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UNIVERSITÉ BATNA 2
FACULTÉ DE MATHÉMATIQUES-INFORMATIQUE
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THÈSE

Présentée en vue d'obtenir du diplôme de doctorat
en sciences en mathématiques
Spécialité: Mathématiques Appliquées

Par

ZAOUIA Saliha

THÈME

Approximation de problèmes fonctionnels et problèmes aux valeurs propres par quasi-interpolants splines

Soutenue le : 19 12 2017

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CONTENTS

INTRODUCTION

It legitimate to say that the first mathematical reference to splines is the paper of Schoenberg. He was one of the first to present and used the spline functions in the approximation theory. The spline theory was developed between 1950 and 1960, be used for scientific calculus (approximation, integral, differential equation). Currently, the splines are considered as an important tool in numerical analysis.

The spline term signify "elastic tongue". In mathematics, a splines are a curves defined piecewise via many polynomials, they have been extensively applied in vector graphics, signal interpolation, animation, computer graphics and other fields of sciences and engineering. A spline function is a function that consists by joining polynomials together at fixed points called knots. They are absolutely generalizations of the notion of polygonal lines. An important class of splines regarding minimal support with respect to a given degree, smoothness, and domain partition is a B-spline, or basis spline. These functions play an essential role in numerical analysis and applied mathematics, namely, the theory of best approximation. Schoenberg was one of the first to introduce spline functions, (cf. [72, 73]). These functions has been the subject of a lot of works over the past decades; see, ([18, 61, 67]). These works discuss all of the usual approximation questions, including existence, smoothness, characterization, stability, uniqueness, strong uniqueness, approximation procedures and convergence analysis. A practical problems in applied mathematics are boundary value problems. Over the years, several papers are developed to numerically solve these types of boundary value problems via spline methods such as finite difference, (cf. [1, 2, 5, 10, 11]), Galerkin, (cf. [13, 16, 23, 15, 26, 39, 43]), orthogonal splines, (cf. [44, 45, 46, 53, 60]), collocation, (cf. [62, 76, 77, 78]) and finite element method, (cf. [79, 80, 81, 82, 83]).

Recently, numerical schemes are constructed to numerically solve different functional equations via spline quasi-interpolants, (cf. [29, 30, 22, 27, 31, 32, 34, 52, 54, 56, 58, 64, 65, 68, 69, 70, 56, 65, 68, 69, 70]).

The purpose of this thesis is to develop the spline quasi-interpolants theory for more general cases to solve some classes of functional problems. The rest of the thesis is organized as follows

The first chapter offers an analysis, definitions and some preliminary results of the concept of spline functions and spline quasi-interpolants. We describe the explicit formulas about quasi-interpolants of degrees $2 \leq d \leq 6$. We give their corresponding norms and approximation orders.

In the second chapter, we present two collocation methods based on spline quasi-interpolants for the approximate solution of integral eigenvalue problems of an integral operator with a regular kernel.

The purpose of the third chapter is to develop the septic quasi-interpolants where we calculate all their coefficients. We we use these results to investigate the generalized Fredholm integral equation of the second kind of the following form

$$u(s) - \sum_{k=1}^{m} \int_a^b H_k(s,t)u(t)dt = f(s), \qquad m \in \mathbb{N}^*, \quad a \leq s \leq b.$$

We introduce three degenerate kernel methods; the latter is a combination of the two previously established methods in the literature. Moreover, we provide a convergence analysis and we give new error bounds. Lastly, we exhibit some numerical examples and compare them with previous results in the literature.

Several works have investigated the self-adjoint singularly-perturbed two-point boundary-value problem:

$$-\epsilon y''(x) + p(x)y(x) = f(x), \quad p(x) > 0, \quad x \in [0,1],$$

with the boundary conditions

$$y(0) = \alpha_0 \quad y(0) = \alpha_1.$$

In the same perspective, the purpose of [50] is to present a method based on sixth degree B-splines for constructing numerical solutions to fifth-order boundary-value problems of the form:

$$y"(x) + p(x)y(x) = f(x), \quad p(x) > 0, \quad x \in [a, b],$$

with the boundary conditions

$$\varphi^{(k)}(a) = \alpha_k, \quad k = 0\ldots 2,$$

$$\varphi^{(k)}(b) = \beta_k, \quad k = 0\ldots 1.$$

More recently, the authors of [51] have studied two sextic-spline collocation methods to compute the numerical solution of nonlinear fifth order boundary-value problems (BVPs) described by a differential equation and boundary conditions of the form

$$\varphi^{(4)}(x) = \psi(x, \varphi(x), \varphi^{(1)}(x), \varphi^{(2)}(x), \varphi^{(3)}(x), \varphi^{(4)}(x), \varphi^{(4)}(x), \varphi^{(4)}(x)) \qquad a \le x \le b,$$

with the boundary conditions

$$\varphi^{(k)}(a) = \alpha_k, \quad k = 0\ldots 2,$$

$$\varphi^{(k)}(b) = \beta_k, \quad k = 0\ldots 1.$$

The last chapter aims to numerically solve the following generalized boundary value problems:

$$\varphi^{(7)}(\tau) = \psi(\tau, \varphi(\tau), \varphi^{(1)}(\tau), \varphi^{(2)}(\tau), \varphi^{(3)}(\tau), \varphi^{(4)}(\tau), \varphi^{(5)}(\tau), \varphi^{(6)}(\tau)) \qquad a \le \tau \le b,$$

with the boundary conditions

$$\varphi^{(k)}(a) = \alpha_k, \quad k = 0\ldots 3,$$

$$\varphi^{(k)}(b) = \beta_k, \quad k = 0\ldots 2.$$

Additionally, we develop two spline quasi-interpolants methods

CHAPTER 1

Spline Quasi-interpolants

1.1 Motivation, Definitions and Properties

Let $n \in \mathbb{N}$, denoting by $\mathbb{P}_n$ the polynomial space with real coefficients and degree less than or equal to n. Recall that $\mathbb{P}_n$ is $\mathbb{R}$-vector space of dimension $n+1$.

Let $a = x_0 < x_1 < \dots x_{N-1} < x_n = b$.

Let us consider a given data

$$(x_i, y_i), \quad i = 1, \dots n, \tag{1.1}$$

the interpolation problem is to find a function $\varphi(.)$ such that

$$y_i = \varphi(x_i), \quad i = 1, \dots n. \tag{1.2}$$

This equation is called the interpolation equation or interpolation condition. It says that the function $\varphi(.)$ passes through the data points.

A function $\varphi(.)$ satisfying the interpolation condition is called an interpolating function for the data.

Such a function is referred to as an *interpolant*.

An interpolation problem differs from the others in many aspects :

- interpolant properties or structures, which may be described in terms of additional interpolation conditions;
- adaptability to changes in sample data;

- computational efficiency in determining an interpolant;
- computational efficiency in evaluating the interpolant at the query point(s);
- computational accuracy if the samples are from an exact function as in traditional mathematics handbooks, or in lookup tables in today's computer systems .

1.2 Lagrange polynomial

For $i \in \{0, \dots, n\}$, consider the following polynomial ℓ_i :

$$\ell_i(x) = \prod_{j=0,\dots,n,\ j\neq i} \frac{x - x_j}{x_i - x_j}.$$

Note that the degree of polynomial ℓ_i is n. Let $k \in \{0, \dots, n\}$, such that

$$\ell_i(x_k) = \begin{cases} 0 \text{ si } k \neq i \\ 1 \text{ si } k = i \end{cases} \tag{1.1}$$

Letting

$$p_n(x) = \sum_{i=0}^{n} y_i \ell_i(x). \tag{1.2}$$

Remark that $p_n \in \mathbb{P}_n$. Let $k \in \{0, \dots, n\}$, with

$$p_n(x_k) = \sum_{i=0}^{n} y_i \delta_{ik} = y_k. \tag{1.3}$$

Theorem 1. *Given data set $D = \{(x_i, y_i)\}_{i=0}^{n}$ in $\mathbb{R}^2$, with distinct x_i's, there exist a unique polynomial $p \in \mathbb{P}_n$ such that*

$$p(x_i) = y_i,\ \forall i \in \{0, \dots, n\}, \tag{1.4}$$

This polynomial p_n is given by (1.2). *which is called* **Lagrange interpolating polynomial** *at (x_i, y_i), $i = 0, \dots, n$.*

Remark 1.2.1. If x_i are distinct,the set $\mathscr{B} = \{\ell_0, \dots, \ell_n\}$ is **basis** of $\mathbb{P}_n$, which is called **Lagrange polynomial basis** associated with interpolation points $x_0, \dots, x_n$.

1.3 Newton polynomial interpolation

Let f be a function on I such that $\{x_i,\ i = 0,\dots,n\} \subset I$.

where no two xj are the same, the Newton interpolation polynomial is a linear combination of Newton basis polynomials

Notation 1.3.1. Let $k \in \{0,\dots,n\}$ and $i \in \mathbb{N}, \quad i \le k$. Denote by

$$p_{i\to k}$$

the Lagrange interpolating polynomial of f at $x_i,\dots,x_k$.

For $i = 0$, we have

$$p_k = p_{0\to k}.$$

Definition 1.3.1. *Let $k \in \{0,\dots,n\}$ and $i \in \mathbb{N}, \quad i \le k$. The polynomial $p_{i\to k}$ hase a degree less than or equal to $k-i$.*

$$p_{i\to k} \in \mathbb{P}_{k-i}.$$

Denoting by $f[x_i,\dots,x_k]$ the coefficient of x^{k-i} in $p_{i\to k}$ which is called the divided difference of order $k-i$.

Remark 1.3.1. The divided difference of order $k-i$

$$f[x_i] = f(x_i),\ \forall i = 0,\dots,n. \tag{1.1}$$

Let $k \in \{1,\dots,n\}$, the polynome $q_k = p_k - p_{k-1} \in \mathbb{P}_k$ verifies $q_k(x_i) = 0,\ \forall i \in \{0,\dots,k-1\}$. Hence

$$q_k(x) = C(x-x_0)\dots(x-x_{k-1}),$$

for some constant C.

Thus,

$$p_k(x) = p_{k-1}(x) + f[x_0,\dots,x_k]\prod_{i=0}^{k-1}(x-x_i). \tag{1.2}$$

Definition 1.3.2. *The Newton formula can be written as*

$$p_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0,\ldots,x_k] \prod_{i=0}^{k-1} (x - x_i). \tag{1.3}$$

The Newton polynomial can be written as

$$N_k(x) = \prod_{i=0}^{k-1} (x - x_i) \tag{1.4}$$

The set $\mathscr{C} = \{N_k,\ k = 0,\ldots,n\}$ *is the* ***basis*** *of* $\mathbb{P}_n$, ***Newton basis***.

Theorem 2. *Let* $k \in \{1,\ldots,n\}$ *and* $i \in \mathbb{N}, \quad i \le k$. *The following formula holds*

$$f[x_i,\ldots,x_k] = \frac{f[x_{i+1},\ldots,x_k] - f[x_i,\ldots,x_{k-1}]}{x_k - x_i}.$$

1.3.1 Cubic splines interpolation

Now, we present an important application of interpolation is which called computer aided geometric design by using cubic splines.

The interpolation spline function of order m is:

1. a polynomial function of order $\leqslant m$ on each interval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, N-1$;
2. interpolate at $x_0, \ldots, x_N$;
3. in C^{m-1} on each interval $[a, b]$.

The most important type of interpolation by smooth functions is cubic spline interpolation.

The values $x_1 < x_2 < \cdots < x_n$ are often called **nodes** or **knots**. We recall that the cubic spline (i.e $m = 3$) is a function with the following properties:

1. On each interval $[x_i, x_{i+1}]$ the function is given by cubic polynomial. The function is defined by *different* cubic polynomials on different intervals.

2. At each interior knot $x_2, \dots x_{n-1}$ the cubic polynomials on adjoining intervals have the property that their values and the values of their first and second derivatives match.

In cubic splines, the interpolant S is comprised of cubic polynomials piecewise subject to the additional interpolation conditions that the first and second derivatives of S are continuous over (a, b).

It is assumed that the number of samples is finite. Let m be the number of intervals determined by the samples. We may describe specifically the pieces of a cubic spline $S(x)$ as follows, for $j = 1, 2, \cdots, m$,

$$S(x) = s_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \quad x \in \Delta_j = [x_j, x_{j+1}].$$

We give the additional smooth conditions

$$s_j^{[k]}(x_{j+1}) = s_{j+1}^{[k]}(x_{j+1}), \quad j = 1, 2, m-1 \quad k = 1, 2 \tag{1.5}$$

1.4 Bernstein polynomials

Let us consider the following Bernstein polynomial

$$b_{i,n}(x) = C_n^i x^i (1-x)^{n-i}, 0 \le i \le n,$$

with the binomial coefficients C_n^i.

We recall that the Bernstein polynomials are basis of the space $\mathbb{P}_n$ of polynomials.

The prototype of polynomial discrete spline quasi-interpolants is the classical Bernstein operator

$$B_n f = \sum_{i=0}^{n} f(\frac{i}{n}) b_{i,n}.$$

Letting

$$\tilde{b}_{i,n} = b_{i,n} / \int_0^1 b_{i,n} = (n+1) b_{i,n}.$$

The prototype of polynomial integral spline quasi-interpolants is the Durrmeyer operator

$$M_n f = \sum_{i=0}^{n} \langle f, \tilde{b}_{i,n} \rangle b_{i,n},$$

where

$$\langle f, g \rangle = \int_0^1 fg.$$

1.5 B-splines on an interval $[a, b]$

Let us turn our attention to the B-splines on an interval $[a, b]$, we begin with some notations and definitions. Let $m \geq 1$ and let us define the following function

$$(. - x)_m^+ : \mathbb{R} \to \mathbb{R}$$

$$t \to \begin{cases} (t-x)^m & \text{if} \quad t \geq x, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.5.1. *Let $X_n = \{x_k, \quad 0 \leq k \leq n\}$ be a partition of the interval $[a, b]$ into n subintervals, i.e. $x_k = a + kh_k$.*
We define

$$S_m(X_n, [a,b]) = \{s \in C^{m-1}([a,b]), \quad s_{/[x_i, x_{i+1}]} \in \mathbb{P}_m, \quad i \doteq 0, ..., n-1\}.$$

The elements of $S_m(X_n, [a,b])$ are called spline functions of order m associated with the subdivision X_n.

Example 1.5.1. *The function:*

$$s(x) = \begin{cases} x^2, & \textit{if} \quad -10 \leq x \leq 0, \\ -x^2, & \textit{if} \quad 0 < x < 1, \\ 1 - 2x, & \textit{if} \quad 1 \leq x \leq 2 \end{cases}$$

is a spline called quadratic spline.

Proposition 3. *[71]*

1. *$S_m(X_n,[a,b])$ content $\mathbb{P}_m[x]$.*

2. *$(t \mapsto (t-x)_m^+) \in S_m(X_n,[a,b])$.*

3. *$S_m(X_n,[a,b])$ is a vector subspace of $C^{m-1}([a,b])$.*

4. *The set of functions*

$$\left\{t \mapsto t^l, \qquad 0 \le l \le m, \quad t \mapsto (t-x_i)_m^+, \qquad 1 \le i \le n-1\right\}$$

forms a basis of $S_m(X_n,[a,b])$. In particular, the dimension of $S_m(X_n,[a,b])$ is equal to $n+m$.

As a consequence of this proposition, any spline S admits the following representation:

$$S(x) = p_m(x) + \sum_{i=1}^{n-1} c_i (x-x_i)_m^+.$$

Example 1.5.2. *The basis of $S_1(\{0,1,2,3,4,5\},[0,5])$ is given by the functions*

$$1; x; (x-1)_+; (x-2)_+; (x-3)_+; (x-4)_+.$$

1.5.1 Construction and Properties of the B-spline

Almost 50 years ago, J. Schoenberg has introduced a class of splines functions called the B-spline, or basis spline, which is characterized by the minimal support with respect to a given degree, smoothness, and domain partition. Moreover, any spline function of given degree can be expressed as a linear combination of B-splines of that degree. Cardinal B-splines have knots that are equidistant from each other. B-splines can be used for curve-fitting and numerical differentiation of experimental data, (see [72]).

Definition 1.5.2. *Let $a < x_1 < x_2 < \ldots < x_{n-1} < b$ be given. Suppose*

$$y_1 \le y_2 \le \ldots \le y_{2m+n-1}$$

is such that

$$y_1 \le \ldots \le y_m \le a, \qquad b \le y_{m+n} \le \ldots \le y_{2m+n-1},$$

and

$$y_{m+1} \le \ldots \le y_{m+n-1} = x_1, \ldots, x_{n-1},$$

then we call $\widetilde{X} = \{y_i\}_1^{2m+n-1}$ *an extended partition associated with* $S_m(X)$.

Let $\widetilde{X} = \{y_i\}_1^{2m+n-1}$ be an extended partition with $S_m(X)$, we assume that $b < y_{2m+n-1}$. For $i = 1, 2, \ldots, m+n-1$, let

$$B_{i,m} = (-1)^m (y_{i+m} - y_i)[y_i, \ldots, y_{i+m}](x-y)_+^{m-1} \qquad a \le x \le b$$

Suppose

$$b = y_{m+n} = \ldots = y_{2m+n-1}$$

and

$$B_{m+n,m}(b) = \lim_{x \to b} B_{m+n,m}(x),$$

then $\{B_{i,m}\}_1^{m+n}$ form a basis for $S_m(X)$. The $B_{1,m}, \ldots, B_{m+n,m}$ are called the B-splines.

Proposition 4. *[17] The B-splines have the following properties:*

1. *Partition of unity:* $\sum_{i=1}^{m+k} B_{i,m}(x) = 1, \qquad$ *for all* $a \le x \le b$.

2. *Positivity:* $B_{i,m}(x) > 0, \qquad$ *for* $x \in]y_i, y_{i+m}[$.

3. *Local support:* $B_{i,m}(x) = 0, \qquad$ *for* $x \notin [y_i, y_{i+m}]$.

Taking into account that the B-splines functions can be defined by induction as follows:

Let

$$a = x_{-m} = \ldots = x_0 < x_1 < \ldots < x_{n-1} < x_n = \ldots = x_{n+m} = b.$$

We have

$$B_{i,0}(x) = \begin{cases} 1 & \text{if} \quad x \in [x_i, x_{i+1}], \\ 0 & \text{otherwise.} \end{cases}$$

For $m \geq 1$

$$B_{i,m}(x) = \gamma_{i,m}(x) B_{i,m-1}(x) + (1 - \gamma_{i+1,m}(x)) B_{i+1,m-1}(x),$$

where

$$\gamma_{i,m}(x) = \begin{cases} \frac{x - x_i}{x_{i+m} - x_i}, & \text{if} \quad x_i \leq x \leq, x_{i+m}. \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.5.3. *Let $x_i = i, \quad 0 \leq i \leq n$.*

- $B_{0,0}(x) = \boldsymbol{1}_{[0,1[}$.
- $B_{0,1}(x) = x\boldsymbol{1}_{[0,1[} + (2-x)\boldsymbol{1}_{[1,2[}$.
- $B_{0,2}(x) = \frac{x^2}{2}\boldsymbol{1}_{[0,1[} + \frac{-2x^2+6x-3}{2}\boldsymbol{1}_{[1,2[} + \frac{(3-x)^2}{2}\boldsymbol{1}_{[2,3[}$.
- $B_{0,3}(x) = \frac{x^3}{6}\boldsymbol{1}_{[0,1[} + \frac{-3x^3+12x^2-12x+4}{6}\boldsymbol{1}_{[1,2[} + \frac{3x^3-24x^2+60x-44}{6}\boldsymbol{1}_{[2,3[} + \frac{(4-x)^3}{6}\boldsymbol{1}_{[3,4[}$.

Let $t_0 \leq \cdots \leq t_{d+1}$ be arbitrary points in $[a,b]$ such that $t_0 < t_{d+1}$.

We give some important properties of B-spline. Following [29]

$$\int_a^b B(t_0, \cdots, t_{d+1}; t)\,dt = \frac{1}{d+1}.$$

We assume that

$$\cdots \leq t_i \leq t_{i+1} \leq \cdots$$

and $t_i < t_{i+d+1}$ for all i. We note that

$$B_{i,d}(t) = (\cdot - t)_+^d [t_i, \cdots, t_{i+d+1}].$$

The spline function $N_{i,d}(t) = (t_{i+d+1} - t_i) B_{i,d}(t)$ is called normalized B-spline .

Theorem 5. *[17] Let $a < t_{d+2} \leq \cdots \leq t_n < b$ be fixed points such that $t_i < t_{i+d+1}$ for all admissible i . Choose arbitrary $2d+2$ additional points $t_1 \leq \cdots t_{d+1} \leq a$ and $b \leq t_{n+1} \leq \cdots \leq t_{n+d+1}$ and define*

$$B_i(t) = B(t_i, \cdots, t_{i+d+1}; t).$$

The B-spline $B_1(t), \cdots, B_n(t)$ constitute a basis for
$S_d(t_{d+2}, \cdots, t_n)$ on $[a,b]$.

1.6 Cardinal splines

In this section, we present the cardinals splines functions, that is to say the polynomial spline functions with equally spaced simple knots. The most results of this section are inspired from [25].

Definition 1.6.1. *For a positive integer m, the cardinals splines space S_m of order m and with knot sequence $\mathbb{Z}$ is defined by the following collection of functions:*

$$S_m = \{f \in C^{m-2}(\mathbb{R}) \quad \textit{such that} \quad f_{|[k,k+1[} \in \mathbb{P}_{m-1}, k \in \mathbb{Z}\}.$$

We note that S_1 is the space of piecewise constant functions.

Now, we adapt the most practical basis for S_m, as in polynomial approximation, the are many orthogonal polynomials basis, where the interpolation polynomial achieve the best approximation.

in this section we discuss the approximation by spline bases.

Let us consider the following associate basis to the space S_m:

$$\tau := \{(x-k)_+^{m-1} : k \in \mathbb{Z}\} \tag{1.1}$$

τ is a bases of space S_m by construction. In fact, we consider the subspace $S_{m,N}$ such that their elements are the restriction of $f \in S_m$ on the interval $[-N, N]$.

