infty(\Omega\times(0,T))$ with compact support in $(\Omega\times(0,T))$, we have

$$\begin{aligned}\int_0^T\int_\Omega u_t\phi\ dxds &+ \int_0^T\int_\Omega \Delta u\Delta\phi\ dxds\\ &= \int_0^T\int_\Omega|\nabla v|^{\alpha_1}\phi\ dxds+\int_0^T\int_\Omega|v|^{\beta_1}\phi\ dxds.\end{aligned}$$

and

$$\begin{aligned}\int_0^T\int_\Omega v_t\phi\ dxds &+ \int_0^T\int_\Omega \Delta v\Delta\phi\ dxds\\ &= \int_0^T\int_\Omega|\nabla u|^{\alpha_2}\phi\ dxds+\int_0^T\int_\Omega|u|^{\beta_2}\phi\ dxds.\end{aligned}$$

Now, we are in a position to state our results.

3. Existence of weak solutions

The main result on the local existence of weak solution to (1.2) is essentially based on suitable conditions on α_1, α_2 and β_1, β_2. It can be read as follows.

THEOREM 31. *Assume that*

$$1 < \alpha_i < \frac{n+8}{n+2}, \qquad 1 < \beta_i < \frac{n+8}{n}, \quad i = 1, 2. \tag{3.1}$$

Then, for all $u_0, v_0 \in L^2(\Omega)$, there exists at least a maximal weak solutions of problem (1.2).

In order to prove this theorem, we need to give some priori estimates on the solutions, and we have the following lemmas.

LEMMA 32. *Assuming that $1 < \alpha_i < \frac{n+8}{n+2}$ and $1 < \beta_i < \frac{n+8}{n}$ and let $(u,v) \in (L^\infty((0,T), L^2(\Omega)))^2$ be a smooth solution to the problem* (1.2). *Then there exists a positive constant C independent of u and v, such that, for all $t \in (0,T)$, we get*

$$\begin{aligned} \int_\Omega \Big(u^2(x,t) + v^2(x,t)\Big)\ dx \ &\le \ \int_\Omega \Big(u_0^2(x) + v_0^2(x)\Big)\ dx \\ &+ \ C\int_0^t \Big(\| u(.,s) \|_{0,2}^2 + \| v(.,s) \|_{0,2}^2 + 1\Big)^\sigma \ ds, \end{aligned} \tag{3.2}$$

where $\sigma > 1$.

PROOF. Multiplying the equations $(1.2)_1, (1.2)_2$ by u and v respectively, and integrating over Ω, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx \ &+ \ \int_\Omega (\Delta u)^2 dx \\ &= \ \int_\Omega u|v|^{\beta_1} dx + \int_\Omega u|\nabla v|^{\alpha_1} dx. \end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\Omega} v^{2}(x,t)dx \quad + \quad & \int_{\Omega}(\Delta v)^{2}dx \\
= \quad & \int_{\Omega} u|v|^{\beta_2}dx + \int_{\Omega} v|\nabla u|^{\alpha_2}dx. \qquad (3.4)
\end{aligned}
$$

$$(3.5)$$

To estimate the nonlinearity terms, we use Sobolev Embedding and interpolation inequalities (see [**175**], pp. 186, 328).

For simplicity, we denote by c a generic positive constant which will change from to line.

We estimate the first term on the right-hand side of (3.3) by Hölder's inequality

$$
\int_{\Omega} u|v|^{\beta_1}dx \quad \leq \quad \left(\int_{\Omega}|u|^{r_1}dx\right)^{\frac{1}{r_1}}\left(\int_{\Omega}|v|^{\beta_1 r_2}dx\right)^{\frac{1}{r_2}}, \quad \frac{1}{r_1}+\frac{1}{r_2}=1.
$$

and interpolation inequality

$$
\int_{\Omega} u|v|^{\beta_1}dx \quad \leq \quad c\,(\|u\|_{0,2}+\|v\|_{0,2})^{(s_1+\beta_1 s_2)}\,(\|\Delta u\|_{0,2}+\|\Delta v\|_{0,2})^{(1-s_1)+\beta_1(1-s_2)}, \quad (3.6)
$$

where

$$
s_1=\frac{n}{2r_1}-\frac{n}{4}+1 \quad \in (0,1), \quad s_2=\frac{n}{2\beta_1 r_2}-\frac{n}{4}+1 \quad \in (0,1).
$$

Making use of Young's inequality, we then have

$$
\begin{aligned}
\int_{\Omega} u|v|^{\beta_1}dx \quad & \leq \quad c\,(\|u\|_{0,2}+\|v\|_{0,2})^{(s_1+\beta_1 s_2)r_3}+\frac{1}{8}\,(\|\Delta u\|_{0,2}+\|\Delta v\|_{0,2})^{((1-s_1)+\beta_1(1-s_2))r_4} \\
& \leq \quad c\left(\|u\|_{0,2}^{2}+\|v\|_{0,2}^{2}\right)^{\sigma_1}+\frac{1}{8}\left(\|\Delta u\|_{0,2}^{2}+\|\Delta v\|_{0,2}^{2}\right), \qquad (3.7)
\end{aligned}
$$

where

$$
r_4=\frac{8}{n(\beta_1-1)}, \quad r_3=\frac{8}{8+n-n\beta_1} \quad ,
$$

and

$$\sigma_1 = 1 + \frac{4(\beta_1 - 1)}{8 + n - n\beta_1} > 1, \qquad 1 < \beta_1 < \frac{n+8}{n}.$$

For the gradient term, we use again Hölder's inequality, Sobolev embedding and interpolation inequality to get

$$\begin{aligned}
\int_\Omega u \mid \nabla v \mid^{\alpha_1} \; dx \;\; &\leq \;\; (\|\nabla u\|_{\alpha_1 r_5} + \|\nabla v\|_{\alpha_1 r_5})^{\alpha_1} ((\|u\|_{r_6} + \|v\|_{r_6}), \quad \frac{1}{r_5} + \frac{1}{r_6} = 1 \\
&\leq \;\; c\left(\|u\|_{0,2}^2 + \|v\|_{0,2}^2\right)^{\alpha_1 s_4 + s_3} \left(\|\Delta u\|_{0,2}^2 + \|\Delta v\|_{0,2}^2\right)^{\alpha_1(1-s_4)+(1-s_3)},
\end{aligned} \tag{3.8}$$

where

$$s_3 = 1 - \frac{n}{4} + \frac{n}{2r_5} \quad \in (0,1), \quad s_4 = \frac{1}{2} - \frac{n}{4} + \frac{n}{2\alpha_1 r_6} \quad \in (0,1).$$

Using Young's inequality $ab \leq \epsilon a^q + C_\epsilon b^{q'}$

$$\begin{aligned}
\int_\Omega u \mid \nabla v \mid^{\alpha_1} \; dx \;\; &\leq \;\; c\left(\|u\|_{0,2}^2 + \|v\|_{0,2}^2\right)^{r_7(\alpha_1 s_4 + s_3)} + \frac{1}{8}\left(\|\Delta u\|_{0,2}^2 + \|\Delta v\|_{0,2}^2\right)^{r_6(\alpha_1(1-s_4)+(1-s_3))} \\
&\leq \;\; c\left(\|u\|_{0,2}^2 + \|v\|_{0,2}^2\right)^{\sigma_2} + \frac{1}{8}\left(\|\Delta u\|_{0,2}^2 + \|\Delta v\|_{0,2}^2\right),
\end{aligned} \tag{3.9}$$

where

$$\sigma_2 = 1 + \frac{4(\alpha_1 - 1)}{8 + n - (n+2)\alpha_1} > 1 \quad \textit{and} \qquad 1 < \alpha_1 < \frac{n+8}{n}.$$

Similarly as above, we get

$$\int_\Omega v|u|^{\beta_2} dx \;\; \leq \;\; c\left(\|u\|_{0,2}^2 + \|v\|_{0,2}^2\right)^{\sigma_3} + \frac{1}{8}\left(\|\Delta u\|_{0,2}^2 + \|\Delta v\|_{0,2}^2\right), \tag{3.10}$$

with $\sigma_3 = 1 + \frac{4(\beta_2-1)}{8+n-n\beta_2} > 1, \qquad 1 < \beta_2 < \frac{n+8}{n}.$

and

$$\int_\Omega v \mid \nabla u \mid^{\alpha_2} \; dx \;\; \leq \;\; c\left(\|u\|_{0,2}^2 + \|v\|_{0,2}^2\right)^{\sigma_4} + \frac{1}{8}\left(\|\Delta u\|_{0,2}^2 + \|\Delta v\|_{0,2}^2\right), \tag{3.11}$$

where

$$\sigma_4 = 1 + \frac{4(\alpha_2 - 1)}{8 + n - (n+2)\alpha_2} > 1 \quad \text{ and } \quad 1 < \alpha_2 < \frac{n+8}{n}.$$

Finally, from (3.7), (3.9), (3.10) and (3.11) we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (u^2(x,t) \;\; &+ \;\; v^2(x,t))\; dx + \frac{1}{2}\int_\Omega (|\Delta(x,t)|^2 + |\Delta v(x,t)|^2) dx \\ &+ \;\; \int_\Omega \left(|u(x,t)|^{\beta_1+1} + |v(x,t)|^{\beta_2+1}\right)\; dx \\ &\leq \;\; c\left(\|u\|_{0,2}^2 + \|v\|_{0,2}^2 + 1\right)^\sigma. \end{aligned}\tag{3.12}$$

where $\sigma = \max\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} > 1.$ □

LEMMA 33. *Assuming that* $1 < \alpha_i < \frac{n+8}{n+2}$, $1 < \beta_i < 1 + \frac{8}{n}$, $\quad i = 1,2$ *and* (u,v) *is a smooth solution of the problem* (1.2)*, and* $(u,v) \in (L^\infty((0,T), L^2(\Omega)))^2$. *Then there exist a positif constant* C *independent of* u *and* v*, and a time* T^* *given by*

$$T^* = \frac{1}{(\sigma-1)\left(\|u_0\|_{0,2}^2 + \|v_0\|_{0,2}^2\right)^{\sigma-1} C} \tag{3.13}$$

such that for all $t \in (0, T^*)$,

$$\max\{\int_\Omega u^2(x,t)dx; \int_\Omega v^2(x,t)dx\} \leq \left(\frac{\left(\|u_0\|_{0,2}^2 + \|v_0\|_{0,2}^2\right)^{(\sigma-1)}}{1 - (\sigma-1)\left(\|u_0\|_{0,2}^2 + \|v_0\|_{0,2}^2\right)^{(\sigma-1)} Ct}\right)^{\frac{1}{(\sigma-1)}} < +\infty \tag{3.14}$$

where C is a generic positif constant and

$$\max\{\int_0^t\int_\Omega |\Delta u|^2 dxds; \int_0^t\int_\Omega |\Delta v|^2 dxds\} \leq \left(\|u_0\|_{0,2}^2 + \|v_0\|_{0,2}^2\right)$$
$$+Ct\left(\frac{\left(\|u_0\|_{0,2}^2 + \|v_0\|_{0,2}^2\right)^{(\sigma-1)}}{1-(\sigma-1)\left(\|u_0\|_{0,2}^2 + \|v_0\|_{0,2}^2\right)^{(\sigma-1)}Ct}\right)^{\frac{\sigma}{(\sigma-1)}} < +\infty. \tag{3.15}$$

Also

$$\max(\int_0^t\int_\Omega |u(x,t)|^{\beta_2+1} dxds; \int_0^t\int_\Omega |v(x,t)|^{\beta_1+1} dxds) \leq \left(\|u_0\|_{0,2}^2 + \|v_0\|_{0,2}^2\right)$$
$$+Ct\left(\frac{\left(\|u_0\|_{0,2}^2 + \|v_0\|_{0,2}^2\right)^{(\sigma-1)}}{1-(\sigma-1)\left(\|u_0\|_{0,2}^2 + \|v_0\|_{0,2}^2\right)^{(\sigma-1)}Ct}\right)^{\frac{\sigma}{(\sigma-1)}} < +\infty, \tag{3.16}$$

PROOF. Using Eq. (3.2) of Lemma(32) and Gronwall′s integral inequality (see, e.g. [**113**], pp. 86]). □

LEMMA 34. *Assuming, for $i = 1, 2$,*

$$\begin{cases} 1 < \alpha_i \leq 2 & \textit{for} \;\; n \leq 3, \\ 1 < \alpha_i < \frac{n+8}{n+2}, & \textit{for} \;\; n \geq 4, \end{cases}$$

and let $(u, v) \in (L^\infty((0,T), L^2(\Omega)))^2$ be a weak solution to problem (1.2).
Then

$$(|\nabla u|^{\alpha_1}, |\nabla v|^{\alpha_2}) \in (L^2((0,T), L^r(\Omega)))^2 \text{ where} \begin{cases} r = 1 & \textit{if} \quad n \leq 3, \\ r \geq \frac{2n}{(n+4)} & \textit{for} \quad n \geq 4. \end{cases} \tag{3.17}$$

Also, there exist positif constants (C_i, q_i) $i = 1, 2$, independent of u and v such that

$$\int_0^T \left(\int_\Omega (|\nabla u|^{\alpha_2})^r dx\right)^{\frac{2}{r}} ds \leq C_1 \sup_{0\leq t\leq T} \|u(.,t)\|_{0,2}^{q_1} \int_0^T \|u(.,s)\|_{2,2}^2 ds, \tag{3.18}$$

$$\int_0^T \left(\int_\Omega (|\nabla v|^{\alpha_1})^r dx\right)^{\frac{2}{r}} ds \leq C_2 \sup_{0\leq t\leq T} \|v(.,t)\|_{0,2}^{q_2} \int_0^T \|v(.,s)\|_{2,2}^2 ds. \tag{3.19}$$

Proof. The proof follows from slight modifications of the proof of lemma 3 (see [**15**]).

□

Proof of existence. In this section, we assume that all assumptions in Theorem (31) are satisfied, we prove the existence of weak solutions to problem (1.2) with the the Faedo-Galerkin method as in [**15**].

Proof. (Proof of existence) Introducing the space $W_m = \{w_1, w_2, ..., w_m\}$ where $(w_j)_{j\geq 1}$ constitutes a basis in $L^2(\Omega)$.
The eigenfunctions of the Laplacien operator in $H_0^1(\Omega)$ given by

$$\begin{cases} -\Delta w_i = \lambda_i w_i & \text{in } \Omega \qquad i = 1, ..., m, \\ w_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, we will try to find an approximate solution of the problem (1.2), for a fixed k, in the form

$$u_k(x,t) = \sum_{j=1}^{k} a_{j,k}(t) w_j(x), \qquad v_k(x,t) = \sum_{j=1}^{k} b_{j,k}(t) w_j(x),$$

where $a_{i,k}(t)$ and $b_{i,k}(t)$ are the solutions of the nonlinear ODE truncated system in the variant t of :

$$\begin{cases} \int_\Omega \frac{\partial u_k}{\partial t} \phi dx + \int_\Omega \Delta u_k \Delta \phi dx = \int_\Omega |\nabla v_k|^{\alpha_1} \phi dx + \int_\Omega |v_k|^{\beta_1} \phi dx, & x \in \Omega, t > 0, \\ a_{j,k}(0) = \int_\Omega u_0 w_j \; dx. \qquad j = 1, ..k \end{cases} \tag{3.20}$$

and

$$\begin{cases} \int_\Omega \frac{\partial v_k}{\partial t}\phi dx + \int_\Omega \Delta v_k \Delta\phi dx = \int_\Omega |\nabla u_k|^{\alpha_2}\phi dx + \int_\Omega |u_k|^{\beta_2}\phi dx, \quad t>0 \ x\in\Omega; \\ \\ b_{j,k}(0) = \int_\Omega u_0 w_j \ dx, \qquad\qquad j=1,..,k, \end{cases} \tag{3.21}$$

for every test function $\phi \in W_k$ and under the initial condition $u_k(0) = u_{0k}$ and $v_k(0) = v_{0k}$.

The System (3.20) and (3.21) are of ordinary differential equations ODE for $2k$ unknown coefficients $a_{j,k}(t)$ and $b_{j,k}(t)$ satisfying the conditions of Picard theorem, hence there exits $T_k^* > 0$, and coefficients $a_{j,k}(t)$ and $b_{j,k}(t)$, $j=1,..,k$, solutions of (3.20) and (3.21) on $(0,T_k^*)$.

Now, we shall prove that these sequences are convergent.

Step 1 (Priori estimates) Using Lemma(33) and following the same steps as in the proof of this lemma to deduce, for a fixed k, the existence of time $T_k^* > 0$,

$$T_k^* = \frac{1}{(\sigma-1)\left(\|u_k(.,0)\|_{0,2}^2 + \|v_k(.,0)\|_{0,2}^2\right)^{\sigma-1} C}, \tag{3.22}$$

such that the sequences $(u_k)_k, (v_k)_k$ are defined on $(0,T_k^*)$.