We assume that

$$p_{m,j} := f|_{[j,j+1[} \in \mathbb{P}_{m-1}, j = -N, \cdots, N-1. \quad \text{as} \quad f \in C^{m-2},$$

so that

$$(p_{m,j}^{(l)} - p_{m,j-1}^{(l)})(j) = 0, \quad l = 0, \cdots, m-2;\ m \geq 2.$$

Hence, by considering the "jumps" of $f^{(m-1)}$ at the knot sequence $\mathbb{Z}$, namely:

$$\begin{aligned} c_j &:= \lim_{\epsilon \to 0^+} [f^{(m-1)}(j+\epsilon) - f^{(m-1)}(j-\epsilon)] \\ &= p_{m,j}^{(m-1)}(j+0) - p_{m,j-1}^{(m-1)}(j-\epsilon), \end{aligned}$$

by applying Taylor formula, we end up with:

$$p_{m,j}(x) = p_{m,j-1}(x) + \frac{c_j}{(m-1)!}(x-j)^{m-1}.$$

Let us introduce the notation:

$$\begin{cases} x_+ := \max(x,0); \\ x_+^{m-1} := (x_+)^{m-1}, \quad m \geq 2. \end{cases} \tag{1.2}$$

Thus, for all $x \in [-N, N]$,

$$f(x) = f|_{-[-N,-N+1}(x) + \sum_{j=-N+1}^{N+1} \frac{c_j}{(m-1)!}(x-j)_+^{m-1}.$$

Therefore, the set of truncated powers,

$$\{(x-k)_+^{m-1} : k = -N-m+1, \cdots, N-1\}, \tag{1.3}$$

which are generated by using integer translates of a single function x_+^{m-1} is a basis of $S_{m,N}$.

In addition, as the space

$$S_m = \bigcup_{N=1}^{\infty} S_{m,N},$$

it follows that the basis in (1.1) can also be extended to be a τ of the infinite dimensional space S_m. In this section, we introduce the cardinal splines on the Hilbert space $L^2(\mathbb{R})$.

Definition 1.6.2. *We define the backward differences recursively by:*

$$\begin{cases} (\Delta f)(x) := f(x) - f(x-1); \\ (\Delta^m f)(s) := (\Delta^{m-1}(\Delta f))(x); \quad m = 2, 3, \cdots \end{cases} \tag{1.4}$$

where,

$$\Delta^m f(x) = \sum_{k=0}^{m} (m\ k)(-1)^k f(x-k) \tag{1.5}$$

Remark 1.6.1. The difference operator of order m applied to a polynomial of order less then or equal $m-1$ is zero ie:

$$\Delta^m f = 0, \quad f \in \mathbb{P}_{m-1}.$$

Definition 1.6.3. *Let $M_1 := N_1$ be the characteristic function define as follows:*

$$N_1(x) := \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \textit{elsewhere}\,. \end{cases}$$

Letting

$$M_m(x) := \frac{1}{(m-1)!}\Delta^m x_+^{m+1} \quad m \ge 2.$$

Remark 1.6.2. According to (1.5) the function M_m written as follows:

$$M_m(x) = \frac{1}{(m-1)!}\sum_{k=0}^{m}(-1)^k \binom{m}{k}(x-k)_+^{m-1}$$

Properties 6. (see [25] pp. 84)

The function M_m is characterized by:

- $\operatorname{Supp} M_m = [0, m]$.
- $M_m \in L^2(\mathbb{R})$.

Now, we define the following set:

$$B := \{M_m(x-k); k \in \mathbb{Z}\}$$

We recall that B is also a bases of S_m; we return to $S_{m,N}$ which is of dimension $m+2N-1$, from (1.3) and by using the property (1.1) the set elements:

$$\{M_m(x-k) : k = -N-m+1, \cdots, N-1\} \tag{1.6}$$

are linearly independent.

For all $f \in S_m$, we define the spline series by:

$$f(x) = \sum_{k=-\infty}^{+\infty} c_k M_m(x-k).$$

Since M_m has compact support, we assure the pointwise convergence of this series.

Theorem 7. *Let $f \in S_m$ and*

$$f(x) = \sum_{k=-\infty}^{+\infty} c_k M_m(x-k).$$

Thus:

$$f \in L^2(\mathbb{R}) \quad \textit{if and only if} \quad (c_k) \in \ell^2.$$

Proof. Assume that: $(c_k) \in \ell^2$, then:

$$|\sum_{k=-\infty}^{+\infty} c_k M_m(x-k)| \leq \sum_{k=-\infty}^{+\infty} |c_k|\, |M_m(x-k)|^{\frac{1}{2}} |M_m(x-k)|^{\frac{1}{2}}.$$

By Cauchy-Schwartz inequality, we obtain:

$$\int_{\mathbb{R}} |\sum_{k=-\infty}^{+\infty} c_k M_m(x-k)|^2 dx \leq \int_{\mathbb{R}} |\{\sum_{k=-\infty}^{+\infty} c_k|^2 |M_m(x-k)|\}\{\sum_{k=-\infty}^{+\infty} |M_m(x-k)|\} dx,$$

with

$$\sum_{k=-\infty}^{+\infty} M_m(x-k)| = 1.$$

Since the Lebesgue Measure is invariant by translation i.e.:

$$\int_{\mathbb{R}} M_m(x-k)dx = \int_{\mathbb{R}} M_m(x)dx$$

we get

$$\int_{\mathbb{R}} |\sum_{k=-\infty}^{+\infty} c_k M_m(x-k)|^2 dx \leq \sum_{k=-\infty}^{+\infty} |c_k|^2 \int_{\mathbb{R}} M_m(x)dx.$$

Since $M_m \in S_m$ have a compact support, the results are achieved. Conversely, for $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\sum_k |c_k|^2 &\leq \sum_k \left(\int_{\mathbb{R}} |\sum_k c_k M_m(x-k)^2| dx \right)^2 \\
&\leq \sum_k \left(\int_{\mathbb{R}} |f(x)| M_m(x-k)| dx \right)^2 \\
&\leq \sum_k \left(\int_{\mathbb{R}} (|f(x)| M_m(x-k)^{\frac{1}{2}}) M_m(x-k)^{\frac{1}{2}} dx \right)^2.
\end{aligned}$$

Since M_m have a compact support and according to Hölder inequality we obtain

$$\sum_k |c_k|^2 \leq \sum_k \left(\int_{\mathbb{R}} |f(x)|^2 M_m(x-k)\,dx \right) \left(\int_{\mathbb{R}} M_m(x-k)\,dx \right)$$
$$\leq \|f\|_{L^2(\mathbb{R})} \int_{\mathbb{R}} M_m(x-k)\,dx < \infty.$$

$\square$

1.6.1 Cardinal B-Spline and their basic properties

B-splines functions are part of regression splines, the B-spline term introduced in 1966 by Curry and Schoenberg which simply means a spline basis.

Definition 1.6.4. *The first order cardinal B-spline N_1 is the characteristic function of the unit interval* $[0,1)$*,*

$$N_1(x) := \begin{cases} 1, & 0 \leq x \leq 1 \\ 0 & elsewhere. \end{cases}$$

For $m \geq 2$, we define recursively by convolution the cardinal B-spline of order m as follows

$$N_m(x) := (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x-t)\,dt, \quad m \geq 2,$$

Theorem 8. *Cardinals B-splines of order m satisfy the following properties:*

(i) For all $f \in C$,

$$\int_{-\infty}^{+\infty} f(x) N_m(x)\,dx = \int_0^1 \cdots \int_0^1 f(x_1 + \cdots + x_m)\,dx_1 \cdots dx_m. \tag{1.7}$$

(ii) For all $g \in C^m$,

$$\int_{-\infty}^{+\infty} g^m(x) N_m(x)\,dx = \sum_{k=0}^{m} (-1)^{m-k} (m \quad k)\, g(k). \tag{1.8}$$

(iii) $N_m(x) = M_m(x)$ for all x.

(iv) $suppN_m = [0,m]$.

(v) $N_m(x) > 0$, for $0 < x < m$.

(vi) For all x

$$\sum_{k=0}^{\infty} N_m(x-k) = 1. \tag{1.9}$$

(vii) $N'_m(x) = (\Delta N_{m-1})(x) = N_{m-1}(x) - N_{m-1}(x-1)$.

(viii) The cardinal B-spline N_m and N_{m-1} are related by:

$$N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1). \tag{1.10}$$

(ix) N_m is symmetric

$$N_m(\frac{m}{2}+x) = N_m(\frac{m}{2}-x), \quad x \in \mathbb{R}$$

Proof. (i) The assertion (1.7) is true for $m = 1$, assume that is true for $m-1$. Then from definition of N_m in (1.5), we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x) N_m(x)\,dx &= \int_{-\infty}^{+\infty} f(x) \left\{ \int_0^1 N_{m-1}(x-t)\,dt \right\} dx \\
&= \int_0^1 \left\{ \int_{-\infty}^{+\infty} f(x) N_{m-1}(x-t)\,dx \right\} dt \\
&= \int_0^1 \left\{ \int_{-\infty}^{+\infty} f(y+t) N_{m-1}(y)\,dx \right\} dt \\
&= \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + \cdots + x_{m-1} + t)\,dx_1 \cdots dx_{m-1}\,dt \\
&= \int_0^1 \cdots \int_0^1 f(x_1 + \cdots + x_m)\,dx_1 \cdots dx_m.
\end{aligned}$$

(ii) The assertion (1.8) comes from (1.7)

$$\begin{aligned}
\int_{-\infty}^{+\infty} g^m(x) N_m(x)\,dx &= \int_0^1 \cdots \int_0^1 g^{(m)}(x_1 + \cdots + x_m)\,dx_1 \cdots dx_m \\
&= \sum_{k=0}^{m} (-1)^{m-k} (m \quad k)\, g(k).
\end{aligned}$$

(iii) For a fixed $x \in \mathbb{R}$, letting:

$$g(t) = \frac{(-1)^m}{(m-1)!} (x-t)_+^{m-1},$$

where:

$$g^{(m)} = \delta(x-t),$$

we replace their quantities in (1.8), we obtain our desire results.

(vi) Let's show by recurrence

It's clear that for $m = 1$ the B-spline forms a partition of unity, suppose that the relation (1.9) takes place up to a certain integer $m-1$

$$\begin{aligned}
\sum_{k=-\infty}^{+\infty} N_m(x-k) &= \sum_{k=-\infty}^{+\infty} \left(\int_0^1 N_{m-1}(x-k-t)\,dt\right) \\
&= \sum_{k=-\infty}^{+\infty} \left(\int_{x-1}^{x} N_{m-1}(y-k)\,dy\right) \\
&= \int_{x-1}^{x} \left(\sum_{k=-\infty}^{+\infty} N_{m-1}(y-k)\,dy\right) \\
&= \int_{x-1}^{x} dy \\
&= 1.
\end{aligned}$$

(vii) It's obvious that:

$$N'_m(x) = \int_0^1 N'_{m-1}(x-t)\,dt = N_{m-1}(x) - N_{m-1}(x-1) = (\Delta N_{m-1})(x).$$

(viii) The idea is to use the definition of M_m and to introduce x_+^{m-1} as following:

$$x_+^{m-1} = x \cdot x_+^{m-1},$$

and we call back the Leibniz rule for difference by:

$$(\Delta^m fg) = \sum_{k=0}^{m} (m \quad k)\,(\Delta^k f)(x)(\Delta^{m-k} g)(x-k),$$

we pose $f(x) = x$ and $g(x) = x_+^{m-2}$ and from the remark (1.1) we have:

$$\begin{aligned}
N_m(x) = M_m(x) &= \frac{1}{(m-1)!}\Delta^m x_+^{m-1} \\
&= \frac{1}{(m-1)!}\{x\Delta^m x_+^{m-2} + m\Delta^{m-1}(x-1)_+^{m-2}\} \\
&= \frac{1}{(m-1)!}\{x[\Delta^{m-1}x_+^{m-2} - \Delta^{m-1}(x-1)_+^{m-2}] + m\Delta^{m-1}(x-1)_+^{m-2}\} \\
&= \frac{x}{(m-1)}M_{m-1}(x) + \frac{m-x}{m-1}M - m - 1(x-1) \\
&= \frac{x}{(m-1)}N_{m-1}(x) + \frac{m-x}{m-1}N - m - 1(x-1).
\end{aligned}$$

(ix) by recurrence on m: for $m = 2$ is verified i.e.: $N_2(1+x) = N_2(1-x),\ x \in \mathbb{R}$.

Assume that $N_{m-1}(\frac{m-1}{2} + x) = N_{m-1}(\frac{m-1}{2} - x),\ x \in \mathbb{R}$.

$$\begin{aligned}
N_m(\frac{m}{2} + x) &= \int_0^1 N_{m-1}(\frac{m}{2} + x - t)dt \\
&= \int_0^1 N_{m-1}(\frac{m-1}{2} + x - t + \frac{1}{2})dt \\
&= \int_0^1 N_{m-1}(\frac{m-1}{2} - (x - t + \frac{1}{2}))dt \\
&= \int_0^1 N_{m-1}(\frac{m}{2} - x - (1-t))dt \\
&= \int_0^1 N_{m-1}(\frac{m}{2} - x - t)dt \\
&= N_m(\frac{m}{2} - x).
\end{aligned}$$

□

Example 1.6.1. *We present five linear cardinal B-spline, quadratic and cubic. Where, the B-spline formulas of order 2 and 3 are:*

$$N_2(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2, \end{cases}$$

$$N_3(x) = \begin{cases} \frac{1}{2}x^2 & 0 \le x \le 1 \\ -\frac{1}{2} + x - (x-1)^2 & 1 \le x \le 2 \\ \frac{5}{2} - x + \frac{5}{2} - x + \frac{1}{2}(x-2)^2 & 2 \le x \le 3 \end{cases}$$

Proposition 9. *The system* $\{N_m(.-k) : k \in \mathbb{Z}\}$ *for all* $m \geq 1$ *is a Riesz basis of* S_m

In order to demonstrate this proposition we need the following theorem:

Theorem 10. *For each function* $\Phi \in L^2$ *and* $0 < A \leq B < \infty$*, the following two assertions are equivalents:*

(i) $\{\phi(.-k) : k \in \mathbb{Z}\}$ *satisfies the Riesz conditions with borders* A *and* B *such that for each* $\{c_k\} \in \ell^2$

$$A\|\{c_k\}\|_{\ell^2}^2 \leq \|\sum_{k=-\infty}^{\infty} c_k\phi(.-k)\|_{L^2}^2 \leq B\|\{c_k\}\|_{\ell^2}^2$$

(ii) *The Fourier transformation* $\hat{\phi}$ *of* ϕ *satisfy*

$$A \leq \sum_{k=-\infty}^{\infty} |\hat{\phi}(x+2\pi k)|^2 \leq B, \text{ almost everywhere} \tag{1.11}$$

Proof. Proposition(2.1)
Firstly, we calculate the Fourier transformation of N_m. From (1.5) we have $\hat{N_m} = (\hat{N_1})^m$ such that:

$$|\hat{N_m}(w)|^2 = |\frac{1-e^{-iw}}{iw}|^{2m} \tag{1.12}$$

and we have

$$\int_{-\infty}^{+\infty} N_m(y+k)\overline{N_m(y)}dy = N_{2m}(m+k). \tag{1.13}$$

Indeed,

$$\begin{aligned}
\int_{-\infty}^{+\infty} N_m(y+k)\overline{N_m(y)}dy &= \int_{-\infty}^{+\infty} N_m(\frac{m}{2}+(y+k-\frac{m}{2}))N_m(y)dy \\
&= \int_{-\infty}^{+\infty} N_m(\frac{m}{2}-(y+k-\frac{m}{2}))N_m(y)dy \\
&= \int_{-\infty}^{+\infty} N_m(m-y-k)N_m(y)dy \\
&= (N_m \star N_m)(m-k) \\
&= N_{2m}(m-k) \\
&= N_{2m}(m+k).
\end{aligned}$$

Since $N_m \in L^2$ we get

$$\sum_{k=-\infty}^{+\infty} |\hat{N}_m(2x+2\pi k)|^2 = \sum_{k=-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} N_m(y+k)\overline{N_m(y)dy} \right\} e^{-ikw},$$

and from (1.13), we find:

$$\sum_{k=-\infty}^{+\infty} |\hat{N}_m(2x+2\pi k)|^2 = \sum_{k=-m+1}^{m-1} N_{2m}(m+k)e^{ikw}.$$

From (v) and (vi) in theorem (2.2), we obtain:

$$\sum_{k=-\infty}^{+\infty} |\hat{N}_m(2x+2\pi k)|^2 \leq 1$$

and the Riesz bound $B = 1$ here is the smallest possible. To determine the greatest lower bound of the expression in (1.13), we consider the so-called "Euler-Frobenius polynomials" of degree $2m-2$

$$E_{2m-1}(z) := (2m-1)!z^{m-1} \sum_{k=-m+1}^{m-1} N_{2m}(m+k)z^k. \tag{1.14}$$

We have

$$\begin{aligned}
\sum_{-\infty}^{+\infty} |\hat{N}_m(w+2\pi k)|^2 &= \frac{1}{(2m-1)!} \Pi_{k=1}^{2m-2} |e^{iw} - \lambda_k| \\
&= \frac{1}{(2m-1)!} \Pi_{k=1}^{m-1} \frac{|1-\lambda_k e^{iw}|\,|1-\lambda_k e^{-iw}|}{|\lambda_k|} \\
&= \frac{1}{(2m-1)!} \Pi_{k=1}^{m-1} \frac{1-2\lambda_k \cos w + \lambda_k^2}{|\lambda_k|}.
\end{aligned}$$

Letting

$$A_m := \frac{1}{(2m-1)!} \Pi_{k=1}^{m-1} \frac{(1+\lambda_k)^2}{|\lambda_k|} > 0,$$

such that

$$\sum_{k=-\infty}^{+\infty} |\hat{N}_m(w+2\pi k)|^2 \geq \sum_{k=-\infty}^{+\infty} |\hat{N}_m(\pi+2\pi k)|^2 = A_m.$$

Thus

$$A_m \leq \sum_{k=-\infty}^{+\infty} |\hat{N}_m(w+2\pi k)|^2 \leq 1.$$

□

Let us define the subspace $V_j^m \subset L^2$ by

$$V_j^m = \{f \in C^{m-2}(\mathbb{R}) \cap L^2(\mathbb{R}) \text{ such that } f_{|[\frac{k}{2^j}, \frac{k+1}{2^j}[} \in P_{m-1}, k \in \mathbb{Z}\} \tag{1.15}$$

Proposition 11. *The following result holds*

$$\begin{cases} Clos_{L^2(\mathbb{R})} U_{j\in\mathbb{Z}} V_j = L^2(\mathbb{R}) \\ \cap_{j\in\mathbb{Z}} V_j = \{0\} \end{cases} \tag{1.16}$$

Corollary 1.6.1. *For any pair of integers m and j, with $m \geq 2$, the family*

$$B_j := \{2^{\frac{j}{2}} N_m(2^j x - k) : k \in \mathbb{Z}\} \tag{1.17}$$

is a Riesz basis of V_j^m with Riesz bounds $A = A_m$ and $B = I$. Furthermore, these bounds are optimal.

Lemma 1.6.1.

$$\begin{cases} N_2(k) = \delta_{k,1}, \quad k \in \mathbb{Z} \\ N_{m+1}(k) = \frac{k}{m} N_m(k) + \frac{m-k+1}{m} N_m(k-1), \quad k = 1, \ldots.., m. \end{cases} \tag{1.18}$$

Theorem 12. *The "two-scale relation" for cardinal B-splines of order m holds:*

$$N_m(x) = \sum_{k=0}^{m} 2^{-m+1} \binom{m}{k} N_m(2x - k). \tag{1.19}$$

Proof. We have

$$N_m(2^j x) = \sum_{k=-\infty}^{\infty} p_{m,k} N_m(2^{j+1} x - k) \tag{1.20}$$

where $\{p_{m,z} : k \in \mathbb{Z}\}$ is a sequence in l^2.

Hence

$$\hat{N}_m(\omega) = \frac{1}{2} (\sum_{k=-\infty}^{\infty} p_{m,k} e^{-ik\omega/2}) \hat{N}_m(\frac{\omega}{2}). \tag{1.21}$$

Since

$$\begin{aligned}\frac{1}{2}\sum_{k=-\infty}^{\infty} p_{m,k}e^{-ikw/2} &= (\frac{1-e^{-iw}}{iw})^m(\frac{iw/2}{1-e^{-iw/2}})^m\\ &= (\frac{1+e^{-iw/2}}{2})^m\\ &= 2^{-m}\sum_{k=0}^{m}\binom{m}{k}e^{-ikw/2},\end{aligned}$$

we get

$$p_{m,k} = \begin{cases} 2^{-m+1}\binom{m}{k} & 0 \le k \le m. \\ 0 & ailleurs. \end{cases} \tag{1.22}$$

□

1.7 Spline quasi-interpolants

In this chapter, we present an important background of spline quasi-interpolants on uniform partitions of bounded intervals. We introduce some results presented by P. Sablonnière and al. concerning a numerical analysis, namely, integration, differentiation and approximation of zeros.

Let us consider the B-spline basis $\{B_j, j = 1\cdots n+d\}$ of some space of splines, say of degree d, on a bounded interval $I = [a,b]$ endowed with some partition X_n of I in n subintervals. Denote by $\mathbb{P}_d$ the space of polynomials of total degree at most d, and h denote the maximum step length of the partition X_n.

Define the univariate spline *quasi-interpolants* operators as follows

$$Qu = \sum_{j\in J}\mu_j(u)B_j, \quad u \in C([a,b]).$$

which are exact on the space $\mathbb{P}_d$, that means

$$Qp = p \quad \text{for all} \quad p \in \mathbb{P}_d.$$

Recall that the approximation order is $O(h^{d+1})$ on smooth functions.

Based on the coefficients μ_j, we distinguish three kinds of spline quasi-interpolant:

1. *Spline differential quasi-interpolant*: it is characterized by the coefficients $\mu_j(u)$ which are a linear combination of values of derivatives of u, of order at most d, at some point in $supp(B_j)$.

2. *Spline integral quasi-interpolant* : it is characterized by the coefficients $\mu_j(u)$ which are a linear combination of weighted mean values of u, i.e. of quantities $\int_a^b f w_j$ where w_j can be, for example, a linear combination of B-splines.

3. *Spline discrete quasi-interpolant*: in this case $\mu_j(u)$ is a linear combination of discrete values of u at some points in the neighbourhood of $supp(B_j)$.

We can consider the operator Q as a projector on the space of splines itself..

Through the use of the spline quasi-interpolants we obtain direct a construction lacking solving any system of linear equations. Also, the value of $Qu(s)$ depends only on values of u in a neighbourhood of s, that means that the spline quasi-interpolants are local. For more details we refer the reader to P. Sablonnière's papers.

1.8 Discrete quasi-interpolants

Let us consider the uniform partition $X_n = \{x_i = a + ih,\ 0 \le i \le n\}$ of the interval $I = [a, b]$ associated with meshlength $h = \frac{b-a}{n}$.

Letting $J = \{1, 2, \dots, n+d\}$. Denote by $S_d(I, X_n)$ the space of splines of degree d and class C^{d-1} and by basis of $\{B_j, j \in J\}$ this space. Moreover,

$$supp(B_j) = [x_{j-d-1}, x_j],$$

and $\mathcal{N}_j = \{x_{j-d}, \dots, x_{j-1}\}$ is the set of the d interior knots in the support of B_j.