Now using the fact that for all k:

$$\|u_k(.,0)\|_{0,2} \le \|u_0(.)\|_{0,2}, \qquad \|v_k(.,0)\|_{0,2} \le \|v_0(.)\|_{0,2}, \tag{3.23}$$

to get the existence a time $T^* > 0, \quad T^* \le T_k^*$, defined by

$$T^* = \frac{1}{(\sigma-1)\left(\|u(.,0)\|_{0,2}^2 + \|v(.,0)\|_{0,2}^2\right)^{(\sigma-1)} C}. \tag{3.24}$$

Step 2 (Boundedness of the approach solutions $(u_k,v_k)_{k\in\mathbb{N}}$)

By repeating the same steps in Lemma (32) and Lemma (33), we observe that for all $k \geqslant 1$

and all $0<t<T^*$,

$$\max(\int_\Omega u_k^2(x,t)dx;\int_\Omega v_k^2(x,t)dx) \le \left(\frac{\left(\|u_0\|_{0,2}^2+\|v_0\|_{0,2}^2\right)^{(\sigma-1)}}{1-(\sigma-1)\left(\|u_0\|_{0,2}^2+\|v_0\|_{0,2}^2\right)^{(\sigma-1)}Ct}\right)^{\frac{1}{(\sigma-1)}} <\infty,$$

and

$$\max(\int_0^t\int_\Omega|\Delta u_k|^2dxds;\int_0^t\int_\Omega|\Delta v_k|^2dxds) \le \left(\|u_0\|_{0,2}^2+\|v_0\|_{0,2}^2\right)$$
$$+Ct\left(\frac{\left(\|u_0\|_{0,2}^2+\|v_0\|_{0,2}^2\right)^{(\sigma-1)}}{1-(\sigma-1)\left(\|u_0\|_{0,2}^2+\|v_0\|_{0,2}^2\right)^{(\sigma-1)}Ct}\right)^{\frac{\sigma}{(\sigma-1)}} <\infty,$$

$$\max(\int_0^t\int_\Omega|u_k(x,t)|^{\beta_2+1}dxds;\int_0^t\int_\Omega|v_k(x,t)|^{\beta_1+1}dxds) \le \left(\|u_0\|_{0,2}^2+\|v_0\|_{0,2}^2\right)$$
$$+Ct\left(\frac{\left(\|u_0\|_{0,2}^2+\|v_0\|_{0,2}^2\right)^{(\sigma-1)}}{1-(\sigma-1)\left(\|u_0\|_{0,2}^2+\|v_0\|_{0,2}^2\right)^{(\sigma-1)}Ct}\right)^{\frac{\sigma}{(\sigma-1)}} <\infty.$$

It then follows that there exists a subsequence still denoting $(u_k,v_k)_k$,

$$\begin{aligned} u_k,v_k &\in L^\infty((0,T),L^2(\Omega)) \,\cap\, L^2((0,T),H^2(\Omega)),\\ u_k,v_k &\in L^{\beta+1}((0,T)\times\Omega), \quad \forall\ 0<t<T^*, \end{aligned} \tag{3.25}$$

and there exists a function (u,v) such that, for any $t<T^*$,

$$u_k \longrightarrow u \ as\ k\longrightarrow\infty \ in \quad L^\infty_{weak-star}((0,t),L^2_{weak}(\Omega)). \tag{3.26}$$

$$u_k \longrightarrow u \ as\ k\longrightarrow\infty \ in \quad L^2_{weak}((0,t),H^2_{weak}(\Omega)). \tag{3.27}$$

also

$$v_k \longrightarrow v \ as \ k \longrightarrow \infty \ in \quad L^{\infty}_{weak-star}((0,t), L^2_{weak}(\Omega)), \tag{3.28}$$

$$v_k \longrightarrow v \ as \ k \longrightarrow \infty \ in \quad L^2_{weak}((0,t), H^2_{weak}(\Omega)). \tag{3.29}$$

Step3 (Analysis of the non-linear term)

Our aim is to prove that $(\frac{\partial u_k}{\partial t}, \frac{\partial v_k}{\partial t}) \in (L^2((0,t), W^{-n-2,2}(\Omega))^2$?

For this purpose, taking $\phi \in H_0^{n+2}(\Omega)$ and by writing $\phi = \phi_k + (\phi - \phi_k)$ where ϕ_k is the $L^2(\Omega)$ projection of ϕ onto the space E_k, we find

$$\begin{aligned} \int_\Omega \frac{\partial u_k}{\partial t}\phi \ dx &= \int_\Omega \frac{\partial u_k}{\partial t}\phi_k \ dx && (3.30) \\ &= -\int_\Omega \Delta u_k \Delta \phi_k \ dx + \int_\Omega |\nabla v_k|^{\alpha_1}\phi_k \ dx + \int_\Omega |v_k|^{\beta_1}\phi_k \ dx. && (3.31) \end{aligned}$$

Applying Sobolev embedding to the right hand side of first and third terms, using interpolation inequality and Young's inequality, for ϵ well chosen, we get

$$\int_\Omega \Delta u_k \Delta \phi_k \ dx + \int_\Omega |v_k|^{\beta_1}\phi_k \ dx \ \leq \ c[\ \| u_k \|_{H^2} + h(t)] \| \phi_k \|_{H^{n+2}}, \tag{3.32}$$

where h is bounded in $L^{\gamma_1}(0,t)$ independent of k, for some $\gamma_1 > 1$.

As proven in [**15**], we have

$$\int_\Omega |\nabla v_k|^{\alpha_1}\phi_k \ dx \leq cF(t) \| \phi_k \|_{n+2,2} . \tag{3.33}$$

where F is bounded in $L^\gamma(0,t)$ independently of k for some $\gamma > 1$.

Thanks to Lions Lemma [**106**],there exists $c > 0$ independently of ϕ such that

$$\| \phi_k \|_{n+2,2} \leq c \ \ \| \phi \|_{n+2,2} .$$

Hence, by (3.32) and (3.33), we obtain

$$\int_\Omega \frac{\partial u_k}{\partial t}\phi \, dx \;\; \leq \;\; c[\| u_k \|_{H^2} + F(t) + h(t)] \;\; \| \phi \|_{H^{n+2}} . \tag{3.34}$$

Following the same steps as previously to get

$$\int_\Omega \frac{\partial v_k}{\partial t}\phi \, dx \;\; \leq \;\; c\left[\| v_k \|_{H^2} + \tilde{F}(t) + \tilde{h}(t)\right] \;\; \| \phi \|_{H^{n+2}}, \tag{3.35}$$

where $\tilde{F}$ and $\tilde{h}$ are bounded in $L^{\gamma_2}(0,t)$ and $L^{\gamma_3}(0,t)$ independent of k with $\gamma_2, \gamma_3 > 1$, that $\frac{\partial u}{\partial t}$ is uniformly bounded in $L^{\gamma}((0,t), W^{-n-2,2}(\Omega))$, where $\tilde{\gamma} = min(\gamma, \gamma_1, \gamma_2, \gamma_3, 2)$.

Combining (3.34) and (3.35) and owing to Aubin-Lions theorem, see Lions ([**106**], pp 57-58), we deduce that there exist subsequence of (u_k) and (v_k) , still represented by the same notation, such that we find that $(u_k)_k$ is compact in the strong topology of $L^2((0,T), W^{1,r}(\Omega))$ and $(v_k)_k$ is compact in the strong topology of $L^2((0,T), W^{1,r}(\Omega))$ with all $1 < r < \infty$ when $n \leq 2$ and all $r < \frac{2n}{n-2}$ when $n > 2$. Then there exists a subsequence denoted again $((u_k, (v_k))_k$ such that ∇u_k converges strongly to ∇u and ∇v_k converges strongly to ∇v in $L^2((0,t), L^r(\Omega))$, i.e when $k \longrightarrow +\infty$

$$\begin{aligned} \nabla u_k &\longrightarrow \nabla u \quad in \quad L^2((0,t), L^r(\Omega)). \\ \nabla v_k &\longrightarrow \nabla v \quad in \quad L^2((0,t), L^r(\Omega)). \end{aligned} \tag{3.36}$$

Basing on [**15**], we have for $\psi \in C_0^\infty(\Omega \times (0,t))$,

$$\begin{aligned} \left| \int_\Omega (|\nabla u_k|^{\alpha_2} - |\nabla u|^{\alpha_2})\psi \, dx \right| \;\; &\leq \;\; \| \psi \|_\infty \| \nabla u_k - \nabla u \|_{L^r} \\ &\times \;\; (1 + \| u_k \|_{H^2} + \| u \|_{H^2}) \end{aligned} \tag{3.37}$$

$$\left|\int_\Omega (|\nabla v_k|^{\alpha_1} - |\nabla v|^{\alpha_1})\psi \ dx\right| \leq \| \psi \|_\infty \| \nabla v_k - \nabla v \|_{L^r} \times (1 + \| v_k \|_{H^2} + \| v \|_{H^2}) \tag{3.38}$$

Thus, from (3.37) and (3.38), we obtain when $k \to \infty$

$$\int_0^t \int_\Omega (|\nabla u_k|^{\alpha_2} - |\nabla u|^{\alpha_2})\psi \ dxds \to 0, \quad \forall \psi \in C^\infty(\Omega \times (0,t)). \tag{3.39}$$
$$\int_0^t \int_\Omega (|\nabla v_k|^{\alpha_2} - |\nabla v|^{\alpha_2})\psi \ dxds \to 0, \quad \forall \psi \in C^\infty(\Omega \times (0,t)).$$

Step 4. (Passage to the limit)

By (3.25)and exploiting Lions Lemma , we get

$$|u_k|^{\beta_2} \longrightarrow |u|^{\beta_2} \quad in \quad L^{\frac{\beta_2+1}{\beta_2}}((\Omega) \times (0,t)). \qquad (weak), \tag{3.40}$$
$$|v_k|^{\beta_1} \longrightarrow |v|^{\beta_1} \quad in \quad L^{\frac{\beta_1+1}{\beta_1}}((\Omega) \times (0,t)). \qquad (weak)$$

Convergences (3.27), (3.29),(3.39) and (3.40) are sufficient to pass to the limit in order to obtain

$$\int_0^t \int_\Omega \frac{\partial u}{\partial t}\psi dxds + \int_0^t \int_\Omega \Delta u \Delta \psi dxds = \int_0^t \int_\Omega |\nabla|^{\alpha_1}\psi dxds + \int_0^t \int_\Omega |v|^{\beta_1}\psi dxds.$$

and

$$\int_0^t \int_\Omega \frac{\partial u}{\partial t}\psi dxds + \int_0^t \int_\Omega \Delta u \Delta \psi dxds = \int_0^t \int_\Omega |\nabla u|^{\alpha_2}\psi dxds + \int_0^t \int_\Omega |u|^{\beta_2}\psi dxds,$$

for all $\psi \in C_0^\infty((0,T) \times \Omega)) \cap L^2((0,t), H_0^1(\Omega)) \cap L^{\beta_1+1}((0,T) \times \Omega) \cap L^{\beta_2+1}((0,T) \times \Omega))$.

Step5 Proving that $|\nabla u_i|^{\alpha_i} \in L^1((0,t) \times \Omega)$ i=1,2 is immediately by simple calculation from Lemma 3.

Step 6 Observe that from the above estimates, we have

$$(u,v) \in (W^{1,2}(0,t;W^{2,-2}(\Omega)))^2,$$

and so its trace at $t=0$ is well defined and $u(x,0) = lim_{k\to+\infty} u_k(x,0) = u_0(x)$ and $v(x,0) = lim_{k\to+\infty} v_k(x,0) = v_0(x)$.

To schow that $\Delta u = \Delta v = 0$ on the boundary $\partial\Omega$, we need the following lemma

LEMMA 35. *Let τ and T fixed such that $0<\tau<T<T^*$.*
Let $(u_{i,k})_{k\geq 0}$ be the sequence of solutions to the Galerkin system (3.20) and (3.21) and , then there exists a positive constant C_τ such that

$$\int_\tau^T \int_\Omega (\Delta^2 u_k)^2 dxds \leq C_\tau, \int_\tau^T \int_\Omega (\Delta^2 v_k)^2 dxds \leq C_\tau.$$

PROOF. See [**15**].
This leads to

$$\int_\tau^T \int_\Omega (\Delta^2 u)^2 dxds \leq K_\tau,$$
$$\int_\tau^T \int_\Omega (\Delta^2 v)^2 dxds \leq K_\tau,$$

where K_τ is a positive constant.
Consequently, $(u,v) \in (L^2(\tau,T), W^{\frac{3}{2},2}(\partial\Omega))^2$ for any $\tau>0$, and so, $\Delta u = \Delta v = 0$ on $\partial\Omega$ on (τ,T),
which achieves the proof of theorem. $\square$

$\square$

4. Blow-up in Finite Time of Solutions

In this section, we present some materials needed in the proof of our results, combining the argument of [15], and prove our main result.

THEOREM 36. *(Blow-up in finite time of solutions)*
Suppose that

$$1 < \alpha_i < \frac{n+8}{n+2}, \qquad 1 < \beta_i < \frac{n+8}{n}, \quad i = 1, 2.$$

Then, for $u_0, v_0 \in L^2(\Omega)$, *and* u_0 *or* v_0 *sufficiently large, the problem* (1.2) *cannot admit a globally defined weak solution.*

Let's recall (see, e.g. [82]), and the references therein the eigenvalue problem

$$-\Delta\ \psi = \lambda\psi \qquad \psi \in H_0^1(\Omega), \tag{4.1}$$

wich has a smallest positive eigenvalue $\lambda = \lambda_1$ and that the associated eigenfunction ϕ does not vanish in Ω. Furthermore, it can be chosen $\phi > 0$.

The proof is based essentially on the well known technique of Kaplan [82] and this important proposition.

PROPOSITION 37. *Assume* $1 < \alpha_i < \frac{n+8}{n+2}$ *and* $1 < \beta_i < \frac{n+8}{n}$, $\quad i = 1, 2$.
Then for $u_0, v_0 \in L^2(\Omega)$ *satisfying* $\int_\Omega \left(u_0(x)^2 + v_0(x)^2\right)\varphi(x)dx > C = C(\Omega, \beta_i)$ *sufficiently large, where* ϕ *is eigne function there exists a finite time* $(T^\sharp = T^\sharp(M) > 0)$ *such that for :*

$$Z(t) = \int_\Omega u(x,t)\varphi(x)dx \quad \textit{we have} \qquad \lim_{t \longrightarrow T^\sharp} Z(t) = +\infty. \tag{4.2}$$

PROOF. Multiplying the equations (1.2) by φ and integration over Ω, we find

$$\begin{aligned} \int_\Omega u_t \varphi dx \quad &+ \quad \int_\Omega (\Delta^2 u)\varphi dx \\ &= \quad \int_\Omega |\nabla v|^{\alpha_1}\varphi dx + \int_\Omega |v|^{\beta_1}\varphi\ dx \end{aligned} \tag{4.3}$$

and

$$\int_\Omega v_t\varphi dx + \int_\Omega (\Delta^2 v)\varphi dx = \int_\Omega |\nabla u|^{\alpha_2}\varphi dx + \int_\Omega |u|^{\beta_2}\varphi\ dx. \tag{4.4}$$

Using Green formula, we see that

$$\int_\Omega (\Delta^2 u)\varphi dx = \int_\Omega \Delta u\ \Delta\varphi dx = \lambda_1^2 \int_\Omega u\varphi dx,, \tag{4.5}$$

where λ_1 is the smallest positive eigenvalue. Using Jensen's inequality, we get

$$\int_\Omega |u|^\rho \varphi\ dx \geq |\int_\Omega u(x,t)\varphi dx|^\rho, \quad \rho > 1. \tag{4.6}$$

Using the elementary inequality : if $q > p$ and $\lambda > 0$, then

$$x^q \geq \lambda^{\frac{q-p}{q-1}} x^p - \lambda x. \qquad x \geq 0.$$

Taking $Z(t) = \int_\Omega (u(x,t) + v(x,t))\varphi(x)dx$, we deduce that :

$$Z'(t) + (1+\lambda_1^2)\ Z(t) \geq c|Z(t)|^\beta, \tag{4.7}$$

where $\beta = \min\{\beta_1, \beta_2\}$. If :

$$Z(0) \geq \left(\frac{\lambda_1^2}{c}\right)^{\frac{1}{\beta-1}}, \tag{4.8}$$

Finally, It follows that $(T^\sharp < +\infty)$ whenever $Z(0)$ is sufficiently large. $\square$

Conclusion

In this thesis, we treated in one hand the existence of solutions and the blow-up of these solutions for two types of evolution equations, fractional ordinary differential equations and Hamilton- Jacobi equations arising in engineering electronic phenomena and hydrodynamics phenomena respectively, and on the other hand with the decay of solutions to a heat hyperbolic system with viscoelastic term, and an abstract hyperbolic system with infinite memory.