In addition, we add multiple knots at the endpoints:

$$a = x_0 = x_{-1} = \dots = x_{-d},$$

and

$$b = x_n = x_{n+1} = \dots = x_{n+d}.$$

We can write any monomial in terms of symmetric functions of knots in $\mathcal{N}_j$

$$e_r(x) = x^r = \sum_{j \in J} \theta_j^{(r)} B_j(x), \quad \theta_j^{(r)} = \binom{d}{r}^{-1} symm_r(\mathcal{N}_j),\ 0 \le r \le d.$$

1.8.1 Quadratic spline discrete quasi-interpolants

For the C^1-quadratic spline discrete quasi-interpolant operator is given by

$$Q_2 u = \sum_{j=1}^{n+2} \mu_j(u) B_j,$$

We recall that

$\mu_1(u) = u_1$, $\mu_2(u) = \frac{1}{6}(-2u_1 + 9u_2 - u_3)$, $\mu_{n+1}(u) = \frac{1}{6}(-u_n + 9u_{n+1} - 2u_{n+2})$, $\mu_{n+2}(u) = u_{n+1}$, and for $3 \le j \le n$,

$$\mu_j(u) = \frac{1}{8}(-u_{j-1} + 10u_j - u_{j+1}).$$

It is easy to show that the exact value $\|Q_2\|_\infty = 1.4734$.

Moreover, the following estimate follows, for any $u \in C^3(I)$

$$\|u - Q_2 u\|_{\infty, I_k} \le \frac{5}{2} d_{\infty, I_k}(u, \Pi_2) \text{ for } 1 \le k \le n,$$

where

$$d_{\infty, I_k} := \inf\left\{ \left\| f - p \right\|_{\infty, I_k} \quad p \in \mathbb{P}_d \right\},$$

and

$$\left\| f - p \right\|_{\infty, I_k} := \max_{s \in I_k} \left| f(s) - p(s) \right|.$$

Thus,

$$\|u - Q_2 u\|_\infty \le A_2 h^3 \quad \text{for some positive constant} \quad A_2.$$

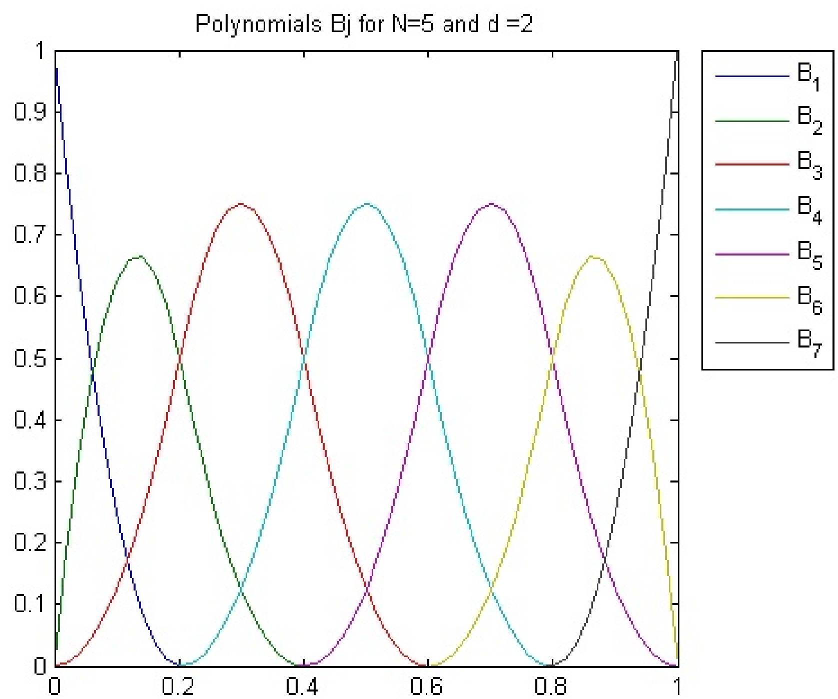
Polynomials Bj for N=5 and d =2
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
B_1
B_2
B_3
B_4
B_5
B_6
B_7

Figure 1.1

1.8.2 Cubic spline discrete quasi-interpolants

The C^2 cubic spline discrete quasi-interpolants have the following form

$$Q_3 u = \sum_{j=1}^{n+3} \mu_j(u) B_j,$$

with the following coefficient functionals:

$$\mu_2(u) = \frac{1}{18}(7u_0 + 18u_1 - 9u_2 + 2u_3),$$

$$\mu_{n+2}(u) = \frac{1}{18}(2u_{n-3} - 9u_{n-2} + 18u_{n-1} + 7u_n),$$

$$\mu_1(u) = u_0, \quad \mu_{n+3}(u) = u_n$$

and for $3 \le j \le n+1$

$$\mu_j(u) = \frac{1}{6}(-u_{j-3} + 8u_{j-2} - u_{j-1}).$$

Following

$$|\mu_2|_\infty = |\mu_{n+2}|_\infty = 2,$$

and

$$|\mu_j|_\infty = \frac{5}{3}, \quad \text{for} \quad 3 \le j \le n+1,$$

we get

$$\|Q_3\|_\infty \le 2.$$

1.8.3 Quartic spline discrete quasi-interpolants

For the C^3, we define the quartic spline discrete quasi-interpolants

$$Q_4 u = \sum_{j=1}^{n+4} \mu_j(u) B_j,$$

with the following coefficient functionals:

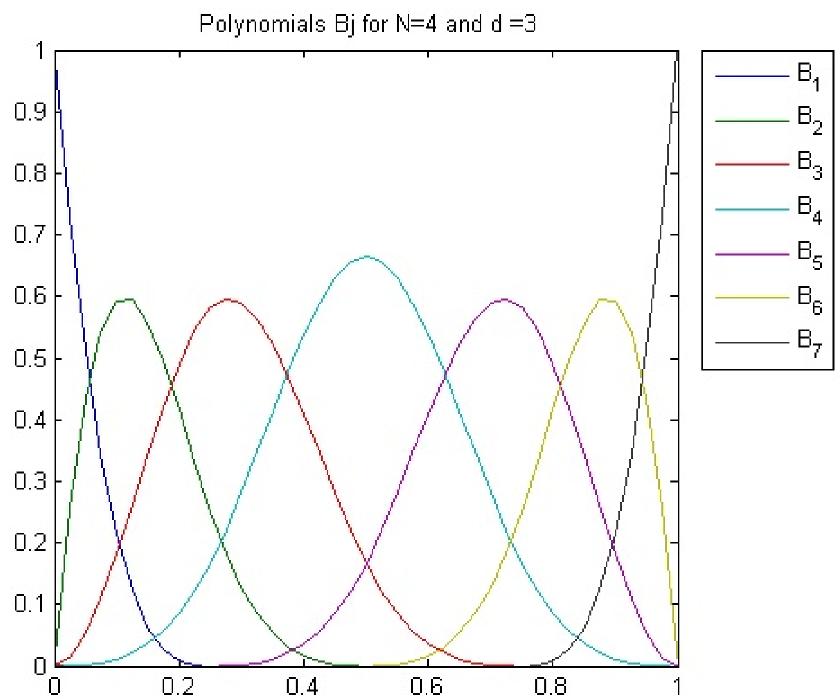
Polynomials Bj for N=4 and d =3
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
B_1
B_2
B_3
B_4
B_5
B_6
B_7

Figure 1.2

$$\mu_1(u) = u_1, \quad \mu_{n+4}(u) = u_{n+2},$$
$$\mu_2(u) = \frac{17}{105}u_1 + \frac{35}{32}u_2 - \frac{35}{96}u_3 + \frac{21}{160}u_4 - \frac{5}{224}u_5,$$
$$\mu_3(u) = -\frac{19}{45}u_1 + \frac{377}{288}u_2 + \frac{61}{288}u_3 - \frac{59}{480}u_4 + \frac{7}{288}u_5,$$
$$\mu_4(u) = \frac{47}{315}u_1 - \frac{77}{144}u_2 + \frac{251}{144}u_3 - \frac{97}{240}u_4 + \frac{47}{1008}u_5,$$
$$\mu_{n+1}(u) = \frac{47}{315}u_{n+2} - \frac{77}{144}u_{n+1} + \frac{251}{144}u_n - \frac{97}{240}u_{n-1} + \frac{47}{1008}u_{n-2},$$
$$\mu_{n+2}(u) = -\frac{19}{45}u_{n+2} + \frac{377}{288}u_{n+1} + \frac{61}{288}u_n - \frac{59}{480}u_{n-1} + \frac{7}{288}u_{n-2},$$
$$\mu_{n+3}(u) = \frac{17}{105}u_{n+2} + \frac{35}{32}u_{n+1} - \frac{35}{96}u_n + \frac{21}{160}u_{n-1} - \frac{5}{224}u_{n-2},$$

and for $5 \le j \le n$

$$\mu_j(u) = \frac{47}{1152}(u_{j-4} + u_j) - \frac{107}{288}(u_{j-3} + u_{j-1}) + \frac{319}{192}u_{j-2}.$$

Now, we want to calculate the functionals $\mu_k, k = 2,3,4$. Following

$$\mu_k(e_r) = \theta_k^{(r)}, \quad 0 \le r \le 4,$$

we obtain

$$\mu_k(u) = \alpha_k u_1 + \beta_k u_2 + \gamma_k u_3 + \delta_k u_4 + \zeta_k u_5,$$

these coefficients are the solutions of the three corresponding linear systems $(2 \le k \le 4)$

$$t_1^r \alpha_k + t_2^r \beta_k + t_3^r \gamma_k + t_4^r \delta_k + t_5^r \zeta_k = \theta_k^{(r)}, \; 0 \le r \le 4,$$

which have the same Vandermonde determinant

$$V_5(t_1, t_2, t_3, t_4, t_5) \ne 0.$$

Thus, the linear systems have unique solutions.

We have

$$|\mu_2|_\infty = |\mu_{n+3}|_\infty \approx 1.77,$$

$$|\mu_3|_\infty = |\mu_{n+2}|_\infty \approx 2.09,$$

$$|\mu_4|_\infty = |\mu_{n+1}|_\infty \approx 2.88,$$

$$|\mu_j|_\infty \approx 2.49, \quad \text{for} \quad 1 \le j \le 5.$$

Thus,

$$\|Q_4\|_\infty \le 2.88.$$

For all $\varphi \in C^5(I)$, the following error estimate holds:

$$\|\varphi - Q_4\varphi\|_{\infty,I_k} \le 4\, d_{\infty,I_k}(\varphi, \Pi_4), \quad \text{for } 1 \le k \le n.$$

Hence,

$$\|\varphi - Q_4\varphi\|_\infty \le A_4 h^5 \quad \text{for some positive constant} \quad A_4.$$

1.8.4 Quintic spline discrete quasi-interpolants

Now, we will discuss about the following C^4 quintic spline discrete quasi-interpolants

$$Q_5 u = \sum_{j=1}^{n+5} \mu_j(u)\, B_j.$$

The the corresponding coefficient functionals are as follows:

$$\begin{aligned}
\mu_1 &= u_0,\ \mu_{n+5} = u_n,\\
\mu_2 &= \frac{163}{300}u_0 + u_1 - u_2 + \frac{2}{3}u_3 - \frac{1}{4}u_4 + \frac{1}{25}u_5,\\
\mu_3 &= \frac{1}{200}u_0 + \frac{103}{60}u_1 - \frac{73}{60}u_2 + \frac{7}{10}u_3 - \frac{29}{120}u_4 + \frac{11}{300}u_5,\\
\mu_4 &= -\frac{41}{400}u_0 + \frac{43}{60}u_1 + \frac{103}{120}u_2 - \frac{7}{10}u_3 + \frac{13}{48}u_4 - \frac{13}{300}u_5.
\end{aligned}$$

We have a symmetric formulas for $n+2 \le j \le n+4$.

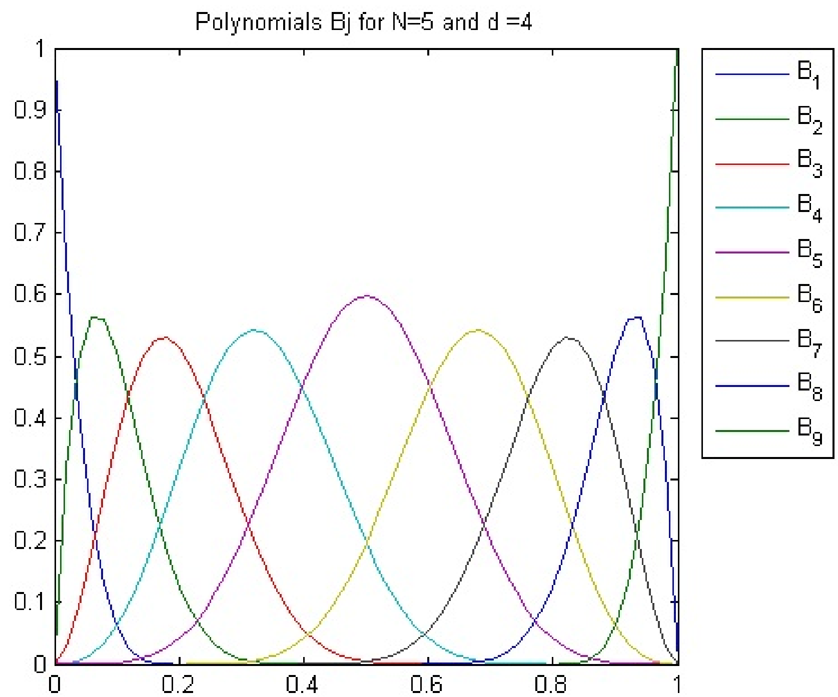
Polynomials Bj for N=5 and d =4
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
B_1
B_2
B_3
B_4
B_5
B_6
B_7
B_8
B_9

Figure 1.3

For $5 \leq j \leq n$:

$$\mu_j = \frac{13}{240}(u_{j-5} + u_{j-1}) - \frac{7}{15}(u_{j-4} + u_{j-2}) + \frac{73}{40}u_{j-3}.$$

In a similar manner, It follows that

$$|\mu_2|_\infty = |\mu_{n+4}|_\infty = 3.5,$$

$$|\mu_3|_\infty = |\mu_{n+3}|_\infty \approx 3.92,$$

$$|\mu_4|_\infty = |\mu_{n+2}|_\infty \approx 2.69,$$

and

$$|\mu_j|_\infty \approx 2.87, \quad \text{for} \quad 5 \leq j \leq n.$$

Thus,

$$\|Q_5\|_\infty \leq 3.92.$$

Moreover, it is easy to prove that

$$\|Q_5\|_\infty \approx 3.106.$$

We get the following error estimate holds

$$\|\varphi - Q_5\varphi\|_{\infty, I_k} \leq 4.5\, d_{\infty, I_k}(\varphi, \Pi_5) \text{ for } 1 \leq k \leq n, \quad \text{and for all} \quad \varphi \in C^6(I).$$

Consequently

$$\|\varphi - Q_5\varphi\|_\infty \leq A_5 h^6 \quad \text{for some positive constant} \quad A_5.$$

Sextic spline discrete quasi-interpolants

Finally, The sextic spline quasi-interpolant is a class c^5 and exact on $\mathbb{P}_6$ and it is defined by

$$Q_6 u = \sum_{k=1}^{n+6} \mu_k(u) B_k$$

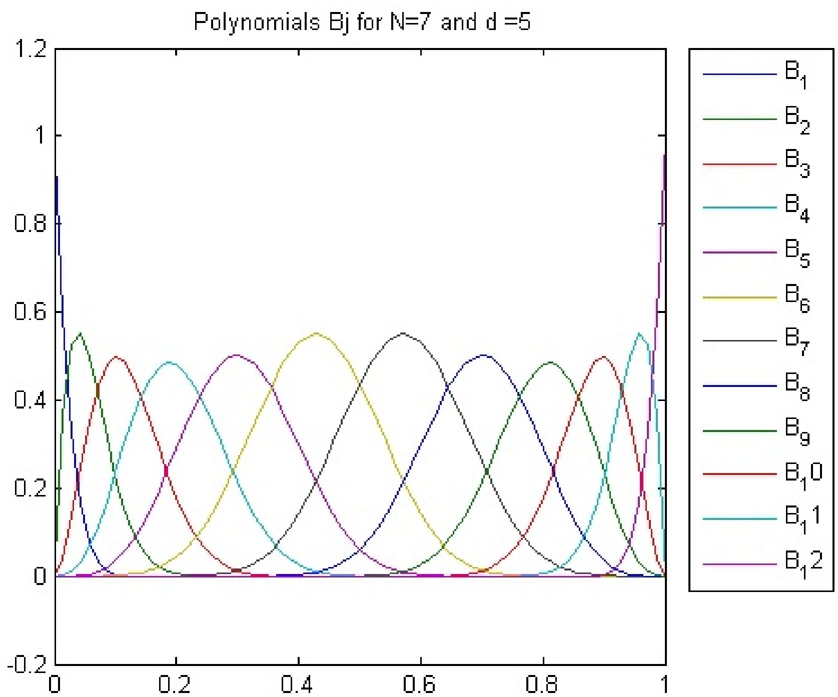
Polynomials Bj for N=7 and d =5
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
B_1
B_2
B_3
B_4
B_5
B_6
B_7
B_8
B_9
B_10
B_11
B_12

Figure 1.4

The coefficient functionals are:

$$\mu_1(u) = u_1,$$

$$\mu_2(u) = \frac{3887}{10395}u_1 + \frac{231}{256}u_2 - \frac{385}{786}u_3 + \frac{231}{640}u_4 - \frac{165}{896}u_5 + \frac{385}{6912}u_6 - \frac{21}{2816}u_7,$$

$$\mu_3(u) = -\frac{5689}{22275}u_1 + \frac{27631}{19200}u_2 - \frac{9151}{34560}u_3 + \frac{1091}{9600}u_4 - \frac{79}{1920}u_5 + \frac{997}{103680}u_6 - \frac{221}{211200}u_7,$$

$$\mu_4(u) = -\frac{20959}{155925}u_1 + \frac{3089}{9600}u_2 + \frac{5015}{3456}u_3 - \frac{4811}{4800}u_4 + \frac{3277}{6720}u_5 - \frac{7381}{51840}u_6 + \frac{1961}{105600}u_7,$$

$$\mu_5(u) = \frac{5821}{31185}u_1 - \frac{1193}{1920}u_2 + \frac{26737}{17280}u_3 + \frac{1}{320}u_4 - \frac{1217}{6720}u_5 + \frac{4001}{51840}u_6 - \frac{83}{7040}u_7,$$

$$\mu_6(u) = -\frac{2156}{31185}u_1 + \frac{2957}{11520}u_2 - \frac{30451}{34560}u_3 + \frac{13673}{5760}u_4 - \frac{33727}{40320}u_5 + \frac{17977}{103680}u_6 - \frac{2159}{126720}u_7,$$

$$\mu_j(u) = -\frac{2159}{138240}u_{j-5} + \frac{751}{4608}u_{j-4} - \frac{37003}{46080}u_{j-3} + \frac{79879}{34560}u_{j-2} - \frac{37003}{46080}u_{j-1} + \frac{751}{4608}u_j - \frac{2159}{138240}u_{j+1},$$

$$\mu_{n+1}(u) = -\frac{2156}{31185}u_{n+2} + \frac{2957}{11520}u_{n+1} - \frac{30451}{34560}u_n + \frac{13673}{5760}u_{n-1} - \frac{33727}{40320}u_{n-2} + \frac{17977}{103680}u_{n-3} - \frac{2159}{126720}u_{n-4},$$

$$\mu_{n+2}(u) = \frac{5821}{31185}u_{n+2} - \frac{1193}{1920}u_{n+1} + \frac{26737}{17280}u_n + \frac{1}{320}u_{n-1} - \frac{1217}{6720}u_{n-2} + \frac{4001}{51840}u_{n-3} - \frac{83}{7040}u_{n-4},$$

$$\mu_{n+3}(u) = -\frac{20959}{155925}u_{n+2} + \frac{3089}{9600}u_{n+1} + \frac{5015}{3456}u_n - \frac{4811}{4800}u_{n-1} + \frac{3277}{6720}u_{n-2} - \frac{7381}{51840}u_{n-3} + \frac{1961}{105600}u_{n-4},$$

$$\mu_{n+4}(u) = -\frac{5689}{22275}u_{n+2} + \frac{27631}{19200}u_{n+1} - \frac{9151}{34560}u_n + \frac{1091}{9600}u_{n-1} - \frac{79}{1920}u_{n-2} + \frac{997}{103680}u_{n-3} - \frac{221}{211200}u_{n-4},$$

$$\mu_{n+5}(u) = \frac{3887}{10395}u_{n+2} + \frac{231}{256}u_{n+1} - \frac{385}{786}u_n + \frac{231}{640}u_{n-1} - \frac{165}{896}u_{n-2} + \frac{385}{6912}u_{n-3} - \frac{21}{2816}u_{n-4},$$

$$\mu_{n+6}(u) = u_{n+2}.$$

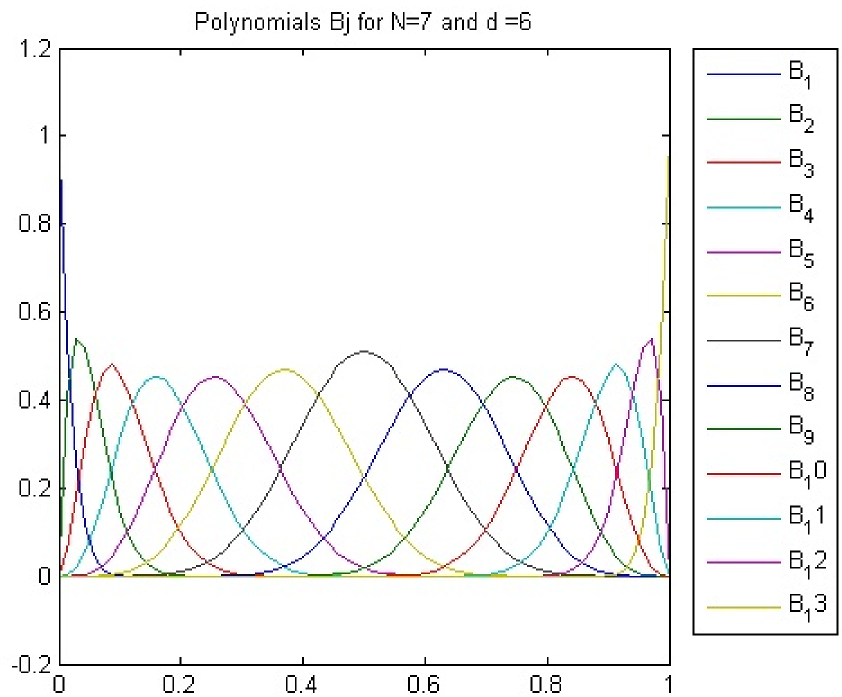
Polynomials Bj for N=7 and d =6
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
B1
B2
B3
B4
B5
B6
B7
B8
B9
B10
B11
B12
B13

Figure 1.5

CHAPTER 2 COLLOCATION METHODS FOR EIGENVALUE PROBLEMS

2.1 Eigenvalue problems

Let us consider a compact linear operator T defined on the space $C[0, 1]$ by

$$(T\varphi)(s) = \int_0^1 k(s,t)\varphi(t)\,dt, \qquad 0 \le s \le 1.$$

We assume that

$$k(.,.) \in C([0,1] \times [0,1]).$$

In this chapter, we will be concerned with the following eigenvalue problem

$$T\phi = \lambda\phi,$$

The compact linear operator T is closely connected with the following linear Fredholm integral equation of the second kind

$$\varphi - T\varphi = f.$$

With recall that

$$\| T\|_\infty = \max_{0\le s\le 1}\int_0^1 |k(s,t)|\,dt < 1$$

We assume that

$$\| T\|_\infty < 1.$$

Let us consider the eigenvalue problem represented by the homogeneous equation

$$T\phi = \lambda\phi, \qquad 0 \neq \lambda \in \mathbb{C}, \qquad \|\phi\| = 1. \tag{2.1}$$

To numerically solve (2.1), we can replace the operator T by a finite rank operator T_n.