In each problem treated in the first part of this work, the nonlinearities are strong. The existence of solutions for the system of fractional ordinary differential equations was established via Shauder's Theorem and the blow-up of solutions in finite time is proved with any conditions on the parameters of the system.

In the second part, we focused on the stabilization of a quasilinear heat hyperbolic system with a viscoelastic term which we established a general and optimal decay result. We also dealt with the general decay of an abstract linear dissipative integrodiffrential equation with infinite memory modelling linear viscoelasticity which we have extended the decay result obtained in [**68**, **69**]. Our results generalizes and improves several stability results in the literature.

For the Hamilton Jacobi system, the existence of weak solutions is done by the Galerkin method and by compactness theorems. The uniqueness of solutions is an open problem. Using interpolation inequalities and Sobolev injections, we established some results on the blow-up of the solutions in finite time.

Bibliography

[1] R.P. Agarwal, M. Benchohra, S. Hamani*A Survey on Existence Results for Boundary Value Problems of Nonlinear Fractional Differential Equations and Inclusions, Acta Applicandae Mathematicae, 109 (2010) 973-1033.*

[2] R.P. Agarwal, M. Benchohra, B.A. Slimani, Existence results for differential equations with fractional order and impulses. Mem. Differ. Equ. Math. Phys. 44 (2008) 1-21.

[3] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Appl. Math. 109(3) 973-1033 (2010)

[4] N. Alaa, I. Mounir, Global existence for reaction-diffusion systems with mass control and critical growth with respect to the gradient, J. Math. Anal. Appl 253 (2001), 532-557.

[5] F. Alabau-Boussouira, P. Cannarsa, A general method for proving sharp energy decay rates for memory-dissipative evolution equations, C. R. Acad. Sci. Paris, Ser. I 347, 867-872 (2009).

[6] F. Alabau-Boussouira, P. Cannarsa, D. Sforza, Decay estimates for second- order evolution equations with memory, J Funct Anal 254 (2008) 1342-1372.

[7] J.A.D. Appleby, D.W. Reynolds, On necessary and sufficient conditions for exponential stability in linear Volterra integro-differential equations , J. Integral Equations Appl. 16 (2004) 221-240.

[8] R.L. Bagley, P.J. Torvik, Fractional calculus-a different approach to the analysis of viscoelastically damped structures, AIAA J. 21 (1983)741-748.

[9] R. Barreto, E.C. Lapa, J.E. Munoz Rivera, Decay rates for viscoelastic plates with memory, J. Elasticity 44 (1) (1996) 61-87.

[10] R.K. Barreto, J.M. Rivera, Uniform rates of decay in nonlinear viscoelasticity for polynomial decaying kernels, Appl Anal 60 (1996) 263-283.

[11] A. B. Basset, A Treatise on Hydrodynamics, vol. 2, Cambridge University Press. England, 1888.

[12] A. B. Basset, On the descent of a sphere in a viscous liquid, Quart. J. Math.41, (1910) 369-381.

[13] S. Berrimi, S.A. Messaoudi, A decay result for a quasilinear parabolic system, Progress in Nonlinear Differential Equations and their Applications. 53 (2005) 43-50.

[14] *S. Berrimi, S.A. Messaoudi, Existence and decay of solutions of a viscoelastic equation with a nonlinear source, Nonlinear Anal TMA 64 (2006)2314-2331.*

[15] *H. Bellout, S. Benachour, E. Titi, Finite-time singularity versus global regularity for hyper-viscous Hamilton-Jacobi-like equations,Nonlinearity 16 (2003) 1967-1989.*

[16] *P. J. Bushell, W. Okrasinski;* On the maximal interval of existence for solutions to some non-linear Volterra integral equations with convolution kernel*, Bull. London Math. Soc.* ***28*** *(1996) 59-65.*

[17] *E.L. Cabanillas, J.E. Munoz Rivera, Decay rates of solutions of an anisotropic inhomogeneous n-dimensional viscoelastic equation with polynomial decaying kernels, Comm. Math. Phys. 177 (1996) 583-602.*

[18] *R. Caponetto, G. Dongola, L. Fortuna, I. Petras;* Fractional Order Systems: Modeling and Control Applications*, World Scientific, Singapore, 2010.*

[19] *M. Caputo, Linear models of dissipation whose Q is almost frequency independent-II, Geophys. J. R. Astr. Soc. 13 (1967) 529-539.*

[20] *S. Carillo, V. Valente, G.V Caffarelli, A result of existence and uniqueness for an integro-differential system in magneto-viscoelasticity, Applicable Analysis 90 (2010)1791-1802.*

[21] *S. Carillo, V. Valente, G.V Caffarelli, An existence theorem for the magnetic viscoelastic problem, Discrete and Continuous Dynamical Systems Series S 5 (2012) 435-447.*

[22] *S. Carillo, V. Valente, G.V Caffarelli, A linear viscoelasticity problem with a singular memory kernel: an existence and uniqueness result. Differential and Integral Equations 26 (2013)1115-1125.*

[23] *M.M. Cavalcanti, V.N.D. Cavalcanti, J.S.P. Filho, J.A Soriano, Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping, Differential Integral Equations 14 (2001) 85-116.*

[24] *M.M. Cavalcanti, V.N.D. Cavalcanti, J.S.P. Filho, J.A Soriano, Exponential decay for the solution of semi linear viscoelastic wave equations with localized damping. Elect J Diff Eqns 44:(2002) 1-14.*

[25] *M.M. Cavalcanti, V.N.D. Cavalcanti, P. Martinez, General decay rate estimates for viscoelastic dissipative systems, Nonlinear Anal 68 (2008) 177-193.*

[26] *M.M. Cavalcanti, H.P. Oquendo, Frictional versus viscoelastic damping in a semi linear wave equation, SIAM J Control Optim 42 (2003) 1310-1324.*

[27] *M.M. Cavalcanti, V.N.D. Cavalcanti, P. Martinez, General decay rate estimates for viscoelastic dissipative systems, Nonlinear Anal 68 (2008) 177-193.*

[28] *M.M. Cavalcanti, A. Guesmia, General decay rates of solutions to a nonlinear wave equation with boundary condition of memory type, Differential and Integral Equations. 18 (2005) 583-600.*

[29] *M.M. Cavalcanti, N. Valéria, D. Cavalcanti, I. Lasiecka, C. M. Webler, Intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density,Advances in Nonlinear Analysis, 6(2) 121-145.*

[30] *P. M. Carvalho Neto, R. Fehlberg Junior* Conditions to the absence of blow up solutions to fractional differential equations*, Submitted exclusively to the London Mathematical Society doi:10.1112/0000/000000*

[31] *M. Chipot, F. B.Weissler, Some blow-up results for a nonlinear parabolic equation with a gradient term, SIAM J. Math. Anal 20 (4), (1989)886-907 .*

[32] *L. Collatz, Funktionalanalysis und numerische Mathematik. Springer, Berlin (1968).*

[33] *P. Collet, J.P Eckmann, H. Epstein, J. Stubbe, A global attracting set for the Kuramoto-Sivashinsky equation, Commun. Math. Phys. 152 (1993)203-14.*

[34] *B.D. Coleman, V.J. Mizel, On the general theory of fading memory, Arch. Ration. Mech. Anal. 29 (1968) 18-31.*

[35] *B.D. Coleman, W. Noll, Foundations of linear viscoelasticity, Rev. Modern Phys. 33 (1961) 239-249.*

[36] *V.V. Chepyzhov, V. Pata, Some remarks on stability of semigroups arising from linear viscoelasticity, Asymptot. Anal. 46 (2006) 251-273.*

[37] *E. Cuesta, J.F. Codes, Image processing by means of a linear integro-differential equationIn: Hamza, M.H. (ed.) Visualization, Imaging, and Image Processing 2003, Paper 91.*

[38] *C.M. Dafermos, Contraction semigroups and trend to equilibrium in continuum mechanics, Applications of Methods of Functional Analysis to Problems in Mechanics (P. Germain and B. Nayroles, Eds.), pp. 295-306, Lecture Notes in Mathematics, no. 503, Springer-Verlag, Berlin-New York, 1976.*

[39] *C.M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Ration. Mech. Anal. 37 (1970) 297-308.*

[40] *C.M. Dafermos, On abstract Volterra equations with applications to linear viscoelasticity, J. Differ. Equations 7 (1970) 554-569.*

[41] *C.M. Dafermos, Development of singularities in the motion of material with fading memory, Arch. Ration. Mech. Anal. 91 (1) (1985) 193-205.*

[42] *G. Dassios, F. Zafiropoulos, Equipartition of energy in linearized 3-d viscoelasticity, Quart. Appl. Math. 48 (4) (1990) 715-730.*

[43] *L. Debnath, Recent applications of fractional calculus to science and engineering, Int. J. Math. Sci. Appl, 54(2003) 3413-3442.*

[44] *D. Delbosco, L. Rodino, Existence and uniqueness for a fractional differnential equation. Journal of Mathematical Analysis and Applications, 204(1996), 609-625.*

[45] M. Du, Z. Wang, H. Hu, Measuring memory with the order of fractional derivativeScientific Reports 3, Article number: 3431 (2013)doi:10.1038/srep03431

[46] M. Eller, J.E. Lagnese, S. Nicaise, Decay rates for solutions of a Maxwell system with nonlinear boundary damping , Comput. Appl. Math. 21 (2002)135-165.

[47] H. Englern, B. Kawohl, S. Luckhaus, Gradient estimates for solutions of parabolic equations and systems, J. Math. Anal. Appl. **147** (1990) 309-329.

[48] M. Fabrizio, A. Morro, Viscoelastic relaxation functions compatible with thermodynamics, J. Elasticity 19 (1988) 63-75.

[49] M. Fabrizio, B. Lazzari, On the existence and asymptotic stability of solutions for linear viscoelastic solids, Arch. Ration. Mech. Anal. 116 (1991) 139-152.

[50] M. Fabrizio, A. Morro, Mathematical problems in linear viscoelasticity, SIAM Studies in Applied Mathematics no.12, SIAM, Philadelphia, 1992.

[51] M. Fabrizio, C. Giorgi, V. Pata, A new approach to equations with memory , Arch. Ration. Mech. Anal. 198 (2010) 189-232.

[52] J. Ferreira, M. Rojas, On global weak solutions of a nonlinear evolution equation in noncylindrical domain , in: Proceedings of the 9th International Colloquium on Differential Equations Ed. VSP, 1999, pp. 155-162.

[53] J. Ferreira, D.C. Pereira, On a hyperbolic degenerate evolution equation with strong dissipation, Int. J. Math. Math. Sci. 15 (1992) 543-552.

[54] M. Fila, Remarks on blow-up for a nonlinear parabolic equation with a gradient term, Proc. Amer. Math. Soc. 111 (3), (1991)795-801.

[55] K. Furati, MD. Kassim, N-e Tatar, Non-existence of global solutions for a differential equation involving Hilfer fractional derivative, Electron. J. Differ. Equ. 2013, Article ID 391062 (2013).

[56] K. Furati, M. Kirane, Necessary conditions for the existence of global solutions to systems of fractional differential equations, Fract. Calc. Appl. Anal. 11 (2008) 281-298.

[57] K.F. Furati, N.E. Tatar, An existence result for a nonlocal fractional differential problem, J. Fract. Calc.,26 (2004) 43-51.

[58] K.F. Furati, N.E Tatar, Behavior of solutions for a weighted Cauchy-type fractional differential problem, J. Fract. Calc., 28 (2005) 23-42.

[59] K.F. Furati, M.D. Kassim, N.E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, Comput. Math. Appl., 64 (2012) 1616-1626.

[60] *V. A. Galaktionov, E. Mitidieri, S. I. Pohozaev , On global solutions and blow-up for Kuramoto-Sivashinsky-type models and well-posed Burnett equations, Nonlinear Anal., Theory, Methods, Appl. 70 (8)(2009)2930-2952.*

[61] *V. A. Galaktionov, S. I. Pokhozhaev, Third-order nonlinear dispersive equations: shocks, rarefaction, and blow-up waves, Zh. Vychisl. Mat. i Mat. Fiz. 48(10), (2008)1819-1846 [Comput. Math. and Math. Phys. 48 (10), 1784-1810 (2008)].*

[62] *L. Gaul, P. Klein, S. Kempfle, Damping description involving fractional operatorsMech. Systems Signal Processing 5 (1991) 81-88.*

[63] *A. George, A. Anastassiou, Fractional Differentiation Inequalities, Springer-Verlag New York,2009*

[64] *C. Giorgi, J.E. Muñoz Revira, V. Pata, Global attractors for a semilinear hyperbolic equation in viscoelasticity, J. Math. Anal. Appl. 260 (2001) 83-99.*

[65] *W. G. Glockle, T.F. Nonnenmacher, A fractional calculus approach of self similar protein dynamics. Biophys. J. 68 (1995) 46-53.*

[66] *R. Gorenflom, F. Mainardi, Fractional calculus, International Centre for Mechanical Sciences (Courses and Lectures), vol 378. Springer, Vienna.*

[67] *M. Grasselli, J.E. Munoz Rivera, V. Pata, On the energy decay of the linear thermoelastic plate with memory, J. Math. Anal. Appl. 309 (2005) 1-14.*

[68] *A. Guesmia, Asymptotic stability of abstract dissipative systems with infinite memory, J. Math. Anal. Appl. 382 (2011) 748-760.*

[69] *A. Guesmia, S.A. Messaoudi, A new approach to the stability of an abstract system in the presence of infinite history, J. Math. Anal. Appl. 416 (2014) 212-228.*

[70] *A. Guesmia, S.A. Messaoudi, On the stabilization of Timoshenko systems with memory cand different speeds of wave propagation, Applied Mathematics and Computation. 219 (2013) 9424-9437.*

[71] *X. Han, M. Wang, General decay estimate of energy for the second order evaluation equation with memory, Acta Applicandae Mathematicae 110 (2010) 195-207.*

[72] *Xi. Han, M. Wang, General decay of energy for a viscoelastic equation with nonlinear damping, Math. Methods Appl. Sci. 32(3), (2009)346-358.*

[73] *N. Hayashi, E. I. Kaikina, P. I. Naumkin, I. A. Shishmarev, Asymptotics for Dissipative Nonlinear Equations, in Lecture Notes in Math. (Springer, New York, 2006), Vol. 1884.*

[74] *D. Henry,* Geometric Theory of Semilinear Parabolic Equations, vol.**40**, of Lectures Notes in Mathematics*, Springer, Berlin, Germany, 1981.*

[75] *R. Hilfer,* Applications of Fractional Calculus in Physics*, Academic Press, Orlando 1999.*

[76] *R. Hilfer, Applications of fractional calculus in physics (Edited by R. Hilfer). World Scientific Publishing Co., Inc., River Edge, NJ, 2000.*

[77] *W.J. Hrusa, Global existence and asymptotic stability for a semi linear Volterra equation with large initial data, SIAM J Math Anal 16 (1985) 110-134.*

[78] *W.J. Hrusa, M. Renardy, On wave propagation in linear viscoelasticity,Quarterly Appl Math 43 (1985) 237-254.*

[79] *W.J. Hrusa, M. Renardy, On a Class of Quasilinear Partial Integrodifferential Equations with Singular Kernels, J Diff Equations 64(1986) 195-220.*

[80] *A. Kadem, M. Kirane, C. M. Kirk, W. E. Olmstead,* Blowing-up Solutions to Systems of Fractional Differential and Integral Equations with Exponential Nonlinearities, *IMA J Appl Math (2014)* ***79****, (6): pp. 1077-1088.*

[81] *M. Kardar, G. Parisi, Y.C. Zhang, Dynamic Scaling of Growing Interfaces,Physical Review Letters, 56(1986)889-892.*

[82] *S.Kaplan, On the growth of solutions of quasi-linear parabolic equations, Commun. Pure Appl. Math, 16 (1963)305-30.*

[83] *M.D. Kassim, K.M. Furati, N-E. Tatar, On a differential equation involving Hilfer-Hadamard fractional derivative. Abstr. Appl. Anal. 2012, Article ID 391062 (2012).*

[84] *M.D. Kassim, N.E. Tatar, Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative, Abstr. Appl. Anal., Article ID 605029(2013) 1-12.*

[85] *M. O. Korpusov, On the Blow-Up of the Solution of an Equation Related to the Hamilton-Jacobi Equation,Matematicheskie Zametki, 93(2013) 81-95.*

[86] *A.A.Kilbas, S.A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions. Differ. Equ. 41 (2005) 84-89.*

[87] *A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.*

[88] *M. Kirane, A. Kadem, A. Debbouche, Blowing-up to times fractional differential equations, Math. Nachr., 1-8 (2013).*

[89] *M. Kirane, S. A. Malik,* Profile of blowing-up solutions to a nonlinear system of fractional differential equations*, Nonlinear Analysis: Theory, Methods & Applications, Volume (73), 2010, pp. 3723-3736.*

[90] *C. M. Kirk, W. E. Olmstead, C. A. Roberts;* A system of nonlinear Volterra equations with blow-up solutions*, J. Integral Equations Appl.* ***25****, (2013), pp. 377-393.*

[91] M.Kirane, A.Kadem, A.Debbouche,Blowing-up solutions to two-times fractional differential equations,Math. Nachr., (2013)1-8.