Let ϕ_n the approximate solution of 2.1, which is the solution of the corresponding matrix eigenvalue problem of the form:

$$T_n\phi_n = \lambda_n\phi_n, \qquad 0 \neq \lambda \in \mathbb{C}, \qquad \|\phi\| = 1. \tag{2.2}$$

More recently, Kulkarni has introduced the following operators of finite rank (see [48])

$$T_n := \pi_n T + T\pi_n - \pi_n T\pi_n$$

for some series of projection operators π_n, with

$$\lim_{n\to+\infty} \|\pi_n x - x\| = 0 \quad \text{for all} \quad x \in C([0,1]).$$

An important example of projection π_n is the interpolation projection at the Gauss points in the space of the polynomial functions by bits and discontinues of degree $\leq r-1$.

Let D_n be the interpolation degenerate kernel operator with respect to the second variable. We replace the 's finite rank operator by D_n we get:

1. If we replace $T\pi_n$ by D_n in

$$T_n := \pi_n T + T\pi_n - \pi_n T\pi_n$$

we get

$$T_n := \pi_n T + D_n - \pi_n D_n.$$

2. If we replace $(T - \pi_n T)\pi_n$ by D_n in

$$\begin{aligned} T_n &:= \pi_n T + T\pi_n - \pi_n T\pi_n, \\ &= \pi_n T + (T - \pi_n T)\pi_n \end{aligned}$$

we get

$$T_n = \pi_n T + D_n.$$

2.2 Piecewise projection operator

In this section, we propose the following partition of $[0,1]$

$$0 = x_0 < x_1 < \ldots < x_n = 1.$$

Letting

$$h_j := x_j - x_{j-1},$$

$$h := \max h_j : j = 1, ..., n.$$

Moreover, we assume that $h \to 0$, as $n \to \infty$.

Let us consider $S_{r,n}$ the space of all piecewise polynomials of order r (i.e., of degree $\leq (r-1)$ with breakpoints at $x_1, ..., x_{n-1}$.

Let $B_r := \tau_1, ..., \tau_r$ denote the set of r zeros of the Legendre polynomial

$$\frac{d^r}{ds^r}(s^2-1)^r$$

in the interval $[-1,1]$, which called the Gauss points.

As in let us define a function $f_j : [-1,1] \to [x_{j-1}, tj]$ as

$$f_j(t) := \frac{1-t}{2} x_{j-1} + \frac{1+t}{2} x_j, \qquad t \in [-1,1].$$

Letting

$$\begin{aligned} A &:= \bigcup_{j=1}^{n} f_j(B_r) \\ &= \left\{\tau_{ij}, \quad f_i(\tau_j), \quad 1 \leq i \leq n, \quad 1 \leq j \leq r\right\}, \end{aligned}$$

the set of nr Gauss points.

Let us consider the following Lagrange polynomials of degree $r-1$ on $[-1,1]$:

$$l_i(s) := \prod_{k=1, k\neq i}^{r} \frac{s-\tau_k}{\tau_i - \tau_k}, \qquad i = 1, 2, \ldots, r, \qquad s \in [-1,1],$$

We recall that

$$l_i(\tau_j) = \delta_{ij}.$$

In addition, we also consider

$$\phi_{jp}(x) := \begin{cases} l_j(f_p^{-1}(x)), & if\, x \in [x_{p-1}, x_p] \\ 0 & \text{otherwise.} \end{cases}$$

We observe that

$$\phi_{jp} \in S_{r,n},$$

and

$$\phi_{jp}(\tau_{ik}) = \delta_{jk}\delta_{ip}, \qquad i, p = 1, 2, \ldots, r, \qquad k, j = 1, 2, \ldots, n.$$

Letting,

$$t_{(k-1)r+i} = \tau_{ik}, \qquad k = 1, 2, \ldots, n$$

$$\psi_{(k-1)r+j} = \phi_{jk} \quad j = 1, 2, \ldots, r$$

Hence

$$A = \{t_i : i = 1, 2, \ldots, nr\}$$

is the set of Gauss points on $[0, 1]$, and the set

$$\{\psi_i : i = 1, 2, \ldots, nr\}$$

forms a base of $S_{r,n}$.

Let us define the operator $\pi_n : C([0, 1]) \to S_{r,n}$ by

$$\pi_n u \in S_{r,n}, (\pi_n u)(t) = u(t), t \in A.$$

We note that $\pi_n u \to u$ as $n \to \infty$ for each $u \in C[0, 1]$, with

$$\pi_n u(s) := \sum_{i=1}^{nr} u(t_i)\psi_i(s).$$

We have

$$\pi_n k(s, t) = k_n(s, t) = \sum_{i=1}^{nr} k(s, t_i)\psi_i(t).$$

The operator D_n can be considered as follows

$$(D_n u)(s) := \sum_{i=1}^{nr} k(s, t_i) \int_0^1 u(t)\psi_i(t)dt.$$

2.3 First collocation method

The goal of this section is to present the collocation method given in literature, especially , in [48]. In this case we consider the following approximate operator

$$T_n = \pi_n T + D_n - \pi_n D_n.$$

The approximate problem is defined as follows

$$(\pi_n T + D_n - \pi_n D_n)\phi_{n1} = \lambda_{n1}\phi_{n1}.$$

We have

$$\begin{aligned}(\pi_n T\phi_{n1})(s) &= \sum_{i=1}^{nr}(\int_0^1 k(t_i,t)\phi_{n1}(t)dt)\psi_i(s)\\ &= \sum_{i=1}^{nr} W_i\psi_i(s),\end{aligned}$$

where

$$W_i := \int_0^1 k(t_i,t)\phi_{n1}(t)dt.$$

Since

$$\begin{aligned}(D_n\phi_{n1})(s) &= \int_0^1 (\sum_{j=1}^{nr} k(s,t_j)\phi_{n1}(t)\psi_j(t))dt\\ &= \sum_{j=1}^{nr} k(s,t_j)\int_0^1 \phi_{n1}(t)\psi_j(t)dt\\ &= \sum_{j=1}^{nr} \overline{k_j}\int_0^1 \phi_{n1}(t)\psi_j(t)dt\\ &= \sum_{j=1}^{nr} Y_j\overline{k_j}(s),\end{aligned}$$

where

$$\overline{k_j}(s) := k(s,t_j).$$

It follows that

$$\begin{aligned}
(\pi_n D_n \phi_{n1})(s) &= \sum_{i=1}^{nr} \left(\int_0^1 \sum_{j=1}^{nr} k(t_i, t_j)\phi_{n1}(t)\psi_j(t))dt\right)\psi_i(s) \\
&= \sum_{i=1}^{nr} \left(\sum_{j=1}^{nr} k(t_i, t_j)\int_0^1 \phi_{n1}(t)\psi_j(t)dt\right)\psi_i(s) \\
&= \sum_{i=1}^{nr} \left(\sum_{j=1}^{nr} Y_j k(t_i, t_j)\right)\psi_i(s).
\end{aligned}$$

Thus,

$$\begin{aligned}
\lambda_{n1}\phi_{n1} &= \sum_{i=1}^{nr} W_i\psi_i + \sum_{j=1}^{nr} Y_j\overline{k_j} + \sum_{i=1}^{nr}\left(\sum_{j=1}^{nr} Y_j k(t_i, t_j)\right)\psi_i \\
&= \sum_{i=1}^{nr} [W_i - \sum_{j=1}^{nr} Y_j k(t_i, t_j)]\psi_i + \sum_{j=1}^{nr} Y_j\overline{k_j} \\
&= \sum_{i=1}^{nr} X_i\psi_i + \sum_{j=1}^{nr} Y_j\overline{k_j}.
\end{aligned}$$

Then we have the following result

$$\begin{aligned}
\lambda_{n1}\pi_n T\phi_{n1} &= \lambda_{n1}\sum_{i=1}^{nr} T\phi_{n1}(t_i)\psi_i \\
&= \sum_{i=1}^{nr} T\lambda_{n1}\phi_{n1}(t_i)\psi_i \\
&= \sum_{i=1}^{nr} T[\sum_{k=1}^{nr} X_k\psi_k(t_i) + \sum_{l=1}^{nr} Y_l\overline{k_l}(t_i)]\psi_i \\
&= \sum_{i=1}^{nr} [\sum_{k=1}^{nr} X_k T\psi_k(t_i) + \sum_{l=1}^{nr} Y_l T\overline{k_l}(t_i)]\psi_i
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{n1} D_n \phi_{n1} &= \lambda_{n1} \sum_{j=1}^{nr} \overline{k_j}) \int_0^1 \psi_j(t)\phi_{n1}(t)dt \\
&= \sum_{j=1}^{nr} \overline{k_j} \langle \psi_j, \lambda_{n1}\phi_{n1} \rangle \\
&= \sum_{j=1}^{nr} \overline{k_j} \int_0^1 \psi_j(t)\lambda_{n1}\phi_{n1}(t)dt \\
&= \sum_{j=1}^{nr} \overline{k_j} \langle \psi_j, \sum_{k=1}^{nr} X_k\psi_k + \sum_{l=1}^{nr} Y_l \overline{k_l} \rangle \\
&= \sum_{j=1}^{nr} [\sum_{k=1}^{nr} X_k \langle \psi_j, \psi_k \rangle + \sum_{l=1}^{nr} Y_l \langle \psi_j, \overline{k_l} \rangle \overline{k_j}] \\
&= \sum_{j=1}^{nr} [\sum_{k=1}^{nr} X_k \langle \psi_k, \psi_j \rangle + \sum_{l=1}^{nr} Y_l \langle \overline{k_l}, \psi_j \rangle] \overline{k_j}.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\lambda_{n1}\pi_n D_n \phi_{n1} &= \lambda_{n1} \sum_{i=1}^{nr} D_n\phi_{n1}(t_i)\psi_i \\
&= \lambda_{n1} \sum_{i=1}^{nr} [\sum_{j=1}^{nr} \overline{k_j} \int_0^1 \psi_j(t_i)\phi_{n1}(t)dt]\psi_i \\
&= \sum_{i=1}^{nr} [\sum_{j=1}^{nr} \overline{k_j} \int_0^1 \psi_j(t_i)\lambda_{n1}\phi_{n1}(t)dt]\psi_i \\
&= \sum_{i=1}^{nr} (\sum_{j=1}^{nr} \overline{k_j} \int_0^1 \psi_j(t_i)[\sum_{k=1}^{nr} X_k\psi_k + \sum_{l=1}^{nr} Y_l\overline{k_l}])\psi_i \\
&= \sum_{i=1}^{nr} [\sum_{j=1}^{nr} (\sum_{k=1}^{nr} X_k \langle \psi_k, \psi_j \rangle + \sum_{l=1}^{nr} Y_l \langle \overline{k_l}, \psi_j \rangle) \overline{k_j}]\psi_i
\end{aligned}$$

we get

$$\lambda_{n1} X_i = \sum_{k=1}^{nr} X_k \widetilde{\psi}_k(t_i) + \sum_{l=1}^{nr} Y_l k_l^*(t_i) - [\sum_{j=1}^{nr} (\sum_{k=1}^{nr} X_k \langle \psi_k, \psi_j \rangle + \sum_{l=1}^{nr} Y_l \langle \overline{k_l}, \psi_j \rangle) \overline{k_j}],$$

also,

$$\lambda_{n1} Y_i = \sum_{k=1}^{nr} X_k \langle \psi_k, \psi_j \rangle + \sum_{l=1}^{nr} Y_l \langle \overline{k_l}, \psi_j \rangle.$$

Then, the above scheme leads to the following linear system series:

$$\lambda_{n1} X = AX + DY - B(CX + EY),$$

$$\lambda_{n1} Y = CX + EY,$$

$$\lambda_{n1} Z = FZ,$$

such as

$$A = (A_{ij}) := (\widetilde{\psi}_j(t_i)),$$

$$B = (B_{ij}) := (\overline{k_j}(t_i)),$$

$$C = (C_{ij}) := (\langle \psi_k, \psi_j \rangle),$$

$$D = (D_{ij}) := (k_j^*(t_i)),$$

$$E = (E_{ij}) = \langle \overline{k_j}, \psi_i \rangle,$$

$$Z = [X, Y]^T,$$

and the matrix F is given by

$$F := \begin{pmatrix} A - BC & D - BE \\ C & E \end{pmatrix}$$

with

$$\overline{k_j}(t_i) := k(t_i, t_j),$$

$$k_j^*(t_i) := T\overline{k_j}(t_i),$$

and

$$\widetilde{\psi}_j(t_i) := T\psi_j(t_i).$$

2.4 Second collocation method

In this section we introduce a new collocation method. In this case we take

$$T_n = \pi_n T + D_n.$$

We have

$$(\pi_n T + D_n)\phi_{n1} = \lambda_{n1}\phi_{n1}.$$

Therefore we obtain

$$\begin{aligned} \pi_n T\phi_{n1})(s) &= \sum_{i=1}^{nr}(\int_0^1 k(t_i,t)\phi_{n1}(t)dt)\psi_i(s) \\ &= \sum_{i=1}^{nr} X_i\psi_i(s), \end{aligned}$$

and

$$\begin{aligned} (D_n\phi_{n1})(s) &= \int_0^1 (\sum_{j=1}^{nr} k(s,t_j)\phi_{n1}(t)\psi_j(t))dt \\ &= \sum_{j=1}^{nr} k(s,t_j)\int_0^1 \phi_{n1}(t)\psi_j(t)dt \\ &= \sum_{j=1}^{nr} \overline{k_j}\int_0^1 \phi_{n1}(t)\psi_j(t)dt = \sum_{j=1}^{nr} Y_j\overline{k_j}, \end{aligned}$$

also

$$\lambda_{n1}\phi_{n1} = \sum_{i=1}^{nr} X_i\psi_i + \sum_{j=1}^{nr} Y_j\overline{k_j}.$$

Similarly,

$$\begin{aligned}
\lambda_{n1}\pi_n T\phi_{n1} &= \lambda_{n1}\sum_{i=1}^{nr} T\phi_{n1}(t_i)\psi_i \\
&= \sum_{i=1}^{nr} T\lambda_{n1}\phi_{n1}(t_i)\psi_i \\
&= \sum_{i=1}^{nr} T[\sum_{k=1}^{nr} X_k\psi_k(t_i) + \sum_{l=1}^{nr} Y_l\overline{k_l}(t_i)]\psi_i \\
&= \sum_{i=1}^{nr} [\sum_{k=1}^{nr} X_k T\psi_k(t_i) + \sum_{l=1}^{nr} Y_l T\overline{k_l}(t_i)]\psi_i \\
&= \sum_{i=1}^{nr} [\sum_{k=1}^{nr} X_k\widetilde{\psi}_k(t_i) + \sum_{l=1}^{nr} Y_l k_l^*(t_i)]\psi_i
\end{aligned}$$

On the other hand

$$\begin{aligned}
\lambda_{n1}D_n\phi_{n1} &= \lambda_{n1}\sum_{j=1}^{nr} \overline{k_j})\int_0^1 \psi_j(t)\phi_{n1}(t)dt \\
&= \sum_{j=1}^{nr} \overline{k_j}\int_0^1 \psi_j(t)\lambda_{n1}\phi_{n1}(t)dt \\
&= \sum_{j=1}^{nr} \overline{k_j}\langle\psi_j, \lambda_{n1}\phi_{n1}\rangle \\
&= \sum_{j=1}^{nr} \overline{k_j}\langle\psi_j, \sum_{k=1}^{nr} X_k\psi_k + \sum_{l=1}^{nr} Y_l\overline{k_l}\rangle \\
&= \sum_{j=1}^{nr} \overline{k_j}\langle\psi_j, \sum_{k=1}^{nr} X_k\psi_k + \sum_{l=1}^{nr} Y_l\overline{k_l}\rangle \\
&= \sum_{j=1}^{nr} [\sum_{k=1}^{nr} X_k\langle\psi_j, \psi_k\rangle + \sum_{l=1}^{nr} Y_l\langle\psi_j, \overline{k_l}\rangle\overline{k_j}
\end{aligned}$$

Hence

$$\lambda_{n1}X_i = \sum_{k=1}^{nr} X_k\widetilde{\psi}_k(t_i) + \sum_{l=1}^{nr} Y_l k_l^*(t_i) - [\sum_{j=1}^{nr}(\sum_{k=1}^{nr} X_k\langle\psi_k, \psi_j\rangle + \sum_{l=1}^{nr} Y_l\langle\overline{k_l}, \psi_j\rangle)\overline{k_j}],$$

$$\lambda_{n1}Y_i = \sum_{k=1}^{nr} X_k\langle\psi_k, \psi_j\rangle + \sum_{l=1}^{nr} Y_l\langle\overline{k_l}, \psi_j\rangle.$$

Consequently:

$$\lambda_{n1} X = AX + DY,$$

$$\lambda_{n1} Y = CX + EY,$$

$$\lambda_{n1} Z = FZ,$$

where

$$A = (A_{ij}) := (\widetilde{\psi}_j(t_i)),$$

$$C = (C_{ij}) := (\langle \psi_k, \psi_j \rangle),$$

$$D = (D_{ij}) := (k_j^*(t_i)),$$

$$E = (E_{ij}) = \langle \overline{k_j}, \psi_i \rangle,$$

$$Z = [XY]^T, \qquad F = \begin{pmatrix} A & D \\ C & E \end{pmatrix}.$$

Moreover,

$$\overline{k_j}(t_i) := k(t_i, t_j),$$

$$k_j^*(t_i) := T\overline{k_j}(t_i),$$

and

$$\widetilde{\psi}_j(t_i) := T\psi_j(t_i).$$

Chapter 3 Generalized Fredholm integral equation

This chapter is the subject of the following paper:

A. Mennouni, S. Zaouia, *Discrete septic spline quasi-interpolants for solving generalized Fredholm integral equation of the second kind via three degenerate kernel methods*, Mathematical Sciences, Springer, 2017, Volume 11, Issue 4, pp 345-357.

abstract

Three main contributions are presented in this paper. First, the septic quasi-interpolants are calculated with all their coefficients. Second, we explore the results to solve a generalized and broad class of Fredholm integral equations of the second kind. Finally, we present three degenerate kernel methods; the latter is a combination of the two previously established methods in the literature. Moreover, we provide a convergence analysis and we give new error bounds. Lastly, we exhibit some numerical examples and compare them with previous results in the literature.

Keywords: Septic spline, quasi-interpolation, integral equations.

3.1 Introduction

In the last years, many authors have presented different approaches to the solution of Fredholm integral equations using spline kernel approximations, in order to limit the number of terms in the approximate kernel. The results of this limitation provide high accuracy excluding unnecessary computational costs due to large linear systems.

The authors of [36] exploit Galerkin method to approximate the solution of the time-dependent Dirac equation in prolate spheroidal coordinates for an electron-molecular two-center system. They use balanced basis of kinetic-tomic to evaluate the initial state from a variational principle. Consequently, they obtain an exact and effective determination of the Dirac spectrum and eigenfunctions.

Using the polar method on parametric cubic spline technique, Prabhakar and Uma get wave resonating quadruplets calculated by the nonlinear source term of the wave model. They decide about the points of the locus for the two spacings, constant spacing and variable spacing. Refer to [66] for details.

The aim of [55] is to numerically solve the time fractional subdiffusion equation with Dirichelt boundary value conditions by use of the collocation method based on quadratic spline. In the paper, the authors explore, in details, the co-efficient matrix of the discretized linear system.

In [24], the authors propose a collocation method based on the quadratic spline in the pricing problem under a finite activity jump diffusion model for the first time.

For the sake of analyzing the nonlinear elastoplastic behavior of prismatic thin-walled members, the authors of [33] found a method of beam finite element working on generalized beam theory.

In another work [49], the authors decide to use quadratic and cubic B-spline quasi-interpolants in order to achieve higher order numerical methods only for a limited equations of Sobolev type in one dimension. Their aim is to compare the accuracy and convergence rate of these methods' performance.

Barton and Calo derive a new way for generating optimal quadrature rules for splines by making an association of source space and a known optimal quadrature and get the rule from the source space to the target one, but they maintain the point number and the optimality of quadrature. The aim of this process is to produce an optimal quadrature rule in a given spline space, (see [14]).

The purpose of [59] is to develop the BS Hermite spline quasi-interpolation scheme, which is related to the continuous extension of the BS linear multistep methods for solving the ordinary differential equations.

The author of [37] explores the quadratic spline quasi-interpolants on bounded domains, and provide some applications to different areas of the approximation theory. In the same scope, a different work gives some significant approximation formulas for derivatives of approximated functions through the application of univariate and multivariate quadratic spline quasi-interpolants. It uses the superconvergence properties of these operators to achieve very accurate derivatives of approximated functions at certain points (see [38]).

In [7], the authors apply a couple of well-known methods in the literature of the integral equations theory (see for instance [3, 12]) to solve a Fredholm integral equation of the second kind, through the discrete quartic spline quasi-interpolant. The result of their work is twofold: on the one hand, they achieve an approximation order $0(h^5)$ for the left method, on the other hand, they achieve another approximation of the order of $0(h^6)$ for the right method. Recently, a paper reached the sextic spline, working on two collocation methods, using both, spline interpolants and spline quasi-interpolants to solve the fifth-order boundary value problems (see [50]).