[92] M. Kirane, Y. Laskri, N-e. Tatar, Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives, J. Math. Anal. Appl., 312 (2005) 488-501.

[93] M. Kirane, SA. Malik, The profile of blowing-up solutions to a nonlinear system of fractional differential equations, Nonlinear Anal., Theory Methods Appl. 73(12) (2010)3723-3736 .

[94] M. Kirane, N-E Tatar, Absence of local and global solutions to an elliptic system with time-fractional dynamical boundary conditions, Siberian J. Math., 48 (2007) 477-488.

[95] M. Kirane, N-e Tatar, Nonexistence of solutions to a hyperbolic equation with a time fractional damping. Z. Anal. Anwend. 25 (2006)131-142.

[96] M. Kirane, M. Medved, N-e Tatar, On the nonexistence of blowing-up solutions to a fractional functional differential equations, Georgian Math. J. 19 (2012)127-144.

[97] V. Kiryakova, Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series, vol. 301. Longman, Harlow (1994). Copublished in the United States with Wiley, New York (1994).

[98] Y. Kuramoto, T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, Prog. Theor. Phys. 55 (1976) 356-69.

[99] V. Lakshmikantham, A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations, Appl. Math. Lett, 21 (2008) 828-834.

[100] Y.Laskri , N-E. Tatar, The critical exponent for an ordinary fractional differential problem. Comput. Math. Appl., 59 (2010) 1266-1270.

[101] J.E. Lagnese, Asymptotic energy estimates for Kirchoff plates subject to weak viscoelastic damping, International Series of Numerical Mathematics, vol. 91. Birkhcauser-Verlag: Bassel, 1989.

[102] A. Larios, E.S. Titi, Global regularity vs. finite-time singularities: some paradigms on the effect of boundary conditions and certain perturbations, "Topics in the Theory of the Navier-Stokes Equations", J.C. Robinson, J.L. Rodigo, W. Sadowski, & A. Vidal- L'opez, editors. London Mathematical Society, Cambridge University Press. (2016)96-125.

[103] I. Lasiecka, S.A. Messaoudi, M.I. Mustafa,Note on intrinsic decay rates for abstract wave equations with memory, J Math Physics 54:(2013)1-18.

[104] G. Lebon, C. Perez-Garcia, J. Casas-Vazquez, On the thermodynamic foundations of viscoelasticity, J. Chem. Phys. 88 (1988) 5068-5075.

[105] *G.W. Leibniz, Letter from Hanover, Germany, September 30, 1695 to G.A. L'Hospital, in J Leibnizen Mathematische Schriften, vol. 2 (Olms Verlag, Hildesheim, Germany, 1962), pp. 301-302. First published in 1849.*

[106] *J.-L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod-Gauthier Villars, Paris, France, 1969.*

[107] *G. Liu, H. Chen, Global and blow-up of solutions for a quasilinear parabolic system with viscoelastic and source terms, Math. Methods Appl. Sci.* ***37*** *(2014) 148-156.*

[108] *W.J. Liu, General decay of solutions to a viscoelastic wave equation with nonlinear localized damping, Ann. Acad. Sci. Fenn. Math. 34(1),(2009)291-302 .*

[109] *W.J. Liu, General decay rate estimate for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms, J. Math. Phys. 50(11)(2009)113506 .*

[110] *Y. Liu, J.Y. Wong,* Global Existence of Solutions for a System of Singular Fractional Differential Equations with Impulse Effects, *J. Appl. Math. Informatics,* ***33****, (2015)(3-4) 32-342.*

[111] *Z. Liu, S. Zheng, On the exponential stability of linear viscoelasticity and thermoviscoelasticity, Quart. Appl. Math. 54 (1996) 21-31.*

[112] *B.N. Lundstrom, M.H. Higgs, W.J. Spain, A.L. Fairhall, Fractional differentiation by neocortical pyramidal neurons. Nat. Neurosci. 11 (2008) 1335-1342.*

[113] *C.Marchioro, M. Pulvirenti, Mathematical theory of incompressible nonviscous fluids Appl, Math. Sci, vol 96 1994 (New York: Springer)*

[114] *M. Marras, S. Vernier Piro, G. Viglialoro, Estimates from below of blow-up in a parabolic system with gradient term, International Journal of Pure and Applied Mathematics, Volume 93 No. 2 2014, 297-306.*

[115] *F.Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics. In: Carpinteri, A., Mainardi, F. (eds.), Fractals and Fractional Calculus in Continuum Mechanics, pp. (1997)291-348. Springer, Wien.*

[116] *F. Mainardi , The fractional relaxation-oscillation and fractional diffusion-wave phenomena, Chaos, Solutons and Fractals, 7(9) (1996) 1461-1477.*

[117] *F. Mainardi, P. PIRONI, F. TAMPIERI, On a generalization of the Basset problem via fractional calculus , in: Tabarrok, B. and DOST, S. (Eds.), 5th Canadian Congress of Applied Mechanics, vol. 2, Victoria, Canada, (1995) 836-837.*

[118] *A. Mennouni, A. Youkana, Finite time blow-up of solutions for a nonlinear system of fractional differential equations , Electronic Journal of Differential Equations, 152 (2017) 1-15.*

[119] *S.A. Messaoudi, B. Said-Houari, N-E Tatar, Global existence and asymptotic behavior for a fractional differential equation, Appl. Math. Comput., 188 (2007) 1955-1962.*

[120] *S.Messaoudi General decay of solutions of a viscoelastic equation, J. Math. Anal. Appl. 341 (2008)1457-1467.*

[121] *S.A. Messaoudi, General decay of the solution energy in a viscoelastic equation with a nonlinear source, Nonlinear Anal. Theory, Methods Appl. 69, (2008)2589-2598.*

[122] *S.A. Messaoudi, H.B. Khenous, A General Stability Result for Viscoelastic Equations with Singular Kernels, J Appl Computat Math 4: (2015)221. doi: 10.4172/2168-9679.1000221*

[123] *S.A. Messaoudi, B.S. Houari, Uniform decay in a Timoshenko-type with past history, J.Math.Anal.Appl. 360 (2009) 459-475.*

[124] *S.A. Messaoudi, B. Tellab, A general decay result in a quasilinear parabolic system with viscoelastic term, Applied Mathematics Letters **25** (2012), 443-447.*

[125] *S.A Messaoudi, W. Al-Khulaifi, General and optimal decay for a quasilinear viscoelastic equation, Applied Mathematics Letters **66** (2017) 16-22.*

[126] *D. Michelson, Steady solutions of the Kuramoto-Sivashinsky equation, Physica D 19 (1986)89-111.*

[127] *K. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, Inc., New York, 1993.*

[128] *J.E. Muñoz Revira, M. Grazia Naso, Optimal energy decay rate for a class of weakly dissipative second-order systems with memory , Appl. Math. Lett. 23 (2010) 743-746.*

[129] *J.E. Munoz Rivera, M. Naso, On the decay of the energy for systems with memory and indefinite dissipation, Asymptotic Analysis 49(3-4) (2006) 189-204.*

[130] *J.E. Munoz Rivera, M.G. Naso, F.M Vegni, Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory,Journal of Mathematical Analysis and Applications, 286(2) (2003)692-704.*

[131] *J.E. Munoz Rivera, H.D. Fernández Sare, Stability of Timoshenko systems with past history, J.Math.Anal.Appl. 339 (2008) 482-502.*

[132] *J.E. Munoz Revira, M. Grazia Naso, Asymptotic stability of semigroups associated with linear weak dissipative systems with memory, J. Math. Anal. Appl. 326 (2007) 691-707.*

[133] *M.I Mustafa, M.Kafini, Decay rates for memory-type plate system with delay and source term, Math. Meth. Appl. Sci., 40 (2017)883-895.*