According to what have been stated so far, we assume the importance of the spline functions in the theory of approximation. Therefore, the present paper builds and carries on previously published works by so many authors. The main idea of this paper offers three contributions. The first one is to build the septic quasi-interpolants with all its coefficients. In the second contribution, we explore the achieved results to solve a generalized and broad class of Fredholm integral equations of the second kind. Finally, we present three degenerate kernel methods, where the latter method is original. We also provide a convergence analysis and we give new error bounds. Moreover, we compare the results of our calculations with other results.

In the following, we start by developing B-splines and monomials of degree seven and then we build the discrete septic quasi-interpolant. Next, we use our new results to solve a class of Fredholm integral equations of the second kind on a broader level using degenerate kernel methods. Then, we provide the order of the convergence for each method. In Section 5 we illustrate our theory by different examples, comparing the results we achieved with already given results in the field to argue that ours are more efficient than the previous ones. Finally, some conclusions are presented.

3.2 Construction of discrete septic quasi-interpolant

3.2.1 B-splines and monomials of degree seven

Let $\mathscr{X} := \mathscr{C}^0([a,b],\mathbb{R})$ be the space of all continuous functions, equipped with the max-norm $\|.\|_\infty$. Let us consider the nodes $x_0, x_1, x_2, ..., x_n$ in the interval $[a,b]$ with

$$x_k = a + kh_k \quad \text{and} \quad h_k = x_k - x_{k-1} \quad \text{for all} \quad 1 \le k \le n.$$

Let $\mathscr{X}_n := \{x_k, 0 \le k \le n\}$ denote the partition of the interval $[a,b]$ into n subintervals.

Define $S_7 := S_7([a,b], X_n)$ to be the space of septic splines of class C^6 on this partition.

Consider the set

$$\Gamma_n := \{1, 2, \ldots, n+7\}.$$

Let a canonical basis of S_7 be $\{B_k, k \in \Gamma_n\}$, which is shaped by the $n+7$ normalized B-splines. We add multiple knots at the endpoints to obtain the support of B_k which is the interval $[x_{k-8}, x_k]$ (see [74]). It is well-known that the representation of monomials using symmetric functions $\text{symm}_r(N_k)$ of interior knots

$$N_k := \{x_{k-7}, x_{k-6}, x_{k-5}, x_{k-4}, x_{k-3}, x_{k-2}, x_{k-1}\} \quad \text{in} \quad \text{Supp}(B_k).$$

Denoting by D^j the derivation operator of the order j, we consider the following function

$$\phi_k(t) := (x_{k-7}-t)(x_{k-6}-t)(x_{k-5}-t)(x_{k-4}-t)(x_{k-3}-t)(x_{k-2}-t)(x_{k-1}-t).$$

For $0 \le r \le 7$, we have:

$$\begin{aligned} m_r(x) &= x^r \\ &= \sum_{k\in\Gamma_n} (-1)^{7-r} \frac{r!}{7!} D^{7-r}\phi_k(0) B_k(x)" \\ &= \sum_{k\in\Gamma_n} \theta_k^r B_k(x), \end{aligned}$$

such that

$$\theta_k^r = \frac{\text{symm}_r(N_k)}{\alpha_r}, \qquad 0 \le r \le 7,$$

where

$$\alpha_r := \binom{7}{r}.$$

Specifically, if $r = 0$, we have $\theta_k^{(0)} = 1$ for all $k \in \Gamma_n$.

For $r = 1$, using the formula

$$\sum_{k\in\Gamma_n} B_k(x) = 1,$$

we obtain the Greville abscissae:

$$\theta_k = \theta_k^{(1)} = \frac{1}{7}\sum_{l=1}^{7} x_{k-l},$$

which are the coefficients of

$$m_1(x) = \sum_{k\in\Gamma_n} \theta_k B_k(x).$$

For $r = 2$, we obtain

$$\theta_k = \theta_k^2 = \binom{7}{2}^{-1} \sum_{1\le l<l'\le 7} x_{k-l}x_{k-l'}.$$

For $r = 3$, we obtain

$$\theta_k = \theta_k^3 = \binom{7}{3}^{-1} \sum_{1\le l<l'<l''\le 7} x_{k-l}x_{k-l'}x_{k-l''}.$$

We have calculated all the coefficients $\theta_j^r, j \in \Gamma_n, 0 \le r \le 7$; the results are given in in tables (3.1)-(3.7) respectively.

j	θ_j^1
1	a
2	$a+\frac{h}{7}$
3	$a+\frac{3h}{7}$
4	$a+\frac{6h}{7}$
5	$a+\frac{10h}{7}$
6	$a+\frac{15h}{7}$
$7 \leq j \leq n+1$	$hj+a-4h$
$n+2$	$b-\frac{15h}{7}$
$n+3$	$b-\frac{10h}{7}$
$n+4$	$b-\frac{6h}{7}$
$n+5$	$b-\frac{3h}{7}$
$n+6$	$b-\frac{h}{7}$
$n+7$	b

Table 3.1: θ_j^1

j	θ_j^2
1	a^2
2	$a^2+\frac{2ha}{7}$
3	$a^2+\frac{6ha}{7}+\frac{2h^2}{21}$
4	$a^2+\frac{12ha}{7}+\frac{11h^2}{21}$
5	$a^2+\frac{20ha}{7}+\frac{5h^2}{3}$
6	$a^2+\frac{30ha}{7}+\frac{85h^2}{21}$
$7 \leq j \leq n+1$	$h^2j^2+(2ah-8h^2)j+a^2-8ah+\frac{46h^2}{3}$
$n+2$	$b^2-\frac{30hb}{7}+\frac{85h^2}{21}$
$n+3$	$b^2-\frac{20hb}{7}+\frac{5h^2}{3}$
$n+4$	$b^2-\frac{12hb}{7}+\frac{11h^2}{21}$
$n+5$	$b^2-\frac{6hb}{7}+\frac{2h^2}{21}$
$n+6$	$b^2-\frac{2hb}{7}$
$n+7$	b^2

Table 3.2: θ_j^2

j	θ_j^3
1	a^3
2	$a^3+\frac{3ha^2}{7}$
3	$a^3+\frac{9ha^2}{7}+\frac{2h^2}{7}$
4	$a^3+\frac{18ha^2}{7}+\frac{11h^2a}{7}+\frac{6h^3}{35}$
5	$a^3+\frac{30ha^2}{7}+5ah^2+\frac{10h^3}{7}$
6	$a^3+\frac{45ha^2}{7}+\frac{85ah^2}{21}+\frac{45h^3}{7}$
$7\leq j\leq n+1$	$h^3j^3+(3ah^2-12h^3)j^2+(3ha^2-24ah^2+46h^3)j+a^3-12a^2h+46ah^2-56h^3$
$n+2$	$b^3-\frac{45hb^2}{7}+\frac{85bh^2}{7}-\frac{45h^3}{7}$
$n+3$	$b^3-\frac{30hb^2}{7}+5bh^2-\frac{10h^3}{7}$
$n+4$	$b^3-\frac{18hb^2}{7}+\frac{11h^2b}{7}-\frac{6h^3}{35}$
$n+5$	$b^3-\frac{9hb^2}{7}+\frac{2bh^2}{7}$
$n+6$	$b^3-\frac{3hb^2}{7}$
$n+7$	b^3

Table 3.3: θ_j^3

j	θ_j^4
1	a^4
2	$a^4+\frac{4ha^3}{7}$
3	$a^4+\frac{12ha^3}{7}+\frac{4a^2h^2}{7}$
4	$a^4+\frac{24ha^3}{7}+\frac{22a^2h^2}{7}+\frac{24ah^3}{35}$
5	$a^4+\frac{40ha^3}{7}+10a^2h^2+\frac{40ah^3}{7}+\frac{24h^4}{35}$
6	$a^4+\frac{60ha^3}{7}+\frac{170a^2h^2}{7}+\frac{180ah^3}{7}+\frac{274h^4}{35}$
	....
$7\leq j\leq n+1$	$h^4j^4+(4ah^3-16h^4)j^3+(6h^2a^2-48ah^3+92h^4)j^2+(4a^3h-48h^2a^2+184ah^3-224h^4)j$ $+a^4-16a^3h+92a^2h^2-224ah^3+\frac{967h^4}{5}$
	....
$n+2$	$b^4-\frac{60hb^3}{7}+\frac{170b^2h^2}{7}-\frac{180bh^3}{7}+\frac{274h^4}{35}$
$n+3$	$b^4-\frac{40hb^3}{7}+10b^2h^2-\frac{40bh^3}{7}+\frac{24h^4}{35}$
$n+4$	$b^4-\frac{24hb^3}{7}+\frac{22b^2h^2}{7}-\frac{24bh^3}{35}$
$n+5$	$b^4-\frac{12hb^3}{7}+\frac{4b^2h^2}{7}$
$n+6$	$b^4-\frac{4hb^3}{7}$
$n+7$	b^4

Table 3.4: θ_j^4

j	θ_j^5
1	a^5
2	$a^5+\frac{5ha^4}{7}$
3	$a^5+\frac{15ha^4}{7}+\frac{20a^3h^2}{7}$
4	$a^5+\frac{30ha^4}{7}+\frac{110a^3h^2}{21}+\frac{12a^2h^3}{7}$
5	$a^5+\frac{50ha^4}{7}+\frac{50a^3h^2}{3}+\frac{100a^2h^3}{7}+\frac{24ah^4}{7}$
6	$a^5+\frac{75ha^4}{7}+\frac{850a^3h^2}{21}+\frac{450a^2h^3}{7}+\frac{274ah^4}{7}+\frac{40h^5}{7}$
	....
$7\le j\le n+1$	$h^5j^5+(5ah^4-20h^5)j^4+(10h^3a^2-80ah^4+\frac{460h^5}{3})j^3$ $+(10h^2a^3-120a^2h^3+460ah^4-560h^5)j^2$ $+(5a^4h-80a^3h^2+460a^2h^3-1120ah^4+967h^5)j+a^5-20a^4h+\frac{460a^3h^2}{3}-560a^2h^3$ $+967ah^4-\frac{1876h^5}{3}$
	....
$n+2$	$b^5-\frac{75hb^4}{7}+\frac{850b^3h^2}{21}-\frac{450b^2h^3}{7}+\frac{274bh^4}{7}-\frac{40h^5}{7}$
$n+3$	$b^5-\frac{50hb^4}{7}+\frac{50b^3h^2}{3}-\frac{100b^2h^3}{7}+\frac{24bh^4}{7}$
$n+4$	$b^5-\frac{30hb^4}{7}+\frac{110b^3h^2}{21}-\frac{12b^2h^3}{7}$
$n+5$	$b^5-\frac{15hb^4}{7}+\frac{20b^3h^2}{7}$
$n+6$	$b^5-\frac{5hb^4}{7}$
$n+7$	b^5

Table 3.5: θ_j^5

j	θ_j^6
1	a^6
2	$a^6+\frac{6ha^5}{7}$
3	$a^6+\frac{18ha^5}{7}+\frac{10a^4h^2}{7}$
4	$a^6+\frac{36ha^5}{7}+\frac{55a^4h^2}{7}+\frac{24a^3h^3}{7}$
5	$a^6+\frac{60ha^5}{7}+25a^4h^2+\frac{200a^3h^3}{7}+\frac{72a^2h^4}{7}$
6	$a^6+\frac{90ha^5}{7}+\frac{4250a^4h^2}{7}+\frac{900a^3h^3}{7}+\frac{822a^2h^4}{7}+\frac{240h^6}{7}$
	....
$7\le j\le n+1$	$h^6j^6+(6ah^5-24h^6)j^5+(15h^4a^2-120ah^5+230h^6)j^4$ $+(20h^3a^3-240a^2h^4+920ah^5-1120h^6)j^3$ $+(15a^4h^2-240a^3h^3+1380a^2h^4-3360ah^5+2901h^6)j^2$ $+(6a^5h-120a^4h^2+920a^3h^3-3360a^2h^4+5802ah^5-3752h^6)j$ $+a^6-24a^5h+230a^4h^2-1120a^3h^3+2901a^2h^4-3752ah^5+\frac{13068h^6}{7}$
	....
$n+2$	$b^6-\frac{90hb^5}{7}+\frac{425b^4h^2}{7}-\frac{900b^3h^3}{7}+\frac{822b^2h^4}{7}-\frac{240h^5}{7}b$
$n+3$	$b^6-\frac{60hb^5}{7}+25b^4h^2-\frac{200b^3h^3}{7}+\frac{72b^2h^4}{7}$
$n+4$	$b^6-\frac{36hb^5}{7}+\frac{55b^4h^2}{7}-\frac{24b^3h^3}{7}$
$n+5$	$b^6-\frac{18hb^5}{7}+\frac{10b^4h^2}{7}$
$n+6$	$b^6-\frac{6hb^5}{7}$
$n+7$	b^6

Table 3.6: θ_j^6

j	θ_j^7
1	a^7
2	a^7+ha^6
3	$a^7+3ha^6+2a^5h^2$
4	$a^7+6ha^6+11a^5h^2+6a^4h^3$
5	$a^7+10ha^6+35a^5h^2+50a^4h^3+24a^3h^4$
6	$a^7+15ha^6+85a^5h^2+225a^4h^3+274a^3h^4+120a^2h^5$
	....
$7\le j\le n+1$	$h^7j^7+(7ah^6-28h^7)j^6+(21h^5a^2-168ah^6+322h^7)j^5$ $+(35h^4a^3-420a^2h^5+1610ah^6-1960h^7)j^4$ $+(35a^4h^3-560a^3h^4+3220a^2h^5-7840ah^6+6769h^7)j^3$ $+(21a^5h^2-420a^4h^3+3220a^3h^4-11760a^2h^5+20307ah^6-13132h^7)j^2$ $+(7a^6h-168a^5h^2+1610a^4h^3-7840a^3h^4+20307a^2h^5-26264ah^6+13068h^7)j$ $+a^7-28a^6h+322a^5h^2-1960a^4h^3+6769a^3h^4-13132a^2h^5+13068ah^6-5040h^7$
	....
$n+2$	$b^7-15hb^6+85b^5h^2-225b^4h^3+274b^3h^4-120b^2h^5$
$n+3$	$b^7-10hb^6+35b^5h^2-50b^4h^3+24b^3h^4$
$n+4$	$b^7-6hb^6+11b^5h^2-6b^4h^3$
$n+5$	$b^7-hb^6+2b^5h^2$
$n+6$	b^7-6hb^6
$n+7$	b^7

Table 3.7: θ_j^7

3.2.2 Discrete quasi-interpolant of degree 7

The discrete septic spline quasi-interpolant, abbreviated as dSSQI, is the operator

$$Qf = \sum_{k\in\Gamma_n} \mu_k(f)B_k,$$

whose coefficients are linear combinations of discrete values of f on the set of data points $\mathscr{X}_n$.

The $dSSQI$ is developed in order to be exact on P_7, i.e.

$$Qp = p \quad \text{for all} \quad p \in P_7,$$

in other words, $Qm_r = m_r$, where

$$m_r(x) = \sum_{k\in\Gamma_n} \mu_k(m_r)B_k(x) = \sum_{k\in\Gamma_n} \theta_k^r B_k(x), \qquad 0 \le r \le 7.$$

Therefore, we obtain the following conditions

$$\mu_k(m_r) = \theta_k^r \qquad \text{for} \quad k \in \Gamma_n, \qquad 0 \le r \le 7.$$

For $7 \le k \le n+1$, the functionals $\mu_k(f)$ merely use values of f in a neighborhood of the support of B_k, that is why it should be expressed as $\mu_k(f)$ thus

$$\mu_k(f) = \alpha_k f_{k-7} + \beta_k f_{k-6} + \gamma_k f_{k-5} + \delta_k f_{k-4} + \lambda_k f_{k-3} + \mu_k f_{k-2} + \nu_k f_{k-1},$$

where $f_k = f(x_k)$. The conditions above are the same as the systems of linear equations:

$$\alpha_k x_{k-7}^r + \beta_k x_{x-6}^r + \gamma_k x_{k-5}^r + \delta_k x_{k-4}^r + \lambda_k x_{k-3}^r + \mu_k x_{k-2}^r + \nu_k x_{k-1}^r = \theta_k^r, \qquad 0 \le r \le 7.$$

For $1 \le k \le 6$ and $n+2 \le k \le n+7$, we get the following equations respectively

$$\mu_k(f) = \alpha_k f_0 + \beta_k f_1 + \gamma_k f_2 + \delta_k f_3 + \lambda_k f_4 + \mu_k f_5 + \nu_k f_6 + \omega_k f_7$$

$$\mu_k(f) = \alpha_k f_{n-7} + \beta_k f_{n-6} + \gamma_k f_{n-5} + \delta_k f_{n-4} + \lambda_k f_{n-3} + \mu_k f_{n-2} + \nu_k f_{n-1} + \omega_k f_n.$$

All these systems have Vandermonde determinants

$$V_8(x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}, x_{k+6}, x_{k+7}) \neq 0.$$

While the x_j, $j \in J_n := \{0, 1, \ldots, n\}$, are different, they have unique solutions. Hence, the existence and uniqueness of the $dSSQI$. In the case of a uniform partition, the coefficient functionals are respectively defined by the following formulæ:

$$\mu_1(f) = f_0.$$

$$\mu_2(f) = \frac{617}{980} f_0 + f_1 - \frac{3}{2} f_2 - \frac{5}{3} f_3 - \frac{5}{4} f_4 + \frac{3}{5} f_5 - \frac{1}{6} f_6 + \frac{1}{49} f_7.$$

$$\mu_3(f) = \frac{3623}{26460} f_0 + \frac{407}{210} f_1 - \frac{337}{140} f_2 + \frac{941}{378} f_3 - \frac{151}{84} f_4 + \frac{59}{70} f_5 - \frac{871}{3780} f_6 + \frac{41}{1470} f_7.$$

$$\mu_4(f) = -\frac{23251}{264600} f_0 + \frac{2887}{2100} f_1 - \frac{207}{700} f_2 - \frac{391}{3780} f_3 + \frac{5}{24} f_4 - \frac{93}{700} f_5 + \frac{401}{9450} f_6 - \frac{83}{14700} f_7.$$

$$\mu_5(f) = -\frac{713}{52920} f_0 - \frac{53}{1260} f_1 + \frac{241}{105} f_2 - \frac{8317}{3780} f_3 + \frac{521}{360} f_4 - \frac{53}{84} f_5 + \frac{617}{3780} f_6 - \frac{167}{8820} f_7.$$

$$\mu_6(f) = \frac{3769}{105840} f_0 - \frac{811}{2520} f_1 + \frac{263}{240} f_2 + \frac{5741}{7560} f_3 - \frac{683}{720} f_4 + \frac{85}{168} f_5 + \frac{311}{2160} f_6 + \frac{311}{7640} f_7.$$

For $7 \leq j \leq n+1$,

$$\mu_j(f) = -\frac{311}{15120}(f_{j-7} + f_{j-1}) + \frac{22}{105}(f_{j-6} + f_{j-2}) - \frac{1657}{1680}(f_{j-5} + f_{j-3}) + \frac{2452}{945} f_{j-4}.$$

$$\begin{aligned}\mu_{n+2}(f) = {} & \frac{3769}{105840} f_n - \frac{811}{2520} f_{n-1} + \frac{263}{240} f_{n-2} + \frac{5741}{7560} f_{n-3} - \frac{683}{720} f_{n-4} \\ & - \frac{85}{168} f_{n-5} - \frac{311}{2160} f_{n-6} + \frac{311}{7640} f_{n-7}.\end{aligned}$$

$$\begin{aligned}\mu_{n+3}(f) = {} & -\frac{713}{52920} f_n - \frac{53}{1260} f_{n-1} + \frac{241}{105} f_{n-2} - \frac{8317}{3780} f_{n-3} + \frac{521}{360} f_{n-4} - \frac{53}{84} f_{n-5} \\ & + \frac{617}{3780} f_{n-6} - \frac{167}{8820} f_{n-7}.\end{aligned}$$

$$\begin{aligned}\mu_{n+4}(f) = {} & -\frac{23251}{264600} f_n + \frac{2887}{2100} f_{n-1} - \frac{207}{700} f_{n-2} - \frac{391}{3780} f_{n-3} + \frac{5}{24} f_{n-4} - \frac{93}{700} f_{n-5} \\ & + \frac{401}{9450} f_{n-6} - \frac{83}{14700} f_{n-7}.\end{aligned}$$

$$\mu_{n+5}(f) = \frac{3623}{26460} f_n + \frac{407}{210} f_{n-1} - \frac{337}{140} f_{n-2} + \frac{941}{378} f_{n-3} - \frac{151}{84} f_{n-4} + \frac{59}{70} f_{n-5}$$
$$- \frac{871}{3780} f_{n-6} + \frac{41}{1470} f_{n-7}.$$
$$\mu_{n+6}(f) = \frac{617}{980} f_n + f_{n-1} - \frac{3}{2} f_{n-2} + \frac{5}{3} f_{n-3} - \frac{5}{4} f_{n-4} + \frac{3}{5} f_{n-5}$$
$$- \frac{1}{6} f_{n-6} + \frac{1}{49} f_{n-7}.$$
$$\mu_{n+7}(f) = f_n.$$

It is noticeable that

$$|\mu_2|_\infty = |\mu_{n+6}|_\infty \approx 6.8333,$$

$$|\mu_3|_\infty = |\mu_{n+5}|_\infty \approx 9.8704,$$

$$|\mu_4|_\infty = |\mu_{n+4}|_\infty \approx 2.2511,$$

$$|\mu_5|_\infty = |\mu_{n+3}|_\infty \approx 6.8114,$$

$$|\mu_6|_\infty = |\mu_{n+2}|_\infty \approx 3.8288.$$

For $7 \le j \le n+1$,

$$|\mu_j|_\infty \approx 5.0275,$$

hence,

$$\|Q\| \le 9.8704.$$

Furthermore, for $f \in C^8(I)$, we also have

$$\|f - Qf\|_{\infty, I_k} \le (1 + 9.87) d_{\infty, I_k}(f, \mathbb{P}_7) \le 11 d_{\infty, I_k}(f, \mathbb{P}_7),$$

where

$$I_k = [x_{k-1}, x_k], \quad 1 \le k \le n,$$

and

$$d_{\infty, I_k}(f, \mathbb{P}_7) = \inf\{\|f - p\|_{\infty, I_k}, p \in \mathbb{P}_7\}.$$

Then

$$\|f - Qf\|_\infty = O(h^8).$$

We can write the quasi-interpolant Q under the quasi-Lagrange form:

$$Qf = \sum_{j\in J_n} f_j L_j,$$

where

$$L_0 = B_1 + \frac{617}{980}B_2 + \frac{3623}{26460}B_3 - \frac{23251}{264600}B_4 - \frac{713}{52920}B_5 + \frac{3769}{105840}B_6 - \frac{311}{15120}B_7.$$

$$L_1 = B_2 + \frac{407}{210}B_3 + \frac{2887}{2100}B_4 - \frac{53}{1260}B_5 - \frac{811}{2520}B_6 + \frac{22}{105}B_7 - \frac{311}{15120}B_8.$$

$$L_2 = -\frac{3}{2}B_2 - \frac{337}{140}B_3 - \frac{207}{700}B_4 + \frac{241}{105}B_5 + \frac{263}{240}B_6 - \frac{1657}{1680}B_7 + \frac{22}{105}B_8 - \frac{311}{15120}B_9.$$

$$\begin{aligned} L_3 = \frac{5}{3}B_2 + \frac{941}{378}B_3 - \frac{391}{3780}B_4 - \frac{8317}{3780}B_5 + \frac{5741}{7560}B_6 + \frac{2452945}{B}{}_7 - \frac{1657}{1680}B_8 + \frac{22}{105}B_9 \\ -\frac{311}{15120}B_{10}. \end{aligned}$$

$$\begin{aligned} L_4 = -\frac{5}{4}B_2 - \frac{151}{84}B_3 + \frac{5}{24}B_4 + \frac{521}{360}B_5 - \frac{683}{720}B_6 - \frac{1657}{1680}B_7 + \frac{2452}{945}B_8 - \frac{1657}{1680}B_9 \\ +\frac{22}{105}B_{10} - \frac{311}{15120}B_{11}. \end{aligned}$$

$$\begin{aligned} L_5 = \frac{3}{5}B_2 + \frac{59}{70}B_3 - \frac{93}{700}B_4 - \frac{53}{84}B_5 + \frac{85}{168}B_6 + \frac{22}{105}B_7 - \frac{1657}{1680}B_8 + \frac{2452}{945}B_9 - \frac{1657}{1680}B_{10} \\ +\frac{22}{105}B_{11} - \frac{311}{15120}B_{12}. \end{aligned}$$

$$\begin{aligned} L_6 &= -\frac{1}{6}B_2 - \frac{871}{3780}B_3 + \frac{401}{9450}B_4 + \frac{617}{3780}B_5 - \frac{311}{2160}B_6 - \frac{311}{15120}B_7 \\ &+ \frac{22}{105}B_8 - \frac{1657}{1680}B_9 + \frac{2452/945}{B}{}_{10} - \frac{1657}{1680}B_{11} + \frac{22}{105}B_{12} - \frac{311}{15120}B_{13}. \end{aligned}$$

$$\begin{aligned} L_7 &= \frac{1}{49}B_2 + \frac{41}{1470}B_3 - \frac{83}{14700}B_4 - \frac{167}{8820}B_5 + \frac{311}{17640}B_6 - \frac{311}{15120}B_8 + \frac{22}{105}B_9 \\ &- \frac{1657}{1680}B_{10} + \frac{2452}{945}B_{11} - \frac{1657}{1680}B_{12} + \frac{22}{105}B_{13} - \frac{311}{15120}B_{14}. \end{aligned}$$

$$L_j = -\frac{311}{15120}(B_{j+1}+B_{j+7}+\frac{22}{105}(B_{j+2}+B_{j+6})-\frac{1657}{1680}(B_{j+3}+B_{j+5})+\frac{2452}{945}B_{j+4},$$
$$8 \le j \le n-8.$$

$$\begin{aligned} L_{n-7} &= \frac{1}{49}B_{n+6}+\frac{41}{1470}B_{n+5}-\frac{83}{14700}B_{n+4}-\frac{167}{8820}B_{n+3}+\frac{311}{17640}B_{n+2}-\frac{311}{15120}B_n \\ &+ \frac{22}{105}B_{n-1}-\frac{1657}{1680}B_{n-2}+\frac{2452}{945}B_{n-3}-\frac{1657}{1680}B_{n-4}+\frac{22}{105}B_{n-5}-\frac{311}{15120}B_{n-6}. \end{aligned}$$

$$\begin{aligned} L_{n-6} &= -\frac{1}{6}B_{n+6}+\frac{871}{3780}B_{n+5}+\frac{401}{9450}B_{n+4}+\frac{617}{3780}B_{n+3}-\frac{311}{2160}B_{n+2}-\frac{311}{15120}B_{n+1} \\ &+ \frac{22}{105}B_n-\frac{1657}{1680}B_{n-1}+\frac{2452}{945}B_{n-2}-\frac{1657}{1680}B_{n-3}+\frac{22}{105}B_{n-4}-\frac{311}{15120}B_{n-5}. \end{aligned}$$

$$\begin{aligned} L_{n-5} &= \frac{3}{5}B_{n+6}+\frac{59}{70}B_{n+5}-\frac{93}{700}B_{n+4}-\frac{53}{84}B_{n+3}-\frac{85}{168}B_{n+2}+\frac{22}{105}B_{n+1} \\ &- \frac{1657}{1680}B_n+\frac{2452}{945}B_{n-1}-\frac{1657}{1680}B_{n-2}+\frac{22}{105}B_{n-3}-\frac{311}{15120}B_{n-4}. \end{aligned}$$

$$\begin{aligned} L_{n-4} &= -\frac{5}{4}B_{n+6}-\frac{151}{84}B_{n+5}+\frac{5}{24}B_{n+4}+\frac{521}{360}B_{n+3}-\frac{683}{720}B_{n+2}-\frac{1657}{1680}B_{n+1} \\ &+ \frac{2452}{945}B_n-\frac{1657}{1680}B_{n-1}+\frac{22}{105}B_{n-2}-\frac{311}{15120}B_{n-3}. \end{aligned}$$

$$L_{n-3} = \frac{5}{3}B_{n+6}-\frac{941}{378}B_{n+5}-\frac{391}{3780}B_{n+4}-\frac{8317}{3780}B_{n+3}+\frac{5741}{7560}B_{n+2}+\frac{2452}{945}B_{n+1}$$
$$-\frac{1657}{1680}B_n+\frac{22}{105}B_{n-1}-\frac{311}{15120}B_{n-2}.$$

$$L_{n-2} = -\frac{3}{2}B_{n+6}-\frac{337}{140}B_{n+5}-\frac{207}{700}B_{n+4}+\frac{241}{105}B_{n+3}+\frac{263}{240}B_{n+2}-\frac{1657}{1680}B_{n+1}$$
$$+\frac{22}{105}B_n-\frac{311}{15120}B_{n-1}.$$

$$L_{n-1} = B_{n+6}+\frac{407}{210}B_{n+5}+\frac{2887}{2100}B_{n+4}-\frac{53}{1260}B_{n+3}-\frac{811}{2520}B_{n+2}+\frac{22}{105}B_{n+1}$$
$$-\frac{311}{15120}B_n.$$

$$L_n = B_{n+7}+\frac{617}{980}B_{n+6}+\frac{3623}{26460}B_{n+5}-\frac{23251}{264600}B_{n+4}-\frac{713}{52920}B_{n+3}$$
$$+\frac{3769}{105840}B_{n+2}-\frac{311}{15120}B_{n+1}.$$

3.3 Solving Fredholm integral equations by degenerate kernel approximations

Consider the following generalized Fredholm integral equation of the second kind

$$u(s) - \sum_{k=1}^{m} \int_a^b H_k(s,t)u(t)dt = f(s), \qquad m \in \mathbb{N}^*, \quad a \leq s \leq b, \tag{3.1}$$

where f is a continuous function. We assume that $H_k(.,.) \in C([a,b]\times[a,b],\mathbb{C}), \quad k = 1,\dots,m$. Then the integral operator

$$Tu(s) := \sum_{k=1}^{m} \int_a^b H_k(s,t)u(t)dt, \quad a \leq s \leq b,$$

is compact from $\mathscr{X}$ into $\mathscr{X}$, we assume that the equation (3.1) has a unique solution. Equation (3.1) can be rewritten in an operator form as:

$$(I - T)u = f.$$

Degenerate kernel methods are crucial in approximation theory and in scientific computing. They have many interesting applications, particularly to solve integral and integro-differential equations. In [41], the authors explore the classical version of the degenerate kernel method to numerically solve the Hammerstein equations. Later, they extend the degenerate kernel method for single-variable Hammerstein equations to include multi-variable Hammerstein equations in [42]. The authors of [63] treat a degenerate approximation of the kernel by using Taylor series and Lagrange interpolation for solving the general nonlinear Fredholm integro-differential equations under mixed conditions. The degenerate kernel in the polar coordinates for two subdomains is adopted in [22] for the closed-form fundamental solution of null-field boundary integral equation method. Majidiana and Babolian [57] apply a degenerate kernel method with piecewise constant interpolation with respect to the second variable in order to approximate isolated eigenvalues of a class of noncompact linear operators. In [28], a new approach to the theory of kernel approximations is developed for the numerical solution of Fredholm integral equations of the second kind using a degenerate-kernel operator of fixed rank. Kalaba and Scott [40] use an initial-value method for integral equations with generalized degenerate kernels. Recently, the authors of [7] rely on two degenerate methods for solving the classical Fredholm integral equation of the second kind, based on (left and right) partial approximations of the kernel through a discrete quartic spline quasi-interpolant. From reviewing the literature, it is noticeable that the degenerate kernel method is commonly used in the development in the theory of approximation on a wide scale, mainly the resolution of integral equations. In a more modern sense, we intend to rely on the same degenerate kernel methods for solving (3.1) through our newly obtained results of discrete septic spline quasi-interpolants, unlike the previously mentioned work [7].

3.3.1 First septic spline degenerate kernel method

We first approximate the given continuous functions $s \mapsto H_k(s,t)$ by the septic spline quasi-interpolant using quasi-Lagrange form:

$$H_k^F(s,t) = \sum_{j\in J_n} H_k(s_j,t)L_j(s).$$

The left degenerate kernel operator is defined by

$$T_n^F u(s) := \sum_{j\in J_n} L_j(s) \sum_{k=1}^{m} \int_a^b H_k(s_j,t)u(t)dt.$$

Approximating the equation (3.1) by

$$u_n^F(s) - \sum_{j\in J_n} L_j(s) \sum_{k=1}^{m} \int_a^b H_k(s_j,t)u_n^F(t)dt = f(s), \tag{3.2}$$

the approximate solution u_n^F of equation (3.2) is given by

$$u_n^F(s) = f(s) + \sum_{j\in J_n} c_j L_j(s),$$

for some scalars c_j.

As a result, we obtain the linear system

$$\sum_{i\in J_n} c_i L_i(s) - \sum_{i\in J_n} L_i(s) \sum_{k=1}^{m} \int_a^b H_k(s_i,t)[f(t) + \sum_{j\in J_n} c_j L_j(t)]dt = 0,$$

hence

$$c_i - \sum_{j\in J_n} c_j \sum_{k=1}^{m} \int_a^b H_k(s_i,t) L_j(t)dt = \sum_{k=1}^{m} \int_a^b H_k(s_i,t) f(t)dt,$$

that is to say, the coefficients c_j are obtained by solving the following linear system

$$(I - F_n)X_n^F = b_n^F,$$

where, for $i \in J_n$ and $j \in J_n$,

$$F_n(i,j) := \sum_{k=1}^{m} \int_a^b H_k(s_i,t) L_j(t)dt,$$

$$b_n^F(i) := \sum_{k=1}^{m} \int_a^b H_k(s_i,t) f(t)dt.$$

3.3.2 Second septic spline degenerate kernel method

Next, we consider the following right degenerate kernel operator

$$T_n^R u(s) := \sum_{j\in J_n} \sum_{k=1}^{m} H_k(s,t_j) \int_a^b L_j(t)u(t)dt.$$

We approximate the given continuous functions $t \mapsto H_k(s,t)$ by a septic spline quasi-interpolant:

$$H_{k_n}^R(s,t) = \sum_{j\in J_n} H_k(s,t_j) L_j(t),$$

and the approximate solution satisfies

$$u_n^R(s) - \sum_{j\in J_n} \sum_{k=1}^{m} H_k(s,t_j) \int_a^b L_j(t) u_n^R(t)dt = f(s).$$

Therefore, u_n^R is of the form:

$$u_n^R(s) = f(s) + \sum_{i\in J_n} r_i \sum_{k=1}^{m} H_{k_i}^R(s) \quad \text{with} \quad H_{k_i}^R(s) := H_k(s,t_i),$$

for some scalars r_j. Hence

$$r_i - \sum_{j\in J_n} r_j \sum_{k=1}^{m} \int_a^b L_j(t) H_{k_i}^{R}(t)\,dt = \int_a^b L_i(t) f(t)\,dt, \quad i \in J_n.$$

The coefficients r_j are obtained by solving the following linear system

$$(I - R_n) X_n^R = b_n^R,$$

where, for $i \in J_n$ and $j \in J_n$,

$$\begin{aligned} R_n(i,j) &:= \sum_{k=1}^{m} \int_a^b L_j(t) H_{k_i}^{R}(t)\,dt, \\ b_n^R(i) &:= \int_a^b L_i(t) f(t)\,dt. \end{aligned}$$

3.3.3 Third septic spline degenerate kernel method

Finally, we approximate the given continuous functions $s \mapsto H_k(s,t)$ by the septic spline quasi-interpolant using quasi-Lagrange form:

$$H_k^{RF}(s,t) = \sum_{j\in J_n} \sum_{i\in J_n} H_k(s_j, t_i) L_j(s) L_i(t).$$

The third degenerate kernel operator is defined by

$$T_n^{RF} u(s) := H_k^{RF}(s,t) = \sum_{j\in J_n} \sum_{i\in J_n} L_j(s) \sum_{k=1}^{m} H_k(s_j, t_i) \int_a^b L_i(t) u(t)\,dt.$$

Approximating the equation (3.1) by

$$u_n^{RF}(s) - \sum_{j\in J_n} \sum_{i\in J_n} L_j(s) \sum_{k=1}^{m} H_k(s_j, t_i) \int_a^b L_i(t) u_n^{RF}(t)\,dt = f(s), \tag{3.3}$$

the approximate solution u_n^{RF} of equation (3.3) is given by

$$u_n^{RF}(s) = f(s) + \sum_{j\in J_n} c_j L_j(s),$$

for some scalars c_j.

As a result, we obtain the linear system

$$\sum_{j\in J_n} c_j L_j(s) - \sum_{j\in J_n}\sum_{i\in J_n} L_j(s)\sum_{k=1}^{m} H_k(s_j,t_i)\int_a^b L_i(t)[f(t)+\sum_{l\in J_n} c_l L_l(t)]dt = 0.$$

Hence

$$c_j - \sum_{i\in J_n}\sum_{l\in J_n}\sum_{k=1}^{m} c_l H_k(s_j,t_i)\int_a^b L_i(t)L_l(t)dt = \sum_{i\in J_n}\sum_{k=1}^{m} H_k(s_j,t_i)\int_a^b L_i(t)f(t)dt,$$

that is to say, the coefficients c_j are obtained by solving the following linear system

$$(I - A_n^{RF})X_n^{RF} = b_n^{RF},$$

where, for $j \in J_n$ and $i \in J_n$,

$$\begin{aligned} A_n^{RF}(j,i) &:= \sum_{l\in J_n}\sum_{k=1}^{m} H_k(s_j,t_i)\int_a^b L_i(t)L_l(t)dt, \\ b_n^{RF}(j) &:= \sum_{i\in J_n}\sum_{k=1}^{m} H_k(s_j,t_i)\int_a^b L_i(t)f(t)dt. \end{aligned}$$

3.4 Error analysis

We will use the following notation:

$$H_k^s(t) = H_k(s,t) = H_k^t(s).$$

The following theorems hold.

Theorem 13. *Let $H_k(.,.) \in C^8([a,b]\times[a,b])$, $\quad k = 1...m$. The following estimate holds:*

$$\|u - u_n^F\|_\infty \le \alpha h^8 \sum_{k=1}^{m}\left\|\frac{\partial^8 H_k}{\partial s^8}\right\|_\infty \|u\|_\infty,$$

for some constant α independent of n.

Proof. Following [12],

$$\|T - T_n^F\| = \max_{a\le s\le b}\sum_{k=1}^{m}\int_a^b |H_k(s,t) - H_{k_n}^F(s,t)|dt,$$

and

$$\|(I-T_n^F)^{-1}\|_\infty \le \frac{\|(I-T)^{-1}\|}{1-\|(I-T)^{-1}\|\|T-T_n^F\|}.$$

Since T is compact, according to [12], the operator $I-T_n^F$ is invertible for n is large enough, and its inverse is uniformly bounded with respect to n. Then there exists $c_1>0$, such as

$$\sup_n \|(I-T_n^F)^{-1}\|_\infty \le c_1.$$

Since

$$\begin{aligned} u-u_n^F &= \left[(I-T)^{-1}-(I-T_n^F)^{-1}\right]f \\ &= (I-T_n^F)^{-1}\left[T-T_n^F\right](I-T)^{-1}f \\ &= (I-T_n^F)^{-1}\left[Tu-T_n^F u\right], \end{aligned}$$

and

$$H_{k_n}^F(s,t)=QH_k^t(s).$$

We obtain

$$\|T^F u-T_n^F u\|_\infty \le c_0 h^8 \sum_{k=1}^m \left\|\frac{\partial^8 H_k}{\partial s^8}\right\|_\infty \|u\|_\infty \quad \text{for some constant } c_0 \text{ independent of } n,$$

hence

$$\|u-u_n^F\|_\infty \le c_1 c_0 h^8 \sum_{k=1}^m \left\|\frac{\partial^8 H_k}{\partial s^8}\right\|_\infty \|u\|_\infty.$$

Letting $\alpha := c_1 c_0$, the desired result is achieved. □

Theorem 14. *Let $H_k(.,.)\in C^8([a,b]\times[a,b])$, $\quad k=1...m$. The following estimate holds:*

$$\|u-u_n^R\|_\infty \le \beta h^8 \sum_{k=1}^m \left\|\frac{\partial^8 H_k}{\partial t^8}\right\|_\infty \|u\|_\infty,$$

where β is a constant independent of n.

Proof. We have

$$\|T-T_n^R\| = \max_{a\le s\le b} \sum_{k=1}^m \int_a^b |H_k(s,t)-H_{k_n}^R(s,t)|\,dt.$$

The operator $I-T_n^R$ is invertible (see [3]), and its inverse is uniformly bounded with respect to n, that is, there exists a constant c_2 such that

$$\sup_n \|(I-T_n^R)^{-1}\|_\infty \leq c_2.$$

Note that

$$u-u_n^R = (I-T_n^R)^{-1}\left[Tu - T_n^R u\right],$$

and

$$H_{k_n}^R(s,t) = QH_k^s(t).$$

It follows that

$$\|T-T_n^R\| \leq h^8 c_3 \sum_{k=1}^m \left\|\frac{\partial^8 H_k}{\partial t^8}\right\|_\infty \|u\|_\infty \quad \text{for some constant } c_3 \text{ independent of } n,$$

hence

$$\|u-u_n^R\|_\infty \leq h^8 c_2 c_3 \sum_{k=1}^m \left\|\frac{\partial^8 H_k}{\partial t^8}\right\|_\infty \|u\|_\infty.$$

Letting $\beta := c_2 c_3$, the desired result is obtained. □

Theorem 15. *Let $H_k(.,.) \in C^8([a,b]\times[a,b])$, $\quad k=1...m$. The following estimate holds:*

$$\|u-u_n^{RF}\|_\infty \leq \gamma h^8 \sum_{k=1}^m \left[\left\|\frac{\partial^8 H_k^t}{\partial s^8}\right\|_\infty + 10\left\|\frac{\partial^8 H_k^s}{\partial t^8}\right\|_\infty\right]\|u\|_\infty,$$

where γ is a constant independent of n.

Proof. Since

$$\|T-T_n^{RF}\|_\infty \leq \sum_{k=1}^m \left[\left\|(I-Q)H_k^t(.)\right\|_\infty + \|(Q\| \left\|(I-Q)H_k^s(.)\right\|_\infty\right],$$

we obtain

$$\|T-T_n^{RF}\|_\infty \leq h^8 \sum_{k=1}^m \left[c_4\left\|\frac{\partial^8 H_k}{\partial s^8}\right\|_\infty + 10 c_5\left\|\frac{\partial^8 H_k^s}{\partial t^8}\right\|_\infty\right].$$

On the other hand,

$$u-u_n^{RF} = (I-T_n^{RF})^{-1}\left[Tu - T_n^{RF}u\right],$$

and

$$\left\|(I - T_n^{RF})^{-1}\right\| \le c_6 \quad \text{for } n \text{ sufficiently large.}$$

Letting $\gamma := c_6 \max c_4 c_5$, the desired result is obtained. □

3.5 Numerical examples

In examples 1 and 2, we compare the results we obtained with previous results presented in [7]. For a high accuracy, we raised the degree to seven, while the mentioned paper worked on degree four. Examples 1 and 2 show the accuracy of our results vis-a-vis of [7]. Denote by R_n^F, R_n^R and R_n^{RF} error terms for the above three septic spline degenerate kernel method respectively. We compare our methods with other methods such as discrete Galerkin methods and discrete collocation methods given in [20], Nyström methods given in [9], Iteration methods given in [8] and Petrov-Galerkin elements via Chebyshev polynomials described in [4].