[134] *MI. Mustafa, S.A. Messaoudi, General stability result for viscoelastic wave equations, J Math Physics 53.(2012)053702 .*

[135] *MI. Mustafa, S.A. Messaoudi, General stability result for viscoelastic wave equations, Journal Of Mathematical Physics 53,053702 (2012).*

[136] *J.E. Munoz Rivera, Asymptotic behavior in linear viscoelasticity, Quart Appl Math 52 (1994)628-648.*

[137] *J.E. Muñoz Revira, M. Grazia Naso, On the decay of the energy for systems with memory and indefinite dissipation, Asympt Anal 49 (2006) 189-204.*

[138] *J.E. Muñoz Revira, M. Grazia Naso, Asymptotic stability of semigroups associated with linear weak dissipative systems with memory , J. Math. Anal. Appl. 326 (2007) 691-707.*

[139] *J.E. Munoz Rivera, A. Peres Salvatierra, Asymptotic behaviour of the energy in partially viscoelastic materials, Quart. Appl. Math. 59 (2001) 557-578.*

[140] *W. Mydlarczyk, W. Okrasinski, C. A. Roberts,* Blow-up solutions to a system of nonlinear Volterra equations,*J. Math. Anal. Appl* ***30*** *(2005) 208-218.*

[141] *J.E. Muñoz Revira, M. Grazia Naso,F.M Vegni, Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory. J Math Anal Appl 286 (2003)692-704.*

[142] *M. Nakao, Y. Ohara, Gradient estimates for a quasilinear parabolic equation of the mean curvature type, J. Math. Soc. Japan* ***48 # 3*** *(1996) 455-466.*

[143] *M. Nakao, C. Chen, Global existence and gradient estimates for the quasilinear parabolic equations of m-Laplacian type with a nonlinear convection term , Journal of Differential Equations* ***162*** *(2000) 224-250.*

[144] *B. Nicolaenko, B. Scheurer, R. Temam, Some global dynamical properties of the Kuramoto-Sivashinsky equations, nonlinear stability and attractors Physica D 16 (1985)155-83.*

[145] *J.A. Nohel, Nonlinear Volterra equations for the heat flow in materials with memory, Integral and functional differential equations, Lecture notes in Pure and Applied Mathematics, Marcel Dekker Inc. 1981*

[146] *K. B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.*

[147] *K. B. Oldham, J. Spanier, The fractional calculus. Theory and applications of differentiation and integration to arbitrary order , Academic Press, New York-London,1974.*

[148] *V. Pata, Stability and exponential stability in linear viscoelasticity, Milan J. Math. 77 (2009) 333-360.*

[149] *A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.*

[150] *V. Pata, Exponential stability in linear viscoelasticity, Quart. Appl. Math. 3 (2006) 499-513.*

[151] *V.Pata Stability and exponential stability in linear viscoelasticity, Milan J. Math. Vol. 77 (2009) 333-360.*

[152] V. Pata, *Exponential stability in linear viscoelasticity with almost flat memory kernels, Commun. Pure Appl. Anal. 9 (2010) 721-730.*

[153] *J.Park, S. Park,General decay for quasilinear viscoelastic equations with nonlinear weak damping, J. Math. Phys. 50(8)(2009)083505 .*

[154] *I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation . Fract. Calc. Appl. Anal. 5 (2002) 367-386 .*

[155] *I. Podlubny, I. Petras, B.M. Vinagre, P. O'Leary, L. Dorcak, Analogue realizations of fractional-order controllers. Fractional order calculus and its applications, Nonlinear Dyn. 29, (2002)281-296.*

[156] *S. I. Pohozaev,Critical nonlinearities in partial differential equations, Milan J.Math. 77 (1), (2009)127-150 .*

[157] *S. I. Pokhozhaev, On blow-up of solutions of the Kuramoto-Sivashinsky equation, Mat. Sb. 199 (9), (2008)97- 106 [Sb.Math. 199 (9), (2008)1355-1365].*

[158] *G. Da Prato, M. Iannelli, Existence and regularity for a class of integro-differential equations of parabolic type, J. Math. Anal. Appl. 112 (1985) 36-55.*

[159] *P. Pucci, J. Serrin, Asymptotic stability for nonlinear parabolic systems, Energy methods in continuum mechanics, Kluwer Acad. Publ., Dordrecht, 1996.*

[160] *D. Qun, L. Huilai,* To study blowing-up of a nonlinear system of fractional differential equations*, Sci Sin Math, 42 (2012) 1205-1212.*

[161] *M. Renardy, W.J. Hrusa, J.A. Nohel, Mathematical problems in viscoelasticity,Harlow John Wiley & Sons, Inc., New York, 1987.*

[162] *M. Renardy, W.J. Hrusa, J.A. Nohel, Mathematical Problems in Viscoelasticity, Pitman Monogr. Pure Appl. Math., vol. 35, Longman Sci. Tech., Harlow, 1988.*

[163] *M. Renardy,Mathematical Analysis of Viscoelastic Flows, CBMS-NSF Conference Series in Applied Mathematics 73, SIAM (2000), 104 pp*

[164] *Yu.A. Rossikhin, M.V. Shitikova, Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results. Appl.Mech. Rev. 63 (2010) 1010801-1-51 .*

[165] *A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P.Mikhailov, Peaking modes in Problems for Quasilinear Parabolic Equations (Nauka, Moscow, 1987) [in Russian].*

[166] *S.G. Samko, A.A. Kilbas, Marichev, O.I.: Fractional Integrals and Derivatives. Theory and Applications. Gordon , Breach, Yverdon (1993).*

[167] *G. Samko, A. A. Kilbas, O. I. Marichev,* Fractional integrals and derivatives: theory and applications*, Gordon and Breach, Yverdon, (1993).*

[168] G.Sivashinsky, On flame propagation under conditions of stoichiometry, SIAM J. Appl. Math. 39 (1980)67-82.

[169] P. Souplet, Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions, Diff. Integral Eqns 15(2002)237-56.

[170] P. Souplet, F. B.Weissler, Poincareś inequality and global solutions of a nonlinear parabolic equation, Ann. Inst. H. Poincare' Anal. Non Liné aire 16 (3),(1999) 335-371 .

[171] M. Stiassnie, On the application of fractional calculus for formulation of viscoelastic models, Appl. Math. Model. 3, (1979)300-302.

[172] E. Tadmor, The well-posedness of the Kuramoto-Sivashinsky equation, SIAM J. Math. Anal, 17 (1986)884-93.

[173] N-e Tatar, Nonexistence results for a fractional problem arising in thermal diffusion in fractal media, Chaos Solitons Fractals 36(5) (2008)1205-1214.

[174] N.E. Tatar, Existence results for an evolution problem with fractional nonlocal conditions, Comput. Math. Appl.,60 (2010) 2971-2982.

[175] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators (Amsterdam: North-Holland)1978.

[176] Han Xi, M. Wang, General decay of energy for a viscoelastic equation with nonlinear damping, Math Meth Appl Sci 32 (2009) 346-358.

[177] H.M. Yin, On parabolic Volterra equations in several space dimensions, SIAM J. of Mathematical Analysis. **22** *(1991) 1723-1737.*

[178] C.Yu, G.Gao, Existence of fractional differential equations, J. Math. Anal. Appl. 310 (2005) 26-29

[179] A. Youkana, S.A. Messaoudi, A.Guesmia, A general decay and optimal decay result in a heat system with a viscoelastic term , (submitted).

[180] A. Youkana, Stability of an abstract system with infinite history, (submitted).

Abstract:

The goal of this thesis is to study on one hand some questions on the existence and the blow-up in finite time of solutions of two type of nonlinear evolution equations, fractional differential equations and Hamilton-Jacobi equations, and on the other hand the decay of solutions for two hyperbolic problems related to viscoelasticity.

Résumé :

L'objectif de cette thèse est d'étudier d'une part l'existence et l'explosion en temps fini des solutions de deux types d'équations d'évolution non linéaires, des équations différentielles fractionnaires et des équations de type Hamilton-Jacobi, et d'autre part la décroissance des solutions de deux différents problèmes hyperboliques intervenant en viscoélasticité.

ملخص:

الهدف من هذه الأطروحة هو، من جهة، دراسة بعض المسائل المتعلقة بوجود الحل وانفجاره لنوعين من معادلات التطور غير الخطية، معادلات تفاضلية كسرية ومعادلات هاملتون-جاكوبي، ومن ناحية أخرى دراسة تخامد الحلول لمسألتي المرونة اللزجة.