Example 1

We consider the following Fredholm integral equation

$$u(s) - \frac{1}{2}\int_0^1 (s+1)e^{-st}u(t)dt = e^{-s} - \frac{1}{2} + \frac{1}{2}e^{-(s+1)}, \qquad 0 \le s \le 1.$$

The exact solution is $u(s) = e^{-s}$. We present in Table (3.8) the corresponding absolute errors R_n^F, R_n^R and R_n^{RF} respectively for this example. We compare our results with the results given in [7].

n	R_n^F	R_n^F in [7]	R_n^R	R_n^R in [7]	R_n^{RF}
8	$1.2978e-10$	$4.5-8$	$4.2236e-11$	$1.2e-8$	$1.2533e-10$
16	$5.0865e-13$	$1.6-9$	$1.4638e-13$	$2.6e-10$	$5.2422e-13$
32	$9.8810e-15$	$4.8e-11$	$5.2736e-15$	$4.6e-12$	$2.0817e-15$

Table 3.8: *Example 1*

Example 2

Consider the following Fredholm integral equation

$$u(s) - \int_0^{\frac{\pi}{2}} \sin(s)\cos(t)u(t)\,dt = \sin s, \qquad 0 \le s \le \frac{\pi}{2}.$$

The exact solution $u(s) = 2\sin(s)$.

n	R_n^F	R_n^F in [7]	R_n^R	R_n^R in [7]	R_n^{RF}
8	$3.0468e-008$	$2.7-6$	$4.2236e-11$	$9.5e-7$	$3.4793e-8$
16	$1.0291e-10$	$8.7e-8$	$5.4732e-11$	$9.5e-8$	$1.1094e-10$
32	$3.8120e-13$	$2.7e-9$	$2.9407e-13$	$2.8e-10$	$5.8328e-13$

Table 3.9: *Example 2*

Example 3

Consider the following Fredholm integral equation

$$u(s) + 2\int_0^1 e^{s-t}u(t)\,dt = 2se^s, \qquad 0 \le s \le 1.$$

The exact solution is $u(s) = 2e^s(s - \frac{1}{3})$.

n	R_n^F	R_n^R	R_n^{RF}
8	$1.0587e-9$	$1.2708e-10$	$1.1810e-9$
16	$5.9771e-12$	$1.6609e-13$	$5.8168e-12$
32	$2.7534e-14$	$1.7764e-15$	$2.2871e-14$

Table 3.10: *Example 3*

Example 4 [cf.[20]]

We consider the following Fredholm integral equation

$$u(s) - \frac{1}{2}\int_0^1 e^{st}u(t)\,dt = f(s), \qquad 0 \le s \le 1.$$

The exact solution is $u(s) = e^{-s}\cos s$. We present in Table (3.11) the corresponding absolute errors R_n^F, R_n^R and R_n^{RF} respectively for this example. We compare our results with the results of discrete Galerkin methods and discrete collocation methods respectively, given in [20].

n	R_n^F	R_n^R	R_n^{RF}	Galerkin method in [20]	collocation method in [20]
8	$2.8380e-11$	$5.2729e-11$	$7.1493e-11$	$7.893595e-05$	$6.855828E-05$
16	$8.5876e-14$	$2.4126e-13$	$3.2456e-13$	$9.778065e-06$	$8.364251e-06$

Table 3.11: *Example 4*

Example 5 [cf.[8]]

We consider the following Fredholm integral equation

$$u(s) - \int_0^1 se^{s-t}u(t)\,dt = se^s, \qquad 0 \le s \le 1.$$

The exact solution is $u(s) = 2se^s$. We present in Table (3.12) the corresponding absolute errors R_n^F, R_n^R and R_n^{RF} respectively for this example. We compare our results with the results given in [8] by using iteration methods based on the classical continuous piecewise linear and quadratic Lagrange interpolants. We give the corresponding absolute errors E_1, E_2 respectively.

n	R_n^F	R_n^R	R_n^{RF}	E_1 in [8]	E_2 in [8]
8	$1.8865e-8$	$7.0747e-10$	$1.9520e-08$	$3.05e-02$	$7.02e-05$
16	$5.5321e-11$	$1.9194e-12$	$5.3475e-11$	$7.99e-03$	$4.38e-06$
32	$1.9158e-13$	$1.4211e-14$	$2.0393e-13$	$2.05e-03$	$2.74e-07$

Table 3.12: *Example 5*

Example 6 [cf.[4]]

We consider the following Fredholm integral equation

$$u(s) - \int_0^1 \frac{e^s \sin s}{1+t^2}u(t)\,dt = f(s), \qquad 0 \le s \le 1.$$

The exact solution is $u(s) = s^3$. We present in Table (3.13) the corresponding absolute errors R_n^F, R_n^R and R_n^{RF} respectively for this example. We compare our results with the results of petrov-Galerkin elements via Chebyshev polynomials described in[4], for $k = 1$ and $n = 10$.

n	R_n^F	R_n^R	R_n^{RF}	E_1 in [4]
10	$4.8528e-10$	$2.0806e-07$	$2.0806e-07$	$6.09910e-03$

Table 3.13: *Example 6*

3.6 Conclusions

In this paper, we present three degenerate kernel methods to numerically solve generalized Fredholm integral equation of the second kind, working on the septic spline quasi-interpolants. These methods are constructed to approach the kernel of the correspondent integral operator. While the first method is an approximation on the left, the second is on right. The last method, nevertheless, combines the former two methods. The strength of this combination lies in the reduction of integrals and calculations in the linear system.

CHAPTER 4
TWO SPLINE COLLOCATION METHODS FOR SEVENTH ORDER BOUNDARY VALUE PROBLEMS

Given a real constants $\alpha_i, i = 0,1,2,3, \beta_i, i = 0,1,2$ and a nonlinear function $\psi := \psi(\tau, \omega_0, \ldots, \omega_6)$ on $\mathbb{R}^7$ sufficiently smooth.

The aim of this chapter is to introduce two collocation methods to numerically solve the following nonlinear seventh order boundary value problem:

$$\varphi^{(7)}(\tau) = \psi(\tau, \varphi(\tau), \varphi^{(1)}(\tau), \varphi^{(2)}(\tau), \varphi^{(3)}(\tau), \varphi^{(4)}(\tau), \varphi^{(5)}(\tau), \varphi^{(6)}(\tau)) \qquad a \le \tau \le b, \tag{4.1}$$

$$\varphi^{(k)}(a) = \alpha_k, \quad k = 0\ldots3,$$

$$\varphi^{(k)}(b) = \beta_k, \quad k = 0\ldots2.$$

4.1 Mathematical background

Let us consider the following subdivision of the interval $[a, b]$

$$a = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_n = b,$$

and let us define eight splines s on $[a, b]$, with equally spaced knots

$$\tau_i := a + ih; i = 0,1,\ldots,n,$$

and

$$h := \frac{(b-a)}{n}.$$

The spline $s(.)$ are constricted through the eight B-splines.

Introduce an additional knots

$$\tau_{-8}, \tau_{-7}, \tau_{-6}, \tau_{-5}, \tau_{-4}, \tau_{-3}, \tau_{-2}, \tau_{-1}, \tau_{n+1}, \tau_{n+2}, \tau_{n+3}, \tau_{n+4}, \tau_{n+5}, \tau_{n+6}, \tau_{n+7}, \tau_{n+8},$$

such that

$$\tau_{-8} \leq \tau_{-7} \leq \tau_{-6} \leq \tau_{-5} \leq \tau_{-4} \leq \tau_{-3} \leq \tau_{-2} \leq \tau_{-1} \leq \tau_0,$$

and

$$\tau_n \leq \tau_{n+1} \leq \tau_{n+2} \leq \tau_{n+3} \leq \tau_{n+4} \leq \tau_{n+5} \leq \tau_{n+6} \leq \tau_{n+7} \leq \tau_{n+8}.$$

Define the eighth degree B-splines $B_i(\tau)$ as follows

For $m = 8$

$$B_{i,m}(\tau) := \gamma_{i,m}(\tau) B_{i,m-1}(\tau) + (1 - \gamma_{i+1,m}(\tau)) B_{i+1,m-1}(\tau),$$

where

$$\gamma_{i,m}(\tau) := \begin{cases} \frac{\tau - \tau_i}{\tau_{i+m} - \tau_i} & \text{if} \quad \tau_i \leq \tau \leq \tau_{i+m}. \\ 0 & \text{otherwise.} \end{cases}$$

Or

$$B_i(\tau) := \begin{cases} \sum_{k=i-4}^{i+4} \frac{(\tau_k - \tau)_+^7}{\pi'(\tau_k)} & \text{if} \tau \in [\tau_{i-4}, \tau_{i+4}] \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\pi(\tau) := \prod_{i-4}^{i+4} (\tau - \tau_j)$$

Recall that

$$B_{-8}(\tau), B_{-7}(\tau), B_{-6}(\tau), B_{-5}(\tau), B_{-4}(\tau), B_{-3}(\tau), B_{-2}(\tau), B_{-1}(\tau), B_0(\tau), B_1(\tau), \ldots, B_{n-1}(\tau)$$

forms a basis for the space of the eighth degree polynomial splines.

Following Schoenberg, the eighth degree B-splines are the unique nonzero splines of smallest compact support with the knots at

$$\tau_{-8} \leq \tau_{-7} \leq \cdots \leq \tau_{-1} \leq \tau_0 \ldots < \tau_n \leq \tau_{n+1} \leq \cdots \leq \tau_{n+7} \leq \tau_{n+8}.$$

Moreover,

$$\frac{d}{d\tau} B_i^m(\tau) = \left(\frac{m}{\tau_{i+m} - \tau_i}\right) B_i^{m-1}(\tau) - \left(\frac{m}{\tau_{i+m+1} - \tau_{i+1}}\right) B_{i+1}^{m-1}(\tau)$$

4.2 Collocation method through the eighth degree spline functions

4.2.1 The eighth degree spline interpolant

Theorem 16. *There exist a unique spline interpolant $s \in S_8$ of φ satisfies*

$$s(t_0) = \varphi(t_0); s^{(1)}(t_0) = \varphi^{(1)}(t_0); s^{(2)}(t_0) = \varphi^{(2)}(t_0); s^{(3)}(t_0) = \varphi^{3}(t_0),$$

$$s(t_i) = \varphi(t_i), \qquad i = 1, \dots n+1,$$

$$s(t_{n+2}) = \varphi(t_{n+2});$$

$$s^{(1)}(t_{n+2}) = \varphi^{(1)}(t_{n+2});$$

$$s^{(2)}(t_{n+2}) = \varphi^{(2)}(t_{n+2}),$$

where φ is the exact solution of the above boundary value problem conditions, and

$$t_0 = \tau_0, t_i = \frac{(\tau_i + \tau_{i-1})}{2}, \qquad i = 1, \dots, n,$$

$$t_{n+1} = \tau_{n-1} \quad \textit{and} \quad t_{n+2} = \tau_n.$$

Proof. For $\tau_{i-2} < t \le \tau_{i-1}$, we have

$$\begin{aligned} B_i(t) &= \frac{1}{h^8}[(\tau_{i+7} - t)^8 - 9(\tau_{i+6} - t)^8 + 36(\tau_{i+5} - t)^8 - 84(\tau_{i+4} - t)^8 + 126(\tau_{i+3} - t)^8 \\ &- 126(\tau_{i+2} - t)^8 + 84(\tau_{i+1} - t)^8 - 36(\tau_i - t)^8 + 9(\tau_{i-1} - t)^8]. \end{aligned}$$

For $\tau_{i-1} < t \le \tau_i$, we have

$$\begin{aligned} B_i(t) &= \frac{1}{h^8}[(\tau_{i+7} - t)^8 - 9(\tau_{i+6} - t)^8 + 36(\tau_{i+5} - t)^8 - 84(\tau_{i+4} - t)^8 + 126(\tau_{i+3} - t)^8 \\ &- 126(\tau_{i+2} - t)^8 + 84(\tau_{i+1} - t)^8 - 36(\tau_i - t)^8]. \end{aligned}$$

For $\tau_i < t \le \tau_{i+1}$, we have

$$\begin{aligned} B_i(t) &= \frac{1}{h^8}[(\tau_{i+7} - t)^8 - 9(\tau_{i+6} - t)^8 + 36(\tau_{i+5} - t)^8 - 84(\tau_{i+4} - t)^8 + 126(\tau_{i+3} - t)^8 \\ &- 126(\tau_{i+2} - t)^8 + 84(\tau_{i+1} - t)^8]. \end{aligned}$$

For $\tau_{i+1} < t \le \tau_{i+2}$, we have

$$
\begin{aligned}
B_i(t) &= \frac{1}{h^8}[(\tau_{i+7}-t)^8 - 9(\tau_{i+6}-t)^8 + 36(\tau_{i+5}-t)^8 - 84(\tau_{i+4}-t)^8 + 126(\tau_{i+3}-t)^8 \\
&- 126(\tau_{i+2}-t)^8].
\end{aligned}
$$

For $\tau_{i+2} < t \le \tau_{i+3}$, we have

$$B_i(t) = \frac{1}{h^8}\left[(\tau_{i+7}-t)^8 - 9(\tau_{i+6}-t)^8 + 36(\tau_{i+5}-t)^8 - 84(\tau_{i+4}-t)^8 + 126(\tau_{i+3}-t)^8\right].$$

For $\tau_{i+3} < t \le \tau_{i+4}$, we have

$$B_i(t) = \frac{1}{h^8}\left[(\tau_{i+7}-t)^8 - 9(\tau_{i+6}-t)^8 + 36(\tau_{i+5}-t)^8 - 84(\tau_{i+4}-t)^8\right].$$

For $\tau_{i+4} < t \le \tau_{i+5}$, we have

$$B_i(t) = \frac{1}{h^8}\left[(\tau_{i+7}-t)^8 - 9(\tau_{i+6}-t)^8 + 36(\tau_{i+5}-t)^8\right].$$

For $\tau_{i+5} < t \le \tau_{i+6}$, we have

$$B_i(t) = \frac{1}{h^8}\left[(\tau_{i+7}-t)^8 - 9(\tau_{i+6}-t)^8\right].$$

For $\tau_{i+6} < t \le \tau_{i+7}$, we have

$$B_i(t) = \frac{1}{h^8}\left[(\tau_{i+7}-t)^8\right].$$

Finally,

$$B_i(t) = 0 \quad \text{otherwise}.$$

Consider the following spline s in S_8

$$s(\tau) = \sum_{j=-8}^{n-1} x_j B_j(\tau),$$

which satisfies the given conditions. Under these conditions, we conclude that

$$B_{-8}(a) = B_{n-1}(b) = 1;$$

$$B_{-8}^{(1)}(a) = -B_{n-1}^{(1)}(b) = \frac{-8}{h};$$

$$B_{-8}^{(2)}(a) = B_{n-1}^{(2)}(b) = \frac{56}{h^2};$$

$$B_{-8}^{(3)}(a) = -B_{n-1}^{(3)}(b) = \frac{-336}{h^3};$$

$$B_{-7}(a) = B_{n-1}(b) = 0;$$

$$B_{-7}^{(1)}(a) = -B_{n-1}^{(1)}(b) = \frac{8}{h};$$

$$B_{-7}^{(2)}(a) = B_{n-1}^{(2)}(b) = \frac{-84}{h^2};$$

$$B_{-7}^{(3)}(a) = -B_{n-1}^{(3)}(b) = \frac{588}{h^3}$$

$$B_{-6}(a) = B_{n-2}(b) = 0; B_{-6}^{(1)}(a) = -B_{n-2}^{(1)}(b) = 0;$$

$$B_{-6}^{(2)}(a) = B_{n-2}^{(2)}(b) = \frac{28}{h^2};$$

$$B_{-6}^{(3)}(a) = -B_{n-1}^{(3)}(b) = \frac{-308}{h^3}$$

$$B_{-5}(a) = B_{n-3}(b) = 0;$$

$$B_{-5}^{(1)}(a) = -B_{n-3}^{(1)}(b) = 0;$$

$$B_{-5}^{(2)}(a) = B_{n-3}^{(2)}(b) = 0;$$

$$B_{-5}^{(3)}(a) = -B_{n-3}^{(3)}(b) = \frac{56}{h^3}.$$

Hence

$$x_{-8} = \varphi(a), x_{-7} = \varphi(a) + \frac{h}{8}\varphi^{(1)}(a);$$

$$x_{-6} = \varphi(a) + \frac{h}{8}\varphi^{(1)}(a) + \frac{h^2}{28}\varphi^{(2)}(a);$$

$$x_{-5} = \varphi(a) + \frac{3h}{4}\varphi^{(1)}(a) + \frac{11h^2}{56}\varphi^{(2)}(a) + \frac{h^3}{56}\varphi^{(3)}(a);$$

$$x_{n-1} = \varphi(b);$$

$$x_{n-2} = \varphi(b) - \frac{h}{8}\varphi^{(1)}(b);$$

$$x_{n-3} = \varphi(b) - \frac{h}{8}\varphi^{(1)}(b) + \frac{h^2}{28}\varphi^{(2)}(b).$$

Once the linear system is solved, the other coefficients x_j, $j = -4, \ldots, n-4$, are obtained as a unique solution of this system. □

4.2.2 The eighth degree spline collocation method

Denoting by D^l the differential operator of order l. We call that for all $\varphi \in \mathscr{C}^9([0,1])$, (cf. [29])

$$\|D^l(\varphi - S)\|_\infty = \mathscr{O}(h^{9-l}) \qquad \text{for} \quad l = 0, 1, \ldots, 7.$$

According to the above, for any $\varphi \in \mathscr{C}^9([0,1])$ we have

$$S^{(7)}(t_i) = \psi(t_i, \varphi(t_i), \varphi^{(1)}(t_i), \varphi^{(2)}(t_i), \varphi^{(3)}(t_i), \varphi^{(4)}(t_i), \varphi^{(5)}(t_i), \varphi^{(6)}(t_i)) + \mathscr{O}(h^2), \qquad i = 1, \ldots, n+1.$$

The goal of this section is to introduce a eighth spline collocation method, that means we find a eighth degree spline

$$S_n(\tau) = \lambda(\tau) + \sum_{j=-4}^{n-4} x_j^n B_j(\tau),$$

such that

$$S_n^{(7)}(t_i) = \psi(t_i, S_n(t_i), S_n^{(1)}(t_i), S_n^{(2)}(t_i), S_n^{(3)}(t_i), S_n^{(4)}(t_i), S_n^{(5)}(t_i), S_n^{(6)}(t_i)), \qquad i = 1, \ldots, n+1,$$

where

$$\lambda = \sum_{j=-8}^{-5} x_j B_j + \sum_{j=n-3}^{n-1} x_j B_j.$$

It is clear that equation is a nonlinear system with respect to $X^n = [x_{-4}^n, x_{-3}^n, \ldots, x_{n-4}^n]^T$. Let N_j, $j = -8 \ldots, n-1$, be the B-spline of degree 8 associated with the uniform partion$\{0 = \tau_{-8} = \ldots = \tau_0 < \tau_1 = 1 < \ldots < \tau_{n-1} = n-1 < \tau_n = \ldots = \tau_{n+8} = n\}$ and defined by

$$B_j(x) = N_j\left(\frac{x-a}{h}\right).$$

Therefore,

$$B_j^{(k)}(t_i) = \frac{1}{h^k} N_j^{(k)}\left(\frac{t_i - a}{h}\right).$$

Letting

$$d_{i,j}^k(h) := B_{j-5}^{(k)}(t_i),$$

$$D_k^h := (d_{i,j}^k(h))_{1 \le i,j \le n+1},$$

$$d_{i,j}^{k} := N_{j-5}^{(k)}(\frac{t_i - a}{h}),$$

$$D_k := (d_{i,j}^{k})_{1\leq i,j\leq n+1},$$

then

$$D_k^h = \frac{1}{h^k} D_k, \qquad k = 0, \ldots, 7.$$

Taking

$$X = [x_{-4}, x_{-3}, \ldots, x_{n-4}]^T,$$

The approximate problem reads in the matrix forms as follows

$$D_7^h X = (H - \Lambda) + r,$$

$$D_7^h X^n = (H_{X^n} - \Lambda),$$

where

$$H := \begin{pmatrix} \psi(t_1, \varphi(t_1), \ldots, \varphi^{(6)}(t_1)) \\ \vdots \\ \vdots \\ \psi(t_{n+1}), \varphi(t_{n+1}), \ldots, \varphi^{(6)}(t_{n+1})) \end{pmatrix}$$

$$H_{X^n} := \begin{pmatrix} \psi(t_1, S_n(t_1), \ldots, S_n^{(6)}(t_1)) \\ \vdots \\ \vdots \\ \psi(t_{n+1}, S_n(t_{n+1}), \ldots, S_n^{(6)}(t_{n+1})) \end{pmatrix}$$

$$\Lambda := [\lambda^{(7)}(t_1), \ldots, \lambda^{(7)}(t_{n+1})]^T,$$

and

$$r_i := O(h^2), i = 1, 2, \ldots n + 1.$$

Hence,

$$D_7 X = h^7 (H - \Lambda) + h^7 r$$

$$D_7 X^n = h^7 (H_{X^n} - \Lambda),$$

We assume that the function ψ is L-Lipschitz, that means that, for all $\tau \in [a,b]$ and all $\varphi_i, \omega_i \in \mathbb{R}, \quad i = 0, \ldots 6$, we have

$$\left|\psi(\tau,\varphi_0,\ldots,\varphi_6) - \psi(\tau,\omega_0,\ldots,\omega_6)\right| \leq L\sum_{i=0}^{6}|\varphi_i - \omega_i|, \quad \text{for some positive constant} L,$$

where $L > 0$ is the Lipschitz constant.

In a similar manner, as in [50, 51] it can be seen that

Proposition 17. *We assume that*

$$L\|D_7^{-1}\|_\infty (\sum_{k=0}^{6} h^{7-k}\|D_k\|_\infty) < 1,$$

then there exists a unique eighth degree spline for the above approximate problem.

Proof. We have

$$X^n = h^7 D_7^{-1}(H_{X^n} - \Lambda)$$

For

$$\Omega := X^n,$$

we obtain

$$h^7 D_7^{-1}(H_\Omega - \Lambda) = \Omega,$$

where

$$H_\Omega := \begin{pmatrix} \psi(t_1, S_\Omega(t_1), \ldots, S_\Omega^{(6)}(t_1)) \\ \vdots \\ \vdots \\ \psi(t_{n+1}, S_\Omega(t_{n+1}), \ldots, S_\Omega^{(6)}(t_{n+1})) \end{pmatrix}$$

$$S_\Omega^n = \lambda + \sum_{j=-4}^{n-4} \omega_{j+5} B_j,$$

and

$$\Omega = [\omega_1, \ldots, \omega_{n+1}]^T.$$

Let us consider the following function

$$\phi(\Omega) := h^7 D_7^{-1}(H_\Omega - \Lambda),$$

Our purpose is to prove that the equation

$$\phi(\Omega) := \Omega$$

has a unique solution, that is to say that the function ϕ has a unique fixed point.

For $\Omega_1, \Omega_2 \in \mathbb{R}^{n+1}$, we have

$$\|\phi(\Omega_1) - \phi(\Omega_2)\|_\infty \leq h^7 \|D_7^{-1}\|_\infty \|H_{\Omega_1} - H_{\Omega_2}\|_\infty.$$

Hence

$$\begin{aligned} |\psi(t_i, S_{\Omega_1}(t_i), \ldots, S_{\Omega_1}^{(6)}(t_i)) - \psi(t_i, S_{\Omega_2}(t_i), \ldots, S_{\Omega_2}^{(6)}(t_i))| &\leq L \sum_{k=0}^{6} |S_{\Omega_1}^{(k)} - S_{\Omega_2}^{(k)}| \\ &\leq L(\sum_{k=0}^{6} h^{-k} \|D_k\|_\infty) \|\Omega_1 - \Omega_2\|_\infty. \end{aligned}$$

Thus,

$$\|H_{\Omega_1} - H_{\Omega_2}\|_\infty \leq L(\sum_{k=0}^{6} h^{-k} \|D_k\|_\infty) \|\Omega_1 - \Omega_2\|_\infty.$$

On the other hand

$$\|\phi(\Omega_1) - \phi(\Omega_2)\|_\infty \leq L \|D_7^{-1}\|_\infty (\sum_{k=0}^{6} h^{7-k} \|D_k\|_\infty) \|\Omega_1 - \Omega_2\|_\infty.$$

Clearly that ϕ is a strong contraction mapping, if

$$L\|D_7^{-1}\|_\infty(\sum_{k=0}^{6} h^{7-k}\|D_k\|_\infty) < 1.$$

□

We have

$$D_7(X - X^n) = h^7(H - F_{X^n}) + h^7 r.$$

Proposition 18. *We assume that*

$$L\|D_7^{-1}\|_\infty(\sum_{k=0}^{6} h^{7-k}\|D_k\|_\infty) < \frac{1}{2},$$

then there exists a constant K which depends only on the function ψ such that

$$\|X - X^n\|_\infty \le K h^2.$$

Proof. Since

$$(X - X^n) \le h^7 D_7^{-1}(H - H_{X^n}) + h^7 D_7^{-1} r.$$

and since $r = O(h^2)$, there exists a constant α such that

$$\|r\|_\infty \le \alpha h^2.$$

Hence

$$\|X - X^n\|_\infty \le h_7\|D_7^{-1}\|_\infty\|(H - H_{X^n})\|_\infty + \alpha h^9\|D_7^{-1}\|_\infty.$$

We also have

$$\begin{aligned}
&|\psi(t_i,\varphi(t_i),\dots,\varphi^{(6)}(t_i)) - \psi(t_i,S_n(t_i),\dots,S_n^{(6)}(t_i))| \\
&\qquad \le L\sum_{k=0}^{6}|\varphi^{(k)}(t_i) - S_n^{(k)}(t_i)| \\
&\qquad \le L\sum_{k=0}^{6}|\varphi^{(k)}(t_i) - S^{(k)}(t_i)| + |S^{(k)}(t_i) - S_n^{(k)}(t_i)|.
\end{aligned}$$

Moreover,

$$\|\varphi^{(k)} - S^{(k)}\|_\infty \le \beta_k h^{9-k} \|\varphi^{(9)}\|_\infty, \qquad \text{for} \quad k = 0, \ldots, 7, \quad \text{and for some constant} \quad \beta_k.$$

In addition

$$S^{(k)}(\tau) - S_n^{(k)}(\tau) = \sum_{j=-4}^{n-4} (x_j - x_j^n) B_j^{(k)}(\tau), \qquad k = 0, \ldots, 7,$$

so

$$|S^{(k)}(t_i) - S_n^{(k)}(t_i)| \le \frac{1}{h^k} \|D_k\|_\infty \|X - X^n\|_\infty,$$

and

$$\|(H - H_{X^n})\|_\infty \le L\|X - X^n\|_\infty (\sum_{k=0}^{6} h^{-k} \|D_k\|_\infty) + L(\sum_{k=0}^{6} \beta_k h^{9-k}) \|y^{(9)}\|_\infty.$$

Thus

$$[I - Lh^7 \|D_7^{-1}\|_\infty (\sum_{k=0}^{6} h^{-k} \|D_k\|_\infty)] \|X - X^n\|_\infty$$

$$\le h^7 \|D_7^{-1}\|_\infty [L(\sum_{k=0}^{6} \beta_k h^{9-k}) \|y^{(9)}\|_\infty + \alpha h^2].$$

Finally, if

$$L\|D_7^{-1}\|_\infty (\sum_{k=0}^{6} h^{7-k} \|D_k\|_\infty) \le \frac{1}{2},$$

we get

$$\|X - X^n\|_\infty \le \frac{L(\sum_{k=0}^{6} \beta_k h^{7-k}) \|\varphi^{(9)}\|_\infty + \alpha}{L(\sum_{k=0}^{6} (b-a)^{-k} \|A_k\|_\infty))} h^2.$$

□

Theorem 19. *The eighth degree spline approximation S_n converges quadratically to the exact solution φ of the boundary value problem. Moreover,*

$$\|\varphi - S_n\|_\infty = O(h^2).$$

Proof. We have

$$\|S - S_n\|_\infty \le K h^2.$$

and

$$\|\varphi - S_n\|_\infty \le \|\varphi - S\|_\infty + \|S - S_n\|_\infty,$$

and the result follows. □

4.3 Collocation method using a spline quasi-interpolant

In this section, we present a collocation method based on the eighth quasi-interpolant spline to approximate the solution of the boundary value problem given above. We will recall some results from [19] concerning the eighth quasi-interpolant spline. Based on these results we develop a new method for solving our problem.

4.3.1 Construction of the eighth quasi-interpolant spline

A quasi-interpolant for know function φ is usually obtained as linear combinations of the elements of a suitable set of positive functions with small local supports (see [50, 51]).
A quasi-interpolant of degree 8 which satisfies exactly the boundary condition is a spline operator of the form

$$Q\varphi = \lambda + \sum_{j=-4}^{n-4} c_j(\varphi) B_j,$$

where the coefficients $c_j(\varphi)$ are linear combination of value of φ at some points lying in the neighborhood of Supp (B_j), and in the set

$$T = \{t_1 = \tau_1, t_{i+1} = (\tau_i + \tau_{i-1})/2, i = 1, \dots n, t_{n+2} = \tau_{n-7}, t_{n+3} = \tau_n\}.$$

Based on the fact that

$$Qp = p, \quad \text{for all} \quad p \in \mathbb{P}_8$$

we can calculate the coefficients $c_j(\varphi)$.

We note that

$$\mathrm{Supp}\,(B_j) = [\tau_j, \tau_{j+9}].$$

Moreover,

$$c_j(x^l) = \theta_j^l = \binom{8}{l}^{-1} symm_l(N_j) \quad 0 \le l \le 8.$$

Letting

$$\varphi_i = \varphi(t_i).$$

Hence, the following linear system

$$\theta_j^l = \alpha_j t_1^l + \beta_j t_2^l + \gamma_j t_3^l + \delta_j t_4^l + \omega_j t_5^l + \varrho_j t_6^l + \nu_j t_7^l + \sigma_j t_8^l + \rho_j t_9^l, \qquad 0 \le l \le 8$$

with the solution

$$c_j(\varphi) = \alpha_j\varphi_1 + \beta_j\varphi_2 + \gamma_j\varphi_3 + \delta_j\varphi_4 + \omega_j\varphi_5 + \varrho_j\varphi_6 + \nu_j\varphi_7 + \sigma_j\varphi_8 + \rho_j\varphi_9, \quad j \in \{-4,-3,-2,-1\}.$$

The corresponding octic quasi-interpolant reads as follows

$$Q\varphi(\tau) = \lambda(\tau) + \sum_{j=2}^{n+2} \varphi_j L_j(\tau), \qquad forall \quad \tau \in [a,b],$$

where the fundamental functions L_j are given by

$$L_2 = \frac{804031}{1003520}B_{-5} - \frac{703811}{2150400}B_{-4} - \frac{20416187}{90316800}B_{-3} + \frac{3997009}{12902400}B_{-2} - \frac{414599}{3225600}B_{-1} + \frac{329389}{51609600}B_0$$

$$L_3 = \frac{179827}{184320}B_{-5} + \frac{40614481}{19353600}B_{-4} + \frac{13552337}{38707200}B_{-3} - \frac{38158213}{38707200}B_{-2} + \frac{4852543}{9676800}B_{-1} - \frac{166603}{2150400}B_0 + \frac{329389}{51609600}B_1$$

$$L_4 = -\frac{829839}{716800}B_{-5} - \frac{720991}{512000}B_{-4} + \frac{39308753}{21504000}B_{-3} - \frac{47833403}{21504000}B_{-2} - \frac{8703173}{5376000}B_{-1} + \frac{5701483}{12902400}B_0 - \frac{166603}{2150400}B_1 + \frac{329389}{51609600}B_2$$

$$L_5 = \frac{200987}{200704}B_{-5} + \frac{76597}{86016}B_{-4} - \frac{6278591}{3612672}B_{-3} - \frac{29381}{73728}B_{-2} + \frac{62983}{18432}B_{-1} - \frac{9982663}{6451200}B_0 + \frac{5701483}{12902400}B_1 - \frac{166603}{2150400}B_2 + \frac{329389}{51609600}B_3$$

$$L_6 = -\frac{470843}{774144}B_{-5} - \frac{5122423}{11612160}B_{-4} + \frac{24024829}{23224320}B_{-3} - \frac{3988961}{23224320}B_{-2} - \frac{9214969}{5806080}B_{-1} + \frac{5768617}{1720320}B_0 - \frac{9982663}{6451200}B_1 + \frac{5701483}{12902400}B_2 - \frac{166603}{2150400}B_3 + \frac{329389}{51609600}B_4$$

$$L_7 = \frac{385437}{1576960}B_{-5} + \frac{1209667}{7884800}B_{-4} - \frac{19011623}{47308800}B_{-3} + \frac{6902227}{47308800}B_{-2} + \frac{5418503}{11827200}B_{-1} - \frac{9982663}{6451200}B_0 + \frac{5768617}{1720320}B_1 - \frac{9982663}{6451200}B_2 + \frac{5701483}{12902400}B_3 - \frac{166603}{2150400}B_4 + \frac{329389}{51609600}B_5$$

$$L_8 = -\frac{108653}{1863680}B_{-5} - \frac{918017}{27955200}B_{-4} + \frac{15571273}{167731200}B_{-3} - \frac{568529}{12902400}B_{-2} - \frac{3413453}{41932800}B_{-1} + \frac{5701483}{12902400}B_0 - \frac{9982663}{6451200}B_1 + \frac{5768617}{1720320}B_2 - \frac{9982663}{6451200}B_3 + \frac{5701483}{12902400}B_4 - \frac{166603}{2150400}B_5 + \frac{166603}{2150400}B_6$$

$$L_9 = \frac{31361}{5017600}B_{-5} + \frac{313123}{96768000}B_{-4} - \frac{13116403}{1354752000}B_{-3} + \frac{1019321}{193536000}B_{-2} + \frac{329389}{48384000}B_{-1} - \frac{166603}{2150400}B_0 \frac{5701483}{12902400}B_1 - \frac{9982663}{6451200}B_2 + \frac{5768617}{1720320}B_3 - \frac{9982663}{6451200}B_4 + \frac{5701483}{12902400}B_5 - \frac{166603}{2150400}B_6 + \frac{329389}{51609600}B_7$$

$$L_j = \frac{329389}{51609600}(B_{j-2} + B_{j-10}) - \frac{166603}{2150400}(B_{j-3} + B_{j-9}) + \frac{5701483}{12902400}(B_{j-4} + B_{j-8}) - \frac{9982663}{6451200}(B_{j-5} + B_{j-7}) + \frac{5768617}{1720320}B_{j-6}, \qquad 10 \le j \le n-7$$

$$L_{n-6} = -\frac{241393}{90316800}B_{n-5} + \frac{31361}{5017600}B_{n-6} + \frac{313123}{96768000}B_{n-7} - \frac{13116403}{1354752000}B_{n-8} + \frac{1019321}{193536000}B_{n-9} + \frac{329389}{48384000}B_{n+10} - \frac{166603}{2150400}B_{n-11} + \frac{5701483}{12902400}B_{n-12} - \frac{9982663}{6451200}B_{n-13} + \frac{5768617}{1720320}B_{n-14} - \frac{9982663}{6451200}B_{n-15} + \frac{5701483}{12902400}B_{n-16} - \frac{166603}{2150400}B_{n-17} + \frac{329389}{51609600}B_{n-18}$$

$$L_{n-5} = \frac{18853}{745472}B_{n-5} - \frac{108653}{1863680}B_{n-6} - \frac{918017}{27955200}B_{n-7} + \frac{15571273}{167731200}B_{n-8} - \frac{568529}{12902400}B_{n-9} - \frac{3413453}{41932800}B_{n-10} + \frac{5701483}{12902400}B_{n-11} - \frac{9982663}{6451200}B_{n-12} + \frac{5768617}{1720320}B_{n-13} - \frac{9982663}{6451200}B_{n-14} + \frac{5701483}{12902400}B_{n-15} - \frac{166603}{2150400}B_{n-16} + \frac{166603}{2150400}B_{n-17}$$

$$L_{n-4} = -\frac{68217}{630784}B_{n-5} + \frac{385437}{1576960}B_{n-6} + \frac{1209667}{7884800}B_{n-7} - \frac{19011623}{47308800}B_{n-8}$$
$$+\frac{6902227}{47308800}B_{n-9} + \frac{5418503}{11827200}B_{n-10} - \frac{9982663}{6451200}B_{n-11} + \frac{5768617}{1720320}B_{n-12} -$$
$$\frac{9982663}{6451200}B_{n-13} + \frac{5701483}{12902400}B_{n-14} - \frac{166603}{2150400}B_{n-15} + \frac{329389}{51609600}B_{n-16}$$
$$L_{n-3} = \frac{143005}{516096}B_{n-5} - \frac{470843}{774144}B_{n-6} - \frac{5122423}{11612160}B_{n-7} + \frac{24024829}{23224320}B_{n-8}$$
$$-\frac{3988961}{23224320}B_{n-9} - \frac{9214969}{5806080}B_{n-10} + \frac{5768617}{1720320}B_{n-11} - \frac{9982663}{6451200}B_{n-12}$$
$$+\frac{5701483}{12902400}B_{n-13} - \frac{166603}{2150400}B_{n-14} + \frac{329389}{51609600}B_{n-15}$$
$$L_{n-2} = -\frac{192035}{401408}B_{n-5} + \frac{200987}{200704}B_{n-6} + \frac{76597}{86016}B_{n-7} - \frac{6278591}{3612672}B_{n-8}$$
$$-\frac{29381}{73728}B_{n-9} + \frac{62983}{18432}B_{n-10} - \frac{9982663}{6451200}B_{n-11}$$
$$+\frac{5701483}{12902400}B_{n-12} - \frac{166603}{2150400}B_{n-13} + \frac{329389}{51609600}B_{n-14}$$
$$L_{n-1} = \frac{868323}{1433600}B_{n-5} - \frac{829839}{716800}B_{n-6} - \frac{720991}{512000}B_{n-7} + \frac{39308753}{21504000}B_{n-8}$$
$$-\frac{47833403}{21504000}B_{n-9} - \frac{8703173}{5376000}B_{n-10}$$
$$+\frac{5701483}{12902400}B_{n-11} - \frac{166603}{2150400}B_{n-12} + \frac{329389}{51609600}B_{n-13}$$
$$L_{n} = -\frac{337489}{516096}B_{n-5} + \frac{179827}{184320}B_{n-6} + \frac{40614481}{19353600}B_{n-7} + \frac{13552337}{38707200}B_{n-8}$$
$$-\frac{38158213}{38707200}B_{n-9} + \frac{4852543}{9676800}B_{n-10} - \frac{166603}{2150400}B_{n-11} + \frac{329389}{51609600}B_{n-12}$$
$$L_{n+1} = \frac{577729}{401408}B_{n-5} + \frac{804031}{1003520}B_{n-6} - \frac{703811}{2150400}B_{n-7} - \frac{20416187}{90316800}B_{n-8}$$
$$+\frac{3997009}{12902400}B_{n-9} - \frac{414599}{3225600}B_{n-10} + \frac{329389}{51609600}B_{n-11}$$
$$L_{n+2} = -\frac{164249}{1576575}B_{n-5} - \frac{968183}{4729725}B_{n-6} + \frac{646049}{10135125}B_{n-7}$$
$$+\frac{4975343}{70945875}B_{n-8} - \frac{66427}{779625}B_{n-9} + \frac{329389}{10135125}B_{n-10}.$$

In addition, the approximate function of the lth derivative of a function φ at the ith point t_i is given by the following

$$\varphi^{(l)}(t_i) \approx Q^{(l)}\varphi(t_i) = \lambda^{(l)}(t_i) + \sum_{j=2}^{n+2} \varphi_j L_j^{(l)}(t_i), \qquad i = 2, \dots, n+2.$$

The following estimate holds for all function $\varphi \in \mathscr{C}^9([a,b])$

$$\|D^l(I-Q)\varphi\|_\infty \leq \delta_l h^{9-l} \|\varphi^{(9)}\|_\infty, \quad \text{for some positive constant} \quad \delta_l, \qquad l = 0, \dots, 7.$$

Let us consider the approximate operator $Q(\varphi^n)$ defined by

$$Q(\varphi^n) = \lambda + \sum_{j=2}^{n+2} \varphi_j^n L_j,$$

and let us consider the following approximate problem

$$D^7 Q\varphi(t_i) = \psi(t_i, \varphi(t_i), \varphi^{(1)}(t_i), \varphi^{(2)}(t_i), \varphi^{(3)}(t_i), \varphi^{(4)}(t_i), \varphi^{(5)}(t_i), \varphi^{(6)}(t_i)) + O(h^2), \qquad i = 2, \dots, n+2.$$

We obtain

$$D^7 Q(\varphi^n)(t_i) = \psi(t_i, Q(\varphi^n)(t_i), Q^{(1)}(\varphi^n)(t_i), \cdots, Q^{(6)}(\varphi^n)(t_i)) + O(h^2),$$

$$i = 2, \dots, n+2.$$

Setting

$$R_k^h := (r_{i,j}^k(h))_{1\leq i,j\leq n+1},$$

$$r_{i,j}^k(h) := L_{1+j}^{(k)}(t_{i+1}),$$

$$R_k = R_k^1$$

$$\Gamma := [\varphi_2, \varphi_3, \dots, \varphi_{n+2}]^T,$$

$$\Gamma^n = [\varphi_2^n, \varphi_3^n, \dots, \varphi_{n+2}^n]^T.$$

$$R_k^h = \frac{1}{h^k} R_k, \qquad k = 0, \dots, 7,$$

$$G = -[\lambda^{(7)}(t_2), \dots, \lambda^{(7)}(t_{n+2})]^T, r_i = O(h^9), i = 1, \dots, n+1.$$

Thus,

$$R_7\Gamma = h^7(H_\Gamma + G) + r,$$

$$R_7\Gamma^n = h^7(H_{\Gamma^n} + G),$$

where

$$H_\Gamma = \begin{pmatrix} \psi(t_2, \varphi(t_2), \ldots, \varphi^{(6)}(t_2) \\ \vdots \\ \vdots \\ \psi(t_{n+2}, \varphi(t_{n+2}), \ldots, \varphi^{(6)}(t_{n+2}) \end{pmatrix}$$

$$H_{\Gamma^n} = \begin{pmatrix} \psi(t_2, \varphi_n(t_2), \ldots, \varphi_n^{(6)}(t_2)) \\ \vdots \\ \vdots \\ \psi(t_{n+2}, \varphi_n(t_{n+2}), \ldots, \varphi_n^{(6)}(t_{n+2})). \end{pmatrix}$$

Proposition 20. *We assume that*

$$L\|R_7^{-1}\|_\infty(\sum_{k=0}^{6} h^{7-k}\|R_k\|_\infty) < 1,$$

then there exists a unique The eighth degree spline that approximates the exact solution φ of the problem () with boundary conditions ().

Proof. We have

$$\Gamma^n = h^7 R_7^{-1}(H_{\Gamma^n} + G).$$

For

$$\Omega = \Gamma^n$$

we get

$$\phi(\Omega) = h^7 R_7^{-1}(H_\Omega + G) = \Omega,$$

with

$$H_\Omega = \begin{pmatrix} \psi(t_2, Q_\omega(t_2), \ldots, Q_\omega^{(6)}(t_2) \\ \vdots \\ \vdots \\ \psi(t_{n+2}, Q_\omega(t_{n+2}), \ldots, Q_\omega^{(6)}(t_{n+2})) \end{pmatrix}$$

and

$$Q_\Omega = \lambda + \sum_{j=2}^{n+2} \omega_{j-1} L_j,$$

and

$$\Omega = [\omega_1, \ldots, \omega_{n+1}]^T.$$

Let $\Omega_1, \Omega_2 \in \mathbb{R}^{n+1}$, hence

$$\|\phi(\Omega_1) - \phi(\Omega_2)\|_\infty \leq h\|R_7^{-1}\|_\infty \|H_{\Omega_1} - H_{\Omega_2}\|_\infty,$$

and hence

$$\begin{aligned} |(\psi(t_i, Q_{\omega_1}(t_i), \ldots, Q_{\omega_1}^{(6)}(t_i)) - (\psi(t_i, Q_{\omega_2}(t_i), \ldots, Q_{\omega_2}^{(6)}(t_i))| \\ \leq L \sum_{k=0}^{6} \|Q_{\Omega_1}^{(k)}(t_i) - Q_{\Omega_2}^{(k)}(t_i\|_\infty \\ \leq L(\sum_{k=0}^{6} h^{-k} \|R_k\|_\infty) \|\Omega_1 - \Omega_2\|_\infty \end{aligned}$$

Thus,

$$\|H_{\Omega_1} - H_{\Omega_2}\|_\infty \leq L(\sum_{k=0}^{6} h^{-k} \|R_k\|_\infty) \|\Omega_1 - \Omega_2\|_\infty,$$

for some constant L such that

$$L\|R_7^{-1}\|_\infty (\sum_{k=0}^{6} h^{7-k} \|R_k\|_\infty) < 1$$

is a strong contraction mapping. Moreover, the function ϕ has a unique fixed point. □

We note that

$$(\Gamma - \Gamma^n) = h^7 R_7^{-1} (H_\Gamma - H_{\Gamma^n}) + r.$$

Conclusions and perspectives

In this thesis, the septic quasi-interpolants are developed with all their coefficients. A new numerical schemes based on quasi-interpolants spline and collocation methods have been constructed and justified to numerically solve some functional problems, especially, Fredholm integral equations of the second kind.

We have presented two methods based on spline interpolants to investigate a generalized boundary value problems. We introduced two computational collocation methods with high accuracy for solving an eigenvalue problems of an integral operator with a regular kernel. These methods are founded on the use of spline quasi-interpolants.

This work may be extended to other type of functional equations. These methods can be applied to nonlinear integral and integro-differential equations, but some modifications are required.

In future works we hope to use spline degenerate kernel methods for approximate the solution of the integral equations of the form

$$u(s) - \sum_{k=1}^{m} \int_a^b H_k(s,t,\psi(t))\,u(t)\,dt = f(s), \qquad m \in \mathbb{N}^*, \quad a \le s \le b,$$

$$u(s) - \sum_{k=1}^{m} \int_a^b H_k(s,t,\psi(t))\ln|s-t|\,u(t)\,dt = f(s), \qquad m \in \mathbb{N}^*, \quad a \le s \le b,$$

$$a(s)u(s) - \frac{b(s)}{\pi}\int_{-1}^{1} \frac{k(s,t,\psi(t))}{s-t}\,u(t)\,dt = f(s), \quad , \quad -1 \le s \le 1.$$

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